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BOUNDEDNESS OF FOURIER INTEGRAL OPERATORS ON $\mathcal{F}L^p$ SPACES

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ABSTRACT. We study the action of Fourier Integral Operators (FIOs) of Hörmander's type on $\mathcal{F}L^p(\mathbb{R}^d)_{\rm comp}$, $1 \le p \le \infty$. We see, from the Beurling-Helson theorem, that generally FIOs of order zero fail to be bounded on these spaces when $p \neq 2$, the counterexample being given by any smooth non-linear change of variable. Here we show that FIOs of order $m = -d|1/2 - 1/p|$ are instead bounded. Moreover, this loss of derivatives is proved to be sharp in every dimension $d \ge 1$, even for phases which are linear in the dual variables. The proofs make use of tools from time-frequency analysis such as the theory of modulation spaces.

1. INTRODUCTION

Consider the spaces $\mathcal{F}L^p(\mathbb{R}^d)_{\rm comp}$ of compactly supported distributions whose Fourier transform is in $L^p(\mathbb{R}^d)$, with the norm $\|f\|_{\mathcal{F}L^p} = \|\hat{f}\|_{L^p}$. Let ϕ be a smooth function. Then it follows from the Beurling-Helson theorem ([1], see also [15]) that the map $Tf(x) = a(x)f(\phi(x))$, $a \in C_0^\infty(\mathbb{R}^d)$, generally fails to be bounded on $\mathcal{F}L^p(\mathbb{R}^d)_{\rm comp}$ if ϕ is non-linear. This is of course a Fourier integral operator of special type, namely $Tf(x) = \int e^{2\pi i \phi(x)\eta} a(x)\hat{f}(\eta)\, d\eta$, by the Fourier inversion formula.

In this paper we study the action on $\mathcal{F}L^p(\mathbb{R}^d)_{\rm comp}$ of Fourier Integral Operators (FIOs) of the form

$$Tf(x) = \int e^{2\pi i \Phi(x,\eta)} \sigma(x,\eta) \hat{f}(\eta)\, d\eta. \tag{1}$$

The symbol σ is in $S^m_{1,0}$, the Hörmander class of order m. Namely, $\sigma \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ and satisfies

$$|\partial_x^\alpha \partial_\eta^\beta \sigma(x,\eta)| \le C_{\alpha,\beta} \langle \eta \rangle^{m-|\beta|}, \quad \forall (x,\eta) \in \mathbb{R}^{2d}. \tag{2}$$

We suppose that σ has compact support with respect to x.

The phase $\Phi(x,\eta)$ is real-valued, positively homogeneous of degree 1 in η, and smooth on $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$. It is actually sufficient to assume a $\Phi(x,\eta)$ that is defined on an open subset $\Lambda \subset \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$, conic in dual variables, and containing the points $(x,\eta) \in \operatorname{supp} \sigma$, $\eta \neq 0$. More precisely, after setting

$$\Lambda' = \{(x,\eta) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) : (x, \lambda\eta) \in \operatorname{supp} \sigma \text{ for some } \lambda > 0\},$$

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we require that Λ contains the closure of Λ' in $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$. We also assume the non-degeneracy condition

$$\det \left(\frac{\partial^2 \Phi}{\partial x_i \partial \eta_l}\Big|_{(x,\eta)} \right) \neq 0 \quad \forall (x,\eta) \in \Lambda. \tag{3}$$

It is easy to see that such an operator maps the space $\mathcal{S}(\mathbb{R}^d)$ of Schwartz functions into the space $\mathcal{C}_0^\infty(\mathbb{R}^d)$ of test functions continuously. We refer to the books [13, 24] for the general theory of FIOs, and especially to [20, 21] for results in L^p.

As the above example shows, general boundedness results in $\mathcal{F}L^p(\mathbb{R}^d)_{\mathrm{comp}}$ are expected only for FIOs of negative order. Our main result deals precisely with the minimal loss of derivatives for boundedness to hold.

Theorem 1.1. *Assume the above hypotheses on the symbol σ and the phase Φ. If*

$$m \leq -d \left| \frac{1}{2} - \frac{1}{p} \right|, \tag{4}$$

then the corresponding FIO T, initially defined on $\mathcal{C}_0^\infty(\mathbb{R}^d)$, extends to a bounded operator on $\mathcal{F}L^p(\mathbb{R}^d)_{\mathrm{comp}}$, whenever $1 \leq p < \infty$. For $p = \infty$, T extends to a bounded operator on the closure of $\mathcal{C}_0^\infty(\mathbb{R}^d)$ in $\mathcal{F}L^\infty(\mathbb{R}^d)_{\mathrm{comp}}$.

The loss of derivatives in (4) is proved to be sharp in any dimension $d \geq 1$, even for phases $\Phi(x,\eta)$ which are linear in η (see Section 6). In contrast, notice that FIOs are continuous on $L^p(\mathbb{R}^d)_{\mathrm{comp}}$ if

$$m \leq -(d-1) \left| \frac{1}{2} - \frac{1}{p} \right|,$$

and this threshold is sharp. In particular, for $d = 1$, the continuity is attained without loss of derivatives, i.e., $m = 0$ ([21, Theorem 2, page 402]; see also [23]).

As a model, consider the following simple example. In dimension $d = 1$, take the phase $\Phi(x,\eta) = \varphi(x)\eta$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism, with $\varphi(x) = x$ for $|x| \geq 1$ and whose restriction to $(-1,1)$ is non-linear. Then consider the FIO

$$Tf(x) = \int_{\mathbb{R}} e^{2\pi i \varphi(x)\eta} G(x) \langle \eta \rangle^m \hat{f}(\eta)\, d\eta, \tag{5}$$

with $G \in \mathcal{C}_0^\infty(\mathbb{R})$, $G(x) = 1$ for $|x| \leq 1$. Moreover, let $1 \leq p \leq 2$. Then Theorem 1.1 and the discussion in Section 6 below show that $T : \mathcal{F}L^p(\mathbb{R})_{\mathrm{comp}} \to \mathcal{F}L^p(\mathbb{R})_{\mathrm{comp}}$ is bounded if and only if m satisfies (4), with $d = 1$.

The techniques employed to prove Theorem 1.1 differ from those used in [21, Theorem 2, page 402] and [23] for the local L^p-boundedness. Indeed, in [21] the main idea was to split the frequency-localized operator, of order $-(d-1)/2$, into a sum of $O(2^{j(d-1)/2})$ FIOs whose phases are essentially linear in η (there $\langle \eta \rangle \asymp 2^j$), hence satisfying the desired estimates without loss of derivatives. Similarly, in [23], the main point was a decomposition of T into a part with phase which is non-degenerate with respect to η (and further factorized) and a degenerate part, with a phase closer to the linear case, fulfilling better estimates. However, as example (5) shows, the case of linear phases in η already contains all the obstructions to the local $\mathcal{F}L^p$-boundedness, so that we cannot here take advantage of this kind of decomposition. Instead, the proof of Theorem 1.1 makes use of the theory of modulation spaces M^p, $1 \leq p \leq \infty$, which are now classical function spaces used in time-frequency analysis (see [8, 9, 12] and Section 2 for the definition and properties).

In short, we say that a temperate distribution f belongs to $M^p(\mathbb{R}^d)$ if its short-time Fourier transform $V_g f(x,\eta)$, defined in (8) below, is in $L^p(\mathbb{R}^{2d})$. Then, $\|f\|_{M^p} := \|V_g f\|_{L^p}$. Here g is a non-zero (so-called window) function in $\mathcal{S}(\mathbb{R}^d)$, and changing $g \in \mathcal{S}(\mathbb{R}^d)$ produces equivalent norms. The space $\mathcal{M}^\infty(\mathbb{R}^d)$ is the closure of $\mathcal{S}(\mathbb{R}^d)$ in the M^∞-norm. For heuristic purposes, distributions in M^p may be regarded as functions which are locally in $\mathcal{F}L^p$ and decay at infinity like a function in L^p.

Modulation spaces are relevant here since, for distributions supported in a fixed compact subset $K \subset \mathbb{R}^d$, there exists $C_K > 0$ such that

$$C_K^{-1}\|u\|_{M^p} \leq \|u\|_{\mathcal{F}L^p} \leq C_K\|u\|_{M^p}$$

(see [8, 9, 11] for even more general embeddings, and the subsequent Lemma 2.1). Then, our framework can be shifted from $\mathcal{F}L^p$ to M^p spaces and, using techniques from time-frequency analysis, Theorem 1.1 shall be proven in a slightly wider generality:

Theorem 1.2. *Assume the above hypotheses on the symbol σ and the phase Φ. Moreover, assume* (4). *Then, if $1 \leq p < \infty$, the corresponding FIO T, initially defined on $\mathcal{S}(\mathbb{R}^d)$, extends to a bounded operator on M^p; if $p = \infty$, the operator T extends to a bounded operator on $\mathcal{M}^\infty$.*

Since the proof of Theorem 1.2 is quite technical, for the benefit of the reader we first exhibit the general pattern. We start by splitting up the symbol $\sigma(x,\eta)$ of (1) into the sum of two symbols supported where $|\eta| \leq 4$ and $|\eta| \geq 2$, respectively. These symbols give rise to two FIOs T_1 and T_2, which will be studied separately.

The operator T_1 carries the singularity of the phase Φ at the origin and is proved to be bounded on M^1 and on $\mathcal{M}^\infty$. The boundedness on M^p, for $1 < p < \infty$, follows from complex interpolation. Precisely, the boundedness on M^1 is straightforward. For the $\mathcal{M}^\infty$-case, we use the fact that $e^{2\pi i\Phi(x,\eta)}\chi(\eta)$, χ being a cutoff function which localizes near the origin, is in $\mathcal{F}L^1_\eta$, uniformly with respect to x (cf. Theorem 4.2).

For the operator T_2, which carries the oscillations at infinity, we still prove the desired boundedness result on M^1 and on $\mathcal{M}^\infty$ in the case $m = -d/2$. Here the general case follows by interpolation with the well-known case $M^2 = L^2$, $m = 0$; see e.g. [21, page 397].

