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SIGN-CHANGING MULTI-BUMP SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH STEEP POTENTIAL WELLS

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Abstract. We study the nonlinear Schrödinger equations:

$$(P_\lambda) \qquad -\Delta u+(\lambda^2 a(x)+1)u = |u|^{p-1}u, \quad u \in H^1(\mathbf{R}^N),$$

where $p > 1$ is a subcritical exponent, $a(x)$ is a continuous function satisfying $a(x) \geq 0$, $0 < \liminf_{|x|\to\infty} a(x) \leq \limsup_{|x|\to\infty} a(x) < \infty$ and $a^{-1}(0)$ consists of 2 connected bounded smooth components Ω_1 and Ω_2.

We study the existence of solutions (u_λ) of (P_λ) which converge to 0 in $\mathbf{R}^N \setminus (\Omega_1 \cup \Omega_2)$ and to a prescribed pair $(v_1(x), v_2(x))$ of solutions of the limit problem:

$$-\Delta v_i + v_i = |v_i|^{p-1}v_i \quad \text{in } \Omega_i$$

$(i = 1, 2)$ as $\lambda \to \infty$.

1. Introduction

In this paper we consider the existence and multiplicity of solutions of the following nonlinear Schrödinger equations:

$$(P_\lambda) \qquad -\Delta u + (\lambda^2 a(x) + 1)u = |u|^{p-1}u \quad \text{in } \mathbf{R}^N, \qquad u(x) \in H^1(\mathbf{R}^N).$$

Here $p \in (1, \frac{N+2}{N-2})$ if $N \geq 3$, $p \in (1, \infty)$ if $N = 1$, 2 and $a(x) \in C(\mathbf{R}^N, \mathbf{R})$ is nonnegative on $\mathbf{R}^N$. We consider multiplicity of solutions (including positive and sign-changing solutions) when the parameter λ is very large.

For $a(x)$, we assume

(a1) $a(x) \in C(\mathbf{R}^N, \mathbf{R})$, $a(x) \geq 0$ for all $x \in \mathbf{R}^N$ and the potential well $\Omega = int\, a^{-1}(0)$ is a nonempty bounded open set with smooth boundary $\partial\Omega$ and $a^{-1}(0) = \overline{\Omega}$.

(a2) $0 < \liminf_{|x|\to\infty} a(x) \leq \sup_{x\in\mathbf{R}^N} a(x) < \infty$.

When λ is large, the potential well Ω plays important roles and the following Dirichlet problem appears as a limit of (P_λ):

$$(1.1) \qquad -\Delta u + u = |u|^{p-1}u \quad \text{in } \Omega, \qquad u = 0 \qquad \text{on } \partial\Omega.$$

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We remark that solutions of (P_λ) and (1.1) can be characterized as critical points of

$$\Psi_\lambda(u) = \int_{\mathbf{R}^N} \frac{1}{2}(|\nabla u|^2 + (\lambda^2 a(x)+1)u^2) - \frac{1}{p+1}|u|^{p+1}\,dx : H^1(\mathbf{R}^N) \to \mathbf{R}, \tag{1.2}$$

$$\Psi_\Omega(u) = \int_{\Omega} \frac{1}{2}(|\nabla u|^2 + u^2) - \frac{1}{p+1}|u|^{p+1}\,dx : H^1_0(\Omega) \to \mathbf{R}, \tag{1.3}$$

and it is known that (1.3) has an unbounded sequence of critical values (see for example [21]).

Bartsch and Wang [3] and Bartsch, Pankov and Wang [4] (see also [2]) studied such a situation first. Their assumptions on $a(x)$ and nonlinearity are more general, and as a special case of their results we have

(i) There exists a least energy solution $u_\lambda(x)$ of (P_λ). Moreover $u_{\lambda_n}(x)$ converges strongly to a least energy solution of (1.3) after extracting a subsequence $\lambda_n \to \infty$ ([3]).

(ii) When $N \geq 3$ and $p \in (1, \frac{N+2}{N-2})$ is close to $\frac{N+2}{N-2}$, there exists at least $\mathrm{cat}(\Omega)$ positive solutions of (P_λ) for large λ ([3]). Here $\mathrm{cat}(\Omega)$ denotes the Lusternik-Schnirelman category of Ω.

(iii) For any $k \in \mathbf{N}$, there exist k pairs of (possibly sign-changing) solutions $\pm u_{1,\lambda}(x), \cdots, \pm u_{k,\lambda}(x)$ of (P_λ) for large $\lambda \geq \lambda(k)$. Moreover they converge to distinct solutions $\pm u_1(x), \cdots, \pm u_k(x)$ of (1.1) after extracting a subsequence $\lambda_n \to \infty$ ([4]).

Here we remark that in [3], [4] they mainly consider the case where Ω is connected.

In this paper we consider the case where Ω consists of multiple connected components, $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_m$, and we consider the multiplicity of positive and sign-changing solutions for large λ.

We studied the multiplicity of positive solutions in our previous paper [16] (see also Cao-Noussair [10]), where it was shown that for any choice of components $\Omega_{k_1}, \cdots, \Omega_{k_\ell}$, there exists a positive solution $u_\lambda(x)$ of (P_λ) for large λ such that after extracting a subsequence $\lambda_n \to \infty$,

$$u_{\lambda_n}(x) \to \begin{cases} u_i(x) & \text{in } \Omega_i \ (i \in \{k_1, \cdots, k_\ell\}), \\ 0 & \text{in } \mathbf{R}^N \setminus (\Omega_{k_1} \cup \cdots \cup \Omega_{k_\ell}) \end{cases}$$

strongly in $H^1(\mathbf{R}^N)$. Here $u_i(x)$ is a least energy solution of

$$-\Delta u + u = |u|^{p-1}u \quad \text{in } \Omega_i, \qquad u = 0 \quad \text{on } \partial\Omega_i. \tag{1.4}$$

In particular, (P_λ) has at least $2^m - 1$ positive solutions for large λ.

We remark that a solution $u_i(x)$ of (1.4) is said to be a least energy solution if and only if

$$\Psi_{i,D}(u_i) = \inf\{\Psi_{i,D}(u);\ u(x) \in H^1_0(\Omega_i) \text{ is a nontrivial solution of (1.4)}\}$$

holds. Here $\Psi_{i,D}(u)$ is defined by

$$\Psi_{i,D}(u) = \int_{\Omega_i} \frac{1}{2}(|\nabla u|^2 + u^2) - \frac{1}{p+1}|u|^{p+1}\,dx : H^1_0(\Omega_i) \to \mathbf{R}\,.$$

("D" stands for Dirichlet boundary conditions.) It is natural to ask about the existence of a sequence of solutions of (P_λ) converging to solutions of (1.4) in each Ω_i, which may not be least energy solutions.

First we study the multiplicity of sign-changing solutions. For the sake of simplicity, we assume that Ω consists of 2 components, that is,

$$\Omega = \Omega_1 \cup \Omega_2. \tag{1.5}$$

In this case we have two limit problems (1.4), which are corresponding to $\Psi_{i,D} : H_0^1(\Omega_i) \to \mathbf{R}$ $(i = 1, 2)$. It is well known that each functional has an unbounded sequence of critical points $(u_j^{(i)}(x))_{j=1}^{\infty} \subset H_0^1(\Omega_i)$ $(i = 1, 2)$. A natural question is to ask, for a given pair $(u_{j_1}^{(1)}(x), u_{j_2}^{(2)}(x))$, whether (P_λ) has a solution $u_\lambda(x) \in H^1(\mathbf{R}^N)$ converging to $u_{j_i}^{(i)}(x)$ in Ω_i and to 0 elsewhere. Here we try to give a partial answer to this problem. More precisely, we try to find a solution $u_\lambda(x) \in H^1(\mathbf{R}^N)$ which converges to $(u_1^{(1)}(x), u_j^{(2)}(x))$ after extracting a subsequence $\lambda_n \to \infty$. Here $u_1^{(1)}(x)$ is a mountain pass solution of (1.4) in Ω_1 and $u_j^{(2)}(x)$ is a minimax solution of (1.4) in Ω_2.

To find an unbounded sequence of critical values of a functional $I(u) \in C^1(E, \mathbf{R})$ defined on an infinite dimensional Hilbert space E, $\mathbf{Z}_2$-symmetry of $I(u)$ — $I(\pm u) = I(u)$ for all $u \in E$—plays an important role. We remark that $\Psi_\lambda(u) \in C^1(H^1(\mathbf{R}^N), \mathbf{R})$ and a functional $\tilde{\Psi}(u_1, u_2) = \Psi_{1,D}(u_1) + \Psi_{2,D}(u_2) \in C^1(H_0^1(\Omega_1) \times H_0^1(\Omega_2), \mathbf{R})$, which is corresponding to (1.4) in $\Omega_1 \cup \Omega_2$, have different symmetries. $\Psi_\lambda(u)$ is $\mathbf{Z}_2$-symmetric and $\tilde{\Psi}(u_1, u_2)$ is $(\mathbf{Z}_2)^2$-symmetric; that is,

$$\begin{aligned} &\Psi_\lambda(su) = \Psi_\lambda(u) \qquad \text{for all } s \in \mathbf{Z}_2 = \{-1, 1\},\ u \in H^1(\mathbf{R}^N), \\ &\tilde{\Psi}(s_1 u_1, s_2 u_2) = \tilde{\Psi}(u_1, u_2) \text{ for all } s_1, s_2 \in \mathbf{Z}_2,\ (u_1, u_2) \in H_0^1(\Omega_1) \times H_0^1(\Omega_2). \end{aligned}$$

Note that $\mathbf{Z}_2$-action on $\Psi_\lambda(u)$ is corresponding to the following $\mathbf{Z}_2$-action on $\tilde{\Psi}(u_1, u_2)$:

$$\tilde{\Psi}(su_1, su_2) = \tilde{\Psi}(u_1, u_2) \quad \text{for all } s \in \{-1, 1\},\ (u_1, u_2) \in H_0^1(\Omega_1) \times H_0^1(\Omega_2),$$

and there are no symmetries of $\Psi_\lambda(u)$ corresponding to the $\mathbf{Z}_2$-symmetry of $\tilde{\Psi}(u_1, u_2)$:

$$\tilde{\Psi}(u_1, \pm u_2) = \tilde{\Psi}(u_1, u_2). \tag{1.6}$$

We also remark that solutions $(u_1^{(1)}(x), u_j^{(2)}(x))$ are obtained using the group action (1.6). Thus to construct solutions $u_\lambda(x)$ converging to $(u_1^{(1)}(x), u_j^{(2)}(x))$ we need to develop a kind of perturbation theory from symmetries, and in this paper we use ideas from Ambrosetti [1], Bahri-Berestycki [5], Struwe [23] and Rabinowitz [20] (see also Bahri-Lions [6], Tanaka [25] and Bolle [8]). In [1, 5, 6, 20, 23, 25], perturbation theories are developed for

$$-\Delta u = |u|^{p-1} u + f(x) \quad \text{in } \Omega, \qquad u = 0 \qquad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain. They successfully showed the existence of an unbounded sequence of solutions for all $f(x) \in L^2(\Omega)$ for a certain range of p.

To state our result, we need some notation: we set

$$\Sigma_{i,D} = \{u \in H_0^1(\Omega_i);\ \|u\|_{H_0^1(\Omega_i)} = 1\}$$

and consider the constraint functional

$$J_{i,D}(v) = \max_{t>0} \Psi_{i,D}(tv) : \Sigma_{i,D} \to \mathbf{R}\,. \tag{1.7}$$

We define

$$c_{min}^{1,D} = \inf_{v\in\Sigma_{1,D}} J_{1,D}(v), \tag{1.8}$$

$$b_n^{2,D} = \inf_{\gamma\in\Gamma_n^{2,D}} \max_{\theta\in S^n} J_{2,D}(\gamma(\theta)), \tag{1.9}$$

where $S^n = \{\theta \in \mathbf{R}^{n+1};\ |\theta| = 1\}$ and

$$\Gamma_n^{2,D} = \{\gamma \in C(S^n, \Sigma_{2,D});\ \gamma(-\theta) = -\gamma(\theta) \text{ for all } \theta \in S^n\}.$$

We will observe that $c_{min}^{1,D}$ and $b_n^{2,D}$ are critical values of $\Psi_{1,D}(u)$, $\Psi_{2,D}(u)$ and

$$b_0^{2,D} \le b_1^{2,D} \le b_2^{2,D} \le \cdots \le b_n^{2,D} \le b_{n+1}^{2,D} \le \cdots, \quad b_n^{2,D} \to \infty\ (n \to \infty).$$

In particular, there exists a sequence $n(1) < n(2) < \cdots < n(k) < n(k+1) < \cdots$ such that $b_{n(k)}^{2,D} < b_{n(k)+1}^{2,D}$. We also define another set of minimax values by

$$c_k^{2,D} = \inf_{\sigma\in\Lambda_k} \max_{\theta\in S_+^{n(k)+1}} J_{2,D}(\sigma(\theta)),$$

where $S_+^{n(k)+1} = \{\theta = (\theta_1, \cdots, \theta_{n(k)+1}, \theta_{n(k)+2});\ \theta \in S^{n(k)+1}, \theta_{n(k)+2} \ge 0\}$ and

$$\Lambda_k = \{\sigma \in C(S_+^{n(k)+1}, \Sigma_{2,D});\ \sigma|_{S^{n(k)}} \in \Gamma_{n(k)}^{2,D},\ \inf_{\theta\in S^{n(k)}} \Psi_{2,D}(\sigma(\theta)) < b_{n(k)}^{2,D} + \delta_k\}.$$

Here $\delta_k = \frac{1}{4}(b_{n(k)+1}^{2,D} - b_{n(k)}^{2,D}) > 0$. We can also see that $c_k^{2,D}$ is a critical value of $\Psi_{2,D}(u)$ and $c_k^{2,D} \to \infty$ as $k \to \infty$.

Now we can give our main result.

Theorem 1.1. *Assume* (a1)–(a2) *and* (1.5). *Then for any $k \in \mathbf{N}$ there exists $\lambda_1(k) \ge 1$ such that for any $\lambda \ge \lambda_1(k)$, (P_λ) has a solution $u_\lambda(x)$ such that*

(i)

$$\Psi_\lambda(u_\lambda) \to c_{min}^{1,D} + c_k^{2,D} \quad \text{as } \lambda \to \infty. \tag{1.10}$$

(ii) *For any given sequence $\lambda_n \to \infty$, we can extract a subsequence $\lambda_{n_\ell} \to \infty$ such that $u_{\lambda_{n_\ell}}$ converges to a function $u(x)$ strongly in $H^1(\mathbf{R}^N)$. Moreover we have*

$$u(x) \text{ satisfies (1.4) in } \Omega_1 \cup \Omega_2, \tag{1.11}$$

$$u|_{\mathbf{R}^N\setminus(\Omega_1\cup\Omega_2)} \equiv 0, \tag{1.12}$$

$$u(x) > 0 \quad \text{in } \Omega_1, \tag{1.13}$$

$$\Psi_{1,D}(u|_{\Omega_1}) = c_{min}^{1,D}, \quad \Psi_{2,D}(u|_{\Omega_2}) = c_k^{2,D}. \tag{1.14}$$

When $N = 1$, we have a stronger result. Namely,

Theorem 1.2. *Assume $N = 1$ and $\Omega_i = (a_i, b_i)$ $(i = 1, 2)$, where $a_1 < b_1 < a_2 < b_2$. Then for any solution $v_1(x)$ of* (1.1) *in Ω_1 and for any solution $v_2(x)$ of* (1.1) *in Ω_2, there exists a solution $u_\lambda(x)$ for large λ such that*

$$u_\lambda(x) \to \begin{cases} v_1(x) & \text{in } \Omega_1, \\ v_2(x) & \text{in } \Omega_2, \\ 0 & \text{in } \mathbf{R}\setminus(\Omega_1\cup\Omega_2) \end{cases}$$

strongly in $H^1(\mathbf{R})$ as $\lambda \to \infty$.

Next we deal with positive solutions. Our result is influenced by the result [3].

Theorem 1.3. *Assume* (a1)–(a2), (1.5) *and* $N \geq 3$. *Then there exists a* $p_1 \in (1, \frac{N+2}{N-2})$ *and* $\lambda_1(p) \geq 1$ *depending on* $p \in (1, \frac{N+2}{N-2})$ *such that for* $p \in (p_1, \frac{N+2}{N-2})$ *and* $\lambda \geq \lambda_1(p)$, (P_λ) *possesses at least* $\mathrm{cat}(\Omega_1) + \mathrm{cat}(\Omega_2) + \mathrm{cat}(\Omega_1 \times \Omega_2)$ *positive solutions.*

Remark 1.4. Since $\mathrm{cat}(\Omega_1 \cup \Omega_2) = \mathrm{cat}(\Omega_1) + \mathrm{cat}(\Omega_2)$, the argument of Bartsch-Wang [3] ensures $\mathrm{cat}(\Omega_1) + \mathrm{cat}(\Omega_2)$ positive solutions, which converge to a positive solution of (1.3) in one of the components and to 0 elsewhere after extracting a subsequence $\lambda_n \to \infty$. We remark that our Theorem 1.3 ensures additional $\mathrm{cat}(\Omega_1 \times \Omega_2)$ positive solutions. We can also observe that these solutions converge to positive solutions in both components Ω_1, Ω_2.

Similarly, we can deal with sign-changing solutions which converge to positive solutions in Ω_1 and negative solutions in Ω_2.

Theorem 1.5. *Assume* (a1)–(a2), (1.5) *and* $N \geq 3$. *Then there exists a* $p_1 \in (1, \frac{N+2}{N-2})$ *and* $\lambda_1(p) \geq 1$ *depending on* $p \in (1, \frac{N+2}{N-2})$ *such that for* $p \in (p_1, \frac{N+2}{N-2})$ *and* $\lambda \geq \lambda_1(p)$, (P_λ) *possesses at least* $\mathrm{cat}(\Omega_1 \times \Omega_2)$ *sign-changing solutions. Moreover, after extracting a subsequence* $\lambda_n \to \infty$, *they converge to positive solutions of* (1.4) *in* Ω_1 *and negative solutions of* (1.4) *in* Ω_2.

In the following sections we will give proofs to our theorems. We will take an approach different from that of Ding and Tanaka [16]. In Section 2, we introduce a reduced functional $J_\lambda(v_1, v_2)$, which is defined on an infinite dimensional torus $\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ — $\Sigma_{1,\lambda}$ and $\Sigma_{2,\lambda}$ will be defined in Section 2 (d). We will observe that for large λ our reduced functional $J_\lambda(v_1, v_2)$ can be approximated by $J_{1,\lambda}(v_1) + J_{2,\lambda}(v_2)$ — $J_{1,\lambda}(v_1)$ and $J_{2,\lambda}(v_2)$ are defined as in (1.7) — in a suitable sense and that it enables us to develop a perturbation theory. In Sections 2, 3 we develop a perturbation theory for $J_\lambda \in C^1(\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}, \mathbf{R})$. The subsolution estimate for Schrödinger operators (cf. Simon [22]) and a perturbation argument under the influence of [1, 5, 20, 23] play essential roles. Section 4 is devoted to showing Theorem 1.1. Isolatedness of critical values of $\Psi_{1,D}(v_1)$ corresponding to positive solutions is important, and we will construct a deformation flow for $J_\lambda \in C^1(\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}, \mathbf{R})$, under which the level set $\{v_1 \in \Sigma_{1,\lambda};\ J_{1,\lambda}(v_1) \leq c\}$ for the unperturbed functional $J_{1,\lambda}(v_1)$ is invariant for a certain range of c. In Section 5 we will give a proof to Theorem 1.2 via Nehari type methods (cf. del Pino, Felmer and Tanaka [15]). In Section 6, we give a proof to our Theorems 1.3, 1.5. Here an estimate of the category of the level set $\{(v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda};\ J_\lambda(v_1, v_2) \leq c\}$ is essential. Sections 7–9 are devoted to proving various propositions and lemmas which are used in previous sections without proofs.

2. Functional setting and variational formulation

(a) Functional setting. To find critical points of $\Psi_\lambda(u)$ in $H^1(\mathbf{R}^N)$, we introduce some notation: for an open set $O \subset \mathbf{R}^N$ and $\lambda \geq 1$ we write

$$\begin{aligned}
\langle u, v \rangle_{\lambda,O} &= \int_O \nabla u \nabla v + (\lambda^2 a(x) + 1) uv \, dx \qquad \text{for } u, v \in H^1(O), \\
\|u\|_{\lambda,O} &= \sqrt{\langle u, u \rangle_{\lambda,O}} \qquad \text{for } u \in H^1(O), \\
\|f\|^*_{\lambda,O} &= \sup_{u \in H^1(O), \|u\|_{\lambda,O} \leq 1} |f(u)| \qquad \text{for } f \in (H^1(O))^*.
\end{aligned}$$

By the assumption $\|\cdot\|_{\lambda,O}$ is equivalent to the standard H^1-norm

$$\|u\|^2_{H^1(O)} = \int_O |\nabla u|^2 + u^2\,dx,$$

and we have

$$\|u\|_{H^1(O)} \le \|u\|_{\lambda,O} \quad \text{for all } u \in H^1(O) \text{ and } \lambda \ge 1. \tag{2.1}$$

We also use

$$\begin{aligned} \|u\|^r_{L^r(O)} &= \int_O |u|^r\,dx \quad \text{for } 1 \le r < \infty, \\ \|u\|_{L^\infty(O)} &= \operatorname*{ess\,sup}_{x\in O} |u(x)|. \end{aligned}$$

For each $i \in \{1,2\}$, we choose bounded open subsets Ω_i', Ω_i'' with smooth boundaries such that

$$\Omega_i \subset\subset \Omega_i' \subset\subset \Omega_i'' \ (i=1,2), \quad \overline{\Omega_1''} \cap \overline{\Omega_2''} = \emptyset.$$

Here and in what follows we write $A \subset\subset B$ if $\overline{A} \subset int(B)$ holds. We will use the following lemma repeatedly with $O = \mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')$, $O = (\Omega_1'' \setminus \Omega_1') \cup (\Omega_2'' \setminus \Omega_2')$, etc.

Lemma 2.1. *For a domain $O \subset \mathbf{R}^N$ with $\overline{O} \cap (\overline{\Omega_1} \cup \overline{\Omega_2}) = \emptyset$ and $r \in [2, \frac{2N}{N-2})$ $(N \ge 3)$ and $r \in [2,\infty)$ $(N = 1,2)$, there exists a constant $C_r(O,\lambda)$ satisfying $C_r(O,\lambda) \to 0$ as $\lambda \to \infty$ such that*

$$\|u\|_{L^r(O)} \le C_r(O,\lambda)\|u\|_{\lambda,O} \quad \textit{for all } u \in H^1(O).$$

Proof. Let $a_\infty(O) = \inf_{x\in O} a(x) > 0$. Then we have

$$\|u\|_{L^2(O)} \le \frac{1}{\sqrt{a_\infty(O)\lambda}}\|u\|_{\lambda,O} \quad \text{for all } u \in H^1(O).$$

On the other hand, by (2.1) we have

$$\|u\|_{L^{\frac{2N}{N-2}}(O)} \le C\|u\|_{\lambda,O} \quad (N \ge 3), \quad \|u\|_{L^q(O)} \le C_q\|u\|_{\lambda,O} \quad (N = 1,2),$$

where $q > r$. Thus by the interpolation we have the conclusion of Lemma 2.1. □

(b) Modification of the nonlinearity $|u|^{p-1}u$. To find solutions with asymptotic behaviors as in Theorems 1.1, 1.3, and 1.5, we take the local mountain pass approach due to del Pino and Felmer [14]. We choose an odd function $f(\xi) \in C^1(\mathbf{R},\mathbf{R})$ such that for some $0 < \ell_1 < \ell_2$

$$\begin{aligned} f(\xi) &= |\xi|^{p-1}\xi \quad \text{for } |\xi| \le \ell_1, \\ f(\xi) &= \frac{1}{2}\xi \quad \text{for } |\xi| \ge \ell_2, \\ 0 &\le f'(\xi) \le \frac{2}{3} \quad \text{for all } \xi \in \mathbf{R}\,. \end{aligned} \tag{2.2}$$

We set

$$g(x,\xi) = \begin{cases} |\xi|^{p-1}\xi & \text{if } x \in \Omega_1' \cup \Omega_2', \\ f(\xi) & \text{if } x \in \mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2'). \end{cases} \tag{2.3}$$

We also use the notation

$$G(x,\xi) = \int_0^\xi g(x,s)\,ds.$$

In what follows we will try to find critical points of

$$\begin{aligned}\Phi_\lambda(u) &= \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + (\lambda^2 a(x)+1)u^2\,dx - \int_{\mathbf{R}^N} G(x,u)\,dx \\ &= \frac{1}{2}\|u\|^2_{\lambda,\mathbf{R}^N} - \int_{\mathbf{R}^N} G(x,u)\,dx.\end{aligned}$$

First we have

Proposition 2.2. *For $\lambda \geq 1$, $\Phi_\lambda(u) \in C^2(H^1(\mathbf{R}^N),\mathbf{R})$ satisfies $(PS)_c$ condition for all $c \in \mathbf{R}$; that is, any sequence $(u_n)_{n=1}^\infty \subset H^1(\mathbf{R}^N)$ satisfying for $c \in \mathbf{R}$*

$$\Phi_\lambda(u_n) \to c, \quad \Phi'_\lambda(u_n) \to 0 \quad \textit{strongly in } H^{-1}(\mathbf{R}^N)$$

has a strongly convergent subsequence in $H^1(\mathbf{R}^N)$.

Proposition 2.3. *Suppose that $(u_\lambda(x))_{\lambda\geq\lambda_0}$ is a family of critical points of $\Phi_\lambda(u)$ and assume that there exist constants m, $M > 0$ independent of λ such that*

$$m \leq \Phi_\lambda(u_\lambda) \leq M \quad \textit{for all } \lambda \geq \lambda_0.$$

Then we have

(i) $\left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} m \leq \|u_\lambda\|^2_{\lambda,\mathbf{R}^N} \leq \left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} M$ *for all* $\lambda \geq \lambda_0$.

(ii) *There exists $\lambda(M) \geq \lambda_0$ such that for $\lambda \geq \lambda(M)$, $u_\lambda(x)$ satisfies*

$$|u_\lambda(x)| \leq \ell_1 \quad \textit{for } x \in \mathbf{R}^N \setminus (\Omega'_1 \cup \Omega'_2).$$

In particular, $g(x,u_\lambda(x)) = |u_\lambda(x)|^{p-1}u_\lambda(x)$ holds in $\mathbf{R}^N$ and $u_\lambda(x)$ is a solution of the original problem (P_λ).

(iii) *After extracting a subsequence $\lambda_n \to \infty$, there exists $u \in H^1(\mathbf{R}^N)$ such that*

$$\|u_{\lambda_n} - u\|_{\lambda_n,\mathbf{R}^N} \to 0 \quad \textit{as } n \to \infty.$$

Moreover, $u(x)$ satisfies $u(x) \equiv 0$ in $\mathbf{R}^N \setminus (\Omega_1 \cup \Omega_2)$ and

$$-\Delta u + u = |u|^{p-1}u \quad \textit{in } \Omega_i, \tag{2.4}$$

$$u = 0 \quad \textit{on } \partial\Omega_i \tag{2.5}$$

for $i = 1,2$. It also holds that $\Phi_{\lambda_n}(u_{\lambda_n}) \to \Psi_{1,D}(u\big|_{\Omega_1}) + \Psi_{2,D}(u\big|_{\Omega_2})$ as $n \to \infty$.

Proofs of Propositions 2.2, 2.3 can be given as in Section 2 of our previous paper [16].

