SIGN-CHANGING MULTI-BUMP SOLUTIONS
FOR NONLINEAR SCHRÖDINGER EQUATIONS
WITH STEEP POTENTIAL WELLS

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ABSTRACT. We study the nonlinear Schrödinger equations:
\[(P_\lambda) \quad -\Delta u + (\lambda^2 a(x) + 1)u = |u|^{p-1}u, \quad u \in H^1(\mathbb{R}^N),\]
where \( p > 1 \) is a subcritical exponent, \( a(x) \) is a continuous function satisfying
\( a(x) \geq 0, \quad 0 < \liminf_{|x| \to \infty} a(x) \leq \limsup_{|x| \to \infty} a(x) < \infty \) and \( a^{-1}(0) \) consists
of 2 connected bounded smooth components \( \Omega_1 \) and \( \Omega_2 \).

We study the existence of solutions \((u_\lambda)\) of \((P_\lambda)\) which converge to 0 in
\( \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2) \) and to a prescribed pair \((v_1(x), v_2(x))\) of solutions of the limit
problem:
\[-\Delta v_i + v_i = |v_i|^{p-1}v_i \quad \text{in } \Omega_i \quad (i = 1, 2) \quad \text{as } \lambda \to \infty.\]

1. INTRODUCTION

In this paper we consider the existence and multiplicity of solutions of the fol-
lowing nonlinear Schrödinger equations:
\[(P_\lambda) \quad -\Delta u + (\lambda^2 a(x) + 1)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N, \quad u(x) \in H^1(\mathbb{R}^N).\]
Here \( p \in (1, \frac{N+2}{N-2}) \) if \( N \geq 3 \), \( p \in (1, \infty) \) if \( N = 1 \), 2 and \( a(x) \in C(\mathbb{R}^N, \mathbb{R}) \) is
nonnegative on \( \mathbb{R}^N \). We consider multiplicity of solutions (including positive and
sign-changing solutions) when the parameter \( \lambda \) is very large.

For \( a(x) \), we assume
\[(a1) \quad a(x) \in C(\mathbb{R}^N, \mathbb{R}), \quad a(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^N \quad \text{and the potential well } \Omega = \text{int } a^{-1}(0) \quad \text{is a nonempty bounded open set with smooth boundary } \partial \Omega \quad \text{and } \quad a^{-1}(0) = \overline{\Omega}.
\]
\[(a2) \quad 0 < \liminf_{|x| \to \infty} a(x) \leq \sup_{x \in \mathbb{R}^N} a(x) < \infty.\]

When \( \lambda \) is large, the potential well \( \Omega \) plays important roles and the following
Dirichlet problem appears as a limit of \((P_\lambda)\):
\[-\Delta u + u = |u|^{p-1}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.\]
We remark that solutions of \((P_\lambda)\) and \(\text{[1.1]}\) can be characterized as critical points of
\[
(1.2) \quad \Psi_\lambda(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + (\lambda^2 a(x) + 1)u^2) - \frac{1}{p+1}|u|^{p+1} \, dx : H^1(\mathbb{R}^N) \to \mathbb{R},
\]
\[
(1.3) \quad \Psi_\Omega(u) = \int_\Omega \frac{1}{2}(|\nabla u|^2 + u^2) - \frac{1}{p+1}|u|^{p+1} \, dx : H^1_0(\Omega) \to \mathbb{R},
\]
and it is known that \(\text{[1.3]}\) has an unbounded sequence of critical values (see for example \(\text{[21]}\)).

Bartsch and Wang \(\text{[3]}\) and Bartsch, Pankov and Wang \(\text{[4]}\) (see also \(\text{[2]}\)) studied such a situation first. Their assumptions on \(a(x)\) and nonlinearity are more general, and as a special case of their results we have

(i) There exists a least energy solution \(u_\lambda(x)\) of \((P_\lambda)\). Moreover \(u_{\lambda_n}(x)\) converges strongly to a least energy solution of \(\text{[1.3]}\) after extracting a subsequence \(\lambda_n \to \infty\) \(\text{[3]}\).

(ii) When \(N \geq 3\) and \(p \in (1, \frac{N+2}{N-2})\) is close to \(\frac{N+2}{N-2}\), there exists at least \(\text{cat}(\Omega)\) positive solutions of \((P_\lambda)\) for large \(\lambda\) \(\text{[3]}\). Here \(\text{cat}(\Omega)\) denotes the Lusternik-Schnirelman category of \(\Omega\).

(iii) For any \(k \in \mathbb{N}\), there exist \(k\) pairs of (possibly sign-changing) solutions \(\pm u_{1,\lambda}(x), \cdots, \pm u_{k,\lambda}(x)\) of \((P_\lambda)\) for large \(\lambda \geq \lambda(k)\). Moreover they converge to distinct solutions \(\pm u_1(x), \cdots, \pm u_k(x)\) of \(\text{[1.1]}\) after extracting a subsequence \(\lambda_n \to \infty\) \(\text{[4]}\).

Here we remark that in \(\text{[3]}, \text{[4]}\) they mainly consider the case where \(\Omega\) is connected.

In this paper we consider the case where \(\Omega\) consists of multiple connected components, \(\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_m\), and we consider the multiplicity of positive and sign-changing solutions for large \(\lambda\).

We studied the multiplicity of positive solutions in our previous paper \(\text{[10]}\) (see also Cao-Noussair \(\text{[10]}\)), where it was shown that for any choice of components \(\Omega_{k_1}, \cdots, \Omega_{k_\ell}\), there exists a positive solution \(u_\lambda(x)\) of \((P_\lambda)\) for large \(\lambda\) such that after extracting a subsequence \(\lambda_n \to \infty\),
\[
u_{\lambda_n}(x) \to \begin{cases} u_i(x) & \text{in } \Omega_i \ (i \in \{k_1, \cdots, k_\ell\}), \\ 0 & \text{in } \mathbb{R}^N \setminus (\Omega_{k_1} \cup \cdots \cup \Omega_{k_\ell}) \end{cases}
\]
strongly in \(H^1(\mathbb{R}^N)\). Here \(u_i(x)\) is a least energy solution of
\[
(1.4) \quad -\Delta u + u = |u|^{p-1}u \quad \text{in } \Omega_i, \quad u = 0 \quad \text{on } \partial \Omega_i.
\]
In particular, \((P_\lambda)\) has at least \(2^m - 1\) positive solutions for large \(\lambda\).

