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ENTROPY SOLUTIONS FOR THE $p(x)$-LAPLACE EQUATION

MANEL SANCHÓN AND JOSÉ MIGUEL URBANO

ABSTRACT. We consider a Dirichlet problem in divergence form with variable growth, modeled on the $p(x)$-Laplace equation. We obtain existence and uniqueness of an entropy solution for L^1 data, as well as integrability results for the solution and its gradient. The proofs rely crucially on *a priori* estimates in Marcinkiewicz spaces with variable exponent, for which we obtain new inclusion results of independent interest.

1. INTRODUCTION

Partial differential equations with nonlinearities involving nonconstant exponents have attracted an increasing amount of attention in recent years. Perhaps the impulse for this comes from the sound physical applications in play, or perhaps it is just the thrill of developing a mathematical theory where PDEs again meet functional analysis in a truly two-way street.

The development, mainly by Růžička [28], of a theory modeling the behavior of electrorheological fluids, an important class of non-Newtonian fluids, seems to have boosted a still far from completed effort to study and understand nonlinear PDEs involving variable exponents. Other applications relate to image processing (*cf.* [8]), elasticity (*cf.* [31]), the flow in porous media (*cf.* [4] and [21]), and problems in the calculus of variations involving variational integrals with nonstandard growth (*cf.* [31], [27], and [1]). This, in turn, gave rise to a revival of the interest in Lebesgue and Sobolev spaces with variable exponent, the origins of which can be traced back to the work of Orlicz in the 1930s. An account of recent advances, some open problems, and an extensive list of references can be found in the interesting survey by Diening *et al.* [14]. Meanwhile, among several other contributions, the introduction by Sharapudinov [29] of the Luxemburg norm and the work of Kováčik and Rákosník [23], where many of the basic properties of these spaces are established, were crucial developments.

In this paper, we consider a problem with potential applications to the modeling of combustion, thermal explosions, nonlinear heat generation, gravitational equilibrium of polytropic stars, glaciology, non-Newtonian fluids, and the flow through

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porous media. Many of these models have already been analyzed for constant exponents of nonlinearity (*cf.* [12], [10], [9], [18], [30], and the references therein), but it seems to be more realistic to assume the exponent to be variable.

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and consider the elliptic problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) &= f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f \in L^1(\Omega)$ and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, $a(\cdot,\xi)$ is measurable in Ω, for every $\xi \in \mathbb{R}^N$, and $a(x,\cdot)$ is continuous in $\mathbb{R}^N$, for almost every $x \in \Omega$), such that the following assumptions hold:

$$a(x,\xi)\cdot\xi \geq b|\xi|^{p(x)}, \tag{1.2}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where b is a positive constant;

$$|a(x,\xi)| \leq \beta(j(x) + |\xi|^{p(x)-1}), \tag{1.3}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where j is a nonnegative function in $L^{p'(\cdot)}(\Omega)$ and $\beta > 0$;

$$(a(x,\xi) - a(x,\xi'))\cdot(\xi - \xi') > 0, \tag{1.4}$$

for almost every $x \in \Omega$ and for every $\xi, \xi' \in \mathbb{R}^N$, with $\xi \neq \xi'$.

Hypotheses (1.2)–(1.4) are the natural extensions of the classical assumptions in the study of nonlinear monotone operators in divergence form for constant $p(\cdot) \equiv p$ (*cf.* [26]).

Concerning the exponent $p(\cdot)$ appearing in (1.2) and (1.3), we assume it is a measurable function $p(\cdot) : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \exists C > 0 \ : \ |p(x) - p(y)| \leq \frac{C}{-\ln|x-y|}, \quad \text{for } |x-y| < \frac{1}{2}; \\ \\ 1 < \operatorname*{ess\,inf}_{x\in\Omega} \ p(x) \leq \operatorname*{ess\,sup}_{x\in\Omega} \ p(x) < N. \end{cases} \tag{1.5}$$

The first condition says that $p(\cdot)$ belongs to the class of log-Hölder continuous functions. These assumptions allow us, in particular, to exploit the functional analytical properties of Lebesgue and Sobolev spaces with variable exponent (see section 2) arising in the study of problem (1.1).

A *weak solution* of (1.1) is a function $u \in W_0^{1,1}(\Omega)$ such that $a(\cdot,\nabla u) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega a(x,\nabla u)\cdot\nabla\varphi \, dx = \int_\Omega f(x)\varphi \, dx, \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{1.6}$$

A *weak energy solution* is a weak solution such that $u \in W_0^{1,p(\cdot)}(\Omega)$.

The model case for (1.1) is the Dirichlet problem for the $p(x)$-Laplacian operator $\Delta_{p(x)}u := \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$,

$$\begin{cases} -\Delta_{p(x)}u &= f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

This and other related problems (where f is replaced by a nonlinear function depending on u) have been studied recently in several papers (*cf.*, for example, [16] for existence and uniqueness or [17] for Hölder continuity) in the framework of weak energy solutions. These results require the assumption that the right-hand side f has enough integrability.

Assuming that f is merely in $L^1(\Omega)$, we need to work with entropy solutions, which are required to be less regular than weak solutions. The notion of entropy solution was introduced by Bénilan *et al.* [5] for problem (1.1) in the framework of a constant $p(\cdot) \equiv p$, and existence and uniqueness were established, together with some estimates for the solution and its weak gradient. Using essentially the same tools, Alvino *et al.* [3] proved existence of an entropy solution for elliptic problems with degenerate coercivity, still in the context of constant exponents.

The main purpose of this paper is to extend the results in [5] to a nonconstant $p(\cdot)$. Defining the truncation function T_t by

$$T_t(s) := \max\{-t, \min\{t, s\}\}, \quad s \in \mathbb{R},$$

we start by extending the notion of entropy solution to problem (1.1) as follows:

Definition 1.1. A measurable function u is an entropy solution to problem (1.1) if, for every $t > 0$, $T_t(u) \in W_0^{1,p(\cdot)}(\Omega)$ and

$$\int_\Omega a(x, \nabla u) \cdot \nabla T_t(u - \varphi)\, dx \le \int_\Omega f(x)\, T_t(u - \varphi)\, dx, \tag{1.8}$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

A function u such that $T_t(u) \in W_0^{1,p(\cdot)}(\Omega)$, for all $t > 0$, does not necessarily belong to $W_0^{1,1}(\Omega)$. However, it is possible to define its weak gradient (see Proposition 3.1 below), still denoted by ∇u.