(c) Reduction to a problem on $H^1(\Omega'_1 \cup \Omega'_2)$. To find critical points of $\Phi_\lambda(u)$, we introduce modified problems. First, we reduce our problem to a problem on $H^1(\Omega'_1 \cup \Omega'_2)$. For a given $u \in H^1(\Omega'_1 \cup \Omega'_2)$ we define

$$I_\lambda(u) = \inf_{w \in A_u} \Phi_\lambda(w),$$

where $A_u = \{w \in H^1(\mathbf{R}^N);\ w = u \text{ on } \Omega'_1 \cup \Omega'_2\}$. First we have

Proposition 2.4. *For any $u \in H^1(\Omega'_1 \cup \Omega'_2)$ and $\lambda \geq 1$ there exists a unique minimizer $w_\lambda(u) \in A_u$. Moreover, $w_\lambda(u)$ satisfies*

(i) *$w_\lambda(u)$ is nondegenerate in the following sense:*

$$\Phi''_\lambda(w_\lambda(u))(h,h) \geq \frac{1}{3}\|h\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega'_1\cup\Omega'_2)} \quad \textit{for all } h \in H^1_0(\mathbf{R}^N \setminus (\Omega'_1 \cup \Omega'_2)).$$

(ii) *$H^1(\Omega'_1 \cup \Omega'_2) \to H^1(\mathbf{R}^N)$; $u \mapsto w_\lambda(u)$ is of class C^1.*

(iii) $I_\lambda(u) = \Phi_\lambda(w_\lambda(u)) : H^1(\Omega_1' \cup \Omega_2') \to \mathbf{R}$ *is of class* C^2.
(iv) $u \in H^1(\Omega_1' \cup \Omega_2')$ *is a critical point of* $I_\lambda(u)$ *if and only if* $w_\lambda(u)$ *is a critical point of* $\Phi_\lambda(u)$.
(v) $I_\lambda(u)$ *satisfies* $(PS)_c$ *condition for all* $c \in \mathbf{R}$.

For a proof of Proposition 2.4, we choose sets $\Omega_1^{(1)}$, $\Omega_2^{(1)}$ such that $\Omega_i \subset\subset \Omega_i^{(1)} \subset\subset \Omega_i'$ for $i = 1, 2$. We need the following lemma.

Lemma 2.5. *There exists a linear operator* $P : H^1(\Omega_1' \cup \Omega_2') \to H^1(\mathbf{R}^N)$ *with the following properties: for* $\lambda \geq 1$ *and* $u \in H^1(\Omega_1' \cup \Omega_2')$

$$(Pu)(x) = u(x) \qquad \text{for all } x \in \Omega_1' \cup \Omega_2',$$
$$\operatorname{supp} Pu \subset \Omega_1'' \cup \Omega_2'',$$
$$\|Pu\|_{H^1(D_{out})} \leq C_1 \|u\|_{H^1(D_{in})}, \tag{2.6}$$
$$\|Pu\|_{L^2(D_{out})} \leq C_1 \|u\|_{L^2(D_{in})}, \tag{2.7}$$
$$\|Pu\|_{L^{p+1}(D_{out})} \leq C_1 \|u\|_{L^{p+1}(D_{in})}, \tag{2.8}$$
$$\|Pu\|_{\lambda, D_{out}} \leq C_1 \|u\|_{\lambda, D_{in}}, \tag{2.9}$$
$$\|Pu\|_{\lambda, \mathbf{R}^N} \leq C_1 \|u\|_{\lambda, \Omega_1' \cup \Omega_2'}. \tag{2.10}$$

Here we use notation

$$D_{out} = (\Omega_1'' \setminus \Omega_1') \cup (\Omega_2'' \setminus \Omega_2'), \quad D_{in} = (\Omega_1' \setminus \Omega_1^{(1)}) \cup (\Omega_2' \setminus \Omega_2^{(1)}),$$

and $C_1 > 0$ *is a constant independent of* $\lambda \geq 1$ *and* $u \in H^1(\Omega_1' \cup \Omega_2')$.

Proof. (2.6)–(2.8) are rather standard. For (2.9) we have from (2.6)–(2.7)

$$\begin{aligned}
\|Pu\|_{\lambda, D_{out}}^2 &\leq \|Pu\|_{H^1(D_{out})}^2 + \lambda^2 \left(\sup_{x \in D_{out}} a(x) \right) \|Pu\|_{L^2(D_{out})}^2 \\
&\leq C_1^2 \|u\|_{H^1(D_{in})}^2 + C_1^2 \lambda^2 \left(\sup_{x \in D_{out}} a(x) \right) \|u\|_{L^2(D_{in})}^2 \\
&\leq C_1^2 \|u\|_{H^1(D_{in})}^2 + C_1^2 \frac{\sup_{x \in D_{out}} a(x)}{\inf_{x \in D_{in}} a(x)} \int_{D_{in}} \lambda^2 a(x) u^2 \, dx \\
&\leq C_1^2 \left(1 + \frac{\sup_{x \in D_{out}} a(x)}{\inf_{x \in D_{in}} a(x)} \right) \|u\|_{\lambda, D_{in}}^2.
\end{aligned}$$

Thus we get (2.9). Property (2.10) also follows from (2.9). □

Proof of Proposition 2.4. First we show the existence and the uniqueness of a minimizer of $\Phi_\lambda(w)$ in A_u. It suffices to show that $A_u \to \mathbf{R}$; $w \mapsto \Phi_\lambda(w)$ is coercive and strictly convex. Since $w = u$ on $\Omega_1' \cup \Omega_2'$ for all $w \in A_u$, we need to show

$$A_u \to \mathbf{R};\ w \mapsto \frac{1}{2} \|w\|_{\lambda, \mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')}^2 - \int_{\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')} G(x, w) \, dx$$

is coercive and strictly convex. We note that

$$|g(x, \xi)| = |f(\xi)| \leq \frac{2}{3} |\xi|, \quad G(x, \xi) \leq \frac{1}{3} |\xi|^2$$

for all $x \in \mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')$ and $\xi \in \mathbf{R}$. We can observe that

$$\begin{aligned} & \frac{1}{2}\|w\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} - \int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} G(x,w)\,dx \\ \geq\ & \frac{1}{2}\|w\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} - \frac{1}{3}\|w\|^2_{L^2(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))} \geq \frac{1}{6}\|w\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}, \end{aligned}$$

and we get coerciveness of $w \mapsto \Phi_\lambda(w)$. On the other hand, since $A_u = \{Pu+h;\ h \in H_0^1(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2'))\}$, for strict convexity of $\Phi_\lambda(w)$ on A_u it suffices to show that $\Phi_\lambda''(w)(h,h) \geq \frac{1}{3}\|h\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}$ for all $w \in H^1(\mathbf{R}^N)$ and $h \in H_0^1(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2'))$. Indeed, we have by (2.2),

$$\begin{aligned} \Phi_\lambda''(w)(h,h) &= \|h\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} - \int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} g'(x,w)h^2\,dx \\ &\geq \|h\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} - \frac{2}{3}\|h\|^2_{L^2(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))} \geq \frac{1}{3}\|h\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}. \end{aligned}$$

Thus $w \mapsto \Phi_\lambda(w)$ is strictly convex. Therefore for any $u \in H^1(\Omega_1' \cup \Omega_2')$ there exists a unique minimizer $w_\lambda(u) \in A_u$ of $\Phi_\lambda(w)$. (i) is also proved.

Next we show (ii). We consider the following mapping:

$$\begin{aligned} F:\quad & H^1(\Omega_1' \cup \Omega_2') \times H_0^1(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')) \to H^{-1}(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')), \\ & (u,\varphi) \quad \mapsto \quad \Phi_\lambda'(Pu+\varphi)|_{H_0^1(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))}. \end{aligned}$$

By the convexity of $\varphi \mapsto \Phi_\lambda(Pu+\varphi)$, $F(u,\varphi) = 0$ holds if and only if $\varphi = \varphi_\lambda(u)$, where $\varphi_\lambda(u) : H^1(\Omega_1' \cup \Omega_2') \to H_0^1(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2'))$ is defined by $\varphi_\lambda(u) = w_\lambda(u) - Pu$. We have for $h_1,\ h_2 \in H_0^1(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2'))$

$$\langle F_\varphi(u,\varphi)h_1, h_2\rangle = \int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} \nabla h_1 \nabla h_2 + (\lambda^2 a(x)+1)h_1h_2 - g'(x,Pu+\varphi)h_1h_2\,dx.$$

By (2.2), we can observe that

$$F_\varphi(u,\varphi) :\ H_0^1(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')) \to H^{-1}(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2'))$$

is invertible for all $(u,\varphi) \in H^1(\Omega_1' \cup \Omega_2') \times H_0^1(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2'))$. Thus, by the implicit function theorem, we can observe $u \mapsto \varphi_\lambda(u)$ is of class C^1. That is, $u \mapsto w_\lambda(u) = Pu + \varphi_\lambda(u)$ is of class C^1.

For (iii), we remark that $I_\lambda(u) = \Phi_\lambda(w_\lambda(u)) = \Phi_\lambda(Pu + \varphi_\lambda(u))$ holds. We also remark that

$$\Phi_\lambda'(Pu+\varphi_\lambda(u))h = 0 \qquad \text{for all } h \in H_0^1(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')). \tag{2.11}$$

Noting $\varphi_\lambda'(u)\zeta \in H_0^1(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2'))$ for all $\zeta \in H^1(\Omega_1' \cup \Omega_2')$, we have

$$I_\lambda'(u)\zeta = \Phi_\lambda'(Pu+\varphi_\lambda(u))(P\zeta) \quad \text{for all } \zeta \in H^1(\Omega_1' \cup \Omega_2'). \tag{2.12}$$

From C^1-continuity of $\varphi_\lambda(u)$ and (2.12), it follows that $I_\lambda(u)$ is of class C^2.

Next we observe (iv). By (2.12) the "if" part clearly holds. Suppose that $I_\lambda'(u) = 0$. Thus by (2.11), (2.12) we have

$$\Phi_\lambda'(w_\lambda(u))(P\zeta + h) = 0 \quad \text{for all } \zeta \in H^1(\Omega_1' \cup \Omega_2') \text{ and } h \in H_0^1(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')).$$

Since $H^1(\mathbf{R}^N) = \{P\zeta + h;\ \zeta \in H^1(\Omega_1' \cup \Omega_2'), h \in H_0^1(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2'))\}$, we have $\Phi_\lambda'(w_\lambda(u)) = 0$. Thus the "only if" part also holds.

For (v), we note that

$$\|\varphi|_{\Omega_1'\cup\Omega_2'}\|_{\lambda,\Omega_1'\cup\Omega_2'} \le \|\varphi\|_{\lambda,\mathbf{R}^N},$$
$$\varphi - P\varphi|_{\Omega_1'\cup\Omega_2'} \in H_0^1(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))$$

for all $\varphi \in H^1(\mathbf{R}^N)$. Thus by (2.11), (2.12)

$$\begin{aligned}\Phi_\lambda'(w_\lambda(u))\varphi &= \Phi_\lambda'(w_\lambda(u))(P\varphi|_{\Omega_1'\cup\Omega_2'} + (\varphi - P\varphi|_{\Omega_1'\cup\Omega_2'}))\\ &= \Phi_\lambda'(w_\lambda(u))(P\varphi|_{\Omega_1'\cup\Omega_2'}) = I_\lambda'(u)(\varphi|_{\Omega_1'\cup\Omega_2'}).\end{aligned}$$

Thus for any $\varphi \in H^1(\mathbf{R}^N)$ with $\|\varphi\|_{\lambda,\mathbf{R}^N} \le 1$

$$|\Phi_\lambda'(w_\lambda(u))\varphi| \le \|I_\lambda'(u)\|^*_{\lambda,\Omega_1'\cup\Omega_2'}\|\varphi|_{\Omega_1'\cup\Omega_2'}\|_{\lambda,\Omega_1'\cup\Omega_2'} \le \|I_\lambda'(u)\|^*_{\lambda,\Omega_1'\cup\Omega_2'}.$$

That is, we have

$$\|\Phi_\lambda'(w_\lambda(u))\|^*_{\lambda,\mathbf{R}^N} \le \|I_\lambda'(u)\|^*_{\lambda,\Omega_1'\cup\Omega_2'}.$$

Thus, if $(u_n)_{n=1}^\infty \subset H^1(\Omega_1'\cup\Omega_2')$ is a $(PS)_c$-sequence for $I_\lambda(u)$, i.e., $I_\lambda(u_n) \to c$ and $\|I_\lambda'(u_n)\|^*_{\lambda,\Omega_1'\cup\Omega_2'} \to 0$, then $(w_\lambda(u_n))_{n=1}^\infty \subset H^1(\mathbf{R}^N)$ is a $(PS)_c$-sequence for $\Phi_\lambda(u)$. By Proposition 2.2, we can see $(w_\lambda(u_n))_{n=1}^\infty$ has a strongly convergent subsequence in $H^1(\mathbf{R}^N)$. Thus $u_n = w_\lambda(u_n)|_{\Omega_1'\cup\Omega_2'}$ has a strongly convergent subsequence in $H^1(\Omega_1'\cup\Omega_2')$ and $(PS)_c$ condition holds for $I_\lambda(u)$. □

For later use, we consider the following linear equation: let

$$V(x) \in L^\infty(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')),$$

and we consider the following linear boundary value problem:

$$-\Delta w + (\lambda^2 a(x)+1)w = V(x)w \quad \text{in } \mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'), \tag{2.13}$$

$$w = u \qquad \text{on } \partial\Omega_1'\cup\partial\Omega_2'. \tag{2.14}$$

We have the following

Lemma 2.6. *Suppose that $V(x) \in L^\infty(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))$ satisfies*

$$\|V(x)\|_{L^\infty(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))} \le \frac{2}{3}. \tag{2.15}$$

Then the linear boundary value problem (2.13)–(2.14) *has a unique solution — we denote it by $Q_{V,\lambda}(u)$ — for all $u \in H^1(\Omega_1'\cup\Omega_2')$. Moreover $Q_{V,\lambda}$ defines a bounded linear operator $Q_{V,\lambda} : H^1(\Omega_1'\cup\Omega_2') \to H^1(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))$, and there exists a constant $C_2 > 0$ independent of $\lambda \ge 1$ such that*

$$\|Q_{V,\lambda}(u)\|_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} \le C_2\|u\|_{\lambda,(\Omega_1'\setminus\Omega_1^{(1)})\cup(\Omega_2'\setminus\Omega_2^{(1)})} \quad \textit{for all } u \in H^1(\Omega_1'\cup\Omega_2'). \tag{2.16}$$

Proof of Lemma 2.6. Set

$$\begin{aligned}F(w) &= \int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} |\nabla w|^2 + (\lambda^2 a(x)+1)w^2 - V(x)w^2\,dx\\ &\quad : H^1(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')) \to \mathbf{R}.\end{aligned}$$

Then solutions of (2.13)–(2.14) can be characterized as critical points of $F(w)$ on $A_u = \{w \in H^1(\mathbf{R}^N);\ w = u \text{ on } \Omega_1'\cup\Omega_2'\}$. By assumption (2.15) we have

$$F(w) \ge \|w\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} - \frac{2}{3}\|w\|^2_{L^2(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))} \ge \frac{1}{3}\|w\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}. \tag{2.17}$$

Thus $F(w)$ is coercive and convex on A_u and we have the existence and the uniqueness of solutions (2.13)–(2.14).

To show (2.16), we note that $Q_{V,\lambda}(u)$ can be characterized as

$$F(Q_{V,\lambda}(u)) = \inf_{w \in A_u} F(w). \tag{2.18}$$

In particular, $F(Q_{V,\lambda}(u)) \le F(Pu)$, where Pu is defined in Lemma 2.5. Thus by (2.17) and (2.9)

$$\begin{aligned} &\frac{1}{3}\|Q_{V,\lambda}(u)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} \le F(Q_{V,\lambda}(u)) \le F(Pu) \\ \le\ & \|Pu\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} + \left|\int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} V(x)(Pu)^2\,dx\right| \\ \le\ & (1+\frac{2}{3})\|Pu\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} \\ \le\ & \frac{5}{3}C_1^2\|u\|^2_{\lambda,(\Omega_1'\setminus\Omega_1^{(1)})\cup(\Omega_2'\setminus\Omega_2^{(1)})}. \end{aligned}$$

Thus we get (2.16). □

Remark 2.7. Since $Q_{V,\lambda}(u)$ is a unique solution of (2.13)–(2.14), we can give another characterization: for $\zeta \in H^1(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))$, $\zeta = Q_{V,\lambda}(u)$ holds if and only if

$$\zeta - Pu \in H_0^1(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))$$

and

$$F'(\zeta)\varphi = 0 \quad \text{for all } \varphi \in H_0^1(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')).$$

We remark that $F'(\zeta)\varphi = 0$ can be rewritten as

$$\langle\zeta,\varphi\rangle_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} - \int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} V(x)\zeta\varphi\,dx = 0 \quad \text{for all } \varphi \in H_0^1(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')).$$

Remark 2.8. (i) For any $u \in H^1(\Omega_1'\cup\Omega_2')$ we remark that $V(x) = f(w_\lambda(u))/w_\lambda(u)$ satisfies $\|V(x)\|_{L^\infty(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))} \le \frac{2}{3}$, that is, $V(x)$ satisfies (2.15). In particular, $w_\lambda(x)$ satisfies $w_\lambda(u) = Q_{f(w_\lambda(u))/w_\lambda(u),\lambda}(u)$.

(ii) For $h \in H^1(\Omega_1'\cup\Omega_2')$, it holds that

$$(w_\lambda'(u)h)(x) = \begin{cases} h(x) & \text{for } x \in \Omega_1'\cup\Omega_2', \\ (Q_{f'(w_\lambda(u)),\lambda}(h))(x) & \text{for } x \in \mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'). \end{cases} \tag{2.19}$$

In fact, $w_\lambda(u)$ satisfies

$$\Phi_\lambda'(w_\lambda(u))\varphi = 0 \quad \text{for all } \varphi \in H_0^1(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')). \tag{2.20}$$

Differentiating with respect to u, we have

$$\Phi_\lambda''(w_\lambda(u))(w_\lambda'(u)h,\varphi) = 0 \quad \text{for all } h \in H^1(\Omega_1'\cup\Omega_2') \text{ and } \varphi \in H_0^1(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')).$$

That is,

$$\langle w_\lambda'(u)h,\varphi\rangle_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} - \int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} f'(w_\lambda(u))(w_\lambda'(u)h)\varphi\,dx = 0. \tag{2.21}$$

Since $w_\lambda(u) - Pu \in H_0^1(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))$, we also have

$$w_\lambda'(u)h - Ph \in H_0^1(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')). \tag{2.22}$$

By Remark 2.7, it follows from (2.21)–(2.22) that

$$(w_\lambda'(u)h)|_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} = Q_{f'(w_\lambda(u)),\lambda}(h).$$

Since $w_\lambda(u)|_{\Omega_1'\cup\Omega_2'} = u$, we have (2.19).

In what follows, we identify $H^1(\Omega_1' \cup \Omega_2')$ and $H^1(\Omega_1') \oplus H^1(\Omega_2')$, and for functions $u_1 \in H^1(\Omega_1')$, $u_2 \in H^1(\Omega_2')$, we write $u = (u_1, u_2)$ if $u \in H^1(\Omega_1' \cup \Omega_2')$ satisfies $u_1 = u|_{\Omega_1'}$, $u_2 = u|_{\Omega_2'}$. When $u = (u_1, u_2)$, we also write

$$\begin{aligned} &I_\lambda(u_1, u_2) \qquad \text{for} \quad I_\lambda(u), \\ &Q_{V,\lambda}(u_1, u_2) \qquad \text{for} \quad Q_{V,\lambda}(u), \\ &\|u\|^2_{\lambda,\Omega_1' \cup \Omega_2'} = \|u\|^2_{\lambda,\Omega_1'} + \|u\|^2_{\lambda,\Omega_2'}. \end{aligned}$$

By (2.19), it is convenient to extend $Q_{V,\lambda}$ as an operator $H^1(\Omega_1' \cup \Omega_2') \to H^1(\mathbf{R}^N)$ by

$$(Q_{V,\lambda} u)(x) = u(x) \quad \text{for } x \in \Omega_1' \cup \Omega_2'.$$

We also denote the extended operator by $Q_{V,\lambda}$. With this notation we have

$$w_\lambda'(u)h = Q_{f'(w_\lambda(u)),\lambda}(h).$$

By (iii) of Proposition 2.4, we have for $h \in H^1(\Omega_1' \cup \Omega_2')$

$$I_\lambda'(u)h = \Phi_\lambda'(w_\lambda(u))(w_\lambda'(u)h) = \Phi_\lambda'(w_\lambda(u)) Q_{f'(w_\lambda(u)),\lambda}(h).$$

Since $Q_{f'(w_\lambda(u)),\lambda}(h) - Q_{0,\lambda}(h) \in H_0^1(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2'))$, it follows from (2.20) that

$$I_\lambda'(u)h = \Phi_\lambda'(w_\lambda(u)) Q_{0,\lambda}(h) \quad \text{for all } h \in H^1(\Omega_1' \cup \Omega_2'). \tag{2.23}$$

Proof of the following estimates will be given in Section 7.

Lemma 2.9. *Let $V(x) : \mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2') \to \mathbf{R}$ be a function satisfying* (2.15). *Then there exist $C_3 > 0$ independent of $\lambda \geq 1$ and $r_\lambda > 0$ satisfying $r_\lambda \to 0$ as $\lambda \to \infty$ such that*

(i) $\|Q_{V,\lambda}(u)\|_{\lambda,\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')} \leq C_3 \|u\|_{\lambda,\Omega_1' \cup \Omega_2'}$ *for all* $u \in H^1(\Omega_1' \cup \Omega_2')$.
(ii) $\|Q_{V,\lambda}(u) - Q_{0,\lambda}(u)\|_{\lambda,\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')} \leq r_\lambda \|u\|_{\lambda,\Omega_1' \cup \Omega_2'}$ *for all* $u \in H^1(\Omega_1' \cup \Omega_2')$.
(iii) $|\langle Q_{0,\lambda}(u_1, 0), Q_{0,\lambda}(0, u_2)\rangle_{\lambda,\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')}| \leq r_\lambda \|u_1\|_{\lambda,\Omega_1'} \|u_2\|_{\lambda,\Omega_2'}$ *for all* $u_1 \in H^1(\Omega_1')$ *and* $u_2 \in H^1(\Omega_2')$.
(iv) $\|w_\lambda(u)\|_{\lambda,\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')} \leq C_3 \|u\|_{\lambda,\Omega_1' \cup \Omega_2'}$,
$\|w_\lambda(u) - Q_{0,\lambda}(u)\|_{\lambda,\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')} \leq r_\lambda \|u\|_{\lambda,\Omega_1' \cup \Omega_2'}$ *for all* $u \in H^1(\Omega_1' \cup \Omega_2')$.

Here and throughout this paper we denote by r_λ various constants independent of u but which depend on λ and satisfy $r_\lambda \to 0$ as $\lambda \to \infty$.

The following functionals correspond to "unperturbed" problems and will be useful later:

$$\begin{aligned} I_{1,\lambda}(u_1) &= \frac{1}{2}\|u_1\|^2_{\lambda,\Omega_1'} - \frac{1}{p+1}\|u_1\|^{p+1}_{L^{p+1}(\Omega_1')} \\ &\quad + \frac{1}{2}\|Q_{0,\lambda}(u_1, 0)\|^2_{\lambda,\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')} \in C^2(H^1(\Omega_1'), \mathbf{R}), \end{aligned} \tag{2.24}$$

$$\begin{aligned} I_{2,\lambda}(u_2) &= \frac{1}{2}\|u_2\|^2_{\lambda,\Omega_2'} - \frac{1}{p+1}\|u_2\|^{p+1}_{L^{p+1}(\Omega_2')} \\ &\quad + \frac{1}{2}\|Q_{0,\lambda}(0, u_2)\|^2_{\lambda,\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')} \in C^2(H^1(\Omega_2'), \mathbf{R}). \end{aligned} \tag{2.25}$$

We will study $I_\lambda(u_1, u_2)$ as a perturbation from $I_{1,\lambda}(u_1) + I_{2,\lambda}(u_2)$.

Lemma 2.10. *There exists $r_\lambda > 0$ such that for $(u_1, u_2) \in H^1(\Omega_1' \cup \Omega_2') = H^1(\Omega_1') \oplus H^1(\Omega_2')$,*

(0) $r_\lambda \to 0$ *as* $\lambda \to \infty$.
(i) $|I_\lambda(u_1, u_2) - I_{1,\lambda}(u_1) - I_{2,\lambda}(u_2)| \leq \frac{1}{2} r_\lambda (\|u_1\|_{\lambda,\Omega_1'} + \|u_2\|_{\lambda,\Omega_2'})^2$.

(ii) $\|\frac{\partial I_\lambda}{\partial u_1}(u_1,u_2) - I'_{1,\lambda}(u_1)\|^*_{\lambda,\Omega'_1} \le r_\lambda(\|u_1\|_{\lambda,\Omega'_1} + \|u_2\|_{\lambda,\Omega'_2})$.

(iii) $\|\frac{\partial I_\lambda}{\partial u_2}(u_1,u_2) - I'_{2,\lambda}(u_2)\|^*_{\lambda,\Omega'_2} \le r_\lambda(\|u_1\|_{\lambda,\Omega'_1} + \|u_2\|_{\lambda,\Omega'_2})$.

We use the notation:

$$\begin{aligned}\|u_1\|^2_{\lambda,\Omega'_1,\#} &= \|u_1\|^2_{\lambda,\Omega'_1} + \|Q_{0,\lambda}(u_1,0)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega'_1\cup\Omega'_2)} \quad \text{for } u_1 \in H^1(\Omega'_1),\\ \|u_2\|^2_{\lambda,\Omega'_2,\#} &= \|u_2\|^2_{\lambda,\Omega'_2} + \|Q_{0,\lambda}(0,u_2)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega'_1\cup\Omega'_2)} \quad \text{for } u_2 \in H^1(\Omega'_2).\end{aligned}$$

Then $\|\cdot\|_{\lambda,\Omega'_i,\#}$ is a norm on $H^1(\Omega'_i)$, and we have from Lemma 2.9 that

$$\|u\|_{\lambda,\Omega'_i} \le \|u\|_{\lambda,\Omega'_i,\#} \le (1+C_3)\|u\|_{\lambda,\Omega'_i} \quad \text{for all } u \in H^1(\Omega'_i) \tag{2.26}$$

and $\|\cdot\|_{\lambda,\Omega'_i,\#}$ is equivalent to $\|\cdot\|_{\lambda,\Omega'_i}$.

