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ON THE CONSTRUCTION OF NADEL MULTIPLIER IDEAL SHEAVES AND THE LIMITING BEHAVIOR OF THE RICCI FLOW

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Abstract. In this note we construct Nadel multiplier ideal sheaves using the Ricci flow on Fano manifolds. This extends a result of Phong, Šešum, and Sturm. These sheaves, like their counterparts constructed by Nadel for the continuity method, can be used to obtain an existence criterion for Kähler-Einstein metrics.

1. Introduction

In this note we construct Nadel multiplier ideal sheaves on Fano manifolds that do not admit Kähler-Einstein metrics, using the Ricci flow. The result is a simple consequence of the uniformity of the Poincaré and Sobolev inequalities along the flow. Aside from giving a description of the limiting behavior of the Ricci flow on such manifolds, this allows one to obtain another proof of the convergence of the Ricci flow on a certain class of Fano manifolds.

The theory of obstructions to the existence of canonical Kähler metrics (see, e.g., [F]) has a long history starting with the observation that a Kähler-Einstein manifold must have a definite or zero first Chern class. Lichnerowicz and Matsushima proved that for a constant scalar curvature Kähler manifold the group of automorphisms $\mathrm{Aut}(M, \mathrm{J})$ is a complexification of the group of isometries. Later it was shown that on such a manifold the closed 1-form on the space of Kähler metrics $\mathcal{H}_\Omega$ (see the end of this section for notation and definitions) defined by the scalar curvature minus its average must be basic with respect to $\mathrm{Aut}(M, \mathrm{J})_0$, that is admit an $\mathrm{Aut}(M, \mathrm{J})_0$-invariant potential function, or equivalently its Futaki character must vanish. Kobayashi and Lübke proved that the tangent bundle of a Kähler-Einstein manifold is semistable.

In contrast to these necessary conditions, there came work on certain sufficient conditions for the existence of Kähler-Einstein metrics on Fano manifolds. Siu, Tian, and Yau showed that certain finite groups of symmetries can be used to this end and produced several examples [Si, T1, TY]. Tian studied the singular locus of sequences of plurisubharmonic functions and introduced a sufficient condition in terms of a related holomorphic invariant [T1]: $\alpha_M(G) > n/(n+1)$, where G is a

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compact subgroup of $\mathrm{Aut}(M,\mathrm{J})$ and

$$\alpha_M(G) = \sup\{\ \alpha\ :\ \int_M e^{-\alpha(\varphi-\sup\varphi)}\omega^n < C_\alpha, \quad \forall\,\varphi\in\mathcal{H}_\omega(G), \quad [\omega]=c_1\}.$$

Then, Nadel introduced the notion of a multiplier ideal sheaf on a compact Kähler manifold and showed that the nonexistence of certain such G-invariant sheaves implies that $\alpha_M(G)\geq 1$ [N]. This construction is related to a theory introduced earlier by Kohn in a different context [K].

Tian translated the failure of the continuity method when a Kähler-Einstein metric does not exist to the statement that a certain subsequence of Kähler potentials along Aubin's continuity path [A2]

$$\omega^n_{\varphi_t} = \omega^n e^{f_\omega - t\varphi_t}, \quad t\in[0,t_0),\ t_0\leq 1, \tag{1}$$

will diverge along a subvariety in a manner that can be ruled out when $\alpha_M(G)$ is large enough. Nadel showed that furthermore the blow-up will occur along a subscheme cut out by a coherent sheaf of multiplier ideals satisfying certain cohomology vanishing conditions. These results provide a powerful tool in showing existence of Kähler-Einstein metrics, since these conditions are often violated in specific examples. This technique was revisited by Demailly and Kollár who also extended it to orbifolds [DK].

Nadel's main result can be stated as follows.[1]

Theorem 1.1 ([N]). *Let (M,J) be a Fano manifold not admitting a Kähler-Einstein metric. Let $\gamma\in(\frac{n}{n+1},\infty)$ and let $\omega\in\mathcal{H}_{c_1}$. Then there exists a subsequence $\{\varphi_{t_j}\}_{j\geq 0}$ of solutions of* (1) *such that $\varphi_{t_j}-\sup\varphi_{t_j}$ converges in the $L^1(M,\omega)$-topology to $\varphi_\infty\in PSH(M,\mathrm{J},\omega)$ and $\mathcal{I}(\gamma\varphi_\infty)$ is a proper multiplier ideal sheaf satisfying*

$$H^r(M,\mathcal{I}(\gamma\varphi_\infty)\otimes K_M^{-\lfloor\gamma\rfloor})=0, \quad \forall\, r\geq 1. \tag{2}$$

The sheaves with $\gamma<1$ will be referred to as Nadel sheaves (see Definition 1.5 below).