Let us introduce the following notation:

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega}\ p(x) \qquad \text{and} \qquad p_+ := \operatorname*{ess\,sup}_{x\in\Omega}\ p(x)\,;$$

given two bounded measurable functions $p(\cdot), q(\cdot) : \Omega \to \mathbb{R}$, we write

$$q(\cdot) \ll p(\cdot) \qquad \text{if } (p - q)_- > 0.$$

Assuming (1.5), the critical Sobolev exponent and the conjugate of $p(\cdot)$ are, respectively,

$$p^*(\cdot) = \frac{Np(\cdot)}{N - p(\cdot)} \quad \text{and} \quad p'(\cdot) = \frac{p(\cdot)}{p(\cdot) - 1}.$$

Our main result is

Theorem 1.2. *Assume* (1.2)–(1.5) *and* $f \in L^1(\Omega)$. *There exists a unique entropy solution u to problem* (1.1). *Moreover,* $|u|^{q(\cdot)} \in L^1(\Omega)$, *for all* $0 \ll q(\cdot) \ll q_0(\cdot)$, *and* $|\nabla u|^{q(\cdot)} \in L^1(\Omega)$, *for all* $0 \ll q(\cdot) \ll q_1(\cdot)$, *where*

$$q_0(\cdot) := \frac{p^*(\cdot)}{p'_+} \quad \text{and} \quad q_1(\cdot) := \frac{q_0(\cdot)}{q_0(\cdot) + 1} p(\cdot). \tag{1.9}$$

The proof of this result will be decomposed into several steps. First, we obtain *a priori* estimates for entropy solutions in Marcinkiewicz spaces with variable exponent. Despite the fact that the theory of functional spaces with variable exponent is developing quickly, the extension of classical Marcinkiewicz spaces is, to the best of our knowledge, undertaken here for the first time. From these estimates, we derive uniform bounds in Lebesgue spaces of variable exponent for an entropy solution and its weak gradient (see Corollaries 3.5 and 3.7 in section 3). The uniqueness follows from choosing adequate test functions in the entropy condition (1.8) and using the *a priori* estimates. Finally, the existence is obtained by passing to the limit in a sequence of weak energy solutions of adequate approximated problems.

Our other theorem concerns weak solutions and extends the results obtained by Boccardo and Gallouët [6, 7] in the context of a constant $p(\cdot) \equiv p$.

Theorem 1.3. *Assume* (1.2)–(1.5) *and* $f \in L^1(\Omega)$. *Let* $q_0(\cdot)$, $q_1(\cdot)$ *be given by* (1.9), *and let* u *be the entropy solution of* (1.1). *If* $2 - 1/N \ll p(\cdot)$, *then* $u \in L^{q(\cdot)}(\Omega)$, *for all* $1 \ll q(\cdot) \ll q_0(\cdot)$, *and* $u \in W_0^{1,q(\cdot)}(\Omega)$, *for all* $1 \ll q(\cdot) \ll q_1(\cdot)$. *If, in addition,* $p(\cdot) - 1 \ll q_1(\cdot)$, *then* u *is a weak solution of* (1.1).

We will see later that $1 \ll q_1(\cdot)$ if and only if $2 - 1/N \ll p(\cdot)$, and hence, by Theorem 1.2, the entropy solution u belongs to $W_0^{1,1}(\Omega)$ if $2-1/N \ll p(\cdot)$. We also remark that, in the constant case, we have

$$q_0 = \frac{N(p-1)}{N-p} \quad \text{and} \quad q_1 = \frac{N(p-1)}{N-1},$$

which coincide with the exponents in [5]. The additional assumption $p(\cdot)-1 \ll q_1(\cdot)$ is needed to show that the entropy solution is indeed a weak solution, *i.e.*, that it satisfies the equation in the distributional sense. Later, we discuss in detail the significance of this assumption and conclude, in particular, that it is not stringent up to dimension $N = 10$ (see Remark 5.7).

In this paper, we always assume that $f \in L^1(\Omega)$; increasing the integrability of f, one expects to obtain more regularity but, for variable exponents, most results in this direction are still missing.

A few comments about known regularity results for the constant exponent case, in terms of the integrability of the right-hand side f, are in order. Assume $p(\cdot) \equiv p$ is constant and the right-hand side $f \in L^m(\Omega)$, for some $m \geq 1$. The existence and uniqueness of an entropy solution u of problem (1.1) is obtained in [5]. Define the numbers

$$\bar{m} := \frac{N}{N(p-1)+1} \qquad \text{and} \qquad \tilde{m} := (p^*)' = \frac{Np}{N(p-1)+p},$$

where $p^* = Np/(N-p)$ is the Sobolev exponent. The following assertions hold:

- **(A1)** If $1 \leq m \leq \max(1, \bar{m})$, then the *entropy* solution u satisfies $|u|^q \in L^1(\Omega)$, for all $0 < q < q_0$, and $|\nabla u|^q \in L^1(\Omega)$, for all $0 < q < q_1$, where

$$q_0 := \frac{Nm(p-1)}{N-mp} \qquad \text{and} \qquad q_1 := \frac{Nm(p-1)}{N-m}$$

  (note that, when $m = 1$, these numbers coincide with the ones defined in (1.9), since we are assuming that $p(\cdot) \equiv p$ is constant).
- **(A2)** If $\max(1, \bar{m}) < m < \tilde{m}$, then u is a *weak* solution and $u \in W_0^{1,q_1}(\Omega)$ (note that $q_1 > 1$).
- **(A3)** If $\tilde{m} \leq m \leq N/p$, then u is a *weak energy* solution and $u \in W_0^{1,q_1}(\Omega)$ (note that $q_1 \geq p$).
- **(A4)** If $m > N/p$, then u is a *bounded weak energy* solution.

The first and last assertions are proved by Alvino *et al.* [3]. The second one follows from the results of Boccardo and Gallouët [6, 7], and the third is a consequence of a result by Kinnunen and Zhou [22, Thm. 1.6]. It is also known that if $m > Np'$, then $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$, a result due to DiBenedetto [10].

For a variable exponent $p(\cdot)$ much less is known. If $f \in W^{-1,p'(\cdot)}(\Omega)$ or, in particular, if $f \in L^{\tilde{m}(\cdot)}(\Omega)$, where $\tilde{m}(\cdot) := (p(\cdot)^*)'$, the existence and uniqueness of a weak energy solution to problem (1.1) is a straightforward generalization of the results obtained by Fan and Zhang [16] for the model problem (1.7).

Recently, Acerbi and Mingione [2] derived Calderón–Zygmund type estimates for (1.1), extending previous results of DiBenedetto and Manfredi [11] for the model problem (1.7) and $p(\cdot) \equiv p$ constant. Using their estimates it is easy to prove the following result.

Proposition 1.4. *Assume* (1.2)–(1.5) *and* $f \in L^{m(\cdot)}_{\mathrm{loc}}(\Omega)$, *where*

$$m(\cdot) := \frac{Np(\cdot)q}{N(p(\cdot)-1)+p(\cdot)q} \qquad \text{with} \quad q \geq 1. \tag{1.10}$$

The unique weak energy solution u of (1.1) *satisfies* $|\nabla u|^{p(\cdot)} \in L^q_{\mathrm{loc}}(\Omega)$.

We note that the function $m(\cdot)$ defined in (1.10) satisfies

$$\tilde{m}(\cdot) < m(\cdot) < N\,, \quad \text{for all } q > 1.$$

As an immediate consequence, one obtains $u \in W^{1,r(\cdot)}_{\mathrm{loc}}(\Omega)$, for all $r(\cdot) \in L^\infty(\Omega)$, if $f \in L^N_{\mathrm{loc}}(\Omega)$. We note that, in the case of constant exponents, Proposition 1.4 states that for $f \in L^m_{\mathrm{loc}}(\Omega)$, with $m \geq \tilde{m}$, we have $u \in W^{1,q_1}_{\mathrm{loc}}(\Omega)$. Moreover, as a consequence of Sobolev embedding, it follows that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ if $m > N/p$. We thus recover local versions of assertions (A3) and (A4). Therefore, to obtain (A3) and (A4) using this reasoning, it would be necessary to prove a global version of Proposition 1.4 for a nonconstant $q(\cdot)$.

