TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 361, Number 11, November 2009, Pages 5629–5646
S 0002-9947(09)04977-0
Article electronically published on June 23, 2009

POLYNOMIAL IDENTITIES IN NIL-ALGEBRAS

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Abstract. We prove that in associative algebras over a field F of characteristic $p \geq 3$ the polynomial identity $x^{2p} = 0$ is not Specht. To prove this we construct a non-finitely based system of polynomial identities which contains the identity $x^{2p} = 0$. We also give an example of a non-Specht polynomial identity of degree $2p$ in unital associative F-algebras.

1. Introduction

Let F be a field, let A be a free associative algebra (without 1) over F on free generators $x_1, x_2, \dots$ and let G be an associative F-algebra (with or without 1). Let $f(x_1, \dots, x_n) \in A$. We say that $f(x_1, \dots, x_n) = 0$ is a *polynomial identity* (or an *identity*) in G if $f(g_1, \dots, g_n) = 0$ for all $g_1, \dots, g_n \in G$. Two systems of polynomial identities $\{u_i = 0 \mid i \in I\}$ and $\{v_j = 0 \mid j \in J\}$ are *equivalent* if every associative F-algebra satisfying all the identities $u_i = 0$ satisfies all the identities $v_j = 0$ and vice versa. If a system $\{u_i = 0 \mid i \in I\}$ is equivalent to some finite system of polynomial identities we say that the system $\{u_i = 0 \mid i \in I\}$ is *finitely based* or *has a finite basis*. We refer to [3], [6], [7], [13] and [16] for further terminology, basic facts and references concerning polynomial identities in associative algebras.

A polynomial identity is called *Specht* if every system containing this identity has a finite basis. Note that over a field of characteristic 0 every system of polynomial identities is finitely based: this is a celebrated result of Kemer [14]. Therefore, over such a field, every polynomial identity is Specht. On the other hand, it has been proved by Belov [4], Grishin [9] and Shchigolev [17] that over a field of a prime characteristic $p > 0$ there are non-finitely based systems of polynomial identities and so there are identities which are not Specht.

We study the following.

Problem 1. For a given field F of characteristic p, find the smallest positive integer $n = n(F)$ such that the identity $x^n = 0$ is not Specht.

Note that over a field F of characteristic p, the identity $x^n = 0$ with $n < p$ is Specht. Indeed, according to the Nagata-Higman-Dubnov-Ivanov theorem ([15], [12], see also [8]) the identity $x^n = 0$ ($n < p$) implies over F the identity of nilpotency $x_1 x_2 \dots x_k = 0$ for some $k = k(n) \in \mathbb{N}$ and the latter identity is well known to be Specht (see, for instance, [2, Theorem 5.1.4]).

Received by the editors June 7, 2006.
2000 *Mathematics Subject Classification.* Primary 16R10.
The first author was partially supported by INTAS.
The second author was partially supported by CNPq/FAPDF/PRONEX-Brazil, CNPq/PADCT-Brazil, FINATEC-Brazil and RFBR-Russia.

On the other hand, the following polynomial identities in associative F-algebras have been proved to be non-Specht:

(1) $x^{32} = 0$ over a field F of characteristic 2 (Grishin, 1999 [9], [10]);
(2) $x^6 = 0$ over a field F of characteristic 2 (Gupta and Krasilnikov, 2002 [11]);
(3) $x^{2p^3(2p+1)} = 0$ over a field F of characteristic $p \geq 3$ (Shchigolev, 1999 [17]);
(4) $x^{2p^3+p^2+1} = 0$ over a field F of characteristic $p \geq 3$ (Shchigolev, 2002 [19]);
(5) $x^{12} = 0$ over a field F of characteristic 3 (Aladova, 2002 [1]);
(6) $x^{6p} = 0$ over a field F of characteristic $p \geq 5$ (Aladova and Krasilnikov, 2003, unpublished).

Our main result is as follows.

Theorem 1.1. *Over a field F of characteristic $p \geq 3$, the identity $x^{2p} = 0$ is not Specht.*

We conjecture that $n(F) = 2p$ for each field F of a prime characteristic $p \geq 3$; that is, the identity $x^n = 0$ is Specht over F if $n < 2p$.

Let p be a prime integer, $p > 2$. Let

$$[x, y] = xy - yx,\ f(x, y) = x^{p-1}y^{p-1}[x, y],$$
$$w_n = w_n(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)$$
$$= [[x_1, x_2], x_3]f(x_3, y_3) \dots f(x_n, y_n)[[y_1, y_2], y_3]\Big([[x_3, x_1], x_2][[y_3, y_1], y_2]\Big)^{p-1}.$$

We obtain Theorem 1.1 as a consequence of the following result.

Theorem 1.2. *Over a field F of characteristic $p \geq 3$ the system of identities*

$$\{w_n = 0 \mid n = 3, 4 \dots\} \ \cup\ \{x^{2p} = 0\}$$

is not equivalent to any finite system of identities in associative F-algebras.

To prove Theorem 1.2 we will construct, for each integer $n \geq 3$, an associative F-algebra B_n such that B_n satisfies the identities $x^{2p} = 0$ and $w_k = 0$ for all $k \leq n$ but does not satisfy the identity $w_{n+1} = 0$.

In the proof we use results of Shchigolev [18]. The algebras B_n and the identities $w_n = 0$ in our paper are similar to, although different from, the corresponding algebras and the identities used by Belov [4]. We also use some ideas of [11].

We also study the following more general problem concerning non-Specht identities:

Problem 2. For a given field F of characteristic p, find the lowest degree $m = m(F)$ of a non-Specht identity for associative F-algebras.

One can consider Problem 2 for unital algebras (that is, for algebras with a unity 1) as well as for algebras which are not necessarily unital. Theorem 1.1 shows that for non-unital associative F-algebras we have $m(F) \leq 2p$. However, this theorem does not give any estimate for $m(F)$ for unital algebras. This is because a unital algebra does not satisfy any identity of the form $x^n = 0$.

Among the polynomial identities in unital associative F-algebras known to be non-Specht, the identity of the lowest degree is probably one that can be obtained from the proof of Belov's result [4]. It has degree $5p + 2$ (where $p =$ char F) and is as follows:

$$[x_1, x_2][x_3, x_4, x_5][x_6, x_7][x_8, x_9, x_{10}] \dots [x_{5p-2}, x_{5p-1}, x_{5p}][x_{5p+1}, x_{5p+2}] = 0.$$

So it was known for unital algebras that $m(F) \leq 5p + 2$.

Let S_{2p} be the permutation group on the set $\{1, 2, \dots, 2p\}$ and let $\lambda(x_1, x_2, \dots, x_{2p})$ be the complete linearization of the polynomial x^{2p},

$$\lambda(x_1, x_2, \dots, x_{2p}) = \sum_{\sigma \in S_{2p}} x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(2p)}.$$

We prove the following theorem.

Theorem 1.3. *Over a field F of characteristic $p \geq 3$ the polynomial identity $\lambda(x_1, x_2, \dots, x_{2p}) = 0$ (of degree $2p$) is not Specht in the class of unital associative F-algebras.*

Thus, for unital associative algebras we have $m(F) \leq 2p$ as well as for non-unital ones.

Theorem 1.3 is an immediate consequence of the following theorem.

Theorem 1.4. *Over a field F of characteristic $p \geq 3$ the system of identities*

$$\{w_n = 0 \mid n = 3, 4, \dots\} \ \cup \ \{\lambda(x_1, x_2, \dots, x_{2p}) = 0\}$$

is not equivalent to any finite system of polynomial identities in unital associative F-algebras.