The Ricci flow, introduced by Hamilton [H1], provides another method for constructing Kähler-Einstein metrics on a Fano manifold, and it is therefore natural to ask whether this method will also yield multiplier ideal obstruction sheaves in the absence of a Kähler-Einstein metric. It may be written as a flow equation on the space of Kähler potentials $\mathcal{H}_\omega$,

$$\omega^n_{\varphi_t} = \omega^n e^{f_\omega-\varphi_t+\dot\varphi_t}, \quad \varphi(0)=\varphi_0. \tag{3}$$

This question was first addressed by Phong, Šešum, and Sturm who proved the following result.

Theorem 1.2 ([PSS]). *Let (M,J) be a Fano manifold not admitting a Kähler-Einstein metric. Let $\gamma\in(1,\infty)$ and let $\omega\in\mathcal{H}_{c_1}$. Then there exist an initial condition φ_0 and a subsequence $\{\varphi_{t_j}\}_{j\geq 0}$ of solutions of* (3) *such that $\varphi_{t_j}-\frac{1}{V}\int_M\varphi_{t_j}\omega^n$ converges in the $L^1(M,\omega)$-topology to $\varphi_\infty\in PSH(M,\mathrm{J},\omega)$ and $\mathcal{I}(\gamma\varphi_\infty)$ is a proper multiplier ideal sheaf satisfying* (2).

[1] This theorem makes use of Nadel's vanishing theorem; see Section 2 for the statement. Also, Nadel's original statement includes (for simplicity) only the case $\gamma\in(\frac{n}{n+1},1)$; however later he comments on the possibility of extending the allowed interval for γ [N, p. 579].

The sheaves thus obtained require the exponent to lie in a smaller interval than in Theorem 1.1; that is, they are not Nadel sheaves. This is a crucial difference between these results. Indeed, the smaller the exponent γ, the stronger the vanishing theorem satisfied by the corresponding multiplier ideal sheaf. The gist of Theorem 1.1 is to use functions invariant under a compact (not necessarily maximally compact) subgroup of automorphisms G in order to obtain G-invariant subschemes satisfying in addition certain cohomology vanishing restrictions. When $\gamma \in (n/(n+1), 1)$, the sheaves constructed by Nadel are not only G-invariant, but also satisfy

$$H^r(M, \mathcal{I}(\gamma\varphi_\infty)) = 0, \quad \forall\, r \geq 0, \tag{4}$$

while this need not hold for sheaves corresponding to exponents $\gamma > 1$. The subschemes cut out by sheaves satisfying (4) satisfy various restrictions, and it is these restrictions that render the continuity method useful in proving existence of Kähler-Einstein metrics on a large class of manifolds. For example, to state the simplest restrictions, Nadel shows that (4) implies that the corresponding subscheme is connected and has arithmetic genus zero and that if it is 1-dimensional, it is a tree of rational curves. It is not clear how to use the sheaves with $\gamma > 1$ to prove such results.

The main result of this note is that the Ricci flow does produce Nadel sheaves, with $\gamma \in (n/(n+1), 1)$.

Theorem 1.3. *Let (M, J) be a Fano manifold not admitting a Kähler-Einstein metric. Let $\gamma \in (n/(n+1), \infty)$ and let $\omega \in \mathcal{H}_{c_1}$. Then there exists an initial condition φ_0 and a subsequence $\{\varphi_{t_j}\}_{j\geq 0}$ of solutions of* (3) *such that $\varphi_{t_j} - \frac{1}{V}\int_M \varphi_{t_j}\omega^n$ converges in the $L^1(M, \omega)$-topology to $\varphi_\infty \in PSH(M, \mathrm{J}, \omega)$ and $\mathcal{I}(\gamma\varphi_\infty)$ is a proper multiplier ideal sheaf satisfying* (2).

Perelman proved that when a Kähler-Einstein metric exists, the Ricci flow will converge to it in the sense of Cheeger-Gromov. Therefore the following corollary, due to Nadel and Perelman, is known:

Corollary 1.4 ([N, TZ])**.** *Let (M, J) be a Fano manifold and let G be a compact subgroup of* $\mathrm{Aut}(M, \mathrm{J})$*. Assume that (M, J) does not admit a G-invariant Nadel sheaf as in Definition* 1.5*. Then the Ricci flow will converge in the sense of Cheeger-Gromov to a Kähler-Einstein metric starting with any G-invariant Kähler metric.*

Observe that Theorem 1.3 allows one to obtain another proof of this corollary. This proof differs from the one obtained by the combination of the results of Nadel and Perelman only in the method of obtaining the L^∞ estimate (and not in the method of obtaining higher order estimates and convergence [CT, PSS, TZ]). It has the virtue of simultaneously proving that a Kähler-Einstein metric exists and that the flow will converge to it (instead of first using the continuity method to prove that a Kähler-Einstein metric exists and then using this fact via properness of an energy functional and Kołodziej's results [Ko] to obtain the L^∞ estimate).

In particular, this gives another proof of the theorem that the two-sphere S^2 (and more generally, complex projective space $\mathbb{P}^n$) may be uniformized using the Ricci flow[2] ([CLT, CT, Ch1, H2] and references therein). Still more generally, Corollary

[2]Indeed, observe that the group of automorphisms generated by the rotations $z \mapsto ze^{\sqrt{-1}\theta}$ and the inversion $z \mapsto 1/z$ acts without fixed points on S^2 while a 0-dimensional Nadel subscheme is a single reduced point.