We remark that a solution \(u_i(x)\) of \(\text{[1.4]}\) is said to be a least energy solution if and only if
\[
\Psi_{i,D}(u_i) = \inf \{ \Psi_{i,D}(u) ; u(x) \in H^1_0(\Omega_i) \} \text{ is a nontrivial solution of } \text{[1.4]}\}
\]
holds. Here \(\Psi_{i,D}(u)\) is defined by
\[
\Psi_{i,D}(u) = \int_{\Omega_i} \frac{1}{2}(|\nabla u|^2 + u^2) - \frac{1}{p+1}|u|^{p+1} \, dx : H^1_0(\Omega_i) \to \mathbb{R},
\]
(“D” stands for Dirichlet boundary conditions.) It is natural to ask about the existence of a sequence of solutions of \((P_\lambda)\) converging to solutions of \(\text{[1.4]}\) in each \(\Omega_i\), which may not be least energy solutions.
First we study the multiplicity of sign-changing solutions. For the sake of simplicity, we assume that $\Omega$ consists of 2 components, that is, 

$$(1.5) \quad \Omega = \Omega_1 \cup \Omega_2.$$ 

In this case we have two limit problems (1.4), which are corresponding to $\Psi_{i,D} : H^1_0(\Omega_i) \to \mathbb{R}$ ($i = 1, 2$). It is well known that each functional has an unbounded sequence of critical points $(u^{1}_{j_i}(x))_{i=1}^{\infty} \subset H^1_0(\Omega_i)$ ($i = 1, 2$). A natural question is to ask, for a given pair $(u^{1}_{j_1}(x), u^{2}_{j_2}(x))$, whether $(P_\lambda)$ has a solution $u_\lambda(x) \in H^1(\mathbb{R}^N)$ converging to $u^{1}_{j_i}(x)$ in $\Omega_i$ and to 0 elsewhere. Here we try to give a partial answer to this problem. More precisely, we try to find a solution $u_\lambda(x) \in H^1(\mathbb{R}^N)$ which converges to $(u^{1}_{1}(x), u^{2}_{1}(x))$ after extracting a subsequence $\lambda_n \to \infty$. Here $u^{1}_{1}(x)$ is a mountain pass solution of (1.4) in $\Omega_1$ and $u^{2}_{1}(x)$ is a minimax solution of (1.4) in $\Omega_2$.

To find an unbounded sequence of critical values of a functional $I(u) \in C^1(E, \mathbb{R})$ defined on an infinite dimensional Hilbert space $E$, $\mathbb{Z}_2$-symmetry of $I(u) - I(\pm u) = I(u)$ for all $u \in E$—plays an important role. We remark that $\Psi_{\lambda}(u) \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and a functional $\tilde{\Psi}(u_1, u_2) = \Psi_{1,D}(u_1) + \Psi_{2,D}(u_2) \in C^1(H^1_0(\Omega_1) \times H^1_0(\Omega_2), \mathbb{R})$, which is corresponding to (1.4) in $\Omega_1 \cup \Omega_2$, have different symmetries. $\Psi_{\lambda}(u)$ is $\mathbb{Z}_2$-symmetric and $\tilde{\Psi}(u_1, u_2)$ is $(\mathbb{Z}_2)^2$-symmetric; that is,

$$\Psi_{\lambda}(su) = \Psi_{\lambda}(u) \quad \text{for all } s \in \mathbb{Z}_2 = \{-1, 1\}, \quad u \in H^1(\mathbb{R}^N),$$

$$\tilde{\Psi}(s_1 u_1, s_2 u_2) = \tilde{\Psi}(u_1, u_2) \quad \text{for all } s_1, s_2 \in \mathbb{Z}_2, \quad (u_1, u_2) \in H^1_0(\Omega_1) \times H^1_0(\Omega_2).$$

Note that $\mathbb{Z}_2$-action on $\Psi_{\lambda}(u)$ is corresponding to the following $\mathbb{Z}_2$-action on $\tilde{\Psi}(u_1, u_2)$:

$$\tilde{\Psi}(su_1, su_2) = \tilde{\Psi}(u_1, u_2) \quad \text{for all } s \in \{-1, 1\}, \quad (u_1, u_2) \in H^1_0(\Omega_1) \times H^1_0(\Omega_2),$$

and there are no symmetries of $\Psi_{\lambda}(u)$ corresponding to the $\mathbb{Z}_2$-symmetry of $\tilde{\Psi}(u_1, u_2)$:

$$(1.6) \quad \tilde{\Psi}(u_1, \pm u_2) = \tilde{\Psi}(u_1, u_2).$$

We also remark that solutions $(u^{1}_{1}(x), u^{2}_{1}(x))$ are obtained using the group action (1.6). Thus to construct solutions $u_\lambda(x)$ converging to $(u^{1}_{1}(x), u^{2}_{1}(x))$ we need to develop a kind of perturbation theory from symmetries, and in this paper we use ideas from Ambrosetti [1], Bahri-Berestycki [5], Struwe [23] and Rabinowitz [20] (see also Bahri-Lions [6], Tanaka [25] and Bolle [8]). In [1, 5, 6, 20, 23, 25], perturbation theories are developed for

$$-\Delta u = |u|^{p-2}u + f(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. They successfully showed the existence of an unbounded sequence of solutions for all $f(x) \in L^2(\Omega)$ for a certain range of $p$.

To state our result, we need some notation: we set

$$\Sigma_{i,D} = \{u \in H^1_0(\Omega_i); \|u\|_{H^1_0(\Omega_i)} = 1\}$$

and consider the constraint functional

$$(1.7) \quad J_{i,D}(v) = \max_{t > 0} \Psi_{i,D}(tv) : \Sigma_{i,D} \to \mathbb{R}.$$
Theorem 1.2. Assume

\[ \lambda \]

Theorem 1.1.

We define

\[ c_{\text{min}}^{1,D} = \inf_{v \in \Omega_{1,D}} J_{1,D}(v), \]
\[ b_n^{2,D} = \max_{\gamma \in \Gamma_n^{2,D}} J_{2,D}(\gamma(\theta)), \]

where \( S_n = \{ \theta \in \mathbb{R}^{n+1}; |\theta| = 1 \} \) and
\[ \Gamma_n^{2,D} = \{ \gamma \in C(S^n, \Sigma_{2,D}); \gamma(-\theta) = -\gamma(\theta) \text{ for all } \theta \in S^n \}. \]

We will observe that \( c_{\text{min}}^{1,D} \) and \( b_n^{2,D} \) are critical values of \( \Psi_{1,D}(u), \Psi_{2,D}(u) \) and
\[ \delta_0 \leq b_0^{2,D} \leq b_1^{2,D} \leq \cdots \leq b_n^{2,D} \leq b_{n+1}^{2,D} \leq \cdots, \quad b_n^{2,D} \to \infty \quad (n \to \infty). \]

In particular, there exists a sequence \( n(1) < n(2) < \cdots < n(k) < n(k+1) < \cdots \) such that \( b_{n(k)}^{2,D} < b_{n(k)+1}^{2,D} \). We also define another set of minimax values by
\[ c_k^{2,D} = \inf_{\sigma \in \Lambda_k} \max_{\theta \in S_n^{(k+1)}} J_{2,D}(\sigma(\theta)), \]

where \( S_n^{(k+1)} = \{ \theta = (\theta_1, \cdots, \theta_{n(k)+1}, \theta_{n(k)+2}); \theta \in S_{n(k)+1}, \theta_{n(k)+2} \geq 0 \} \) and
\[ \Lambda_k = \{ \sigma \in C(S_n^{(k+1)}, \Sigma_{2,D}); \sigma|_{S_n^{(k)}} \in \Gamma_n^{2,D}, \inf_{\theta \in S_n^{(k)}} \Psi_{2,D}(\sigma(\theta)) < b_{n(k)}^{2,D} + \delta_k \}. \]

Here \( \delta_k = \frac{1}{4}(b_{n(k)}^{2,D} - b_{n(k)+1}^{2,D}) > 0 \). We can also see that \( c_k^{2,D} \) is a critical value of \( \Psi_{2,D}(u) \) and \( c_k^{2,D} \to \infty \) as \( k \to \infty \).

Now we can give our main result.