2. Auxiliary results

Recall that $\binom{k}{l} = \frac{k!}{l!(k-l)!}$ if $k \geq l$. Put $\binom{k}{l} = 0$ if $k < l$. Let $[x, y, z] = [[x, y], z]$.

Lemma 2.1. *Let G be an associative ring satisfying the identity $[x, y, z] = 0$. Let t and k be integers such that $t \geq 1$, $k \geq 0$ and $t + k > 1$. Then, for any $a_1, a_2, \dots, a_{t-1}, r \in G$, we have*

$$\sum_{\substack{i_1+i_2+\dots+i_t=k \\ i_1,\dots,i_t \geq 0}} r^{i_1} a_1 r^{i_2} a_2 r^{i_3} \dots r^{i_{t-1}} a_{t-1} r^{i_t} = \binom{k+t-1}{t-1} r^k a_1 a_2 \dots a_{t-1}$$

(2.1)

$$+ \binom{k+t-1}{t} r^{k-1} \Big(\sum_{j=1}^{t-1} (t-j) a_1 \dots a_{j-1} a_{j+1} \dots a_{t-1} [a_j, r] \Big).$$

Proof. We will prove the equation (2.1) by induction on t and k. If $t = 1$, then (2.1) reduces to the equation $r^k = r^k$ and if $k = 0$, then (2.1) reduces to $a_1 a_2 \dots a_{t-1} = a_1 a_2 \dots a_{t-1}$. Therefore, (2.1) holds for $t = 1$ and any $k > 0$ as well as for $k = 0$ and any $t > 1$.

Let $t > 1$ and $k > 0$. Suppose that (2.1) holds for the pairs of integers $(t, k-1)$ and $(t-1, k)$. We are going to prove that it holds also for the pair (t, k). Let

$$\sigma = \sum_{\substack{i_1+\dots+i_t=k \\ i_1,\dots,i_t \geq 0}} r^{i_1} a_1 r^{i_2} a_2 r^{i_3} \dots r^{i_{t-1}} a_{t-1} r^{i_t}.$$

It is clear that $\sigma = \sigma_1 + \sigma_2$, where

$$\sigma_1 = \Big(\sum_{\substack{i_1+\dots+i_t=k-1 \\ i_1,\dots,i_t\geq 0}} r^{i_1}a_1r^{i_2}a_2r^{i_3}\dots r^{i_{t-1}}a_{t-1}r^{i_t} \Big) \cdot r,$$

$$\sigma_2 = \Big(\sum_{\substack{i_1+\dots+i_{t-1}=k \\ i_1,\dots,i_{t-1}\geq 0}} r^{i_1}a_1r^{i_2}a_2r^{i_3}\dots r^{i_{t-2}}a_{t-2}r^{i_{t-1}} \Big) \cdot a_{t-1}.$$

Note that, for all $a, b, r \in G$, we have $[a, r]b = b[a, r]$ and $[a, r][b, r] = 0$. Using these equations and the inductive hypothesis it is straightforward to check that

$$\sigma_1 = \binom{k+t-2}{t-1} r^k a_1 \dots a_{t-1} + \binom{k+t-2}{t-1} r^{k-1} \Big(\sum_{j=1}^{t-1} a_1 \dots a_{j-1}a_{j+1}\dots a_{t-1}[a_j, r] \Big) + \binom{k+t-2}{t} r^{k-1} \Big(\sum_{j=1}^{t-1} (t-j) a_1 \dots a_{j-1}a_{j+1}\dots a_{t-1}[a_j, r] \Big)$$

and

$$\sigma_2 = \binom{k+t-2}{t-2} r^k a_1 \dots a_{t-1} + \binom{k+t-2}{t-1} r^{k-1} \Big(\sum_{j=1}^{t-2} (t-j-1) a_1 \dots a_{j-1}a_{j+1}\dots a_{t-1}[a_j, r] \Big).$$

The result follows. □

Lemma 2.2. *Let F be a field of characteristic $p \geq 3$ and let G be an associative F-algebra which satisfies the identities $[x, y, z] = 0$ and $x^p = 0$. Let t and k be integers such that $1 \leq t \leq p-1$, $k \geq p-t+1$. Then, for all $a_1, a_2, \dots, a_{t-1}, r \in G$, we have*

$$\sum_{\substack{i_1+i_2+\dots+i_t=k \\ i_1,\dots,i_t\geq 0}} r^{i_1}a_1r^{i_2}a_2\dots r^{i_{t-1}}a_{t-1}r^{i_t} = 0. \tag{2.2}$$

Proof. Let $t = 1$. Then $k \geq p$ and the equation (2.2) reduces to the equation $r^k = 0$, which holds since G satisfies the identity $x^p = 0$.

Let $2 \leq t \leq p-1$. Then, by Lemma 2.1,

$$\sum_{\substack{i_1+\dots+i_t=k \\ i_1,\dots,i_t\geq 0}} r^{i_1}a_1r^{i_2}a_2\dots r^{i_{t-1}}a_{t-1}r^{i_t} = \binom{k+t-1}{t-1} r^k a_1a_2\dots a_{t-1} + \binom{k+t-1}{t} r^{k-1} f_t,$$

where $f_t \in G$. If $k \geq p+1$, then the equation (2.2) holds because G satisfies the identity $x^p = 0$ and so $r^k = r^{k-1} = 0$. If $p-t+1 \leq k \leq p-1$, then $\binom{k+t-1}{t-1} = \frac{(k+t-1)\dots(k+1)}{(t-1)!}$ and $\binom{k+t-1}{t} = \frac{(k+t-1)\dots(k+1)k}{t!}$ are multiples of p. Since $\operatorname{char} F = p$, the equation (2.2) holds for such a k. Finally, if $k = p$, then $r^p = 0$ because G satisfies the identity $x^p = 0$ and $\binom{k+t-1}{t} = \binom{p+t-1}{t} = \frac{(p+t-1)\dots(p+1)p}{t!}$ is a multiple of p, so for $k = p$ and $2 \leq t \leq p-1$ the equation (2.2) holds as well. □

Lemma 2.3. *Let F be a field of characteristic $p \geq 3$ and let k, l be integers such that $k \geq 1$, $0 \leq l \leq p-1$. Let G be an associative F-algebra which satisfies the identities $[x,y,z]=0$ and $x^p=0$. Then, for any $a_0, a_1, r \in G$, we have*

$$\sum_{\substack{j_1+\cdots+j_{p-1}=l\\ 0\leq j_1,\ldots,j_{p-1}\leq 1}} \sum_{\substack{i_1+i_2+\cdots+i_p=k\\ i_1,\ldots,i_p\geq 0}} r^{i_1}a_{j_1}r^{i_2}a_{j_2}\ldots r^{i_{p-1}}a_{j_{p-1}}r^{i_p}=0.$$

Proof. Note that, for all $k \geq 1$,

$$\binom{k+p-1}{p-1} r^k a_{j_1}\ldots a_{j_{p-1}} = 0. \tag{2.3}$$

Indeed, if $1 \leq k \leq p-1$, then $\binom{k+p-1}{p-1}a = 0$ for each $a \in G$ because $\operatorname{char} F = p$. On the other hand, if $k \geq p$, then $r^k = 0$ because G satisfies the identity $x^p = 0$. It follows from (2.3) and Lemma 2.1 that

$$\sum_{\substack{i_1+i_2+\cdots+i_p=k\\ i_1,\ldots,i_p\geq 0}} r^{i_1}a_{j_1}r^{i_2}a_{j_2}\ldots r^{i_{p-1}}a_{j_{p-1}}r^{i_p}$$
$$=\binom{k+p-1}{p} r^{k-1}\Big(\sum_{h=1}^{p-1}(p-h)a_{j_1}\ldots a_{j_{h-1}}a_{j_{h+1}}\ldots a_{j_{p-1}}[a_{j_h},r]\Big).$$