To chase our goal, we perform a dyadic decomposition of the symbol (of order $m = -d/2$) in shells where $|\eta| \asymp 2^j$, $j \geq 0$, obtaining the representation $T_2 = \sum_{j=1}^\infty T^{(j)}$. Each dyadically localized operator $T^{(j)}$ is then conjugated with the dilation operators $U_{2^{j/2}}f(x) = f(2^{j/2}x)$, so that

$$T^{(j)} = U_{2^{j/2}}\tilde{T}^{(j)}U_{2^{-j/2}}, \tag{6}$$

where $\tilde{T}^{(j)}$ is an FIO with phase

$$\Phi_j(x,\eta) = \Phi(2^{-j/2}x, 2^{j/2}\eta) = 2^{j/2}\Phi(2^{-j/2}x,\eta). \tag{7}$$

Now, $\Phi_j(x,\eta)$ has derivatives of order ≥ 2 bounded on the support of the corresponding symbol, and, as a consequence, the operators $\tilde{T}^{(j)}$ are bounded on M^1 and $\mathcal{M}^\infty$, with operator norm $\lesssim 2^{-jd/2}$. These results are established and proved in Propositions 3.2 and 3.3; they can be seen as a microlocal version of [4, 6], where M^p-boundedness (without loss of derivatives) was proved for FIOs with phase function possessing globally bounded derivatives of order ≥ 2 (extending results for

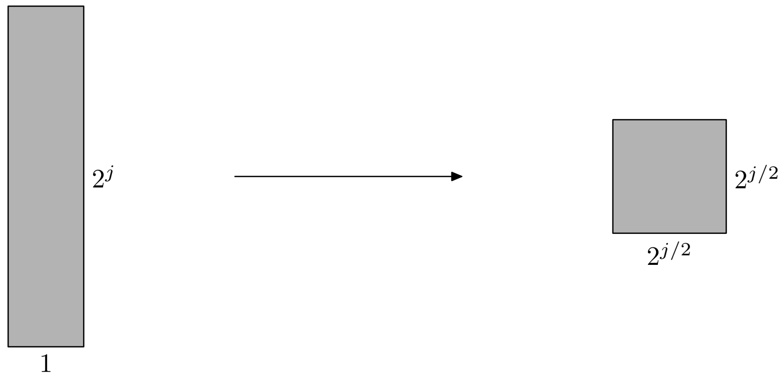

FIGURE 1

$p = 2$ in [3]). Combining this with boundedness results for dilation operators on modulation spaces [22], we obtain the boundedness of $T^{(j)}$ uniformly with respect to j. In this way, the assumption (4) is used to compensate the bound for the norm of the dilation operator.

A last non-trivial technical problem is summing (on $j \geq 1$) the corresponding estimates. For this, we make use of the almost orthogonality of the $T^{(j)}$. As a tool, for M^1 we will need an equivalent characterization of the M^1-norm; see (11) below. On the other hand, for the $\mathcal{M}^\infty$-case, we essentially use the following property:

If $\hat{u}$ is localized in the shell $|\eta| \asymp 2^j$, then $\widehat{Tu}$ is localized in the neighbor shells.

In practice, this is achieved by means of another dyadic partition in the frequency domain and the composition formula for a pseudodifferential operator and an FIO (see Theorem 3.1).

Finally, we observe that the above trick of conjugating with dilations has a nice interpretation in terms of the geometry of the symbol classes, namely, the associated partition of the phase space by suitable boxes. The symbol estimates for the Hörmander's classes correspond to a partition of the phase space in boxes of size 1×2^j, $j \geq 0$. Instead, the estimates satisfied by the phases $\Phi_j(x, \eta)$ in (7) are of Shubin type [19] and correspond to a partition of the phase space by boxes of size $2^{j/2} \times 2^{j/2}$. This enters the general philosophy of [7]. Figure 1 shows the passage from Hörmander's geometry to Shubin's one, performed in (6). As a motivation of our study we recall that the solutions of the Cauchy problem for a strictly hyperbolic equation can be described by Fourier integral operators. If the equation has constant coefficients, then T reduces to a Fourier multiplier and acts continuously on M^p without loss of derivatives; cf. [2, 4, 6]. In the case of variable coefficients, our Theorem 1.2 gives the optimal regularity results on M^p. One could be disappointed then, since the loss $d|1/2-1/p|$ is even larger than $(d-1)|1/2-1/p|$ in L^p; cf. [20, 21]. However, when passing to treat FIOs whose symbols are not compactly supported in x, cf. [18], modulation spaces seem really to be the right function spaces to control both local regularity and decay at infinity. We plan to devote a subsequent paper to such topics.

The paper is organized as follows. In Section 2 the definitions and basic properties of the spaces $\mathcal{F}L^p$ and the modulation spaces M^p are reviewed. Section 3 contains preliminaries on FIOs and the proof of the above-mentioned microlocal version of results in [6]. In Section 4 we prove Theorem 1.2 for a symbol $\sigma(x,\eta)$, which, in addition, vanishes for η large. Section 5 is devoted to the proof of Theorem 1.2 for a symbol $\sigma(x,\eta)$ vanishing for η small. Finally, Section 6 exhibits the optimality of both Theorems 1.1 and 1.2.

Notation. We define $|x|^2 = x\cdot x$, for $x \in \mathbb{R}^d$, where $x\cdot y = xy$ is the scalar product on $\mathbb{R}^d$. The space of smooth functions with compact support is denoted by $\mathcal{C}_0^\infty(\mathbb{R}^d)$, the Schwartz class is $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\eta) = \mathcal{F}f(\eta) = \int f(t)e^{-2\pi i t\eta}dt$. Translation and modulation operators (*time and frequency shifts*) are defined, respectively, by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\eta f(t) = e^{2\pi i \eta t} f(t).$$

We have the formulas $(T_x f)\hat{} = M_{-x}\hat{f}$, $(M_\eta f)\hat{} = T_\eta \hat{f}$, and $M_\eta T_x = e^{2\pi i x\eta} T_x M_\eta$. The inner product of two functions $f, g \in L^2(\mathbb{R}^d)$ is $\langle f, g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t)}\,dt$, and its extension to $\mathcal{S}' \times \mathcal{S}$ will be also denoted by $\langle \cdot,\cdot\rangle$. The notation $A \lesssim B$ means $A \leq cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ means $c^{-1}A \leq B \leq cA$, for some $c \geq 1$. The symbol $B_1 \hookrightarrow B_2$ denotes the continuous embedding of the space B_1 into B_2.

2. Function spaces and preliminaries

2.1. $\mathcal{F}L^p$ **spaces.** For every $1 \leq p \leq \infty$, we define by $\mathcal{F}L^p$ the Banach space of all distributions $\hat{f} \in \mathcal{S}'$ such that $f \in L^p(\mathbb{R}^d)$ is endowed with the norm

$$\|\hat{f}\|_{\mathcal{F}L^p} = \|f\|_{L^p};$$

for details see, e.g., [14].

The space $\mathcal{F}L^p(\mathbb{R}^d)_{\mathrm{comp}}$ consists of the distributions in $\mathcal{F}L^p(\mathbb{R}^d)$ having compact support. It is the inductive limit of the Banach spaces $\mathcal{F}L^p(K_n) := \{f \in \mathcal{F}L^p(\mathbb{R}^d) : \operatorname{supp} f \subset K_n\}$, where $\{K_n\}$ is any increasing sequence of compacts whose union is $\mathbb{R}^d$.

Since the operator T in the Introduction has a symbol compactly supported in x, we see that the conclusion of Theorem 1.1 is equivalent to an estimate of the type

$$\|Tu\|_{\mathcal{F}L^p} \leq C_K \|u\|_{\mathcal{F}L^p}, \quad \forall u \in \mathcal{C}_0^\infty(K),$$

for every compact $K \subset \mathbb{R}^d$.

2.2. **Modulation spaces** [8, 9, 12]**.** Let $g \in \mathcal{S}$ be a non-zero window function. The short-time Fourier transform (STFT) $V_g f$ of a function/tempered distribution f with respect to the window g is defined by

$$V_g f(x,\eta) = \langle f, M_\eta T_x g\rangle = \int e^{-2\pi i \eta y} f(y)\overline{g(y-x)}\,dy, \tag{8}$$

i.e., the Fourier transform $\mathcal{F}$ applied to $f\overline{T_z g}$.

There is also an inversion formula for the STFT (see e.g. [12, Corollary 3.2.3]). Namely, if $\|g\|_{L^2} = 1$ and, for example, $u \in L^2(\mathbb{R}^d)$, it turns out that

$$u = \int_{\mathbb{R}^{2d}} V_g u(y,\eta) M_\eta T_y g\,dy\,d\eta. \tag{9}$$

For $1 \leq p \leq \infty$, $s \in \mathbb{R}$, the modulation space $M^p_s(\mathbb{R}^n)$ is defined as the space of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the norm

$$\|f\|_{M^p_s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^{2d}} |V_g f(x,\eta)|^p \langle \eta \rangle^{sp} \, dx d\eta \right)^{1/p}$$

is finite (with the obvious changes if $p = \infty$). If $s = 0$ we simply write M^p in place of M^p_0. This definition is independent of the choice of the window g in the sense of equivalent norms. Moreover, if $1 \leq p < \infty$, then M^1_s is densely embedded into M^p_s, as is the Schwartz class $\mathcal{S}$. Among the properties of modulation spaces, we record that $M^2 = L^2$, $M^1 \subset L^1 \cap \mathcal{F}L^1$. For $p \neq 2$, M^p does not coincide with any Lebesgue space. Indeed, for $1 \leq p < 2$, $M^p \subset L^p$, whereas, for $2 < p \leq \infty$, $L^p \subset M^p$. Differently from the L^p spaces, they enjoy the embedding property: $M^p \hookrightarrow M^q$ if $p \leq q$. If $p < \infty$, the dual of M^p_s is $(M^p_s)' = M^{p'}_{-s}$.

Let us define by $\mathcal{M}^p_s(\mathbb{R}^d)$ the completion of $\mathcal{S}(\mathbb{R}^d)$ under the norm $\|\cdot\|_{M^p_s}$. Then the following are true [10]:

(i) If $1 \leq p < \infty$, then $\mathcal{M}^p_s(\mathbb{R}^d) = M^p_s(\mathbb{R}^d)$.

(ii) If $1 \leq p_1, p_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}$, and $0 < \theta < 1$, $1 \leq p \leq \infty$, $s \in \mathbb{R}$ satisfy $1/p = (1-\theta)/p_1 + \theta/p_2$, $s = (1-\theta)s_1 + \theta s_2$, then

$$(\mathcal{M}^{p_1}_{s_1}, \mathcal{M}^{p_2}_{s_2})_{[\theta]} = \mathcal{M}^p_s. \tag{10}$$

(iii) $(\mathcal{M}^\infty_s(\mathbb{R}^d))' = M^1_{-s}$.

In the sequel, the following characterization of M^p_s spaces will be useful (see, e.g., [9, 25]): let $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^d)$, $\varphi \geq 0$, such that $\sum_{m \in \mathbb{Z}^d} \varphi(\eta - m) \equiv 1$, for all $\eta \in \mathbb{R}^d$. Then

$$\|u\|_{M^p_s} \asymp \left(\sum_{m \in \mathbb{Z}^d} \|\varphi(D-m)u\|^p_{L^p} \langle m \rangle^{ps} \right)^{1/p}, \tag{11}$$

where $\varphi(D-m)u = \mathcal{F}^{-1}[\varphi(\cdot - m)\hat{u}]$ (with the obvious changes if $p = \infty$).