**Theorem 1.1.** Assume (a1)-(a2) and (1.5). Then for any \( k \in \mathbb{N} \) there exists \( \lambda_1(k) \geq 1 \) such that for any \( \lambda \geq \lambda_1(k) \), \((P_{\lambda})\) has a solution \( u_{\lambda}(x) \) such that

(i) \[ \psi_{\lambda}(u_{\lambda}) \to c_{\text{min}}^{1,D} + c_k^{2,D} \quad \text{as } \lambda \to \infty. \]

(ii) For any given sequence \( \lambda_n \to \infty \), we can extract a subsequence \( \lambda_{n_1} \to \infty \) such that \( u_{\lambda_{n_1}}(x) \) converges to a function \( u(x) \) strongly in \( H^1(\mathbb{R}^N) \). Moreover we have

\[ u(x) \text{ satisfies } (1.4) \text{ in } \Omega_1 \cup \Omega_2, \]
\[ u|_{\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)} = 0, \]
\[ u(x) > 0 \quad \text{in } \Omega_1, \]
\[ \psi_{1,D}(u|_{\Omega_1}) = c_{\text{min}}^{1,D}, \quad \psi_{2,D}(u|_{\Omega_2}) = c_k^{2,D}. \]

When \( N = 1 \), we have a stronger result. Namely,

**Theorem 1.2.** Assume \( N = 1 \) and \( \Omega_1 = (a_1, b_1) \) \((i = 1, 2)\), where \( a_1 < b_1 < a_2 < b_2 \). Then for any solution \( v_1(x) \) of (1.1) in \( \Omega_1 \) and for any solution \( v_2(x) \) of (1.1) in \( \Omega_2 \), there exists a solution \( u_{\lambda}(x) \) for large \( \lambda \) such that

\[ u_{\lambda}(x) \to \begin{cases} v_1(x) & \text{in } \Omega_1, \\ v_2(x) & \text{in } \Omega_2, \\ 0 & \text{in } \mathbb{R} \setminus (\Omega_1 \cup \Omega_2) \end{cases} \]

strongly in \( H^1(\mathbb{R}) \) as \( \lambda \to \infty \).

Next we deal with positive solutions. Our result is influenced by the result [3].
Theorem 1.3. Assume (a1)–(a2), \( (1.5) \) and \( N \geq 3 \). Then there exists a \( p_1 \in (1, \frac{N+2}{N-2}) \) and \( \lambda_1(p) \geq 1 \) depending on \( p \in (1, \frac{N+2}{N-2}) \) such that for \( p \in (p_1, \frac{N+2}{N-2}) \) and \( \lambda \geq \lambda_1(p) \), \( (P_\lambda) \) possesses at least \( \text{cat}(\Omega_1) + \text{cat}(\Omega_2) + \text{cat}(\Omega_1 \times \Omega_2) \) positive solutions.

Remark 1.4. Since \( \text{cat}(\Omega_1 \cup \Omega_2) = \text{cat}(\Omega_1) + \text{cat}(\Omega_2) \), the argument of Bartsch-Wang \( 3 \) ensures \( \text{cat}(\Omega_1) + \text{cat}(\Omega_2) \) positive solutions, which converge to a positive solution of \( (1.3) \) in one of the components and to 0 elsewhere after extracting a subsequence.

In Section 6, we give a proof to our Theorems 1.3, 1.5. Here we will give a proof to Theorem 1.2 via Nehari type methods (cf. del Pino, Felmer \( 8 \)).

Similarly, we can deal with sign-changing solutions which converge to positive solutions in \( \Omega_1 \) and negative solutions in \( \Omega_2 \).

Theorem 1.5. Assume (a1)–(a2), \( (1.5) \) and \( N \geq 3 \). Then there exists a \( p_1 \in (1, \frac{N+2}{N-2}) \) and \( \lambda_1(p) \geq 1 \) depending on \( p \in (1, \frac{N+2}{N-2}) \) such that for \( p \in (p_1, \frac{N+2}{N-2}) \) and \( \lambda \geq \lambda_1(p) \), \( (P_\lambda) \) possesses at least \( \text{cat}(\Omega_1 \times \Omega_2) \) sign-changing solutions. Moreover, after extracting a subsequence \( \lambda_n \to \infty \), they converge to positive solutions of \( (1.3) \) in \( \Omega_1 \) and negative solutions of \( (1.3) \) in \( \Omega_2 \).

In the following sections we will give proofs to our theorems. We will take an approach different from that of Ding and Tanaka \( 16 \). In Section 2 we introduce a reduced functional \( J_\lambda(v_1, v_2) \), which is defined on an infinite dimensional torus \( \Sigma_{1,\lambda} \times \Sigma_{2,\lambda} \) and \( \Sigma_{2,\lambda} \) will be defined in Section 2 (d). We will observe that for large \( \lambda \) our reduced functional \( J_\lambda(v_1, v_2) \) can be approximated by \( J_{1,\lambda}(v_1) + J_{2,\lambda}(v_2) \) and \( J_{2,\lambda}(v_2) \) are defined as in \( (1.7) \) — in a suitable sense and that it enables us to develop a perturbation theory. In Sections 2–3 we develop a perturbation theory for \( J_\lambda \in C^1(\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}, \mathbb{R}) \). The subsolution estimate for Schrödinger operators (cf. Simon \( 22 \)) and a perturbation argument under the influence of \( 11 \) \( 15 \) \( 20 \) \( 23 \) play essential roles. Section 4 is devoted to showing Theorem 1.1. Isolatedness of critical values of \( \Psi_{1,D}(v_1) \) corresponding to positive solutions is important, and we will construct a deformation flow for \( J_\lambda \in C^1(\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}, \mathbb{R}) \), under which the level set \( \{v_1 \in \Sigma_{1,\lambda}; J_1(v_1) \leq c\} \) for the unperturbed functional \( J_1(v_1) \) is invariant for a certain range of \( c \). In Section 5 we will give a proof to Theorem 1.2 via Nehari type methods (cf. del Pino, Felmer and Tanaka \( 15 \)). In Section 6 we give a proof to our Theorems 1.3, 1.5. Here an estimate of the category of the level set \( \{(v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}; J_\lambda(v_1, v_2) \leq c\} \) is essential. Sections 7–9 are devoted to proving various propositions and lemmas which are used in previous sections without proofs.

