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FLUCTUATIONS OF THE FRONT IN A ONE DIMENSIONAL MODEL OF $X + Y \to 2X$

FRANCIS COMETS, JEREMY QUASTEL, AND ALEJANDRO F. RAMÍREZ

Abstract. We consider a model of the reaction $X + Y \to 2X$ on the integer lattice in which Y particles do not move while X particles move as independent continuous time, simple symmetric random walks. Y particles are transformed instantaneously to X particles upon contact. We start with a fixed number $a \geq 1$ of Y particles at each site to the right of the origin. We prove a central limit theorem for the rightmost visited site of the X particles up to time t and show that the law of the environment as seen from the front converges to a unique invariant measure.

1. Introduction

Consider the following microscopic model of a combustive reaction or epidemic on the integer lattice $\mathbb{Z}$. There are two types of particles: X particles, which move as independent, continuous–time, symmetric, nearest neighbor random walks of total jump rate 2; and Y particles, which do not move. Initially there are $a \geq 1$ particles of type Y at sites $1, 2, \ldots$, and at least one X particle at 0. When an X particle jumps to a site where there are Y particles, all a of them immediately become X particles and start moving as rate-2 continuous-time symmetric random walks.

We are interested in the asymptotic behavior of the rightmost site r_t visited by the X particles up to time t, which we call the *front*.

Let $\eta(t, x)$ denote the number of X particles at $x \in \mathbb{Z}$ at time $t \geq 0$. Since there are always exactly a of the Y particles at each $x > r_t$ we do not really have to keep track of them and we can just think of an X particle as branching into $a + 1$ particles when it jumps to $r + 1$, with the result that there are $a + 1$ particles at the new rightmost visited site, $r + 1$.

The state space of our process will be

(1) $\mathbb{S} = \{(r, \eta) \ : \ r \in \mathbb{Z}, \eta \in \mathbb{N}^{\{\ldots,r-1,r\}} \text{ such that } 0 < \sum_{x \leq r} e^{\theta(x-r)}\eta(x) < \infty\},$

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where $\theta > 0$ is chosen sufficiently small; see (7). The infinitesimal generator is

$$\mathcal{L}f(r,\eta) = \sum_{x,x+e\leq r} \eta(x)(f(r,\eta-\delta_x+\delta_{x+e}) - f(r,\eta))$$
$$+\eta(r)(f(r+1,\eta-\delta_r+(a+1)\delta_{r+1}) - f(r,\eta)),$$

where δ_x denotes the configuration with one particle at x. In Section 6 we will define the process with state space $\mathbb{S}$ and show that the process is strong Markov when $\mathbb{S}$ is endowed with the appropriate metric.

In [13] it is shown (see also [1] and [2]) that there exists $v \in (0,\infty)$, such that a.s.,

$$\lim_{t\to\infty} r_t/t = v.$$

A shape theorem was proved in higher dimensions. These proofs are based on the subadditive ergodic theorem.

However, such methods do not go beyond law of large numbers type results. Here we are interested in the fluctuations of r_t. The main result is:

Theorem 1. *There exists $\sigma^2 \in (0,\infty)$, nonrandom, and independent of initial conditions in $\mathbb{S}$, such that*

$$B_t^\epsilon := \epsilon^{1/2}\left(r_{\epsilon^{-1}t} - \epsilon^{-1}vt\right), \qquad t \geq 0, \tag{2}$$

converges in law as $\epsilon \to 0$ to Brownian motion with variance σ^2.

The method is based on a renewal structure. There exists a sequence of regeneration times $\{\kappa_n : n \geq 1\}$, with bounded second moments, with the property that for each $n \geq 1$ the particles $\eta(x,\kappa_n)$, $x < r_{\kappa_n}$ do not affect the behaviour of r_t for $t \geq \kappa_n$. While it does give a central limit theorem, the disadvantage of the method is that it appears to be restricted to one dimensional systems.

As a side benefit we are able to study the ergodic theory of the process as observed from the front.

Theorem 2. *Consider the process as seen from the front, $\tau_{-r_t}\eta(t)$, and denote its distribution by μ_t. There is a unique invariant measure μ_∞ and $\mu_t \Rightarrow \mu_\infty$.*

The model we are studying has been considered in the physics literature (see [11] and the references therein). Recently there has been a resurgence of interest in such models because, especially in one and two dimensions, strong deviations from mean field behavior were detected experimentally.

Mathematically much less is known. In [5] a model is studied in which particles perform random walks with exclusion and particles are created at empty sites by contact. The position of the rightmost particle satisfies a law of large numbers with a computable speed.

Kesten and Sidoravicius [8] consider a similar model in which the Y particles move as well. Let D_X and D_Y denote the jump rates of the two types. If $D_X = D_Y > 0$, they prove a shape theorem in $\mathbb{Z}^d$. When $D_X \neq D_Y$, they can only obtain a linear upper bound. Some related results, within the context of diffusion limited aggregation on a tree, were previously obtained by Barlow, Pemantle and Perkins [3].

One thing which makes these problems difficult is the slow convergence of the process as seen from the front to its equilibrium. There is no gap and the front is of *pulled* type in the physics jargon (see [14]). Decomposing $r_t = M_t + \int_0^t \eta(s,r_s)ds$, where M_t is a martingale, we see that to prove a central limit theorem requires time integrability of the correlations, $\int_0^\infty \langle \eta(t, r_t); \eta(0, r_0)\rangle dt < \infty$. To see that the problem is subtle, start, for example, with one X particle at the origin. The probability that the front has not moved by time t is that of a random walk not having hit 1 by time t which is $\mathcal{O}(t^{-1/2})$. Hence

$$||\mu_t - \mu_\infty||_{TV} \geq \mathcal{O}(t^{-1/2}),$$

and we see that the central limit theorem can only hold because we are *typically* in a much better situation.

In [6] we considered a preliminary model in which any particle which jumps to a site with M particles is immediately killed. This was done to simplify the renewal structure. In the unbounded case (with no killing) one has to show that at the regeneration times, the configuration behind the front is not uncontrollably bad; otherwise there is no way to bound moments of the regeneration times. The difficulty in constructing regeneration times appears to be very common when dealing with dynamic environments (see [4] for an example in which the environment is assumed to be rapidly mixing). The only way we have been able to control this problem is by introducing the series in (1), which is essentially a Lyapunov function. One of the basic ideas of the proof is that the front moves ballistically while the individual walks move diffusively. An individual walk starting behind the front then has a nonzero probability of never hitting it. If there is an exponential bound as in (1), it is not hard to see that there is a nonzero probability that none of the particles behind the front ever hit it, and hence one can expect to find regeneration times with good control on their tails.

In Section 2, we construct the process out of a collection of random walks. This allows us to follow various groups of particles. Section 2 also contains a lower bound on the front velocity produced by the first few particles. The renewal structure is constructed in Section 3, following the approach of [16]. This is used in Section 4 to prove the law of large numbers for the front, the central limit theorem for the front, Theorem 1, and the asymptotics of the environment viewed from the front, Theorem 2. In Section 5 we prove the main estimates, providing the finiteness of second moments of the regeneration times. Finally, in Section 6, it is proved that the process is strong Markov on $\mathbb{S}$.

2. Setup and preliminary definitions

We start with a construction of the process out of a large collection of independent, continuous-time, symmetric, simple rate-2 random walks: $\{Y_{x,i}(t)\}$, $x \in \mathbb{Z}$, $i = 1, 2, \ldots$, with $Y_{x,i}(0) = x$. Consider an initial condition $(0, \eta) \in \mathbb{S}$ with $\sum_{x \leq r} \eta(x) < \infty$. For each $x \leq 0$, and $1 \leq i \leq \eta(x)$, let $Z_{x,i}(t) := Y_{x,i}(t)$, $t \geq 0$. Let τ_1 be the first time that one of the random walks $Z_{x,i}(t)$, $x \leq 0$, hits 1. For $0 \leq t < \tau_1$, let $r_t := 0$ and $\eta(z, t) = \sum_{x \leq 0,\ i} 1(Z_{x,i}(t) = z)$. At time τ_1 we add particles $\{Z_{1,1}, \ldots, Z_{1,a}\}$, which will then have trajectories $Z_{1,i}(t) := Y_{1,i}(t - \tau_1)$, $t \geq \tau_1$. Let τ_2 be the first time that one of the random walks $Z_{x,i}(t)$, $x \leq 1$, hits 2. For $\tau_1 \leq t < \tau_2$, let $r_t := 1$ and $\eta(z, t) := \sum_{x \leq 1,\ i} 1(Z_{x,i}(t) = z)$.

Continuing in this way, we define the process $\{(r_t, \eta(t)) : t \geq 0\}$ for finite initial conditions. In Section 6 we will show that the definition actually makes sense for initial data in $\mathbb{S}$.

To construct the regeneration times, we will need to look at the process as defined above at certain stopping times, applying the strong Markov property. If we look at our process at a stopping time $T \geq 0$, at each site $z \leq r_T$ we will have some particles, each one carrying a *label* (x, i), x indicating the site where the branching that created it took place, some time in the past. If we make a time shift, setting $t = T \mapsto t = 0$, each particle will have a starting position $z \leq r$ and label (x, i) describing its birthplace. Clearly $x \leq r$ as well, but there is no reason to expect that $z = x$. We have explicitly ruled out the trivial case in which there are no X particles at all, so at least one of these labels x has $x = r$. Let us call $\mathcal{Y}$ this configuration of particle positions with the corresponding labels and define the ordered pair $w := (r, \mathcal{Y})$. We will denote by $\mathbb{P}_w$ the law of the process in the Skorohod space of trajectories taking values in the space of front positions on $\mathbb{Z}$ and labeled particle configurations at the left of the front. Throughout, with a slight abuse of notation, we write $w \in \mathbb{S}$ to indicate that the initial data w of particle positions and labels is such that $(r, \eta) \in \mathbb{S}$.

Now we construct an auxiliary process using only the first M particles, which provides a simple lower bound on the front velocity. Take

$$M = 4(a + 5). \tag{3}$$

Define stopping times $\nu_0 := 0$, and ν_1 as the first time one of the original random walks $\{Y_{r,i} : 1 \leq i \leq a\}$ hits the site $r + 1$. Next, define ν_2 as the first time one of the random walks $\{Y_{z,i} : r \leq z \leq r + 1, 1 \leq i \leq a\}$ hits the site $r + 2$. In general, for $k \geq 2$, we define ν_k as the first time one of the random walks $\{Y_{z,i} : r \vee (r + k - M) \leq z \leq r + k - 1, 1 \leq i \leq a\}$ hits the site $r + k$. For $n \in \mathbb{N}$, let

$$\tilde{r}_t^r := r + n, \qquad \text{if } \textstyle\sum_{k=0}^{n} \nu_k \leq t < \sum_{k=0}^{n+1} \nu_k.$$

Now, observing that for each $0 \leq j \leq M - 1$, the random variables $\{\nu_{Mk+j} : k \geq 1\}$ are independent and have finite moments since $M \geq 3$, we see that

$$\lim_{t \to \infty} \tilde{r}_t^r / t =: \alpha > 0, \qquad \text{a.s.} \tag{4}$$

where α depends only on a and M.

Letting $\tau_0 = 0$, the next lemma is immediate (see Lemma 3 of [6]).