Finally, since Theorem 1.2 guarantees the existence and uniqueness of an entropy solution for (1.1), the extension of (A1) and (A2) for variable exponents only requires *a priori* estimates for such a solution. We feel that the techniques needed to obtain such estimates are slight modifications of the ones used in section 3 in the L^1 case but this extension remains open.

The paper is organized as follows. In section 2, we recall the definitions of Lebesgue and Sobolev spaces with variable exponent and some of their properties. Then, we introduce Marcinkiewicz spaces with variable exponent and establish their relation with Lebesgue spaces. In section 3, we obtain *a priori* estimates for an entropy solution and its weak gradient. In section 4, we prove uniqueness of entropy solutions. Finally, in section 5, we consider approximate problems and, using the *a priori* estimates, we establish the existence results.

2. Marcinkiewicz spaces with variable exponent

In this section, we define Marcinkiewicz spaces with variable exponent and investigate their relation with Lebesgue spaces. To the best of our knowledge, this definition is considered here for the first time and the properties obtained are new.

We start with a brief overview of the state of the art concerning Lebesgue spaces with variable exponent, and Sobolev spaces modeled upon them. We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\varrho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\, dx$$

is finite. If the exponent is bounded, *i.e.*, if $p_+ < \infty$, then the expression

$$\|u\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leq 1\right\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. One central property of $L^{p(\cdot)}(\Omega)$ is that the norm and the modular topologies coincide; *i.e.*, $\varrho_{p(\cdot)}(u_n) \to 0$ if and only if $\|u_n\|_{p(\cdot)} \to 0$. The space $\left(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}\right)$ is a separable Banach space. Moreover, if $p_- > 1$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. Finally, we have the Hölder inequality:

$$\left|\int_\Omega uv\ dx\right| \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}, \tag{2.1}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

Now, let

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

By $W_0^{1,p(\cdot)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

The proofs of the following two results can be found in [20] and [13], respectively.

Proposition 2.1 (The $p(\cdot)$-Poincaré inequality)**.** *Let Ω be a bounded open set and let $p(\cdot) : \Omega \to [1, \infty)$ satisfy* (1.5)*. There exists a constant C, depending only on $p(\cdot)$ and Ω, such that the inequality*

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)} \tag{2.2}$$

holds for every $u \in W_0^{1,p(\cdot)}(\Omega)$.

Proposition 2.2 (Sobolev embedding)**.** *Let Ω be a bounded open set, with a Lipschitz boundary, and let $p(\cdot) : \Omega \to [1, \infty)$ satisfy* (1.5)*. Then we have the following continuous embedding:*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega), \tag{2.3}$$

where $p^*(\cdot) = \frac{Np(\cdot)}{N-p(\cdot)}$.

Now, we give a useful result in order to apply the Sobolev inequality (*cf.* [15]).

Lemma 2.3. *Let $p(\cdot)$ and $q(\cdot)$ be measurable functions such that $p(\cdot) \in L^\infty(\Omega)$ and $1 \leq p(x)q(x) \leq +\infty$, for a.e. $x \in \Omega$. Let $f \in L^{q(\cdot)}(\Omega)$, $f \not\equiv 0$. Then*

$$\begin{aligned} &\|f\|_{p(\cdot)q(\cdot)}^{p_+} \leq \||f|^{p(\cdot)}\|_{q(\cdot)} \leq \|f\|_{p(\cdot)q(\cdot)}^{p_-} \quad \text{if } \|f\|_{p(\cdot)q(\cdot)} \leq 1, \\ &\|f\|_{p(\cdot)q(\cdot)}^{p_-} \leq \||f|^{p(\cdot)}\|_{q(\cdot)} \leq \|f\|_{p(\cdot)q(\cdot)}^{p_+} \quad \text{if } \|f\|_{p(\cdot)q(\cdot)} \geq 1. \end{aligned} \tag{2.4}$$

In particular, if $p(\cdot) \equiv p$ is constant, then

$$\||f|^p\|_{q(\cdot)} = \|f\|_{pq(\cdot)}^p.$$

This closes our brief tour of Lebesgue and Sobolev spaces with variable exponent. Let's now consider Marcinkiewicz spaces with variable exponent. To the best of our knowledge, the next definition is new.

Definition 2.4. Let $q(\cdot)$ be a measurable function such that $q_- > 0$. We say that a measurable function u belongs to the Marcinkiewicz space $M^{q(\cdot)}(\Omega)$ if there exists a positive constant M such that

$$\int_{\{|u|>t\}} t^{q(x)}\ dx \le M, \quad \text{for all} \ \ t>0.$$

We remark that for $q(\cdot) \equiv q$ constant this definition coincides with the classical definition of the Marcinkiewicz space $M^q(\Omega)$ (*cf.* [25]). Moreover, it is clear that $u \in M^{q(\cdot)}(\Omega)$ if $|u|^{q(\cdot)} \in L^1(\Omega)$. Indeed,

$$\int_{\{|u|>t\}} t^{q(x)}\ dx \le \int_\Omega |u|^{q(x)}\ dx, \quad \text{for all } t>0.$$

In particular, $L^{q(\cdot)}(\Omega) \subset M^{q(\cdot)}(\Omega)$, for all $q(\cdot) \ge 1$.

For constant exponents it is straightforward to prove some sort of reciproque: if $u \in M^r(\Omega)$, then $|u|^q \in L^1(\Omega)$, for all $0 < q < r$. The following result extends this assertion to the nonconstant setting; unlike the constant case, the proof presents some difficulties.

Proposition 2.5. *Let $r(\cdot)$ and $q(\cdot)$ be bounded functions such that $0 \ll q(\cdot) \ll r(\cdot)$ and let $\epsilon := (r-q)_- > 0$. If $u \in M^{r(\cdot)}(\Omega)$, then*

$$\int_\Omega |u|^{q(x)}\ dx \le 2|\Omega| + (r_+ - \epsilon)\frac{M}{\epsilon},$$

where M is the constant appearing in the definition of $M^{r(\cdot)}(\Omega)$. In particular, $M^{r(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega)$, for all $1 \le q(\cdot) \ll r(\cdot)$.

Proof. Noting that $0 \ll q(\cdot) \le r(\cdot) - \epsilon$, we define the a.e. differentiable function

$$\varphi(t) := \int_{\{|u|>t\}} t^{r(x)-\epsilon}\ dx, \quad \text{for all } t>0.$$

Writing its derivative as

$$\varphi'(t) = \int_{\{|u|>t\}} (r(x)-\epsilon) t^{r(x)-\epsilon-1}\ dx - \lim_{h\downarrow 0} \frac{1}{h} \int_{\{t-h<|u|\le t\}} t^{r(x)-\epsilon}\ dx,$$

we obtain

$$\begin{aligned} -\frac{d}{dt}\int_{\{|u|>t\}} |u|^{r(x)-\epsilon}\ dx &= \lim_{h\downarrow 0} \frac{1}{h} \int_{\{t-h<|u|\le t\}} |u|^{r(x)-\epsilon}\ dx \\ &\le \lim_{h\downarrow 0} \frac{1}{h} \int_{\{t-h<|u|\le t\}} t^{r(x)-\epsilon}\ dx \\ &= \int_{\{|u|>t\}} (r(x)-\epsilon) t^{r(x)-\epsilon-1}\ dx - \varphi'(t). \end{aligned}$$