2. Functional setting and variational formulation

(a) Functional setting. To find critical points of \( \Psi_\lambda(u) \) in \( H^1(\mathbb{R}^N) \), we introduce some notation: for an open set \( O \subset \mathbb{R}^N \) and \( \lambda \geq 1 \) we write

\[
\langle u, v \rangle_{\lambda,O} = \int_O \nabla u \nabla v + (\lambda^2 a(x) + 1)uv \, dx \quad \text{for } u, v \in H^1(O),
\]

\[
\|u\|_{\lambda,O} = \sqrt{\langle u, u \rangle_{\lambda,O}} \quad \text{for } u \in H^1(O),
\]

\[
\|f\|_{\lambda,O}^* = \sup_{u \in H^1(O), \|u\|_{\lambda,O} \leq 1} |f(u)| \quad \text{for } f \in (H^1(O))^*.
\]
By the assumption $\| \cdot \|_{\lambda, O}$ is equivalent to the standard $H^1$-norm
\[
\| u \|_{H^1(O)}^2 = \int_O |\nabla u|^2 + u^2 \, dx,
\]
and we have
\[
\| u \|_{H^1(O)} \leq \| u \|_{\lambda, O} \quad \text{for all } u \in H^1(O) \text{ and } \lambda \geq 1.
\]
(2.1)
We also use
\[
\| u \|_{L^r(O)} = \int_O |u|^r \, dx \quad \text{for } 1 \leq r < \infty,
\]
\[
\| u \|_{L^\infty(O)} = \sup_{x \in O}|u(x)|.
\]
For each $i \in \{1, 2\}$, we choose bounded open subsets $\Omega'_i$, $\Omega''_i$ with smooth boundaries such that
\[
\Omega_i \subset \subset \Omega'_i \subset \subset \Omega''_i \quad (i = 1, 2), \quad \overline{\Omega''_i} \cap \overline{\Omega'''_i} = \emptyset.
\]
Here and in what follows we write $A \subset \subset B$ if $\overline{A} \subset int(B)$. We will use the following lemma repeatedly with $O = \mathbb{R}^N \setminus (\Omega'_1 \cup \Omega'_2)$, $\Omega = (\Omega''_1 \setminus \Omega'_1) \cup (\Omega''_2 \setminus \Omega'_2)$, etc.

**Lemma 2.1.** For a domain $O \subset \mathbb{R}^N$ with $\overline{O} \cap (\overline{\Omega'_1} \cup \overline{\Omega''_2}) = \emptyset$ and $r \in [2, \frac{2N}{N-2})$ $(N \geq 3)$ and $r \in [2, \infty)$ $(N = 1, 2)$, there exists a constant $C_r(O, \lambda)$ satisfying $C_r(O, \lambda) \to 0$ as $\lambda \to \infty$ such that
\[
\| u \|_{L^r(O)} \leq C_r(O, \lambda) \| u \|_{\lambda, O} \quad \text{for all } u \in H^1(O).
\]

**Proof.** Let $a_\infty(O) = \inf_{x \in O} a(x) > 0$. Then we have
\[
\| u \|_{L^2(O)} \leq \frac{1}{\sqrt{a_\infty(O)}} \| u \|_{\lambda, O} \quad \text{for all } u \in H^1(O).
\]
On the other hand, by (2.1) we have
\[
\| u \|_{L^{\frac{2N}{N-2}}(O)} \leq C \| u \|_{\lambda, O} \quad (N \geq 3), \quad \| u \|_{L^r(O)} \leq C_q \| u \|_{\lambda, O} \quad (N = 1, 2),
\]
where $q > r$. Thus by the interpolation we have the conclusion of Lemma 2.1.

**(b) Modification of the nonlinearity** $|u|^{p-1}u$. To find solutions with asymptotic behaviors as in Theorems 1.1, 1.3 and 1.5 we take the local mountain pass approach due to del Pino and Felmer [14]. We choose an odd function $f(\xi) \in C^1(\mathbb{R}, \mathbb{R})$ such that for some $0 < \ell_1 < \ell_2$
\[
f(\xi) = |\xi|^{p-1}\xi \quad \text{for } |\xi| \leq \ell_1,
\]
\[
f(\xi) = \frac{1}{2}\xi \quad \text{for } |\xi| \geq \ell_2,
\]
(2.2)
\[
0 \leq f'(\xi) \leq \frac{2}{3} \quad \text{for all } \xi \in \mathbb{R}.
\]
We set
\[
g(x, \xi) = \begin{cases} |\xi|^{p-1}\xi & \text{if } x \in \Omega'_1 \cup \Omega'_2, \\
f(\xi) & \text{if } x \in \mathbb{R}^N \setminus (\Omega'_1 \cup \Omega'_2).
\end{cases}
\]
We also use the notation
\[
G(x, \xi) = \int_0^\xi g(x, s) \, ds.
\]
In what follows we will try to find critical points of

$$\Phi_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + (\lambda^2 a(x) + 1)u^2 \, dx - \int_{\mathbb{R}^N} G(x, u) \, dx$$

First we have

**Proposition 2.2.** For $\lambda \geq 1$, $\Phi_\lambda(u) \in C^2(\mathbb{H}^1(\mathbb{R}^N), \mathbb{R})$ satisfies (PS)$_c$ condition for all $c \in \mathbb{R}$; that is, any sequence $(u_n)_{n=1}^\infty \subset \mathbb{H}^1(\mathbb{R}^N)$ satisfying for $c \in \mathbb{R}$

$$\Phi_\lambda(u_n) \to c, \quad \Phi'_\lambda(u_n) \to 0 \quad \text{strongly in } \mathbb{H}^{-1}(\mathbb{R}^N)$$

has a strongly convergent subsequence in $\mathbb{H}^1(\mathbb{R}^N)$.

**Proposition 2.3.** Suppose that $(u_\lambda(x))_{\lambda \geq \lambda_0}$ is a family of critical points of $\Phi_\lambda(u)$ and assume that there exist constants $m, M > 0$ independent of $\lambda$ such that

$$m \leq \Phi_\lambda(u_\lambda) \leq M \quad \text{for all } \lambda \geq \lambda_0.$$

Then we have

(i) $\left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} m \leq \|u_\lambda\|_{L^p, \mathbb{R}^N}^2 \leq \left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} M$ for all $\lambda \geq \lambda_0$.

(ii) There exists $\lambda(M) \geq \lambda_0$ such that for $\lambda \geq \lambda(M)$, $u_\lambda(x)$ satisfies

$$|u_\lambda(x)| \leq \ell_1 \quad \text{for } x \in \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2).$$

In particular, $g(x, u_\lambda(x)) = |u_\lambda(x)|^{p-1}u_\lambda(x)$ holds in $\mathbb{R}^N$ and $u_\lambda(x)$ is a solution of the original problem $(P_\lambda)$.

(iii) After extracting a subsequence $\lambda_n \to \infty$, there exists $u \in \mathbb{H}^1(\mathbb{R}^N)$ such that

$$\|u_{\lambda_n} - u\|_{L^p, \mathbb{R}^N} \to 0 \quad \text{as } n \to \infty.$$

Moreover, $u(x)$ satisfies $u(x) \equiv 0$ in $\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)$ and

$$-\Delta u + u = |u|^{p-1}u \quad \text{in } \Omega_i,$$

$$u = 0 \quad \text{on } \partial \Omega_i$$

for $i = 1, 2$. It also holds that $\Phi_{\lambda_n}(u_{\lambda_n}) \to \Psi_{1,D}(u_{|\Omega_1}) + \Psi_{2,D}(u_{|\Omega_2})$ as $n \to \infty$.

Proofs of Propositions 2.2, 2.3 can be given as in Section 2 of our previous paper.