As a corollary to Lemma 2.10, we have

Corollary 2.11. *For all $u = (u_1,u_2) \in H^1(\Omega'_1) \oplus H^1(\Omega'_2)$*

(i)

$$\begin{aligned}&\frac{1}{2}(1-r_\lambda)\left(\|u_1\|^2_{\lambda,\Omega'_1,\#} + \|u_2\|^2_{\lambda,\Omega'_2,\#}\right) - \frac{1}{p+1}\left(\|u_1\|^{p+1}_{L^{p+1}(\Omega'_1)} + \|u_2\|^{p+1}_{L^{p+1}(\Omega'_2)}\right)\\ &\quad\le I_\lambda(u_1,u_2)\\ &\quad\le \frac{1}{2}(1+r_\lambda)\left(\|u_1\|^2_{\lambda,\Omega'_1,\#} + \|u_2\|^2_{\lambda,\Omega'_2,\#}\right) - \frac{1}{p+1}\left(\|u_1\|^{p+1}_{L^{p+1}(\Omega'_1)} + \|u_2\|^{p+1}_{L^{p+1}(\Omega'_2)}\right).\end{aligned}$$

(ii)

$$\begin{aligned}&(1-r_\lambda)\|u_1\|^2_{\lambda,\Omega'_1,\#} - \|u_1\|^{p+1}_{L^{p+1}(\Omega'_1)} + (1-r_\lambda)\|u_2\|^2_{\lambda,\Omega'_2,\#} - \|u_2\|^{p+1}_{L^{p+1}(\Omega'_2)}\\ \le\ & I'_\lambda(u_1,u_2)(u_1,u_2)\\ \le\ & (1+r_\lambda)\|u_1\|^2_{\lambda,\Omega'_1,\#} - \|u_1\|^{p+1}_{L^{p+1}(\Omega'_1)} + (1+r_\lambda)\|u_2\|^2_{\lambda,\Omega'_2,\#} - \|u_2\|^{p+1}_{L^{p+1}(\Omega'_2)}.\end{aligned}$$

We also remark that $I_{i,\lambda}(u)$ $(i=1,2)$ has the following properties:

Lemma 2.12. *For $i = 1,2$ we have*

(i) *$I_{i,\lambda}(u_i) : H^1(\Omega'_i) \to \mathbf{R}$ has a mountain pass geometry. That is, there are constants r_0, $\delta_0 > 0$ and $e_i \in H^1_0(\Omega'_i)$ independent of $\lambda \ge 1$ such that*

$$\begin{aligned}&I_{i,\lambda}(u_i) \ge \delta_0 \quad \textit{for all } u_i \in H^1(\Omega'_i) \textit{ with } \|u_i\|_{\lambda,\Omega'_i} = r_0,\\ &\|e_i\|_{\lambda,\Omega'_i} > r_0 \quad \textit{and} \quad I_{i,\lambda}(e_i) < 0.\end{aligned}$$

(ii) *$I_{i,\lambda}(u_i)$ satisfies the $(PS)_c$-condition for all $c \in \mathbf{R}$.*

Proof of Lemma 2.12. For $i = 1,2$, we remark that $I_{i,\lambda}(u)$ can be written as

$$I_{i,\lambda}(u) = \frac{1}{2}\|u\|^2_{\lambda,\Omega'_i,\#} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega'_i)},$$

and we can check (i), (ii) in a rather standard way. □

(d) Reduction to a problem on $\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$. Next we reduce our problem on an infinite dimensional torus $\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$, where

$$\Sigma_{i,\lambda} = \{v \in H^1(\Omega'_i);\ \|v\|_{\lambda,\Omega'_i} = 1\} \quad \text{for } i = 1,2.$$

We define $J_{i,\lambda} : \Sigma_{i,\lambda} \to (0,\infty]$ $(i=1,2)$, $J_\lambda : \Sigma_{1,\lambda} \times \Sigma_{2,\lambda} \to (0,\infty]$ by

$$J_{i,\lambda}(v_i) = \sup_{s\ge 0} I_{i,\lambda}(sv_i), \quad J_\lambda(v_1,v_2) = \sup_{s,t\ge 0} I_\lambda(sv_1,tv_2).$$

For $J_{i,\lambda}(v_i)$ $(i=1,2)$, we can easily see that $I_{i,\lambda}(sv_i):[0,\infty)\to\mathbf{R}$ takes a global maximum at

$$s=s_{i,\lambda}(v_i)=\|v_i\|_{\lambda,\Omega_i',\#}^{\frac{2}{p-1}}\|v_i\|_{L^{p+1}(\Omega_i')}^{-\frac{p+1}{p-1}} \tag{2.27}$$

and

$$J_{i,\lambda}(v_i)=I_{i,\lambda}(s_{i,\lambda}(v_i)v_i)=\left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{\|v_i\|_{\lambda,\Omega_i',\#}}{\|v_i\|_{L^{p+1}(\Omega_i')}}\right)^{\frac{2(p+1)}{p-1}}. \tag{2.28}$$

Recalling $\|v_i\|_{H^1(\Omega_i')}\le\|v_i\|_{\lambda,\Omega_i'}\le\|v_i\|_{\lambda,\Omega_i',\#}$ for $v_i\in\Sigma_{i,\lambda}$, we also have

$$\begin{aligned}\inf_{v_i\in\Sigma_{i,\lambda}}J_{i,\lambda}(v_i) &= \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\inf_{v_i\in H^1(\Omega_i'),v_i\neq0}\frac{\|v_i\|_{\lambda,\Omega_i',\#}}{\|v_i\|_{L^{p+1}(\Omega_i')}}\right)^{\frac{2(p+1)}{p-1}}\\ &\ge m_i\equiv\left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\inf_{v_i\in H^1(\Omega_i'),v_i\neq0}\frac{\|v_i\|_{H^1(\Omega_i')}}{\|v_i\|_{L^{p+1}(\Omega_i')}}\right)^{\frac{2(p+1)}{p-1}}. \end{aligned}\tag{2.29}$$

Remark 2.13. In a similar way to (2.28) we have for $J_{i,D}(v_i)$ defined in (1.7)

$$J_{i,D}(v_i)=\left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{1}{\|v_i\|_{L^{p+1}(\Omega_i)}}\right)^{\frac{2(p+1)}{p-1}}\quad\text{for } v_i\in\Sigma_{i,D}.$$

For $J_\lambda(v_1,v_2)$ we have

Lemma 2.14. (i) *There exists $r_\lambda>0$ satisfying $r_\lambda\to0$ as $\lambda\to\infty$ such that*

$$(1-r_\lambda)^{\frac{2(p+1)}{p-1}}(J_{1,\lambda}(v_1)+J_{2,\lambda}(v_2))\le J_\lambda(v_1,v_2)\le(1+r_\lambda)^{\frac{2(p+1)}{p-1}}(J_{1,\lambda}(v_1)+J_{2,\lambda}(v_2))$$

for all $(v_1,v_2)\in\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}$.

(ii) $\inf_{(v_1,v_2)\in\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}J_\lambda(v_1,v_2)\ge m_1+m_2$, *where m_1, $m_2>0$ are defined in* (2.29).

Proof. (i) easily follows from (i) of Corollary 2.11. For (ii), since $\|u_i\|_{\lambda,\Omega_i'}\ge\|u_i\|_{H^1(\Omega_i')}$ we have

$$I_\lambda(u_1,u_2)\ge\frac{1}{2}\left(\|u_1\|_{H^1(\Omega_1')}^2+\|u_2\|_{H^1(\Omega_2')}^2\right)-\frac{1}{p+1}\left(\|u_1\|_{L^{p+1}(\Omega_1')}^{p+1}+\|u_2\|_{L^{p+1}(\Omega_2')}^{p+1}\right).$$

Thus (ii) follows. □

In what follows, we use the following notation: for $M\in\mathbf{R}$

$$[J_\lambda\le M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}=\{(v_1,v_2)\in\Sigma_{1,\lambda}\times\Sigma_{2,\lambda};\ J_\lambda(v_1,v_2)\le M\},$$
$$[J_\lambda<M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}=\{(v_1,v_2)\in\Sigma_{1,\lambda}\times\Sigma_{2,\lambda};\ J_\lambda(v_1,v_2)<M\}.$$

The following propositions state fundamental properties of $J_\lambda(v_1,v_2)$, which we will use repeatedly in the rest of the paper.

Proposition 2.15. *For any $M>0$ there exist constants $\lambda(M)$, $\delta_M>0$ such that for $\lambda\ge\lambda(M)$*

(i) $\|v_1\|_{L^{p+1}(\Omega_1')}^{p+1}\ge\delta_M$, $\|v_2\|_{L^{p+1}(\Omega_2')}^{p+1}\ge\delta_M$ *for all $(v_1,v_2)\in[J_\lambda\le M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$.*

(ii) *For any $(v_1,v_2)\in[J_\lambda\le M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$, $(s,t)\mapsto J_\lambda(sv_1,tv_2)$ has a unique maximizer — we denote $(s_\lambda(v_1,v_2),t_\lambda(v_1,v_2))$ — and it is nondegenerate.*

(iii) *There exist constants R_0, $R_1(M) > 0$ — $R_1(M) > 0$ depends on $M > 0$ but is independent of (v_1, v_2), and R_0 is independent of M, (v_1, v_2) — such that*

$$s_\lambda(v_1, v_2)^2 + t_\lambda(v_1, v_2)^2 \le R_1(M)^2, \tag{2.30}$$

$$s_\lambda(v_1, v_2), t_\lambda(v_1, v_2) \ge R_0. \tag{2.31}$$

(iv) $(s_\lambda(v_1, v_2), t_\lambda(v_1, v_2)) : [J_\lambda < M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}} \to \mathbf{R}^2$ *is of class* C^1 *and* $J_\lambda : [J_\lambda < M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}} \to \mathbf{R}$ *is of class* C^1.

(v) *There exists $r_\lambda > 0$ such that*

$$r_\lambda \to 0 \qquad \text{as } \lambda \to \infty,$$

$$|J_\lambda(v_1, v_2) - J_{1,\lambda}(v_1) - J_{2,\lambda}(v_2)| < r_\lambda, \tag{2.32}$$

$$\begin{aligned} &\left|J'_\lambda(v_1, v_2)(h_1, h_2) - J'_{1,\lambda}(v_1)h_1 - J'_{2,\lambda}(v_2)h_2\right| \\ &\qquad < r_\lambda(\|h_1\|_{\lambda,\Omega'_1} + \|h_2\|_{\lambda,\Omega'_2}) \end{aligned} \tag{2.33}$$

for all $(v_1, v_2) \in [J_\lambda < M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$ and $(h_1, h_2) \in H^1(\Omega'_1) \oplus H^1(\Omega'_2)$ satisfying $\langle v_1, h_1\rangle_{\lambda,\Omega'_1} = \langle v_2, h_2\rangle_{\lambda,\Omega'_2} = 0$.

Proposition 2.16. *For $M > 0$ suppose $\lambda \ge \lambda(M)$. Then $J_\lambda(v_1, v_2)$ satisfies the $(PS)_c$-condition for $c \in (-\infty, M)$.*

Propositions 2.15 and 2.16 enable us to construct deformation flows to ensure the existence of critical points $J_\lambda(v_1, v_2)$. In Section 9 we will construct deformation flows in various settings.

(e) Convergence and positivity of critical points. The following proposition describes the behavior of critical points $v_\lambda = (v_{1\lambda}, v_{2\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ as $\lambda \to \infty$.

Proposition 2.17. *Suppose that $v_\lambda = (v_{1\lambda}, v_{2\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ satisfies for some $c > 0$*

$$\|J'_\lambda(v_\lambda)\|_{(T_{v_\lambda}(\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}))^*} \to 0, \quad J_\lambda(v_\lambda) \to c \qquad \text{as } \lambda \to \infty. \tag{2.34}$$

Then after extracting a subsequence $\lambda_n \to \infty$ we have

(i) *There exists $v_\infty = (v_{1\infty}, v_{2\infty}) \in \Sigma_{1,D} \times \Sigma_{2,D}$ such that for $i = 1, 2$*

$$\|v_{i\lambda_n} - v_{i\infty}\|_{H^1(\Omega_i)} \to 0, \quad \|v_{i\lambda_n}\|_{\lambda_n,\Omega'_i\setminus\Omega_i} \to 0 \quad \text{as } n \to \infty, \tag{2.35}$$

$$J'_{1,D}(v_{1\infty}) = 0, \ J'_{2,D}(v_{2\infty}) = 0, \tag{2.36}$$

$$J_{i,D}(v_{i\infty}) = \lim_{n\to\infty} J_{i,\lambda_n}(v_{i\lambda_n}), \quad J_{1,D}(v_{1\infty}) + J_{2,D}(v_{2\infty}) = c. \tag{2.37}$$

(ii) *Define $u_\lambda = (u_{1\lambda}, u_{2\lambda}) \in H^1(\Omega'_1 \cup \Omega'_2) = H^1(\Omega'_1) \oplus H^1(\Omega'_2)$ and $\tilde{u}_\lambda \in H^1(\mathbf{R}^N)$ by*

$$u_\lambda = (u_{1\lambda}, u_{2\lambda}) = (s_\lambda(v_\lambda)v_{1\lambda}, t_\lambda(v_\lambda)v_{2\lambda}), \quad \tilde{u}_\lambda = w_\lambda(u_\lambda). \tag{2.38}$$

Then

$$\Phi_\lambda(\tilde{u}_\lambda) = I_\lambda(u_\lambda) = J_\lambda(v_\lambda) \to c, \quad \Phi'_\lambda(\tilde{u}_\lambda) \to 0, \quad I'_\lambda(u_\lambda) \to 0$$

as $\lambda \to \infty$. After extracting a subsequence $\lambda_n \to \infty$, there exists $u_\infty = (u_{1\infty}, u_{2\infty}) \in H^1_0(\Omega_1 \cup \Omega_2) = H^1_0(\Omega_1) \oplus H^1_0(\Omega_2)$ such that

1° $\|\tilde{u}_{\lambda_n} - u_\infty\|_{\lambda_n,\mathbf{R}^N} \to 0$ *as $n \to \infty$.*

2° $u_{i\infty}$ *solves* (2.4)–(2.5).

3° $\Psi_{1,D}(u_{1\infty}) + \Psi_{2,D}(u_{2\infty}) = c$.

4° $u_{1\infty} \not\equiv 0$ *in* Ω_1, $u_{2\infty} \not\equiv 0$ *in* Ω_2 *and* $v_{i\infty} = \frac{u_{i\infty}}{\|u_{i\infty}\|_{H^1(\Omega_i)}}$ *for* $i = 1, 2$.

Proof. First we show 1°–4° of the second statement (ii). Suppose that $v_\lambda = (v_{1\lambda}, v_{2\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ satisfies (2.34). Defining u_λ, $\tilde{u}_\lambda$ by (2.38), we can see that there exist R_0, $R_1 > 0$ such that

$$s_\lambda(v_\lambda), t_\lambda(v_\lambda) \in [R_0, R_1] \quad \text{for large } \lambda. \tag{2.39}$$

By Proposition 2.3, there exists a subsequence $\lambda_n \to \infty$ and $u_\infty = (u_{1\infty}, u_{2\infty}) \in H_0^1(\Omega_1) \oplus H_0^1(\Omega_2)$ such that 1°–2° hold. By (2.39), we have $u_{1\infty} \not\equiv 0$, $u_{2\infty} \not\equiv 0$ and 3°.

Setting $v_{i\infty} = u_{i\infty}/\|u_{i\infty}\|_{H^1(\Omega_i)} \in \Sigma_{i,D}$ for $i = 1, 2$, we can deduce (2.35)–(2.37) from (2.39) and (ii). □

In a way similar to Proposition 2.17, we have

Lemma 2.18. *Let $i \in \{1, 2\}$ and $v_\lambda \in \Sigma_{i,\lambda}$ satisfies for some $c > 0$*

$$\|J_{i,\lambda}(v_\lambda)\|_{(T_{v_\lambda}(\Sigma_{i,\lambda}))^*} \to 0, \quad J_{i,\lambda}(v_\lambda) \to c \quad \text{as } \lambda \to \infty.$$

Then after extracting a subsequence $\lambda_n \to \infty$, there exists $v_\infty \in \Sigma_{i,D}$ such that

$$\|v_{\lambda_n} - v_\infty\|_{H^1(\Omega_i)} \to 0, \quad \|v_{\lambda_n}\|_{\lambda_n, \Omega_i' \setminus \Omega_i} \to 0, \tag{2.40}$$

$$J_{i,D}'(v_\infty) = 0, \quad J_{i,D}(v_\infty) = c. \tag{2.41}$$

To see the positivity of the limit $v_{1\infty}$ in Ω_1 (equivalently $u_{1\infty}$), the following proposition is useful.

Proposition 2.19. *For any $M > 0$ there exists $\delta_0(M) > 0$ such that for $i = 1, 2$*

(i) *If $v \in \Sigma_{i,D}$ satisfies*

$$J_{i,D}'(v) = 0, \quad J_{i,D}(v) \le M, \quad \|v_-\|_{L^{p+1}(\Omega_i)}^{p+1} \le \delta_0(M),$$

then $v > 0$ in Ω_1.

(ii) *If $v_\lambda = (v_{1\lambda}, v_{2\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ satisfies*

$$J_\lambda'(v_\lambda) = 0, \quad J_\lambda(v_\lambda) \le M, \quad \|(v_{1\lambda})_-\|_{L^{p+1}(\Omega_1)}^{p+1} \le \delta_0(M),$$

then, after extracting a subsequence $\lambda_n \to \infty$ there exists $v_\infty = (v_{1\infty}, v_{2\infty}) \in \Sigma_{1,D} \times \Sigma_{2,D}$ such that the conclusion of Proposition 2.17 holds, and moreover $v_{1\infty} > 0$ in Ω_1.

(iii) *If $v_\lambda = (v_{1\lambda}, v_{2\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ satisfies*

$$J_\lambda'(v_\lambda) = 0, \quad J_\lambda(v_\lambda) \le M,$$
$$\|(v_{1\lambda})_-\|_{L^{p+1}(\Omega_1')}^{p+1} \le \delta_0(M), \quad \|(v_{2\lambda})_-\|_{L^{p+1}(\Omega_2')}^{p+1} \le \delta_0(M),$$

then there exists $\lambda_M \ge 1$ such that for large $\lambda \ge \lambda_M$, $\tilde{u}_\lambda > 0$ in $\mathbf{R}^N$. Here $\tilde{u}_\lambda$ is defined in (2.38).

Proof. First we remark that there exists $\delta_1 > 0$ such that for $u \in H_0^1(\Omega_i)$

$$-\Delta u + u = |u|^{p-1}u \quad \text{in } \Omega_i, \quad \|u_-\|_{L^{p+1}(\Omega_i)} \le \delta_1 \tag{2.42}$$

implies $u_- \equiv 0$. In fact, multiplying $u_-(x)$ to (2.42) and integrating over Ω_i, we have

$$\|u_-\|_{H^1(\Omega_i)}^2 = \|u_-\|_{L^{p+1}(\Omega_i)}^{p+1} \le \delta_1^{p-1}\|u_-\|_{L^{p+1}(\Omega_i)}^2 \le C\delta_1^{p-1}\|u_-\|_{H^1(\Omega_i)}^2.$$

Thus, choosing $\delta_1 > 0$ small so that $C\delta_1^{p-1} < 1$, we have $u_- \equiv 0$.

The statements (i), (ii) easily follow from the above property and Proposition 2.17. In a similar way, (iii) follows from the following fact: there exists $\delta_2 > 0$ independent of λ such that for $u \in H^1(\mathbf{R}^N)$

$$-\Delta u + (\lambda^2 a(x) + 1)u = g(x,u) \quad \text{in } \mathbf{R}^N, \quad \|u_-\|_{L^{p+1}(\mathbf{R}^N)} \le \delta_2$$

implies $u_- \equiv 0$. □

3. Sign-changing solutions

We devote Sections 3–4 for a proof of Theorem 1.1. We develop a perturbation theory, and we study $J_\lambda(v_1, v_2)$ as a perturbation from $J_{1,\lambda}(v_1) + J_{2,\lambda}(v_2)$. First we study basic properties of our "unperturbed" functionals $J_{i,\lambda}(v_i)$.

(a) Minimax methods for $J_{i,\lambda}(v_i)$. As we observed in Section 2, $I_{1,\lambda}(u_1)$ and $I_{2,\lambda}(u_2)$ have a mountain pass geometry and we can apply the mountain pass theorem. Moreover $I_{2,\lambda}(u_2)$ possesses $\mathbf{Z}_2$-symmetry: $I_{2,\lambda}(-u_2) = I_{2,\lambda}(u_2)$, and it enables us to apply symmetric mountain pass theorems. Here we introduce related minimax methods for $J_{1,\lambda}(v_1)$ and $J_{2,\lambda}(v_2)$.

We define minimax values $c^{1,\lambda}_{min}$, $b^{2,\lambda}_n$ ($n \in \mathbf{N}$) by

$$c^{1,\lambda}_{min} = \inf_{v_1 \in \Sigma_{1,\lambda}} J_{1,\lambda}(v_1), \quad b^{2,\lambda}_n = \inf_{\gamma \in \Gamma^{2,\lambda}_n} \max_{\theta \in S^n} J_{2,\lambda}(\gamma(\theta)),$$

where $S^n = \{\theta = (\theta_1, \cdots, \theta_{n+1});\ |\theta| = 1\}$ and

$$\Gamma^{2,\lambda}_n = \{\gamma \in C(S^n, \Sigma_{2,\lambda});\ \gamma(-\theta) = -\gamma(\theta) \text{ for all } \theta \in S^n\}.$$

Since $J_{1,\lambda}(v_1)$, $J_{2,\lambda}(v_2)$ satisfy the $(PS)_c$-condition for all $c \in \mathbf{R}$, we have

Lemma 3.1. (i) *$c^{1,\lambda}_{min}$ is a critical value of $J_{1,\lambda}(v_1)$.*
(ii) *$b^{2,\lambda}_n$ is a critical value of $J_{2,\lambda}(v_2)$.*
(iii) *$b^{2,\lambda}_1 \le b^{2,\lambda}_2 \le \cdots \le b^{2,\lambda}_n \le b^{2,\lambda}_{n+1} \le \cdots$.*
(iv) *$b^{2,\lambda}_n \to \infty$ as $n \to \infty$.*

The main result of this subsection is the following

Proposition 3.2. *Let $c^{1,D}_{min}$ ($b^{2,D}_n$, respectively) be a critical value of $J_{1,D}(v_1) \in C^1(\Sigma_{1,D}, \mathbf{R})$ ($J_{2,D}(v_2) \in C^1(\Sigma_{2,D}, \mathbf{R})$, respectively) defined in* (1.8) *(*(1.9)*, respectively). Then we have*

(i) *$c^{1,\lambda}_{min} \to c^{1,D}_{min}$ as $\lambda \to \infty$.*
(ii) *$b^{2,\lambda}_n \to b^{2,D}_n$ as $\lambda \to \infty$.*

Proof. We deal with (ii). (i) can be proved in a similar way. We regard $H^1_0(\Omega_2) \subset H^1(\Omega'_2)$, and thus we have $\Sigma_{2,D} \subset \Sigma_{2,\lambda}$ and $J_{2,\lambda}(v) = J_{2,D}(v)$ for all $v \in \Sigma_{2,D}$. We also remark that $\Gamma^{2,D}_n \subset \Gamma^{2,\lambda}_n$. Thus we have

$$b^{2,\lambda}_n \le b^{2,D}_n \quad \text{for all } n \in \mathbf{N} \text{ and } \lambda \ge 1. \tag{3.1}$$

We will show that

$$\liminf_{\lambda \to \infty} b^{2,\lambda}_n \ge b^{2,D}_n. \tag{3.2}$$

To show (3.2), we will show the following.

Lemma 3.3. *For any $m > 0$, $\delta > 0$ and any open set $O \subset\subset \Omega_2$, there exists $\lambda_{m,\delta,O} \geq 1$ with the following property: for $\lambda \geq \lambda_{m,\delta,O}$ there exists an odd continuous operator $R_\lambda : \{v \in \Sigma_{2,\lambda};\ J_{2,\lambda}(v) \leq m\} \to \Sigma_{2,D}$ such that*

$$J_{2,D}(R_\lambda v) \leq (1+\delta) J_{2,\lambda}(v) \quad \text{for all } v \in \Sigma_{2,\lambda} \text{ with } J_{2,\lambda}(v) \leq m \tag{3.3}$$

and

$$R_\lambda v = v \quad \text{for all } v \in \Sigma_{2,\lambda} \cap H_0^1(O) \text{ with } J_{2,\lambda}(v) \leq m. \tag{3.4}$$

We postpone a proof of Lemma 3.3, and we give a proof of Proposition 3.2.

Proof of Proposition 3.2. We show (3.2). Set $m = b_n^{2,D} + 1$ and choose $\delta \in (0,1)$ arbitrary. (Property (3.4) is not used here and $O \subset\subset \Omega_2$ can be fixed arbitrarily. (3.4) will be used later in the proof of Lemma 3.6.) By Lemma 3.3, there exists an odd continuous mapping R_λ satisfying (3.3). Choose $\gamma(\theta) \in \Gamma_n^{2,\lambda}$ satisfying $\max_{\theta \in S^n} J_{2,\lambda}(\gamma(\theta)) \leq b_n^{2,\lambda} + \delta$. Recalling (3.1), we have $J_{2,\lambda}(\gamma(\theta)) \leq m$ for all $\theta \in S^n$. We set $\tilde{\gamma}(\theta) = R_\lambda(\gamma(\theta)) \in \Gamma_n^{2,D}$. By (3.3),

$$J_{2,D}(\tilde{\gamma}(\theta)) = J_{2,D}(R_\lambda(\gamma(\theta))) \leq (1+\delta) J_{2,\lambda}(\gamma(\theta)) \leq (1+\delta)(b_n^{2,\lambda} + \delta) \quad \text{for all } \theta \in S^n.$$

Thus

$$b_n^{2,D} \leq \max_{\theta \in S^n} J_{2,D}(\tilde{\gamma}(\theta)) \leq (1+\delta)(b_n^{2,\lambda} + \delta) \quad \text{for } \lambda \geq \lambda_{m,\delta}.$$

Therefore $b_n^{2,D} \leq (1+\delta)(\liminf_{\lambda \to \infty} b_n^{2,\lambda} + \delta)$. Since $\delta \in (0,1)$ is arbitrary, (3.2) holds. □

Next we prove Lemma 3.3.

Proof of Lemma 3.3. We use the following notation: for an open set $\tilde{\Omega} \subset \mathbf{R}^N$ we write

$$\begin{aligned} \Sigma_{\tilde{\Omega}} &= \{v \in H_0^1(\tilde{\Omega});\ \|v\|_{H^1(\tilde{\Omega})} = 1\}, \\ J_{\tilde{\Omega}}(v) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{1}{\|v\|_{L^{p+1}(\tilde{\Omega})}}\right)^{\frac{2(p+1)}{p-1}} \in C^1(\Sigma_{\tilde{\Omega}}, \mathbf{R}). \end{aligned}$$

Proof of Lemma 3.3 consists of 3 steps.

Step 1: For any $\tilde{\Omega}$ satisfying $\Omega_2 \subset\subset \tilde{\Omega} \subset\subset \Omega_2'$ and $m > 0$, $\epsilon > 0$, there exists $\lambda_{\tilde{\Omega},\epsilon,m} \geq 1$ and an odd continuous operator

$$R_\lambda^1 : \{v \in \Sigma_{2,\lambda};\ J_{2,\lambda}(v) \leq m\} \to \Sigma_{\tilde{\Omega}} \quad (\lambda \geq \lambda_{\tilde{\Omega},\epsilon,m}),$$

such that

$$J_{\tilde{\Omega}}(R_\lambda^1 v) \leq (1+\epsilon) J_{2,\lambda}(v) \quad \text{for all } v \in \Sigma_{2,\lambda} \text{ with } J_{2,\lambda}(v) \leq m, \tag{3.5}$$

$$R_\lambda^1 v = v \qquad \text{for all } v \in \Sigma_{2,\lambda} \cap H_0^1(\Omega_2) \text{ with } J_{2,\lambda}(v) \leq m. \tag{3.6}$$

Step 2: For any $\epsilon' > 0$ and any open set $O \subset\subset \Omega_2$ there exist a smooth open set $\tilde{\Omega}$ satisfying $\Omega_2 \subset\subset \tilde{\Omega} \subset\subset \Omega_2'$ and an odd continuous operator $R^2 : \Sigma_{\tilde{\Omega}} \to \Sigma_{2,D}$ such that

$$J_{2,D}(R^2 v) \leq (1+\epsilon') J_{\tilde{\Omega}}(v) \quad \text{for all } v \in \Sigma_{\tilde{\Omega}}, \tag{3.7}$$

$$R^2 v = v \qquad \text{for all } v \in \Sigma_{\tilde{\Omega}} \cap H_0^1(O). \tag{3.8}$$

Step 3: Conclusion.

Step 1: Existence of R^1_λ satisfying (3.5), (3.6).
First we remark that $J_{2,\lambda}(v) \le m$ implies

$$\left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{\|v\|_{\lambda,\Omega_2',\#}}{\|v\|_{L^{p+1}(\Omega_2')}}\right)^{\frac{2(p+1)}{p-1}} \le m.$$

Recalling $\|v\|_{\lambda,\Omega_2',\#} \ge \|v\|_{\lambda,\Omega_2'} = 1$, we have

$$\|v\|_{L^{p+1}(\Omega_2')} \ge \tilde{m} \equiv \left(\frac{\frac{1}{2} - \frac{1}{p+1}}{m}\right)^{\frac{p-1}{2(p+1)}}.$$

The following proposition will be proved in Section 8, in which the subsolution estimate (cf. Simon [22]) plays an important role.