Let

$$\sigma=\sum_{\substack{j_1+\cdots+j_{p-1}=l\\ 0\leq j_1,\ldots,j_{p-1}\leq 1}} \sum_{\substack{i_1+i_2+\cdots+i_p=k\\ i_1,\ldots,i_p\geq 0}} r^{i_1}a_{j_1}r^{i_2}a_{j_2}\ldots r^{i_{p-1}}a_{j_{p-1}}r^{i_p}.$$

Then

$$\sigma=\binom{k+p-1}{p} r^{k-1}\Big(\sum_{\substack{j_1+\cdots+j_{p-1}=l\\ 0\leq j_1,\ldots,j_{p-1}\leq 1}} \sum_{h=1}^{p-1}(p-h)a_{j_1}\ldots a_{j_{h-1}}a_{j_{h+1}}\ldots a_{j_{p-1}}[a_{j_h},r]\Big)$$
$$=\binom{k+p-1}{p} r^{k-1}\sum_{h=1}^{p-1}(p-h)\Big(\sum_{\substack{j_1+\cdots+j_{p-1}=l\\ 0\leq j_1,\ldots,j_{p-1}\leq 1}} a_{j_1}\ldots a_{j_{h-1}}a_{j_{h+1}}\ldots a_{j_{p-1}}[a_{j_h},r]\Big).$$

Let

$$\tau=\sum_{\substack{j_1+\cdots+j_{p-1}=l\\ 0\leq j_1,\ldots,j_{p-1}\leq 1}} a_{j_1}\ldots a_{j_{p-2}}[a_{j_{p-1}},r].$$

It can be easily seen that

$$\tau=\sum_{\substack{j_1+\cdots+j_{p-1}=l\\ 0\leq j_1,\ldots,j_{p-1}\leq 1}} a_{j_1}\ldots a_{j_{m-1}}a_{j_{m+1}}\ldots a_{j_{p-1}}[a_{j_m},r]$$

for all m, $1 \leq m \leq p-2$. It follows that

$$\sigma=\binom{k+p-1}{p} r^{k-1}\Big((p-1)+(p-2)+\cdots+2+1\Big)\tau=\binom{k+p-1}{p} r^{k-1}\cdot\frac{p(p-1)}{2}\tau.$$

Since $\operatorname{char} F = p$, we have $\sigma = 0$. □

Lemma 2.4. *Let F be a field of characteristic $p \geq 3$ and let k be an integer, $k \geq 0$. Let G be an associative F-algebra which satisfies the identity $[x, y, z] = 0$. Then, for any $a, d_0, \ldots, d_k \in G$, we have*

$$\sum_{\substack{i_1+\cdots+i_{p-1}=k \\ i_1,\ldots,i_{p-1}\geq 0}} \sum_{\substack{j_1+\cdots+j_p=1 \\ j_1,\ldots,j_p\geq 0}} a^{j_1} d_{i_1} a^{j_2} \ldots a^{j_{p-1}} d_{i_{p-1}} a^{j_p} = 0.$$

Proof. Since char $F = p$, it follows from Lemma 2.1 that

$$\sum_{\substack{j_1+\cdots+j_p=1 \\ j_1,\ldots,j_p\geq 0}} a^{j_1} d_{i_1} a^{j_2} \ldots a^{j_{p-1}} d_{i_{p-1}} a^{j_p} = \sum_{s=1}^{p-1} (p-s) d_{i_1} \ldots d_{i_{s-1}} d_{i_{s+1}} \ldots d_{i_{p-1}} [d_{i_s}, a].$$

Let

$$\sigma = \sum_{\substack{i_1+\cdots+i_{p-1}=k \\ i_1,\ldots,i_{p-1}\geq 0}} \sum_{\substack{j_1+\cdots+j_p=1 \\ j_1,\ldots,j_p\geq 0}} a^{j_1} d_{i_1} a^{j_2} \ldots a^{j_{p-1}} d_{i_{p-1}} a^{j_p}.$$

Then

$$\sigma = \sum_{\substack{i_1+\cdots+i_{p-1}=k \\ i_1,\ldots,i_{p-1}\geq 0}} \sum_{s=1}^{p-1} (p-s) d_{i_1} \ldots d_{i_{s-1}} d_{i_{s+1}} \ldots d_{i_{p-1}} [d_{i_s}, a]$$

$$= \sum_{s=1}^{p-1} (p-s) \Big(\sum_{\substack{i_1+\cdots+i_{p-1}=k \\ i_1,\ldots,i_{p-1}\geq 0}} d_{i_1} \ldots d_{i_{s-1}} d_{i_{s+1}} \ldots d_{i_{p-1}} [d_{i_s}, a] \Big).$$

Let

$$\tau = \sum_{\substack{i_1+\cdots+i_{p-1}=k \\ i_1,\ldots,i_{p-1}\geq 0}} d_{i_1} d_{i_2} \ldots d_{i_{p-2}} [d_{i_{p-1}}, a].$$

It can be easily seen that

$$\tau = \sum_{\substack{i_1+\cdots+i_{p-1}=k \\ i_1,\ldots,i_{p-1}\geq 0}} d_{i_1} \ldots d_{i_{p-3}} d_{i_{p-1}} [d_{i_{p-2}}, a] = \ldots$$

$$\cdots = \sum_{\substack{i_1+\cdots+i_{p-1}=k \\ i_1,\ldots,i_{p-1}\geq 0}} d_{i_2} \ldots d_{i_{p-1}} [d_{i_1}, a].$$

Therefore,

$$\sigma = (p-1)\tau + \cdots + (p-s)\tau + \cdots + \tau = \frac{p(p-1)}{2} \tau = 0$$

since char $F = p$. □

3. Construction of the algebra B_n

The lemma below follows immediately from a result of Shchigolev [18, Lemma 13]. Recall that $f(x_1, x_2) = x_1^{p-1} x_2^{p-1} [x_1, x_2]$.

Lemma 3.1. *Let F be a field of characteristic $p \geq 3$. Then there exists a unital associative F-algebra R such that the following conditions are satisfied:*

1. *R as a vector space over F is a direct sum of its two-sided ideal I and the one-dimensional subspace generated by 1;*
2. *for each $h \in I$, we have $h^p = 0$;*

3. *R as an F-algebra with unity is generated by certain elements $z_i \in I$ ($i \in \mathbb{N}$) such that, for each integer $n > 0$, the product*

$$f(z_1, z_2)f(z_3, z_4)f(z_5, z_6)\dots f(z_{2n+1}, z_{2n+2})$$

is not contained in the linear span of the set

$$\{1\} \cup \{f(u_1, u_2)\dots f(u_{2k-1}, u_{2k}) \mid 1 \le k \le n, u_1, \dots, u_{2k} \in R\}; \tag{3.1}$$

4. *$[u, v, w] = 0$ for all $u, v, w \in R$.* □

Notice that in [18] Shchigolev's result was stated for an infinite field F. However, Shchigolev observed in [19] that the result remains valid for a finite field F as well: it is sufficient just to replace the expression "T-space" by "homogeneous T-space" in the proof.