If we consider the space of functions/distributions u in $M^p(\mathbb{R}^d)$ that are supported in any fixed compact set, then their M^p-norm is equivalent to the $\mathcal{F}L^p$-norm. More precisely, we have the following result [8, 9, 11, 17]:

Lemma 2.1. *Let $1 \leq p \leq \infty$. For every $u \in \mathcal{S}'(\mathbb{R}^d)$, supported in a compact set $K \subset \mathbb{R}^d$, we have $u \in M^p \Leftrightarrow u \in \mathcal{F}L^p$, and*

$$C_K^{-1} \|u\|_{M^p} \leq \|u\|_{\mathcal{F}L^p} \leq C_K \|u\|_{M^p}, \tag{12}$$

where $C_K > 0$ depends only on K.

In order to state the dilation properties for modulation spaces, we introduce the indices:

$$\mu_1(p) = \begin{cases} -1/p' & \text{if} \quad 1 \leq p \leq 2, \\ -1/p & \text{if} \quad 2 \leq p \leq \infty, \end{cases}$$

and

$$\mu_2(p) = \begin{cases} -1/p & \text{if} \quad 1 \leq p \leq 2, \\ -1/p' & \text{if} \quad 2 \leq p \leq \infty. \end{cases}$$

For $\lambda > 0$, we define the dilation operator $U_\lambda f(x) = f(\lambda x)$. Then, the dilation properties of M^p are as follows (see [22, Theorem 3.1]).

Theorem 2.1. *We have: (i) For $\lambda \geq 1$,*

$$\|U_\lambda f\|_{M^p} \lesssim \lambda^{d\mu_1(p)} \|f\|_{M^p}, \quad \forall\, f \in M^p(\mathbb{R}^d).$$

(ii) For $0 < \lambda \leq 1$,

$$\|U_\lambda f\|_{M^p} \lesssim \lambda^{d\mu_2(p)} \|f\|_{M^p}, \quad \forall\, f \in M^p(\mathbb{R}^d).$$

These dilation estimates are sharp, as discussed in [22]; see also [5].

3. Preliminary results on FIOs

In this section we recall the composition formula of a pseudodifferential operator and an FIO. Then we prove some auxiliary results for FIOs with phases having bounded derivatives of order ≥ 2.

3.1. Composition of pseudodifferential and Fourier integral operators.

First, recall the general Hörmander symbol class $S^m_{\rho,\delta}$ of smooth functions on $\mathbb{R}^{2d}$ such that

$$|\partial^\alpha_x \partial^\beta_\eta \sigma(x,\eta)| \leq C_{\alpha,\beta} \langle \eta \rangle^{m-\rho|\beta|+\delta|\alpha|}, \quad (x,\eta) \in \mathbb{R}^{2d}.$$

A regularizing operator is a pseudodifferential operator

$$Ru = \int e^{2\pi i x\eta} r(x,\eta) \hat{u}(\eta) d\eta,$$

with a symbol r in the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$ (equivalently, an operator with kernel in $\mathcal{S}(\mathbb{R}^{2d})$, which maps $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$). Then, the composition formula for a pseudodifferential operator and an FIO is as follows (see, e.g., [13], [16, Theorem 4.1.1], [19, Theorem 18.2], [24]; we limit ourselves to recalling what is needed in the subsequent proofs).

Theorem 3.1. *Let the symbol σ and the phase Φ satisfy the assumptions in the Introduction. Assume, in addition, $\sigma(x,\eta) = 0$ for $|\eta| \leq 1$, if $\Phi(x,\eta)$ is not linear in η. Let $a(x,\eta)$ be a symbol in $S^{m'}_{1,0}$. Then,*

$$a(x,D)T = S + R,$$

where S is an FIO with the same phase Φ and symbols $s(x,\eta)$, of order $m+m'$, satisfying

$$\operatorname{supp} s \subset \operatorname{supp} \sigma \cap \{(x,\eta) \in \Lambda : \ (x, \nabla_x \Phi(x,\eta)) \in \operatorname{supp} a\},$$

and R is a regularizing operator with symbol $r(x,\eta)$ satisfying

$$\Pi_\eta(\operatorname{supp} r) \subset \Pi_\eta(\operatorname{supp} \sigma),$$

where Π_η is the orthogonal projection on $\mathbb{R}^d_\eta$.

Moreover, the symbol estimates satisfied by s and the seminorm estimates of r in the Schwartz space are uniform when σ and a vary in bounded subsets of $S^m_{1,0}$ and $S^{m'}_{1,0}$, respectively.

3.2. **FIOs with phases having bounded derivatives of order ≥ 2.** In what follows we present a micro-localized version of [6, Theorems 3.1, 4.1], where the hypotheses of such theorems are satisfied only in the ϵ-neighborhood Σ_ϵ of the support of σ, σ being the FIO's symbol. Namely, set

$$\Sigma_\epsilon = \bigcup_{(x_0,\eta_0)\in \operatorname{supp}\sigma} B_\epsilon(x_0,\eta_0).$$

Proposition 3.2. *Let $\sigma \in S^0_{0,0}$ and Σ_ϵ be as above. Let Φ be a real-valued function defined and smooth on Σ_ϵ. Suppose that*

$$|\partial_z^\alpha \Phi(z)| \leq C_\alpha \quad \text{for } |\alpha|\geq 2,\ z=(x,\eta)\in\Sigma_\epsilon. \tag{13}$$

Let $g,\gamma \in \mathcal{S}(\mathbb{R}^d)$, $\|g\|_{L^2}=\|\gamma\|_{L^2}=1$, with $\operatorname{supp}\gamma\subset B_{\epsilon/4}(0)$, $\operatorname{supp}\hat{g}\subset B_{\epsilon/4}(0)$. Then, for every $N\geq 0$, there exists a constant $C>0$ such that

$$|\langle T(M_\omega T_y g), M_{\omega'}T_{y'}\gamma\rangle| \leq C\,\mathbb{1}_{\Sigma_{\epsilon/2}}(y',\omega)\langle \nabla_x\Phi(y',\omega)-\omega'\rangle^{-N}\langle\nabla_\eta\Phi(y',\omega)-y\rangle^{-N}.$$

The constant C only depends on N, g,γ, and upper bounds for a finite number of derivatives of σ and on a finite number of constants in (13).

Proof. We can write

$$\begin{aligned}
&\langle T(M_\omega T_y g), M_{\omega'}T_{y'}\gamma\rangle\\
&=\int_{\mathbb{R}^d} TM_\omega T_y g(x)\overline{M_{\omega'}T_{y'}\gamma(x)}\,dx\\
&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{2\pi i\Phi(x,\eta)}\sigma(x,\eta)T_\omega M_{-y}\hat{g}(\eta)M_{-\omega'}T_{y'}\bar{\gamma}(x)\,dxd\eta\\
&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} M_{(0,-y)}T_{(0,-\omega)}\left(e^{2\pi i\Phi(x,\eta)}\sigma(x,\eta)\right)\hat{g}(\eta)M_{-\omega'}T_{y'}\bar{\gamma}(x)\,dxd\eta\\
&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} T_{(-y',0)}M_{-(\omega',0)}M_{(0,-y)}T_{(0,-\omega)}\left(e^{2\pi i\Phi(x,\eta)}\sigma(x,\eta)\right)\bar{\gamma}(x)\hat{g}(\eta)\,dxd\eta\\
&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{2\pi i[\Phi(x+y',\eta+\omega)-(\omega',y)\cdot(x+y',\eta)]}\sigma(x+y',\eta+\omega)\bar{\gamma}(x)\hat{g}(\eta)\,dxd\eta\\
&=\int_{B_{\epsilon/4}(0)}\int_{B_{\epsilon/4}(0)} e^{2\pi i[\Phi((x+y',\eta+\omega))-(\omega',y)\cdot(x+y',\eta)]}\sigma(x+y',\eta+\omega)\bar{\gamma}(x)\hat{g}(\eta)\,dxd\eta.
\end{aligned}$$

Observe that, if $(y',\omega)\notin\Sigma_{\epsilon/2}$, by the assumptions on the support of γ and $\hat{g}$ and the triangle inequality, this integral vanishes.

Hence, assume $(y',\omega)\in\Sigma_{\epsilon/2}$. Since Φ is smooth on Σ_ϵ, we perform a Taylor expansion of $\Phi(x,\eta)$ at (y',ω) and obtain

$$\Phi(x+y',\eta+\omega)=\Phi(y',\omega)+\nabla_z\Phi(y',\omega)\cdot(x,\eta)+\Phi_{2,(y',\omega)}(x,\eta),$$

for $z=(x,\eta)\in B_{\epsilon/4}(0)\times B_{\epsilon/4}(0)$, where the remainder is given by

$$\Phi_{2,(y',\omega)}(x,\eta)=2\sum_{|\alpha|=2}\int_0^1(1-t)\partial^\alpha\Phi((y',\omega)+t(x,\eta))\,dt\frac{(x,\eta)^\alpha}{\alpha!}. \tag{14}$$

Notice that the segment $(y',\omega)+t(x,\eta)$, $0\leq t\leq 1$, belongs entirely to Σ_ϵ if $(x,\eta)\in B_{\epsilon/4}(0)\times B_{\epsilon/4}(0)$.

Hence, we can write

$$|\langle T(M_\omega T_y g), M_{\omega'} T_{y'}\gamma\rangle| = \Big| \int_{B_{\epsilon/4}(0)} \int_{B_{\epsilon/4}(0)} e^{2\pi i\{[\nabla_z \Phi(y',\omega)-(\omega',y)]\cdot(x,\eta)\}} \times e^{2\pi i \Phi_{2,(y',\omega)}(x,\eta)} \sigma(x+y',\eta+\omega)\bar{\gamma}(x)\hat{g}(\eta)\, dx d\eta \Big|.$$

For $N \in \mathbb{N}$, using the identity:

$$(1-\Delta_x)^N(1-\Delta_\eta)^N e^{2\pi i\{[\nabla_z \Phi(y',\omega)-(\omega',y)]\cdot(x,\eta)\}} = \langle 2\pi(\nabla_x\Phi(y',\omega)-\omega')\rangle^{2N} \times \langle 2\pi(\nabla_\eta\Phi(y',\omega)-y)\rangle^{2N} e^{2\pi i\{[\nabla_z \Phi(y',\omega)-(\omega',y)]\cdot(x,\eta)\}},$$

we integrate by parts and obtain

$$|\langle T(M_\omega T_y g), M_{\omega'} T_{y'}\gamma\rangle| = \langle 2\pi(\nabla_x\Phi(y',\omega)-\omega')\rangle^{-2N} \langle 2\pi(\nabla_\eta\Phi(y',\omega)-y)\rangle^{-2N} \times \Big| \int_{B_{\epsilon/4}(0)} \int_{B_{\epsilon/4}(0)} e^{2\pi i\{[\nabla_z \Phi(y',\omega)-(\omega',y)]\cdot(x,\eta)\}} \times (1-\Delta_x)^N(1-\Delta_\eta)^N [e^{2\pi i \Phi_{2,(y',\omega)}(x,\eta)} \sigma(x+y',\eta+\omega)\bar{\gamma}(x)\hat{g}(\eta)]\, dx d\eta \Big|.$$

Hence it suffices to apply the Leibniz formula taking into account that, as a consequence of (13), we have the estimates $\partial_z^\alpha \Phi_{2,(y',\omega)}(z) = O(\langle z\rangle^2)$ for $z=(x,\eta) \in B_{\epsilon/4}(0) \times B_{\epsilon/4}(0)$, uniformly with respect to (y',ω). □

In the following proposition, where supp σ and Σ_ϵ are understood to be bounded in the η variables, we prove the M^p-continuity of T with uniform norm bound with respect to the constants C_α in (13).