Using the previous inequality and remarking that $0 \leq \varphi(t) \leq M/t^\epsilon$, for all $t > 0$, since $u \in M^{r(\cdot)}(\Omega)$, we derive the estimate

$$\begin{aligned}
\int_\Omega |u|^{q(x)}\,dx &\leq |\Omega| + \int_{\{|u|>1\}} |u|^{r(x)-\epsilon}\,dx \\
&= |\Omega| + \int_1^\infty \left(-\frac{d}{dt} \int_{\{|u|>t\}} |u|^{r(x)-\epsilon}\,dx \right) dt \\
&\leq |\Omega| + \int_1^\infty \left(\int_{\{|u|>t\}} (r(x)-\epsilon) t^{r(x)-\epsilon-1}\,dx - \varphi'(t) \right) dt \\
&\leq |\Omega| + (r^+ - \epsilon) \int_1^\infty \frac{1}{t^{\epsilon+1}} \left(\int_{\{|u|>t\}} t^{r(x)}\,dx \right) dt + \varphi(1) \\
&\leq 2|\Omega| + (r^+ - \epsilon) \int_1^\infty \frac{M}{t^{\epsilon+1}}\,dt \\
&= 2|\Omega| + (r^+ - \epsilon) \frac{M}{\epsilon}
\end{aligned}$$

and the result follows. □

3. A priori estimates

We start with the existence of the weak gradient for every measurable function u such that $T_t(u) \in W_0^{1,p(\cdot)}(\Omega)$, for all $t > 0$.

Proposition 3.1. *If u is a measurable function such that $T_t(u) \in W_0^{1,p(\cdot)}(\Omega)$, for all $t > 0$, then there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$v\chi_{\{|u|<t\}} = \nabla T_t(u) \quad \text{for a.e. } x \in \Omega, \text{ and for all } t > 0,$$

where χ_E denotes the characteristic function of a measurable set E. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v coincides with the standard distributional gradient of u.

Proof. The result follows from [3, Theorem 1.5], since

$$T_t(u) \in W_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,p_-}(\Omega), \quad \text{for all } t > 0. \qquad \square$$

The next result provides estimates in Marcinkiewicz spaces (and hence, by Proposition 2.5, in Lebesgue spaces) for an entropy solution of (1.1).

Proposition 3.2. *Assume* (1.2)–(1.5) *and $f \in L^1(\Omega)$. If u is an entropy solution of* (1.1)*, then there exists a positive constant M, depending only on $p(\cdot)$, N, and Ω, such that*

$$\int_{\{|u|>t\}} t^{p^*(x)/p'_+}\,dx \leq M \left(\frac{\|f\|_1}{b} + 1 \right)^{p^*_+/p_-}, \quad \text{for all } t > 0.$$

Proof. Taking $\varphi = 0$ in the entropy inequality (1.8) and using (1.2), we obtain

$$\begin{aligned}
b \int_\Omega |\nabla T_t(u)|^{p(x)}\,dx &\leq \int_{\{|u|\leq t\}} a(x,\nabla u)\cdot\nabla u\,dx \\
&\leq \int_\Omega f(x)\,T_t(u)\,dx \leq t\|f\|_1,
\end{aligned}$$

for all $t > 0$. Therefore, defining $\psi := T_t(u)/t$, we have, for all $t > 0$,

$$\int_\Omega t^{p(x)-1}|\nabla\psi|^{p(x)}\, dx = \frac{1}{t}\int_\Omega |\nabla T_t(u)|^{p(x)}\, dx \le M_1 := \frac{\|f\|_1}{b}. \tag{3.1}$$

Let $\gamma > 0$ be a number to be chosen later. Using the Sobolev inequality (2.3) and Lemma 2.3, we estimate

$$\begin{aligned}
\int_{\{|u|>t\}} t^{p^*(x)/\gamma}\, dx &= \int_{\{|\psi|=1\}} t^{p^*(x)/\gamma}|\psi|^{p^*(x)}\, dx \\
&\le \int_\Omega \left(t^{1/\gamma}|\psi|\right)^{p^*(x)}\, dx \\
&\le \left\|t^{1/\gamma}\psi\right\|_{p^*(\cdot)}^{\alpha_1} \\
&\le C^{\alpha_1}\left\|\nabla(t^{1/\gamma}\psi)\right\|_{p(\cdot)}^{\alpha_1} \\
&\le C^{\alpha_1}\left(\int_\Omega |\nabla(t^{1/\gamma}\psi)|^{p(x)}\, dx\right)^{\alpha_1/\beta_1} \\
&= C^{\alpha_1}\left(\int_\Omega t^{p(x)/\gamma+1-p(x)}\frac{1}{t}|\nabla T_t(u)|^{p(x)}\, dx\right)^{\alpha_1/\beta_1}, \qquad (3.2)
\end{aligned}$$

where

$$\alpha_1 = \begin{cases} p_+^* & \text{if} \quad \|t^{1/\gamma}\psi\|_{p^*(\cdot)} \ge 1 \\ p_-^* & \text{if} \quad \|t^{1/\gamma}\psi\|_{p^*(\cdot)} \le 1 \end{cases} \quad \text{and } \beta_1 = \begin{cases} p_- & \text{if} \quad \|\nabla(t^{1/\gamma}\psi)\|_{p(\cdot)} \ge 1 \\ p_+ & \text{if} \quad \|\nabla(t^{1/\gamma}\psi)\|_{p(\cdot)} \le 1. \end{cases}$$

Now, choosing $\gamma = p'_+$ in (3.2), noting that $t^{p(x)/\gamma+1-p(x)} \le 1$ for all $t \ge 1$, and using (3.1), we obtain

$$\int_{\{|u|>t\}} t^{p^*(x)/p'_+}\, dx \le C^{\alpha_1}\left(M_1+1\right)^{p_+^*/p_-}$$

for all $t \ge 1$. Finally, for $0 < t < 1$, we have

$$\int_{\{|u|>t\}} t^{p^*(x)/p'_+}\, dx \le |\Omega|.$$

Combining both estimates, the result follows. $\square$

Remark 3.3. Recalling from (1.9) that

$$q_0(\cdot) = \frac{p^*(\cdot)}{p'_+},$$

Proposition 3.2 yields $u \in M^{q_0(\cdot)}(\Omega)$. We note that for $p(\cdot) \equiv p$ we have that $u \in M^{q_0}(\Omega)$, with

$$q_0 = \frac{N(p-1)}{N-p} = \frac{p^*}{p'},$$

recovering the result obtained in [5]. For the nonconstant case, it remains an open problem to show that $u \in M^{q(\cdot)}(\Omega)$, with $q(\cdot) = p^*(\cdot)/p'(\cdot)$.

Remark 3.4. We stress that the dependence of the constant M on $p(\cdot)$ occurs solely through the constants p_- and p_+.

As a consequence of Proposition 2.5 and Proposition 3.2 we obtain the following result.