Proposition 3.4. *For any $\tilde{m} > 0$, $\tilde{\epsilon} > 0$ and any open sets $\tilde{\Omega}$, $\tilde{\tilde{\Omega}}$ with smooth boundaries and $\Omega_2 \subset\subset \tilde{\tilde{\Omega}} \subset\subset \tilde{\Omega} \subset\subset \Omega_2'$, there exists $\lambda(\tilde{m}, \tilde{\epsilon}, \tilde{\Omega}, \tilde{\tilde{\Omega}}) \ge 1$ with the following properties: For any $\lambda \ge \lambda(\tilde{m}, \tilde{\epsilon}, \tilde{\Omega}, \tilde{\tilde{\Omega}})$ there exists an odd continuous mapping*

$$F_\lambda : \{v \in \Sigma_{2,\lambda};\ \|v\|_{L^{p+1}(\Omega_2')} \ge \tilde{m}\} \to H^1_0(\tilde{\Omega})$$

such that for any $v \in \{v \in \Sigma_{2,\lambda};\ \|v\|_{L^{p+1}(\Omega_2')} \ge \tilde{m}\}$,

1° $(F_\lambda v)(x) = v(x)$ *for all* $x \in \tilde{\tilde{\Omega}}$.
2° $\|F_\lambda v\|_{\lambda,\Omega_2'} \le 1 + \tilde{\epsilon}$.
3° $\|F_\lambda v\|_{L^{p+1}(\Omega_2')} \ge (1 - \tilde{\epsilon})\|v\|_{L^{p+1}(\Omega_2')}$.

Now we can show Step 1. Let F_λ be an operator given in the above proposition. We define R^1_λ by

$$R^1_\lambda v = \frac{F_\lambda v}{\|F_\lambda v\|_{H^1(\tilde{\Omega})}}.$$

By 1°–3° of Proposition 3.4 and (2.28),

$$\begin{aligned}
J_{\tilde{\Omega}}(R^1_\lambda v) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{1}{\|R^1_\lambda v\|_{L^{p+1}(\Omega_2')}}\right)^{\frac{2(p+1)}{p-1}} \\
&\le \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{\|R^1_\lambda v\|_{\lambda,\Omega_2'}}{\|R^1_\lambda v\|_{L^{p+1}(\Omega_2')}}\right)^{\frac{2(p+1)}{p-1}} \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{\|F_\lambda v\|_{\lambda,\Omega_2'}}{\|F_\lambda v\|_{L^{p+1}(\Omega_2')}}\right)^{\frac{2(p+1)}{p-1}} \\
&\le \left(\frac{1+\tilde{\epsilon}}{1-\tilde{\epsilon}}\right)^{\frac{2(p+1)}{p-1}}\left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{1}{\|v\|_{L^{p+1}(\Omega_2')}}\right)^{\frac{2(p+1)}{p-1}} \\
&\le \left(\frac{1+\tilde{\epsilon}}{1-\tilde{\epsilon}}\right)^{\frac{2(p+1)}{p-1}} J_{2,\lambda}(v) \qquad \text{for all } v \in \Sigma_{2,\lambda}.
\end{aligned}$$

Thus we have (3.5) for sufficiently small $\tilde{\epsilon} > 0$. (3.6) also clearly holds. □

Step 2: Existence of $\tilde{\Omega}$ and R^2 satisfying (3.7), (3.8).
Since $\partial\Omega_2$ is smooth, for any $\nu > 0$ and $O \subset\subset \Omega_2$ we can find a smooth open set $\Omega_2^{(\nu)}$ satisfying $\Omega_2 \subset\subset \Omega_2^{(\nu)} \subset\subset \Omega_2'$ and a diffeomorphism $\varphi^{(\nu)} = (\varphi_1^{(\nu)}, \cdots, \varphi_N^{(\nu)}) : \overline{\Omega_2} \to \overline{\Omega_2^{(\nu)}}$ such that

$$\begin{aligned} &\varphi^{(\nu)}(x) = x \qquad \text{for all } x \in O, \\ &\max_{x\in\overline{\Omega_2}} \left| \frac{\partial \varphi_i^{(\nu)}}{\partial x_j}(x) - \delta_{ij} \right| < \nu \qquad \text{for } i, j \in \{1, 2, \cdots, N\}. \end{aligned} \tag{3.9}$$

We remark that $\Omega_2^{(\nu)}$ approaches Ω_2 as $\nu \to 0$.

We define $\widetilde{\varphi}^{(\nu)} : H_0^1(\Omega_2^{(\nu)}) \to H_0^1(\Omega_2)$ by $(\widetilde{\varphi}^{(\nu)}u)(x) = u(\varphi^{(\nu)}(x))$. We can observe that there exists a constant $c_\nu > 0$ such that for $u \in H_0^1(\Omega_2^{(\nu)})$

$$c_\nu \to 0 \quad \text{as } \nu \to 0,$$

$$(1 - c_\nu)\|u\|_{L^{p+1}(\Omega_2^{(\nu)})} \le \|\widetilde{\varphi}^{(\nu)}u\|_{L^{p+1}(\Omega_2)} \le (1 + c_\nu)\|u\|_{L^{p+1}(\Omega_2^{(\nu)})}, \tag{3.10}$$

$$(1 - c_\nu)\|u\|_{H^1(\Omega_2^{(\nu)})} \le \|\widetilde{\varphi}^{(\nu)}u\|_{H^1(\Omega_2)} \le (1 + c_\nu)\|u\|_{H^1(\Omega_2^{(\nu)})}. \tag{3.11}$$

Now we can prove (3.7). For a given $\epsilon > 0$, we choose $\nu \in (0, 1)$ such that $\left(\frac{1+c_\nu}{1-c_\nu}\right)^{\frac{2(p+1)}{p-1}} < 1 + \epsilon$. We will show that (3.7) holds for $\tilde{\Omega} = \Omega_2^{(\nu)}$.

For $v \in \Sigma_{\tilde{\Omega}}$ we define $R^2 v \in \Sigma_{2,D}$ by $R^2 v = \frac{\tilde{\varphi}^{(\nu)}v}{\|\tilde{\varphi}^{(\nu)}v\|_{H^1(\Omega_2)}}$. Then by (3.10)–(3.11)

$$\begin{aligned} J_{2,D}(R^2 v) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{\|\tilde{\varphi}^{(\nu)}v\|_{H^1(\Omega_2)}}{\|\tilde{\varphi}^{(\nu)}v\|_{L^{p+1}(\Omega_2)}}\right)^{\frac{2(p+1)}{p-1}} \\ &\le \left(\frac{1+c_\nu}{1-c_\nu}\right)^{\frac{2(p+1)}{p-1}} \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{\|v\|_{H^1(\Omega_2^{(\nu)})}}{\|v\|_{L^{p+1}(\Omega_2^{(\nu)})}}\right)^{\frac{2(p+1)}{p-1}} \\ &= \left(\frac{1+c_\nu}{1-c_\nu}\right)^{\frac{2(p+1)}{p-1}} J_{\tilde{\Omega}}(v). \end{aligned}$$

By the choice of ν, we have (3.7). Equation (3.8) also follows from (3.9). □

Step 3: Conclusion.
For a given $\delta > 0$ we choose $\epsilon, \epsilon' > 0$ such that $(1+\epsilon)(1+\epsilon') < 1+\delta$. By Step 2, there exist $\tilde{\Omega}$ and R^2 such that (3.7) holds. Next, using Step 1, we find $\lambda_{\tilde{\Omega},\epsilon,m} \ge 1$ and R_λ^1 ($\lambda \ge \lambda_{\tilde{\Omega},\epsilon,m}$) such that (3.5) holds. Defining $R_\lambda : \{v \in \Sigma_{2,\lambda};\ J_{2,\lambda}(v) \le m\} \to \Sigma_{2,D}$ by $R_\lambda = R^2 \circ R_\lambda^1$, we have (3.3)–(3.4). □

As stated in the Introduction, since $b_n^{2,D} \to \infty$ as $n \to \infty$, there exists a sequence $n(1) < n(2) < \cdots < n(k) < n(k+1) < \cdots$ such that

$$b_{n(k)}^{2,D} < b_{n(k)+1}^{2,D}. \tag{3.12}$$

We set

$$\delta_k = \frac{1}{4}(b_{n(k)+1}^{2,D} - b_{n(k)}^{2,D}) > 0 \tag{3.13}$$

and choose $\gamma_{0k} \in \Gamma^{2,D}_{n(k)}$ such that

$$\max_{\theta \in S^{n(k)}} J_{2,D}(\gamma_{0k}(\theta)) < b^{2,D}_{n(k)} + \delta_k, \tag{3.14}$$

$$\operatorname{supp} \gamma_{0k} \subset \Omega_2 \text{ is compact in } \Omega_2 \text{ for all } \theta \in S^{n(k)}. \tag{3.15}$$

We define a minimax value $c^{2,D}_k$ by

$$c^{2,D}_k = \inf_{\sigma \in \Lambda^{2,D}_k} \max_{\theta \in S^{n(k)+1}_+} J_{2,D}(\sigma(\theta)), \tag{3.16}$$

where

$$\Lambda^{2,D}_k = \{\sigma \in C(S^{n(k)+1}_+, \Sigma_{2,D});\ \sigma(\theta) = \gamma_{0k}(\theta) \text{ for all } \theta \in S^{n(k)}\}. \tag{3.17}$$

Here we regard $S^{n(k)} = \{(\theta_1, \cdots, \theta_{n(k)+1}, 0);\ \theta_1^2 + \cdots + \theta^2_{n(k)+1} = 1\} = \partial S^{n(k)+1}_+ \subset S^{n(k)+1}_+$.

This type of minimax method is introduced in Ambrosetti [1], Bahri-Berestycki [5], Struwe [23] and Rabinowitz [20]. In the definition (3.16)–(3.17) of $c^{2,D}_k$ we don't require evenness, and it enables us to detect critical points of perturbed functionals.

As a fundamental property of $c^{2,D}_k$ we have

Lemma 3.5. (i) $c^{2,D}_k \geq b^{2,D}_{n(k)+1}$ $(> b^{2,D}_{n(k)})$.

(ii) $c^{2,D}_k$ *is a critical value of* $J_{2,D}(v_2)$.

Proof. (i) For a given $\sigma \in \Lambda^{2,D}_k$ we define $\gamma_\sigma \in \Gamma^{2,D}_{n(k)+1}$ by

$$\gamma_\sigma(\theta) = \begin{cases} \sigma(\theta) & \text{if } \theta \in S^{n(k)+1}_+, \\ -\sigma(-\theta) & \text{if } \theta \in S^{n(k)+1} \setminus S^{n(k)+1}_+. \end{cases}$$

Since $J_{2,D}$ is even, we have $\max_{\theta \in S^{n(k)+1}} J_{2,D}(\gamma_\sigma(\theta)) = \max_{\theta \in S^{n(k)+1}_+} J_{2,D}(\sigma(\theta))$. Thus we have $c^{2,D}_k \geq b^{2,D}_{n(k)+1}$.

(ii) By (i), we can see that $\Lambda^{2,D}_k$ is stable under a suitable class of deformations. Thus we can observe that $c^{2,D}_k$ is a critical point value of $J_{2,D}(v_2)$. □

Next we introduce related minimax methods to $J_{2,\lambda}(v_2)$. By Proposition 3.2, (3.1), and (3.12), for any $k \in \mathbf{N}$ there exists $\lambda(k)$ such that for $\lambda \geq \lambda(k)$

$$b^{2,\lambda}_{n(k)} \leq b^{2,D}_{n(k)} \leq b^{2,\lambda}_{n(k)} + \delta_k, \quad b^{2,\lambda}_{n(k)} \leq b^{2,\lambda}_{n(k)+1} - 3\delta_k. \tag{3.18}$$

We define for $\lambda \geq \lambda(k)$

$$c^{2,\lambda}_k = \inf_{\sigma \in \Lambda^{2,\lambda}_k} \max_{\theta \in S^{n(k)+1}_+} J_{2,\lambda}(\sigma(\theta)),$$

where $\Lambda^{2,\lambda}_k = \{\sigma \in C(S^{n(k)+1}_+, \Sigma_{2,\lambda});\ \sigma(\theta) = \gamma_{0k}(\theta) \text{ for all } \theta \in S^{n(k)}\}$. Here $\gamma_{0k} \in \Gamma^{2,D}_{n(k)} \subset \Gamma^{2,\lambda}_{n(k)}$ is chosen in (3.14)–(3.15).

As in Lemma 3.5 and Proposition 3.2, we have

Lemma 3.6. (i) $c^{2,\lambda}_k \geq b^{2,\lambda}_{n(k)+1}$ *for* $\lambda \geq \lambda(k)$.

(ii) $c^{2,\lambda}_k$ *is a critical value of* $J_{2,\lambda}(v_2)$ *for* $\lambda \geq \lambda(k)$.

(iii) $c^{2,\lambda}_k \to c^{2,D}_k$ *as* $\lambda \to \infty$.

Proof. (i) As in (i) of Lemma 3.5, we have (i).

(ii) By (3.14) and (3.18),

$$\begin{aligned}\max_{\theta\in S^{n(k)}} J_{2,\lambda}(\gamma_{0k}(\theta)) &= \max_{\theta\in S^{n(k)}} J_{2,D}(\gamma_{0k}(\theta)) < b^{2,D}_{n(k)} + \delta_k \\ &\le b^{2,\lambda}_{n(k)} + 2\delta_k \le b^{2,\lambda}_{n(k)+1} - \delta_k < c^{2,\lambda}_k.\end{aligned}$$

Thus we can see that $c^{2,\lambda}_k$ is a critical value of $J_{2,\lambda}(v_2)$.

(iii) We recall (3.15) and choose $O \subset\subset \Omega_2$ such that $\overline{\bigcup_{\theta\in S^{n(k)}} \operatorname{supp}\gamma_{0k}(\theta)} \subset O$. Applying Lemma 3.3, we can show (iii) as in Proposition 3.2. □

(b) Minimax methods for $J_\lambda(v_1, v_2)$. Let $(n(k))_{k=1}^\infty$, $(\gamma_{0k})_{k=1}^\infty$ be sequences obtained in (3.12)–(3.15) and let $v_{10} \in \Sigma_{1,D}$ be a minimizer of $J_{1,D}(v_1)$, that is,

$$J_{1,D}(v_{10}) = c^{1,D}_{min}.$$

We remark that we may assume $v_{10} > 0$ in Ω_1. Using $(n(k))_{k=1}^\infty$, $(\gamma_{0k})_{k=1}^\infty$ and v_{10}, we introduce minimax values c^λ_k for $J_\lambda(v_1, v_2)$ by

$$c^\lambda_k = \inf_{\tilde\gamma\in\Lambda^\lambda_k} \max_{\theta\in S^{n(k)+1}_+} J_\lambda(\tilde\gamma(\theta)),$$

where

$$\begin{aligned}\Lambda^\lambda_k = \{\tilde\gamma = (\tilde\gamma_1, \tilde\gamma_2) \ \in\ & C(S^{n(k)+1}_+, \Sigma_{1,\lambda}\times\Sigma_{2,\lambda}); \\ & \tilde\gamma(\theta) = (v_{10}, \gamma_{0k}(\theta)) \text{ for all } \theta\in S^{n(k)}, \\ & \|\tilde\gamma_1(\theta)_-\|^{p+1}_{L^{p+1}(\Omega'_1)} \le \nu_{M_k} \text{ for all } \theta\in S^{n(k)+1}_+\}.\end{aligned}$$

Here $M_k = c^{1,D}_{min} + c^{2,D}_k + 1 > 0$ and $\nu_{M_k} > 0$ is given in Proposition 9.1 and Corollary 9.3 in Section 9. First we remark

$$c^\lambda_k \le c^{1,D}_{min} + c^{2,D}_k \qquad \text{for all } \lambda \ge 1. \tag{3.19}$$

In fact, for any $\epsilon > 0$ there exists $\gamma_\epsilon \in \Lambda^{2,D}_k$ such that $\max_{\theta\in S^{n(k)+1}_+} J_{2,D}(\gamma_\epsilon(\theta)) < c^{2,D}_k + \epsilon$. Setting $\tilde\gamma_\epsilon(\theta) = (v_{10}, \gamma_\epsilon(\theta)) \in \Lambda^\lambda_k$ and noting $J_\lambda(\tilde\gamma_\epsilon(\theta)) = J_\lambda(v_{10}, \gamma_\epsilon(\theta)) = J_{1,D}(v_{10}) + J_{2,D}(\gamma_\epsilon(\theta))$, we have $c^\lambda_k \le c^{1,D}_{min} + c^{2,D}_k + \epsilon$ for all $\lambda \ge 1$. Since $\epsilon > 0$ is arbitrary, we have (3.19). We choose $\lambda_{M_k} \ge 1$ such that (2.32) holds for $\lambda \ge \lambda_{M_k}$ and $v \in [J_\lambda \le M_k]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$.

Now we can prove

Proposition 3.7. (i) *For $\lambda \ge \lambda_{M_k}$,*

$$c^\lambda_k \ge c^{1,\lambda}_{min} + c^{2,\lambda}_k - r_\lambda. \tag{3.20}$$

(ii)

$$\lim_{\lambda\to\infty} c^\lambda_k = c^{1,D}_{min} + c^{2,D}_k. \tag{3.21}$$

(iii) *For sufficiently large λ, c^λ_k is a critical value of $J_\lambda(v_1, v_2)$.*

Proof. (i) For any $\tilde\gamma(\theta) = (p(\theta), \gamma(\theta)) \in \Lambda^\lambda_k$, we observe $\gamma(\theta) \in \Lambda^{2,\lambda}_k$. Thus for any $\tilde\gamma \in \Lambda^\lambda_k$ satisfying $\max_{\theta\in S^{n(k)+1}_+} J_\lambda(\tilde\gamma(\theta)) \le M_k$, we have by (2.32) that

$$J_\lambda(\tilde\gamma(\theta)) \ge J_{1,\lambda}(p(\theta)) + J_{2,\lambda}(\gamma(\theta)) - r_\lambda \ge c^{1,\lambda}_{min} + J_{2,\lambda}(\gamma(\theta)) - r_\lambda \quad \text{for all } \theta\in S^{n(k)+1}_+.$$

Thus we get (3.20).

(ii) (3.21) follows easily from (3.19), (3.20), (i) of Proposition 3.2 and (iii) of Lemma 3.6.

(iii) For $\theta \in S^{n(k)}$, we have

$$J_\lambda(v_{10}, \gamma_{0k}(\theta)) = J_{1,\lambda}(v_{10}) + J_{2,\lambda}(\gamma_{0k}(\theta)) \le c^{1,D}_{min} + b^{2,D}_{n(k)} + \delta_k.$$

Thus we have from (3.21), (3.13) and (i) of Lemma 3.5 that

$$\max_{\theta \in S^{n(k)}} J_\lambda(v_{10}, \gamma_{0k}(\theta)) \le c^\lambda_k - 2\delta_k$$

for sufficiently large λ, and we can show Λ^λ_k is stable under a suitable class of deformations. Thus c^λ_k is a critical value of $J_\lambda(v_1, v_2)$ for sufficiently large λ. □

At this stage we remark that if critical point $v_\lambda = (v_{1\lambda}, v_{2\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ satisfies

$$J_\lambda(v_\lambda) \to c^{1,D}_{min} + c^{2,D}_k, \tag{3.22}$$

$$\|(v_{1\lambda})_-\|^{p+1}_{L^{p+1}(\Omega'_1)} < \delta_0(M_k), \tag{3.23}$$

then by Proposition 2.17 and Proposition 2.19 a sequence of solutions $u_\lambda(x)$ corresponding to v_λ satisfies (1.10)–(1.13). We remark that by Corollary 9.3 in Section 9 there exists a critical point v_λ corresponding to the critical value c^λ_k and v_λ satisfies (3.22)–(3.23).

In the next section we show (1.14).

4. Proof of Theorem 1.1

(a) Construction of a pseudo-gradient vector field. By the argument in the previous sections, to prove Theorem 1.1 we need to show (1.14). For that purpose, we need to construct a deformation flow $\eta(t)$ under which the level set $\{v_1 \in \Sigma_{1,\lambda};\ J_{1,\lambda}(v_1) \le c^{1,D}_{min} + \delta\}$ is invariant for small $\delta > 0$. The following isolatedness of critical values is important for the construction of such a flow.

Proposition 4.1 ([Dancer [12, 13]; cf. Cao-Noussair [10]). *Let*

$$\widetilde{\Psi}_{1,D}(u) = \int_{\Omega_1} \frac{1}{2}(|\nabla u|^2 + u^2) - \frac{1}{p+1} u_+^{p+1}\, dx : H^1_0(\Omega_1) \to \mathbf{R}$$

and suppose that $u(x)$ is a positive solution of (1.4) *in Ω_1. Then $\widetilde{\Psi}_{1,D}(u)$ is an isolated critical value of $\widetilde{\Psi}_{1,D}$.*

Proof. See the Appendix of [10] and [12, 13]. Here the analyticity of the map $u \mapsto (-\Delta)^{-1}(u^p)$ in $C^1(\Omega)$ near a positive solution is essential. □

By Proposition 4.1, we can see $c^{1,D}_{min}$ is an isolated critical value of $\Psi_{1,D}(u)$ (equivalently of $J_{1,D}(v)$).

Remark 4.2. For the following arguments, we need only nowhere denseness of critical values in a neighborhood of $c^{1,D}_{min}$ or $c^{2,D}_k$. We remark that isolatedness implies nowhere denseness. We refer to Fučík-Kučera-Nečas-Souček-Souček [17] for the Morse-Sard theorem in infinite dimensional setting. We also remark that Sard's property is also used effectively in the study of the Hamiltonian system in Cieliebak-Séré [11].

First we have

Lemma 4.3. *Suppose that for* $v \in \Sigma_{1,D}$

$$J_{1,D}(v) \in [\alpha, \beta] \quad \textit{implies} \quad J'_{1,D}(v) \neq 0 \tag{4.1}$$

for some $0 < \alpha < \beta < \infty$. *Then there exist* $\rho_0 > 0$ *and* $\lambda_0 \geq 1$ *such that for* $\lambda \geq \lambda_0$, $v \in \Sigma_{1,\lambda}$

$$J_{1,\lambda}(v) \in [\alpha, \beta] \quad \textit{implies} \quad \|J'_{1,\lambda}(v)\|_{T_v(\Sigma_{1,\lambda})^*} \geq \rho_0. \tag{4.2}$$

Proof. We argue indirectly, and we assume that there exist $\lambda_n \geq 1$, $v_n \in \Sigma_{1,\lambda_n}$ $(n = 1, 2, \cdots)$ satisfying

$$\lambda_n \to \infty, \quad J_{1,\lambda_n}(v_n) \to c \in [\alpha, \beta], \quad \|J'_{1,\lambda_n}(v_n)\|_{T_{v_n}(\Sigma_{1,\lambda_n})^*} \to 0 \quad \text{as } n \to \infty.$$

By Lemma 2.18, after extracting a subsequence — still denoted by n —, there exists $v_\infty \in \Sigma_{1,D}$ such that (2.40)–(2.41) hold. In particular, we have that c is a critical value of $J_{1,D}(v)$, which is in contradiction to assumption (4.1). □

We also have

Lemma 4.4. *Suppose that* (4.1) *holds. Then for any* $M > 0$ *there exists* $\lambda_1(M) \geq \lambda_0$ *such that for any* $\lambda \geq \lambda_1(M)$ *and* $v = (v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ *satisfying* $J_\lambda(v_1, v_2) \leq M$ *and* $J_{1,\lambda}(v_1) \in [\alpha, \beta]$, *there exists*

$$X = (X_1, X_2) \in T_v(\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}) = (T_{v_1}(\Sigma_{1,\lambda})) \oplus (T_{v_2}(\Sigma_{2,\lambda}))$$

such that

$$\|X_1\|_{T_{v_1}(\Sigma_{1,\lambda})} = 1, \quad X_2 = 0,$$
$$J'_{1,\lambda}(v_1)X_1 \geq \frac{\rho_0}{2}, \quad J'_\lambda(v_1, v_2)(X_1, X_2) \geq \frac{\rho_0}{2},$$

where $\rho_0 > 0$ *is a constant appearing in* (4.2).

Proof. This is a consequence from (4.2) and (2.32)–(2.33). □

To prove Theorem 1.1, it suffices to show for any $\delta > 0$ there exists a critical point $v_\lambda = (v_{1\lambda}, v_{2\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ of $J_\lambda(v)$ for sufficiently large λ with the following properties:

$$\left| J_{1,\lambda}(v_{1\lambda}) - c^{1,D}_{min} \right| < \delta, \quad \left| J_{2,\lambda}(v_{2\lambda}) - c^{2,D}_k \right| < \delta. \tag{4.3}$$

In fact, if the above statement is true, then there exists a sequence $v_\lambda = (v_{1\lambda}, v_{2\lambda})$ such that

$$J_{1,\lambda}(v_{1\lambda}) \to c^{1,D}_{min}, \quad J_{2,\lambda}(v_{2\lambda}) \to c^{2,D}_k. \tag{4.4}$$

Clearly it follows from (4.4) that

$$J_\lambda(v_\lambda) \to c^{1,D}_{min} + c^{2,D}_k \quad \text{as } \lambda \to \infty.$$

Thus by Proposition 2.17, after extracting a subsequence, we have $v_{i\lambda_n} \to v_{i\infty}$ $(i = 1, 2)$ and

$$J_{1,D}(v_{1\infty}) = \lim_{n\to\infty} J_{1,\lambda_n}(v_{1\lambda_n}) = c^{1,D}_{min}.$$

Thus $v_{1\infty}(x)$ has a constant sign, and we may assume $v_{1\infty}(x) > 0$ and Theorem 1.1 follows. We remark here that critical point $v_\lambda(x)$ obtained in this section may not satisfy $J_\lambda(v_\lambda) = c^\lambda_k$.

In what follows, we argue indirectly and assume for some $\delta_0 > 0$ that

(4.5) there are no critical points $v = (v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ of $J_\lambda(v)$ satisfying (4.3) with $\delta = \delta_0$.

For $\epsilon_0 \in (0, \delta_0)$ we choose a neighborhood $N_{\epsilon_0,\lambda}$ of $(c_{min}^{1,D}, c_k^{2,D})$ in $\mathbf{R}^2$ in the following way: By the isolatedness of $c_{min}^{1,D}$, we can choose an interval $[\alpha, \beta]$ such that

$$[\alpha, \beta] \subset (c_{min}^{1,D} + \frac{1}{2}\delta_0, c_{min}^{1,D} + \delta_0)$$

and $J_{1,D}(v_1)$ has no critical values in $[\alpha, \beta]$. We set

$$\begin{aligned} N_{\epsilon_0,\lambda} \quad = \quad & \{(j_1, j_2) \in \mathbf{R}^2;\, c_{min}^{1,D} - \frac{1}{2}\delta_0 \le j_1 \le \beta, \\ & (1 + r_\lambda)^{-\frac{2(p+1)}{p-1}}(c_{min}^{1,\lambda} + c_k^{2,\lambda} - \epsilon_0) \le j_1 + j_2 \\ & \le (1 - r_\lambda)^{-\frac{2(p+1)}{p-1}}(c_{min}^{1,\lambda} + c_k^{2,\lambda} + \epsilon_0)\}. \end{aligned}$$

Since $c_{min}^{1,\lambda} \to c_{min}^{1,D}$, $c_k^{2,\lambda} \to c_k^{2,D}$ as $\lambda \to \infty$, we can see for $\epsilon_0 > 0$ small and $\lambda > 0$ large

$$(c_{min}^{1,D}, c_k^{2,D}) \in N_{\epsilon_0,\lambda}, \tag{4.6}$$

$$N_{\epsilon_0,\lambda} \subset (c_{min}^{1,D} - \delta_0, c_{min}^{1,D} + \delta_0) \times (c_k^{2,D} - \delta_0, c_k^{2,D} + \delta_0), \tag{4.7}$$

$$0 < \epsilon_0 < \delta_k = \frac{1}{4}(b_{n(k)+1}^{2,D} - b_{n(k)}^{2,D}). \tag{4.8}$$

By assumption (4.5), we have the following.