Lemma 1. *Suppose that at $t = 0$ the rightmost visited site is r and $(r, 1), \ldots, (r, a)$ are within the set of labels. Then for $k \geq 1$, $\tau_k - \tau_{k-1} \leq \nu_k$.*

For each $t \geq 0$ and $y \leq r_t$, let

$$\zeta(t, y) := \sum_{(x,i), x < r} 1(Z_{x,i}(t) = y). \tag{5}$$

$\zeta(t, y)$ are the particles at y which were originally produced by branching at $x < r$ where r was the front at time 0. Also for $z_1 < z_2 < r$ we can follow the particles which originated from a branching at $z_1 < x \leq z_2$,

$$\eta_{z_1,z_2}(t, y) := \sum_{(x,i): z_1 < x \leq z_2} 1(Z_{x,i}(t) = y).$$

We will also write $\eta_z(t, y)$ for $\eta_{-\infty,z}(t, y)$. We will use the notation

$$m_{z_1,z_2}(t) := \sum_{x=z_1+1}^{z_2} \eta_{z_1,z_2}(x, t)$$

to denote the total number of such particles that find themselves again in $\{z_1 + 1, \dots, z_2\}$ at time t. Finally for $\theta > 0$ and $t \geq 0$ the *exponential density norm* of particles is given by

$$\phi_z(t) = \phi_z(t, w) := \sum_{x \in \mathbb{Z}} e^{\theta(x - r_t)} \eta_z(t, x).$$

3. Renewal structure

Fix some integer L satisfying

$$aL^{1/4} \geq M, \tag{6}$$

and real numbers θ, α_1 and α_2 satisfying

$$0 < 2\sinh 2\theta < \alpha_1 < \alpha_2 < \alpha = \lim_{t \to \infty} \tilde{r}_t^r / t. \tag{7}$$

Define stopping times

$$W := \inf\{t \geq 0 : \phi_{r-L}(t, r, \eta(0)) \geq e^{\theta(\lfloor \alpha_1 t \rfloor - (r_t - r))}\},$$

the first time that the exponential density norm of the particles which originated from a branching at a site at a distance greater than or equal to L from the initial position of the front increases beyond $e^{\theta(\lfloor \alpha_1 t \rfloor - (r_t - r))}$; and

$$V := \inf\{t \geq 0 : \max_{r-L<z<r} \max_{1 \leq i \leq a} Z_{z,i}(t) > \lfloor \alpha_1 t \rfloor + r\}, \tag{8}$$

the first time some of the particles originating from a branching at a site at a distance smaller than L from the initial position of the front hit the line $\lfloor \alpha_1 t \rfloor + r$. The front r_t is bounded below by $\tilde{r}_t$, which has speed $\alpha > \alpha_1$. Hence $e^{\theta(\lfloor \alpha_1 t \rfloor - (r_t - r))}$ should be small for large times t. When $W = \infty$, none of the particles initially to the left of $r - L$ ever touches $\lfloor \alpha_1 t \rfloor + r$. The separation of the particles behind the front, in terms of the stopping time V related to the closest ones and W related to the farthest ones, is necessary to be able to obtain good tail estimates for the regeneration times. In particular we will need to take L large: Choose p such that

$$0 < pe^{\theta} < 1, \tag{9}$$

and take L so that

$$(a-1)e^{-L\theta} < p. \tag{10}$$

Next define

$$U := \inf\{t \geq 0 : \tilde{r}_t^r - r < \lfloor \alpha_2 t \rfloor\},$$

a slow-down time for the auxiliary front. When $U = \infty$ we know that the real front r_t stays ahead of $\lfloor \alpha_2 t \rfloor$. Because U is defined in terms of $\tilde{r}_t$ instead of r_t, U is independent of V and W. Let

$$D := \min\{U, V, W\}.$$

Also $U \circ \theta_s, V \circ \theta_s$ and $W \circ \theta_s$ are the first times U, V or W happen after time $s \geq 0$, and $D \circ \theta_s := \min\{U \circ \theta_s, V \circ \theta_s, W \circ \theta_s\}$. For each $y \in \mathbb{Z}$, let

$$T_y := \inf\{t \geq 0 : r_t \geq y\},$$

and define for $x \geq r$,

$$J_x := \inf\{j \geq 1 : \phi_{x+(j-1)L}(T_{x+jL}) \leq p \text{ and } m_{x+jL-L^{1/4}, x+jL}(T_{x+jL}) \geq aL^{1/4}/2\}, \tag{11}$$

the first trial after the front visits site x, such that the exponential density norm of particles originating at sites at a distance greater than L from the front, decreases to a quantity smaller than p and such that there are sufficiently many particles originating from sites close to the front which are again there at time T_{x+jL} when the front advances L steps.

Define sequences of $\mathcal{F}_t$-stopping times, $\{S_k : k \geq 0\}$ and $\{D_k : k \geq 1\}$ as follows: $S_0 := 0$, $R_0 := r$, and for $k \geq 0$,

$$S_{k+1} := T_{R_k + J_{R_k} L} \qquad D_{k+1} := D \circ \theta_{S_{k+1}} + S_{k+1}, \qquad R_{k+1} := r_{D_{k+1}}.$$

The $S_k, k \geq 1$ are good times when there is control on the cloud of particles originating from sites far from the front, in the sense that the exponential norm is small enough, and there are enough particles originating from sites close to the front.

For $k \geq 1$, define $U_k := U \circ \theta_{S_k} + S_k$, $V_k := V \circ \theta_{S_k} + S_k$ and $W_k := W \circ \theta_{S_k} + S_k$. Let

$$K := \inf\{k \geq 1 : S_k < \infty, D_k = \infty\},$$

and define the *regeneration time*

$$\kappa := S_K, \tag{12}$$

if $K < \infty$ and $\kappa = \infty$ otherwise. κ is *not* a stopping time.

$\mathcal{G}$, the information up to time κ, is the completion with respect to $\mathbb{P}_w$ of the smallest σ-algebra containing all sets of the form $\{\kappa \leq t\} \cap A, A \in \mathcal{F}_t$.

The following proposition is one of the main steps in the proofs of Theorems 1 and 2 and will be proven in Section 5.

Proposition 1. *For every initial data $w \in \mathbb{S}$,*

$$\kappa < \infty, \qquad\qquad \mathbb{P}_w\text{-a.s.} \tag{13}$$

Let $a\delta_0$ denote initial data with $r = 0$, $\eta(r) = a$ and $\eta(x) = 0$, $x < 0$. Then

$$\mathbb{E}_{a\delta_0}\left[\kappa^2 | U = \infty\right] < \infty \qquad \textit{and} \qquad \mathbb{E}_{a\delta_0}\left[r_\kappa^2 | U = \infty\right] < \infty. \tag{14}$$

Recall the definition (5) of ζ. A second key observation is

Proposition 2. *Let A be a Borel subset of $D([0,\infty); \mathbb{S})$ and let $w \in \mathbb{S}$. Then*

$$\mathbb{P}_w[\tau_{-r_\kappa}\zeta(\kappa + \cdot) \in A | \mathcal{G}] = \mathbb{P}_{a\delta_0}[\eta(\cdot) \in A | U = \infty].$$

Proof. As in [6, p.154], the event $\{\kappa < \infty\}$ is the disjoint union of $\{S_k < \infty, D_k = \infty\}$, $k \geq 1$. Then, the proof can be completed by applying the strong Markov property together with the independence of U_k and $V_k \wedge W_K$ given $\mathcal{F}_{S_k}$ and translation invariance. □

Define the sequence of regeneration times $\kappa_1 \leq \kappa_2 \leq \cdots$ by $\kappa_1 := \kappa$ and for $n \geq 1$,

$$\kappa_{n+1} := \kappa_n + \kappa(w_{\kappa_n + \cdot}),$$

where $\kappa(w_{\kappa_n+\cdot})$ is the regeneration time starting from $w_{\kappa_n+\cdot}$ and we set $\kappa_{n+1} = \infty$ on $\kappa_n = \infty$ for $n \geq 1$. κ_1 is the *first regeneration time* and κ_n is the *n-th regeneration time*.

For each $n \geq 1$, $\mathcal{G}_n$ is the completion with respect to $\mathbb{P}_w$ of the smallest σ-algebra containing all sets of the form $\{\kappa_1 \leq t_1\} \cap \cdots \cap \{\kappa_n \leq t_n\} \cap A$, $A \in \mathcal{F}_{t_n}$. Noting that $\{\kappa_1 = \infty\}$ is a null event for $\mathbb{P}_w$ (Proposition 1) one can see that $\{U < \infty\} \cap \{\kappa_1 < \infty\} = \{\tilde{r}_U \leq r_{\kappa_1}\} \cap \{\kappa_1 < \infty\} \in \mathcal{G}_1$ (see Lemma 5 of [6]). Hence

$\{U = \infty\} \in \mathcal{G}_1$. So we have the following general version of Proposition 2 (see Proposition 2 of [6]),

Proposition 3. *Let A be a Borel subset of $D([0,\infty);\mathbb{S})$ and $w \in \mathbb{S}$. Then*

$$\mathbb{P}_w[\tau_{-r_{\kappa_n}}\zeta(\kappa_n + \cdot) \in A \mid \mathcal{G}_n] = \mathbb{P}_{a\delta_0}[\eta(\cdot) \in A \mid U = \infty].$$

Corollary 1 (The renewal structure). *Let $w \in \mathbb{S}$. (i) Under $\mathbb{P}_w$, $\kappa_1, \kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \ldots$ are independent, and $\kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \ldots$ are identically distributed with law identical to that of κ_1 under $\mathbb{P}_{a\delta_0}[\cdot|U = \infty]$. (ii) Under $\mathbb{P}_w$, $r_{\cdot\wedge\kappa_1}, r_{(\kappa_1+\cdot)\wedge\kappa_2} - r_{\kappa_1}, r_{(\kappa_2+\cdot)\wedge\kappa_3} - r_{\kappa_2}, \ldots$ are independent, and $r_{(\kappa_1+\cdot)\wedge\kappa_2} - r_{\kappa_1}, r_{(\kappa_2+\cdot)\wedge\kappa_3} - r_{\kappa_2}, \ldots$ are identically distributed with law identical to that of r_{κ_1} under $\mathbb{P}_{a\delta_0}[\cdot|U = \infty]$.*

4. Limit theorems

4.1. **Law of large numbers.** We will prove that for every $w \in \mathbb{S}$,

$$\lim_{t\to\infty} \frac{r_t}{t} = v := \frac{\mathbb{E}_{a\delta_0}[r_{\kappa_1}|U = \infty]}{\mathbb{E}_{a\delta_0}[\kappa_1|U = \infty]}. \tag{15}$$

Note that we have that $\kappa_1 < \infty$, $\mathbb{P}_w$-a.s. Hence by Corollary 1 a.s.,

$$\lim_{n\to\infty} \frac{\kappa_n}{n} = \mathbb{E}_{a\delta_0}[\kappa_1|U = \infty] \quad \text{and} \quad \lim_{n\to\infty} \frac{r_{\kappa_n}}{n} = \mathbb{E}_{a\delta_0}[r_{\kappa_1}|U = \infty]. \tag{16}$$

Now, for $t \geq 0$, define $n_t := \sup\{n \geq 0 : \kappa_n \leq t\}$, with the convention $\kappa_0 = 0$. From (16) we see that a.s. $n_t < \infty$. Also, $\lim_{t\to\infty} r_{\kappa_{n_t}}/t = v$. The limit (15) now follows from the observation

$$\lim_{t\to\infty} t^{-1}|r_t - r_{\kappa_{n_t}}| = 0,$$

which is a consequence of the inequality $|r_t - r_{\kappa_{n_t}}| \leq |r_{\kappa_{n_t+1}} - r_{\kappa_{n_t}}|$ and the fact that $\lim_{t\to\infty} r_{\kappa_{n_t}}/t = v$ a.s.

4.2. **Central limit theorem.** Consider the quantity B_t^ϵ defined in (2) and

$$\Sigma_m := \textstyle\sum_{j=1}^m R_j, \tag{17}$$

where $R_j := r_{\kappa_{j+1}} - r_{\kappa_j} - (\kappa_{j+1} - \kappa_j)v$. Now, for $0 \leq t \leq T < \infty$,

$$|B_t^\epsilon - \epsilon^{1/2}\Sigma_{n_{t/\epsilon}}| \leq 2\epsilon^{1/2} \sup_{0\leq n\leq n_{\lfloor\epsilon^{-1}T\rfloor}} (r_{\kappa_{n+1}} - r_{\kappa_n}) + 2v\epsilon^{1/2} \sup_{0\leq n\leq n_{\lfloor\epsilon^{-1}T\rfloor}} (\kappa_{n+1} - \kappa_n).$$

On the other hand, from Proposition 1, we can conclude that for every $u > 0$,

$$\lim\nolimits_{\epsilon\to 0} \mathbb{P}_{a\delta_0}[\epsilon^{1/2} \sup\nolimits_{0\leq n\leq n_{\lfloor\epsilon^{-1}T\rfloor}} (\kappa_{n+1} - \kappa_n) > u] = 0.$$

Hence in probability,

$$\sup_{0\leq n\leq n_{\lfloor\epsilon^{-1}T\rfloor}} \epsilon^{1/2}(\kappa_{n+1} - \kappa_n) \to 0 \quad \text{and} \quad \sup_{0\leq n\leq n_{\lfloor\epsilon^{-1}T\rfloor}} \epsilon^{1/2}(r_{\kappa_{n+1}} - r_{\kappa_n}) \to 0. \tag{18}$$