Corollary 3.5. *Assume* (1.2)–(1.5) *and* $f \in L^1(\Omega)$. *Let*

$$q_0(\cdot) = \frac{p^*(\cdot)}{p'_+}. \tag{3.3}$$

If u *is an entropy solution to problem* (1.1), *then* $|u|^{q(\cdot)} \in L^1(\Omega)$, *for all* $q(\cdot)$ *such that* $0 \ll q(\cdot) \ll q_0(\cdot)$. *Moreover, there exists a constant* M_0, *depending only on* $p(\cdot)$, $q(\cdot)$, N *and* Ω, *such that*

$$\int_\Omega |u|^{q(x)}\ dx \le 2|\Omega| + M_0 \left(\frac{\|f\|_1}{b} + 1\right)^{p_+^*/p_-}. \tag{3.4}$$

Proof. Let $0 \ll q(\cdot) \ll q_0(\cdot)$ and define $\delta := (q_0 - q)_- > 0$. By Proposition 3.2,

$$\int_{\{|u|>t\}} t^{q_0(x)}\ dx \le M \left(\frac{\|f\|_1}{b} + 1\right)^{p_+^*/p_-}, \quad \text{for all } t > 0,$$

where M is a positive constant, depending only on $p(\cdot)$, N and Ω. From Proposition 2.5, we have

$$\int_\Omega |u|^{q(x)}\ dx \le 2|\Omega| + (q_0 - \delta)_+ \frac{M}{\delta} \left(\frac{\|f\|_1}{b} + 1\right)^{p_+^*/p_-};$$

estimate (3.4) now follows with

$$M_0 = (q_0 - \delta)_+ \frac{M}{\delta}.$$

□

Now, we prove *a priori* estimates in Marcinkiewicz spaces for the weak gradient of an entropy solution.

Proposition 3.6. *Assume* (1.2)–(1.5) *and* $f \in L^1(\Omega)$. *Let* u *be an entropy solution of* (1.1). *If there exists a positive constant* M *such that*

$$\int_{\{|u|>t\}} t^{q(x)}\ dx \le M, \quad \text{for all } t > 0, \tag{3.5}$$

then $|\nabla u|^{\alpha(\cdot)} \in M^{q(\cdot)}(\Omega)$, *where* $\alpha(\cdot) = p(\cdot)/(q(\cdot)+1)$. *Moreover,*

$$\int_{\{|\nabla u|^{\alpha(\cdot)}>t\}} t^{q(x)}\ dx \le \frac{\|f\|_1}{b} + M, \quad \text{for all } t > 0.$$

Proof. Using (3.5), the definition of $\alpha(\cdot)$, and (3.1), which still holds in this setting, we have

$$\begin{aligned}
\int_{\{|\nabla u|^{\alpha(x)}>t\}} t^{q(x)}\ dx &\le \int_{\{|\nabla u|^{\alpha(x)}>t\}\cap\{|u|\le t\}} t^{q(x)}\ dx + \int_{\{|u|>t\}} t^{q(x)}\ dx \\
&\le \int_{\{|u|\le t\}} t^{q(x)} \left(\frac{|\nabla u|^{\alpha(x)}}{t}\right)^{p(x)/\alpha(x)} dx + M \\
&= \frac{1}{t}\int_{\{|u|\le t\}} |\nabla T_t(u)|^{p(x)}\ dx + M \\
&\le \frac{\|f\|_1}{b} + M, \quad \text{for all } t > 0.
\end{aligned}$$

□

As a consequence of Proposition 2.5, Proposition 3.2, and Proposition 3.6, we obtain the following result.

Corollary 3.7. *Assume* (1.2)–(1.5) *and* $f \in L^1(\Omega)$. *Let* $q_0(\cdot)$ *be defined in* (3.3) *and let*

$$q_1(\cdot) = \frac{q_0(\cdot)}{q_0(\cdot)+1} p(\cdot).$$

If u is an entropy solution of problem (1.1), *then* $|\nabla u|^{q(\cdot)} \in L^1(\Omega)$, *for all* $q(\cdot)$ *such that* $0 \ll q(\cdot) \ll q_1(\cdot)$. *Moreover, there exist constants* M_2 *and* M_3, *depending only on* $p(\cdot)$, $q(\cdot)$, N *and* Ω, *such that*

$$\int_\Omega |\nabla u|^{q(x)}\, dx \le 2|\Omega| + M_2 \frac{\|f\|_1}{b} + M_3 \left(\frac{\|f\|_1}{b} + 1\right)^{p_+^*/p_-}. \tag{3.6}$$

Proof. By Proposition 3.6 (and using also Proposition 3.2), we have

$$|\nabla u|^{\alpha(\cdot)} \in M^{q_0(\cdot)}(\Omega), \quad \text{with} \quad \alpha(\cdot) = p(\cdot)/(q_0(\cdot)+1),$$

and

$$\int_{\left\{|\nabla u|^{\alpha(\cdot)} > t\right\}} t^{q_0(x)}\, dx \le \frac{\|f\|_1}{b} + M\left(\frac{\|f\|_1}{b} + 1\right)^{p_+^*/p_-}, \quad \text{for all } t > 0,$$

where M is a positive constant, depending only on $p(\cdot)$, N and Ω.

Let $0 \ll q(\cdot) \ll q_1(\cdot)$, note that $r(\cdot) := q(\cdot)/\alpha(\cdot) \ll q_0(\cdot)$ and define $\varrho := (q_0 - r)_- > 0$. From Proposition 2.5 applied to $|\nabla u|^{\alpha(\cdot)}$, we have

$$\begin{aligned} \int_\Omega |\nabla u|^{q(x)}\, dx &= \int_\Omega |\nabla u|^{\alpha(x) r(x)}\, dx \\ &\le 2|\Omega| + \frac{(q_0-\varrho)_+}{\varrho}\left[\frac{\|f\|_1}{b} + M\left(\frac{\|f\|_1}{b}+1\right)^{p_+^*/p_-}\right], \end{aligned}$$

and the result follows with

$$M_2 = \frac{(q_0-\varrho)_+}{\varrho} \quad \text{and} \quad M_3 = M M_2. \qquad \square$$

4. Uniqueness of entropy solutions

In this section we establish the uniqueness of an entropy solution, extending the result obtained in [5] for a constant exponent.

Theorem 4.1. *Assume* (1.2)–(1.5) *and* $f \in L^1(\Omega)$. *If u and v are entropy solutions of* (1.1), *then* $u = v$, *a.e. in* Ω.

Proof. Let $h > 0$. We write the entropy inequality (1.8) corresponding to the solution u, with $T_h v$ as test function, and to the solution v, with $T_h u$ as test function. Upon addition, we get

$$\begin{aligned} &\int_{\{|u - T_h v| \le t\}} a(x, \nabla u)\cdot \nabla(u - T_h v)\, dx + \int_{\{|v - T_h u| \le t\}} a(x, \nabla v)\cdot\nabla(v - T_h u)\, dx \\ &\le \int_\Omega f(x)\Big(T_t(u - T_h v) + T_t(v - T_h u)\Big)\, dx. \end{aligned} \tag{4.1}$$

Define

$$E_1 := \{|u-v| \le t,\ |v| \le h\},$$
$$E_2 := E_1 \cap \{|u| \le h\}, \qquad \text{and} \qquad E_3 := E_1 \cap \{|u| > h\}.$$

We start with the first integral in (4.1). Using assumption (1.2), we obtain

$$\int_{\{|u-T_h v| \le t\}} a(x,\nabla u)\cdot\nabla(u-T_h v)\,dx \ge \int_{E_1} a(x,\nabla u)\cdot\nabla(u-v)\,dx$$
$$\ge \int_{E_2} a(x,\nabla u)\cdot\nabla(u-v)\,dx - \int_{E_3} a(x,\nabla u)\cdot\nabla v\,dx. \tag{4.2}$$