Lemma 4.5. *Let*

$$A_\lambda = \{(v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda};\, (J_{1,\lambda}(v_1), J_{2,\lambda}(v_2)) \in N_{\epsilon_0,\lambda}\},$$

where $\epsilon_0 > 0$ is fixed so that (4.6)–(4.8) hold. Then for sufficiently large λ there exists a vector field X on A_λ such that

(i) *$X(v) = (X_1(v), X_2(v)) \in T_v(\Sigma_{1,\lambda} \times \Sigma_{2,\lambda})$, $\|X(v)\|_{T_v(\Sigma_{1,\lambda}\times\Sigma_{2,\lambda})} \in [\frac{1}{2}, 1]$ for all $v \in A_\lambda$ and $v \mapsto X(v)$ is locally Lipschitz continuous.*

(ii) *There is a constant $\mu_\lambda > 0$ such that $J'_\lambda(v)X(v) \ge \mu_\lambda$ for all $v \in A_\lambda$.*

(iii) *If $v = (v_1, v_2) \in A_\lambda$ satisfies $J_{1,\lambda}(v_1) \in [\alpha, \beta]$, then $X(v) = (X_1(v), 0)$ and it satisfies $J'_{1,\lambda}(v_1)X_1(v) \ge \frac{\rho_0}{2}$ and $J'_\lambda(v)X(v) \ge \frac{\rho_0}{2}$.*

Proof. First we remark that $v = (v_1, v_2) \in A_\lambda$ implies $J_{1,\lambda}(v_1) \le c_{min}^{1,D} + \delta_0$, $J_{2,\lambda}(v_2) \le c_k^{2,D} + \delta_0$. Thus by (i) of Lemma 2.14, we have

$$J_\lambda(v) \le (1 + r_\lambda)^{\frac{2(p+1)}{p-1}}(c_{min}^{1,D} + c_k^{2,D} + 2\delta_0).$$

Choosing $M = 2^{\frac{2(p+1)}{p-1}}(c_{min}^{1,D} + c_k^{2,D} + 2\delta_0)$, we can apply Lemma 4.4 in the setting (iii). We also remark that $J_\lambda(v)$ satisfies the Palais-Smale condition. Thus under assumption (4.5), we have $\inf_{v\in A_\lambda} \|J'_\lambda(v)\|_{T_v(\Sigma_{1,\lambda}\times\Sigma_{2,\lambda})^*} > 0$. In a standard way, we can construct a locally Lipschitz continuous pseudo-gradient vector field satisfying (i)–(iii). $\square$

Remark 4.6. It is easily seen that there exists a constant $\nu_0 > 0$ such that for large λ

$$(4.9)\quad v = (v_1, v_2) \in A_\lambda \quad \text{implies} \quad \begin{aligned} &\|J'_\lambda(v)\|_{T_v(\Sigma_{1,\lambda}\times\Sigma_{2,\lambda})^*},\ \|J'_{1,\lambda}(v_1)\|_{T_{v_1}(\Sigma_{1,\lambda})^*},\\ &\|J'_{2,\lambda}(v_2)\|_{T_{v_2}(\Sigma_{2,\lambda})^*} \le \nu_0,\\ &J_{1,\lambda}(v_1) \ge c^{1,\lambda}_{min} \ge c^{1,D}_{min} - \delta_0. \end{aligned}$$

We also have the following.

Lemma 4.7. *For sufficiently large λ, we have for all $v = (v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$*

(i) $J_{1,\lambda}(v_1) \ge c^{1,\lambda}_{min}$.

(ii) $J_{1,\lambda}(v_1) + J_{2,\lambda}(v_2) \ge (1 - r_\lambda)^{-\frac{2(p+1)}{p-1}}(c^{1,\lambda}_{min} + c^{2,\lambda}_k + \epsilon_0)$ *implies* $J_\lambda(v_1, v_2) \ge c^{1,D}_{min} + c^{2,D}_k + \frac{2}{3}\epsilon_0$.

(iii) $J_{1,\lambda}(v_1) + J_{2,\lambda}(v_2) \le (1 + r_\lambda)^{-\frac{2(p+1)}{p-1}}(c^{1,\lambda}_{min} + c^{2,\lambda}_k - \epsilon_0)$ *implies* $J_\lambda(v_1, v_2) \le c^{1,D}_{min} + c^{2,D}_k - \frac{2}{3}\epsilon_0$.

Proof. (i) is obvious. (ii), (iii) follow from (i) of Lemma 2.14 since $c^{1,\lambda}_{min} \to c^{1,D}_{min}$, $c^{2,\lambda}_k \to c^{2,D}_k$ as $\lambda \to \infty$. □

We choose a function $\psi \in C^\infty(\mathbf{R}, [0, 1])$ such that

$$\psi(s) = \begin{cases} 1 & \text{for } s \ge c^{1,D}_{min} + c^{2,D}_k - \frac{1}{3}\epsilon_0,\\ 0 & \text{for } s \le c^{1,D}_{min} + c^{2,D}_k - \frac{2}{3}\epsilon_0. \end{cases}$$

We define a vector field $\widetilde{X}$ on A_λ:

$$\widetilde{X}(v_1, v_2) = \psi(J_\lambda(v_1, v_2))X(v_1, v_2)$$

and consider the following ODE in A_λ:

$$(4.10)\qquad \frac{d\eta}{dt} = -\widetilde{X}(\eta), \quad \eta(0, u_0) = u_0.$$

(b) End of the proof of Theorem 1.1. To prove Theorem 1.1, we argue as in Section 3 and choose $\gamma_\epsilon \in \Lambda^{2,D}_k$ (defined in (3.17)) such that

$$\max_{\theta \in S^{n(k)+1}_+} J_{2,D}(\gamma_\epsilon(\theta)) < c^{2,D}_k + \epsilon.$$

We consider a deformation of $\tilde{\gamma}_\epsilon(\theta) = (v_{01}, \gamma_\epsilon(\theta)) \in \Lambda^\lambda_k$ through the deformation flow (4.10). In this section we write

$$\begin{aligned} \Lambda^\lambda_k \quad = \quad &\{\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2) \in C(S^{n(k)+1}_+, \Sigma_{1,\lambda} \times \Sigma_{2,\lambda});\\ &\tilde{\gamma}(\theta) = (v_{10}, \gamma_{0k}(\theta)) \text{ for all } \theta \in S^{n(k)}\}. \end{aligned}$$

Choosing $\epsilon \in (0, \epsilon_0)$ small, we have

$$(4.11)\qquad \begin{aligned} J_\lambda(v_{01}, \gamma_\epsilon(\theta)) \quad &= \quad J_{1,\lambda}(v_{01}) + J_{2,\lambda}(\gamma_\epsilon(\theta)) \le c^{1,D}_{min} + c^{2,D}_k + \epsilon\\ &\le \quad (1 - r_\lambda)^{-\frac{2(p+1)}{p-1}}(c^{1,\lambda}_{min} + c^{2,\lambda}_k + \epsilon_0) \end{aligned}$$

for all $\theta \in S^{n(k)+1}_+$.

We will observe that

(i) $\tilde{\gamma}_\epsilon(\theta) \notin A_\lambda$ implies $J_\lambda(\tilde{\gamma}_\epsilon(\theta)) < c^{1,D}_{min} + c^{2,D}_k - \frac{2}{3}\epsilon_0$ (Lemma 4.8).

(ii) For $\tilde{\gamma}_\epsilon(\theta) \in A_\lambda$, we consider $\eta(t) = \eta(t, \tilde{\gamma}_\epsilon(\theta))$, where $\eta(t, u_0)$ is a solution of (4.10). Then
  (a) $\frac{d}{dt} J_\lambda(\eta(t)) \le 0$ for all $t \ge 0$.
  (b) $J_\lambda(\eta(t)) \ge c_{min}^{1,D} + c_k^{2,D} - \frac{1}{3}\epsilon_0$ implies $J_{1,\lambda}(\eta(t)) \in [c_{min}^{1,D} - \delta_0, \beta)$ and $\frac{d}{dt} J_\lambda(\eta(t)) \le -\mu_\lambda$ (Lemma 4.9).

First we have

Lemma 4.8. *For sufficiently large λ*

$$J_\lambda(v_{01}, \gamma_\epsilon(\theta)) \ge c_{min}^{1,D} + c_k^{2,D} - \frac{2}{3}\epsilon_0 \tag{4.12}$$

implies

$$(v_{01}, \gamma_\epsilon(\theta)) \in A_\lambda. \tag{4.13}$$

Proof. First we recall $J_{1,\lambda}(v_{01}) = c_{min}^{1,D} < \alpha < \beta$. Suppose that $(v_{01}, \gamma_\epsilon(\theta))$ ($\theta \in S_+^{n(k)+1}$) satisfies (4.12). By (iii) of Lemma 4.7, we have

$$J_{1,\lambda}(v_{01}) + J_{2,\lambda}(\gamma_\epsilon(\theta)) \ge (1 + r_\lambda)^{-\frac{2(p+1)}{p-1}} (c_{min}^{1,\lambda} + c_k^{2,\lambda} - \epsilon_0).$$

Thus recalling (4.11), we can see (4.13) holds for all $\theta \in S_+^{n(k)+1}$ satisfying (4.12). □

Now we define for $\theta \in S_+^{n(k)+1}$ and $L > 0$

$$\widetilde{\gamma}_L(\theta) = \begin{cases} (v_{01}, \gamma_\epsilon(\theta)) & \text{if } (v_{01}, \gamma_\epsilon(\theta)) \notin A_\lambda, \\ \eta(L, (v_{01}, \gamma_\epsilon(\theta))) & \text{if } (v_{01}, \gamma_\epsilon(\theta)) \in A_\lambda, \end{cases}$$

where $\eta(t, u_0)$ is a solution of (4.10).

The following lemma is the key of the proof; in particular it ensures $\widetilde{\gamma}_L(\theta)$ is well-defined for all $L > 0$ provided $\epsilon > 0$ is sufficiently small.

Lemma 4.9. *Suppose that $(v_{01}, \gamma_\epsilon(\theta))$ satisfies $(v_{01}, \gamma_\epsilon(\theta)) \in A_\lambda$. Let $\eta(t) = \eta(t, (v_{01}, \gamma_\epsilon(\theta)))$ be the maximal solution of (4.10) and $[0, \ell_\theta]$ be its maximal interval of existence satisfying $\eta(t) \in A_\lambda$ for all $t \in [0, \ell_\theta]$. Then for $\epsilon \in (0, \epsilon_0)$ small and λ large we have*

(i) $\|\frac{d\eta}{dt}\|_{T_{\eta(t)}(\Sigma_{1,\lambda} \times \Sigma_{2,\lambda})} \le 1$ *for all* $t \in [0, \ell_\theta]$.
(ii) $\frac{d}{dt} J_\lambda(\eta(t)) \le 0$ *for all* $t \in [0, \ell_\theta]$.
(iii) $J_\lambda(\eta(t)) \le c_{min}^{1,D} + c_k^{2,D} + \epsilon$ *for all* $t \in [0, \ell_\theta]$.
(iv) $(J_{1,\lambda} + J_{2,\lambda})(\eta(t)) < (1 - r_\lambda)^{-\frac{2(p+1)}{p-1}} (c_{min}^{1,\lambda} + c_k^{2,\lambda} + \epsilon_0)$ *for all* $t \in [0, \ell_\theta]$.
(v) *If $\eta(t) = \eta(t, (v_{01}, \gamma_\epsilon(\theta)))$ satisfies $\eta(t_0) \in \partial A_\lambda$ for some $t_0 \in [0, \ell_\theta]$, then*

$$\eta(t) \equiv (v_{01}, \gamma_\epsilon(\theta)) \quad \text{for all } t \in [0, \ell_\theta],$$
$$J_\lambda(v_{01}, \gamma_\epsilon(\theta)) \le c_{min}^{1,D} + c_k^{2,D} - \frac{2}{3}\epsilon_0.$$

(vi) $\ell_\theta = \infty$ *for all* $(v_{01}, \gamma_\epsilon(\theta)) \in A_\lambda$. *Moreover* $(v_{01}, \gamma_\epsilon(\theta)) \in int A_\lambda$ *implies* $\eta(t) \in int A_\lambda$ *for all* $t \ge 0$ *and* $(v_{01}, \gamma_\epsilon(\theta)) \notin int A_\lambda$ *implies* $\eta(t) \equiv (v_{01}, \gamma_\epsilon(\theta))$ *for all* $t \ge 0$.
(vii) *If $\eta(t) = \eta(t, (v_{01}, \gamma_\epsilon(\theta)))$ satisfies*

$$J_\lambda(\eta(t)) \ge c_{min}^{1,D} + c_k^{2,D} - \frac{1}{3}\epsilon_0, \tag{4.14}$$

then $\eta(t) \in A_\lambda$ and $\frac{d}{dt} J_\lambda(\eta(t)) \le -\mu_\lambda$.

Proof. (i)–(ii) follow from the definition of $\widetilde{X}(v)$.

(iii) Since $J_\lambda(v_{01}, \gamma_\epsilon(\theta)) = J_{1,\lambda}(v_{01}) + J_{2,\lambda}(\gamma_\epsilon(\theta)) \le c_{min}^{1,D} + c_k^{2,D} + \epsilon$, (iii) follows from (ii).

(iv) By (i) of Lemma 2.14, it follows from (iii) that

$$\begin{aligned}(J_{1,\lambda} + J_{2,\lambda})(\eta(t)) &\le (1 - r_\lambda)^{-\frac{2(p+1)}{p-1}}(c_{min}^{1,D} + c_k^{2,D} + \epsilon)\\ &\le (1 - r_\lambda)^{-\frac{2(p+1)}{p-1}}(c_{min}^{1,\lambda} + c_k^{2,\lambda} + \epsilon_0).\end{aligned}$$

Thus for sufficiently large λ, we have (iv).

(v) By (iv) and Remark 4.6, $\eta(t) = (\eta_1(t), \eta_2(t))$ never touches a boundary $\{(v_1, v_2);\ J_{1,\lambda}(v_1)+J_{2,\lambda}(v_2) = (1-r_\lambda)^{-\frac{2(p+1)}{p-1}}(c_{min}^{1,\lambda}+c_k^{2,\lambda}+\epsilon_0)\}\cup\{(v_1, v_2);\ J_{1,\lambda}(v_1) = c_{min}^{1,D} - \frac{1}{2}\delta_0\}$. We also remark that $\eta(t)$ never touches a boundary $\{(v_1, v_2);\ J_{1,\lambda}(v_1) = \beta\}$ by Lemma 4.5(iii) and (4.10).

Thus $\eta(t) \in \partial A_\lambda$ implies

$$(J_{1,\lambda} + J_{2,\lambda})(\eta(t)) = (1 + r_\lambda)^{-\frac{2(p+1)}{p-1}}(c_{min}^{1,\lambda} + c_k^{2,\lambda} - \epsilon_0). \tag{4.15}$$

We remark that (4.15) implies

$$J_\lambda(\eta(t)) \le c_{min}^{1,\lambda} + c_k^{2,\lambda} - \epsilon_0 \le c_{min}^{1,D} + c_k^{2,D} - \frac{2}{3}\epsilon_0. \tag{4.16}$$

Since $\widetilde{X}(\eta(t)) = 0$ follows from (4.16), we have (v) from the uniqueness of solutions of (4.10). (vi)–(vii) also follow from the above arguments. □

Proof of Theorem 1.1. We choose $\epsilon \in (0, \epsilon_0)$ and let $L_\lambda = \frac{2\epsilon_0}{\mu_\lambda} > 0$. We consider $\widetilde{\gamma}(\theta) = \eta(L_\lambda, (v_{01}, \gamma_\epsilon(\theta))) : S_+^{n(k)+1} \to \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$. Then we can easily see $\widetilde{\gamma} \in \Lambda_k^\lambda$ and

$$\sup_{\theta \in S_+^{n(k)+1}} J_\lambda(\widetilde{\gamma}(\theta)) \le c_{min}^{1,D} + c_k^{2,D} - \frac{1}{3}\epsilon_0. \tag{4.17}$$

In fact, if $(v_{01}, \gamma_\epsilon(\theta)) \notin A_\lambda$, then by Lemma 4.8 we have

$$J_\lambda(\widetilde{\gamma}(\theta)) \le J_\lambda(v_{01}, \gamma_\epsilon(\theta)) \le c_{min}^{1,D} + c_k^{2,D} - \frac{2}{3}\epsilon_0.$$

So we consider the case where $(v_{01}, \gamma_\epsilon(\theta)) \in A_\lambda$. If $J_\lambda(\eta(L_\lambda, (v_{01}, \gamma_\epsilon(\theta)))) \le c_{min}^{1,D} + c_k^{2,D} - \frac{1}{3}\epsilon_0$, then (4.17) clearly holds. On the other hand, if $J_\lambda(\eta(L_\lambda, (v_{01}, \gamma_\epsilon(\theta)))) \ge c_{min}^{1,D} + c_k^{2,D} - \frac{1}{3}\epsilon_0$, then by (vii) of Lemma 4.9,

$$\begin{aligned}J_\lambda(\eta(L_\lambda, (v_{01}, \gamma_\epsilon(\theta)))) &= J_\lambda(v_{01}, \gamma_\epsilon(\theta)) + \int_0^{L_\lambda} \frac{d}{dt} J_\lambda(\eta(t))\, dt\\ &\le c_{min}^{1,D} + c_k^{2,D} + \epsilon - \mu_\lambda L_\lambda\\ &\le c_{min}^{1,D} + c_k^{2,D} - \epsilon_0,\end{aligned}$$

which is a contradiction.

Now we remark that (4.17) implies

$$c_k^\lambda = \inf_{\gamma \in \Lambda_k^\lambda} \sup_{\theta \in S_+^{n(k)+1}} J_\lambda(\gamma(\theta)) \le \sup_{\theta \in S_+^{n(k)+1}} J_\lambda(\widetilde{\gamma}(\theta)) \le c_{min}^{1,D} + c_k^{2,D} - \frac{1}{3}\epsilon_0.$$

However, by Proposition 3.7, we have $c_k^\lambda \to c_{min}^{1,D} + c_k^{2,D}$. This is a contradiction and $J_\lambda(v)$ has a critical point in A_λ. Thus Theorem 1.1 is proved. □

5. One dimensional case

In this section we consider 1 dimensional case:

$$(P_\lambda) \qquad -u'' + (\lambda^2 a(x) + 1)u = |u|^{p-1}u \quad \text{in } \mathbf{R}, \quad u(x) \in H^1(\mathbf{R}).$$

For $a(x)$, we assume

(a1) $a(x) \in C(\mathbf{R}, \mathbf{R})$, $a(x) \geq 0$ for all $x \in \mathbf{R}$ and the potential well consists of 2 intervals, that is, $int\, a^{-1}(0) = \Omega_1 \cup \Omega_2$, $\Omega_1 = (a_1, b_1)$ and $\Omega_2 = (a_2, b_2)$ $(a_1 < b_1 < a_2 < b_2)$.

(a2) $0 < \liminf_{|x|\to\infty} a(x) \leq \sup_{x\in\mathbf{R}} a(x) < \infty$.

(a) Limit equations. For $-\infty < a < b < \infty$, we consider the following equation:

$$(5.1) \qquad -u'' + u = |u|^{p-1}u \quad \text{in } (a, b), \qquad u(a) = u(b) = 0.$$

For any $n \in \mathbf{N}$, there exists a unique solution $\widetilde{w}_n(a, b; x)$ of (5.1) which has the following property (cf. [9]):

$$\widetilde{w}_n(a, b; x) \text{ possesses exactly } n-1 \text{ zeros in } (a, b) \text{ and } \widetilde{w}'_n(a, b; a) > 0.$$

Moreover we have

1° $\widetilde{w}_n(a, b; x) = \widetilde{w}_n(0, b-a; x-a)$.

2° $\widetilde{w}'_n(0, \ell_1; 0) > \widetilde{w}'_n(0, \ell_2; 0)$ for $\ell_2 > \ell_1 > 0$.

3° $\widetilde{w}_n(0, \ell; x) = (-1)^{k-1}\widetilde{w}_1(0, \frac{\ell}{n}; x - (k-1)\frac{\ell}{n})$ for $x \in ((k-1)\frac{\ell}{n}, k\frac{\ell}{n})$, $\ell > 0$, $k = 1, \cdots, n$.

We remark that the set of all solutions of (5.1) can be described as $\{\pm\widetilde{w}_n(a, b; x);\ n \in \mathbf{N}\} \cup \{0\}$. In what follows, we define

$$w_n(a, b; x) = \begin{cases} \widetilde{w}_n(a, b; x) & \text{in } (a, b), \\ 0 & \text{otherwise.} \end{cases}$$

(b) The proof of Theorem 1.2. We have two limit problems for (P_λ): for $i = 1, 2$

$$-u'' + u = |u|^{p-1}u \quad \text{in } (a_i, b_i), \quad u(a_i) = u(b_i) = 0.$$

In order to prove Theorem 1.2, for any $n_1, n_2 \in \mathbf{N}, s_1, s_2 \in \{-1, +1\}$ we will construct solutions of (P_λ) which converge to $s_i w_{n_i}(a_i, b_i; x)$ in (a_i, b_i) and to 0 elsewhere. We use the broken-geodesic method which is used in [15]. See also [19] and [26] for related arguments. We consider the case $n_1, n_2 \geq 3$. We can deal with the other cases in a similar way.

We set $t_1^0 = a_1 + \frac{b_1-a_1}{n_1}, t_2^0 = b_1 - \frac{b_1-a_1}{n_1}, t_3^0 = a_2 + \frac{b_2-a_2}{n_2}, t_4^0 = b_2 - \frac{b_2-a_2}{n_2}$, $\mathbf{t}^0 = (t_1^0, t_2^0, t_3^0, t_4^0)$. For small $\epsilon \in (0, \min\{\frac{b_1-a_1}{n_1}, \frac{b_2-a_2}{n_2}\})$ we set

$$U = \prod_{i=1}^{4} (t_i^0 - \epsilon, t_i^0 + \epsilon).$$

To find solutions of (P_λ), first we fix $\mathbf{t} = (t_1, t_2, t_3, t_4) \in U$, $t_0 = -\infty$, $t_5 = \infty$ and we solve (P_λ) in the intervals (t_i, t_{i+1}) for $i = 0, 1, \cdots, 4$:

$$(P_{\lambda,i}) \qquad -u'' + (\lambda^2 a(x) + 1)u = |u|^{p-1}u \quad \text{in } (t_i, t_{i+1}), \quad u(x) \in H_0^1(t_i, t_{i+1}).$$

Remark 5.1. We remark that $(P_{\lambda,1})$ and $(P_{\lambda,3})$ are independent of λ. Thus $(P_{\lambda,1})$ ($(P_{\lambda,3})$ resp.) has a unique solution $w_{n_1-2}(t_1, t_2; x)$ ($w_{n_2-2}(t_3, t_4; x)$ resp.), which has $n_1 - 3$ ($n_2 - 3$ resp.) zeros and $w'_{n_1-2}(t_1, t_2; t_1) > 0$ ($w'_{n_2-2}(t_3, t_4; t_3) > 0$ resp.).

The following lemma deals with $(P_{\lambda,i})$ $(i = 0, 2, 4)$, which can be shown just as in [15].

Lemma 5.2. *There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, there exists $\Lambda_0 = \Lambda_0(\delta) > 0$ such that for $\lambda > \Lambda_0$ and $(t_1, t_2, t_3, t_4) \in U$,*

(i) *$(P_{\lambda,0})$ has a unique solution $u_{0,\lambda}(t_1; x)$ which satisfies*

$$\|u_{0,\lambda}(t_1; x) - w_1(a_1, t_1; x)\|_{L^\infty(-\infty,t_1)} < \delta.$$

(ii) *$(P_{\lambda,2})$ has a unique solution $u_{2,\lambda}(t_2, t_3; x)$ which satisfies*

$$\|u_{2,\lambda}(t_2, t_3; x) - w_1(t_2, b_1; x) - w_1(a_2, t_3; x)\|_{L^\infty(t_2,t_3)} < \delta.$$

Similarly, $(P_{\lambda,2})$ has a unique solution $\widehat{u}_{2,\lambda}(t_2, t_3; x)$ which satisfies

$$\|\widehat{u}_{2,\lambda}(t_2, t_3; x) - w_1(t_2, b_1; x) + w_1(a_2, t_3; x)\|_{L^\infty(t_2,t_3)} < \delta.$$

(iii) *$(P_{\lambda,4})$ has a unique solution $u_{4,\lambda}(t_4; x)$ which satisfies*

$$\|u_{4,\lambda}(t_4; x) - w_1(t_4, b_2; x)\|_{L^\infty(t_4,\infty)} < \delta.$$

Remark 5.3. From Lemma 5.2, $u_{0,\lambda}(t_1; x)$ satisfies

$$\|u_{0,\lambda}(t_1; x) - w_1(a_1, t_1; x)\|_{L^\infty(-\infty,t_1)} \to 0 \quad \text{as } \lambda \to \infty. \tag{5.2}$$

Since $(P_{\lambda,0})$ is independent of λ in $[a_1, t_1]$, (5.2) implies

$$\|u_{0,\lambda}(t_1; x) - w_1(a_1, t_1; x)\|_{C^2([a_1,t_1])} \to 0 \quad \text{as } \lambda \to \infty.$$

By uniqueness of $u_{0,\lambda}(t_1; x)$, we find $u_{0,\lambda}(t_1; t_1)$ and $u'_{0,\lambda}(t_1; t_1)$ are continuous with respect to $t_1 \in (t_1^0 - \epsilon, t_1^0 + \epsilon)$. Similar results hold for $u_{2,\lambda}(t_2, t_3; x)$, $\widehat{u}_{2,\lambda}(t_2, t_3; x)$ and $u_{4,\lambda}(t_4; x)$.