Proposition 3.3. *Consider a symbol $\sigma \in S^0_{0,0}$, with $\sigma(x,\eta) = 0$ for $|\eta| \le 2$. Moreover, let $\Omega \subset \mathbb{R}^d$ be open and $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ be conic and open, such that $\Omega \times \Gamma$ contains the ϵ-neighborhood Σ_ϵ of* supp σ.

Then consider a phase $\Phi \in C^\infty(\Omega\times\Gamma)$, positively homogeneous of degree 1 in η, satisfying

$$|\partial_z^\alpha \Phi(x,\eta)| \le C_\alpha \quad \text{for } |\alpha| \ge 2,\ (x,\eta) \in \Sigma_\epsilon, \tag{15}$$

$$\left|\det\left(\frac{\partial^2\Phi}{\partial x_i \partial \eta_l}\Big|_{(x,\eta)}\right)\right| \ge \delta > 0, \quad \forall (x,\eta) \in \Omega\times\Gamma, \tag{16}$$

and such that

(17) $\forall x \in \Omega$, *the map* $\Gamma \ni \eta \mapsto \nabla_x\Phi(x,\eta)$ *is a diffeomorphism onto the range,*

(18) $\forall \eta \in \Gamma$, *the map* $\Omega \ni x \mapsto \nabla_\eta\Phi(x,\eta)$ *is a diffeomorphism onto the range.*

Then, for every $1 \le p \le \infty$ it turns out that

$$\|Tu\|_{M^p} \le C\|u\|_{M^p}, \quad \forall u \in \mathcal{S}(\mathbb{R}^d),$$

where the constant C depends only on ϵ, δ, on upper bounds for a finite number of derivatives of σ and a finite number of the constants in (15).

Proof. Let $g,\gamma \in \mathcal{S}(\mathbb{R}^d)$, with $\|g\|_{L^2} = \|\gamma\|_{L^2} = 1$, supp $\gamma \subset B_{\epsilon/4}(0)$, supp $\hat{g} \subset B_{\epsilon/4}(0)$. Let $u \in \mathcal{S}(\mathbb{R}^d)$. The inversion formula (9) for the STFT gives

$$V_\gamma(Tu)(y',\omega') = \int_{\mathbb{R}^{2d}} \langle T(M_\omega T_y g), M_{\omega'}T_{y'}\gamma\rangle V_g u(y,\omega) dy\, d\omega.$$

The desired estimate follows if we prove that the map K_T, defined by

$$K_T G(y',\omega') = \int_{\mathbb{R}^{2d}} \langle T(M_\omega T_y g), M_{\omega'} T_{y'}\gamma\rangle G(y,\omega)dy\, d\omega,$$

is continuous on $L^p(\mathbb{R}^{2d})$. By Schur's test (see, e.g., [12, Lemma 6.2.1]) it suffices to prove that its integral kernel

$$K_T(y',\omega';y,\omega) = \langle T(M_\omega T_y g), M_{\omega'} T_{y'}\gamma\rangle$$

satisfies

$$K_T \in L^\infty_{y',\omega'}(L^1_{y,\omega}) \tag{19}$$

and

$$K_T \in L^\infty_{y,\omega}(L^1_{y',\omega'}). \tag{20}$$

Let us verify (19). By Proposition (3.2) and the fact that $\mathbb{1}_{\Sigma_{\epsilon/2}}(y',\omega) \le \mathbb{1}_\Omega(y')\mathbb{1}_\Gamma(\omega)$ we have

$$|K_T(y',\omega';y,\omega)| \le C\,\mathbb{1}_\Omega(y')\mathbb{1}_\Gamma(\omega)\langle\nabla_x\Phi(y',\omega)-\omega'\rangle^{-N}\langle\nabla_\eta\Phi(y',\omega)-y\rangle^{-N} \quad \forall N\in\mathbb{N}.$$

Hence (19) will be proved if we verify that there exists a constant $C>0$ such that

$$\int \mathbb{1}_\Gamma(\omega)\langle\nabla_x\Phi(y',\omega)-\omega'\rangle^{-N}\, d\omega \le C, \quad \forall (y',\omega')\in\Omega\times\mathbb{R}^d.$$

In order to prove this estimate we perform the change of variable

$$\beta_{y'}: \Gamma \ni \omega \longmapsto \nabla_x\Phi(y',\omega),$$

which is a diffeomorphism on the range by (17). The Jacobian determinant of its inverse is homogeneous of degree 0 in ω and uniformly bounded with respect to y' by the hypotheses (15) and (16). Hence, the last integral is, for $N>d$,

$$\begin{aligned} &\lesssim \int_{\beta_{y'}(\Gamma)} \langle\tilde\omega-\omega'\rangle^{-N}\, d\tilde\omega \\ &\le \int_{\mathbb{R}^d} \langle\tilde\omega-\omega'\rangle^{-N}\, d\tilde\omega = C. \end{aligned}$$

The proof of (20) is analogous and is left to the reader.

Finally, the uniformity of the norm of T as a bounded operator, established in the last part of the statement, follows from the proof itself. $\square$

4. Singularity at the origin

In this section we prove Theorem 1.2 for an operator satisfying the assumptions stated there and whose symbol σ satisfies, in addition,

$$\sigma(x,\eta) = 0 \quad \text{for } |\eta|\ge 4. \tag{21}$$

Here we do not use the hypothesis (3). Indeed, we will deduce the desired result from the following one, after extending $\Phi|_{\Lambda'}$ to a phase function, still denoted by $\Phi(x,\eta)$, positively homogeneous of degree 1 in η (possibly degenerate) and everywhere defined in $\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})$.

Proposition 4.1. *Let $\sigma(x,\eta)$ be a smooth symbol satisfying*

$$\sigma(x,\eta)=0 \quad \text{for } |x|+|\eta|\geq R, \tag{22}$$

for some $R>0$. Let $\Phi(x,\eta)$, $x\in\mathbb{R}^d$, $\eta\in\mathbb{R}^d\setminus\{0\}$, be a smooth phase function, positively homogeneous of degree 1 in η. Then the corresponding FIO T extends to a bounded operator on M^p, for every $1\leq p<\infty$, and on $\mathcal{M}^\infty$.

In order to prove Proposition 4.1, we use the complex interpolation method between the spaces M^1 and $\mathcal{M}^\infty$. Indeed, using (10),

$$(M^1,\mathcal{M}^\infty)_{[\theta]}=M^p, \quad \frac{1}{p}=\theta, \quad 0<\theta<1,$$

we attain the boundedness of T on every M^p, $1<p<\infty$, if we prove the boundedness on M^1 and $\mathcal{M}^\infty$. The rest of this section is devoted to that.

4.1. **Boundedness on M^1.** For every $u\in\mathcal{S}(\mathbb{R}^d)$, we have

$$\partial^\alpha(Tu)(x)=\int_{\mathbb{R}^d}e^{2\pi i\Phi(x,\eta)}\left(\sum_{\beta\leq\alpha}p_\beta(\partial^{|\beta|}\Phi(x,\eta))\partial_x^{\alpha-\beta}\sigma(x,\eta)\right)\hat{u}(\eta)\,d\eta,$$

where $p_\beta(\partial^{|\beta|}\Phi(x,\eta))$ is a polynomial of order $|\beta|$ in the derivatives of Φ with respect to x of order at most $|\beta|$.

Hence, because of the homogeneity of Φ and (22),

$$\|\partial^\alpha(Tu)\|_{L^1}\leq\left(\sum_{\beta\leq\alpha}\int_{|x|\leq R}\tilde{C}_\beta\sup_{|\eta|\leq R}\langle\eta\rangle^{|\beta|}|\partial_x^{\alpha-\beta}\sigma(x,\eta)|dx\right)\|\hat{u}\|_{L^1}\leq C_\alpha\|u\|_{\mathcal{F}L^1}.$$

Using the relation (12), the previous estimate, and the inclusion $M^1\hookrightarrow\mathcal{F}L^1$, the result is easily attained:

$$\|Tu\|_{M^1}\asymp\|Tu\|_{\mathcal{F}L^1}\leq C\sup_{|a|\leq d+1}\|\partial^\alpha(Tu)\|_{L^1}\leq C\sup_{|a|\leq d+1}C_\alpha\|u\|_{\mathcal{F}L^1}\leq\tilde{C}\|u\|_{M^1}.$$

4.2. **Boundedness on $\mathcal{M}^\infty$.** First, we recall a slight variant of [2, Theorem 9]:

Theorem 4.2. *Let Φ be a phase function as in Proposition 4.1. Let χ be a smooth function satisfying $\chi(\eta)=1$ for $|\eta|\leq R$, $\chi(\eta)=0$ for $|\eta|\geq 2R$, for some $R>0$. Then for every compact subset $K\subset\mathbb{R}^d$ there exists a constant $C_K>0$ such that*

$$\sup_{x\in K}\|e^{2\pi i\Phi(x,\cdot)}\chi\|_{\mathcal{F}L^1}<C_K. \tag{23}$$

The proof is a straightforward generalization of [2, Theorem 9], where the case of phases independent of x was considered. Namely, since the parameter x varies in a compact set K, all the estimates given there hold uniformly with respect to x (more generally, Theorem 4.2 holds for phases positively homogeneous of order $\alpha>0$ with respect to η, but here we are only interested in the case $\alpha=1$).

We also observe that, if $\phi\in\mathcal{C}_0^\infty(\mathbb{R}^d)$, then

$$\|\phi(D)u\|_{L^\infty}\leq C\|u\|_{L^\infty}, \qquad \forall u\in\mathcal{S}(\mathbb{R}^d), \tag{24}$$

where $\phi(D)u=\mathcal{F}^{-1}(\phi\hat{u})=\hat{\phi}*u$. This is a consequence of Young's inequality, since $\hat{\phi}\in L^1$.