Let M_n be the F-linear span of the set (3.1). Define $\mathcal{R}_n$ to be the quotient algebra of the algebra of $(2p + 1) \times (2p + 1)$ matrices

$$\begin{pmatrix} 0 & R & R & R & R & \dots & R & R & R \\ 0 & R & R & R & R & \dots & R & R & R \\ 0 & 0 & 0 & R & R & \dots & R & R & R \\ 0 & 0 & 0 & R & R & \dots & R & R & R \\ 0 & 0 & 0 & 0 & 0 & \dots & R & R & R \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & R & R \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & R & R \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

over the ideal

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & M_n \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

that is,

$$\mathcal{R}_n = \begin{pmatrix} 0 & R & R & R & R & \dots & R & R & R/M_n \\ 0 & R & R & R & R & \dots & R & R & R \\ 0 & 0 & 0 & R & R & \dots & R & R & R \\ 0 & 0 & 0 & R & R & \dots & R & R & R \\ 0 & 0 & 0 & 0 & 0 & \dots & R & R & R \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & R & R \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & R & R \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

Define B_n to be the subalgebra of $\mathcal{R}_n$ generated by the matrix

$$D = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

and all the matrices

$$\mathbf{r} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & r & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}, \tag{3.2}$$

where $r \in I$ and I is the ideal of R mentioned in Lemma 3.1.

We need the following observation.

Remark 3.2. Let A, B and C be $k \times k$, $k \times l$ and $l \times l$ matrices, respectively. Then

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \tag{3.3}$$

is a $(k+l) \times (k+l)$ matrix. Note that the mappings

$$\psi^{(1)} : \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \to A \qquad \text{and} \qquad \psi^{(2)} : \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \to C$$

define homomorphisms of the algebra of the matrices of the form (3.3) onto the algebras of $k \times k$ and $l \times l$ matrices, respectively. It follows that the mapping

$$\psi : \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} \to A_{22}$$

also defines a homomorphism of the corresponding associative algebra. Here A_{11}, A_{22}, A_{33} are $k \times k$, $l \times l$ and $m \times m$ matrices, respectively, A_{12} and A_{23} are $k \times l$ and $l \times m$ matrices and A_{13} is a $k \times m$ matrix.

Lemma 3.3. *Every element of the algebra B_n is a matrix of the form*

$$\begin{pmatrix} 0 & u & v & * & * & * & \dots & * & * & * & * & * \\ 0 & r & w & h & * & * & \dots & * & * & * & * & * \\ 0 & 0 & 0 & u & v & * & \dots & * & * & * & * & * \\ 0 & 0 & 0 & r & w & h & \dots & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & u & \dots & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & r & \dots & * & * & * & * & * \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & u & v & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & r & w & h & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & u & v \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & r & w \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.4}$$

where $u, v, w, h \in R$ and $r \in I$.

Proof. Let X be an arbitrary element of $\mathcal{R}_n$. Let

$$X = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & \dots & * & * & * \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & \dots & * & * & * \\ 0 & 0 & 0 & a_{34} & a_{35} & \dots & * & * & * \\ 0 & 0 & 0 & a_{44} & a_{45} & \dots & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \dots & * & * & * \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & a_{2p-1,2p} & a_{2p-1,2p+1} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & a_{2p,2p} & a_{2p,2p+1} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

(entries denoted by $*$ are not important for the argument). Define mappings of the algebra $\mathcal{R}_n$ into the algebra of 5×5 matrices over R as follows:

$$\psi_k(X) = \begin{pmatrix} 0 & a_{2k-1,2k} & a_{2k-1,2k+1} & a_{2k-1,2k+2} & a_{2k-1,2k+3} \\ 0 & a_{2k,2k} & a_{2k,2k+1} & a_{2k,2k+2} & a_{2k,2k+3} \\ 0 & 0 & 0 & a_{2k+1,2k+2} & a_{2k+2,2k+3} \\ 0 & 0 & 0 & a_{2k+2,2k+2} & a_{2k+2,2k+3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (1 \le k \le p-1).$$

By Remark 3.2, $\psi_1,\ \psi_2, \dots,\ \psi_{p-1}$ are homomorphisms of the F-algebra $\mathcal{R}_n$.

Note that

$$\psi_1(D) = \psi_2(D) = \dots = \psi_{p-1}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\psi_1(\mathbf{r}) = \psi_2(\mathbf{r}) = \dots = \psi_{p-1}(\mathbf{r}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for all matrices $\mathbf{r}$ ($r \in I$) of the form (3.2). Since the algebra B_n is generated by D and the matrices of the form (3.2), we have

$$\psi_1(X) = \psi_2(X) = \cdots = \psi_{p-1}(X)$$

for each $X \in B_n$. Comparing the matrices $\psi_1(X) = \psi_2(X) = \cdots = \psi_{p-1}(X)$, we have

$$a_{22} = a_{44} = \cdots = a_{(2p-2)(2p-2)}$$

and also

$$a_{44} = a_{66} = \cdots = a_{2p,2p},$$

so

$$a_{22} = a_{44} = \cdots = a_{2p,2p}.$$

Similarly, $a_{12} = a_{34} = \cdots = a_{(2p-1)2p}$, $a_{23} = a_{45} = \cdots = a_{2p(2p+1)}$, $a_{13} = a_{35} = \cdots = a_{(2p-1)(2p+1)}$ and $a_{24} = a_{46} = \cdots = a_{(2p-2)2p}$. □

4. Proof of Theorem 1.2

First we are going to check that the algebra B_n satisfies the identities $w_k = 0$ for all $k \leq n$ but does not satisfy the identity $w_{n+1} = 0$.

Let $X_i^{(j)}$ ($i = 1, 2, \ldots, k,\ j = 1, 2$) be arbitrary elements of B_n. Let

$$X_i^{(j)} = \begin{pmatrix} 0 & u_i^{(j)} & * & * & * & \dots & * & * & * \\ 0 & r_i^{(j)} & v_i^{(j)} & * & * & \dots & * & * & * \\ 0 & 0 & 0 & u_i^{(j)} & * & \dots & * & * & * \\ 0 & 0 & 0 & r_i^{(j)} & v_i^{(j)} & \dots & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \dots & * & * & * \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & u_i^{(j)} & * \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & r_i^{(j)} & v_i^{(j)} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}, \tag{4.1}$$

where $r_i^{(j)} \in I$, $u_i^{(j)}, v_i^{(j)} \in R$. Then

$$[X_1^{(j)}, X_2^{(j)}, X_3^{(j)}] = \begin{pmatrix} 0 & \bar{u}^{(j)} & * & * & \dots & * & * & * \\ 0 & 0 & \bar{v}^{(j)} & * & \dots & * & * & * \\ 0 & 0 & 0 & \bar{u}^{(j)} & \dots & * & * & * \\ 0 & 0 & 0 & 0 & \dots & * & * & * \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \bar{u}^{(j)} & * \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \bar{v}^{(j)} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

$$[X_3^{(j)}, X_1^{(j)}, X_2^{(j)}] = \begin{pmatrix} 0 & \tilde{u}^{(j)} & * & * & \dots & * & * & * \\ 0 & 0 & \tilde{v}^{(j)} & * & \dots & * & * & * \\ 0 & 0 & 0 & \tilde{u}^{(j)} & \dots & * & * & * \\ 0 & 0 & 0 & 0 & \dots & * & * & * \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \tilde{u}^{(j)} & * \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \tilde{v}^{(j)} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}
\bar{u}^{(j)} &= (u_1^{(j)} r_2^{(j)} - u_2^{(j)} r_1^{(j)}) r_3^{(j)} - u_3^{(j)} [r_1^{(j)}, r_2^{(j)}],\\
\bar{v}^{(j)} &= [r_1^{(j)}, r_2^{(j)}] v_3^{(j)} - r_3^{(j)} (r_1^{(j)} v_2^{(j)} - r_2^{(j)} v_1^{(j)}),\\
\tilde{u}^{(j)} &= (u_3^{(j)} r_1^{(j)} - u_1^{(j)} r_3^{(j)}) r_2^{(j)} - u_2^{(j)} [r_3^{(j)}, r_1^{(j)}],\\
\tilde{v}^{(j)} &= [r_3^{(j)}, r_1^{(j)}] v_2^{(j)} - r_2^{(j)} (r_3^{(j)} v_1^{(j)} - r_1^{(j)} v_3^{(j)}).
\end{aligned} \tag{4.2}$$

The matrices $[X_1^{(j)}, X_2^{(j)}, X_3^{(j)}]$ and $[X_3^{(j)}, X_1^{(j)}, X_2^{(j)}]$ are nil-triangular since the algebra R satisfies the identity $[x, y, z] = 0$.