End of the proof of Theorem 1.2. Let us outline the proof for the case $n_1, n_2 \geq 3$. We can deal with the other cases in a similar way. We define functions $u_\lambda(\mathbf{t}; x), \widehat{u}_\lambda(\mathbf{t}; x)$ for $\mathbf{t} = (t_1, \cdots, t_4) \in U$ by

$$\begin{aligned} u_\lambda(\mathbf{t}; x) &= u_{0,\lambda}(t_1; x) - w_{n_1-2}(t_1, t_2; x) + (-1)^{n_1-1} u_{2,\lambda}(t_2, t_3; x) \\ &\quad + (-1)^{n_1} w_{n_2-2}(t_3, t_4; x) + (-1)^{n_1+n_2} u_{4,\lambda}(t_4; x) \end{aligned}$$

or

$$\begin{aligned} \widehat{u}_\lambda(\mathbf{t}; x) &= u_{0,\lambda}(t_1; x) - w_{n_1-2}(t_1, t_2; x) + (-1)^{n_1-1} \widehat{u}_{2,\lambda}(t_2, t_3; x) \\ &\quad + (-1)^{n_1-1} w_{n_2-2}(t_3, t_4; x) + (-1)^{n_1+n_2-1} u_{4,\lambda}(t_4; x). \end{aligned}$$

We deal with $u_\lambda(\mathbf{t}; x)$. We can deal with $\widehat{u}_\lambda(\mathbf{t}; x)$ in a similar way. We observe that $u_\lambda(\mathbf{t}; x)$ is a solution of (P_λ) if and only if $u_\lambda(\mathbf{t}; x)$ is of class C^1 with respect to x. We define a function $F_\lambda(\mathbf{t}) \in C(U, \mathbf{R}^4)$ by

$$F_\lambda(\mathbf{t}) = \begin{bmatrix} F_{1,\lambda}(\mathbf{t}) \\ F_{2,\lambda}(\mathbf{t}) \\ F_{3,\lambda}(\mathbf{t}) \\ F_{4,\lambda}(\mathbf{t}) \end{bmatrix} = \begin{bmatrix} w'_{n_1-2}(t_1, t_2; t_1) + u'_{0,\lambda}(t_1; t_1) \\ u'_{2,\lambda}(t_2, t_3; t_2) + (-1)^{n_1-1} w'_{n_1-2}(t_1, t_2; t_2) \\ w'_{n_2-2}(t_3, t_4; t_3) + u'_{2,\lambda}(t_2, t_3; t_3) \\ u'_{2,\lambda}(t_4; t_4) + (-1)^{n_2-1} w'_{n_2-2}(t_3, t_4; t_4) \end{bmatrix} : U \to \mathbf{R}^4 .$$

Then $u_\lambda(\mathbf{t}; x)$ is of class C^1 if and only if $F_\lambda(\mathbf{t}) = 0$. We will show $deg(F_\lambda, U, 0) = 1$. We remark that $\partial U = \bigcup_{i=1}^4 (\{t_i = t_i^0 - \epsilon\} \cup \{t_i = t_i^0 + \epsilon\})$. For $i = 1, 2, 3, 4$, we have

$$F_{i,\lambda}(\mathbf{t}) < 0 \quad \text{if} \quad t_i = t_i^0 - \epsilon, \qquad F_{i,\lambda}(\mathbf{t}) > 0 \quad \text{if} \quad t_i = t_i^0 + \epsilon \tag{5.3}$$

for large λ. In fact, when $t_1 = t_1^0 - \epsilon$, by the properties of $w_n(a, b; x)$, we find

$$\begin{aligned} F_{1,\lambda}(\mathbf{t}) &= w'_{n_1-2}(t_1^0 - \epsilon, t_2; t_1^0 - \epsilon) + u'_{0,\lambda}(t_1^0 - \epsilon; t_1^0 - \epsilon) \\ &\leq w'_{n_1-2}(t_1^0 - \epsilon, t_2^0 - \epsilon; t_1^0 - \epsilon) + u'_{0,\lambda}(t_1^0 - \epsilon; t_1^0 - \epsilon) \\ &= w'_1(a_1, t_1^0; a_1) + u'_{0,\lambda}(t_1^0 - \epsilon; t_1^0 - \epsilon). \end{aligned}$$

By Remark 5.3, we observe

$$w'_1(a_1, t_1^0; a_1) + u'_{0,\lambda}(t_1^0 - \epsilon; t_1^0 - \epsilon) \to w'_1(a_1, t_1^0; a_1) + w'_1(a_1, t_1^0 - \epsilon; t_1^0 - \epsilon) < 0 \text{ as } \lambda \to \infty.$$

We can show the other properties in (5.3) in a similar way. By (5.3), $deg(F_\lambda, U, 0)$ is well-defined for large λ and $deg(F_\lambda, U, 0) = 1$. Therefore there exists $\mathbf{t}_\lambda^* \in U$ such that $u_\lambda(\mathbf{t}_\lambda^*; x)$ is a solution of (P_λ). Since $\epsilon > 0$ is arbitrary, we can show $\mathbf{t}_\lambda^* \to \mathbf{t}^0$ as $\lambda \to \infty$; that is, $u_\lambda(\mathbf{t}_\lambda^*; x)$ converges to $w_{n_1}(a_1, b_1; x)$ in (a_1, b_1), to $(-1)^{n_1-1} w_{n_2}(a_2, b_2; x)$ in (a_2, b_2) and to 0 elsewhere. □

6. Proofs of Theorems 1.3 and 1.5

In this section we give proofs of Theorem 1.3 and Theorem 1.5. Since the exponent p plays an important role, we write dependence of $J_\lambda, J_{i,D}$ on p explicitly in this section and use the notation

$$\begin{aligned} J_\lambda(p; v_1, v_2) &= J_\lambda(v_1, v_2) \quad \text{for } (v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}, \\ J_{i,D}(p; v_i) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{1}{\|v_i\|_{L^{p+1}(\Omega_i)}}\right)^{\frac{2(p+1)}{p-1}} \quad \text{for } v_i \in \Sigma_{i,D}. \end{aligned}$$

We define

$$c_p^\lambda = \inf_{(v_1,v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} J_\lambda(p; v_1, v_2), \quad c_p(\Omega_i) = \inf_{v_i \in \Sigma_{i,D}} J_{i,D}(p; v_i).$$

Since $J_\lambda(p; v_1, v_2)$, $J_{i,D}(p; v_i)$ satisfy the Palais-Smale condition, c_p^λ and $c_p(\Omega_i)$ are critical values of $J_\lambda(p; v_1, v_2)$ and $J_{i,D}(p; v_i)$. First we observe

Lemma 6.1. (i) $c_p^\lambda < c_p(\Omega_1) + c_p(\Omega_2)$ *for all* $\lambda \geq 1$.
(ii) $c_p^\lambda \to c_p(\Omega_1) + c_p(\Omega_2)$ *as* $\lambda \to \infty$.

Proof. Let $(v_{1,\lambda}, v_{2,\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ be a minimizer of $J_\lambda(p; v_1, v_2)$, that is, $J_\lambda(p; v_{1,\lambda}, v_{2,\lambda}) = c_p^\lambda$. (i) Since $\Sigma_{1,D} \times \Sigma_{2,D} \subset \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$, we have

$$\begin{aligned} c_p^\lambda &= \inf_{(v_1,v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} J_\lambda(p; v_1, v_2) \leq \inf_{(v_1,v_2) \in \Sigma_{1,D} \times \Sigma_{2,D}} J_\lambda(p; v_1, v_2) \\ &= \inf_{(v_1,v_2) \in \Sigma_{1,D} \times \Sigma_{2,D}} \big(J_{1,D}(p; v_1) + J_{2,D}(p; v_2)\big) = c_p(\Omega_1) + c_p(\Omega_2). \end{aligned}$$

Since $(v_{1,\lambda}, v_{2,\lambda})$ corresponds to a positive solution $u(x)$ of $-\Delta u + (\lambda^2 a(x) + 1)u = g(x, u)$ in $\mathbf{R}^N$, we have $v_{i,\lambda} \in \Sigma_{i,\lambda} \setminus \Sigma_{i,D}$ and we have strict inequality.

(ii) By Proposition 2.17, there exist a sequence $\lambda_n \to \infty$ and critical points $v_i \in \Sigma_{i,D}$ of $J_{i,D}$ $(i = 1, 2)$ such that

$$(v_{1,\lambda_n}, v_{2,\lambda_n}) \to (v_1, v_2) \quad \text{strongly in } H^1(\Omega_1') \oplus H^1(\Omega_2')$$

and

$$J_{\lambda_n}(p; v_{1,\lambda_n}, v_{2,\lambda_n}) \to J_{1,D}(p; v_1) + J_{2,D}(p; v_2) \geq c_p(\Omega_1) + c_p(\Omega_2).$$

Therefore by (i), $c_p^{\lambda_n} \to c_p(\Omega_1) + c_p(\Omega_2)$. Since the limit does not depend on the subsequence, conclusion (ii) holds. □

To study the case $p \sim \frac{N+2}{N-2}$, we introduce the notation

$$c_p(Q) = \inf_{u \in H_0^1(Q), \|u\|_{H^1(Q)}=1} \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{1}{\|u\|_{L^{p+1}(Q)}}\right)^{\frac{2(p+1)}{p-1}} \quad \text{for } Q \subset \mathbf{R}^N .$$

$c_p(Q)$ has the following properties:

Lemma 6.2. (0) *$Q \subset Q'$ implies $c_p(Q') \le c_p(Q)$.*

(i) *$c_{\frac{N+2}{N-2}} \equiv c_{\frac{N+2}{N-2}}(Q)$ does not depend on Q.*

(ii) *$c_{\frac{N+2}{N-2}} = c_{\frac{N+2}{N-2}}(Q)$ is never achieved in a proper subset of $\mathbf{R}^N$.*

(iii) *For any $Q \subset \mathbf{R}^N$, $c_p(Q) \to c_{\frac{N+2}{N-2}}$ as $p \to \frac{N+2}{N-2} - 0$.*

Proof. Introducing

$$d_p(Q) = \inf_{u \in H_0^1(Q), \|u\|_{H^1(Q)}=1} \frac{1}{\|u\|^2_{L^{p+1}(Q)}} \quad \text{for } Q \subset \mathbf{R}^N,$$

(0)–(iii) are equivalent to

(0) $Q \subset Q'$ implies $d_p(Q') \le d_p(Q)$.

(i) $d_{\frac{N+2}{N-2}} \equiv d_{\frac{N+2}{N-2}}(Q)$ does not depend on Q.

(ii) $d_{\frac{N+2}{N-2}} = d_{\frac{N+2}{N-2}}(Q)$ is never achieved in a proper subset of $\mathbf{R}^N$.

(iii) For any $Q \subset \mathbf{R}^N$, $d_p(Q) \to d_{\frac{N+2}{N-2}}$ as $p \to \frac{N+2}{N-2} - 0$,

which can be derived as in [7] (see page 81 and Lemma 4.1). □

For $r > 0$ we set

$$\Omega_i^+ = \{x \in \mathbf{R}^N; dist(x, \Omega_i) < r\}, \quad \Omega_i^- = \{x \in \Omega_i; dist(x, \partial\Omega_i) > r\}.$$

We choose $r > 0$ small enough so that

(i) The inclusions $\Omega_i^- \subset \Omega_i$ and $\Omega_i \subset \Omega_i^+$ are homotopy equivalent.

(ii) $B_r(x) \subset \Omega_i$ for all $x \in \Omega_i^-$.

By the choice of $r > 0$, we have

Corollary 6.3. (i) $c_p^\lambda < c_p(\Omega_1) + c_p(\Omega_2) < 2c_p(B_r)$.

(ii) *There exists $p_1 \in (1, \frac{N+2}{N-2})$ such that for any $p \in (p_1, \frac{N+2}{N-2})$, there exists $\Lambda_1(p) > 0$ such that*

$$c_p(B_r) < c_p^\lambda \quad \textit{for all } \lambda \ge \Lambda_1(p).$$

Proof. (i) follows from Lemma 6.1 (i) and Lemma 6.2 (0). (ii) By Lemma 6.2 (iii), we have $c_p(\Omega_1) + c_p(\Omega_2) \to 2c_{\frac{N+2}{N-2}}$, $c_p(B_r) \to c_{\frac{N+2}{N-2}}$. Thus (ii) follows from Lemma 6.1 (ii). □

For $v \in \Sigma_{i,\lambda}$ we set

$$\beta_i(p; v) = \frac{\int_{\Omega_i'} |v|^{p+1} x \, dx}{\int_{\Omega_i'} |v|^{p+1} \, dx} : \Sigma_{i,\lambda} \to \mathbf{R}^N .$$

We have the following.

Lemma 6.4. *There exists $\delta > 0$ such that if $p \in (\frac{N+2}{N-2} - \delta, \frac{N+2}{N-2})$ and $v \in \Sigma_{i,D}$ satisfies*

$$J_{i,D}(p; v) \le c_{\frac{N+2}{N-2}} + \delta,$$

then $\beta_i(p; v) \in \Omega_i^+$.

We remark that

$$\inf_{v\in\Sigma_{i,D}} J_{i,D}(p;v) = c_p(\Omega_i) \to c_{\frac{N+2}{N-2}} \quad \text{as} \quad p \to \frac{N+2}{N-2} - 0.$$

Thus $\{v \in \Sigma_{i,D};\ J_{i,D}(p;v) \le c_{\frac{N+2}{N-2}} + \delta\}$ is not empty for p close to $\frac{N+2}{N-2}$.

Proof. We argue indirectly and assume that there exist p_n, $v_n \in \Sigma_{i,D}$ such that

$$p_n \to \frac{N+2}{N-2}, \quad J_{i,D}(p_n;v_n) \to c_{\frac{N+2}{N-2}}, \quad \beta_i(p_n;v_n) \notin \Omega_i^+. \tag{6.1}$$

By Hölder's inequality,

$$\begin{aligned} & J_{i,D}(\tfrac{N+2}{N-2};v_n) \\ = \ & \frac{1}{N}\frac{1}{\|v_n\|^N_{L^{\frac{2N}{N-2}}(\Omega_i)}} \le \frac{1}{N}|\Omega_i|^{-\frac{\frac{N+2}{N-2}-p_n}{p_n+1}}\frac{1}{\|v_n\|^N_{L^{p_n+1}(\Omega_i)}} \\ = \ & \frac{1}{N}|\Omega_i|^{-\frac{\frac{N+2}{N-2}-p_n}{p_n+1}}\left(\frac{1}{2}-\frac{1}{p_n+1}\right)^{-\frac{p_n-1}{p_n+1}\frac{N}{2}} J_{i,D}(p_n;v_n)^{\frac{p_n-1}{p_n+1}\frac{N}{2}} \\ \to \ & c_{\frac{N+2}{N-2}}. \end{aligned} \tag{6.2}$$

Since $J_{i,D}(\frac{N+2}{N-2};v) \ge c_{\frac{N+2}{N-2}}$ for all $v \in \Sigma_{i,D}$, it follows from (6.2) that

$$\frac{1}{\|v_n\|^2_{L^{\frac{2N}{N-2}}(\Omega_i)}} \to d_{\frac{N+2}{N-2}},$$

where $d_{\frac{N+2}{N-2}}$ is defined in the proof of Lemma 6.2. From the well-known compactness result (cf. Struwe [24] and Lions [18]), it follows that there exist $r_n \to 0$, $(x_n)_{n=1}^\infty \subset \Omega_i$ and a function $w_0(x) \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ satisfying $\|\nabla w_0\|_{L^2(\mathbf{R}^N)} = 1$ and $\|w_0\|^2_{L^{\frac{2N}{N-2}}(\mathbf{R}^N)} = d^{-1}_{\frac{N+2}{N-2}}$ such that $r_n^{\frac{N-2}{2}} v_n(r_n x + x_n) \to w_0(x)$ strongly in $\mathcal{D}^{1,2}(\mathbf{R}^N)$. Thus

$$\beta_i(p_n;v_n) \in \Omega_i^+ \quad \text{for large } n.$$

This is a contradiction to assumption (6.1). □

A following proposition is a key of the proof of Theorem 1.3.

Proposition 6.5. *There exists $p_2 \in (1, \frac{N+2}{N-2})$ such that for any $p \in (p_2, \frac{N+2}{N-2})$, there exists $\Lambda_2(p) \ge 1$ such that $(\beta_1(p;v_1), \beta_2(p;v_2)) \in \Omega_1^+ \times \Omega_2^+$ for all $\lambda \ge \Lambda_2(p)$ and $(v_1,v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ satisfying $J_\lambda(p;v_1,v_2) \le 2c_p(B_r)$.*

We remark by Corollary 6.3 (i) that the level set

$$\begin{aligned} & [J_\lambda(p;v_1,v_2) \le 2c_p(B_r)]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}} \\ = \ & \{(v_1,v_2) \in \Sigma_{1,\lambda}\times\Sigma_{2,\lambda}; J_\lambda(p;v_1,v_2) \le 2c_p(B_r)\} \end{aligned}$$

is not empty.

Proof. We choose $p_2 \in (p_1, \frac{N+2}{N-2})$ such that

$$2c_p(B_r) - c_p(\Omega_i) \le c_{\frac{N+2}{N-2}} + \delta \quad \text{for } i = 1,2 \text{ and } p \in (p_2, \frac{N+2}{N-2}), \tag{6.3}$$

where $\delta > 0$ is given in Lemma 6.4. We will show the conclusion of Proposition 6.5 holds for p_2.

Again we argue indirectly, and we assume that there exist $p \in (p_2, \frac{N+2}{N-2})$, $\lambda_n \to \infty$ and $(v_{1n}, v_{2n}) \in \Sigma_{1,\lambda_n} \times \Sigma_{2,\lambda_n}$ such that

$$J_{\lambda_n}(p; v_{1n}, v_{2n}) \le 2c_p(B_r), \quad (\beta_1(p; v_{1n}), \beta_2(p; v_{2n})) \notin \Omega_1^+ \times \Omega_2^+.$$

By Proposition 2.15, (v_{1n}, v_{2n}) satisfies $\|v_{in}\|_{L^{p+1}(\Omega_i')} \ge \delta_M$, where $\delta_M > 0$ is given in Proposition 2.15 with $M = 2c_{\frac{N+2}{N-2}} + 1$. Extracting a subsequence if necessary, we may assume

$$v_{in} \to v_{i0} \quad \text{strongly in } L^{p+1}(\Omega_i') \text{ and weakly in } H^1(\Omega_i').$$

Thus we have $\|v_{i0}\|_{L^{p+1}(\Omega_i')} \ge \delta_M$ and

$$(\beta_1(p; v_{10}), \beta_2(p; v_{20})) \notin \Omega_1^+ \times \Omega_2^+. \tag{6.4}$$

We also remark that for any compact set $D \subset \Omega_i' \setminus \overline{\Omega_i}$

$$\|v_{in}\|^2_{L^2(D)} \le \frac{1}{\lambda_n^2 \inf_{x\in D} a(x)} \|v_{in}\|^2_{\lambda_n, \Omega_i'} \to 0.$$

In particular, we have $v_{i0} \in H_0^1(\Omega_i)$. Thus we have $\frac{v_{i0}}{\|v_{i0}\|_{H^1(\Omega_i)}} \in \Sigma_{i,D}$, and it follows from $\|v_{i0}\|_{H^1(\Omega_i)} \le \liminf_{n\to\infty} \|v_{in}\|_{H^1(\Omega_i')} \le \liminf_{n\to\infty} \|v_{in}\|_{\lambda_n, \Omega_i'} = 1$ that

$$\begin{aligned} J_{i,D}\Big(\frac{v_{i0}}{\|v_{i0}\|_{H^1(\Omega_i)}}\Big) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{\|v_{i0}\|_{H^1(\Omega_i)}}{\|v_{i0}\|_{L^{p+1}(\Omega_i)}}\right)^{\frac{2(p+1)}{p-1}} \\ &\le \left(\frac{1}{2} - \frac{1}{p+1}\right) \|v_{i0}\|_{L^{p+1}(\Omega_i)}^{-\frac{2(p+1)}{p-1}}. \end{aligned}$$

Thus we have

$$\begin{aligned} & J_{1,D}\Big(\frac{v_{10}}{\|v_{10}\|_{H^1(\Omega_1)}}\Big) + J_{2,D}\Big(\frac{v_{20}}{\|v_{20}\|_{H^1(\Omega_2)}}\Big) \\ \le\ & \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\|v_{10}\|_{L^{p+1}(\Omega_1')}^{-\frac{2(p+1)}{p-1}} + \|v_{20}\|_{L^{p+1}(\Omega_2')}^{-\frac{2(p+1)}{p-1}}\right) \\ =\ & \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\|v_{1n}\|_{L^{p+1}(\Omega_1')}^{-\frac{2(p+1)}{p-1}} + \|v_{2n}\|_{L^{p+1}(\Omega_2')}^{-\frac{2(p+1)}{p-1}}\right) \\ \le\ & \lim_{n\to\infty} J_{\lambda_n}(p; v_{1n}, v_{2n}) \le 2c_p(B_r). \end{aligned}$$

Here we used the fact that

$$\begin{aligned} J_\lambda(p; v_1, v_2) &= \sup_{s,t\ge 0} I_\lambda(p; sv_1, tv_2) \ge \sup_{s,t\ge 0} \widetilde{I}_\lambda(p; sv_1, tv_2) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\|v_1\|_{L^{p+1}(\Omega_1')}^{-\frac{2(p+1)}{p-1}} + \|v_2\|_{L^{p+1}(\Omega_2')}^{-\frac{2(p+1)}{p-1}}\right), \end{aligned}$$

where $\widetilde{I}_\lambda(p; v_1, v_2) = \frac{1}{2}\|v_1\|^2_{\lambda,\Omega_1'} + \frac{1}{2}\|v_2\|^2_{\lambda,\Omega_2'} - \frac{1}{p+1}\|v_1\|^{p+1}_{L^{p+1}(\Omega_1')} - \frac{1}{p+1}\|v_2\|^{p+1}_{L^{p+1}(\Omega_2')}$. Since $c_p(\Omega_i) \le J_{i,D}(\frac{v_{i0}}{\|v_{i0}\|_{H^1(\Omega_i)}})$, we have

$$J_{1,D}\Big(\frac{v_{10}}{\|v_{10}\|_{H^1(\Omega_1)}}\Big) \le 2c_p(B_r) - c_p(\Omega_2), \quad J_{2,D}\Big(\frac{v_{20}}{\|v_{20}\|_{H^1(\Omega_2)}}\Big) \le 2c_p(B_r) - c_p(\Omega_1).$$

By (6.3), we have $J_{i,D}(\frac{v_{i0}}{\|v_{i0}\|_{H^1(\Omega_i)}}) \le c_{\frac{N+2}{N-2}} + \delta$. Thus by Lemma 6.4 we have $(\beta_1(p; v_{10}), \beta_2(p; v_{20})) \in \Omega_1^+ \times \Omega_2^+$, which is in contradiction to (6.4). □

In order to prove Theorem 1.3, we need the following lemma.

Lemma 6.6. *Let A, B, X be topological spaces and suppose that there exist maps $\alpha : A \to X$ and $\beta : X \to B$ such that $\beta \circ \alpha : A \to B$ is a homotopy equivalence. Then* $\mathrm{cat}(A) \le \mathrm{cat}(X)$.

Proof. Suppose that $\mathrm{cat}(X) = k$. Then there exist closed sets $X_1, \cdots, X_k \subset X$ such that $X \subset X_1 \cup \cdots \cup X_k$ and each X_i are contractible in X. We set $A_i = \alpha^{-1}(X_i) \subset A$. It follows that

$$\mathrm{cat}(A) \le \sum_{i=1}^{k} \mathrm{cat}(A_i).$$

We claim that, if $A_i \ne \emptyset$, A_i is contractible in A; that is, $\mathrm{cat}(A_i) = 1$. Since X_i are contractible in X, there exist $H_i \in C([0,1] \times X_i, X)$ and $x_{i0} \in X$ such that $H_i(0,x) = x$, $H_i(1,x) = x_{i0}$ for all $x \in X_i$. Furthermore, since $\beta \circ \alpha : A \to B$ is a homotopy equivalence, there exist continuous maps $\varphi : B \to A$ and $G \in C([0,1] \times A, A)$ such that $G(0,x) = x$, $G(1,x) = \varphi(\beta(\alpha(x)))$ for all $x \in X_i$. We define $F_i \in C([0,2] \times A_i, A)$ by

$$F_i(t,x) = \begin{cases} G(t,x) & \text{if } t \in [0,1] \text{ and } x \in A_i, \\ \varphi(\beta(H_i(t-1, \alpha(x)))) & \text{if } t \in [1,2] \text{ and } x \in A_i. \end{cases}$$

Then F_i satisfies $F_i(0,x) = x$, $F_i(2,x) = \varphi(\beta(x_{i0}))$ for all $x \in A_i$. Therefore, A_i is contractible in A; that is, $\mathrm{cat}(A_i) = 1$. Consequently we get $\mathrm{cat}(A) \le k = \mathrm{cat}(X)$. □

Set $M = 2c_{\frac{N+2}{N-2}} + 1$ and let $\nu_M > 0$ be a constant given in Proposition 9.1 and Corollary 9.3. We set

$$X_\lambda^{++} = \{(v_1,v_2) \in [J_\lambda(p; v_1, v_2) \le 2c_p(B_r)]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}}; \varphi_-(v_1) \le \nu_M,\ \psi_-(v_2) \le \nu_M\},$$
$$X_\lambda^{+-} = \{(v_1,v_2) \in [J_\lambda(p; v_1, v_2) \le 2c_p(B_r)]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}}; \varphi_-(v_1) \le \nu_M,\ \psi_+(v_2) \le \nu_M\},$$
$$X_\lambda^{-+} = \{(v_1,v_2) \in [J_\lambda(p; v_1, v_2) \le 2c_p(B_r)]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}}; \varphi_+(v_1) \le \nu_M,\ \psi_-(v_2) \le \nu_M\},$$
$$X_\lambda^{--} = \{(v_1,v_2) \in [J_\lambda(p; v_1, v_2) \le 2c_p(B_r)]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}}; \varphi_+(v_1) \le \nu_M,\ \psi_+(v_2) \le \nu_M\},$$

where $\varphi_\pm(v_1) = \|v_{1\pm}\|^{p+1}_{L^{p+1}(\Omega_1')}$, $\psi_\pm(v_2) = \|v_{2\pm}\|^{p+1}_{L^{p+1}(\Omega_2')}$.

Proposition 6.7. *Assume* (a1)–(a2), (1.5) *and* $N \ge 3$. *Then there exist* $p_2 \in (1, \frac{N+2}{N-2})$ *and* $\Lambda_3(p) \ge 1$ *such that for* $p \in (p_2, \frac{N+2}{N-2})$ *and* $\lambda \ge \Lambda_3(p)$,

$$\mathrm{cat}(X_\lambda^{\sigma_1\sigma_2}) \ge \mathrm{cat}(\Omega_1 \times \Omega_2) \qquad \textit{for all } \ \sigma_1, \sigma_2 \in \{+,-\}.$$

Moreover J_λ *has at least* $\mathrm{cat}(\Omega_1 \times \Omega_2)$ *critical points in* X_λ^{++} (X_λ^{+-}, X_λ^{-+}, X_λ^{--} *respectively).*

Proof. Let $\widetilde{U} \in H_0^1(B_r)$ be a unique solution of

$$-\Delta u + u = u^p \quad \text{in } B_r, \qquad u > 0 \quad \text{in } B_r, \qquad u = 0 \quad \text{on } \partial B_r.$$

We set

$$U_y(x) = \frac{\widetilde{U}(x-y)}{\|\widetilde{U}\|_{\lambda, B_r}} \in H_0^1(B_r(y)).$$

We note that for all $(y,z) \in \Omega_1^- \times \Omega_2^-$,

$$J_\lambda(p; \sigma_1 U_y, \sigma_2 U_z) = 2c_p(B_r) \qquad \text{for all } \ \sigma_1, \sigma_2 \in \{+,-\},$$
$$(\beta_1(p; \sigma_1 U_y), \beta_2(p; \sigma_2 U_z)) = (y,z) \qquad \text{for all } \ \sigma_1, \sigma_2 \in \{+,-\}.$$

Let p_2 and $\Lambda_2(p)$ be constants given in Proposition 6.5. For any $p \in [p_2, \frac{N+2}{N-2})$ and $\lambda \geq \Lambda_2(p)$, we define two maps by

$$\alpha^{\sigma_1\sigma_2}(y,z) = (\sigma_1 U_y, \sigma_2 U_z) : \Omega_1^- \times \Omega_2^- \to X_\lambda^{\sigma_1\sigma_2},$$
$$\beta^{\sigma_1\sigma_2}(v_1,v_2) = (\beta_1(p;v_1), \beta_2(p;v_2)) : X_\lambda^{\sigma_1\sigma_2} \to \Omega_1^+ \times \Omega_2^+.$$

By Proposition 6.5, these maps are well defined and $\beta^{\sigma_1\sigma_2} \circ \alpha^{\sigma_1\sigma_2}(y,z) : \Omega_1^- \times \Omega_2^- \to \Omega_1^+ \times \Omega_2^+$ is an identity. Therefore, from Lemma 6.6 we find

$$\mathrm{cat}(X_\lambda^{\sigma_1\sigma_2}) \geq \mathrm{cat}(\Omega_1^- \times \Omega_2^-) = \mathrm{cat}(\Omega_1 \times \Omega_2).$$

Moreover by Corollary 9.3 there exists $\Lambda_3(p) \geq \Lambda_2(p)$ such that for $\lambda \geq \Lambda_3(p)$ there exists a deformation $\eta(t,v)$ in $X_\lambda^{\sigma_1\sigma_2}$ with the following properties:

(i) $\eta(t,v) \in X_\lambda^{\sigma_1\sigma_2}$ for all $(t,v) \in [0,1] \times X_\lambda^{\sigma_1\sigma_2}$,
(ii) $J_\lambda(\eta(t,v)) \leq J_\lambda(v)$ for all $(t,v) \in [0,1] \times X_\lambda^{\sigma_1\sigma_2}$,
(iii) If $b < M \equiv 2c_{\frac{N+2}{N-2}} + 1$ is not a critical value of J_λ, then there exists $\epsilon > 0$ such that

$$J_\lambda(\eta(1,v)) < b - \epsilon \qquad \text{for } v \in [J_\lambda < b + \epsilon].$$

Thus J_λ has at least $\mathrm{cat}(\Omega_1 \times \Omega_2)$ critical points in $X_\lambda^{\sigma_1\sigma_2}$. □

End of the proof of Theorem 1.3. By Proposition 6.7, J_λ has at least $\mathrm{cat}(\Omega_1 \times \Omega_2)$ critical points in X_λ^{++}. By Proposition 2.19 (iii) those critical points are positive, and by Corollary 6.3 we have $\inf J_\lambda(p; v_1, v_2) = c_p^\lambda \geq c_p(B_r)$. In [3], they showed that (P_λ) possesses at least $\mathrm{cat}(\Omega_1 \cup \Omega_2) = \mathrm{cat}(\Omega_1) + \mathrm{cat}(\Omega_2)$ positive solutions. We remark that their solutions $u(x)$ satisfy $\Psi_\lambda(u) < c_p(B_r)$. Thus their solutions are different from those obtained in Proposition 6.7. Thus (P_λ) has at least $\mathrm{cat}(\Omega_1) + \mathrm{cat}(\Omega_2) + \mathrm{cat}(\Omega_1 \times \Omega_2)$ positive solutions, and the proof of Theorem 1.3 is completed. □

End of the proof of Theorem 1.5. By Proposition 6.7, J_λ has at least $\mathrm{cat}(\Omega_1 \times \Omega_2)$ critical points $v_\lambda = (v_{1,\lambda}, v_{2,\lambda})$ in X_λ^{+-}. From Propositions 2.17 and 2.19, extracting subsequence $\lambda_n \to \infty$, v_{i,λ_n} converges to a critical point $v_{i\infty} \in \Sigma_{i,D}$ of $J_{i,D}$ $(i = 1, 2)$ and $v_{i\infty}$ satisfies $v_{1\infty} > 0$ in Ω_1, $v_{2\infty} < 0$ in Ω_2. Thus Theorem 1.5 is proved. □

7. Proofs of Lemmas 2.9, 2.10 and Propositions 2.15, 2.16

In this section we give proofs of various lemmas stated in Section 2 without proofs.