1.4 applies also to symmetric toric Fano manifolds (see [So]), and actually the proof of Theorem 1.3 implies that whenever $\alpha_M(G) > \frac{n}{n+1}$ the same is true.

We emphasize that no new techniques are needed beyond those in the continuity method setting described above and our purpose in this note is solely to point out this similarity between the limiting behavior of the Ricci flow and that of the more classical continuity method. We believe Theorem 1.3 adds important information regarding the limiting behavior of the Ricci flow beyond that in Theorem 1.2.

The proof of Theorem 1.3 differs from that of Theorem 1.2 in that it follows the lines of the original continuity method results [A2, N, Si, T1] instead of appealing to results of Kołodziej. This is also the key to obtaining the result for singularity exponents in the whole interval $(n/(n+1), \infty)$. The crucial ingredient that makes this possible is that the relevant estimates on the Ricci flow established by Perelman, Ye, and Zhang, some of which appeared after the work of Phong-Šešum-Sturm, allow one to adapt the continuity method arguments to the setting of the flow.[3] This is described in Section 2. Section 3 contains some remarks, including a brief conjectural discussion on a possible extension of the result to the setting of constant scalar curvature metrics.

Setup and notation. Let (M, J) be a connected compact closed Kähler manifold of complex dimension n and let $\Omega \in H^2(M, \mathbb{R})$ be a Kähler class. Let $d = \partial + \bar\partial$ and define the Laplacian $\Delta = -\bar\partial \circ \bar\partial^\star - \bar\partial^\star \circ \bar\partial$ with respect to a Riemannian metric g on M and assume that J is compatible with g and parallel with respect to its Levi-Cività connection. Let $\omega := \omega_g = \sqrt{-1}/2\pi \cdot g_{i\bar j}(z) dz^i \wedge d\bar z^j$ denote its corresponding Kähler form, a closed positive $(1,1)$-form on (M, J). Let H_g denote the Hodge projection operator from the space of closed forms onto the kernel of Δ. Let $V = \Omega^n([M]) = \int_M \omega^n$. Denote by $\mathcal{H}_\Omega$ the space of Kähler forms representing Ω.

Let $PSH(M, \mathrm{J}, \omega) \subseteq L^1_{\mathrm{loc}}(M)$ denote the set of ω-plurisubharmonic functions. Define the space of smooth strictly ω-plurisubharmonic functions (Kähler potentials)

$$\mathcal{H}_\omega = \{\varphi \in C^\infty(M) \,:\, \omega_\varphi := \omega + \sqrt{-1}\partial\bar\partial\varphi > 0\}.$$

Let $\mathrm{Ric}\,\omega = -\sqrt{-1}/2\pi \cdot \partial\bar\partial \log\det(g_{i\bar j})$ denote the Ricci form of ω. It is well-defined globally and represents the first Chern class $c_1 := c_1(T^{1,0}M, \mathrm{J}) \in H^2(M, \mathbb{Z})$. Alternatively it may be viewed as minus the curvature form of the canonical line bundle K_M, the top exterior product of the holomorphic cotangent bundle $T^{1,0\,\star}M$. One calls (M, J) Fano when $c_1 > 0$. One calls ω Kähler-Einstein if $\mathrm{Ric}\,\omega = a\omega$ for some real a. Let $f_\omega \in \varphi \in C^\infty(M)$ denote the unique function satisfying $\sqrt{-1}\partial\bar\partial f_\omega = \mathrm{Ric}\,\omega - \omega$ and $V^{-1}\int_M e^{f_\omega}\omega^n = 1$.

Let $\mathrm{Aut}(M, \mathrm{J})_0$ denote the identity component of the complex Lie group $\mathrm{Aut}(M, \mathrm{J})$ of automorphisms (biholomorphisms) of (M, J) and denote by $\mathrm{aut}(M, \mathrm{J})$ its Lie algebra of infinitesimal automorphisms composed of real vector fields X satisfying $\mathcal{L}_X \mathrm{J} = 0$. Denote by $\mathcal{H}_\omega(G) \subseteq \mathcal{H}_\omega$ the subspace of G-invariant potentials.

For $\varphi \in PSH(M, \mathrm{J}, \omega)$ define the multiplier ideal sheaf associated to φ to be the sheaf $\mathcal{I}(\varphi)$ defined for each open set $U \subseteq M$ by local sections

$$\mathcal{I}(\varphi)(U) = \{h \in \mathcal{O}_M(U) : |h|^2 e^{-\varphi} \in L^1_{\mathrm{loc}}(M)\}. \tag{5}$$

[3] With the exception of the case $n = 1$ ($M = S^2$) that does not make use of Ye and Zhang's estimate.

Such sheaves are coherent [D, p. 73], [N]. Such a sheaf is called proper if it is neither zero nor the structure sheaf $\mathcal{O}_M$.