This proves that $B_t^\epsilon - \epsilon^{1/2}\Sigma_{n_{\epsilon^{-1}t}}$ converges to 0 in probability, uniformly on compact sets of t. From Donsker's invariance principle, we know that $\sqrt{\epsilon}\Sigma_{\cdot/\epsilon}$ converges in law to a Brownian motion with variance $\mathbb{E}_{a\delta_0}[(r_{\kappa_1} - \kappa_1 v)^2|U = \infty]$, where $\Sigma_s, s \geq 0$ now stands for the linear interpolation of $\Sigma_m, m \geq 0$. Using that $\lim_{t\to\infty} n_t/t = 1/\mathbb{E}_{a\delta_0}[\kappa_1|U = \infty]$ we can conclude that as $\epsilon \to 0$, B_t^ϵ converges to a Brownian motion with variance

$$\sigma^2 := \frac{\mathbb{E}_{a\delta_0}[(r_{\kappa_1} - \kappa_1 v)^2|U = \infty]}{\mathbb{E}_{a\delta_0}[\kappa_1|U = \infty]}. \tag{19}$$

4.3. **Nondegeneracy of the variance.** We will show that $\sigma^2 > 0$. It is enough to show that there exists some β, $0 < \beta < v$ such that

$$\mathbb{P}_{a\delta_0}[r_{\kappa_1} = L, L\beta^{-1} \leq \kappa_1 | U = \infty] > 0.$$

Now

$$\mathbb{P}_{a\delta_0}[r_{\kappa_1} = L, L\beta^{-1} \leq \kappa_1, U = \infty] \geq \mathbb{P}_{a\delta_0}[L\beta^{-1} < S_1 < U, D \circ \theta_{S_1} = \infty].$$

But the right-hand side can be written as

$$\mathbb{E}_{a\delta_0}[1(L\beta^{-1} < S_1 < U)\mathbb{E}_{a\delta_0}[1(\min\{V \circ \theta_{S_1}, W \circ \theta_{S_1}\} = \infty)1(U \circ \theta_{S_1} = \infty) \mid \mathcal{F}_{S_1}]].$$

Now note that given $\mathcal{F}_{S_1}$, $U \circ \theta_{S_1}$, $V \circ \theta_{S_1}$ and $W \circ \theta_{S_1}$ are independent. Hence

(20)

$$\begin{aligned}&\mathbb{E}_{a\delta_0}[1(\min\{V \circ \theta_{S_1}, W \circ \theta_{S_1}\} = \infty)1(U \circ \theta_{S_1} = \infty) \mid \mathcal{F}_{S_1}] \\ &\quad = \mathbb{P}_{a\delta_0}[V \circ \theta_{S_1} = \infty \mid \mathcal{F}_{S_1}]\mathbb{P}_{a\delta_0}[W \circ \theta_{S_1} = \infty \mid \mathcal{F}_{S_1}]\mathbb{P}_{a\delta_0}[U \circ \theta_{S_1} = \infty \mid \mathcal{F}_{S_1}].\end{aligned}$$

This implies that

$$\mathbb{P}_{a\delta_0}[L\beta^{-1} < S_1 < U, D \circ \theta_{S_1} = \infty] \geq C\mathbb{P}_{a\delta_0}[L\beta^{-1} < S_1 < U],$$

for some constant $C > 0$. Now, we have to show that $\mathbb{P}_{a\delta_0}[L\beta^{-1} < S_1 < U] > 0$. Note that the event $\{L\beta^{-1} < S_1 < U\}$ contains the following event: one of the initial a particles at 0 jumps to site 1 at some time v_1, such that $\beta^{-1} < v_1 < 2\beta^{-1}$; the other $a-1$ particles initially at 0 stay at the same site during the time interval $[0, 2L\beta^{-1}]$; at time v_1, one of the a particles originating at site 1 jumps to site 2 at some time $v_2 + v_1$ such that $\beta^{-1} < v_2 < 2\beta^{-1}$; the other $a-1$ particles born at site 1 stay at the same site during the time interval $[0, 2L\beta^{-1}]$; in general, if k is such that $3 \leq k \leq L$, at time $v_k + v_{k-1} + \cdots + v_1$ one of the particles born at site k moves to site $k+1$, and $\beta^{-1} < v_k < 2\beta^{-1}$; all other $a-1$ particles born at site k stay at the same site during the time interval $[0, 2L\beta^{-1}]$. Note that $T_L = v_1 + \cdots + v_L$ and at this time we have $\phi_0(T_L) \leq (a-1)e^{-L\theta}$. By (10) this quantity is smaller than p. It is easy to see that the above described event has positive probability.

4.4. **Ergodic theorem.** Let μ_t be the law under $\mathbb{P}_w$ of the process as seen from the front

$$\tau_{-r_t}\eta(t) \in \tilde{\Omega} := \{0, 1, 2, \ldots\}^{\mathbb{Z}_-}.$$

$\tau_{-r_t}\eta(t)$ is itself a Markov process with infinitesimal generator

$$\tilde{\mathcal{L}}f(\eta) = \eta(0)[f(\tau_{-1}(\eta - \delta_0) + a\delta_0) - f(\eta)] + \sum_{x,y \leq 0, |x-y|=1} \eta(x)[f(\eta - \delta_x + \delta_y) - f(\eta)].$$

Let f be a bounded continuous local function f on $\tilde{\Omega}$. Denote by $\ell(f)$ the smallest integer ℓ such that $f(\eta)$ does not depend on $\eta(x), x < -\ell$. The formula

$$\int_{\tilde{\Omega}} f d\mu_\infty = \frac{\mathbb{E}_{a\delta_0}[\int_{\kappa_N}^{\kappa_{N+1}} f(\tau_{-r_s}\eta(s))ds \mid U = \infty]}{\mathbb{E}_{a\delta_0}[\kappa_1 \mid U = \infty]}, \quad N(\alpha_2 - \alpha_1) > \ell(f) \tag{21}$$

defines a probability measure μ_∞ on $\tilde{\Omega}$. The right-hand side of (21) does not depend on N provided that condition $N(\alpha_2 - \alpha_1) > \ell(f)$ holds. This shows that the family of probability measures defined on finite cylinders by this formula is consistent.

Theorem 3. *$\mu_t \to \mu_\infty$ weakly as $t \to \infty$, and μ_∞ is invariant for $\tilde{\mathcal{L}}$.*

Proof. Let f be bounded and continuous on $\tilde{\Omega}$. To prove convergence, first note that the last term in the decomposition

$$\int_{\tilde{\Omega}} f d\mu_t = \mathbb{E}_w[\kappa_{N+1} \leq t, f(\tau_{-r_t}\eta(t))] + \mathbb{E}_w[\kappa_{N+1} > t, f(\tau_{-r_t}\eta(t))]$$

vanishes as $t \to \infty$. Also

$$\begin{aligned}(22)\quad & \mathbb{E}_w[\kappa_{N+1} \leq t, f(\tau_{-r_t}\eta(t))] \\ &= \sum_{k\geq 1, x\in\mathbb{Z}} \mathbb{E}_w[\kappa_{N+k} \leq t < \kappa_{N+k+1}, r_{\kappa_k} = x, f(\tau_{-r_t}\eta(t))] \\ &= \sum_{k\geq 1, x\in\mathbb{Z}} \mathbb{E}_w\Big[r_{\kappa_k} = x,\ \mathbb{E}_w[\kappa_{N+k} \leq t < \kappa_{N+k+1}, f(\tau_{-r_t}\eta(t))|\mathcal{G}_k]\Big] \\ &= \sum_{k\geq 1, x\in\mathbb{Z}} \mathbb{E}_w\Big[r_{\kappa_k}{=}x, \mathbb{E}_w\Big[\kappa_{N+k} \leq t < \kappa_{N+k+1}, f\Big(\tau_{-r_t}\zeta^{(k)}(t-\kappa_k)\Big)\ |\mathcal{G}_k\Big]\Big],\end{aligned}$$

where $\zeta^{(k)}$ is a short notation for $\zeta(\kappa_k + \cdot)$. Note that we have used that $N(\alpha_2 - \alpha_1) > \ell(f)$. By Proposition 3,

$$\begin{aligned}&\mathbb{E}_w\Big[r_{\kappa_k}{=}x, \mathbb{E}_w\Big[\kappa_{N+k} \leq t < \kappa_{N+k+1}, f\Big(\tau_{-r_t}\zeta^{(k)}(t-\kappa_k)\Big)\ |\mathcal{G}_k\Big]\Big] \\ &= \int_0^t \mathbb{P}_w[r_s = x, \kappa_k \in ds]\mathbb{E}_{a\delta_0}\left[\kappa_N \leq t-s < \kappa_{N+1}, f\left(\tau_{-r_{t-s}}\eta(t-s)\right)|U = \infty\right] \\ &\overset{u=t-s}{=} \int_0^t \mathbb{P}_w[r_{t-u} = x, t-\kappa_k \in du]\mathbb{E}_{a\delta_0}\left[\kappa_N \leq u < \kappa_{N+1}, f\left(\tau_{-r_u}\eta(u)\right)|U = \infty\right].\end{aligned}$$

Hence the final term of (22) can be written $\int_0^t \mathcal{N}_t(du)F_f(u)$, where

$$\mathcal{N}_t([0,u]) := \sum_{k\geq 1} \mathbb{P}_w[\kappa_k \in [t-u, t]]$$

and

$$F_f(u) := \mathbb{E}_{a\delta_0}\left[\kappa_N \leq u < \kappa_{N+1}, f\left(\tau_{-r_u}\eta(u)\right)\ |\ U = \infty\right].$$

We will use the following renewal theorem (Theorem 6.2 in [17]): Let $X_1, X_2, \ldots$ be i.i.d. and independent of S_0. The random walk $S_n = S_0 + X_1 + \cdots + X_n$, $n = 0, 1, 2, \ldots$ is a *renewal process* if S_0 is nonnegative and X_k are strictly positive. It has *spread-out* step-lengths if there exists an $r \geq 1$ and a nonnegative measurable function m such that $\int_{\mathbb{R}} m(x)dx > 0$ and for all Borel sets A,

$$P(X_1 + \cdots + X_r \in A) \geq \int_A m(x)dx.$$

Theorem 4 (Renewal theorem)**.** *Let S be a renewal process with spread-out step-lengths and $E[X_1] < \infty$. For Borel sets B, let $N(B) := \sum_{k=0}^{\infty} \mathbf{1}_{\{S_k \in B\}}$. Then for each $h \in [0, \infty)$,*

$$E[N(t+B)] \to |B|/E[X_1]$$

uniformly over Borel sets $B \subset [0,h]$. Here $|B|$ is the Lebesgue measure of B.

We check the spread-out assumption as follows: With T_L the time of the L-th jump for the particle with label $(0,1)$, A the event that all these L jumps are to the right, B the event that no other particle moves between times 0 and 1, we have for $0 < s < t < 1$,

$$\begin{aligned} \mathbb{P}_w[\kappa_2 - \kappa_1 \in (s,t]] &= \mathbb{P}_{a\delta_0}[\kappa_1 \in (s,t] \mid U = \infty] \\ &\geq \frac{\mathbb{P}_{a\delta_0}[T_L \in (s,t], A, B, U \circ \theta_1 = \infty, V \circ \theta_1 = \infty]}{\mathbb{P}_{a\delta_0}[U = \infty]} \\ &= C \int_s^t f_L(u) du, \end{aligned} \tag{23}$$

with f_L the L-fold convolution of the exponential density with rate 2 and C is a constant that (we can check using independence) satisfies $C > 0$. This shows that $\kappa_2 - \kappa_1$ is spread-out. Hence from the renewal theorem,

$$\mathcal{N}_t(B) \to |B| / \mathbb{E}_{a\delta_0}[\kappa_1 \mid U = \infty] \quad \text{as } t \to \infty \tag{24}$$

uniformly over Borel sets B in any finite interval. Since $F_f(u)$ is bounded and measurable, we have

$$\int_{\tilde{\Omega}} f d\mu_t \to \int_{\tilde{\Omega}} f d\mu_\infty.$$

Because the process preserves the set of bounded uniformly continuous functions which separate measures (Proposition 4), any limit measure is invariant. □

5. Expectations and variances of the regeneration times

Throughout, $w \in \mathbb{S}$ will be the initial data of particle positions and labels. We will call r the corresponding front position and η the particle count. C and δ will denote constants which do not depend on the initial data w.

5.1. **Bounds on W.** For each (x,i) that has branched at times less than or equal to t, let

$$M_{x,i}(t) := Z_{x,i}(0) + \sup_{0 \leq s \leq t} |Z_{x,i}(s) - Z_{x,i}(0)|.$$

For each $z \leq r - 1$, let

$$\psi_z(t) = \psi_z(t, r, \eta) := \textstyle\sum_{(x,i), x \leq z} e^{\theta(M_{x,i}(t) - r_t)}.$$

Note that for $t \geq 0$ and $z \leq r - 1$,

$$\phi_z(t) \leq \psi_z(t). \tag{25}$$

By (7) and the intermediate value theorem,

$$\mu := \theta\alpha_1 - 2(\cosh\theta - 1) > 0. \tag{26}$$

This enables us to obtain the following exponential bound. We will use $\phi_z(t, r, \eta) := \phi_z(t, w)$.