(c) Reduction to a problem on $\mathbb{H}^1(\Omega'_1 \cup \Omega'_2)$. To find critical points of $\Phi_\lambda(u)$, we introduce modified problems. First, we reduce our problem to a problem on $\mathbb{H}^1(\Omega'_1 \cup \Omega'_2)$. For a given $u \in \mathbb{H}^1(\Omega'_1 \cup \Omega'_2)$ we define

$$I_\lambda(u) = \inf_{w \in A_u} \Phi_\lambda(w),$$

where $A_u = \{w \in \mathbb{H}^1(\mathbb{R}^N); w = u \text{ on } \Omega'_1 \cup \Omega'_2\}$. First we have

**Proposition 2.4.** For any $u \in \mathbb{H}^1(\Omega'_1 \cup \Omega'_2)$ and $\lambda \geq 1$ there exists a unique minimizer $w_\lambda(u) \in A_u$. Moreover, $w_\lambda(u)$ satisfies

(i) $w_\lambda(u)$ is nondegenerate in the following sense:

$$\Phi''_{\lambda}(w_\lambda(u))(h, h) \geq \frac{1}{3} \|h\|_{L^p, (\Omega'_1 \cup \Omega'_2)}^2 \quad \text{for all } h \in \mathbb{H}^1_0(\mathbb{R}^N \setminus (\Omega'_1 \cup \Omega'_2)).$$

(ii) $\mathbb{H}^1(\Omega'_1 \cup \Omega'_2) \to \mathbb{H}^1(\mathbb{R}^N); u \mapsto w_\lambda(u)$ is of class $C^1$.  

Lemma 2.5. There exists a linear operator \( \Omega \) following properties: for \( u \in H^1(\Omega_1 \cup \Omega_2) \) is of class \( C^2 \),

\[
\int \mathcal{L}(u) = \Phi_\lambda(w(u)) : H^1(\Omega_1' \cup \Omega_2') \to \mathbb{R} \text{ is of class } C^2.
\]

Thus we get (2.9). Property (2.10) also follows from (2.9). \( \square \)

Proof of Proposition 2.4. We choose sets \( \Omega_1 \), \( \Omega_2 \) such that \( \Omega_i \subset \subset \Omega_i' \subset \subset \Omega_i'' \) for \( i = 1, 2 \). We need the following lemma.

Lemma 2.5. There exists a linear operator \( P : H^1(\Omega_1'' \cup \Omega_2') \to H^1(\mathbb{R}^N) \) with the following properties: for \( \lambda \geq 1 \) and \( u \in H^1(\Omega_1'' \cup \Omega_2') \)

\[
(2.6) \quad ||P_u||_{H^1(D_{out})} \leq C_1 ||u||_{H^1(D_{in})},
\]

\[
(2.7) \quad ||P_u||_{L^2(D_{out})} \leq C_1 ||u||_{L^2(D_{in})},
\]

\[
(2.8) \quad ||P_u||_{L^{p+1}(D_{out})} \leq C_1 ||u||_{L^{p+1}(D_{in})},
\]

\[
(2.9) \quad ||P_u||_{L^\infty(D_{out})} \leq C_1 ||u||_{L^\infty(D_{in})},
\]

\[
(2.10) \quad ||P_u||_{L^\infty(R^N)} \leq C_1 ||u||_{L^\infty(\Omega_1'' \cup \Omega_2')}.
\]

Here we use notation \( D_{out} = (\Omega_1'' \setminus \Omega_1') \cup (\Omega_2' \setminus \Omega_2'), \ D_{in} = (\Omega_1' \setminus \Omega_1) \cup (\Omega_2' \setminus \Omega_2), \) and \( C_1 > 0 \) is a constant independent of \( \lambda \geq 1 \) and \( u \in H^1(\Omega_1'' \cup \Omega_2') \).

Proof. (2.6)–(2.9) are rather standard. For (2.9) we have from (2.6)–(2.7)

\[
||P_u||_{L^\infty(D_{out})} \leq C_1 ||u||_{L^\infty(D_{in})} + \sum_{x \in D_{out}} a(x) \int_{D_{out}} \lambda^2 ||P_u||_{L^2(D_{out})}^2
\]

\[
\leq C_1 ||u||_{L^\infty(D_{in})} + C_1^2 \lambda^2 \sum_{x \in D_{out}} a(x) \int_{D_{out}} \lambda^2 a(x) ||P_u||_{L^2(D_{out})}^2 \int_{D_{out}} a(x) \int_{D_{out}} \lambda^2 a(x) u^2 dx
\]

\[
\leq C_1 ||u||_{L^\infty(D_{in})} + C_1^2 \lambda^2 \frac{\sup_{x \in D_{out}} a(x)}{\inf_{x \in D_{in}} a(x)} \int_{D_{in}} \lambda^2 a(x) u^2 dx
\]

Thus we get (2.9). Property (2.10) also follows from (2.9). \( \square \)

Proof of Proposition 2.4. First we show the existence and the uniqueness of a minimizer of \( \Phi_\lambda(w) \) in \( A_u \). It suffices to show that \( A_u \to \mathbb{R} : w \mapsto \Phi_\lambda(w) \) is coercive and strictly convex. Since \( w = u \) on \( \Omega_1' \cup \Omega_2' \) for all \( w \in A_u \), we need to show

\[
A_u \to \mathbb{R} : w \mapsto \int_{\Omega_1'' \setminus (\Omega_1' \cup \Omega_2')} G(x,w) dx
\]

is coercive and strictly convex. We note that

\[
|g(x,\xi)| = |f(\xi)| \leq \frac{2}{3} |\xi|, \quad G(x,\xi) \leq \frac{1}{3} |\xi|^2
\]
for all \( x \in \mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2') \) and \( \xi \in \mathbb{R} \). We can observe that
\[
\frac{1}{2} \| \lambda \|_{L^2(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2'))}^2 - \int_{\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')} G(x, w) \, dx \\
\geq \left( \frac{1}{2} \right) \| w \|_{L^2(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2'))}^2 - \frac{1}{3} \| w \|_{L^2(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2'))}^2 \geq \frac{1}{6} \| w \|_{L^2(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2'))}^2.
\]
and we get coerciveness of \( w \mapsto \Phi_\lambda(w) \). On the other hand, since \( A_w = \{ Pu + h; \, h \in H_0^1(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')) \} \), for strict convexity of \( \Phi_\lambda(w) \) on \( A_w \) it suffices to show that \( \Phi'_\lambda(w)(h, h) \geq \frac{1}{2} \| h \|_{L^2(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2'))}^2 \) for all \( w \in H^1(\mathbb{R}^N) \) and \( h \in H_0^1(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')) \). Indeed, we have by (2.21),
\[
\Phi'_\lambda(w)(h, h) = \left\| h \right\|_{L^2(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2'))}^2 - \int_{\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')} g'(x, w)h^2 \, dx \\
\geq \left\| h \right\|_{L^2(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2'))}^2 - \frac{1}{3} \| h \|_{L^2(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2'))}^2 \geq \frac{1}{6} \| h \|_{L^2(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2'))}^2.
\]
Thus \( w \mapsto \Phi_\lambda(w) \) is strictly convex. Therefore for any \( u \in H^1(\Omega_1' \cup \Omega_2') \) there exists a unique minimizer \( w_\lambda(u) \in A_w \) of \( \Phi_\lambda(w) \). (i) is also proved.