Definition 1.5 ([N, Definition 2.4])**.** A proper multiplier ideal sheaf $\mathcal{I}(\varphi)$ will be called a Nadel sheaf whenever there exists $\epsilon > 0$ such that $(1+\epsilon)\varphi \in PSH(M, \mathrm{J}, \omega)$.

According to Theorem 2.4 such sheaves satisfy (4). Define the complex singularity exponent $c_\omega(\varphi)$ of a function $\varphi \in PSH(M, \mathrm{J}, \omega)$ by

$$c_\omega(\varphi) = \sup\{\, \gamma \,:\, \int_M e^{-\gamma\varphi}\omega^n < \infty\}. \tag{6}$$

Note that $\int_M e^{-\gamma\varphi}\omega^n = \infty$ implies that $\int_M e^{-(\gamma+\epsilon)\varphi}\omega^n = \infty$ for any $\epsilon > 0$.

Denote by $\lfloor x \rfloor$ the largest integer not larger than x.

2. Proof of the main result

The proof is split into steps similarly to the setting of the continuity method. First, we obtain an upper bound for $\frac{1}{V}\int_M -\varphi_t\omega^n_{\varphi_t}$ in terms of $\frac{1}{V}\int_M \varphi_t\omega^n$. Second, we show that if the complex singularity exponents of the functions φ_t are uniformly bigger than $n/(n+1)$, then one has a uniform upper bound on $\frac{1}{V}\int_M \varphi_t\omega^n$, and hence on $\sup\varphi_t$. Third, we show that $-\inf\varphi_t$ is uniformly bounded from above in terms of $\frac{1}{V}\int_M -\varphi_t\omega^n_{\varphi_t}$. Fourth, we construct the Nadel multiplier ideal sheaf.

We turn to the proof. We assume throughout that $n > 1$ (the remaining case will be handled separately at the end). First we recall some necessary facts and estimates.

Consider the Ricci flow on a Fano manifold

$$\begin{aligned} \frac{\partial\omega(t)}{\partial t} &= -\operatorname{Ric}\omega(t) + \omega(t), \quad t \in \mathbb{R}_+, \\ \omega(0) &= \omega, \end{aligned} \tag{7}$$

and a corresponding flow equation on the space of Kähler potentials $\mathcal{H}_\omega$,

$$\omega^n_{\varphi_t} = \omega^n e^{f_\omega - \varphi_t + \dot\varphi_t}, \quad \varphi(0) = c_0, \quad t \in \mathbb{R}_+. \tag{8}$$

This flow exists for all $t > 0$ [Ca]. Here c_0 is a constant uniquely determined by ω [PSS, (2.10)]. This choice is necessary in order to have the second estimate of Theorem 2.1(i) below [CT, §10.1], [PSS, §2].

Let

$$||\psi||_{L^p(M,\omega)} = \Big(\frac{1}{V}\int_M \psi^p\omega^n\Big)^{\frac{1}{p}},$$

and let

$$\begin{aligned} ||\psi||^2_{W^{1,2}(M,\omega)} &= ||\nabla\psi||^2_{L^2(M,\omega)} + ||\psi||^2_{L^2(M,\omega)} \\ &= \frac{1}{V}\int_M n\sqrt{-1}\partial\psi\wedge\bar\partial\psi\wedge\omega^{n-1} + \frac{1}{V}\int_M \psi^2\omega^n. \end{aligned}$$

We will make essential use of the following estimates of Perelman, Ye, and Zhang.

Theorem 2.1. *Let (M, J) be a Fano manifold of complex dimension $n > 1$ and let φ_t satisfy* (8)*. There exist $C_1, C_2 > 0$ depending only on (M, J, ω) such that:*

(i) [ST, TZ] *One has*

$$||f_{\omega_{\varphi_t}}||_{L^\infty(M)} \le C_1, \quad ||\dot\varphi_t||_{L^\infty(M)} \le C_1, \quad \forall\, t > 0. \tag{9}$$

(ii) [Ye, Z] *For all $\psi \in W^{1,2}(M,\omega)$ one has*

$$||\psi||_{L^{\frac{2n}{n-1}}(M,\omega_{\varphi_t})} \le C_2 ||\psi||_{W^{1,2}(M,\omega_{\varphi_t})}, \quad \forall\, t > 0. \tag{10}$$

Following Aubin [A2], define functionals on $\mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$ by

$$I(\omega,\omega_\varphi) = V^{-1}\int_M \sqrt{-1}\partial\varphi \wedge \bar\partial\varphi \wedge \sum_{l=0}^{n-1} \omega^{n-1-l} \wedge \omega_\varphi^l = V^{-1}\int_M \varphi(\omega^n - \omega_\varphi^n), \tag{11}$$

$$J(\omega,\omega_\varphi) = \frac{V^{-1}}{n+1}\int_M \sqrt{-1}\partial\varphi \wedge \bar\partial\varphi \wedge \sum_{l=0}^{n-1} (n-l)\omega^{n-l-1} \wedge \omega_\varphi^l. \tag{12}$$

Following Ding [Di], define a functional on $\mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$ by

$$F(\omega,\omega_\varphi) = -(I-J)(\omega,\omega_\varphi) - \frac{1}{V}\int_M \varphi\omega_\varphi^n - \log\frac{1}{V}\int_M e^{f_\omega - \varphi}\omega^n. \tag{13}$$

It is exact; that is to say it satisfies the cocycle condition $F(\omega_1,\omega_2) + F(\omega_2,\omega_3) = F(\omega_1,\omega_3)$. Its critical points are precisely Kähler-Einstein metrics. The following monotonicity result is well-known (see, e.g., [CT, Lemma 3.7]).