Lemma 2. *There exists $C < \infty$ such that for all (r, η) with $\phi_{r-L}(0, r, \eta) < \infty$ and all $t \geq 0$,*

$$\mathbb{P}_w[t < W < \infty] \leq C \phi_{r-L}(0, r, \eta) \exp\{-\mu t\}.$$

Proof. Without loss of generality, $r = 0$. Note that

$$\mathbb{P}_w[t < W < \infty] \leq \mathbb{P}_w\left[\bigcup_{s \geq t} \left\{\phi_{-L}(s) \geq e^{\theta(\lfloor \alpha_1 s \rfloor - r_s)}\right\}\right].$$

Also, from (25), recall that

$$\mathbb{P}_w\left[t < W < \infty\right] \leq \sum_{n=[t]}^{\infty} \mathbb{P}_w\left[\sum_{(x,i),x\leq -L} e^{\theta M_{x,i}(n+1)} \geq e^{\theta\lfloor \alpha_1 n\rfloor}\right]. \tag{27}$$

If $\{X_t : t \geq 0\}$ is a simple symmetric continuous-time rate-2 random walk on $\mathbb{Z}$ with $X_0 = x$, using the reflection principle we can deduce that

$$E\left[\exp\{\theta(x + \sup_{0\leq s\leq t} |X_s - x|)\}\right] \leq 3\exp\{\theta x + 2(\cosh\theta - 1)t\}. \tag{28}$$

So the right-hand side of (27) is bounded by

$$\begin{aligned} &3\sum_{n=[t]}^{\infty} \exp\{2(\cosh\theta - 1)(n+1) - \theta\lfloor\alpha_1 n\rfloor\} \sum_{(x,i),x\leq -L} \exp\{\theta Z_{x,i}(0)\} \\ &\quad\leq 3\phi_{-L}(0)\sum_{n=[t]}^{\infty} \exp\{2(\cosh\theta - 1)(n+1) - \theta\lfloor\alpha_1 n\rfloor\}. \end{aligned} \tag{29}$$

Summing the last expression over n we obtain the lemma. □

Define for $t \geq 0$, and $z \leq r$,

$$N_z(t) := e^{\theta r_t - 2(\cosh\theta - 1)t}\phi_z(t).$$

Lemma 3. *Suppose that (r, η) and z are such that $z \leq r$ and $\phi_z(0, r, \eta) < \infty$. Then, $\{N_z(t) : t \geq 0\}$ is an $\mathcal{F}_t$-martingale.*

Proof. Note that

$$N_z(t) = \sum_{(x,i),x\leq z} e^{\theta Z_{x,i}(t) - 2(\cosh\theta - 1)t}.$$

Each term in the sum is an $\mathcal{F}_t$-martingale and since $\phi_z(0) < \infty$, the martingales $\sum_{(x,i),-n\leq x\leq z} e^{\theta Z_{x,i}(t)-2(\cosh\theta-1)t}$ converge in $L^1(\mathbb{P}_w)$ to $N_z(t)$ as $n \to \infty$. □

Lemma 4. *There is a $\delta > 0$ such that for all (r, η) with $\phi_{r-L}(0, r, \eta) \leq p$,*

$$\mathbb{P}_w\left[W < \infty\right] < 1 - \delta.$$

Proof. Without loss of generality, we can assume that $r = 0$. By (26),

$$\mathbb{P}_w\left[W < \infty\right] \leq \mathbb{E}_w\left[e^{(\theta\alpha_1 - 2(\cosh\theta - 1))W} 1(W < \infty)\right]. \tag{30}$$

Now, from the definition of the exponential density norm and of the stopping time W, the a.s. right-continuity of the trajectories of the random walks, and Fatou's Lemma, it follows that $e^{\theta(\lfloor\alpha_1 W\rfloor - r_W)} \leq \phi_{-L}(W)$. Hence from inequality (30) we conclude that $\mathbb{P}_w\left[W < \infty\right]$ is bounded by

$$e^{\theta}\mathbb{E}_w\left[e^{\theta r_W - 2(\cosh\theta - 1)W}\phi_{-L}(W)1(W < \infty)\right] = e^{\theta}\mathbb{E}_w[N_{-L}(W)1(W < \infty)]. \tag{31}$$

Now, note that $\mathbb{E}[N_{-L}(W)1(W < n)] \leq \mathbb{E}_w[N_{-L}(n\wedge W)]$. By the optional stopping theorem and Fatou's Lemma,

$$\mathbb{E}_w[N_{-L}(W)1(W < \infty)] \leq \lim_{n\to\infty} \mathbb{E}_w[N_{-L}(n \wedge W)] = N_{-L}(0) \leq p.$$

This and condition (9) combined with the fact that the right-hand side of (31) is an upper bound for $\mathbb{P}_w\left[W < \infty\right]$ proves the lemma. □

5.2. **Bounds on V.**

Lemma 5. *There is a $0 < C < \infty$ such that for all all $t \geq 0$,*

$$\mathbb{P}_w\left[t < V < \infty\right] \leq CL \exp\{-t/C\}.$$

Proof. Letting $r = 0$, $P_w\left[t < V < \infty\right]$ is bounded by the probability that one of the random walks born at a site between $-L$ and -1 is at the right of $\lfloor \alpha_1 s \rfloor$ at some time $s \geq t$. The worst case is when they all start at 0, in which case the probability is $aLP[t < \tau < \infty]$, where $\tau := \inf\{t \geq 0 : X_t > \lfloor \alpha_1 t \rfloor\}$ and where $\{X_t : t \geq 0\}$ is a single random walk starting from 0. One easily checks that $P[t < \tau < \infty] \leq C \exp\{-t/C\}$ for some constant C. □

Lemma 6. *There is a $\delta > 0$ such that*

$$\mathbb{P}_w\left[V < \infty\right] < 1 - \delta.$$

Proof. Without loss of generality $r = 0$. Note that the probability $\mathbb{P}_w[V < \infty]$ is upper bounded by the probability that a random walk within a group of aL independent ones all initially at site $x = 0$, at some time $t \geq 0$ is at the right of $\lfloor \alpha_1 t \rfloor$. But this probability is $1 - \gamma^{aL}$, where $\gamma < 1$ is the probability that a single random walk starting form $x = 0$ never beats $\{\lfloor \alpha_1 t \rfloor : t \geq 0\}$. □

5.3. **Bounds on U.** The following lemma can be proved by observing that at each $t \geq \nu_j$, with $j \geq M + 1$, the auxiliary process has at least $M \geq 20$ particles behind the front (see also [6]).

Lemma 7. *There exists $0 < C < \infty$ and $\delta > 0$ such that for all η with particles $\eta(r) \geq a$ and all $t > 0$,*

(1) *for all $M' < M$, we have $\mathbb{P}_w\left[t < U < \infty\right] \leq Ct^{-M'/4}$;*
(2) *$\mathbb{P}_w\left[U < \infty\right] < 1 - \delta$.*

5.4. **Bounds on D.** The following lemma is elementary.

Lemma 8. *There exists $0 < C < \infty$ such that for every $t > 0$, it is true that $\mathbb{P}_{a\delta_0}[\nu_1 > t] \leq Ct^{-a/2}$ and for $j \geq M + 1$, $\mathbb{P}_{a\delta_0}[\nu_j > t] \leq Ct^{-M/2}$. In particular, for every $M' < M$ we have that $\mathbb{E}_{a\delta_0}[\nu_j^{M'/2}] < \infty$.*

From here we obtain the following estimate.

Lemma 9. *Let $\beta \in (0, \alpha)$. Assume that $r = 0$. There exists $0 < C < \infty$ such that*

a) *If $\eta(0) \geq a$, then $\mathbb{P}_w\left[T_n > n/\beta\right] \leq Cn^{-a/2}$.*
b) *If $m_{-L^{1/4},0}(0) \geq aL^{1/4}/2$, then $\mathbb{P}_w\left[T_n > n/\beta\right] \leq (CL^{1/4})^{aL^{1/4}/2} n^{-aL^{1/4}/2} + Cn^{-(M-1)/4}$.*
c) *For all $k \geq M$, $\mathbb{P}_w\left[T_{n+k} - T_k > n/\beta\right] \leq Cn^{-(M-1)/4}$.*

Proof. a) First remark that $\mathbb{P}_w[T_n > n/\beta] \leq \mathbb{P}_{a\delta_0}[T_n > n/\beta]$. Now, $T_n = \sum_{i=1}^n \tau_i$. Hence by Lemma 1 we have $T_n \leq \sum_{j=1}^n \nu_j$. Therefore,

$$\mathbb{P}_{a\delta_0}\left[T_n > n/\beta\right] \leq \mathbb{P}_{a\delta_0}\left[\sum_{i=1}^n \nu_i > n/\beta\right].$$

Choose β' such that $\beta < \beta' < \alpha$. Since $1/\beta = (1/\beta - 1/\beta') + 1/\beta'$ and ν_1 stochastically dominates ν_j for $j \geq 2$, we have, for $n \geq M + 1$,

$$\mathbb{P}_{a\delta_0}\left[T_n > n/\beta\right] \leq M\mathbb{P}_{a\delta_0}\left[\nu_1 > \tfrac{n}{M}\left(\tfrac{1}{\beta} - \tfrac{1}{\beta'}\right)\right] + \mathbb{P}_{a\delta_0}\left[\tfrac{1}{n}\textstyle\sum_{i=M+1}^n \nu_i > \tfrac{1}{\beta'}\right]. \tag{32}$$

But $\mathbb{P}_{a\delta_0}\left[\frac{1}{n}\sum_{i=M+1}^{n}\nu_i > 1/\beta'\right] \le \mathbb{P}_{a\delta_0}\left[\frac{1}{n}\sum_{i=M+1}^{n}\gamma_i > c\right]$, where $\gamma_j := \nu_j - 1/\alpha$ and $c := 1/\beta' - 1/\alpha > 0$. On the other hand, for each $0 \le i < l$, $l = \lfloor (M+1)/(a+1)\rfloor$, $\{\gamma_{kl+i}\}$, $k = 0, 1, 2, \dots$ are independent. Thus,

$$\mathbb{P}_{a\delta_0}\left[\frac{1}{n}\sum_{i=M+1}^{n}\nu_i > \frac{1}{\beta'}\right] \le \sum_{i=0}^{l-1}\mathbb{P}_{a\delta_0}\left[\frac{1}{n}\sum_{k=(M+1-i)/l}^{n}\gamma_{kl+i} > (a+1)c/M\right].$$

For $q \ge 2$, if $X_1, X_2, \dots$ are i.i.d. with mean zero and $E[|X_i|^q] < \infty$, then $E[|\sum_{i=1}^{n} X_i|^q] \le Cn^{q/2}$ for some $C < \infty$ (see, for example, [12], page 60). Since, by Lemma 8, we have $\mathbb{E}_{a\delta_0}[\gamma_j^{(M-1)/2}] < \infty$, it follows that the last expression of the above display is bounded by $Cn^{-(M-1)/4}$ for some new $C < \infty$. Finally observe that $M = 4(a+5)$, and use Lemma 8 to bound the first term of inequality (32) to finish the proof.

b) Note that for $n \ge L^{1/4}$, $T_n \le \rho_0 + \sum_{j=L^{1/4}}^{n}\nu_j$, where ρ_0 is the first time that any random walk within a set of $aL^{1/4}/2$ simple independent random walks starting from site $-L^{1/4}$ hits site $\lfloor L^{1/4}\rfloor$. Choosing $\alpha < \beta' < \beta$, it follows that

$$\mathbb{P}_w\left[T_n > n/\beta\right] \le \mathbb{P}_w\left[\rho_0 > n\left(\frac{1}{\beta} - \frac{1}{\beta'}\right)\right] + \mathbb{P}_w\left[\frac{1}{n}\sum_{i=L^{1/4}}^{n}\nu_i > \frac{1}{\beta'}\right].$$

Now, by (6) and an argument analogous to the one used in part (a), we know that the second term in the right-hand side of the above display is bounded by $C/n^{(M-1)/4}$ for some constant C. Note that there exists a constant C such that the probability that a random walk starting from a site x does not hit 0 before time t is bounded by $C|x|/t^{1/2}$. It follows that

$$\mathbb{P}_w\left[\rho_0 > n\left(\frac{1}{\beta} - \frac{1}{\beta'}\right)\right] \le \left(C\frac{L^{1/4}}{n^{1/2}}\right)^{CL^{1/4}}.$$

c) This follows in analogy to the previous arguments again using (6). □

Corollary 2. *There exist $C = C(p)$, $0 < C < \infty$, and $\delta > 0$ such that whenever $\phi_{r-L}(0, r, \eta) \le p$ and $\eta(r) \ge a$,*

a) $\mathbb{P}_w\left[t < D < \infty\right] \le Ct^{-(M-1)/2} + CLe^{-t/C}$, $t > 0$,

b) $\mathbb{P}_w\left[D < \infty\right] < 1 - \delta$.