We now have all the pieces in place to prove the boundedness on $\mathcal{M}^\infty$ of the FIO T. Since its symbol σ vanishes for $|\eta| \geq R$, taking χ as in Theorem 4.2, we have $\sigma(x,\eta) = \sigma(x,\eta)\chi(\eta)$ and, for every $v \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned} Tv(x) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{2\pi i(\Phi(x,\eta)-y\eta)}\sigma(x,\eta)\chi(\eta)v(y)\,d\eta dy \\ &= \int_{\mathbb{R}^d} \mathcal{F}[(e^{2\pi i(\Phi(x,\cdot)}\chi)(\sigma(x,\cdot))](y)v(y)\,dy. \end{aligned}$$

If we set $K = \{x \in \mathbb{R}^d : \ |x| \leq R\}$, since $\mathcal{F}L^1$ is an algebra under pointwise multiplication and using the majorization (23), we obtain

$$\|Tv\|_{L^\infty} \leq \|v\|_{L^\infty} \sup_K (\|e^{2\pi i\Phi(x,\cdot)}\chi\|_{\mathcal{F}L^1}\|\sigma(x,\cdot)\|_{\mathcal{F}L^1}) \leq C\|v\|_{L^\infty}. \tag{25}$$

Since $\sigma(x,\eta) = 0$ for $|\eta| \geq R$, for every $\phi \in \mathcal{S}(\mathbb{R}^d)$, with $\phi(\eta) \equiv 1$ for $|\eta| \leq R$, we have as well

$$Tu = T(\phi(D)u),$$

so that, using the embeddings $L^\infty \hookrightarrow M^\infty$, and (25) for $v = \phi(D)u$,

$$\|Tu\|_{M^\infty} \leq \|Tu\|_{L^\infty} = \|T(\phi(D)u)\|_{L^\infty} \leq C\|\phi(D)u\|_{L^\infty}.$$

Now, choose a function φ as in (11). Then φ satisfies $\operatorname{supp}(T_m\varphi) \cap \operatorname{supp}\phi \neq \emptyset$ for finitely many $m \in \mathbb{Z}^d$ only. Hence, the estimate (24) yields, for $u \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned} \|\phi(D)u\|_{L^\infty} &\lesssim \sum_{m\in\mathbb{Z}^d} \|\phi(D)\varphi(D-m)u\|_{L^\infty} \lesssim \sup_{m\in\mathbb{Z}^d} \|\phi(D)\varphi(D-m)u\|_{L^\infty} \\ &\lesssim \sup_{m\in\mathbb{Z}^d} \|\varphi(D-m)u\|_{L^\infty} \\ &\asymp \|u\|_{M^\infty}. \end{aligned}$$

So the FIO T is bounded on $\mathcal{M}^\infty$. This concludes the proof of Proposition 4.1.

5. Oscillations at infinity

In this section we prove Theorem 1.2 for an operator satisfying the assumptions stated there and and whose symbol σ satisfies, in addition,

$$\sigma(x,\eta) = 0 \quad \text{for } |\eta| \leq 2.$$

We first perform a further reduction.

For every $(x_0,\eta_0) \in \Lambda'$, $|\eta_0| = 1$, there exist an open neighborhood $\Omega \subset \mathbb{R}^d$ of x_0, an open conic neighborhood $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ of η_0 and $\delta > 0$ such that

$$|\det \partial_{x,\eta}\Phi(x,\eta)| \geq \delta > 0, \quad \forall (x,\eta) \in \Omega \times \Gamma, \tag{26}$$

and

$$\forall x \in \Omega,\ \text{the map } \Gamma \ni \eta \mapsto \nabla_x\Phi(x,\eta) \text{ is a diffeomorphism onto the range,} \tag{27}$$

$$\forall \eta \in \Gamma,\ \text{the map } \Omega \ni x \mapsto \nabla_\eta\Phi(x,\eta) \text{ is a diffeomorphism onto the range.} \tag{28}$$

Hence, by a compactness argument and a finite partition of unity we can assume that σ itself is supported in a cone of the type $\Omega' \times \Gamma'$, for some open $\Omega' \subset \mathbb{R}^d$, $\Gamma' \subset \mathbb{R}^d \setminus \{0\}$ conic, $\Omega' \Subset \Omega$, $\Gamma' \Subset \Gamma$, with Φ satisfying the above conditions on $\Omega \times \Gamma$.

We now prove the boundedness of an operator T of order $m = -d/2$ on M^1 and on $\mathcal{M}^\infty$. Since it is a classical fact that FIOs of order 0 are continuous on

$L^2 = M^2$ (see, e.g., [21, page 402]), the desired continuity result on M^p when $m = -d|1/2 - 1/p|$, $1 < p < \infty$, follows by complex interpolation.

In detail, the interpolation step goes as follows. Observe first that, for every $s \in \mathbb{R}$, the operator $\langle D\rangle^s$ defines an isomorphism of $\mathcal{M}^p_s$ onto $\mathcal{M}^p$. This follows easily from the characterization of the M^p_s-norm in (11), after writing $\varphi = \tilde\varphi\varphi$, for some $\tilde\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $\tilde\varphi \equiv 1$ on $\operatorname{supp}\varphi$, combined with the fact that the multiplier $\langle\eta\rangle^s\tilde\varphi(\eta - m)\langle m\rangle^{-s}$ is in $\mathcal{F}L^1$ uniformly with respect to m.

Hence, the operator $T = T\langle D\rangle^{-s}\langle D\rangle^s$ is bounded $\mathcal{M}^p_s \to \mathcal{M}^p$ if and only if $T\langle D\rangle^{-s}$ is bounded on $\mathcal{M}^p$. Observe moreover that $T\langle D\rangle^{-s}$ is an FIO with the same phase as T, and symbol $\sigma(x,\eta)\langle\eta\rangle^{-s}$, which has order $m - s$.

Now suppose that the desired result is already obtained for $p = 1, 2$. Take $1 < p < 2$ and consider an FIO T of order $m = -d(1/p - 1/2)$. Then, taking the above remarks into account, T extends to a bounded operator $M^1_{m+d/2} \to M^1$ and $M^2_m \to M^2$. Hence, the boundedness on M^p follows by complex interpolation, i.e. (10), because, if $\theta \in (0,1)$ satisfies $(1-\theta)/1 + \theta/2 = 1/p$, one has $(m + d/2)(1 - \theta) + m\theta = 0$. The proof for $2 < p < \infty$ is similar.

Of course, when in (4) there is a strict inequality, the desired result follows from the equality case, for an operator with order $m' < m$ also has order m.

Hence, from now on, we assume $m = -d/2$ and prove the boundedness of T on M^1 and on $\mathcal{M}^\infty$.

5.1. **Boundedness on M^1.** We need the following result (cf. [11, 22]).

Lemma 5.1. *Let χ be a smooth function supported where $B_0^{-1} \le |\eta| \le B_0$, for some $B_0 > 0$. Then, for every $u \in \mathcal{S}(\mathbb{R}^d)$,*

$$\sum_{j=1}^{\infty} \|\chi(2^{-j}D)u\|_{M^1} \lesssim \|u\|_{M^1},$$

where $\chi(2^{-j}D)u = \mathcal{F}^{-1}[\chi(2^{-j}\cdot)\hat u]$.

Proof. Let $u \in \mathcal{S}(\mathbb{R}^d)$. By using the characterization of the M^1-norm in (11), we have

$$\sum_{j=1}^{\infty} \|\chi(2^{-j}D)u\|_{M^1} \asymp \sum_{j=1}^{\infty}\sum_{m\in\mathbb{Z}^d} \|\varphi(D-m)\chi(2^{-j}D)u\|_{L^1} \tag{29}$$

$$= \sum_{m\in\mathbb{Z}^d}\sum_{j=1}^{\infty} \|\chi(2^{-j}D)\varphi(D-m)u\|_{L^1} \tag{30}$$

$$\lesssim \sum_{m\in\mathbb{Z}^d}\sup_{j\ge 1} \|\chi(2^{-j}D)\varphi(D-m)u\|_{L^1}. \tag{31}$$

In the last inequality we used the fact that, for any m, the number of indices $j \ge 1$ for which $\operatorname{supp}\,\chi(2^{-j}\,\cdot) \cap \operatorname{supp}\,\varphi(\cdot - m) \ne \emptyset$ is uniformly bounded with respect to m.

The Fourier multiplier $\chi(2^{-j}D)$ is bounded on L^1, uniformly with respect to j. Indeed, for every $f \in \mathcal{S}$,

$$\begin{aligned}\|\chi(2^{-j}D)f\|_{L^1} &= \|\mathcal{F}^{-1}(\chi(2^{-j}\cdot)) * f\|_{L^1} \le \|\mathcal{F}^{-1}(\chi(2^{-j}\cdot))\|_{L^1}\|f\|_{L^1}\\ &= 2^{jd}\|\mathcal{F}^{-1}(\chi)(2^j\cdot)\|_{L^1}\|f\|_{L^1} = \|\mathcal{F}^{-1}(\chi)(\cdot)\|_{L^1}\|f\|_{L^1}.\end{aligned}$$

Hence, $\|\chi(2^{-j}D)\varphi(D-m)u\|_{L^1} \lesssim \|\varphi(D-m)u\|_{L^1}$. Finally, using (11),

$$\sum_{j=1}^{\infty} \|\chi(2^{-j}D)u\|_{M^1} \lesssim \sum_{m\in\mathbb{Z}^d} \|\varphi(D-m)u\|_{L^1} \asymp \|u\|_{M^1}.$$

□

Consider now the usual Littlewood-Paley decomposition of the frequency domain. Namely, fix a smooth function $\psi_0(\eta)$ such that $\psi_0(\eta) = 1$ for $|\eta| \leq 1$ and $\psi_0(\eta) = 0$ for $|\eta| \geq 2$. Set $\psi(\eta) = \psi_0(\eta) - \psi_0(2\eta)$, $\psi_j(\eta) = \psi(2^{-j}\eta)$, $j \geq 1$. Then

$$1 = \sum_{j=0}^{\infty} \psi_j(\eta), \quad \forall \eta \in \mathbb{R}^d.$$

Notice that, if $j \geq 1$, ψ_j is supported where $2^{j-1} \leq |\eta| \leq 2^{j+1}$. Since $\sigma(x,\eta) = 0$, for $|\eta| \geq 2$, we can write

$$T = \sum_{j\geq 1} T^{(j)},$$

where $T^{(j)}$ has symbol $\sigma_j(x,\eta) := \sigma(x,\eta)\psi_j(\eta)$. Moreover, we observe that

$$T^{(j)} = U_{2^{j/2}} \tilde{T}^{(j)} U_{2^{-j/2}},$$

where $\tilde{T}^{(j)}$ is the FIO with phase

$$\Phi_j(x,\eta) := \Phi(2^{-j/2}x, 2^{j/2}\eta) = 2^{j/2}\Phi(2^{-j/2}x, \eta) \tag{32}$$

and symbol

$$\tilde{\sigma}_j(x,\eta) := \sigma_j(2^{-j/2}x, 2^{j/2}\eta),$$

and $U_\lambda f(y) = f(\lambda y)$, $\lambda > 0$, is the dilation operator. From Theorem 2.1 we have

$$\|U_\lambda f\|_{M^1} \lesssim \|f\|_{M^1}, \quad \lambda \geq 1, \tag{33}$$

and

$$\|U_\lambda f\|_{M^1} \lesssim \lambda^{-d}\|f\|_{M^1}, \quad 0 < \lambda \leq 1. \tag{34}$$

Assume for a moment that

$$\|\tilde{T}^{(j)}u\|_{M^1} \lesssim 2^{-jd/2}\|u\|_{M^1}. \tag{35}$$

Then, using (33) and (34), we obtain

$$\|T^{(j)}u\|_{M^1} \leq 2^{jd/2}2^{-jd/2}\|u\|_{M^1} = \|u\|_{M^1}.$$

Actually, for the frequency localization of $T^{(j)}$, the following finer estimate holds:

$$\|T^{(j)}u\|_{M^1} = \|T^{(j)}(\chi(2^{-j}D)u)\|_{M^1} \leq \|\chi(2^{-j}D)u\|_{M^1},$$

where χ is a smooth function satisfying $\chi(\eta) = 1$ for $1/2 \leq |\eta| \leq 2$ and $\chi(\eta) = 0$ for $|\eta| \leq 1/4$ and $|\eta| \geq 4$ (so that $\chi\psi = \psi$). Summing this last estimate on j with the aid of Lemma 5.1, we obtain

$$\|Tu\|_{M^1} \lesssim \|u\|_{M^1},$$

which is the desired estimate.