Lemma 2.2. *The functional F is monotonically decreasing along the flow* (7).

(i) First step. As a consequence of Lemma 2.2 and (9) we have

$$\begin{aligned} 0 \ge F(\omega,\omega_{\varphi_t}) &= -(I-J)(\omega,\omega_{\varphi_t}) - \frac{1}{V}\int_M \varphi_t\omega_{\varphi_t}^n - \log\frac{1}{V}\int_M e^{-\dot\varphi_t}\omega_{\varphi_t}^n \\ &\ge -(I-J)(\omega,\omega_{\varphi_t}) - \frac{1}{V}\int_M \varphi_t\omega_{\varphi_t}^n - C_1. \end{aligned} \tag{14}$$

From (11)-(12) it follows that $\frac{1}{n+1}I \le J$. We then have

$$\begin{aligned} \frac{1}{V}\int_M -\varphi_t\omega_{\varphi_t}^n &\le (I-J)(\omega,\omega_{\varphi_t}) + C_1 \\ &\le \frac{n}{n+1}I(\omega,\omega_{\varphi_t}) + C_1 = \frac{n}{n+1}\frac{1}{V}\int_M \varphi_t(\omega^n - \omega_{\varphi_t}^n) + C_1. \end{aligned}$$

Hence,

$$\frac{1}{V}\int_M -\varphi_t\omega_{\varphi_t}^n \le \frac{n}{V}\int_M \varphi_t\omega^n + (n+1)C_1. \tag{15}$$

This completes the first step of the proof.

(ii) Second step. Assume $\gamma > 0$ is such that

$$\frac{1}{V}\int_M e^{-\gamma(\varphi_t - \frac{1}{V}\int_M \varphi_t\omega^n)}\omega^n \le C,$$

with C independent of t. Using the flow equation, rewrite this as

$$\frac{1}{V}\int_M e^{(1-\gamma)\varphi_t + \gamma\frac{1}{V}\int_M \varphi_t\omega^n - \dot\varphi_t - f_\omega}\omega_{\varphi_t}^n \le C.$$

Jensen's inequality gives

$$\frac{1}{V}\int_M \Big((1-\gamma)\varphi_t + \gamma\frac{1}{V}\int_M \varphi_t\omega^n - \dot\varphi_t - f_\omega\Big)\omega_{\varphi_t}^n \le \log C.$$

Using (9) and (15), we obtain

$$\gamma \frac{1}{V}\int_M \varphi_t \omega^n \le (1-\gamma)\frac{1}{V}\int_M -\varphi_t \omega_{\varphi_t}^n + C' \le n(1-\gamma)\frac{1}{V}\int_M \varphi_t\omega^n + C''.$$

Whenever $\gamma \in (\frac{n}{n+1}, 1)$, this yields an a priori estimate on $\frac{1}{V}\int_M \varphi_t \omega^n$.

Under this assumption this also implies an a priori upper bound on $\sup \varphi_t$. Indeed, let G_ω be a Green function for $-\Delta = -\Delta_\omega$ satisfying $\int_M G_\omega(\cdot, y)\omega^n(y) = 0$. Set $A_\omega = -\inf_{M\times M} G_\omega$. Recall the sub-mean value property of ω-plurisubharmonic functions:

$$\varphi(p) \le \frac{1}{V}\int_M \varphi\omega^n + nA_\omega, \quad \forall\, p \in M. \tag{16}$$

In fact, by assumption $-n < \Delta\varphi$. The Green formula gives

$$\begin{aligned}\varphi(p) - \frac{1}{V}\int_M \varphi\omega^n &= -\frac{1}{V}\int_M G_\omega(p,y)\Delta\varphi(y)\omega^n(y)\\ &= \frac{1}{V}\int_M (G_\omega(p,y)+A_\omega)(-\Delta\varphi(y))\omega^n(y) \le nA_\omega,\end{aligned}$$

by the normalization of G_ω. This completes the second step.

(iii) Third step. This step follows from an argument used by Tian [T1] for the continuity method. It adapts to our setting thanks to Theorem 2.1.