By assumption (1.3) and the Hölder inequality (2.1), we estimate the last integral in the above expression as follows:

$$\left|\int_{E_3} a(x,\nabla u)\cdot\nabla v\,dx\right| \le \beta\int_{E_3}\left(j(x)+|\nabla u|^{p(x)-1}\right)|\nabla v|\,dx$$
$$\le 2\beta\left(\|j\|_{p'(\cdot)} + \left\||\nabla u|^{p(x)-1}\right\|_{p'(\cdot),\{h<|u|\le h+t\}}\right)\|\nabla v\|_{p(\cdot),\{h-t<|v|\le h\}}. \tag{4.3}$$

The last expression converges to zero as h tends to infinity, by Proposition 3.2, inequality (2.4), and the following bound for an entropy solution w:

$$\int_{\{h<|w|\le h+t\}} |\nabla w|^{p(x)}\,dx \le \frac{1}{b}\int_{\{h<|w|\le h+t\}} a(x,\nabla w)\cdot\nabla w\,dx \le \frac{t}{b}\,\|f\|_1,$$

which follows from taking $\varphi = T_h(w)$ as a test function in the entropy inequality (1.8). Therefore, from (4.2) and (4.3), we obtain

$$\int_{\{|u-T_h v|\le t\}} a(x,\nabla u)\cdot\nabla(u-T_h v)\,dx \ge I + \int_{E_2} a(x,\nabla u)\cdot\nabla(u-v)\,dx, \tag{4.4}$$

where I converges to zero as h tends to infinity. We may adopt the same procedure to treat the second integral in (4.1) and obtain

$$\int_{\{|v-T_h u|\le t\}} a(x,\nabla v)\cdot\nabla(v-T_h u)\,dx \ge II - \int_{E_2} a(x,\nabla v)\cdot\nabla(u-v)\,dx, \tag{4.5}$$

where II converges to zero as h tends to infinity.

Next, we consider the right-hand side of inequality (4.1). Noting that

$$T_t(u-T_h v) + T_t(v-T_h u) = 0 \quad \text{in} \quad \{|u|\le h,\ |v|\le h\},$$

we obtain

$$\left|\int_\Omega f(x)\left(T_t(u-T_h v)+T_t(v-T_h u)\right)dx\right|$$
$$\le 2t\left(\int_{\{|u|>h\}}|f|\,dx + \int_{\{|v|>h\}}|f|\,dx\right).$$

Since both meas $\{|u|>h\}$ and meas $\{|v|>h\}$ tend to zero as h goes to infinity (by Proposition 3.2), the right-hand side of inequality (4.1) tends to zero as h goes to infinity. From this assertion, (4.1), (4.4), and (4.5) we obtain, letting $h\to+\infty$,

$$\int_{\{|u-v|\le t\}} (a(x,\nabla u)-a(x,\nabla v))\cdot\nabla(u-v)\,dx \le 0, \quad \text{for all } t>0.$$

By assumption (1.4), we conclude that $\nabla u = \nabla v$, a.e. in Ω.

Finally, from the Poincaré inequality (2.2), we have

$$\|T_t(u-v)\|_{p(\cdot)} \leq C\|\nabla(T_t(u-v))\|_{p(\cdot)} = 0, \quad \text{for all } t > 0,$$

and hence $u = v$, a.e. in Ω. □

5. Existence of entropy and weak solutions

Let $(f_n)_n$ be a sequence of bounded functions, strongly converging to $f \in L^1(\Omega)$ and such that

$$\|f_n\|_1 \leq \|f\|_1, \quad \text{for all } n. \tag{5.1}$$

We consider the problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) &= f_n(x) & \text{in} & \Omega, \\ u &= 0 & \text{on} & \partial\Omega. \end{cases} \tag{5.2}$$

It follows from a standard modification of the arguments in [16, Theorem 4.2] that problem (5.2) has a unique weak energy solution $u_n \in W_0^{1,p(\cdot)}(\Omega)$. Our aim is to prove that these approximate solutions u_n tend, as n goes to infinity, to a measurable function u which is an entropy solution of the limit problem (1.1). We will divide the proof into several steps and use as the main tool the *a priori* estimates for u_n and its gradient obtained in section 3. Much of the reasoning is based on the ideas developed in [7], [5], and [3]; although some of the arguments are not new, we have decided to present a self-contained proof for the sake of clarity and readability.

We start by proving that the sequence $(u_n)_n$ of solutions of problem (5.2) converges in measure to a measurable function u.

Proposition 5.1. *Assume* (1.2)–(1.5), $f \in L^1(\Omega)$ *and* (5.1). *Let* $u_n \in W_0^{1,p(\cdot)}(\Omega)$ *be the solution of* (5.2). *The sequence* $(u_n)_n$ *is Cauchy in measure. In particular, there exists a measurable function* u *such that* $u_n \to u$ *in measure.*

Proof. Let $s > 0$ and define

$$E_1 := \{|u_n| > t\}\,, \; E_2 := \{|u_m| > t\}\,, \text{ and } \; E_3 := \{|T_t(u_n) - T_t(u_m)| > s\}\,,$$

where $t > 0$ is to be fixed. We note that

$$\{|u_n - u_m| > s\} \subset E_1 \cup E_2 \cup E_3,$$

and hence,

$$\mathrm{meas}\,\{|u_n - u_m| > s\} \leq \mathrm{meas}\,(E_1) + \mathrm{meas}\,(E_2) + \mathrm{meas}\,(E_3). \tag{5.3}$$

Let $\epsilon > 0$. Using (5.1) and the uniform bound given by Proposition 3.2, we choose $t = t(\epsilon)$ such that

$$\mathrm{meas}\,(E_1) \leq \epsilon/3 \quad \text{and} \quad \mathrm{meas}\,(E_2) \leq \epsilon/3. \tag{5.4}$$

On the other hand, taking $\varphi = 0$ in the entropy condition (1.8) for u_n yields

$$\int_\Omega |\nabla T_t(u_n)|^{p(x)}\,dx \leq \frac{\|f\|_1}{b}t, \quad \text{for all } n \geq 0, \tag{5.5}$$

using (1.2) and (5.1). Therefore, we can assume, by the Sobolev embedding (2.3), that $(T_t(u_n))_n$ is a Cauchy sequence in $L^{q(\cdot)}(\Omega)$, for all $1 \leq q(\cdot) \ll p^*(\cdot)$. Consequently, there exists a measurable function u such that

$$T_t(u_n) \to T_t(u), \quad \text{in } L^{q(\cdot)}(\Omega) \; \text{ and a.e.}$$

Thus,

$$\text{meas}\,(E_3) \leq \int_\Omega \left(\frac{|T_t(u_n) - T_t(u_m)|}{s} \right)^{q(x)} dx \leq \frac{\epsilon}{3}$$

for all $n, m \geq n_0(s, \epsilon)$.

Finally, from (5.3), (5.4), and the last estimate, we obtain that

$$\text{meas}\,\{|u_n - u_m| > s\} \leq \epsilon, \quad \text{for all } n, m \geq n_0(s, \epsilon); \tag{5.6}$$

i.e., $(u_n)_n$ is a Cauchy sequence in measure. □

In order to prove that the sequence $(\nabla u_n)_n$ converges in measure to the weak gradient of u we need the following standard fact in measure theory (*cf.* [19]).