Let $H = [X_3^{(1)}, X_1^{(1)}, X_2^{(1)}][X_3^{(2)}, X_1^{(2)}, X_2^{(2)}]$. Then

$$H = \begin{pmatrix}
0 & 0 & \tilde{u}^{(1)}\tilde{v}^{(2)} & * & * & \dots & * & * & * \\
0 & 0 & 0 & \tilde{v}^{(1)}\tilde{u}^{(2)} & * & \dots & * & * & * \\
0 & 0 & 0 & 0 & \tilde{u}^{(1)}\tilde{v}^{(2)} & \dots & * & * & * \\
0 & 0 & 0 & 0 & 0 & \dots & * & * & * \\
0 & 0 & 0 & 0 & 0 & \dots & * & * & * \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \tilde{u}^{(1)}\tilde{v}^{(2)} \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0
\end{pmatrix},$$

$$H^{p-1} = \begin{pmatrix}
0 & \cdots & 0 & (\tilde{u}^{(1)}\tilde{v}^{(2)})^{p-1} & * & * \\
0 & \cdots & 0 & 0 & (\tilde{v}^{(1)}\tilde{u}^{(2)})^{p-1} & * \\
0 & \cdots & 0 & 0 & 0 & (\tilde{u}^{(1)}\tilde{v}^{(2)})^{p-1} \\
0 & \cdots & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}$$

and

$$[X_1^{(2)}, X_2^{(2)}, X_3^{(2)}] H^{p-1} = \begin{pmatrix}
0 & \cdots & 0 & \bar{u}^{(2)}(\tilde{v}^{(1)}\tilde{u}^{(2)})^{p-1} & * \\
0 & \cdots & 0 & 0 & \bar{v}^{(2)}(\tilde{u}^{(1)}\tilde{v}^{(2)})^{p-1} \\
0 & \cdots & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots \\
0 & \cdots & 0 & 0 & 0
\end{pmatrix}.$$

Further,

$$f(X_3^{(1)}, X_3^{(2)}) \dots f(X_k^{(1)}, X_k^{(2)}) = \begin{pmatrix}
0 & * & * & \dots & * & * & * \\
0 & \tilde{r} & * & \dots & * & * & * \\
0 & 0 & 0 & \dots & * & * & * \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & 0 & * & * \\
0 & 0 & 0 & \dots & 0 & \tilde{r} & * \\
0 & 0 & 0 & \dots & 0 & 0 & 0
\end{pmatrix},$$

where $\tilde{r} = f(r_3^{(1)}, r_3^{(2)}) \dots f(r_k^{(1)}, r_k^{(2)})$. It follows that

$$f(X_3^{(1)}, X_3^{(2)}) \dots f(X_k^{(1)}, X_k^{(2)})[X_1^{(2)}, X_2^{(2)}, X_3^{(2)}]H^{p-1} = \begin{pmatrix} 0 & \cdots & 0 & * \\ 0 & \cdots & 0 & y \\ 0 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

where $y = f(r_3^{(1)}, r_3^{(2)}) \cdots f(r_k^{(1)}, r_k^{(2)})\bar{v}^{(2)}(\tilde{u}^{(1)}\tilde{v}^{(2)})^{p-1}$. Therefore, if

$$\begin{aligned} W &= w_k(X_1^{(1)}, X_2^{(1)}, \dots, X_k^{(1)}, X_1^{(2)}, X_2^{(2)}, \dots, X_k^{(2)}) \\ &= [X_1^{(1)}, X_2^{(1)}, X_3^{(1)}]f(X_3^{(1)}, X_3^{(2)}) \cdots f(X_k^{(1)}, X_k^{(2)})[X_1^{(2)}, X_2^{(2)}, X_3^{(2)}] \\ &\qquad \times \Big([X_3^{(1)}, X_1^{(1)}, X_2^{(1)}][X_3^{(2)}, X_1^{(2)}, X_2^{(2)}]\Big)^{(p-1)}, \end{aligned}$$

then

$$W = \begin{pmatrix} 0 & 0 & \dots & 0 & d + M_n \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{4.3}$$

with $d = \bar{u}^{(1)}y = \bar{u}^{(1)}f(r_3^{(1)}, r_3^{(2)}) \dots f(r_k^{(1)}, r_k^{(2)})\bar{v}^{(2)}\big(\tilde{u}^{(1)}\tilde{v}^{(2)}\big)^{p-1}$. By (4.2), we have

$$\begin{aligned} d = &\big((u_1^{(1)}r_2^{(1)} - u_2^{(1)}r_1^{(1)})r_3^{(1)} - u_3^{(1)}[r_1^{(1)}, r_2^{(1)}]\big)f(r_3^{(1)}, r_3^{(2)}) \dots f(r_k^{(1)}, r_k^{(2)}) \\ &\cdot \big([r_1^{(2)}, r_2^{(2)}]v_3^{(2)} - r_3^{(2)}(r_1^{(2)}v_2^{(2)} - r_2^{(2)}v_1^{(2)})\big) \\ \cdot &\Big(\big((u_3^{(1)}r_1^{(1)} - u_1^{(1)}r_3^{(1)})r_2^{(1)} - u_2^{(1)}[r_3^{(1)}, r_1^{(1)}]\big)\big([r_3^{(2)}, r_1^{(2)}]v_2^{(2)} - r_2^{(2)}(r_3^{(2)}v_1^{(2)} - r_1^{(2)}v_3^{(2)})\big)\Big)^{p-1}. \end{aligned}$$

Since the identity $[x, y, z] = 0$ implies $[x, z][y, z] = 0$, the element $f(r_i, r_j)$ belongs to the centre of R for all $r_i, r_j \in R$. Also, $r_i \in I$, so, by Lemma 3.1, $(r_i)^p = 0$. Therefore,

$$\begin{aligned} d &= -u_3^{(1)}[r_1^{(1)}, r_2^{(1)}]f(r_3^{(1)}, r_3^{(2)}) \dots f(r_k^{(1)}, r_k^{(2)}) \cdot [r_1^{(2)}, r_2^{(2)}]v_3^{(2)} \\ &\qquad \cdot \Big((u_3^{(1)}r_1^{(1)} - u_1^{(1)}r_3^{(1)})r_2^{(1)}r_2^{(2)}(r_3^{(2)}v_1^{(2)} - r_1^{(2)}v_3^{(2)})\Big)^{p-1} \\ &= -u_3^{(1)}v_3^{(2)}\big(r_2^{(1)}\big)^{p-1}[r_1^{(1)}, r_2^{(1)}]\big(r_2^{(2)}\big)^{p-1}[r_1^{(2)}, r_2^{(2)}]f(r_3^{(1)}, r_3^{(2)}) \dots f(r_k^{(1)}, r_k^{(2)}) \\ &\qquad \cdot \Big((u_3^{(1)}r_1^{(1)} - u_1^{(1)}r_3^{(1)})(r_3^{(2)}v_1^{(2)} - r_1^{(2)}v_3^{(2)})\Big)^{p-1} \\ &= -u_3^{(1)}v_3^{(2)}\big(r_2^{(1)}\big)^{p-1}[r_1^{(1)}, r_2^{(1)}]\big(r_2^{(2)}\big)^{p-1}[r_1^{(2)}, r_2^{(2)}] \\ &\qquad \cdot f(r_3^{(1)}, r_3^{(2)}) \dots f(r_k^{(1)}, r_k^{(2)})\Big(u_3^{(1)}r_1^{(1)}r_1^{(2)}v_3^{(2)}\Big)^{p-1} \\ &= -\big(r_1^{(1)}\big)^{p-1}\big(r_2^{(1)}\big)^{p-1}[r_1^{(1)}, r_2^{(1)}]\big(r_1^{(2)}\big)^{p-1}\big(r_2^{(2)}\big)^{p-1}[r_1^{(2)}, r_2^{(2)}] \\ &\qquad \cdot f(r_3^{(1)}, r_3^{(2)}) \dots f(r_k^{(1)}, r_k^{(2)})\Big(u_3^{(1)}v_3^{(2)}\Big)^{p}. \end{aligned}$$