Put $\eta = \sup_M \varphi_t - \varphi_t + 1$ and let $p > 0$. The first part of the argument involves reducing the $L^\infty(M)$ estimate for η to an $L^2(M, \omega_{\varphi_t})$ estimate. First,

$$\begin{aligned}\int_M \eta^p \omega_{\varphi_t}^n &\ge \int_M \eta^p(\omega_{\varphi_t}^n - \omega_{\varphi_t}^{n-1}\wedge\omega) = -\int_M \eta^p \sqrt{-1}\partial\bar\partial\eta\wedge\omega_{\varphi_t}^{n-1}\\ &= \int_M \sqrt{-1}\partial(\eta^p)\wedge\bar\partial\eta\wedge\omega_{\varphi_t}^{n-1}\\ &= \frac{4p}{(p+1)^2}\int_M \sqrt{-1}\partial\eta^{\frac{p+1}{2}}\wedge\bar\partial\eta^{\frac{p+1}{2}}\wedge\omega_{\varphi_t}^{n-1}.\end{aligned} \tag{17}$$

Combined with (10), this gives

$$\frac{1}{C_2^2}||\eta||^{p+1}_{L^{\frac{(p+1)n}{n-1}}(M,\omega_{\varphi_t})} \le \frac{n(p+1)^2}{4p}||\eta||^p_{L^p(M,\omega_{\varphi_t})} + ||\eta||^{p+1}_{L^{p+1}(M,\omega_{\varphi_t})}. \tag{18}$$

Following Tian, a Moser iteration argument [T1, pp. 235-236] now allows us to conclude that there exists a constant C depending only on C_2 and (M, J, ω) such that

$$\sup\eta \le C||\eta||_{L^2(M,\omega_{\varphi_t})}.$$

The second part of the argument requires a uniform Poincaré inequality in order to bound the $L^2(M, \omega_{\varphi_t})$ norm of η in terms of its $L^1(M, \omega_{\varphi_t})$ norm. Recall the following weighted Poincaré inequality [F, §2.4] (see also [TZ]).

Lemma 2.3. *Let (M, J) be a Fano manifold. Then for any function $\psi \in W^{1,2}(M, \omega)$,*

$$\frac{1}{V}\int_M \big(\psi - \frac{1}{V}\int_M \psi e^{f_\omega}\omega^n\big)^2 e^{f_\omega}\omega^n \le \frac{1}{V}\int_M n\sqrt{-1}\partial\psi\wedge\bar\partial\psi\wedge e^{f_\omega}\omega^{n-1}.$$

As a corollary we have, thanks to (9), a uniform Poincaré-type inequality along the flow:

$$e^{-2C_1}||\eta||^2_{L^2(M,\omega_{\varphi_t})} - e^{C_1}||\eta||^2_{L^1(M,\omega_{\varphi_t})} \le ||\nabla\eta||^2_{L^2(M,\omega_{\varphi_t})}. \tag{19}$$

Therefore, applying (17), now with $p = 1$, combined with (19), we obtain

$$e^{-2C_1}||\eta||^2_{L^2(M,\omega_{\varphi_t})} - e^{C_1}||\eta||^2_{L^1(M,\omega_{\varphi_t})} \le n||\eta||_{L^1(M,\omega_{\varphi_t})},$$

which completes the second part of the argument.

Finally,

$$||\eta||_{L^1(M,\omega_{\varphi_t})} = 1 + \sup \varphi_t + \frac{1}{V}\int_M -\varphi_t \omega^n_{\varphi_t}, \tag{20}$$

and this is uniformly bounded in terms of $\frac{1}{V}\int_M \varphi_t \omega^n$ thanks to (15) and (16). This completes the third step.

(iv) Fourth step. Assume that (M, J) does not admit a Kähler-Einstein metric. The relevant theory of complex Monge-Ampère equations due to Aubin and Yau [A1, Y] now implies that $\{||\varphi_t||_{L^\infty(M)}\}_{t\in[0,\infty)}$ is unbounded. If not, one would have uniform higher-order estimates on φ_t, and properties of Mabuchi's K-energy [M] (equivalent in many ways to F) then show that a subsequence will converge to a smooth Kähler-Einstein metric (see [PSS, §2]). Combining the first three steps, this implies that for each $m \in \mathbb{N}$ we may find an unbounded increasing subsequence of times $\{t_{j(m)}\}_{j(m)\ge 1} \subseteq [0, \infty)$ for which

$$\lim_{j\to\infty} \int_M e^{-(\frac{n}{n+1}+\frac{1}{m})\left(\varphi_{t_{j(m)}} - \frac{1}{V}\int_M \varphi_{t_{j(m)}}\omega^n\right)}\omega^n = \infty.$$

Hence, by the diagonal argument, there exists a subsequence of potentials $\{\varphi_{t_j}\}_{j\ge 1}$ for which (one may equivalently work throughout with $\sup \varphi_{t_j}$ instead of $\frac{1}{V}\int_M \varphi_{t_j}\omega^n$)

$$\lim_{j\to\infty} \int_M e^{-\gamma\left(\varphi_{t_j} - \frac{1}{V}\int_M \varphi_{t_j}\omega^n\right)}\omega^n = \infty, \quad \forall\, \gamma \in (n/(n+1), \infty). \tag{21}$$