Lemma 5.2. *Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < +\infty$. Consider a measurable function $\gamma : X \to [0, +\infty]$ such that*

$$\mu(\{x \in X : \gamma(x) = 0\}) = 0.$$

Then, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mu(A) < \epsilon, \quad \textit{for all } A \in \mathcal{M} \quad \textit{with} \quad \int_A \gamma \, d\mu < \delta.$$

Proposition 5.3. *Assume* (1.2)–(1.5), *$f \in L^1(\Omega)$ and* (5.1). *Let $u_n \in W_0^{1,p(\cdot)}(\Omega)$ be the solution of* (5.2). *Then ∇u_n converges in measure to the weak gradient of u.*

Proof. We claim that $(\nabla u_n)_n$ is Cauchy in measure. Indeed, let $s > 0$, and consider

$$E_1 := \{|\nabla u_n| > h\} \cup \{|\nabla u_m| > h\}, \quad E_2 := \{|u_n - u_m| > t\},$$

and

$$E_3 := \{|\nabla u_n| \leq h, |\nabla u_m| \leq h, |u_n - u_m| \leq t, |\nabla u_n - \nabla u_m| > s\},$$

where h and t will be chosen later. We note that

$$\{|\nabla u_n - \nabla u_m| > s\} \subset E_1 \cup E_2 \cup E_3. \tag{5.7}$$

Let $\epsilon > 0$. By Proposition 3.6, we may choose $h = h(\epsilon)$ large enough such that $\text{meas}(E_1) \leq \epsilon/3$ for all $n, m \geq 0$. On the other hand, by Proposition 5.1 (see (5.6)), we have that $\text{meas}(E_2) \leq \epsilon/3$ for all $n, m \geq n_0(t, \epsilon)$. Moreover, by assumption (1.4), there exists a real-valued function $\gamma : \Omega \to [0, +\infty]$ such that $\text{meas}\{x \in \Omega : \gamma(x) = 0\} = 0$ and

$$(a(x, \xi) - a(x, \xi')) \cdot (\xi - \xi') \geq \gamma(x), \tag{5.8}$$

for all ξ, $\xi' \in \mathbb{R}^N$ such that $|\xi|$, $|\xi'| \leq h$, $|\xi - \xi'| \geq s$, for a.e. $x \in \Omega$ (*cf.* [7]). Let $\delta = \delta(\epsilon)$ be given from Lemma 5.2, replacing ϵ and A by $\epsilon/3$ and E_3, respectively. Using (5.8), the equation, and (5.1), we obtain

$$\int_{E_3} \gamma(x) \, dx \leq \int_{E_3} (a(x, \nabla u_n) - a(x, \nabla u_m)) \cdot \nabla(u_n - u_m) \, dx \leq 2\|f\|_1 t < \delta,$$

choosing $t = \delta/(4\|f\|_1)$. From Lemma 5.2, it follows that $\text{meas}(E_3) < \epsilon/3$. Thus, using (5.7) and the estimates obtained for E_1, E_2, and E_3, it follows that $\text{meas}(\{|\nabla u_n - \nabla u_m| \geq s\}) \leq \epsilon$, for all $n, m \geq n_0(s, \epsilon)$, proving the claim.

As a consequence, $(\nabla u_n)_n$ converges in measure to some measurable function v. Finally, since $(\nabla T_t u_n)_n$ is uniformly bounded in $L^{p(\cdot)}(\Omega)$, for all $t > 0$, it converges weakly to $\nabla(T_t u)$ in $L^1(\Omega)$. Therefore, v coincides with the weak gradient of u (see Proposition 3.1). □

We now prove the main theorem in this paper.

Proof of Theorem 1.2. Fix $t > 0$, $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, and choose $T_t(u_n - \varphi)$ as a test function in (1.6), with u replaced by u_n, to obtain

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla T_t(u_n - \varphi)\ dx = \int_\Omega f_n(x)\, T_t(u_n - \varphi)\ dx.$$

We note that this choice can be made using a standard density argument. We now pass to the limit in the previous identity. Concerning the right-hand side, the convergence is obvious since f_n converges strongly in L^1 to f and $T_t(u_n - \varphi)$ converges weakly-$*$ in L^∞, and a.e. to $T_t(u - \varphi)$.

Next, we write the left-hand side as

$$\int_{\{|u_n - \varphi| \le t\}} a(x, \nabla u_n) \cdot \nabla u_n\ dx - \int_{\{|u_n - \varphi| \le t\}} a(x, \nabla u_n) \cdot \nabla \varphi\ dx \tag{5.9}$$

and note that $\{|u_n - \varphi| \le t\}$ is a subset of $\{|u_n| \le t + \|\varphi\|_\infty\}$. Hence, taking $s = t + \|\varphi\|_\infty$, we rewrite the second integral in (5.9) as

$$\int_{\{|u_n - \varphi| \le t\}} a(x, \nabla T_s(u_n)) \cdot \nabla \varphi\ dx.$$

Since $a(x, \nabla T_s(u_n))$ is uniformly bounded in $(L^{p'(\cdot)}(\Omega))^N$ (by assumption (1.3) and (5.5)) and by Proposition 5.3, we have that it converges weakly to $a(x,\nabla T_s(u))$ in $(L^{p'(\cdot)}(\Omega))^N$. Therefore the last integral converges to

$$\int_{\{|u - \varphi| \le t\}} a(x, \nabla u) \cdot \nabla \varphi\ dx.$$

The first integral in (5.9) is nonnegative, by (1.2), and it converges a.e. by Proposition 5.3. It follows from Fatou's lemma that

$$\int_{\{|u - \varphi| \le t\}} a(x, \nabla u) \cdot \nabla u\ dx \le \liminf_{n \to +\infty} \int_{\{|u_n - \varphi| \le t\}} a(x, \nabla u_n) \cdot \nabla u_n\ dx.$$

Gathering results, we obtain

$$\int_\Omega a(x, \nabla u) \cdot \nabla T_t(u - \varphi)\ dx \le \int_\Omega f(x) T_t(u - \varphi)\ dx;$$

i.e., u is an entropy solution of (1.1).

The uniqueness follows from Theorem 4.1 and the regularity properties from Corollaries 3.5 and 3.7. $\square$

To obtain Theorem 1.3 we need to prove, in particular, that u satisfies the equation in the distributional sense, *i.e.*, that (1.6) holds. For this, we need two technical lemmas. The first one is an extension of Lemma 6.1 in [5].

Lemma 5.4. *Let $(v_n)_n$ be a sequence of measurable functions. If v_n converges in measure to v and is uniformly bounded in $L^{q(\cdot)}(\Omega)$, for some $1 \ll q(\cdot) \in L^\infty(\Omega)$, then $v_n \to v$ strongly in $L^1(\Omega)$.*

Proof. Note first that $L^{q(\cdot)}(\Omega) \subset L^{q_-}(\Omega)$, and hence we may assume $(v_n)_n$ to be uniformly bounded in $L^{q_-}(\Omega)$. Using this fact and the Hölder inequality, we obtain

$$\begin{aligned} \int_\Omega |v_m - v_n| \, dx &= \int_{\{|v_m - v_n| \le s\}} |v_m - v_n| \, dx + \int_{\{|v_m - v_n| > s\}} |v_m - v_n| \, dx \\ &\le |\Omega| s + \mathrm{meas}(\{|v_m - v_n| > s\})^{1/q'_-} \|v_m - v_n\|_{q_-} \\ &\le |\Omega| s + C \, \mathrm{meas}(\{|v_m - v_n| > s\})^{1/q'_-}, \end{aligned} \tag{5.10}$$

for all $s > 0$.