Next we show (ii). We consider the following mapping:
\[
F : \quad H^1(\Omega_1' \cup \Omega_2') \times H_0^1(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')) \to H^{-1}(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')) \\
\quad (u, \varphi) \mapsto \Phi'_\lambda(Pu + \varphi)|_{H_0^1(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2'))}.
\]
By the convexity of \( \varphi \mapsto \Phi_\lambda(Pu + \varphi) \), \( F(u, \varphi) = 0 \) holds if and only if \( \varphi = \varphi_\lambda(u) \), where \( \varphi_\lambda(u) : H^1(\Omega_1' \cup \Omega_2') \to H_0^1(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')) \) is defined by \( \varphi_\lambda(u) = w_\lambda(u) - Pu \). We have for \( h_1, h_2 \in H_0^1(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')) \)
\[
\langle F_\varphi(u, \varphi)h_1, h_2 \rangle = \int_{\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')} \nabla h_1 \nabla h_2 + (\lambda^2 a(x) + 1)h_1h_2 - g'(x, Pu + \varphi)h_1h_2 \, dx.
\]
By (2.22), we can observe that
\[
F_\varphi(u, \varphi) : H_0^1(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')) \to H^{-1}(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2'))
\]
is invertible for all \( u, \varphi \in H^1(\Omega_1' \cup \Omega_2') \times H_0^1(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')) \). Thus, by the implicit function theorem, we can observe \( u \mapsto \varphi_\lambda(u) \) is of class \( C^1 \). That is, \( u \mapsto w_\lambda(u) = Pu + \varphi_\lambda(u) \) is of class \( C^1 \).

For (iii), we remark that \( I_\lambda(u) = \Phi_\lambda(w_\lambda(u)) = \Phi_\lambda(Pu + \varphi_\lambda(u)) \) holds. We also remark that
\[
(2.11) \quad \Phi'_\lambda(Pu + \varphi_\lambda(u))h = 0 \quad \text{for all } h \in H_0^1(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')).
\]
Noting \( \varphi'_\lambda(\zeta) \in H_0^1(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')) \) for all \( \zeta \in H^1(\Omega_1' \cup \Omega_2') \), we have
\[
(2.12) \quad I_\lambda'(u) = \Phi'_\lambda(Pu + \varphi_\lambda(u))(P\zeta) \quad \text{for all } \zeta \in H^1(\Omega_1' \cup \Omega_2').
\]
From \( C^1 \)-continuity of \( \varphi_\lambda(u) \) and (2.12), it follows that \( I_\lambda(u) \) is of class \( C^2 \).

Next we observe (iv). By (2.22) the “if” part clearly holds. Suppose that \( I_\lambda'(u) = 0 \). Thus by (2.11), (2.12) we have
\[
\Phi'_\lambda(w_\lambda(u))(P\zeta + h) = 0 \quad \text{for all } \zeta \in H^1(\Omega_1' \cup \Omega_2') \text{ and } h \in H_0^1(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')).
\]
Since \( H^1(\mathbb{R}^N) = \{ Pu + h; \, \zeta \in H^1(\Omega_1' \cup \Omega_2'), h \in H_0^1(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')) \} \), we have \( \Phi'_\lambda(w_\lambda(u)) = 0 \). Thus the “only if” part also holds.
For (v), we note that
\[ \|\varphi\|_{L^2(\Omega_1 \cup \Omega_2)} \leq \|\varphi\|_{\lambda, R^N}, \]
\[ \varphi - P\varphi|_{\Omega_1 \cup \Omega_2} \in H_0^1(\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)) \]
for all \( \varphi \in H^1(\mathbb{R}^N) \). Thus by (2.11, 2.12)
\[ \Phi_\lambda(w_{\lambda}(u)) - \Phi_\lambda(w_{\lambda}(u))(P\varphi|_{\Omega_1 \cup \Omega_2} + (\varphi - P\varphi|_{\Omega_1 \cup \Omega_2})) = \Phi_\lambda(w_{\lambda}(u))(P\varphi|_{\Omega_1 \cup \Omega_2}) = I_\lambda(u)(\varphi|_{\Omega_1 \cup \Omega_2}). \]
Thus for any \( \varphi \in H^1(\mathbb{R}^N) \) with \( \|\varphi\|_{\lambda, R^N} \leq 1 \)
\[ |\Phi_\lambda(w_{\lambda}(u))\varphi| \leq \|I_\lambda(u)\|_{\lambda, \Omega_1 \cup \Omega_2} \|\varphi\|_{\lambda, \Omega_1 \cup \Omega_2} \leq \|I_\lambda(u)\|_{\lambda, \Omega_1 \cup \Omega_2}. \]
That is, we have
\[ \|\Phi_\lambda(w_{\lambda}(u))\|_{\lambda, R^N} \leq \|I_\lambda(u)\|_{\lambda, \Omega_1 \cup \Omega_2}. \]
Thus, if \( (u_n)_{n=1}^\infty \subset H^1(\Omega_1 \cup \Omega_2) \) is a \((PS)_c\)-sequence for \( I_\lambda(u) \), i.e., \( I_\lambda(u_n) \to c \) and \( \|I'_\lambda(u_n)\|_{\lambda, \Omega_1 \cup \Omega_2} \to 0 \), then \( (w_n(u_n))_{n=1}^\infty \subset H^1(\mathbb{R}^N) \) is a \((PS)_c\)-sequence for \( \Phi_\lambda(u) \).
By Proposition 2.2 we can see \( (w_{\lambda}(u_n))_{n=1}^\infty \) has a strongly convergent subsequence in \( H^1(\mathbb{R}^N) \). Thus \( u_n = w_{\lambda}(u_n)|_{\Omega_1 \cup \Omega_2} \) has a strongly convergent subsequence in \( H^1(\Omega_1 \cup \Omega_2) \) and \((PS)_c\) condition holds for \( I_\lambda(u) \).

For later use, we consider the following linear equation: let
\[ V(x) \in L^\infty(\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)), \]
and consider the following linear boundary value problem:
\begin{align}
(2.13) & \quad -\Delta w + (\lambda^2 a(x) + 1)w = V(x)w \quad \text{in } \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2), \\
(2.14) & \quad w = u \quad \text{on } \partial \Omega_1 \cup \partial \Omega_2.
\end{align}
We have the following

**Lemma 2.6.** Suppose that \( V(x) \in L^\infty(\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)) \) satisfies
\[ (2.15) \quad \|V(x)\|_{L^\infty(\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2))} \leq \frac{2}{3}. \]
Then the linear boundary value problem (2.13–2.14) has a unique solution — we denote it by \( Q_{V, \lambda}(u) \) — for all \( u \in H^1(\Omega_1 \cup \Omega_2) \). Moreover \( Q_{V, \lambda} \) defines a bounded linear operator \( Q_{V, \lambda} : H^1(\Omega_1 \cup \Omega_2) \to H^1(\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)) \), and there exists a constant \( C_2 > 0 \) independent of \( \lambda \geq 1 \) such that
\[ (2.16) \quad \|Q_{V, \lambda}(u)\|_{\lambda, R^N \setminus (\Omega_1 \cup \Omega_2)} \leq C_2\|u\|_{\lambda, (\Omega_1 \cup \Omega_2)} \quad \text{for all } u \in H^1(\Omega_1 \cup \Omega_2). \]

**Proof of Lemma 2.6.** Set
\[ F(w) = \int_{\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)} |\nabla w|^2 + (\lambda^2 a(x) + 1)w^2 - V(x)w^2 \, dx \quad : \, H^1(\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)) \to \mathbb{R}. \]
Then solutions of (2.13–2.14) can be characterized as critical points of \( F(w) \) on \( A_u = \{ w \in H^1(\mathbb{R}^N); w = u \text{ on } \Omega_1 \cup \Omega_2 \} \). By assumption (2.15) we have
\[ (2.17) \quad F(w) \geq \|w\|_{L^2(\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2))}^2 - \frac{2}{3}\|w\|_{L^2(\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2))}^2 \geq \frac{1}{3}\|w\|_{L^2(\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2))}^2. \]
Thus \( F(w) \) is coercive and convex on \( A_u \) and we have the existence and the uniqueness of solutions (2.13–2.14).
To show (2.10), we note that \( Q_{V, \lambda}(u) \) can be characterized as

\[
F(Q_{V, \lambda}(u)) = \inf_{w \in A_u} F(w).
\]