Proof. Item a) follows from Lemmas 2, 5 and 7. b) Since W, V and U are independent,

$$\mathbb{P}_w\left[D < \infty\right] = 1 - \mathbb{P}_w\left[W = \infty\right]\mathbb{P}_w\left[V = \infty\right]\mathbb{P}_w\left[U = \infty\right]. \tag{33}$$

So item b) follows from Lemmas 4, 6 and 7. □

We finish this subsection with three lemmas and a corollary which will be subsequently used to obtain estimates for the stopping time S. The following lemma will be proved in Section 6.

Lemma 10. *There exist $0 < C < \infty$ and $\gamma_0 > 0$, not depending on w, such that for all $\gamma \ge \gamma_0$,*

$$\mathbb{P}_w[r_t - r \ge \gamma t] \le \phi_r(0, w)e^{-Ct}, \qquad t \ge 0. \tag{34}$$

Lemma 11. *There exists $C = C(p)$, $0 < C < \infty$, such that whenever $\phi_{r-L}(0, w) \le p$,*

$$\mathbb{P}_w\left[r_D - r > t, D < \infty\right] \le Ct^{-(M-1)/2} + CLe^{-t/C}, \qquad t > 0.$$

Proof. Note that

$$\begin{aligned}\mathbb{P}_w\left[r_D - r > \gamma t, D < \infty\right] &\le \mathbb{P}_w\left[r_D - r > \gamma t, D \le t\right] + \mathbb{P}_w\left[t < D < \infty\right]\\ &\le \mathbb{P}_w\left[r_t - r > \gamma t\right] + \mathbb{P}_w\left[t < D < \infty\right]. \qquad (35)\end{aligned}$$

The statement follows from (34), $\phi_r(0,w) \le \phi_{r-L}(0,w) + aL$ and part (a) of Corollary 2. ☐

Lemma 12. *Suppose $\eta(r) \ge a$ and $\phi_{r-L}(0,w) \le p$. Then, $\mathbb{P}_w$-a.s. on $\{D < \infty\}$,*

$$\phi_{r-L}(D) \le e^{\theta}.$$

Proof. Without loss of generality assume $r = 0$. From $\phi_{-L}(0,w) \le p < 1$, we have $D > 0$. By the definition of U, note that whenever $t \le U < \infty$, we have $\tilde{r}_t \ge \lfloor \alpha_2 t \rfloor$. By Lemma 1 we have $r_t \ge \tilde{r}_t$. It follows that $r_t \ge \lfloor \alpha_2 t \rfloor$. Therefore, if $t \le U < \infty$, we have

$$\lfloor \alpha_1 t \rfloor - r_t \le -(\lfloor \alpha_2 t \rfloor - \lfloor \alpha_1 t \rfloor) \le 0. \qquad (36)$$

Therefore, if $D = U$, inequality (36) shows that $\lfloor \alpha_1 D \rfloor - r_D \le 0$. Hence, since in this case with probability one $D < W$, it follows that

$$\phi_{-L}(D) \le e^{\theta(\lfloor \alpha_1 D \rfloor - r_D)} \le 1. \qquad (37)$$

Similarly, if $D = V$, since $D < U$ and $D < W$ happen with probability one, inequality (37) still holds a.s. On the other hand, if $W < \infty$ we have

$$\phi_{-L}(W) \le e^{\theta} e^{\theta(\alpha_1 W - r_W)}, \qquad (38)$$

since in the worst case scenario at time W all particles jump one step to the right. Hence if $D = W$, since with probability one we have $D < U$, by inequality (36), the exponent on the right-hand side of (38) is nonpositive, so that $\phi_{-L}(W) \le e^{\theta}$. ☐

Corollary 3. *There exists $0 < C < \infty$, such that whenever $\phi_{r-L}(0,w) \le p$ and $\eta(r) \ge a$,*

$$\mathbb{E}_w[\phi_{r_D}(D), D < \infty] < LC.$$

Proof. In the worst case scenario, all particles born between sites $r - L$ and r_D are at site r_D at time D. Then $\phi_{r_D}(D) \le \phi_{r-L}(D) + a(L - r + r_D)$. Hence by Lemma 12, $\phi_{r_D}(D) \le e^{\theta} + a(L + r_D - r)$. Lemma 11 together with $M \ge 4$ finishes the proof. ☐

5.5. **Bounds on S.**

Lemma 13. *There exists $0 < C < \infty$ such that*

a) *Whenever $\eta(r) \ge a$, for all $n \ge 1$, $\mathbb{P}_w[m_{r,r+n}(T_{r+n}) < an/2] \le Cn^{-a/2}$.*

b) *Whenever $\eta(r) \ge a$, for all $n \ge 1$,*

$$\mathbb{P}_w[m_{r_D+n-n^{1/4},r_D+n}(T_{r_D+n}) < an^{1/4}/2] \le Cn^{-a(M-1)/(8(M+1))} + CLe^{-n^{a/(4(M+1))}}. \qquad (39)$$

c) $\lim_{n\to\infty} \mathbb{P}_w[m_{r-n,r+L}(T_{r+L}) < aL/2] = 0.$

Proof. Without loss of generality assume $r = 0$ in the proofs of (a) and (c).

a) Choose $0 < \beta < \alpha$. Then

$$\mathbb{P}_w\left[m_{0,n}(T_n) < an/2\right] \le \mathbb{P}_w\left[m_{0,n}(T_n) < an/2, T_n \le \beta^{-1} n\right] + \mathbb{P}_w\left[T_n > \beta^{-1} n\right]. \qquad (40)$$

Note that the event $\{m_{0,n}(T_n) < an/2, T_n \leq n/\beta\}$ is contained in the event that at least one particle born at any of the sites $\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1, \dots, n$ hits some site $x \leq 0$ in a time shorter than or equal to n/β. Hence we can conclude that

$$\mathbb{P}_w\left[m_{0,n}(T_n) < an/2, T_{0,n} \leq \beta^{-1}n\right] \leq a(n - \lfloor n/2 \rfloor)P[M'_{n/\beta} \geq n/2], \tag{41}$$

where P is the law of a simple symmetric rate-2 random walk $\{X_t : t \geq 0\}$ on $\mathbb{Z}$ starting from 0 and $M'_t := \sup_{0 \leq s \leq t} X_s$. Now, by the reflection principle,

$$P[M'_t \geq x] \leq 2P[X_t \geq x].$$

Hence from inequality (41), we see that

$$\mathbb{P}_w\left[m_{0,n}(T_n) < an/2, T_{0,n} \leq \beta{-}1n\right] \leq a(n+1)P[X_{n/\beta} \geq n/2].$$

But, for every $t \geq 0$ and positive integer x, $P[X_t \geq x] \leq e^{-2tI(x/(2t))}$, where $I(u) = u \sinh^{-1} u - \sqrt{1+u^2} + 1$. Hence

$$a(n+1)P[X_{n/\beta} \geq n/2] \leq (a+1)(n+1)\exp\left\{-\beta^{-1}2nI(\beta/4)\right\}.$$

Finally, using the inequality

$$\mathbb{P}_w[T_n > n/\beta] \leq \mathbb{P}_{a\delta_0}[T_n > n/\beta],$$

part (a) of Lemma 9 to bound the second term of inequality (40) and the fact that $(a+1)(n+1)\exp\{-2n\beta^{-1}I(\beta/4)\} \leq C/n^{a/2}$ for n large enough, we conclude the proof.

b) By a) and Lemma 11, and the strong Markov property, the expression

$$\mathbb{P}_w[m_{r_D+n-n^{1/4}, r_D+n}(T_{r_D+n}) < an/2]$$

is upper-bounded by

$$\begin{aligned}\textstyle\sum_{k=1}^{m} \mathbb{P}_w[m_{k+n-n^{1/4},k+n}(T_{k+n}) < an^{1/4}/2] + \mathbb{P}_w[r_D > m, D < \infty]\\ \leq Cmn^{-a/8} + Cm^{-(M-1)/2} + CLe^{-Cm},\end{aligned}$$

for some $C > 0$ for every $m \geq 1$. Choosing $m^{a/(4(M+1))}$ we obtain (39).

c) Note that

$$\mathbb{P}_w\left[m_{-n,L}(T_L) < aL/2\right] \leq \mathbb{P}_w\left[m_{-n,L}(T_L) < aL/2, T_L \leq n\right] + \mathbb{P}_w\left[T_L > n\right]. \tag{42}$$

Clearly $\lim_{n\to\infty} \mathbb{P}_w\left[T_L > n\right] = 0$. On the other hand, an argument similar to the one used to derive (41), shows that the first term of the right-hand side of (42) is bounded by $aLP[M'_n \geq n]$, which tends to 0 as n tends to ∞. □

To simplify notation, on $\{D < \infty\}$, define

$$F_n := T_{r_D+Ln} - D, \qquad n \geq 1.$$

Lemma 14. *For every $0 < \beta < \alpha$, there exists $C < \infty$ depending only on β, such that whenever $\sum_{i=r-L^{1/4}}^{r} \eta_i \geq aL^{1/4}/2$ and $\phi_{r-L}(0,w) \leq p$, for all $n \geq 1$,*

$$\mathbb{P}_w\left[F_n > \beta^{-1}Ln, D < \infty\right] \leq (CnL^{3/4})^{-(M-1)/4+1} + CLe^{-nL^{3/4}/C}.$$

Proof. Without loss of generality let $r = 0$. Note that $\mathbb{P}_w\left[F_n > \beta^{-1}Ln, D < \infty\right]$ is upper-bounded by

$$\textstyle\sum_{k=1}^{L^{3/4}n} \mathbb{P}_w\left[F_n > \beta^{-1}Ln, r_D = k, D < \infty\right] + \mathbb{P}_w\left[r_D > L^{3/4}n, D < \infty\right]. \tag{43}$$

On $\{D < \infty\}$ we have $T_{r_D} \leq D$, so $F_n \leq T_{r_D+Ln} - T_{r_D}$. Hence

$$\mathbb{P}_w\left[F_n > \beta^{-1}Ln, r_D = k, D < \infty\right] \leq \mathbb{P}_w\left[T_{k+Ln} - T_k > \beta^{-1}Ln\right].$$

Now, by part (c) of Lemma 9, for all $k > M$ we have $\mathbb{P}_w\left[T_{k+Ln} - T_k > \beta^{-1}Ln\right] \le C(nL)^{-(M-1)/4}$, for some $C < \infty$. On the other hand, for $1 \le k \le M$, $\mathbb{P}_w\left[T_{k+Ln} - T_k > \beta^{-1}Ln\right] \le \mathbb{P}_w\left[T_{M+Ln} > \beta^{-1}Ln\right]$. Thus, by part ($b$) of Lemma 9, since the initial condition w has at least $aL^{1/4}/2$ particles to the right of $r = 0$ at a distance strictly less than $L^{1/4}$ to the origin, we know that

$$\mathbb{P}_w\left[T_{M+Ln} > \beta^{-1}Ln\right] \le \left(\frac{CL^{1/4}}{nL}\right)^{aL^{1/4}/2} + C(nL)^{-(M-1)/4},$$

for some other constant $C < \infty$. From (6), we therefore conclude that

$$\sum_{k:1\le k\le L^{3/4}n} \mathbb{P}_w\left[F_n > \beta^{-1}Ln, r_D = k, D < \infty\right] \le (CnL^{3/4})^{-(M-1)/4+1}. \tag{44}$$

Using Lemma 11 to estimate the second term of display (43) and combining this with inequality (44) we finish the proof. □

Next we prove that given $D_{k-1} < \infty$, $S_k < \infty$ a.s. and has tails with appropriate decay. We need the following simple lemma:

Lemma 15. *Let $q \ge 1$ be an integer. Consider a sequence $\{a_k : k \ge 1\}$ of non-negative real numbers such that $\sum_{k=1}^{\infty} a_k < 1$ and $\sum_{k=1}^{\infty} k^q a_k < \infty$. Assume that $\{c_m : m \ge 1\}$ is a sequence such that $c_1 \le a_1$ and*

$$c_m \le a_m + \sum_{k=1}^{m-1} a_{m-k}c_k \tag{45}$$

for every $m \ge 2$. Then $\sum_{k=1}^{\infty} k^q c_k < \infty$.