Since R is a direct sum of the vector spaces I and $\langle 1 \rangle_F$, for each $r \in R$ we have $r = \alpha + \rho$, where $\alpha \in F$, $\rho \in I$. It follows that $r^p = (\alpha + \rho)^p = \alpha^p \in F$ (recall that $\rho^p = 0$ for each $\rho \in I$). Then

$$d = -\alpha^p f(r_1^{(1)}, r_2^{(1)})f(r_1^{(2)}, r_2^{(2)})f(r_3^{(1)}, r_3^{(2)}) \dots f(r_k^{(1)}, r_k^{(2)}).$$

If $k \le n$, then, by the definition of M_n, we have

$$f(r_1^{(1)}, r_2^{(1)})f(r_1^{(2)}, r_2^{(2)})f(r_3^{(1)}, r_3^{(2)}) \dots f(r_k^{(1)}, r_k^{(2)}) \in M_n,$$

so $d \in M_n$ and $W = 0$, that is,

$$w_k(X_1^{(1)}, X_2^{(1)}, \dots, X_k^{(1)}, X_1^{(2)}, X_2^{(2)}, \dots, X_k^{(2)}) = 0$$

for all $X_i^{(j)} \in B_n$ $(i = 1, 2, \dots, k,\ j = 1, 2)$.

Thus, the algebra B_n satisfies the identities $w_k = 0$ for all $k \le n$.

In order to prove that B_n does not satisfy the identity $w_{n+1} = 0$, consider the element

$$\begin{aligned} W' &= w_{n+1}(\mathbf{z_1}, \mathbf{z_2}, D + \mathbf{z_5}, \mathbf{z_7}, \dots, \mathbf{z_{2n+1}}, \mathbf{z_3}, \mathbf{z_4}, D + \mathbf{z_6}, \mathbf{z_8}, \dots, \mathbf{z_{2n+2}}) \\ &= [\mathbf{z_1}, \mathbf{z_2}, D + \mathbf{z_5}] f(D + \mathbf{z_5}, D + \mathbf{z_6}) f(\mathbf{z_7}, \mathbf{z_8}) \dots f(\mathbf{z_{2n+1}}, \mathbf{z_{2n+2}}) \\ &\qquad \times [\mathbf{z_3}, \mathbf{z_4}, D + \mathbf{z_6}]\big([D + \mathbf{z_5}, \mathbf{z_1}, \mathbf{z_2}][D + \mathbf{z_6}, \mathbf{z_3}, \mathbf{z_4}]\big)^{(p-1)}, \end{aligned}$$

where, for each i, $\mathbf{z_i}$ is the matrix of the form (3.2) corresponding to the generator z_i of R mentioned in Lemma 3.1. By the calculations above, W' is of the form (4.3) with

$$d = -f(z_1, z_2)f(z_3, z_4)f(z_5, z_6) \dots f(z_{2n+1}, z_{2n+2}).$$

By Lemma 3.1, $d \notin M_n$. Thus, $W' \ne 0$, so the algebra B_n does not satisfy the identity $w_{n+1} = 0$.

Now to complete the proof of Theorem 1.2 it remains to check that B_n satisfies the identity $x^{2p} = 0$, where $p =$ char F.

Let E_{ij} be the matrix units $(1 \le i, j \le 2p+1)$; that is, E_{ij} has 1 at the position (i, j) and 0 elsewhere. It is clear that

$$E_{ij}E_{kl} = \begin{cases} 0, & \text{if } j \ne k; \\ E_{il}, & \text{if } j = k. \end{cases}$$

For each $X \in B_n$,

$$X = \sum_{1 \le i \le j \le 2p+1} a_{ij} E_{ij},$$

where $a_{ii} \in I$ $(i = 2, 4, \dots, 2p)$, $a_{ii} = 0$ $(i = 1, 3, 5, \dots, 2p+1)$, $a_{ij} \in R$ if $i < j$ and $(i, j) \ne (1, 2p+1)$ and $a_{1,2p+1} \in R/M_n$. Since I satisfies the identity $x^p = 0$, the matrix X^{2p} is nil-triangular. It can be written as follows:

$$X^{2p} = \sum_{1 \le i' < j' \le 2p+1} m_{i'j'} E_{i'j'},$$

where each $m_{i'j'}$ is a non-commutative polynomial of degree $2p$ on a_{ij} $(1 \le i \le j \le 2p+1)$. Each polynomial $m_{i'j'}$ is the sum of monomials of the form

$$a_{l_1 l_2} a_{l_2 l_3} \dots a_{l_{2p} l_{2p+1}}, \tag{4.4}$$

where $i' = l_1 \le l_2 \le \dots \le l_{2p} \le l_{2p+1} = j'$ and, for each s, either $l_s < l_{s+1}$ or $l_s = l_{s+1}$ and the latter number is even. Recall that $a_{22} = a_{44} = \dots = a_{(2p)(2p)}$. Let $r = a_{22}$. The monomial (4.4) can be written as follows:

$$b_0 r^{i_1} b_1 r^{i_2} b_2 \dots b_{t-1} r^{i_t} b_t, \tag{4.5}$$

where $1 \le t \le p$, $i_1, \dots, i_t \ge 0$. Here

$$b_0 = a_{s_1^{(0)} s_2^{(0)}} a_{s_2^{(0)} s_3^{(0)}} \dots a_{s_{c_0}^{(0)} s_{c_0+1}^{(0)}},$$

where $0 \le c_0 \le p$ (if $c_0 = 0$, then $b_0 = 1$), $s_q^{(0)} < s_{q'}^{(0)}$ if $q < q'$, $s_1^{(0)}, s_2^{(0)}, \dots, s_{c_0}^{(0)}$ are odd and $s_{c_0+1}^{(0)}$ is even;

$$b_j = a_{s_1^{(j)} s_2^{(j)}} a_{s_2^{(j)} s_3^{(j)}} \dots a_{s_{c_j}^{(j)} s_{c_j+1}^{(j)}},$$

where $1 \le j \le t-1$, $1 \le c_j \le p-1$, $s_1^{(j)} = s_{c_{j-1}+1}^{(j-1)}$ ($s_1^{(1)} = i'$ if $c_0 = 0$), $s_q^{(j)} < s_{q'}^{(j)}$ if $q < q'$, $s_1^{(j)}$, $s_{c_j+1}^{(j)}$ are even and $s_2^{(j)}, \dots, s_{c_j}^{(j)}$ are odd;

$$b_t = a_{s_1^{(t)} s_2^{(t)}} a_{s_2^{(t)} s_3^{(t)}} \dots a_{s_{c_t}^{(t)} s_{c_t+1}^{(t)}},$$

where $0 \le c_t \le p$ (if $c_t = 0$, then $b_t = 1$), $s_q^{(t)} < s_{q'}^{(t)}$ if $q < q'$, $s_2^{(t)}, \dots, s_{c_t+1}^{(t)}$ are odd and $s_1^{(t)} = s_{c_{t-1}+1}^{(t-1)}$ is even.