Taking s small enough in (5.10) and using the convergence in measure of $(v_n)_n$, we obtain that, for all $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that $\|v_m - v_n\|_1 < \epsilon$, for all $m, n \ge n_0(\epsilon)$. □

The second technical lemma will be the key point to prove that the entropy solution satisfies the equation in the sense of distributions.

Proposition 5.5. *Assume* (1.2)–(1.5), $f \in L^1(\Omega)$ *and* (5.1). *Let* $u_n \in W_0^{1,p(\cdot)}(\Omega)$ *be the solution of* (5.2). *If* $p(\cdot) - 1 \ll q_1(\cdot)$, *then the following assertions hold:*

(i) $a(x, \nabla u_n)$ *converges to* $a(x, \nabla u)$ *strongly in* $L^1(\Omega)$.

(ii) $a(x, \nabla u) \in L^{q(\cdot)}(\Omega)$, *for some* $1 \ll q(\cdot)$.

(iii) u *and* ∇u *satisfy* (3.4) *and* (3.6).

Proof. (i) – (ii) By Proposition 5.3 and the Nemitskii Theorem (*cf.* [24, p. 20]), we obtain that $a(x, \nabla u_n)$ converges to $a(x, \nabla u)$ in measure. Moreover, using (1.3), we have

$$|a(x, \nabla u_n)| \le \beta \left(j(x) + |\nabla u_n|^{p(x)-1} \right),$$

with $j \in L^{p'(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega)$, for all $1 \le q(\cdot) \ll N/(N-1)$. By Corollary 3.7 applied to u_n, (5.1) and the assumption that $p(\cdot) - 1 \ll q_1(\cdot)$, we have that $(|\nabla u_n|^{p(\cdot)-1})_n$ is uniformly bounded in $L^{q(\cdot)}(\Omega)$, for some $1 \ll q(\cdot)$. Hence, using Lemmas 5.3 and 5.4, we obtain that $a(x, \nabla u_n)$ converges to $a(x, \nabla u)$ strongly in $L^1(\Omega)$, and $a(x, \nabla u) \in L^{q(\cdot)}(\Omega)$.

(iii) It follows from taking the limit as $n \to +\infty$ in Corollaries 3.5 and 3.7 applied to u_n and using (5.1). □

Proof of Theorem 1.3. Let $u_n \in W_0^{1,p(\cdot)}(\Omega)$ be the solution of (5.2) and let u be given by Proposition 5.1. Using Proposition 5.5 (i) and the strong convergence in L^1 of the f_n to f, we obtain (1.6) by passing to the limit in

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla \varphi \, dx = \int_\Omega f_n(x) \varphi \, dx,$$

for all $\varphi \in C_0^\infty(\Omega)$.

Now, we claim that $1 \ll q_1(\cdot)$ and $1 \ll q_0(\cdot)$ under the assumption $2 - 1/N \ll p(\cdot)$. By definition,

$$q_1(\cdot) = \frac{p(\cdot)^2}{p(\cdot) + p'_+(1 - p(\cdot)/N)};$$

therefore, $1 \ll q_1(\cdot)$ if and only if

$$\frac{p'_+/p(\cdot) - (p(\cdot) - 1)}{p'_+} \ll \frac{1}{N}. \tag{5.11}$$

Noting that the left-hand side is decreasing in $p(\cdot)$ and that $p'_+ = p_-/(p_- - 1)$, (5.11) is equivalent to

$$(2-p)_+ = 2 - p_- = \frac{p'_+/p_- - (p_- - 1)}{p'_+} < \frac{1}{N},$$

i.e., $(p - 2 + 1/N)_- > 0$, proving the first assertion. On the other hand, $1 \ll q_0(\cdot)$ is equivalent to

$$1 \ll (p_- - 1)\frac{Np(\cdot)}{p_-(N - p(\cdot))}.$$

Since the right-hand side is increasing in $p(\cdot)$, it is sufficient to prove that

$$1 < (p_- - 1)\frac{N}{(N - p_-)},$$

which follows easily from the assumption $2 - 1/N \ll p(\cdot)$.

As a consequence, from Corollaries 3.5 and 3.7,

$$u \in L^{q(\cdot)}(\Omega), \quad \text{for all } \ 1 \le q(\cdot) \ll q_0(\cdot)$$

and

$$u \in W_0^{1,q(\cdot)}(\Omega), \quad \text{for all } \ 1 \le q(\cdot) \ll q_1(\cdot). \qquad \square$$

Remark 5.6. Assumption $p(\cdot)-1 \ll q_1(\cdot)$, which is obviously satisfied for p constant, is equivalent to the condition

$$\frac{Np'(\cdot)}{N - p(\cdot)} \gg {p'}_+. \tag{5.12}$$

The analysis of the behaviour of the function on the left-hand side of this inequality leads to the following conclusions:

(i) if $p_+ < \sqrt{N}$, then (5.12) is satisfied for any function $p(\cdot)$ such that

$$\frac{1}{p_-} - \frac{1}{p_+} < \frac{p_+ - 1}{N};$$

(ii) if $p_- \le \sqrt{N} \le p_+$, then (5.12) is satisfied for any function $p(\cdot)$ such that

$$p_- > \frac{N}{2\sqrt{N} - 1};$$

(iii) if $p_- > \sqrt{N}$, then (5.12) is satisfied for any function $p(\cdot)$.

The condition in case (i) only holds if p_- is *close* to p_+, so it forces a modest variation in the field of values of $p(\cdot)$.

Remark 5.7. We finally comment on the significance of assumption $p(\cdot) - 1 \ll q_1(\cdot)$, under $2 - 1/N \ll p(\cdot)$. Observe that:

- It always holds if $p(\cdot) \ll 2$ since then $p(\cdot) - 1 \ll 1 \ll q_1(\cdot)$.
- The condition in case (i) above is only pertinent for dimensions $N \ge 5$ since $p(\cdot) \ll 2$ if $p_+ < \sqrt{N}$, for $N = 2, 3, 4$. Moreover, for $N \ge 5$, the condition is satisfied if

$$\sup_{p_- > 2 - \frac{1}{N}} \frac{1}{p_-} = \frac{N}{2N - 1} < \frac{2\sqrt{N} - 1}{N} = \inf_{2 \le p_+ < \sqrt{N}} \left\{ \frac{1}{p_+} + \frac{p_+ - 1}{N} \right\},$$

  which holds for $N \le 10$.
- The condition in case (ii) above always holds for dimensions $N \le 10$, since then $\frac{N}{2\sqrt{N}-1} < 2 - \frac{1}{N}$.

Therefore, up to dimension $N = 10$, assumption $p(\cdot) - 1 \ll q_1(\cdot)$ is automatically satisfied when we assume $2 - 1/N \ll p(\cdot)$.

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CMUC, Department of Mathematics, University of Coimbra, 3001–454 Coimbra, Portugal

E-mail address: msanchon@mat.uc.pt

Current address: Departament de Matemàtica Aplicada i Anàlisi, Facultat de Matemàtiques, Universitat de Barcelona, Gran Via 585, E-08007 Barcelona, Spain

CMUC, Department of Mathematics, University of Coimbra, 3001–454 Coimbra, Portugal

E-mail address: jmurb@mat.uc.pt