It remains to prove (35). This follows from Proposition 3.3 applied to the operator $2^{jd/2}\tilde{T}^{(j)}$. Indeed, it is easy to see that the hypotheses are satisfied uniformly with respect to j. Precisely, we observe that, for every $j \geq 1$,

$$|\partial_x^\alpha \partial_\eta^\beta \tilde{\sigma}_j(x,\eta)| \lesssim 2^{-j\frac{d}{2} - j\frac{|\alpha|+|\beta|}{2}},$$

and $\tilde{\sigma}_j(x,\eta)$ is supported where $2^{j/2-1} \le |\eta| \le 2^{j/2+1}$, $x \in \Omega'_j := \{2^{j/2}x,\ x \in \Omega'\}$, $\eta \in \Gamma'$. Moreover, after setting $\Omega_j := \{2^{j/2}x,\ x \in \Omega\}$, we see that

$$|\partial_x^\alpha \partial_\eta^\beta \Phi_j(x,\eta)| \lesssim 2^{j\left(1-\frac{|\alpha|+|\beta|}{2}\right)}, \tag{36}$$

for (x,η) in the set $\Omega_j \times \Gamma, 2^{j/2-2} \le |\eta| \le 2^{j/2+2}$, which contains an ϵ-neighborhood of $\operatorname{supp} \tilde{\sigma}_j$, with ϵ independent of j. Finally, (26), (27), and (28) give

$$\left|\det\left(\frac{\partial^2 \Phi_j}{\partial x_i \partial \eta_l}\Big|_{(x,\eta)}\right)\right| \ge \delta > 0, \quad \forall (x,\eta) \in \Omega_j \times \Gamma, \tag{37}$$

and

$$\forall x \in \Omega_j,\ \text{the map } \Gamma \ni \eta \mapsto \nabla_x \Phi_j(x,\eta) \text{ is a diffeomorphism onto the range,} \tag{38}$$

$$\forall \eta \in \Gamma,\ \text{the map } \Omega_j \ni x \mapsto \nabla_\eta \Phi_j(x,\eta) \text{ is a diffeomorphism onto the range.} \tag{39}$$

Hence Proposition 3.3 applies and gives (35).

5.2. **Boundedness on $\mathcal{M}^\infty$.** We need the following result (cf. [11, 22]).

Lemma 5.2. *For $k \ge 0$, let $f_k \in \mathcal{S}(\mathbb{R}^d)$ satisfy $\operatorname{supp} \hat{f}_0 \subset B_2(0)$ and*

$$\operatorname{supp} \hat{f}_k \subset \{\eta \in \mathbb{R}^d :\ 2^{k-1} \le |\eta| \le 2^{k+1}\}, \quad k \ge 1.$$

Then, if the sequence f_k is bounded in $M^\infty(\mathbb{R}^d)$, then the series $\sum_{k=0}^\infty f_k$ converges in $M^\infty(\mathbb{R}^d)$ and

$$\|\sum_{k=0}^\infty f_k\|_{M^\infty} \lesssim \sup_{k\ge 0} \|f_k\|_{M^\infty}. \tag{40}$$

Proof. The convergence of the series $\sum_{k=0}^\infty f_k$ in $M^\infty(\mathbb{R}^d)$ is straightforward.
We then prove the desired estimate. Choose a window function g with $\operatorname{supp} \hat{g} \subset B_{1/2}(0)$. We can write

$$V_g(f_k)(x,\omega) = (\hat{f}_k * M_{-x}\hat{g})(\omega).$$

Hence, $\operatorname{supp} V_g(f_0) \subset B_{5/2}(0) \subset B_{2^2}(0)$, and

$$\begin{aligned}\operatorname{supp} V_g(f_k) &\subset \{\eta \in \mathbb{R}^d :\ 2^{k-1} - 2^{-1} \le |\eta| \le 2^{k+1} + 2^{-1}\}\\ &\subset \{\eta \in \mathbb{R}^d :\ 2^{k-2} \le |\eta| \le 2^{k+2}\},\end{aligned}$$

for $k \ge 1$. Hence, for each (x,ω), there are at most four non-zero terms in the sum $\sum_{k=0}^\infty V_g(f_k)(x,\omega)$. Using this fact we obtain

$$\begin{aligned}\|\sum_{k=0}^\infty f_k\|_{M^\infty} &\asymp \|\sum_{k=0}^\infty V_g(f_k)\|_{L^\infty} \le \|\sum_{k=0}^\infty |V_g(f_k)|\|_{L^\infty}\\ &\le 4\|\sup_{k\ge 0} |V_g(f_k)|\|_{L^\infty} = 4 \sup_{k\ge 0} \|V_g(f_k)\|_{L^\infty} \asymp \sup_{k\ge 0} \|f_k\|_{M^\infty}.\end{aligned}$$

□

We now proceed with the proof of the boundedness of T (of order $m = -d/2$) on $\mathcal{M}^\infty$. The Littlewood-Paley decomposition of the frequency domain, introduced at the beginning of this section, and Lemma 5.2 yield

$$\begin{aligned}\|Tu\|_{M^\infty} &= \|\sum_{k\geq 0}\psi_k(D)Tu\|_{M^\infty}\\ &\lesssim \sup_{k\geq 0}\|\psi_k(D)Tu\|_{M^\infty}\\ &\leq \sup_{k\geq 0}\sum_{j=1}^{\infty}\|\psi_k(D)T^{(j)}u\|_{M^\infty}, \qquad (41)\end{aligned}$$

where, as before, $T=\sum_{j=1}^\infty T^{(j)}$, with $T^{(j)}$ having symbol $\sigma_j(x,\eta):=\sigma(x,\eta)\psi_j(\eta)$ and the same phase Φ. Notice that the sequence of symbols $\sigma_j(x,\eta)$ is bounded in $S^{-d/2}_{1,0}$, whereas the sequence of symbols $\psi_k(\eta)$ is bounded in $S^0_{1,0}$.

Applying Theorem 3.1 to each product $\psi_k(D)T^{(j)}$, we have

$$\psi_k(D)T^{(j)}=S_{k,j}+R_{k,j},$$

where $S_{k,j}$ are FIOs with the same phase Φ and symbols $\sigma_{k,j}$ belonging to a bounded subset of $S^{-d/2}_{1,0}$, supported in

$$\{(x,\eta)\in\Omega\times\Gamma:\ |\nabla_x\Phi(x,\eta)|\leq 2, 2^{j-1}\leq|\eta|\leq 2^{j+1}\},\quad \text{if } k=0, \qquad (42)$$

and in

$$\{(x,\eta)\in\Omega\times\Gamma:\ 2^{k-1}\leq|\nabla_x\Phi(x,\eta)|\leq 2^{k+1}, 2^{j-1}\leq|\eta|\leq 2^{j+1}\},\quad \text{if } k\geq 1. \qquad (43)$$

The operators $R_{k,j}$ are smoothing operators whose symbols $r_{k,j}$ are in a bounded subset of $\mathcal{S}(\mathbb{R}^{2d})$, supported where $2^{j-1}\leq|\eta|\leq 2^{j+1}$.

Observe that, by the Euler's identity and (26),

$$|\nabla_x\Phi(x,\eta)|=|\partial^2_{x,\eta}\Phi(x,\eta)\eta|\asymp|\eta|,\quad \forall(x,\eta)\in\Omega\times\Gamma.$$

Inserting this equivalence in (42) and (43), we obtain that there exists $N_0>0$ such that $\sigma_{k,j}$ vanishes identically if $|j-k|>N_0$, whence, the right-hand side in (41) is seen to be

$$\leq \sup_{k\geq 0}\sum_{j\geq 1:|j-k|\leq N_0}\|S_{k,j}u\|_{M^\infty}+\sup_{k\geq 0}\sum_{j=1}^{\infty}\|R_{k,j}u\|_{M^\infty}.$$

This expression will be dominated by the M^∞-norm of u if we prove that[1]

$$\|S_{k,j}u\|_{M^\infty}\lesssim\|u\|_{M^\infty} \qquad (44)$$

and

$$\|R_{k,j}u\|_{M^\infty}\lesssim 2^{-j}\|u\|_{M^\infty}. \qquad (45)$$

This last estimate is easy to obtain. Namely, observe that the symbols $2^j r_{k,j}(x,\eta)$ are still in a bounded subset of $\mathcal{S}(\mathbb{R}^{2d})$: since $|\eta|\asymp 2^j$, on the support of $r_{k,j}$, for every $\alpha\in\mathbb{Z}^d_+$, $N\geq 0$,

$$|\partial^\alpha_x\partial^\beta_\eta r_{k,j}(x,\eta)|\lesssim_{\alpha,\beta,N}(1+|x|+|\eta|)^{-N}\lesssim 2^{-j}(1+|x|+|\eta|)^{-N+1}.$$

This implies that the corresponding operators $2^jR_{k,j}$ are uniformly bounded on $\mathcal{M}^\infty$, i.e. (45).

---

[1] Of course, as always, we mean that the constant which is implicit in the notation $\lesssim$ is independent of k,j.