One may further extract an unbounded increasing sub-subsequence of times such that $\{\varphi_{t_{j_k}} - V^{-1}\int_M \varphi_{t_{j_k}}\omega^n\}_{k\ge 1}$ converges in the $L^1(M, \omega)$-topology to a limit $\varphi_\infty \in PSH(M, \mathrm{J}, \omega)$ [DK, pp. 549-550]. The Demailly-Kollár lower semi-continuity of complex singularity exponents then implies

$$c_\omega(\varphi_\infty) \le \liminf_{k\to\infty} c_\omega(\varphi_{t_{j_k}} - V^{-1}\textstyle\int_M \varphi_{t_{j_k}}\omega^n) \le \dfrac{n}{n+1}. \tag{22}$$

Equivalently,

$$||e^{-\gamma\varphi_\infty}||_{L^1(M,\omega)} = \infty, \quad \forall\, \gamma \in (n/(n+1), \infty),$$

and the multiplier ideal sheaf $\mathcal{I}(\gamma\varphi_\infty)$ defined for each open set $U \subseteq M$ by local sections $\mathcal{I}(\gamma\varphi_\infty)(U) = \{h \in \mathcal{O}_M(U) : |h|^2 e^{-\gamma\varphi_\infty} \in L^1_{\mathrm{loc}}(M)\}$ is not $\mathcal{O}_M$. It is also not zero since φ_∞ is not identically $-\infty$ as its average is zero.

Finally, we recall a version of Nadel's vanishing theorem.

Theorem 2.4 ([N]). *Let (M, J, ω) be a Hodge manifold and (L, h) an ample holomorphic line bundle over M equipped with a smooth Hermitian metric with positive curvature form Ψ_h given locally by $-\sqrt{-1}/2\pi \cdot \partial\bar\partial \log h$. Assume that $(1+\epsilon)\varphi \in PSH(M, \mathrm{J}, \Psi_h)$ for some $\epsilon > 0$. Then*

$$H^r(M, \mathcal{I}(\varphi) \otimes K_M \otimes L) = 0, \quad \forall\, r \ge 1. \tag{23}$$

The cohomology vanishing statement (2) for the sheaf $\mathcal{I}(\gamma\varphi_\infty)$ just constructed is a consequence of Theorem 2.4 with $L = K_M^{-\lfloor\gamma\rfloor-1}$ since

$$(\lfloor\gamma\rfloor + 1)\omega + \gamma\sqrt{-1}\partial\bar\partial\varphi_\infty \ge (\lfloor\gamma\rfloor + 1 - \gamma)\omega.$$

This concludes the proof of Theorem 1.3 for $n > 1$.

To treat the remaining case $n = 1$ ($M = S^2$), replace the third step with the following argument. First, one has a uniform lower bound for the scalar curvature along the flow (see, e.g., [C]), i.e., a lower bound on the Ricci curvature when $n = 1$. Next, by Perelman [ST] a uniform diameter bound holds along the flow. Therefore the quantity $\operatorname{diam}(M, g_{\omega_{\varphi_t}})^2 \cdot \inf\{\operatorname{Ric}\omega_{\varphi_t}(v,v) \,:\, ||v||_{g_{\omega_{\varphi_t}}} = 1\}$ is uniformly bounded from below. Applying the Bando-Mabuchi Green's function estimate [BM, Theorem 3.2] now implies that $A_{\omega_{\varphi_t}}$ (see (16) for notation) is uniformly bounded from above (here we invoked Perelman's diameter bound again). Since $\Delta_{\omega_{\varphi_t}}\varphi_t < 1$, Green's formula gives a uniform bound for $-\inf\varphi_t$ in terms of $\frac{1}{V}\int_M -\varphi_t\omega_{\varphi_t}$. Now the first, second, and fourth steps apply without change to give the desired result. □

3. Remarks and further study

We end with some remarks.

We saw that the uniform Sobolev inequality of Ye and Zhang can be used instead of the Kołodziej theorem in order to obtain the L^∞ estimate for Corollary 1.4. Observe that this also applies to the proof of the main theorem in [TZ], at least in the case of no holomorphic vector fields. Indeed, by (16), (20) and (15), $\sup\varphi_t$ and $-\inf_{\varphi_t}$ are bounded in terms of $\frac{1}{V}\int\varphi_t\omega^n$. By (8) and (9) one has $\frac{1}{V}\int\varphi_t\omega^n \le I(\omega,\omega_{\varphi_t}) + C_1$. Hence, if a Kähler-Einstein metric exists, the properness of F in the sense of Tian [T3] implies a uniform L^∞ bound on φ_t (compare [PSS, §3]). The same applies also to Pali's theorem [P].