Proof of Lemma 2.9. Let $V(x)$ be a function satisfying (2.15). (i) is nothing but (2.16) in Lemma 2.6.

(ii) Set $v = Q_{V,\lambda}(u)$, $w = Q_{0,\lambda}(u)$. Since $v - w \in H_0^1(\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2'))$, we have from the variational characterization (2.18) of $v = Q_{V,\lambda}(u)$ that

$$\langle v, v - w \rangle_{\lambda, \mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')} - \int_{\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')} V(x) v (v - w)\, dx = 0. \tag{7.1}$$

Similarly from the variational characterization for $w = Q_{0,\lambda}(u)$, we have

$$\langle w, v - w \rangle_{\lambda, \mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')} = 0. \tag{7.2}$$

Subtracting (7.2) from (7.1),

$$\begin{aligned}\|v-w\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} &= \int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} V(x)v(v-w)\,dx\\ \le\ & \left(\int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}|V(x)|v^2\,dx\right)^{\frac12}\left(\int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}|V(x)||v-w|^2\,dx\right)^{\frac12}.\end{aligned}$$

By (2.15)

$$\begin{aligned}&\le \frac49\|v\|_{L^2(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))}\|v-w\|_{L^2(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))}\\ &\le r_\lambda\|v\|_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}\|v-w\|_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}.\end{aligned}$$

Therefore by (i) we have

$$\|v-w\|_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}\le r_\lambda\|v\|_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}\le r_\lambda C_3\|u\|_{\lambda,\Omega_1'\cup\Omega_2'}.$$

Thus we get (ii).

(iii) For (iii) we need the following proposition.

Proposition 7.1. *For any open sets O_1, O_2 satisfying $\Omega_1'\subset\subset O_1$, $\Omega_2'\subset\subset O_2$, there exist $c_\lambda(O_1)$, $c_\lambda(O_2)>0$ such that*

$$c_\lambda(O_1),\ c_\lambda(O_2)\to 0\quad \text{as } \lambda\to\infty,$$

$$\|Q_{0,\lambda}(u_1,0)\|_{\lambda,\mathbf{R}^N\setminus(O_1\cup\Omega_2')}\le c_\lambda(O_1)\|u_1\|_{\lambda,\Omega_1'}\quad \text{for all } u_1\in H^1(\Omega_1'), \tag{7.3}$$

$$\|Q_{0,\lambda}(0,u_2)\|_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup O_2)}\le c_\lambda(O_2)\|u_2\|_{\lambda,\Omega_2'}\quad \text{for all } u_2\in H^1(\Omega_2'). \tag{7.4}$$

We will give a proof of Proposition 7.1 in Section 8.

For a proof of (iii) we take open sets O_1, O_2 such that $\Omega_1'\subset\subset O_1$, $\Omega_2'\subset\subset O_2$ and $O_1\cap O_2=\emptyset$. By Proposition 7.1, there exist constants $c_\lambda(O_1)$, $c_\lambda(O_2)>0$ satisfying (7.3)–(7.4).

We have

$$\begin{aligned}&\big\langle Q_{0,\lambda}(u_1,0),Q_{0,\lambda}(0,u_2)\big\rangle_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}\\ =\ &\big\langle Q_{0,\lambda}(u_1,0),Q_{0,\lambda}(0,u_2)\big\rangle_{\lambda,O_1\setminus\Omega_1'}+\big\langle Q_{0,\lambda}(u_1,0),Q_{0,\lambda}(0,u_2)\big\rangle_{\lambda,O_2\setminus\Omega_2'}\\ &+\big\langle Q_{0,\lambda}(u_1,0),Q_{0,\lambda}(0,u_2)\big\rangle_{\lambda,\mathbf{R}^N\setminus(O_1\cup O_2)}.\end{aligned}$$

By (7.3)–(7.4) and (i)

$$\begin{aligned}&\Big|\big\langle Q_{0,\lambda}(u_1,0),Q_{0,\lambda}(0,u_2)\big\rangle_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}\Big|\\ \le\ &\|Q_{0,\lambda}(u_1,0)\|_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}\|Q_{0,\lambda}(0,u_2)\|_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup O_2)}\\ &+\|Q_{0,\lambda}(u_1,0)\|_{\lambda,\mathbf{R}^N\setminus(O_1\cup\Omega_2')}\|Q_{0,\lambda}(0,u_2)\|_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}\\ &+\|Q_{0,\lambda}(u_1,0)\|_{\lambda,\mathbf{R}^N\setminus(O_1\cup\Omega_2')}\|Q_{0,\lambda}(0,u_2)\|_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup O_2)}\\ \le\ &(C_3c_\lambda(O_1)+C_3c_\lambda(O_2)+c_\lambda(O_1)c_\lambda(O_2))\|u_1\|_{\lambda,\Omega_1'}\|u_2\|_{\lambda,\Omega_2'}.\end{aligned}$$

Thus we get (iii).

(iv) Recalling Remark 2.8 (i), (iv) follows from (ii). □

Next we show

Proof of Lemma 2.10. (i) We can write

$$\begin{aligned} I_\lambda(u_1,u_2) &= \frac{1}{2}\|u_1\|^2_{\lambda,\Omega_1'} + \frac{1}{2}\|u_2\|^2_{\lambda,\Omega_2'} - \frac{1}{p+1}\|u_1\|^{p+1}_{L^{p+1}(\Omega_1')} - \frac{1}{p+1}\|u_2\|^{p+1}_{L^{p+1}(\Omega_2')} \\ &\quad + \frac{1}{2}\|w_\lambda(u_1,u_2)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} - \int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} G(x,w_\lambda(u_1,u_2))\,dx. \end{aligned}$$

Recalling (2.24)–(2.25), we have

$$\begin{aligned} & I_\lambda(u_1,u_2) - I_{1,\lambda}(u_1) - I_{2,\lambda}(u_2) \\ = \; & \frac{1}{2}\|w_\lambda(u_1,u_2)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} - \frac{1}{2}\|Q_{0,\lambda}(u_1,0)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} \\ & - \frac{1}{2}\|Q_{0,\lambda}(0,u_2)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} - \int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} G(x,w_\lambda(u_1,u_2))\,dx. \end{aligned}$$

By (2.2)–(2.3),

$$\begin{aligned} & |I_\lambda(u_1,u_2) - I_{1,\lambda}(u_1) - I_{2,\lambda}(u_2)| \\ \le \; & \frac{1}{2}\Big|\|w_\lambda(u_1,u_2)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} - \|Q_{0,\lambda}(u_1,0)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} \\ & - \|Q_{0,\lambda}(0,u_2)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}\Big| + \frac{2}{3}\|w_\lambda(u_1,u_2)\|^2_{L^2(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))} \\ = \; & \frac{1}{2}\,|(I)| + (II). \end{aligned}$$

We also recall Remark 2.8 (i), that is,

$$w_\lambda(u) = Q_{V,\lambda}(u) \quad \text{for} \quad V(x) = g(x,w_\lambda(u))/w_\lambda(u),$$

and we have

$$\begin{aligned} (I) &= \|Q_{V,\lambda}(u) - Q_{0,\lambda}(u) + Q_{0,\lambda}(u)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} \\ &\quad - \|Q_{0,\lambda}(u_1,0)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} - \|Q_{0,\lambda}(0,u_2)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} \\ &= \|Q_{V,\lambda}(u) - Q_{0,\lambda}(u)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} \\ &\quad + 2\big\langle Q_{V,\lambda}(u) - Q_{0,\lambda}(u), Q_{0,\lambda}(u)\big\rangle_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} \\ &\quad + 2\big\langle Q_{0,\lambda}(u_1,0), Q_{0,\lambda}(0,u_2)\big\rangle_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}. \end{aligned}$$

Thus by Lemma 2.9,

$$\begin{aligned} |(I)| &\le r_\lambda^2\|u\|^2_{\lambda,\Omega_1'\cup\Omega_2'} + 2r_\lambda C_3\|u\|^2_{\lambda,\Omega_1'\cup\Omega_2'} + 2r_\lambda\|u_1\|_{\lambda,\Omega_1'}\|u_2\|_{\lambda,\Omega_2'} \\ &\le r_\lambda'\|u\|^2_{\lambda,\Omega_1'\cup\Omega_2'}. \end{aligned} \tag{7.5}$$

On the other hand, using Lemma 2.1,

$$\begin{aligned} |(II)| &= \frac{2}{3}\|Q_{V,\lambda}(u)\|^2_{L^2(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))} \\ &\le \frac{2}{3}C_2(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'),\lambda)^2\|Q_{V,\lambda}(u)\|^2_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} \\ &\le \frac{2}{3}C_2(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'),\lambda)^2C_3^2\|u\|^2_{\lambda,\Omega_1'\cup\Omega_2'}. \end{aligned} \tag{7.6}$$

(i) follows from (7.5)–(7.6).

(ii) For $\varphi \in H_0^1(\Omega_1')$, we have

$$
\begin{aligned}
I_{1,\lambda}'(u_1)\varphi &= \langle u_1, \varphi\rangle_{\lambda,\Omega_1'} - \int_{\Omega_1'} |u_1|^{p-1}u_1\varphi\,dx \\
&\quad + \langle Q_{0,\lambda}(u_1,0), Q_{0,\lambda}(\varphi,0)\rangle_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')}, \qquad (7.7)\\
\frac{\partial I_\lambda}{\partial u_1}(u_1,u_2)\varphi &= \langle u_1, \varphi\rangle_{\lambda,\Omega_1'} - \int_{\Omega_1'} |u_1|^{p-1}u_1\varphi\,dx \\
&\quad + \langle w_\lambda(u_1,u_2), \frac{\partial w_\lambda}{\partial u_1}(u_1,u_2)\varphi\rangle_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} \\
&\quad - \int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} g(x,w_\lambda(u_1,u_2))\frac{\partial w_\lambda}{\partial u_1}(u_1,u_2)\varphi\,dx. \qquad (7.8)
\end{aligned}
$$

By Remark 2.8, we have $w_\lambda(u) = Q_{f(w_\lambda)/w_\lambda,\lambda}(u)$ and $\frac{\partial w_\lambda}{\partial u_1}(u_1,u_2)\varphi = Q_{f'(w_\lambda(u)),\lambda}(\varphi,0)$ for $\varphi \in H^1(\Omega_1')$. Thus from (7.7)–(7.8),

$$
\begin{aligned}
&\frac{\partial I_\lambda}{\partial u_1}(u_1,u_2)\varphi - I_{1,\lambda}(u_1)\varphi \\
=\ & -\langle Q_{0,\lambda}(u_1,0), Q_{0,\lambda}(\varphi,0)\rangle_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} \\
&+\langle Q_{f(w_\lambda)/w_\lambda,\lambda}(u_1,u_2), Q_{f'(w_\lambda),\lambda}(\varphi,0)\rangle_{\lambda,\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} \\
&+\int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} g(x,w_\lambda(u_1,u_2))Q_{f'(w_\lambda),\lambda}(\varphi,0)\,dx \\
=\ & (I)+(II)+(III).
\end{aligned}
$$

By Lemma 2.9,

$$|(I)+(II)| \le r_\lambda \|u\|_{\lambda,\Omega_1'\cup\Omega_2'}\|\varphi\|_{\lambda,\Omega_1'\cup\Omega_2'}.$$

We also have from Lemma 2.1

$$
\begin{aligned}
&\left|\int_{\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2')} g(x,w_\lambda(u))Q_{f'(w_\lambda),\lambda}(\varphi,0)\,dx\right| \\
\le\ & \frac{2}{3}\|w_\lambda(u)\|_{L^2(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))}\|Q_{f'(w_\lambda),\lambda}(\varphi,0)\|_{L^2(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'))} \\
\le\ & \frac{2}{3}C_2(\mathbf{R}^N\setminus(\Omega_1'\cup\Omega_2'),\lambda)^2\|u\|_{\lambda,\Omega_1'\cup\Omega_2'}\|\varphi\|_{\lambda,\Omega_1'\cup\Omega_2'}.
\end{aligned}
$$

Thus we have for some $r_\lambda' > 0$ satisfying $r_\lambda' \to 0$ as $\lambda \to \infty$,

$$\left|\frac{\partial I_\lambda}{\partial u_1}(u_1,u_2)\varphi - I_{1,\lambda}(u_1)\varphi\right| \le r_\lambda'\|u\|_{\lambda,\Omega_1'\cup\Omega_2'}\|\varphi\|_{\lambda,\Omega_1'} \quad \text{for all } \varphi \in H^1(\Omega_1').$$

Therefore (ii) holds. (iii) can be shown in a similar way. □

Next we give a proof of Proposition 2.15. First we observe some fundamental properties of $J_{i,\lambda}(v)$: We can find an M-independent constant R_0 and constants $R_1(M)$, $\delta_M > 0$, which depend on $M > 0$ but are independent of λ such that for $i = 1,2$ and $v_i \in \Sigma_{i,\lambda}$

(i) $J_{i,\lambda}(v_i) \le M$ implies $\|v_i\|_{L^{p+1}(\Omega_i')} \ge \delta_M$.

(ii) Suppose that $J_{i,\lambda}(v_i) \le M$. Then for $s \ge 0$

$$\frac{m_i}{2} \le I_{i,\lambda}(sv_i) \quad \text{implies} \quad R_0 \le s \le R_1(M).$$

Here $m_i > 0$ is defined in (2.29). This fact is easily seen from the definition of $J_{i,\lambda}(v)$ and (2.26).

For a proof of Proposition 2.15, we first observe that $J_\lambda(v_1, v_2)$ has properties similar to (i)–(ii). In what follows we fix $\lambda_0 \geq 1$ such that

$$r_\lambda \leq \frac{1}{2} \qquad \text{for } \lambda \geq \lambda_0.$$

We can observe the following lemma easily from Corollary 2.11.

Lemma 7.2. *There exist M-independent constants R_0, $\delta_0 > 0$ and constants $R_1(M)$, $\delta_M > 0$, which depend on $M > 0$ but are independent of λ such that for $(v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$*

(i) *$J_\lambda(v_1, v_2) \leq M$ implies $\|v_1\|_{L^{p+1}(\Omega_1')}$, $\|v_2\|_{L^{p+1}(\Omega_2')} \in [\delta_M, \delta_0]$.*

(ii) *Suppose $\lambda \geq \lambda_0$ and $J_\lambda(v_1, v_2) \leq M$. Then for $(s, t) \in [0, \infty)^2$*

$$\frac{1}{2}(m_1 + m_2) \leq I_\lambda(sv_1, tv_2) \quad \textit{implies} \quad R_0^2 \leq s^2 + t^2 \leq R_1(M)^2.$$

(iii) *There exists $\lambda_{1M} \geq \lambda_0$ such that for $\lambda \geq \lambda_{1M}$ and $(v_1, v_2) \in [J_\lambda \leq M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}}$*

(a) *$s^2 + t^2 \leq R_0^2$ implies $I_\lambda'(sv_1, tv_2)(sv_1, tv_2) > 0$.*

(b) *$s^2 + t^2 \geq R_1(M)^2$ implies $I_\lambda'(sv_1, tv_2)(sv_1, tv_2) < 0$.*

(c) *$s = R_0$ and $t \in [0, R_1(M)]$ implies $I_\lambda'(sv_1, tv_2)(v_1, 0) > 0$.*

(d) *$s \in [0, R_1(M)]$ and $t = R_0$ implies $I_\lambda'(sv_1, tv_2)(0, v_2) > 0$.*

(e) *$J_\lambda(v_1, v_2) > \max\{\max_{0 \leq s \leq R_0, t \geq 0} I_\lambda(sv_1, tv_2), \max_{s \geq 0, 0 \leq t \leq R_0} I_\lambda(sv_1, tv_2)\}$.*

We remark that (i) of Proposition 2.15 follows from Lemma 7.2. By the above Lemma 7.2, $(s, t) \mapsto I_\lambda(sv_1, tv_2)$ takes its maximum in $\{(s, t);\ s^2 + t^2 \leq R_1(M)^2,\ s, t \geq R_0\}$. More precisely we have

Lemma 7.3. *For any $M > 0$ there exists $\delta_{\lambda,M} > 0$ such that*

(i) *$\delta_{\lambda,M} \to 0$ as $\lambda \to \infty$.*

(ii) *For any $(v_1, v_2) \in [J_\lambda \leq M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}}$,*

$$I_\lambda(sv_1, tv_2) = J_\lambda(v_1, v_2) \textit{ implies } |s - s_{1,\lambda}(v_1)| \leq \delta_{\lambda,M}, |t - t_{2,\lambda}(v_2)| \leq \delta_{\lambda,M}. \tag{7.9}$$

Here $s_{1,\lambda}(v_1)$ ($t_{2,\lambda}(v_2)$ respectively) is a unique critical point of $s \mapsto I_{i,\lambda}(sv_i)$ ($t \mapsto I_{2,\lambda}(tv_2)$ respectively), which is given in (2.27).

We can show Lemma 7.3 without difficulty, and we omit the proof. We have the following uniqueness and regularity result for the maximizer.

Lemma 7.4. *For any $M > 0$ there exists $\lambda_M \geq 1$ such that*

(i) *For any $\lambda \geq \lambda_M$ and $(v_1, v_2) \in [J_\lambda \leq M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}}$*

$$(s, t) \mapsto I_\lambda(sv_1, tv_2)$$

has a unique maximum and is nondegenerate.

(ii) *We denote the unique maximizer $(s_\lambda(v_1, v_2), t_\lambda(v_1, v_2))$. Then for $\lambda \geq \lambda_M$*

$$[J_\lambda \leq M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} \to \mathbf{R}^2; (v_1, v_2) \mapsto (s_\lambda(v_1, v_2), t_\lambda(v_1, v_2))$$

is of class C^1.

(iii) *$[J_\lambda < M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} \to \mathbf{R}; (v_1, v_2) \mapsto J_\lambda(v_1, v_2) = I_\lambda(s_\lambda(v_1, v_2)v_1, t_\lambda(v_1, v_2)v_2)$ is of class C^1 for $\lambda \geq \lambda_M$.*

Proof. For $(v_1, v_2) \in [J_\lambda \le M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$, we set $f_\lambda(s,t) = I_\lambda(sv_1, tv_2)$. By (2.23),

$$\partial_s f_\lambda(s,t) = I'_\lambda(sv_1, tv_2)(v_1, 0) = \Phi'_\lambda(w_\lambda(sv_1, tv_2))Q_{0,\lambda}(v_1, 0).$$

Thus we have

$$\begin{aligned}
\partial_s^2 f_\lambda(s,t) &= \Phi''_\lambda(w_\lambda(sv_1, tv_2))(w'_\lambda(sv_1, tv_2)(v_1, 0), Q_{0,\lambda}(v_1, 0)) \\
&= \Phi''_\lambda(w_\lambda(sv_1, tv_2))(Q_{f'(w_\lambda(sv_1,tv_2)),\lambda}(v_1, 0), Q_{0,\lambda}(v_1, 0)), \\
\partial_t\partial_s f_\lambda(s,t) &= \Phi''_\lambda(w_\lambda(sv_1, tv_2))(Q_{f'(w_\lambda(sv_1,tv_2)),\lambda}(0, v_2), Q_{0,\lambda}(v_1, 0)).
\end{aligned}$$

Analogously,

$$\partial_t^2 f_\lambda(s,t) = \Phi''_\lambda(w_\lambda(sv_1, tv_2))(Q_{f'(w_\lambda(sv_1,tv_2)),\lambda}(0, v_2), Q_{0,\lambda}(0, v_2)).$$

Since

$$\Phi''_\lambda(u)(h_1, h_2) = \langle h_1, h_2\rangle_{\lambda,\mathbf{R}^N} - p\int_{\Omega'_1\cup\Omega'_2} |u|^{p-1}h_1h_2\,dx - \int_{\mathbf{R}^N\setminus(\Omega'_1\cup\Omega'_2)} f'(u)h_1h_2\,dx,$$

we have by Lemma 2.9

$$\begin{aligned}
\partial_s^2 f_\lambda(s,t) &= \langle Q_{f'(w_\lambda(sv_1,tv_2)),\lambda}(v_1, 0), Q_{0,\lambda}(v_1, 0)\rangle_{\lambda,\mathbf{R}^N} - ps^{p-1}\|v_1\|^{p+1}_{L^{p+1}(\Omega'_1)} \\
&\quad - \int_{\mathbf{R}^N\setminus(\Omega'_1\cup\Omega'_2)} f'(w_\lambda(sv_1, tv_2))Q_{f'(w_\lambda(sv_1,tv_2)),\lambda}(v_1, 0)Q_{0,\lambda}(v_1, 0)\,dx \\
&= \|v_1\|^2_{\lambda,\Omega'_1,\#} - ps^{p-1}\|v_1\|^{p+1}_{L^{p+1}(\Omega'_1)} + r_\lambda(s,t,v_1,v_2).
\end{aligned}$$

Here $r_\lambda(s,t,v_1,v_2)$ satisfies $r_\lambda(s,t,v_1,v_2) \to 0$ as $\lambda \to \infty$ uniformly in $(v_1, v_2) \in [J_\lambda \le M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$ and $(s,t) \in \{(s,t);\, s^2 + t^2 \le R_1(M)^2, s, t \ge R_0\}$.

Now suppose that $(v_1, v_2) \in [J_\lambda \le M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$ and $(s,t) \in \{(s,t);\, s^2 + t^2 \le R_1(M)^2,\ s, t \ge R_0\}$ is a critical point of $f_\lambda(s,t)$. By (7.9), for any $\epsilon > 0$ there exists $\lambda_{\epsilon,M} \ge 1$ such that

$$\left|\partial_s^2 f_\lambda(s,t) - (1-p)\|v_1\|^{p+1}_{L^{p+1}(\Omega'_1)}\right| < \epsilon \quad \text{for } \lambda \ge \lambda_{\epsilon,M}.$$

In a similar way, we have

$$|\partial_t\partial_s f_\lambda(s,t)| < \epsilon, \quad \left|\partial_t^2 f_\lambda(s,t) - (1-p)\|v_2\|^{p+1}_{L^{p+1}(\Omega'_2)}\right| < \epsilon$$

for $\lambda \ge \lambda_{\epsilon,M}$ and for any critical point (s,t) in $\{(s,t);\, s^2+t^2 \le R_1(M)^2, s, t \ge R_0\}$.

Recalling (i) of Lemma 7.2 and choosing $\epsilon > 0$ small, we have

$$\det\begin{bmatrix} \partial_s^2 f_\lambda(s,t) & \partial_t\partial_s f_\lambda(s,t) \\ \partial_t\partial_s f_\lambda(s,t) & \partial_t^2 f_\lambda(s,t) \end{bmatrix} > 0$$

for all critical points in $\{(s,t);\, R_0^2 \le s^2+t^2 \le R_1(M)^2, s, t \ge R_0\}$. Thus uniqueness of critical points follows from (iii) of Lemma 7.2 and we get (i). Since unique maximizer is nondegenerate, the conclusion (ii) follows from the implicit function theorem. (iii) follows from (ii). □

Proof of (ii)–(v) of Proposition 2.15. (ii) and (iv) are nothing but (i), (iii) of Lemma 7.4. (iii) follows from Lemma 7.2.

Next we prove (v). For (2.32), we have

$$\begin{aligned}
& |J_\lambda(v_1,v_2) - J_{1,\lambda}(v_1) - J_{2,\lambda}(v_2)| \\
= \; & |I_\lambda(s_\lambda(v_1,v_2)v_1, t_\lambda(v_1,v_2)v_2) - I_{1,\lambda}(s_{1,\lambda}(v_1)v_1) - I_{2,\lambda}(t_{2,\lambda}(v_2)v_2)| \\
\le \; & |I_\lambda(s_\lambda(v_1,v_2)v_1, t_\lambda(v_1,v_2)v_2) - I_{1,\lambda}(s_\lambda(v_1,v_2)v_1) - I_{2,\lambda}(s_\lambda(v_1,v_2)v_2)| \\
& +|I_{1,\lambda}(s_\lambda(v_1,v_2)v_1) - I_{1,\lambda}(s_{1,\lambda}(v_1)v_1)| \\
& +|I_{2,\lambda}(s_\lambda(v_1,v_2)v_2) - I_{2,\lambda}(t_{2,\lambda}(v_2)v_2)|.
\end{aligned}$$

Thus by Lemma 2.10 and Lemma 7.3, we get (2.32). We can show (2.33) in a similar way. □

Finally we give a proof of Proposition 2.16.