It remains to prove (44). To this end, we make use of the dilation operator U_λ, as before. Precisely, we recall from Theorem 2.1 that

$$\|U_\lambda f\|_{M^\infty} \lesssim \|f\|_{M^\infty}, \quad \lambda \geq 1, \tag{46}$$

and

$$\|U_\lambda f\|_{M^\infty} \lesssim \lambda^{-d}\|f\|_{M^\infty}, \quad 0 < \lambda \leq 1. \tag{47}$$

Write

$$S_{k,j} := U_{2^{j/2}}\tilde{S}_{k,j}U_{2^{-j/2}},$$

where $\tilde{S}_{k,j}$ is the FIO with phase $\Phi_j(x,\eta)$ defined in (32), and symbol

$$\tilde{\sigma}_{k,j}(x,\eta) := \sigma_{k,j}(2^{-j/2}x, 2^{j/2}\eta).$$

Hence, taking into account (46), (47), we see that (44) will follow from

$$\|\tilde{S}_{k,j}u\|_{M^\infty} \lesssim 2^{-jd/2}\|u\|_{M^\infty}.$$

This last estimate is a consequence of Proposition 3.3 applied to $2^{jd/2}\tilde{S}_{k,j}$. Indeed, using the notation above, namely $\Omega'_j := \{2^{j/2}x,\ x \in \Omega'\}$, $\Omega_j := \{2^{j/2}x,\ x \in \Omega\}$, we have already observed that (36), (37), (38), (39) hold. Moreover, $\tilde{\sigma}_{k,j}(x,\eta)$ is supported where $2^{j/2-1} \leq |\eta| \leq 2^{j/2+1}$, $x \in \Omega'_j$, $\eta \in \Gamma'$, and satisfies

$$|\partial_x^\alpha\partial_\eta^\beta\tilde{\sigma}_{k,j}(x,\eta)| \lesssim 2^{-j\frac{d}{2}-j\frac{|\alpha|+|\beta|}{2}}.$$

This concludes the proof of Theorem 1.2.

6. Sharpness of the results

In this section we prove the sharpness of Theorems 1.1 and 1.2. Precisely, for every $m > -d\left|\frac{1}{2}-\frac{1}{p}\right|$, $1 \leq p \leq \infty$, there are FIOs of the type (1), satisfying the assumptions in the Introduction, which do not extend to bounded operators on $\mathcal{F}L^p_{\mathrm{comp}}$, $1 \leq p < \infty$, nor on the closure of $\mathcal{C}_0^\infty(\mathbb{R}^d)$ in $\mathcal{F}L^\infty_{\mathrm{comp}}$, and therefore do not extend to bounded operators on M^p, $1 \leq p < \infty$, nor on $\mathcal{M}^\infty$.

The key idea is that the composition operator $f \to f \circ \varphi$, with $\varphi : \mathbb{R} \to \mathbb{R}$ being a non-linear C^1 change of variables, is unbounded on the space $\mathcal{F}L^p(\mathbb{R})_{\mathrm{loc}}$ [1, 15].

6.1. **Some auxiliary results.**

We need to recall the van der Corput Lemma (see, e.g., [21, Proposition 2, page 332]).

Lemma 6.1. *Suppose ϕ is real-valued and smooth in $(a,b) \subset \mathbb{R}$ and that $|\phi^{(k)}(t)| \geq 1$ for all $t \in (a,b)$ and for some $k \geq 2$. Then, for every $\lambda > 0$,*

$$\left|\int_a^b e^{i\lambda\phi(t)}\,dt\right| \leq c_k\lambda^{-1/k}, \tag{48}$$

where the bound c_k is independent of ϕ and λ.

Proposition 6.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ diffeomorphism, whose restriction to the interval $(0,1)$ is a non-linear diffeomorphism on $(0,1)$. This means that there exists an interval $I \subset (0,1)$ such that $|\varphi''(t)| \geq \rho > 0$, for all $t \in I$. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\chi \geq 0$, with supp $\chi \subset (0,1)$ and $|(\mathrm{supp}\chi) \cap \varphi(I)| > 0$. Then, if we set*

$$f_n(t) = \chi(t)e^{2\pi i n t}, \quad n \in \mathbb{N}, \tag{49}$$

for $1 \leq p \leq 2$, *we have*

$$\|f_n \circ \varphi\|_{\mathcal{F}L^p} \geq c(p, \varphi, \chi)\, n^{1/p-1/2}, \qquad \forall n \in \mathbb{N}. \tag{50}$$

Proof. The proof follows the pattern of [15, page 219]. In place of the sequence $\{e^{int}\}_{n\geq 1}$ used there, here we consider the smooth compactly supported sequence of functions $\{f_n\}_{n\geq 1}$ in (49).

The assumptions on χ, Parseval's identity, and Hölder's inequality yield

$$\begin{aligned} 0 < C = \int_I \chi(\varphi(t))\, dt &= \left| \int_{\mathbb{R}} f_n(\varphi(t))(\mathbb{1}_I(t) e^{-2\pi i n \varphi(t)})\, dt \right| \\ &= \left| \int_{\mathbb{R}} \mathcal{F}^{-1}(f_n \circ \varphi)(\eta) \mathcal{F}(\mathbb{1}_I e^{-2\pi i n \varphi})(\eta)\, d\eta \right| \\ &\leq \|\mathcal{F}(f_n \circ \varphi)\|_{L^p} \|\mathcal{F}(\mathbb{1}_I e^{-2\pi i n \varphi})\|_{L^{p'}} = \|f_n \circ \varphi\|_{\mathcal{F}L^p} \|\mathbb{1}_I e^{-2\pi i n \varphi}\|_{\mathcal{F}L^{p'}}, \end{aligned} \tag{51}$$

for $1/p + 1/p' = 1$.

Let us estimate $\|\mathbb{1}_I e^{-2\pi i n \varphi}\|_{\mathcal{F}L^{p'}}$. The van der Corput Lemma (Lemma 6.1), for $\lambda = 2\pi\rho n$, $\phi(t) = \phi_{n,\eta}(t) = -(\varphi(t)/\rho + t\eta/(\rho n))$, hence $|\phi''(t)| = \left| \frac{\varphi''(t)}{\rho} \right| \geq 1$, $[a,b] = \overline{I}$, gives

$$\left| \mathcal{F}(\mathbb{1}_I e^{-2\pi i n \varphi})(\eta) \right| = \left| \int_a^b e^{2\pi i \rho n \phi(t)}\, dt \right| \leq c_2 (2\pi\rho n)^{-1/2}, \quad \forall n \in \mathbb{N},\ \forall \eta \in \mathbb{R},$$

with the constant c_2 independent of n, η. Taking the L^∞-norm,

$$\|\mathbb{1}_I e^{-2\pi i n \varphi}\|_{\mathcal{F}L^\infty} \leq c(\rho) n^{-1/2}. \tag{52}$$

On the other hand,

$$\|\mathbb{1}_I e^{-2\pi i n \varphi}\|_{\mathcal{F}L^2} = \|\mathbb{1}_I e^{-2\pi i n \varphi}\|_{L^2} = |I|^{1/2}. \tag{53}$$

For $1 < p < 2$ (hence $2 < p' < \infty$), Hölder's inequality gives

$$\begin{aligned} \int_{\mathbb{R}} \left| \mathcal{F}(\mathbb{1}_I e^{-2\pi i n \varphi})(\eta) \right|^{p'} d\eta &\leq \| \left| \mathcal{F}(\mathbb{1}_I e^{-2\pi i n \varphi}) \right|^{p'-2} \|_{L^\infty} \| \left| \mathcal{F}(\mathbb{1}_I e^{-2\pi i n \varphi}) \right|^2 \|_{L^1} \\ &= \|\mathbb{1}_I e^{-2\pi i n \varphi}\|_{\mathcal{F}L^\infty}^{p'-2} \|\mathbb{1}_I e^{-2\pi i n \varphi}\|_{L^2}^2, \end{aligned}$$

whence

$$\|\mathbb{1}_I e^{-2\pi i n \varphi}\|_{\mathcal{F}L^{p'}} \leq \|\mathbb{1}_I e^{-2\pi i n \varphi}\|_{\mathcal{F}L^\infty}^{1-2/p'} \|\mathbb{1}_I e^{-2\pi i n \varphi}\|_{L^2}^{2/p'} \leq c(p, \varphi) n^{-1/2+1/p'}.$$

This last estimate also holds for $p = 1, 2$ (because of (52) and (53)), and inserted in (51) gives

$$\|f_n \circ \varphi\|_{\mathcal{F}L^p} \geq c(p, \varphi, \chi)\, n^{1/p-1/2},$$

for $1 \leq p \leq 2$, as desired. $\square$

The generalization to dimension $d \geq 1$ reads as follows.

Corollary 6.2. *Let* φ *be as in Proposition* 3.2 *and* f_n *as defined in* (49). *We define*

$$\tilde{f}_n(t_1, \ldots, t_d) = f_n(t_1) \cdots f_n(t_d), \quad \tilde{\varphi}(t_1, \ldots, t_d) = (\varphi(t_1), \ldots, \varphi(t_d)). \tag{54}$$

Then

$$\|\tilde{f}_n \circ \tilde{\varphi}\|_{\mathcal{F}L^p(\mathbb{R}^d)} \geq c(p, \varphi, \chi)\, n^{d(1/p-1/2)}, \tag{55}$$

for $1 \leq p \leq 2$.

We also need the following result.

Lemma 6.2. *Let $h \in \mathcal{S}(\mathbb{R}^d)$, $\langle t\rangle^m = (1+|t|^2)^{m/2}$, $m \in \mathbb{R}$. Then, for $y \in \mathbb{R}^d$ and $1 \le p \le \infty$,*

$$\|hT_y\langle\cdot\rangle^m\|_{M^p} \le c(h,p)\langle y\rangle^m. \tag{56}$$

Proof. For a non-zero window function $g \in \mathcal{S}(\mathbb{R}^d)$, we have

$$V_g(hT_y\langle\cdot\rangle^m)(x,\eta) = \int_{\mathbb{R}^d} e^{-2\pi i\eta t}\langle t-y\rangle^m h(t)\overline{g(t-x)}\,dt.$$

Let us show that the STFT $V_g(hT_y\langle\cdot\rangle^m)$ is in $L^p(\mathbb{R}^{2d})$ with the majorization (56). For $N_1 \in \mathbb{N}$, an integration by parts gives

$$V_g(hT_y\langle\cdot\rangle^m)(x,\eta) = (1+|2\pi\eta|^2)^{-N_1}\int_{\mathbb{R}^d} e^{-2\pi i\eta t}(1-\Delta_t)^{N_1}[\langle t-y\rangle^m h(t)\overline{g(t-x)}]dt.$$

By Petree's inequality,

$$|\partial_t^\alpha\langle t-y\rangle^m| \lesssim \langle t-y\rangle^{m-|\alpha|} \lesssim \langle y\rangle^{m-|\alpha|}\langle t\rangle^{|m-|\alpha||}, \quad \forall\alpha\in\mathbb{Z}_+^d,\ m\in\mathbb{R}.$$

The functions g,h are in $\mathcal{S}(\mathbb{R}^d)$, so that

$$|\partial_t^\beta g(t-x)| \lesssim \langle t-x\rangle^{-N_2} \lesssim \langle x\rangle^{-N_2}\langle t\rangle^{N_2},$$
$$|\partial_t^\gamma h(t)| \lesssim \langle t\rangle^{-N_3},\ \forall N_2,N_3\in\mathbb{N},\ \forall\beta,\gamma\in\mathbb{Z}_+^d.$$