The statement of Theorem 1.1 can be refined to hold for all $\gamma \in (t_0 n/(n+1), \infty)$ with $t_0 = t_0(\omega) \le 1$ defined to be the first time for which $\{||\varphi_t||_{L^\infty(M)}\}_{t\in[0,t_0)}$ is unbounded, with φ_t a solution of (1) [T1, p. 234] (see also [N, p. 582]). One has the inequality $t_0 \le \sup\{\, b \,:\, \operatorname{Ric}\omega \ge b\omega,\ \omega \in \mathcal{H}_{c_1},\ b \in \mathbb{R}\}$, the right hand side a holomorphic invariant studied by Tian; on some Fano manifolds it is smaller than 1 [T2]. We do not know whether for these manifolds the exponent in Theorem 1.3 can be lowered as well. Perhaps the difference here is related to the fact that the flow always exists for all time unlike the continuity path. Yet, as far as the usefulness of the sheaves is concerned, this does not seem to be crucial since they all satisfy the same vanishing conditions (4) for exponents smaller than 1. It is worth mentioning that this invariant is smaller than 1 only when the functional F is not bounded from below [BM, DT]. It would be very interesting to know what can be said regarding the converse (compare [R1, §1]). We are therefore led to pose the following problem:

Problem 3.1. On a Fano manifold, determine whether the lower boundedness of the functional F (equivalently, of Mabuchi's K-energy) on $\mathcal{H}_{c_1}$ is equivalent to $\sup\{\, b \,:\, \operatorname{Ric}\omega \ge b\omega,\ \omega \in \mathcal{H}_{c_1},\ b \in \mathbb{R}\} = 1$.

The Kähler-Ricci flow has been widely studied for manifolds whose first Chern class is definite or zero (see, e.g., [Ch2]). This flow is simply the dynamical system induced by integrating minus the Ricci potential vector field $-f$ on the space of Kähler forms $\mathcal{H}_\Omega$. A vector field ψ on the space of Kähler forms is an assignment $\omega \mapsto \psi_\omega \in \varphi \in C^\infty(M)/\mathbb{R}$. The vector field f is the assignment $\omega \mapsto f_\omega$ with f_ω defined by $\operatorname{Ric}\omega - \mu\omega = \sqrt{-1}\partial\bar\partial f_\omega,\ \mu \in \mathbb{R}$.

Motivated by this observation, one is naturally led to extend the definition of the Kähler-Ricci flow to an arbitrary Kähler manifold, simply by defining the flow lines to be integral curves of minus the Ricci potential vector field $-f$ on $\mathcal{H}_\Omega$, with Ω an arbitrary Kähler class. Recall that the Ricci potential is defined in general by $\operatorname{Ric}\omega - H_\omega \operatorname{Ric}\omega = \sqrt{-1}\partial\bar\partial f_\omega$. The resulting flow equation can also be written as

$$\begin{aligned} \frac{\partial\omega(t)}{\partial t} &= -\operatorname{Ric}\omega(t) + H_t \operatorname{Ric}\omega(t), \quad t \in \mathbb{R}_+, \\ \omega(0) &= \omega, \end{aligned} \tag{24}$$

for each t for which a solution exists in $\mathcal{H}_\Omega$. This flow, introduced by Guan, is part of the folklore in the field although it has not been much studied.[4]

Several authors have raised the question whether Nadel's construction can be extended to the study of constant scalar curvature Kähler metrics. We believe that Nadel-type obstruction sheaves should arise from this dynamical system (as well as from its discretization [R2, R3]; in these two references a "discrete" analogue of Theorem 1.2 was shown to hold; see [R3, Theorem 10.15], and note "ψ_{j_k}" there should be typed "$\psi_{j_k} - \frac{1}{V}\int_M \psi_{j_k}\omega^n$") in the absence of fixed points.

As we saw, it is important to make a choice of how to induce a flow on $\mathcal{H}_\omega$ from that on $\mathcal{H}_\Omega$, and different normalizations give rise to different sheaves. The flow equation (8) corresponds to restricting the evolution to a certain codimension one submanifold of $\mathcal{H}_\omega$. For a general Kähler class one may define an operator on the space of Kähler forms $\mathcal{H}_\Omega$, identified as an open subset of $\varphi \in C^\infty(M)/\mathbb{R}$, by

$$h : \varphi \in \mathcal{H}_\Omega \mapsto h(\varphi) \in C^\infty(M)/R, \tag{25}$$

with

$$H_{\omega_\varphi} \operatorname{Ric}\omega_\varphi - H_\omega \operatorname{Ric}\omega = \sqrt{-1}\partial\bar\partial h(\varphi). \tag{26}$$

By choosing such an appropriate submanifold, analogously to (8), one may consider the flow

$$\omega^n_{\varphi_t} = \omega^n e^{f_\omega - h(\varphi_t) + \dot\varphi_t}\omega^n. \tag{27}$$

The difficulty lies in the fact that this operator is in general no longer a multiple of the identity.

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[4] It seems that Guan first considered this flow in unpublished work in the 1990s (see references to [G1]). After posting the first version of this note, I became aware, thanks to G. Székelyhidi, of a recent paper [G2] by Guan in which this flow is studied. For a different but related flow see [S].

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