Proof of Proposition 2.16. Suppose $(v_{1n}, v_{2n}) \in [J_\lambda \le M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$ satisfies

$$J_\lambda(v_{1n}, v_{2n}) \to c \in (-\infty, M), \quad \|J'_\lambda(v_{1n}, v_{2n})\|_{(T_{(v_{1n},v_{2n})}(\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}))^*} \to 0,$$

where

$$\begin{aligned}
\|J'_\lambda(v_1,v_2)\|_{(T_{(v_1,v_2)}(\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}))^*} = \sup\{&|J'_\lambda(v_1,v_2)(h_1,h_2)|; \\
&(h_1,h_2) \in H^1(\Omega'_1) \oplus H^1(\Omega'_2),\ \langle v_1, h_1\rangle_{\lambda,\Omega'_1} = \langle v_2, h_2\rangle_{\lambda,\Omega'_2} = 0\}.
\end{aligned}$$

We note

$$J'_\lambda(v_1,v_2)(h_1,h_2) = I'_\lambda(s_\lambda(v_1,v_2)v_1, t_\lambda(v_1,v_2)v_2)(s_\lambda(v_1,v_2)h_1, t_\lambda(v_1,v_2)h_2)$$

and

$$I'_\lambda(s_\lambda(v_1,v_2)v_1, t_\lambda(v_1,v_2)v_2)(\alpha v_1, \beta v_2) = 0 \quad \text{for all } (\alpha,\beta) \in \mathbf{R}^2.$$

Thus

$$\begin{aligned}
& \|I'_\lambda(s_\lambda(v_1,v_2)v_1, t_\lambda(v_1,v_2)v_2)\|^*_{\lambda,\Omega'_1\cup\Omega'_2} \\
\le \; & \sqrt{\frac{1}{s_\lambda(v_1,v_2)^2} + \frac{1}{t_\lambda(v_1,v_2)^2}}\,\|J'_\lambda(v_1,v_2)\|_{(T_{(v_1,v_2)}(\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}))^*}.
\end{aligned}$$

Recalling (2.30)–(2.31), we have

$$\|I'_\lambda(s_\lambda(v_{1n},v_{2n})v_{1n}, t_\lambda(v_{1n},v_{2n})v_{2n})\|^*_{\lambda,\Omega'_1\cup\Omega'_2} \to 0.$$

Thus $(s_\lambda(v_{1n},v_{2n})v_{1n}, t_\lambda(v_{1n},v_{2n})v_{2n})$ is a $(PS)_c$-sequence for $I_\lambda(u_1,u_2)$. By Proposition 2.4, it has a strongly convergent subsequence in $H^1(\Omega'_1) \oplus H^1(\Omega'_2)$. Recalling (2.30)–(2.31) again, we can see that (v_{1n}, v_{2n}) has a strongly convergent subsequence in $\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$, and the $(PS)_c$-condition holds for $J_\lambda(v_1,v_2)$. □

8. Proofs of Propositions 3.4 and 7.1

The aim of this section is to give proofs to Proposition 3.4 and Proposition 7.1.

Suppose that a compact set K and open sets O_1, O_2, O_3 satisfy the following condition:

(8.1) $K \subset\subset O_1 \subset\subset O_2 \subset\subset O_3$ and ∂K, ∂O_i $(i = 1,2,3)$ are compact and smooth.

(8.2) $\inf_{x\in O_3\setminus K} a(x) > 0.$

For a given $\varphi(x) \in H^1(O_1 \setminus K)$ and $\lambda \geq 1$ we consider the following linear elliptic problem:

$$-\Delta u + (\lambda^2 a(x) + 1)u = 0 \quad \text{in } O_3 \setminus \overline{O_1}, \tag{8.3}$$

$$u(x) = \varphi(x) \ \text{ on } \partial O_1, \quad u(x) = 0 \ \text{ on } \partial O_3, \tag{8.4}$$

$$u(x) \in H^1(O_3 \setminus \overline{O_1}). \tag{8.5}$$

Obviously (8.3)–(8.5) has a unique solution — we denote it by $(S_\lambda \varphi)(x)$ — and $(S_\lambda \varphi)(x)$ can be characterized as

$$\|S_\lambda \varphi\|_{\lambda, O_3 \setminus \overline{O_1}} \leq \|u\|_{\lambda, O_3 \setminus \overline{O_1}} \tag{8.6}$$

for all $u \in H^1(O_3 \setminus \overline{O_1})$ satisfying (8.4).

The following proposition gives estimates of the solution $S_\lambda \varphi$:

Proposition 8.1. *Assume that K, O_1, O_2, O_3 satisfy* (8.1)–(8.2). *Then there exist constants a, C_4, $C_5 > 0$ such that*

$$\|S_\lambda \varphi\|_{\lambda, O_3 \setminus \overline{O_1}} \leq C_4 \|\varphi\|_{\lambda, O_1 \setminus K}, \tag{8.7}$$

$$\|S_\lambda \varphi\|_{\lambda, O_3 \setminus \overline{O_2}} \leq C_5 e^{-a\lambda} \|\varphi\|_{\lambda, O_1 \setminus K}, \tag{8.8}$$

for all $\lambda \geq 1$ and $\varphi \in H^1(O_1 \setminus K)$.

We postpone the proof of Proposition 8.1 until the end of this section. As an immediate corollary to Proposition 8.1, we have

Corollary 8.2. *Assume* (8.1)–(8.2). *Then there exists a family of linear bounded operators $P_\lambda : H^1(O_3 \setminus K) \to H^1(O_3 \setminus K)$ such that*

(i) *For all $\lambda \geq 1$,*

$$(P_\lambda u)(x) = u(x) \quad \textit{for all } x \in O_1 \setminus K, \tag{8.9}$$

$$(P_\lambda u)(x) = 0 \quad \textit{for all } x \in O_3 \setminus \overline{O_2}. \tag{8.10}$$

(ii) *There exists a family of positive constants $\epsilon_\lambda > 0$ independent of u such that*

$$\epsilon_\lambda \to 0 \quad \textit{as } \lambda \to \infty, \tag{8.11}$$

$$\|P_\lambda u\|_{\lambda, O_3 \setminus K} \leq (1 + \epsilon_\lambda) \|u\|_{\lambda, O_3 \setminus K} \quad \textit{for all } \lambda \geq 1 \textit{ and } u \in H^1(O_3 \setminus K). \tag{8.12}$$

Proof. Choose an open set D such that $O_1 \subset\subset D \subset\subset O_2 \subset\subset O_3$. Applying Proposition 8.1 to K, O_1, D, O_3, we find a solution operator $S_\lambda : H^1(O_1 \setminus K) \to H^1(O_3 \setminus K)$ such that

$$\|S_\lambda u\|_{\lambda, O_3 \setminus K} \leq \|u\|_{\lambda, O_3 \setminus K}, \tag{8.13}$$

$$\|S_\lambda u\|_{\lambda, O_3 \setminus \overline{D}} \leq C_5 e^{-a\lambda} \|u\|_{\lambda, O_3 \setminus K}, \tag{8.14}$$

for some constant $a > 0$. Here we use the same notation S_λ for an operator $u \mapsto S_\lambda(u|_{O_1 \setminus K})$; $H^1(O_3 \setminus K) \to H^1(O_3 \setminus K)$.

We choose a function $\zeta(x) \in C^\infty(\mathbf{R}^N, [0,1])$ such that

$$\zeta(x) = \begin{cases} 1 & \text{for all } x \in D, \\ 0 & \text{for all } x \in \mathbf{R}^N \setminus \overline{O_2}. \end{cases}$$

Setting $(P_\lambda u)(x) = \zeta(x)(S_\lambda u)(x)$, we have (8.9)–(8.12). In fact, (8.9)–(8.10) are clear. For (8.11)–(8.12), first we observe by (8.14)

$$\begin{aligned}
(8.15)\qquad \|\zeta S_\lambda u\|^2_{\lambda,O_3\setminus\overline{D}} &= \int_{O_3\setminus\overline{D}} |\nabla(\zeta S_\lambda u)|^2 + (\lambda^2 a(x)+1)\zeta^2|S_\lambda u|^2\,dx \\
&\le 2\int_{O_3\setminus\overline{D}} |\nabla\zeta|^2|S_\lambda u|^2 + |\zeta|^2|\nabla(S_\lambda u)|^2\,dx \\
&\quad + \int_{O_3\setminus\overline{D}} (\lambda^2 a(x)+1)|S_\lambda u|^2\,dx \\
&\le 2(1+\|\nabla\zeta\|^2_{L^\infty(\mathbf{R}^N)})\|S_\lambda u\|^2_{\lambda,O_3\setminus\overline{D}} \\
&\le 2(1+\|\nabla\zeta\|^2_{L^\infty(\mathbf{R}^N)})C_5^2 e^{-2a\lambda}\|u\|^2_{\lambda,O_3\setminus K}.
\end{aligned}$$

Thus by (8.13) and (8.15)

$$\begin{aligned}
\|P_\lambda u\|^2_{\lambda,O_3\setminus K} &= \|S_\lambda u\|^2_{\lambda,D\setminus K} + \|\zeta S_\lambda u\|^2_{\lambda,O_3\setminus\overline{D}} \\
&\le \|u\|^2_{\lambda,O_3\setminus K} + 2(1+\|\nabla\zeta\|^2_{L^\infty(\mathbf{R}^N)})C_5^2 e^{-2a\lambda}\|u\|^2_{\lambda,O_3\setminus K}.
\end{aligned}$$

Thus we get (8.11)–(8.12). □

Now we are in a position to prove Proposition 3.4.

Proof of Proposition 3.4. We choose an open set D such that

$$\Omega_2 \subset\subset D \subset\subset \tilde{\tilde{\Omega}} \subset\subset \tilde{\Omega} \subset\subset \Omega_2'.$$

We apply Corollary 8.2 to $(K, O_1, O_2, O_3) = (\overline{D}, \tilde{\tilde{\Omega}}, \tilde{\Omega}, \Omega_2')$, and we find a family of operators $P_\lambda : H^1(\Omega_2'\setminus\overline{D}) \to H^1(\Omega_2'\setminus\overline{D})$ satisfying (8.9)–(8.12). Defining $F_\lambda : \{v\in\Sigma_{2,\lambda};\ \|v\|_{L^{p+1}(\Omega_2')} \ge \tilde{m}\} \to H^1_0(\tilde{\Omega})$ by

$$(F_\lambda u)(x) = \begin{cases} u(x) & \text{for } x\in\overline{D}, \\ (P_\lambda u)(x) & \text{for } x\in\tilde{\Omega}\setminus\overline{D}, \end{cases}$$

1°, 2° of Proposition 3.4 follow from (8.9)–(8.12).

For 3°, first we remark for $v\in\Sigma_{2,\lambda}$ that $\|v\|_{\lambda,\Omega_2'\setminus\tilde{\tilde{\Omega}}} \le \|v\|_{\lambda,\Omega_2'} = 1$. Thus

$$\begin{aligned}
\|F_\lambda v\|^{p+1}_{L^{p+1}(\Omega_2')} &= \|F_\lambda v\|^{p+1}_{L^{p+1}(\tilde{\Omega})} \ge \|v\|^{p+1}_{L^{p+1}(\tilde{\tilde{\Omega}})} = \|v\|^{p+1}_{L^{p+1}(\Omega_2')} - \|v\|^{p+1}_{L^{p+1}(\Omega_2'\setminus\tilde{\tilde{\Omega}})} \\
&\ge \|v\|^{p+1}_{L^{p+1}(\Omega_2')} - C_{p+1}(\Omega_2'\setminus\tilde{\tilde{\Omega}},\lambda)\|v\|^2_{\lambda,\Omega_2'\setminus\tilde{\tilde{\Omega}}} \\
&\ge \|v\|^{p+1}_{L^{p+1}(\Omega_2')} - C_{p+1}(\Omega_2'\setminus\tilde{\tilde{\Omega}},\lambda) \\
&\ge \left(1 - \frac{C_{p+1}(\Omega_2'\setminus\tilde{\tilde{\Omega}},\lambda)}{\tilde{m}}\right)\|v\|^{p+1}_{L^{p+1}(\Omega_2')}.
\end{aligned}$$

Therefore, 3° also holds for large λ. □

Proof of Proposition 7.1. We choose an open set D such that $\Omega_1 \subset\subset D \subset\subset \Omega_1' \subset\subset O_1$. Applying Proposition 8.1 with $(K, O_1, O_2, O_3) = (\overline{D}, \Omega_1', O_1, \mathbf{R}^N\setminus\Omega_2')$, we have for some constant $a>0$

$$\|Q_{0,\lambda}(u_1,0)\|_{\lambda,\mathbf{R}^N\setminus(O_1\cup\Omega_2')} \le C_5 e^{-a\lambda}\|u_1\|_{\lambda,\Omega_1'\setminus\overline{D}} \le C_5 e^{-a\lambda}\|u_1\|_{\lambda,\Omega_1'}$$

for all $\lambda\ge 1$ and $u_1\in H^1(\Omega_1')$. Thus we have (7.3). We can show (7.4) analogously. □

Proof of Proposition 8.1. We choose open sets D_1, D_2, E_1, E_2, E_3 such that

$$K \subset\subset O_1 \subset\subset D_1 \subset\subset D_2 \subset\subset O_2 \subset\subset E_3 \subset\subset E_2 \subset\subset E_1 \subset\subset O_3.$$

In what follows, for $\varphi \in H^1(O_1 \setminus K)$ we denote $u(x) = (S_\lambda \varphi)(x)$, that is, $u(x)$ solves (8.3)–(8.5). First we remark that using the characterization (8.6) we can prove (8.7) as in Lemma 2.5.

The proof of (8.8) consists of 4 steps. As the first step we show

Step 1: There exists a constant $C_6 > 0$ independent of $\varphi \in H^1(O_1 \setminus K)$ such that

$$\max_{x \in \overline{E}_1 \setminus D_1} |u(x)| \le C_6 \|\varphi\|_{\lambda, O_1 \setminus K}. \tag{8.16}$$

We choose $r > 0$ such that $\overline{B(x_0, r)} \subset O_3 \setminus \overline{O_1}$ for all $x_0 \in \overline{E_1} \setminus D_1$, where $B(x_0, r) = \{y \in \mathbf{R}^N; |y - x_0| < r\}$. By the subsolution estimate ([22], Theorem C.1.2), we have for some constant $C_7 > 0$ independent of λ, x_0, φ

$$\begin{aligned} |u(x_0)| &\le C_7 \int_{B(x_0,r)} |u(x)|\, dx \le C_7 \sqrt{\mathrm{meas}(B(x_0,r))} \|u\|_{L^2(B(x_0,r))} \\ &\le C_7 \sqrt{\mathrm{meas}(B(x_0,r))} \|u\|_{\lambda, O_3 \setminus \overline{O_1}} \le C_4 C_7 \sqrt{\mathrm{meas}(B(x_0,r))} \|\varphi\|_{\lambda, O_1 \setminus K}. \end{aligned}$$

Thus (8.16) holds. □

Step 2: There exists $\mu_1 > 0$ independent of λ and φ such that

$$\max_{x \in \overline{E}_2 \setminus D_2} |u(x)| \le e^{-\mu_1 \lambda} \|\varphi\|_{\lambda, O_1 \setminus K}. \tag{8.17}$$

We choose $b \in (0, \inf_{x \in O_3 \setminus K} a(x))$. From (8.3) it holds that

$$(-\Delta + 2\lambda^2 b) u^2 = -2(\lambda^2 a(x) + 1 - \lambda^2 b) u^2 - 2|\nabla u|^2.$$

In particular, we have

$$(-\Delta + 2\lambda^2 b) u^2 \le 0. \tag{8.18}$$

We choose $r > 0$ such that $\overline{B(x_0, r)} \subset \overline{E_1} \setminus D_1$ for all $x_0 \in \overline{E}_2 \setminus D_2$. Now we consider a solution $w(x)$ of

$$-\Delta w + 2\lambda^2 b w = 0 \quad \text{in } B(x_0, r), \quad w = (C_6 \|\varphi\|_{\lambda, O_1 \setminus K})^2 \quad \text{on } \partial B(x_0, r). \tag{8.19}$$

Then we have for some $b' > 0$

$$|w(x_0)| \le C_6^2 e^{-b' \lambda} \|\varphi\|^2_{\lambda, O_1 \setminus K} \qquad \text{for all } \lambda \ge 1.$$

Comparing (8.18) and (8.19), the conclusion (8.17) follows from Step 1. □

Step 3: There exists $\mu_2 > 0$ independent of λ and φ such that

$$\|u\|_{\lambda, E_3 \setminus \overline{O_2}} \le e^{-\mu_2 \lambda} \|\varphi\|_{\lambda, O_1 \setminus K}. \tag{8.20}$$

Choose $\psi(x) \in C_0^\infty(\mathbf{R}^N, [0,1])$ such that

$$\psi(x) = \begin{cases} 1 & \text{for } x \in E_3 \setminus \overline{O_2}, \\ 0 & \text{for } x \in \mathbf{R}^N \setminus (\overline{E_2} \setminus D_2). \end{cases}$$

Multiplying ψu to (8.3) and integrating, we have

$$\int_{O_3 \setminus K} \nabla u \nabla(\psi u) + (\lambda^2 a(x) + 1) \psi u^2\, dx = 0.$$

Thus, by (8.17) and (8.7),

$$\begin{aligned}\|u\|^2_{\lambda,E_3\setminus\overline{O_2}} &\le \int_{E_2\setminus\overline{D_2}} |\nabla\psi||u||\nabla u|\,dx\\ &\le C_8\|\nabla\psi\|_{L^\infty(\mathbf{R}^N)}\left(\max_{x\in\overline{E_2}\setminus D_2}|u(x)|\right)\|u\|_{\lambda,O_3\setminus\overline{O_1}}\\ &\le C_8C_4e^{-\mu_1\lambda}\|\nabla\psi\|_{L^\infty(\mathbf{R}^N)}\|\varphi\|^2_{\lambda,O_1\setminus K}.\end{aligned}$$

Thus we get (8.20). □

Step 4: Conclusion.

From the characterization (8.6), it follows that

$$\|u\|_{\lambda,O_3\setminus\overline{O_2}} \le \|w\|_{\lambda,O_3\setminus\overline{O_2}}$$

for all $w \in H^1(O_3\setminus\overline{O_2})$ satisfying $w = u$ on ∂O_2 and $w = 0$ on ∂O_3. In particular, setting $w = \zeta u$, where $\zeta(x) \in C^\infty(\mathbf{R}^N, [0,1])$ is a function such that

$$\zeta(x) = \begin{cases} 0 & \text{for } x \in O_3\setminus\overline{E_3},\\ 1 & \text{in a neighborhood of } \partial O_2,\end{cases}$$

we can deduce as in the proof of Corollary 8.2 that

$$\|u\|_{\lambda,O_3\setminus\overline{O_2}} \le \|\zeta u\|_{\lambda,E_3\setminus\overline{O_2}} \le C_9\|u\|_{\lambda,E_3\setminus\overline{O_2}} \le C_9e^{-\mu_2\lambda}\|\varphi\|_{\lambda,O_1\setminus K}.$$

Thus the proof of Proposition 8.1 is completed. □

9. Construction of a deformation flow

The aim of this section is to construct a deformation flow which enables us to find critical points whose limits are positive or negative in Ω_i.

We define

$$\begin{aligned}\varphi_\pm(v_1) &= \|v_{1\pm}\|^{p+1}_{L^{p+1}(\Omega_1')} : \Sigma_{1,\lambda} \to \mathbf{R},\\ \psi_\pm(v_2) &= \|v_{2\pm}\|^{p+1}_{L^{p+1}(\Omega_2')} : \Sigma_{2,\lambda} \to \mathbf{R},\end{aligned}$$

and for $a < b$, we use the following notation:

$$\begin{aligned}[a \le \varphi_\pm \le b]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}} &= \{(v_1,v_2) \in \Sigma_{1,\lambda}\times\Sigma_{2,\lambda};\ a \le \varphi_\pm(v_1) \le b\},\\ [a \le \psi_\pm \le b]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}} &= \{(v_1,v_2) \in \Sigma_{1,\lambda}\times\Sigma_{2,\lambda};\ a \le \psi_\pm(v_2) \le b\}.\end{aligned}$$

First we have

Proposition 9.1. *For $M > 0$ and $\sigma_1, \sigma_2 \in \{+,-\}$, there exist $\lambda_M > 1$, $\nu_M \in (0, \delta_0(M)]$ such that for any $\lambda \ge \lambda_M$, if $b < M$ is not a critical value of J_λ, then there exist ϵ, $\mu > 0$ and a vector field $X(v)$ on $[J_\lambda \le M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$ with the following properties:*

(i) *$X(v) = (X_1(v), X_2(v)) \in T_v(\Sigma_{1,\lambda}\times\Sigma_{2,\lambda})$ for all $v \in [J_\lambda \le M]$ and $v \mapsto X(v)$ is locally Lipschitz continuous.*
(ii) *$J'_\lambda(v)X(v) \le -\mu$ for all $v \in [b-\epsilon \le J_\lambda \le b+\epsilon]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$.*
(iii) *$X(v) = 0$ for all $v \in [J_\lambda \le b - 2\epsilon]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$.*
(iv) *$J'_\lambda(v)X(v) \le 0$ for all $v \in [J_\lambda \le M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$.*

(v)

$$\varphi'_{\sigma_1}(v_1)X_1(v) \le 0 \text{ for all } v \in [J_\lambda \le M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}} \cap [\nu_M \le \varphi_{\sigma_1} \le 2\nu_M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}},$$
$$\psi'_{\sigma_2}(v_2)X_2(v) \le 0 \text{ for all } v \in [J_\lambda \le M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}} \cap [\nu_M \le \psi_{\sigma_2} \le 2\nu_M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}.$$

Here we remark that $\delta_0(M) > 0$ is given in Proposition 2.19.

To prove Proposition 9.1 we define

$$\begin{aligned} Y_1^{\pm}(v_1) &= -v_{1\pm} + \|v_{1\pm}\|^2_{\lambda,\Omega'_1} v_1 \in T_{v_1}(\Sigma_{1,\lambda}) \quad \text{for } v_1 \in \Sigma_{1,\lambda}, \\ Y_2^{\pm}(v_2) &= -v_{2\pm} + \|v_{2\pm}\|^2_{\lambda,\Omega'_2} v_2 \in T_{v_2}(\Sigma_{2,\lambda}) \quad \text{for } v_2 \in \Sigma_{2,\lambda}. \end{aligned}$$

We have the following.

Lemma 9.2. *For $M > 0$, there exist $\lambda_M > 1$, $\nu_M \in (0, \delta_0(M)]$ and $\rho_M > 0$ such that for any $\lambda \ge \lambda_M$, σ_1, $\sigma_2 \in \{+,-\}$,*

(i) *for $v \in [J_\lambda \le M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}} \cap [\frac{1}{2}\nu_M \le \varphi_{\sigma_1} \le 2\nu_M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$,*

$$J'_\lambda(v)(Y_1^{\sigma_1}(v), 0) \le -\rho_M, \quad \varphi'_{\sigma_1}(v_1)Y_1^{\sigma_1}(v) \le -\rho_M; \tag{9.1}$$

(ii) *for $v \in [J_\lambda \le M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}} \cap [\frac{1}{2}\nu_M \le \psi_{\sigma_2} \le 2\nu_M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$,*

$$J'_\lambda(v)(0, Y_2^{\sigma_2}(v)) \le -\rho_M, \quad \psi'_{\sigma_2}(v_2)Y_2^{\sigma_2}(v) \le -\rho_M.$$

Proof. We only prove (i) with $\sigma_1 = +$. We can deal with other cases in a similar way. For $v \in [J_\lambda \le M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$,

$$\begin{aligned} \varphi'_+(v_1)Y_1^+(v) &= (p+1)\int_{\Omega'_1} (v_{1+})^p Y_1^+(v)dx \\ &= -(p+1)(1 - \|v_{1+}\|^2_{\lambda,\Omega'_1})\|v_{1+}\|^{p+1}_{L^{p+1}(\Omega'_1)} \\ &\le -(p+1)(1 - C^{-2}\|v_{1+}\|^2_{L^{p+1}(\Omega'_1)})\|v_{1+}\|^{p+1}_{L^{p+1}(\Omega'_1)}. \end{aligned} \tag{9.2}$$

Here we use the inequality $\|v_{1+}\|_{L^{p+1}(\Omega'_1)} \le C\|v_{1+}\|_{\lambda,\Omega'_1}$.

Writing $s = s_\lambda(v_1, v_2), t = t_\lambda(v_1, v_2)$, by Lemma 2.10 and Proposition 2.15 we have

$$\begin{aligned} & J'_\lambda(v_1, v_2)(Y_1^+(v), 0) \\ = \;& I'_\lambda(sv_1, tv_2)(sY_1^+(v), 0) = -I'_\lambda(sv_1, tv_2)(sv_{1+}, 0) \\ \le \;& -I'_{1,\lambda}(sv_1)(sv_{1+}) + r_\lambda(\|sv_1\|_{\lambda,\Omega'_1} + \|tv_2\|_{\lambda,\Omega'_2})\|sv_{1+}\|_{\lambda,\Omega'_1} \\ = \;& -s^2\|v_{1+}\|^2_{\lambda,\Omega'_1,\#} + s^{p+1}\|v_{1+}\|^{p+1}_{L^{p+1}(\Omega'_1)} + r_\lambda(s+t)s \\ \le \;& -s^2C^{-2}\|v_{1+}\|^2_{L^{p+1}(\Omega'_1)} + s^{p+1}\|v_{1+}\|^{p-1}_{L^{p+1}(\Omega'_1)} + r_\lambda s(s+t) \\ \le \;& -R_0^2C^{-2}\|v_{1+}\|^2_{L^{p+1}(\Omega'_1)} + R_1(M)^{p+1}\|v_{1+}\|^{p-1}_{L^{p+1}(\Omega'_1)} + 2r_\lambda R_1(M)^2. \end{aligned} \tag{9.3}$$

From (9.2) and (9.3), we can find $\nu_M > 0$ and $\rho_M > 0$ such that if $\varphi_+(v_1) = \|v_{1+}\|^{p+1}_{L^{p+1}(\Omega'_1)} \in [\frac{\nu_M}{2}, 2\nu_M]$, (9.1) holds. □

Proof of Proposition 9.1. Assume that $b < M$ is not a critical value. In a standard way, we can find a pseudo-gradient vector field $\tilde{X}(v)$ satisfying (i)–(iv) of Proposition 9.1.

We choose smooth functions $\zeta : \mathbf{R} \to [0,1]$ such that $\zeta(s) = 0$ for $s \le \frac{1}{2}\nu_M$,$\zeta(s) = 1$ for $s \ge \nu_M$ and $\xi : \mathbf{R} \to [0,1]$ such that $\xi(s) = 0$ for $s \le b - 2\epsilon$, $\xi(s) = 1$ for

$s \geq b - \epsilon$ and set

$$\begin{aligned} X(v) &= (1-\zeta(\varphi_{\sigma_1}(v_1)))\,(1-\zeta(\psi_{\sigma_2}(v_2)))\,\tilde{X}(v) \\ &\quad +\xi(J_\lambda(v_1,v_2))(\zeta(\varphi_{\sigma_1}(v_1))(Y_1^{\sigma_1}(v_1),0)+\zeta(\psi_{\sigma_2}(v_2))(0,Y_2^{\sigma_1}(v_2))). \end{aligned}$$

Then we can easily observe that $X(v)$ has the desired properties (i)–(v) of Proposition 9.1. □

As an immediate corollary to Proposition 9.1, we have

Corollary 9.3. *For $M > 0$ and $\sigma_1, \sigma_2 \in \{+,-\}$, there exist $\lambda_M > 1$, $\nu_M \in (0, \delta_0(M)]$ such that for any $\lambda \geq \lambda_M$, if $b < M$ is not a critical value of J_λ, then there exist $\epsilon > 0$ and a continuous mapping $\eta(t,v) : [0,1] \times [J_\lambda \leq M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}} \to [J_\lambda \leq M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$ such that*

(i) *$J_\lambda(\eta(t,v)) \leq J_\lambda(v)$ for all $(t,v) \in [0,1] \times [J_\lambda \leq M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$,*
(ii) *$J_\lambda(\eta(1,v)) \leq b - \epsilon$ for all $v \in [J_\lambda \leq b+\epsilon]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$,*
(iii) *$\eta(t,v) \in [J_\lambda \leq M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}} \cap [\varphi_{\sigma_1} \leq \nu_M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$ for all $(t,v) \in [0,1] \times ([J_\lambda \leq M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}} \cap [\varphi_{\sigma_1} \leq \nu_M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}})$,*
(iv) *$\eta(t,v) \in [J_\lambda \leq M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}} \cap [\psi_{\sigma_2} \leq \nu_M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}}$ for all $(t,v) \in [0,1] \times ([J_\lambda \leq M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}} \cap [\psi_{\sigma_2} \leq \nu_M]_{\Sigma_{1,\lambda}\times\Sigma_{2,\lambda}})$.*

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