Hence

$$\begin{aligned}|V_g(hT_y\langle\cdot\rangle^m)(x,\eta)| &\lesssim (1+|\eta|^2)^{-N_1}\langle x\rangle^{-N_2}\sup_{|\alpha|\le N_1}\langle y\rangle^{m-|\alpha|}\langle t\rangle^{|m-|\alpha||+N_2-N_3}\\ &\lesssim \langle y\rangle^m\langle t\rangle^{|m|+N_1+N_2-N_3}\langle\eta\rangle^{-2N_1}\langle x\rangle^{-N_2}, \quad \forall N_1,N_2,N_3\in\mathbb{N}.\end{aligned}$$

Choosing N_1, N_2 such that $2pN_1 > d$, $pN_2 > 2$, and N_3 such that $p(N_3-N_1-N_2-|m|) > d$, we attain the desired result. □

We use the previous lemma to compute the action of the multiplier $\langle D\rangle^m$ on the functions $\tilde{f}_n$. Precisely,

Corollary 6.3. *Let $m\in\mathbb{R}$ and $\tilde{f}_n$ be as defined in* (54)*. Then,*

$$\|\langle D\rangle^m\tilde{f}_n\|_{M^p} \le c(\chi,p)n^m. \tag{57}$$

Proof. Using the invariance of the modulation spaces M^p under Fourier transform, we have

$$\|\langle D\rangle^m\tilde{f}_n\|_{M^p} \asymp \|\langle\cdot\rangle^m\widehat{\tilde{f}_n}\|_{M^p} = \|\langle\cdot\rangle^m T_{\tilde{n}}\widehat{\tilde{\chi}}\|_{M^p} = \|(T_{-\tilde{n}}\langle\cdot\rangle^m)\widehat{\tilde{\chi}}\|_{M^p},$$

with $\tilde{\chi}(t_1,\dots,t_d) = (\chi\otimes\cdots\otimes\chi)(t_1,\dots,t_d)$, χ defined in Proposition 6.1, and $\tilde{n} = (n,\dots,n)$. The previous lemma and the estimate $\langle\tilde{n}\rangle \lesssim d^{1/2}n$ yield the majorization (57). □

We can now prove the sharpness of Theorems 1.1 and 1.2. It is clear that it would be sufficient to prove the sharpness of Theorem 1.1, because of Lemma 2.1 and the fact that our operators have symbols compactly supported in x. However, we start with showing the optimality of Theorem 1.2, and then we show how the argument above in fact gives the optimality of Theorem 1.1 as well.

6.2. **Sharpness of Theorem 1.2.**

Sharpness for $1 \leq p \leq 2$. Consider the FIO

$$T_{\tilde\varphi} f(x) = f \circ \tilde\varphi(x) = \int_{\mathbb{R}^d} e^{2\pi i \tilde\varphi(x)\eta} \hat f(\eta)\, d\eta,$$

where $\tilde\varphi$ is defined in (54). We require that the one-dimensional diffeomorphism φ satisfies the assumptions of Proposition 6.1 and the additional hypothesis

$$0 < c \leq |\varphi'(x)| \leq C, \quad \forall x \in \mathbb{R}, \tag{58}$$

Then, the phase $\Phi(x,\eta) = \tilde\varphi(x)\eta$ fulfills the standard assumptions in the Introduction; in particular it is non-degenerate. Notice that $T_{\tilde\varphi}$ maps $\mathcal{C}_0^\infty(\mathbb{R}^d)$ into itself and $\operatorname{supp} T_{\tilde\varphi} f \subset (0,1)^d$ if $\operatorname{supp} f \subset (0,1)^d$.

We are interested in an FIO with symbol σ of order m and with compact support with respect to the x-variable. So, let $G \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $G \geq 0$ and $G \equiv 1$ on $[0,1]^d$, and consider the FIO F defined by

$$Ff(x) = G(x)[(T_{\tilde\varphi}\langle D\rangle^m) f](x) = \int_{\mathbb{R}^d} e^{2\pi i \tilde\varphi(x)\eta} G(x)\langle \eta\rangle^m \hat f(\eta)\, d\eta. \tag{59}$$

The symbol $\sigma(x,\eta) = G(x)\langle\eta\rangle^m$ is of order m with compact support in x. So, if m satisfies (4), Theorem 1.2 assures the boundedness of F on M^p. We now show that this threshold is sharp for $1 \leq p \leq 2$. Indeed, consider the functions $\tilde f_n$ in (54). They are supported in $(0,1)^d$, so $T_{\tilde\varphi}\tilde f_n$ are. Hence, applying the estimate (55) and Lemma 2.1, we obtain

$$\begin{aligned} n^{d(1/p-1/2)} &\lesssim \|\tilde f_n \circ \tilde\varphi\|_{\mathcal{F}L^p(\mathbb{R}^d)} = \|T_{\tilde\varphi}\tilde f_n\|_{\mathcal{F}L^p(\mathbb{R}^d)} = \|G T_{\tilde\varphi}\tilde f_n\|_{\mathcal{F}L^p(\mathbb{R}^d)} \\ &\asymp \|G T_{\tilde\varphi}\tilde f_n\|_{M^p(\mathbb{R}^d)} = \|G T_{\tilde\varphi}\langle D\rangle^m \langle D\rangle^{-m}\tilde f_n\|_{M^p(\mathbb{R}^d)} \\ &\lesssim \|F\|_{M^p\to M^p}\|\langle D\rangle^{-m}\tilde f_n\|_{M^p(\mathbb{R}^d)} \lesssim \|F\|_{M^p\to M^p}\, n^{-m}, \end{aligned}$$

where the last inequality is due to (57). For $n \to \infty$, we obtain $-m \geq d(1/p - 1/2)$, i.e., (4).

Sharpness for $2 < p \leq \infty$**.** Observe that the adjoint operator $T^*_{\tilde\varphi}$ of the above FIO $T_{\tilde\varphi}$ is still an FIO given by

$$T^*_{\tilde\varphi} f(x) = \frac{1}{|J_{\tilde\varphi}(\tilde\varphi^{-1}(x))|}\int_{\mathbb{R}^d} e^{2\pi i \widetilde{\varphi}^{-1}(x)\eta} f(\eta)\, d\eta,$$

with $\widetilde{\varphi}^{-1}(x_1,\dots,x_d) = (\varphi^{-1}(x_1),\dots,\varphi^{-1}(x_d))$ and $|J_{\tilde\varphi}|$ the Jacobian of $\tilde\varphi$. Its phase $\Phi(x,\eta) = \widetilde{\varphi}^{-1}(x)\eta$ still fulfills the standard assumptions.

Now, let $H \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $H \geq 0$, and $H(x) \equiv 1$ on supp $(G\circ\varphi^{-1})$. We define the operator

$$\tilde F f(x) = H(x)[\langle D\rangle^m T^*_{\tilde\varphi}(Gf)](x). \tag{60}$$

Using Theorem 3.1, it is easily seen that $\tilde F$ is an FIO of order m, with symbol compactly supported in the x-variable. Its adjoint is given by

$$\tilde F^* = G T_{\tilde\varphi}\langle D\rangle^m H = F + R, \tag{61}$$

where F is defined in (59) and the remainder R is given by

$$Rf(x) = G(x)[T_{\tilde\varphi}\langle D\rangle^m((H-1)f)](x).$$

If we choose a function $\tilde{G} \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $\tilde{G} \equiv 1$ on supp G, we can write

$$\begin{aligned} Rf &= \tilde{G}(x)G(x)[T_{\tilde{\varphi}}\langle D\rangle^m((H-1)f)](x) \\ &= \tilde{G}(x)T_{\tilde{\varphi}}[(G\circ\tilde{\varphi}^{-1})\langle D\rangle^m((H-1)f)](x). \end{aligned}$$

By assumptions, supp $(G\circ\varphi^{-1})\cap$ supp $(H-1)=\emptyset$, so that the pseudodifferential operator

$$f \longmapsto (G\circ\varphi^{-1})\langle D\rangle^m((H-1)f)$$

is a regularizing operator (this immediately follows by the composition formula of pseudodifferential operators; see, e.g., [13, Theorem 18.1.8, Vol. III]): this means that it maps $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$. The operator $T_{\tilde{\varphi}}$ is a smooth change of variables, so $\tilde{G}(x)T_{\tilde{\varphi}}$ maps $\mathcal{S}(\mathbb{R}^d)$ into itself. To sum up, the remainder operator R maps $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$; hence it is bounded on M^p. This means that $\tilde{F}^*$ is continuous on some M^p iff F is.

The operator $\tilde{F}$ is an FIO of the type (1), with symbol of order m and compactly supported in the x variable. Hence it is bounded on M^p if m fulfills (4). We now show that this threshold is sharp for $2<p<\infty$. Indeed, if $\tilde{F}$ were bounded on M^p, then its adjoint $\tilde{F}^*$ would be bounded on $(M^p)'=M^{p'}$, with $1<p'<2$, and the same for F. But the former case gives the boundedness of F on $M^{p'}$ iff $-m\geq d(1/p'-1/2)=d(1/2-1/p)$, which is the desired threshold. For $p=\infty$, if $\tilde{F}$ were bounded on $\mathcal{M}^\infty$, its adjoint $\tilde{F}^*$ would be bounded on $(\mathcal{M}^\infty)'=M^1$ and the former argument applies.

6.3. **Sharpness of Theorem 1.1.** We start with an elementary remark. Consider an FIO T satisfying the hypotheses in the Introduction. Suppose that it does *not* satisfy an estimate of the type

$$\|Tu\|_{M^p}\leq C\|u\|_{M^p}, \quad \forall u\in\mathcal{S}(\mathbb{R}^d)$$

(hence $m>-d\left|\frac{1}{2}-\frac{1}{p}\right|$). Suppose, in addition, that the distribution kernel of $K(x,y)$ of T has the property that the two projections of $\operatorname{supp} K$ on $\mathbb{R}^d_x$ and $\mathbb{R}^d_y$ are bounded sets. Then, by Lemma 2.1, one sees that there exist compact subsets $K,K'\subset\mathbb{R}^d$ and a sequence of Schwartz functions u_n, $n\in\mathbb{N}$, such that

$$\operatorname{supp} u_n\subset K, \quad \operatorname{supp} Tu_n\subset K', \quad \forall n\in\mathbb{N},$$

and

$$\|Tu_n\|_{\mathcal{F}L^p}\geq n\|u_n\|_{\mathcal{F}L^p}, \quad \forall n\in\mathbb{N}.$$

Hence T does not extend to a bounded operator on $\mathcal{F}L^p_{\mathrm{comp}}$ if $1\leq p<\infty$, nor on the closure of the test functions in $\mathcal{F}L^\infty_{\mathrm{comp}}$ if $p=\infty$.

Taking this fact into account, we see that the operator $\tilde{F}$ in (60) provides the desired counterexample for $2<p\leq\infty$ if $m>-d\left|\frac{1}{2}-\frac{1}{p}\right|$.

Similarly, the operator $\tilde{F}^*$ in (61) provides the counterexample for $1\leq p\leq 2$.

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