Let $\deg(b_0 b_1 \dots b_t)$ be the degree of the monomial $b_0 b_1 \dots b_t$ with respect to a_{ij} ($1 \le i < j \le 2p+1$). It is clear that

$$\deg(b_0 b_1 \dots b_t) = c_0 + c_1 + \dots + c_t. \tag{4.6}$$

Note that

$$s_1^{(0)}, s_2^{(0)}, \dots, s_{c_0}^{(0)}, s_{c_0+1}^{(0)} (= s_1^{(1)}), s_2^{(1)}, \dots, s_{c_{t-1}+1}^{(t-1)} (= s_1^{(t)}), \dots, s_{c_t}^{(t)}, s_{c_t+1}^{(t)} \tag{4.7}$$

is a subsequence of the sequence $1, 2, \dots, 2p+1$ containing $c_0 + \dots + c_t + 1$ terms. In this subsequence the terms $s_1^{(j)} (= s_{c_{j-1}+1}^{(j-1)})$, for $1 \le j \le t$, are even and all the other terms $s_i^{(j)}$ are odd. Therefore, (4.7) contains t even numbers. Since the sequence $1, 2, \dots, 2p+1$ contains p even numbers, the subsequence (4.7) does not contain $p - t$ even elements of the set $\{1, 2, \dots, 2p+1\}$. It follows that (4.7) contains at most $2p + 1 - (p - t) = p + t + 1$ terms, that is

$$c_0 + c_1 + \dots + c_t + 1 \le p + t + 1.$$

By (4.6), we have

$$\deg(b_0 b_1 \dots b_t) \le p + t. \tag{4.8}$$

Note that $\deg(b_0 b_1 \dots b_t) = p + t$ (or, equivalently, $c_0 + c_1 + \dots + c_t + 1 = p + t + 1$) if and only if (4.7) contains all the odd elements of the set $\{1, 2, \dots, 2p+1\}$. In particular, in this case $i' = s_1^{(0)} = 1$ and $j' = s_{c_t+1}^{(t)} = 2p + 1$.

Consider $m_{i'j'}$. For each t, $1 \le t \le p$, and each set $\{b_0, b_1, \dots, b_t\}$ of the products described above, we can collect the monomials of the form (4.5) of the polynomial $m_{i'j'}$ to the sum

$$\sum_{\substack{i_1 + \dots + i_t = k \\ i_1, \dots, i_t \ge 0}} b_0 r^{i_1} b_1 r^{i_2} \dots r^{i_t} b_t, \tag{4.9}$$

where $k = 2p - \deg(b_0 b_1 \dots b_t)$. Note that, by (4.8),

$$k \ge 2p - (p + t) = p - t.$$

Thus, $m_{i'j'}$ is a sum of elements of the form (4.9) with $1 \le t \le p$ and $k \ge p - t$.

Suppose that either $i' \ne 1$ or $j' \ne 2p + 1$. Then $\deg(b_0 b_1 \dots b_t) < p + t$ because the equation $\deg(b_0 b_1 \dots b_t) = p + t$ implies $i' = 1$, $j' = 2p + 1$. Therefore,

$$k > 2p - (p + t) = p - t,$$

where $1 \le t \le p$. If $1 \le t \le p - 1$, then, by Lemma 2.2, the sums of the form (4.9) are equal to 0. Let $t = p$. Then each term b_j ($1 \le j \le p - 1$) is equal either to $a_{2j,2j+2}$ or to $a_{2j,2j+1} a_{2j+1,2j+2}$, so, by Lemma 3.3, we have either $b_j = a_{24}$ or

$b_j = a_{23}a_{12}$. For a given l, $p-1 \le l \le 2p-2$, and given b_0, b_t, all the sums of the form (4.9) of the polynomial $m_{i'j'}$ with $\deg(b_1 \dots b_{p-1}) = l$ can be collected into the sum

$$\sum_{\substack{j_1+\dots+j_{p-1}=l\\ 1\le j_1,\dots,j_{p-1}\le 2}} \sum_{\substack{i_1+\dots+i_p=k\\ i_1,\dots,i_p\ge 0}} b_0 r^{i_1} b'_{j_1} r^{i_2} \dots b'_{j_{p-1}} r^{i_p} b_p, \tag{4.10}$$

where $b'_1 = a_{24}$, $b'_2 = a_{23}a_{12}$. Since $t = p$, we have $k > p - t = 0$. By Lemma 2.3, if $k \ge 1$ and $p-1 \le l \le 2p-2$, then every sum of the form (4.10) is equal to 0. Therefore, $m_{i'j'} = 0$ if either $i' \ne 1$ or $j' \ne 2p+1$.

Now consider $m_{1,2p+1}$. It is clear that $m_{1,2p+1} = d_1 + d_2 + M_n$, where d_1 is the sum of all the elements of the form (4.9) such that $\deg(b_0b_1\dots b_t) < p+t$ and d_2 is the sum of all such elements with $\deg(b_0b_1\dots b_t) = p+t$. Using the argument above it is easy to prove that $d_1 = 0$. Hence, $m_{1,2p+1} = d_2 + M_n$. For every element of the form (4.9) with $\deg(b_0b_1\dots b_t) = p+t$, the corresponding sequence (4.7) contains all the odd numbers from 1 to $2p+1$. Since, by Lemma 3.3, $a_{2i-1,2i} = a_{12}$, $a_{2i-1,2i+1} = a_{13}$, $a_{2i,2i+1} = a_{23}$ for all i, $1 \le i \le p$, the terms b_j in the sum (4.9) are of the form

$$\begin{aligned} b_0 &= a_{s_1^{(0)}s_2^{(0)}} \dots a_{s_{c_0}^{(0)}s_{c_0+1}^{(0)}} = a_{13}^{c_0-1}a_{12},\\ b_j &= a_{s_1^{(j)}s_2^{(j)}} \dots a_{s_{c_j}^{(j)}s_{c_j+1}^{(j)}} = a_{23}a_{13}^{c_j-2}a_{12} \qquad (1 \le j \le t-1),\\ b_t &= a_{s_1^{(t)}s_2^{(t)}} \dots a_{s_{c_t}^{(t)}s_{c_t+1}^{(t)}} = a_{23}a_{13}^{c_t-1}. \end{aligned}$$

It follows that the sum (4.9) reduces to

$$\sum_{\substack{i_1+\dots+i_t=k\\ i_1,\dots,i_t\ge 0}} a_{13}^{c_0-1}a_{12}r^{i_1}a_{23}a_{13}^{c_1-2}a_{12}\dots r^{i_t}a_{23}a_{13}^{c_t-1}, \tag{4.11}$$

where $k = 2p - (p+t) = p - t$, $1 \le t \le p$. Since $c_0 + \dots + c_t = p + t$, we have

$$(c_0-1)+(c_1-2)+\dots+(c_{t-1}-2)+(c_t-1) = p+t-2-2(t-1) = p-t. \tag{4.12}$$

Let σ_{t_0} be the sum of all the elements (4.11) of d_2 with $t = t_0$. Then

$$m_{1,2p+1} = d_2 + M_n = \sum_{1\le t\le p} \sigma_t + M_n$$

and, by (4.12),

$$\begin{aligned} \sigma_t &= \sum_{\substack{j_0+\dots+j_t=p-t\\ j_0,\dots,j_t\ge 0}} \sum_{\substack{i_1+\dots+i_t=p-t\\ i_1,\dots,i_t\ge 0}} a_{13}^{j_0}(a_{12}r^{i_1}a_{23})a_{13}^{j_1}\dots(a_{12}r^{i_t}a_{23})a_{13}^{j_t}\\ &= \sum_{\substack{i_1+\dots+i_t=p-t\\ i_1,\dots,i_t\ge 0}} \sum_{\substack{j_0+\dots+j_t=p-t\\ j_0,\dots,j_t\ge 0}} a_{13}^{j_0}(a_{12}r^{i_1}a_{23})a_{13}^{j_1}\dots(a_{12}r^{i_t}a_{23})a_{13}^{j_t}, \end{aligned}$$

where $1 \le t \le p$.