Proof. We use induction on $0 \le q' \le q$. Let $A_{q'} := \sum_{k=1}^{\infty} k^{q'} a_k$ and $C_{q'} := \sum_{k=1}^{\infty} k^{q'} c_k$. Let us first show that if $A_0 < \infty$, then $C_0 < \infty$. Let $n \ge 2$. Summing (45) from $m = 2$ to m,

$$\sum_{k=1}^{n-1} c_k\left(1 - \sum_{j=1}^{n-k} a_j\right) + c_n \le A_0.$$

Letting $n \to \infty$ we have $\sum_{k=1}^{\infty} c_k \le \frac{A_0}{1-A_0} < \infty$. Now assume that $C_{q'-1} < \infty$ for some $1 \le q' \le q$. We will show that then $C_{q'} < \infty$. Summing (45), multiplied by $m^{q'}$, from $m = 2$ to m we see that

$$\sum_{m=1}^{n} m^{q'} c_m \le A_{q'} + \sum_{m=2}^{n} \sum_{k=1}^{m-1} m^{q'} a_{m-k} c_k. \tag{46}$$

Substituting the binomial expansion $m^{q'} = \sum_{i=0}^{q'} \binom{q'}{i} (m-k)^i k^{q'-i}$ in (46) and interchanging the order of summation, we have

$$\sum_{m=2}^{n-1} m^{q'} c_m \left(1 - \sum_{j=1}^{n-m} a_j\right) + n^{q'} c_n \le A_{q'} + \sum_{i=1}^{q'} \binom{q'}{i} \sum_{k=1}^{n-1} k^{q'-1} c_k \sum_{m=1}^{n-k} m^i a_m.$$

Letting $n \to \infty$, we obtain $C_{q'} \le (1 - A_0)^{-1}\left(A_{q'} + \sum_{i=1}^{q'} \binom{q'}{i} C_{q'-i} A_i\right) < \infty$. □

Lemma 16. *Assume $r = 0$.*

a) *For every $h > 0, s > 0$ and $n \ge 1$ we have*

$$\mathbb{P}_w\left[\psi_0(T_n) > h, T_n < s\right] \le 3h^{-1}\psi_0(0, w)e^{2(\cosh\theta - 1)s - \theta n}. \tag{47}$$

b) *For every $h > 0, s > 0$, $k \ge 1$ and $n \ge k$ we have*

$$\mathbb{P}_w\left[\psi_k(T_n) - \psi_{k-L}(T_n) > h, T_n - T_k < s \middle| \mathcal{F}_{T_k}\right] \le 3h^{-1}aLe^{2(\cosh\theta - 1)s - \theta(n-k)}. \tag{48}$$

Proof. Note that $e^{\theta M_{x,i}(T_n)} \leq e^{\theta M_{x,i}(s)}$ on $T_n < s$. Therefore, since $\psi_0(T_n) = e^{-\theta n} \sum_{(x,i),x\leq 0} e^{\theta M_{x,i}(T_n)}$, we have

$$\psi_0(T_n) \leq e^{-\theta n} \sum_{(x,i),x\leq 0} e^{\theta M_{x,i}(s)}.$$

Now, by (28) we have $\mathbb{E}_w[\sum_{(x,i),x\leq 0} e^{\theta M_{x,i}(s)}] \leq 3\psi_0(0,w)e^{2(\cosh\theta-1)s}$. Hence

$$\mathbb{E}_w\left[\psi_0(T_n)\right] \leq 3\psi_0(0,w)e^{2(\cosh\theta-1)s-\theta n}.$$

Using Tchebyshev's inequality we obtain (47). A similar argument, using the fact that $\psi_k(T_k) - \psi_{k-L}(T_k) \leq aL$, proves (48). □

For $n \geq 1$ let

$$F'_n := T_{r_D+nL} = F_n + D.$$

For $t \geq 0$ we will use the notation w_t for the data of particle positions and labels at time t.

Corollary 4. *There exists $C < \infty$ such that for every $\lambda > 0$ and $n \geq 1$,*

$$\mathbb{P}_w\left[\psi_{r_D}(F'_n, w_D) > \lambda, F_n < \alpha_1^{-1}nL, D < \infty\right] \leq \lambda^{-1}CLe^{-\alpha_1^{-1}nL(\alpha_1\theta-2(\cosh\theta-1))}. \tag{49}$$

Proof. By the strong Markov property, part (a) of Lemma 16, and translation invariance, the left-hand side of (49) is bounded by

$$\begin{aligned}&\mathbb{E}_w\left[\mathbb{P}_{\tau_{-r_D}w_D}\left[\psi_0(T_{nL}) > \lambda, T_{nL} < \alpha_1^{-1}nL\right]1(D<\infty)\right]\\ &\leq \lambda^{-1}CLe^{-\alpha_1^{-1}nL(\alpha_1\theta-2(\cosh\theta-1))},\end{aligned}$$

where we have used in the last inequality,

$$\sup_w \mathbb{E}_w[\psi_{r_D}(D, w_D), D < \infty] = \sup_w \mathbb{E}_w[\phi_{r_D}(D), D < \infty]$$

and Corollary 3. □

Corollary 5. *Assume that $\eta(r) \geq a$, and $m_{r-L^{1/4},r}(0) \geq aL^{1/4}/2$. Then, there is a $C < \infty$ such that*

$$\begin{aligned}&\mathbb{P}_w\left[\{\psi_{r_D}(T_{r_D+L}, w_D) > p\} \cup \left\{m_{r_D+L-L^{1/4},r_D+L}(T_{r_D+L}) < aL^{1/4}/2\right\}, D < \infty\right]\\ &\leq p^{-1}CLe^{-\alpha_1^{-1}L(\alpha_1\theta-2(\cosh\theta-1))} + \left(C/L^{3/4}\right)^{-\frac{M-1}{4}+1} + CL^{-a\frac{M-1}{8(M+1)}}.\end{aligned} \tag{50}$$

Proof. By Corollary 4 with $n = 1$ and $\lambda = p$, and Lemma 14,

$$\begin{aligned}&\mathbb{P}_w[\psi_{r_D}(T_{r_D+L}) > p, D < \infty]\\ &\leq p^{-1}CLe^{-\alpha_1^{-1}L(\alpha_1\theta-2(\cosh\theta-1))} + \left(C/L^{3/4}\right)^{-(M-1)/4+1} + CLe^{-L^{3/4}/C},\end{aligned} \tag{51}$$

for some constant $C > 0$. Therefore, from (51) and part (b) of Lemma 13,

$$\begin{aligned}&\mathbb{P}_w[\psi_{r_D}(T_{r_D+L}) > p, D < \infty] + \mathbb{P}_w[m_{r_D+L-L^{1/4},r_D+L}(T_{r_D+L}) < aL^{1/4}/2]\\ &\leq p^{-1}CLe^{-\alpha_1^{-1}L(\alpha_1\theta-2(\cosh\theta-1))} + \left(C/L^{3/4}\right)^{-(M-1)/4+1} + CLe^{-L^{3/4}/C}\\ &\quad +CL^{-a(M-1)/(8(M+1))} + CLe^{-L^{a/(4(M+1))}}\end{aligned} \tag{52}$$

for some $C > 0$. □

We end this section with the following key result, providing a tail estimate for the law of J_{r_D} (with J_x for x integer, defined in (11)).

Lemma 17. *Assume that L satisfies* (6), $\theta, \alpha_1, \alpha_2$ *satisfy* 7 *and* p *satisfies* (9). *Then, there exists* $0 < C < \infty$, *and an integer* L_0 *such that if* $L \geq L_0$,

a) *If* $\eta(r) = a$ *and* $m_{r-L,r}(0) \geq aL/2$, *for every* $t \geq 0$,

$$\mathbb{P}_w[J_{r_D} > t, D < \infty] < Ct^{3-(M-1)/4}, \qquad t > 0.$$

b) *For every* $t \geq 0$,

$$\mathbb{P}_w[J_r > t, U = \infty] < Ct^{3-(M-1)/4}, \qquad t > 0.$$

Furthermore,

$$J_r < \infty, \qquad \mathbb{P}_w\text{-a.s.} \tag{53}$$

Proof. Without loss of generality we assume $r = 0$.

a) For $n = 1, 2, \ldots$,

$$\mathbb{P}[J > n, D < \infty] \leq \mathbb{P}\left[B_n, D < \infty\right], \tag{54}$$

where we have dropped the subscripts on $\mathbb{P}_w$ and J_{r_D} and defined

$$B_n := \bigcap_{i=1}^n \left\{\psi_{r_D+(i-1)L}(F'_i, w_D) > p\right\} \cup B'_i,$$
$$B'_i := \left\{m_{r_D+iL-L^{1/4}, r_D+iL}(F'_i) < aL^{1/4}/2\right\}.$$

We have used here that $\phi_z(t) \leq \psi_z(t)$ (see (25)). Let us examine now the terms with $n \geq 2$ in (54). Note that in this case, $\psi_{r_D+(n-1)L} = \psi_{r_D} + \sum_{k=1}^{n-1} \Delta_k$, where

$$\Delta_k := \psi_{r_D+kL} - \psi_{r_D+(k-1)L}.$$

Since $\frac{1}{2^{n-1}} + \sum_{k=1}^{n-1} \frac{1}{2^{n-k}} = 1$, we have

$$\left\{\psi_{r_D+(n-1)L} > p\right\} \subset \left\{\psi_{r_D} > p/2^{n-1}\right\} \cup \left[\bigcup_{k=1}^{n-1} \left\{\Delta_k > p/2^{n-k}\right\}\right]. \tag{55}$$

Let

$$A_0^n := \left\{\psi_{r_D}(F'_n) > p/2^{n-1}\right\}, \quad A_k^n := \left\{\Delta_k(F'_n) > p/2^{n-k}\right\},$$

for $1 \leq k \leq n-1$. From (55), for $n \geq 2$,

$$B_n \subset B_{n-1} \cap \left(B'_n \cup A_0^n \cup A_1^n \cup \cdots \cup A_{n-1}^n\right).$$

So for $n \geq 2$,

$$\mathbb{P}\left[B_n, D < \infty\right] \quad \leq \quad \sum_{k=0}^{n-1} \mathbb{P}\left[A_k^n, B_{n-1}, D < \infty\right] + \mathbb{P}\left[B'_n, B_{n-1}, D < \infty\right]. \tag{56}$$

By the strong Markov property, we have for any $\lambda \in \mathbb{R}$ and $1 \leq k \leq n-1$,

$$\mathbb{P}\left[F'_n - F'_k \geq \lambda,\ D < \infty \middle| \mathcal{F}_{F'_{k-1}}\right] \leq \mathbb{P}_{w_{F'_{k-1}}}\left[T_{(n-k+1)L} - T_L \geq \lambda\right]. \tag{57}$$

Hence by c) of Lemma 9 and the fact that $\alpha_1 < \alpha$, we have

$$\mathbb{P}\left[F'_n - F'_k \geq (n-k)L/\alpha_1 \middle| \mathcal{F}_{F'_{k-1}}\right] \leq C((n-k)L)^{-(M-1)/4}.$$

By the strong Markov property again, and by b) of Lemma 16,

$$\begin{aligned} &\mathbb{P}\left[\Delta_k(F'_n) > p/2^{n-k}, F'_n - F'_k < (n-k)L/\alpha_1 \middle| \mathcal{F}_{F'_{k-1}}\right] \\ &\qquad \leq 3p^{-1}aL2^{n-k}e^{-(n-k)L\left(\theta - 2\alpha_1^{-1}(\cosh\theta - 1)\right)}. \end{aligned} \tag{58}$$

Therefore, for $n \geq 2$ and $1 \leq k \leq n-1$,

$$\begin{aligned}\mathbb{P}\left[A_k^n|\mathcal{F}_{F'_{k-1}}\right] &\leq \mathbb{P}\left[\Delta_k(F'_n) > p2^{-n+k}, F'_n - F'_k < (n-k)L/\alpha_1 \middle| \mathcal{F}_{F'_{k-1}}\right] \\ &\quad + \mathbb{P}\left[F'_n - F'_k \geq (n-k)L/\alpha_1 | \mathcal{F}_{F'_{k-1}}\right] \\ &\leq 3p^{-1}aLp2^{n-k}e^{-(n-k)L\left(\theta-2\alpha_1^{-1}(\cosh\theta-1)\right)} + C((n-k)L)^{-(M-1)/4}. \end{aligned} \tag{59}$$