In particular, \( F(Q_{V, \lambda}(u)) \leq F(Pu) \), where \( Pu \) is defined in Lemma 2.6. Thus by (2.17) and (2.20)

\[
\frac{1}{3} \|Q_{V, \lambda}(u)\|_\lambda^2_{R^N \setminus (\Omega_1 \cup \Omega_2)} \leq F(Q_{V, \lambda}(u)) \leq F(Pu)
\]

\[
\leq \|Pu\|_{\lambda}^2_{R^N \setminus (\Omega_1 \cup \Omega_2)} + \int_{R^N \setminus (\Omega_1 \cup \Omega_2)} V(x)(Pu)^2 \, dx
\]

\[
\leq (1 + \frac{2}{3}) \|Pu\|_{\lambda}^2_{R^N \setminus (\Omega_1 \cup \Omega_2)}
\]

\[
\leq \frac{5}{3} C^2 \|u\|_{\lambda}^2_{(\Omega_1 \setminus \Omega_0) \cup (\Omega_2 \setminus \Omega_0)}.
\]

Thus we get (2.16). □

**Remark 2.7.** Since \( Q_{V, \lambda}(u) \) is a unique solution of (2.13)–(2.14), we can give another characterization: for \( \zeta \in H^1(R^N \setminus (\Omega_1 \cup \Omega_2)) \), \( \zeta = Q_{V, \lambda}(u) \) holds if and only if

\[
\zeta - Pu \in \tilde{H}_0^1(R^N \setminus (\Omega_1 \cup \Omega_2))
\]

and

\[
F'(\zeta) \phi = 0 \quad \text{for all } \phi \in \tilde{H}_0^1(R^N \setminus (\Omega_1 \cup \Omega_2)).
\]

We remark that \( F'(\zeta) \phi = 0 \) can be rewritten as

\[
\langle \zeta, \phi \rangle_{\lambda, R^N \setminus (\Omega_1 \cup \Omega_2)} - \int_{R^N \setminus (\Omega_1 \cup \Omega_2)} V(x) \zeta \phi \, dx = 0 \quad \text{for all } \phi \in \tilde{H}_0^1(R^N \setminus (\Omega_1 \cup \Omega_2)).
\]

**Remark 2.8.** (i) For any \( u \in H^1(\Omega_1 \cup \Omega_2) \) we remark that \( V(x) = f(w_{\lambda}(u))/w_{\lambda}(u) \) satisfies \( \|V(x)\|_{L^\infty(\Omega_1 \setminus (\Omega_1 \cup \Omega_2))} \leq \frac{2}{3} \), that is, \( V(x) \) satisfies (2.15). In particular, \( w_{\lambda}(x) \) satisfies \( w_{\lambda}(u) = Q_{f(w_{\lambda}(u))/w_{\lambda}(u), \lambda}(u) \).

(ii) For \( h \in H^1(\Omega_1 \cup \Omega_2) \), it holds that

\[
w_{\lambda}(u)(h)(x) = \begin{cases} h(x) & \text{for } x \in \Omega_1 \cup \Omega_2, \\ (Q_{f(w_{\lambda}(u)), \lambda}(h))(x) & \text{for } x \in R^N \setminus (\Omega_1 \cup \Omega_2). \end{cases}
\]

In fact, \( w_{\lambda}(u) \) satisfies

\[
\Phi'_{\lambda}(w_{\lambda}(u)) \phi = 0 \quad \text{for all } \phi \in \tilde{H}_0^1(R^N \setminus (\Omega_1 \cup \Omega_2)).
\]

Differentiating with respect to \( u \), we have

\[
\Phi''_{\lambda}(w_{\lambda}(u))(w_{\lambda}'(u))h, \phi = 0 \quad \text{for all } h \in H^1(\Omega_1 \cup \Omega_2) \text{ and } \phi \in \tilde{H}_0^1(R^N \setminus (\Omega_1 \cup \Omega_2)).
\]

That is,

\[
\langle w_{\lambda}'(u)h, \phi \rangle_{\lambda, R^N \setminus (\Omega_1 \cup \Omega_2)} - \int_{R^N \setminus (\Omega_1 \cup \Omega_2)} f'(w_{\lambda}(u))(w_{\lambda}'(u))h \phi \, dx = 0.
\]

Since \( w_{\lambda}(u) - Pu \in \tilde{H}_0^1(R^N \setminus (\Omega_1 \cup \Omega_2)) \), we also have

\[
w_{\lambda}'(u)h - Ph \in \tilde{H}_0^1(R^N \setminus (\Omega_1 \cup \Omega_2)).
\]

By Remark 2.7 it follows from (2.21)–(2.22) that

\[
(w_{\lambda}'(u)h)|_{R^N \setminus (\Omega_1 \cup \Omega_2)} = Q_{f(w_{\lambda}(u)), \lambda}(h).
\]

Since \( w_{\lambda}(u)|_{\Omega_1 \cup \Omega_2} = u \), we have (2.19).
In what follows, we identify \( H^1(\Omega_1' \cup \Omega_2') \) and \( H^1(\Omega_1') \oplus H^1(\Omega_2') \), and for functions \( u_1 \in H^1(\Omega_1') \), \( u_2 \in H^1(\Omega_2') \), we write \( u = (u_1, u_2) \) if \( u \in H^1(\Omega_1' \cup \Omega_2') \) satisfies \( u_1 = u|_{\Omega_1'} \), \( u_2 = u|_{\Omega_2'} \). When \( u = (u_1, u_2) \), we also write
\[
I_\lambda(u_1, u_2) \quad \text{for} \quad I_\lambda(u),
\]
\[
Q_{V,\lambda}(u_1, u_2) \quad \text{for} \quad Q_{V,\lambda}(u),
\]
\[
||u||_{L^2(\Omega_1' \cup \Omega_2')}^2 = ||u||_{L^2(\Omega_1')}^2 + ||u||_{L^2(\Omega_2')}^2.
\]
By \( 2.19 \), it is convenient to extend \( Q_{V,\lambda} \) as an operator \( H^1(\Omega_1' \cup \Omega_2') \to H^1(\mathbb{R}^N) \)
by
\[
(Q_{V,\lambda}(u))(x) = u(x) \quad \text{for} \quad x \in \Omega_1' \cup \Omega_2'.
\]
We also denote the extended operator by \( Q_{V,\lambda} \). With this notation we have
\[
w_\lambda'(u)(h) = Q_{f'(w_\lambda(u)),\lambda}(h).
\]
By (iii) of Proposition \( 2.4 \), we have for \( h \in H^1(\Omega_1' \cup \Omega_2') \)
\[
I_\lambda'(u)(h) = \Phi_\lambda'(w_\lambda(u))(w_\lambda'(u)h) = \Phi_\lambda'(w_\lambda(u))Q_{f'(w_\lambda(u)),\lambda}(h).
\]
Since \( Q_{f'(w_\lambda(u)),\lambda}(h) - Q_0(h) \in H^1_0(\mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2')) \), it follows from \( 2.20 \) that
\[
I_\lambda'(u)h = \Phi_\lambda'(w_\lambda(u))Q_0(h) \quad \text{for all} \quad h \in H^1(\Omega_1' \cup \Omega_2').
\]
Proof of the following lemma will be given in Section \( 7 \).