If $1 \le t \le p-2$, then it follows from Lemma 2.1 and the equation $\operatorname{char} F = p$ that

$$\sum_{\substack{j_0+\dots+j_t=p-t\\ j_0,\dots,j_t\ge 0}} a_{13}^{j_0}(a_{12}r^{i_1}a_{23})a_{13}^{j_1}\dots(a_{12}r^{i_t}a_{23})a_{13}^{j_t} = 0.$$

(Note that to apply Lemma 2.1 to the sum above it is convenient to replace $j_0, \dots, j_t$ with $j_1, \dots, j_{t+1}$.) Hence, we have $\sigma_1 = \dots = \sigma_{p-2} = 0$. By Lemma 2.4, $\sigma_{p-1} = 0$. Let $t = p$. Then $k = p - t = 0$, so

$$\sigma_p = a_{12}a_{23}a_{12}a_{23}\dots a_{23} = \big(a_{12}a_{23}\big)^p.$$

Therefore,

$$m_{1,2p+1} = \big(a_{12}a_{23}\big)^p + M_n.$$

By Lemma 3.1, for each $w \in R$ there are $\alpha \in F$ and $r \in I$ such that $w = \alpha(1+r)$. Then $w^p = \alpha^p(1 + r^p) = \alpha^p$ since $r^p = 0$. Hence, by the definition of M_n, we have $w^p \in M_n$ for any $w \in R$ and any $n \geq 1$. In particular, $(a_{12}a_{23})^p \in M_n$, so, for any $n \geq 1$, we have $m_{1,2p+1} = M_n$, that is, $m_{1,2p+1} = 0 \in R/M_n$. It follows that $X^{2p} = 0$, where X is an arbitrary element of B_n.

Thus, for each $n \geq 1$, the algebra B_n satisfies the identity $x^{2p} = 0$. This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.4

Let B_n^+ be the F-algebra obtained from B_n by formally adjoining a unity element 1. Then as a vector space over F the algebra B_n^+ is a direct sum of (its two-sided ideal) B_n and the one-dimensional subspace generated by 1. We will prove Theorem 1.4 by checking that, for each $n \geq 3$, the unital algebra B_n^+ satisfies the identities $\lambda = 0$ and $w_k = 0$ for $k \leq n$ but does not satisfy the identity $w_{n+1} = 0$.

First we check that B_n^+ satisfies the identity $\lambda = 0$. We need the following two observations.

1. Since B_n satisfies the polynomial identity $x^{2p} = 0$ and $\lambda = 0$ is the complete linearization of $x^{2p} = 0$, the algebra B_n satisfies the identity $\lambda = 0$.

2. Since F is a field of characteristic p, we have

$$\lambda(x_1, \dots, x_{2p-1}, 1) = 2p \sum_{\tau \in S_{2p-1}} x_{\tau(1)} \dots x_{\tau(2p-1)} = 0.$$

Similarly, for each i,

$$\lambda(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_{2p}) = 0.$$

Now let Y_i $(i = 1, \dots, 2p)$ be arbitrary elements of B_n^+. Then $Y_i = \alpha_i + X_i$, where $\alpha_i \in F$ and $X_i \in B_n$ for all i. The polynomial λ is multilinear, so, by the observations above,

$$\lambda(Y_1, \dots, Y_{2p}) = \lambda(\alpha_1 + X_1, \dots, \alpha_{2p} + X_{2p}) = \lambda(X_1, \dots, X_{2p}) = 0.$$

Thus, the algebra B_n^+ satisfies the identity $\lambda = 0$, as required.

Now we need to check that, for each $n \geq 3$, the algebra B_n^+ satisfies $w_k = 0$ for $k \leq n$ but does not satisfy $w_{n+1} = 0$. Since the identity $w_{n+1} = 0$ is not satisfied in B_n, it obviously is not satisfied in B_n^+. Therefore, to complete the proof of Theorem 1.4 it suffices to prove that B_n^+ satisfies the identity $w_k = 0$ for $k \leq n$.

Let $Y_i^{(j)}$ $(i = 1, 2, \dots, k, j = 1, 2)$ be arbitrary elements of B_n^+. Then $Y_i^{(j)} = \alpha_i^{(j)} + X_i^{(j)}$, where $\alpha_i^{(j)} \in F$ and $X_i^{(j)} \in B_n$ for all i, j. Let

$$W = w_k(Y_1^{(1)}, Y_2^{(1)}, \dots, Y_k^{(1)}, Y_1^{(2)}, Y_2^{(2)}, \dots, Y_k^{(2)}).$$

Then

$$\begin{aligned} W &= [Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}] f(Y_3^{(1)}, Y_3^{(2)}) \cdots f(Y_k^{(1)}, Y_k^{(2)}) [Y_1^{(2)}, Y_2^{(2)}, Y_3^{(2)}] \\ &\qquad \times \Big([Y_3^{(1)}, Y_1^{(1)}, Y_2^{(1)}][Y_3^{(2)}, Y_1^{(2)}, Y_2^{(2)}] \Big)^{(p-1)} \\ &= [X_1^{(1)}, X_2^{(1)}, X_3^{(1)}] f(\alpha_3^{(1)} + X_3^{(1)}, \alpha_3^{(2)} + X_3^{(2)}) \cdots f(\alpha_k^{(1)} + X_k^{(1)}, \alpha_k^{(2)} + X_k^{(2)}) \\ &\qquad \times [X_1^{(2)}, X_2^{(2)}, X_3^{(2)}] \Big([X_3^{(1)}, X_1^{(1)}, X_2^{(1)}][X_3^{(2)}, X_1^{(2)}, X_2^{(2)}] \Big)^{(p-1)}. \end{aligned}$$

Suppose that $X_i^{(j)}$ are as in (4.1). It follows from the calculations in the proof of Theorem 1.2 that W is of the form (4.3) with

$$d = -\alpha^p f(r_1^{(1)}, r_2^{(1)}) f(r_1^{(2)}, r_2^{(2)}) f(\alpha_3^{(1)} + r_3^{(1)}, \alpha_3^{(2)} + r_3^{(2)}) \dots f(\alpha_k^{(1)} + r_k^{(1)}, \alpha_k^{(2)} + r_k^{(2)}).$$

If $k \le n$, then, by the definition of M_n, we have

$$f(s_1^{(1)}, s_2^{(1)}) f(s_1^{(2)}, s_2^{(2)}) f(s_3^{(1)}, s_3^{(2)}) \dots f(s_k^{(1)}, s_k^{(2)}) \in M_n$$

for all $s_i^{(j)} \in R$. Therefore, $d \in M_n$ and $W = 0$, that is,

$$w_k(Y_1^{(1)}, Y_2^{(1)}, \dots, Y_k^{(1)}, Y_1^{(2)}, Y_2^{(2)}, \dots, Y_k^{(2)}) = 0$$

for all $Y_i^{(j)} \in B_n^+$ $(i = 1, 2, \dots, k,\ j = 1, 2)$.

Thus, the algebra B_n^+ satisfies the identities $w_k = 0$ for all $k \le n$, as required. This completes the proof of Theorem 1.4.

Acknowledgement

Thanks are due to the referee whose valuable remarks and suggestions improved the paper considerably.

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