From inequality (49) of Corollary 4 with $\lambda = 2^n/p$, Lemma 14 and the assumption that initially $m_{-L^{1/4},0}(0) \geq aL^{1/4}/2$, we also obtain that for $n \geq 2$,

$$\mathbb{P}[A_0^n, D < \infty] \leq Cp^{-1}2^nLe^{-nL\left(\theta-2\alpha_1^{-1}(\cosh\theta-1)\right)} + (CnL^{-\frac{3}{4}})^{\frac{M-1}{4}-1} + CLe^{-n\frac{L^{\frac{3}{4}}}{C}}. \tag{60}$$

Now, for $n \geq 2$, by a) of Lemma 13, the strong Markov property, and the fact that there are a particles at the rightmost visited site at time $T_{r_D+nL-L^{1/4}}$,

$$\mathbb{P}\left[B'_n|\,\mathcal{F}_{F'_{n-1}}\right] \leq CL^{-a/8}. \tag{61}$$

Define a sequence

$$a_1 := 3CL^{-a(M-1)/(8(M+1))}, \quad a_n := 4(C(n-1)L^{-3/4})^{(M-1)/4-1}, \quad n \geq 2. \tag{62}$$

Note that there is an $L_0 \geq C$, such that if $L \geq L_0$, for $n = 1, 2, \ldots$, we have that

$$CLe^{-\frac{nL^{3/4}}{C}} + L(3a+C)\frac{2^n}{p}e^{-nL\left(\theta-2\alpha_1^{-1}(\cosh\theta-1)\right)} \leq (CnL^{-\frac{3}{4}})^{\frac{M-1}{4}-1} \leq \frac{a_n}{4}, \tag{63}$$

$$(CL^{-3/4})^{(M-1)/4-1} \leq CL^{-a(M-1)/(8(M+1))} \tag{64}$$

(which is possible by inequality (26) and (3)) and that $\sum_{n=1}^{\infty} a_n < 1$. Let us now define $c_n := \mathbb{P}[B_n, D < \infty]$ for $n \geq 1$. We want to prove that the sequence $\{c_n : n \geq 1\}$ satisfies

$$c_1 \leq a_1, \qquad c_n \leq a_n + \textstyle\sum_{k=1}^{n-1} a_{n-k}c_k, \qquad n \geq 2. \tag{65}$$

From (50) of Corollary 5, (63) and (64) note that the first inequality of display (65) is satisfied. Now note that by inequality (61) and (3), whenever $L \geq L_0$, for $n \geq 2$,

$$\mathbb{P}[B'_n, B_{n-1}, D < \infty] \leq CL^{-a/8}\mathbb{P}[B_{n-1}, D < \infty] \leq a_1\mathbb{P}[B_{n-1}, D < \infty]. \tag{66}$$

Inequality (60) and condition (63) imply $\mathbb{P}[A_0^n, D < \infty] \leq a_n/2$, and inequality (59) implies that $\mathbb{P}[A_1^n, D < \infty] \leq a_n/2$. Hence for $n \geq 2$,

$$\mathbb{P}[A_0^n, B_{n-1}, D < \infty] + \mathbb{P}[A_1^n, B_{n-1}, D < \infty] \leq a_n. \tag{67}$$

Similarly for $2 \leq k \leq n-2$ we have $\mathbb{P}[A_k^n|\mathcal{F}_{F'_{k-1}}] \leq a_{n-k+1}$. Thus, since $B_{n-1} \subset B_{k-1}$ for $2 \leq k \leq n-2$,

$$\mathbb{P}[A_k^n, B_{n-1}, D < \infty] \leq \mathbb{P}[A_k^n, B_{k-1}, D < \infty] \leq a_{n-k+1}\mathbb{P}[B_{k-1}, D < \infty]. \tag{68}$$

Also, by inequality (59) and condition (63) for $n \geq 2$, we have $\mathbb{P}[A_{n-1}^n|\mathcal{F}_{F'_{n-2}}] \leq a_2/2$. Thus, since $B_{n-1} \subset B_{n-2}$, for $n \geq 3$,

$$\mathbb{P}[A_{n-1}^n, B_{n-1}, D < \infty] \leq \mathbb{P}[A_{n-1}^n, B_{n-2}, D < \infty] \leq a_2\mathbb{P}[B_{n-2}, D < \infty]. \tag{69}$$

For $n = 2$, (65) now follows after substituting estimates (66) and (67) in inequality (56), for $n = 3$ after substituting (66), (67) and (69) in inequality (56) while for $n \geq 4$ it follows after substituting (66), (67), (68) and (69) in inequality (56).

But recall that

$$\sum_{n=1}^{\infty} n^{(M-1)/4-3} a_n < \infty. \tag{70}$$

Hence by Lemma 15, (65), and inequality (54) we conclude that

$$\sum_{n=1}^{\infty} n^{(M-1)/4-3} \mathbb{P}\left[J_{r_D} > n, D < \infty\right] < \infty.$$

This implies that $\limsup_{n\to\infty} n^{(M-1)/4-3}\mathbb{P}\left[J_{r_D} > n, D < \infty\right] = 0$. Thus, there exists $C < \infty$ such that for every $n \geq 1$, $\mathbb{P}\left[J_{r_D} > n, D < \infty\right] \leq Cn^{-(M-1)/4+3}$. This together with the monotonicity in t of the expression $\mathbb{P}\left[J_{r_D} > t, D < \infty\right]$ finishes the proof of (a).

b) By analogy with (54), note that for $n = 1, 2, \ldots$,

$$\mathbb{P}[J_0 > n, U = \infty] \leq \mathbb{P}\left[B_n, U = \infty\right], \tag{71}$$

where again we have dropped the subscript on $\mathbb{P}_w$, but now

$$B_n := \bigcap_{i=1}^{n} \left\{\psi_{(i-1)L}(T_{iL}, w) > p\right\} \cup B'_i, \quad B'_i := \left\{m_{iL-L^{1/4},iL}(T_{iL}) < aL^{1/4}/2\right\}.$$

An analysis similar to that of part (a) proves (65) and (70) with $c_n := \mathbb{P}\left[B_n, U = \infty\right]$ for $n \geq 1$ and $\{a_n : n \geq 1\}$ as in (62). Part (b) now follows by Lemma 15 as in part (a).

c) Note that $\{J_0 = \infty\} \subset \{J_L = \infty\}$. Now, for every $n \geq 1$,

$$\mathbb{P}_w[J_L = \infty] \leq \mathbb{P}_w[J_L = \infty, m_{-n,L}(T_L) \geq aL/2] + \mathbb{P}_w[m_{-n,L}(T_L) < aL/2].$$

Now, following the proof of part (a), it is possible to show that $\mathbb{P}_w[J_L > t, m_{-n,L}(T_L) \geq aL/2] \leq Ct^{3-(M-1)/4}$, for some constant $C > 0$. Hence for every $n \geq 1$,

$$\mathbb{P}_w[J_L = \infty] \leq \mathbb{P}_w[m_{-n,L}(T_L) < aL/2].$$

Letting $n \to \infty$ and using part (c) of Lemma 13 we finish the proof. $\square$

Corollary 6. *For every* $k \geq 2$,

$$S_k < \infty \quad \mathbb{P}_w\text{-a.s. } \textit{on the event } \{D_{k-1} < \infty\}. \tag{72}$$

Furthermore,

$$S_1 < \infty \qquad \mathbb{P}_w\text{-a.s.} \tag{73}$$

Proof. Assertion (73) is a consequence of (53) of Lemma 17 and the fact that $\{S_1 < \infty\} = \{J_0 < \infty\}$. Similarly, assertion (72) follows directly from part (a) of Lemma 17 and the fact that $\{D_{k-1} < \infty, S_k < \infty\} = \{D_{k-1} < \infty, J_{r_{D_{k-1}}} < \infty\}$. $\square$

5.6. **Variance bounds for the regeneration times and positions.** In this subsection we will prove Proposition 1. Let us first prove assertion (13) of Proposition 1. By Corollary 6, note that for every $k \geq 1$,

$$\mathbb{P}_w[\kappa = \infty] \leq \mathbb{P}_w[D_k < \infty].$$

But by the strong Markov property and part (b) of Corollary 2, the right-hand side of the above inequality is bounded by $(1-\delta)^k$. It follows that

$$\mathbb{P}_w[\kappa = \infty] \leq (1-\delta)^k,$$

for every $k \geq 1$. Letting $k \to \infty$ gives (13).

To prove (14) we will need the following lemma.

Lemma 18. *For every $\epsilon > 0$, there is a $0 < C < \infty$ such that*

$$\mathbb{P}_{a\delta_0}\left[\kappa > t \middle| U = \infty\right] \leq C t^{-(M-1)/4+3+\epsilon}. \tag{74}$$

Proof. Since $\kappa < \infty$ a.s., we can write (dropping the subscript $a\delta_0$)

$$\mathbb{P}\left[\kappa > t \middle| U = \infty\right] = \sum_{k=1}^{\infty} \mathbb{P}\left[S_k > t, K = k \middle| U = \infty\right].$$

Applying recursively the strong Markov property to the stopping times $\{S_j : j \geq 1\}$ we see that for every $k \geq 1$,

$$\mathbb{P}\left[S_k > t, K = k \middle| U = \infty\right] \leq (1-\delta)^{k-1},$$

where $\delta > 0$ is given by part (b) of Corollary 2. Let $0 < \beta < 1/2$. For any $l > 0$ we therefore have

$$\mathbb{P}\left[\kappa > t \middle| U = \infty\right] \leq \sum_{k=1}^{l} \mathbb{P}\left[t < S_k < \infty \middle| U = \infty\right] + \delta^{-1}(1-\delta)^{l}. \tag{75}$$

Let $0 < \gamma < 1$ and consider the event

$$A_k := \{r_{D_1} - r_{S_1} < t^\gamma, r_{D_2} - r_{S_2} < t^\gamma, \ldots, r_{D_{k-1}} - r_{S_{k-1}} < t^\gamma, S_k < \infty\}.$$

On A_k we have $r_{S_k} \leq kt^\gamma + L\sum_{j=0}^{k-1} J_{r_{D_j}}$, where we adopt the convention $D_0 := 0$, so that $r_{D_0} = 0$. Since $\tilde{r}_t \leq r_t$, if $U = \infty$, then $r_t \geq \lfloor \alpha_2 t \rfloor$ for all $t > 0$. Therefore, on $A_k \cap \{U = \infty\}$,

$$\lfloor \alpha_2 S_k \rfloor \leq kt^\gamma + L\textstyle\sum_{j=0}^{k-1} J_{r_{D_j}}.$$

Now define the event

$$B_k := \{J_{r_{D_0}} < t^\gamma, J_{r_{D_1}} < t^\gamma, \ldots, J_{r_{D_{k-1}}} < t^\gamma, S_k < \infty\}. \tag{76}$$

On $A_k \cap B_k \cap \{U = \infty\}$, $\lfloor \alpha_2 S_k \rfloor \leq kt^\gamma(1+L)$. Hence for $t > (lt^\gamma(1+L)+1)/\alpha_2$ and $k \leq l$,

$$\mathbb{P}[t < S_k < \infty, A_k, B_k | U = \infty] = 0$$

and therefore

$$\mathbb{P}[t < S_k < \infty | U = \infty] \leq \mathbb{P}[A_k^c, S_k < \infty | U = \infty] + \mathbb{P}[B_k^c, S_k < \infty | U = \infty]. \tag{77}$$

Using part (a) of Lemma 17 to bound the probability of the event $\{J_{r_0} \geq t^\gamma\} = \{J_0 \geq t^\gamma\}$ and part (b) to bound the probability of the events $\{J_{r_{D_j}} \geq t^\gamma\}$, for $1 \leq j \leq k-1$, we can see that the second term of the right-hand side of inequality (77) is bounded by $Ckt^{-\gamma((M-1)/4-3)}$. On the other hand, by Lemma 11, the first term is bounded by $Ckt^{-\gamma(M-1)/2}$. Choosing $l = C_1 \log t$ with $C_1 := ((M-1)/4-3)\left(\log(1-\delta)^{-1}\right)^{-1}$ and γ close enough to 1, we obtain (74). □

Proof of (14) *of Proposition* 1. The assertion for κ of (14) follows from Lemma 18 noting that $M \geq 21$ (by condition (3)) and that for r_κ follows from Lemma 18 and (34).

6. Construction and strong Markov property

In this section we will construct the process and we will prove that it satisfies the strong Markov property. We were not able to prove that the process is Feller. Note that the state space $\mathbb{S} \cup \{0\}$ is a Polish space, but it is not locally compact. Therefore, if one wants to prove the Feller property, it is not useful to consider the set of continuous functions vanishing at infinity. On the other hand, we are able to prove that the semigroup maps the set of uniformly continuous bounded functions into itself. This class separates probability measures, and our process is right continuous. Therefore the standard derivation (see for example pages 216-217 of Lamperti [9]) goes through, and this implies the strong Markov property.

Throughout, $\theta > 0$ is arbitrary, P is the joint law of the independent random walks used to define the process for finite initial conditions and E is the corresponding expectation.

For general $(r, \eta) \in \mathbb{S}$, we construct the process by taking limits of approximations with finite initial conditions. For each $\ell = 1, 2, \ldots$, let $\eta^\ell(x) = 0$ if $x \leq r - \ell$, and $\eta^\ell(x) = \eta(x)$ if $r - \ell < x \leq r$. Consider the process $\{(r_t^\ell, \eta^\ell(t)) : t \geq 0\}$ starting from this finite initial condition, defined at the beginning of Section 2.

By our construction note that r_t^ℓ is increasing in ℓ and hence we can define

$$r_t := \lim_{\ell \to \infty} r_t^\ell.$$

We will see that for every $t \geq 0$, a.s. $r_t < \infty$.

Consider $f_\theta(\eta^\ell)$, where

$$f_\theta(\eta) := \sum_x \eta(x) e^{\theta x}.$$

We compute

$$\mathcal{L} f_\theta(\eta^\ell) = \sum_x \eta^\ell(x) e^{\theta x} \left[e^\theta + e^{-\theta} - 2 + ((a+1)e^\theta - 1) 1(x = r) \right].$$

Hence if we let $\lambda_{1,\theta} := e^\theta + e^{-\theta} - 2$ and $\lambda_{2,\theta} := (a+1)e^\theta + e^{-\theta} - 2$, then

$$\lambda_{1,\theta} f_\theta \leq \mathcal{L} f_\theta \leq \lambda_{2,\theta} f_\theta.$$

In particular,

$$E[f_\theta(\eta^\ell(t)) \mid \mathcal{F}_0] \leq e^{\lambda_{2,\theta} t} f_\theta(\eta^\ell(0)). \tag{78}$$

In addition, $f_\theta(\eta^\ell(t))$ is a nonnegative submartingale and therefore, by Doob's inequality,

$$P[\sup_{0 \leq s \leq t} f_\theta(\eta^\ell(s)) \geq e^{\gamma \theta t} \mid \mathcal{F}_0] \leq e^{-\gamma \theta t} E[f_\theta(\eta^\ell(t)) \mid \mathcal{F}_0]. \tag{79}$$

Since r_t^ℓ is the rightmost site which has been occupied up to time t we have $\sup_{0 \leq s \leq t} f_\theta(\eta^\ell(s)) \geq e^{\theta r_t^\ell}$. Hence from (78) and (79) we have

$$P[r_t^\ell \geq \gamma t \mid \mathcal{F}_0] \leq e^{-c_{\gamma,\theta} t} f_\theta(\eta^\ell(0)), \tag{80}$$

where $c_{\gamma,\theta} := \gamma \theta - \lambda_{2,\theta}$. This proves that for each ℓ and $t \geq 0$, a.s. $r_t^\ell < \infty$ and hence $\lim_{n \to \infty} \tau_n = \infty$. Also, taking the limit when $\ell \to \infty$ in (80), we obtain

$$P[r_t \geq \gamma t \mid \mathcal{F}_0) \leq e^{-c_{\gamma,\theta} t} f_\theta(\eta(0)]. \tag{81}$$

This proves Lemma 10 of Section 5.4. Furthermore, if $(r, \eta) \in \mathbb{S}$, then $f_\theta(\eta) < \infty$, so we have $r_t < \infty$ a.s.

Now choose γ large enough so that we have $c_{\gamma,\theta} > 0$. Define for each $y = 1, 2, \ldots$,

$$T_y := \inf\{t \geq 0 : r_t = y\}. \tag{82}$$

We have $r_{T_y \wedge t} = \lim_{\ell\to\infty} r^\ell_{T_y\wedge t}$. Let ℓ_k be the smallest natural number such that $r_{T_k\wedge t} = r^\ell_{T_k\wedge t}$ for all $\ell \geq \ell_k$. Then if $\bar\ell := \max\{\ell_1, \ldots, \ell_{r_t}\}$, the front $r_{\cdot\wedge t}$ generated by the initial condition η^ℓ up to time t does not depend on ℓ if $\ell \geq \bar\ell$. This means that particles that are initially at any site $x \leq r - \bar\ell$ never visit any site to the right of the front before time t. Using attractiveness, we can then conclude that the sequence $\eta^\ell(s)$ is increasing for $s \leq t$ and $\ell \geq \bar\ell$. Therefore

$$\eta(t) := \lim_{\ell\to\infty} \eta^\ell(t)$$

exists almost surely. Taking the limit when $\ell \to \infty$ in (78) and using Fatou's Lemma we see that

$$E[f_\theta(\eta(t)) \mid \mathcal{F}_0] \leq e^{\lambda_{2,\theta}t} f_\theta(\eta(0)). \tag{83}$$

Noting that r_t is increasing, this shows that $(r_t, \eta(t))$ stays in $\mathbb{S}$. Hence we have shown that $(r_t, \eta(t))$ is a Markov process on $\mathbb{S}$.

We next want to show that it satisfies the strong Markov property. We need some more preliminary estimates. Note that

$$\begin{aligned} &\mathcal{L}f_\theta^2 - 2f_\theta\mathcal{L}f_\theta \\ &= \sum_x \eta(x)e^{2\theta x}\left[(e^\theta - 1)^2 + (e^{-\theta} - 1)^2 + ((a^2 - 1)e^{2\theta} - 2(a-1)e^\theta)\right] \\ &\leq \lambda_{3,\theta} f_{2\theta}, \end{aligned} \tag{84}$$

for some $\lambda_{3,\theta} < \infty$. Hence if $M_\theta(t) := f_\theta(\eta^\ell(t)) - \int_0^t \mathcal{L}f_\theta(\eta^\ell(s))ds$, then $\langle M_\theta(t)\rangle = \int_0^t \left(\mathcal{L}f_\theta^2 - 2f_\theta\mathcal{L}f_\theta\right)(\eta^\ell(s))ds$, and then

$$E[M_\theta^2(t) \mid \mathcal{F}_0] \leq 2\lambda_{3,\theta}\int_0^t E[f_{2\theta}(\eta^\ell(s)) \mid \mathcal{F}_0]ds \leq 2\lambda_{3,\theta}\lambda_{2,2\theta}^{-1}e^{\lambda_{2,2\theta}t}f_\theta(\eta^\ell(0)).$$

Here we used in the last inequality that $f_{2\theta}(\eta^\ell(0)) \leq f_\theta(\eta^\ell(0))$, since there are no particles initially to the right of the origin. In particular, by Tchebyshev's inequality,

$$P[f_\theta(\eta^\ell(t)) < e^{\lambda_{1,\theta}t}f_\theta(\eta^\ell(0)) - A \mid \mathcal{F}_0] \leq A^{-2}e^{\lambda_{4,\theta}t}f_\theta(\eta^\ell(0)), \tag{85}$$

for some $\lambda_{4,\theta} < \infty$. We can pass to the limit to obtain

$$P[f_\theta(\eta(t)) < e^{\lambda_{1,\theta}t}f_\theta(\eta(0)) - A \mid \mathcal{F}_0] \leq A^{-2}e^{\lambda_{4,\theta}t}f_\theta(\eta(0)). \tag{86}$$

This proves that we have a well-defined Markov process starting from any initial data in $\mathbb{S}$. To prove the strong Markov property, we show that the semigroup of the process preserves the set of bounded uniformly continuous functions.

Lemma 19. *For each $\epsilon > 0$ and $t > 0$ there exists $\delta > 0$ such that if (r, η) and (r, η') are any two configurations of particles on $\mathbb{S}$ with $\sum_{x\leq r} e^{\theta x}|\eta(x) - \eta'(x)| < \delta$, then there is a stopping time τ and a coupling of two copies $(r_s, \eta(s))$ and $(r'_s, \eta'(s))$ of our Markov process with generator $\mathcal{L}$ for $0 \leq s \leq \tau$ satisfying*

(1) $\bar P[\tau < t] < \epsilon$,
(2) $\bar E[d((r_t, \eta(t)), (r'_t, \eta'(t)))\mathbf{1}_{\tau>t}] < \epsilon$,
(3) $\bar P[r_0 = r'_0 = r, \eta(0) = \eta, \eta'(0) = \eta'] = 1$.

Here $\bar P$ is the coupling measure and $\bar E$ is the corresponding expectation.

Proof. Consider the difference $\zeta = \eta - \eta'$. We have $\sum_{x\le 0} e^{\theta x}|\zeta(x)| < \delta$, so choosing δ sufficiently small we have $\zeta(x) = 0$ for $x \in \{-L, \ldots, 0\}$ for some large L. We attempt to couple the two processes by moving the particles together whenever possible. Then positive and negative parts of ζ move as independent random walks of positive and negative type, the two types annihilating on contact and the coupling succeeds up to the first time τ when a particle of either type hits r_s. It is easy to choose δ small enough, and therefore L large enough, so that (1) is satisfied. To prove (2), note that up to time τ, $d((r_t, \eta(t)), (r'_t, \eta'(t))) = \sum_{x\le r_t} e^{\theta(x-r_t)}|\zeta(t,x)|$ and we can get an easy upper bound by using $r_t \ge 0$ and letting $\bar\zeta(t)$ be the process obtained by starting with $|\zeta(0)|$ and using the same random walks, but dropping the signs and the annihilations. Then $\sum_x e^{\theta(x-r_t)}|\zeta(t,x)| \le \sum_x e^{\theta x}\bar\zeta(t,x)$. (2) follows since $e^{-\lambda_{1,\theta}t}\sum_x e^{\theta x}\bar\zeta(t,x)$ is a martingale. □

Proposition 4. *Let $(r,\eta) \in \mathbb{S}$ and $P_{r,\eta}$ be the law of the process $\{(r_t, \eta(t)) : t \ge 0\}$ with initial condition (r,η) under P. Assume that g is a bounded uniformly continuous function on $\mathbb{S}$. Then, for each $t \ge 0$, $P_t g(r,\eta) := E_{r,\eta}[g(r_t, \eta(t))]$, where $E_{r,\eta}$ is the expectation associated to $P_{r,\eta}$, is a bounded uniformly continuous function.*

Proof. Since g is uniformly continuous, we can choose $\epsilon_0 > 0$ so that $d((r,\eta),(r',\eta')) < \epsilon_0$ implies $|g(r,\eta) - g(r',\eta')| < \epsilon/3$. By Lemma 19, there exists δ such that if $d((r,\eta),(r',\eta')) < \delta$, then there is a stopping time $\tau \ge 0$ such that $\bar P[d((r_t,\eta(t)),(r'_t,\eta'(t))) > \epsilon_0, \tau > t] < \epsilon/(6B)$ and $\bar P[\tau < t] \le \epsilon/(3B)$. Hence $|E_{r,\eta}[g(r_t,\eta(t))] - E_{r',\eta'}[g(r'_t,\eta'(t))]| \le \bar E[|g(r_t,\eta(t)) - g(r'_t,\eta'(t))|\mathbf{1}_{\tau>t}] + 2BP[\tau < t] \le \epsilon$. This proves that $P_t g$ is uniformly continuous as well. Obviously it is bounded. □

Corollary 7. *The process $\{(r_t, \eta(t) : t \ge 0\}$ satisfies the strong Markov property.*

Proof. It is enough to notice that the process is right-continuous, that the bounded uniformly continuous functions on $\mathbb{S}$ separate probability measures, and apply Proposition 4. We then follow the standard derivation (see for example pages 216-217 of Lamperti [9]) proving first the strong Markov property for stopping times with a countable range, and then approximating any arbitrary stopping time by a decreasing sequence of stopping times with a countable range. □

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Laboratoire de Probabilités et Modèles Aléatoires, Université Paris 7- Denis Diderot, 2, Place Jussieu, F-75251 Paris Cedex 05, France
E-mail address: comets@math.jussieu.fr

Departments of Mathematics and Statistics, University of Toronto, 40 St. George Street, Toronto, Ontario, Canada M5S 1L2
E-mail address: quastel@math.toronto.edu

Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Macul, Santiago, Chile
E-mail address: aramirez@mat.puc.cl