**Lemma 2.9.** Let \( V(x) : \mathbb{R}^N \setminus (\Omega_1' \cup \Omega_2') \to \mathbb{R} \) be a function satisfying \( 2.15 \). Then there exist \( C_3 > 0 \) independent of \( \lambda \geq 1 \) and \( r_\lambda > 0 \) satisfying \( r_\lambda \to 0 \) as \( \lambda \to \infty \) such that
\[
\begin{align*}
(\text{i}) \quad ||Q_{V,\lambda}(u)||_{L^\infty(\Omega_1' \cup \Omega_2')} & \leq C_3 ||u||_{L^\infty(\Omega_1' \cup \Omega_2')} \quad \text{for all} \quad u \in H^1(\Omega_1' \cup \Omega_2'), \\
(\text{ii}) \quad ||Q_{V,\lambda}(u) - Q_0(\lambda)(u)||_{L^\infty(\Omega_1' \cup \Omega_2')} & \leq r_\lambda ||u||_{L^\infty(\Omega_1' \cup \Omega_2')} \quad \text{for all} \quad u \in H^1(\Omega_1' \cup \Omega_2'), \\
(\text{iii}) \quad ||Q_0(\lambda)(u_1, 0), Q_0(\lambda)(0, u_2)||_{L^\infty(\Omega_1' \cup \Omega_2')} & \leq r_\lambda ||u_1||_{L^\infty(\Omega_1')} ||u_2||_{L^\infty(\Omega_2')} \quad \text{for all} \quad u \in H^1(\Omega_1' \cup \Omega_2'), \\
(\text{iv}) \quad ||w_\lambda(\lambda)(u)||_{L^\infty(\Omega_1' \cup \Omega_2')} & \leq C_3 ||u||_{L^\infty(\Omega_1' \cup \Omega_2')} , \\
||w_\lambda(\lambda)(u) - Q_0(\lambda)(u)||_{L^\infty(\Omega_1' \cup \Omega_2')} & \leq r_\lambda ||u||_{L^\infty(\Omega_1' \cup \Omega_2')} \quad \text{for all} \quad u \in H^1(\Omega_1' \cup \Omega_2').
\end{align*}
\]
Here and throughout this paper we denote by \( r_\lambda \) various constants independent of \( u \) but which depend on \( \lambda \) and satisfy \( r_\lambda \to 0 \) as \( \lambda \to \infty \).

The following functionals correspond to “unperturbed” problems and will be useful later:
\[
I_{1,\lambda}(u_1) = \frac{1}{2} ||u_1||_{L^2(\Omega_1')}^2 - \frac{1}{p + 1} ||u_1||_{L^{p+1}(\Omega_1')}^{p+1} \\
+ \frac{1}{2} ||Q_0(\lambda)(u_1, 0)||_{L^2(\Omega_1' \cup \Omega_2')}^2 \in C^2(H^1(\Omega_1'), \mathbb{R}),
\]
\[
I_{2,\lambda}(u_2) = \frac{1}{2} ||u_2||_{L^2(\Omega_2')}^2 - \frac{1}{p + 1} ||u_2||_{L^{p+1}(\Omega_2')}^{p+1} \\
+ \frac{1}{2} ||Q_0(\lambda)(0, u_2)||_{L^2(\Omega_1' \cup \Omega_2')}^2 \in C^2(H^1(\Omega_2'), \mathbb{R}).
\]
We will study \( I_\lambda(u_1, u_2) \) as a perturbation from \( I_{1,\lambda}(u_1) + I_{2,\lambda}(u_2) \).

**Lemma 2.10.** There exists \( r_\lambda > 0 \) such that for \( (u_1, u_2) \in H^1(\Omega_1' \cup \Omega_2') = H^1(\Omega_1') \oplus H^1(\Omega_2') \),
\[
(0) \quad r_\lambda \to 0 \quad \text{as} \quad \lambda \to \infty.
\]
\[
(1) \quad |I_\lambda(u_1, u_2) - I_{1,\lambda}(u_1) - I_{2,\lambda}(u_2)| \leq \frac{1}{2} r_\lambda (||u_1||_{L^\infty(\Omega_1')} + ||u_2||_{L^\infty(\Omega_2')}^2).
\]
(ii) \( \frac{\partial I}{\partial u_1}(u_1, u_2) - I_{1, \lambda}(u_1) \|_{L^p(\Omega')} \leq r \lambda \| u_1 \|_{L^p(\Omega')} + \| u_2 \|_{L^p(\Omega')} \).

(iii) \( \frac{\partial I}{\partial u_2}(u_1, u_2) - I_{2, \lambda}(u_2) \|_{L^p(\Omega')} \leq r \lambda \| u_1 \|_{L^p(\Omega')} + \| u_2 \|_{L^p(\Omega')} \).

We use the notation:

\[
\| u_1 \|_{L^p(\Omega')}^2 = \| u_1 \|_{L^p(\Omega')}^2 + \| Q_{0, \lambda}(u_1, 0) \|_{L^p(\Omega')}^2 \quad \text{for } u_1 \in H^1(\Omega'),
\]

\[
\| u_2 \|_{L^p(\Omega')}^2 = \| u_2 \|_{L^p(\Omega')}^2 + \| Q_{0, \lambda}(0, u_2) \|_{L^p(\Omega')}^2 \quad \text{for } u_2 \in H^1(\Omega').
\]

Then \( \| \cdot \|_{L^p(\Omega')} \) is a norm on \( H^1(\Omega') \), and we have from Lemma 2.10 that

\[
\| u \|_{L^p(\Omega')} \leq (1 + C_3) \| u \|_{L^p(\Omega')} \quad \text{for all } u \in H^1(\Omega'),
\]

and \( \| \cdot \|_{L^p(\Omega')} \) is equivalent to \( \| \cdot \|_{L^p(\Omega')} \). As a corollary to Lemma 2.10, we have

**Corollary 2.11.** For all \( u = (u_1, u_2) \in H^1(\Omega') \oplus H^1(\Omega') \)

(i)

\[
\frac{1}{2} (1 - r \lambda) \left( \| u_1 \|_{L^p(\Omega')}^2 + \| u_2 \|_{L^p(\Omega')}^2 \right) - \frac{1}{p + 1} \left( \| u_1 \|_{L^{p+1}(\Omega')}^p + \| u_2 \|_{L^{p+1}(\Omega')}^p \right) 
\leq I_{1, \lambda}(u_1, u_2)
\]

\[
\leq \frac{1}{2} (1 + r \lambda) \left( \| u_1 \|_{L^p(\Omega')}^2 + \| u_2 \|_{L^p(\Omega')}^2 \right) - \frac{1}{p + 1} \left( \| u_1 \|_{L^{p+1}(\Omega')}^p + \| u_2 \|_{L^{p+1}(\Omega')}^p \right).
\]

(ii)

\[
(1 - r \lambda) \| u_1 \|_{L^p(\Omega')}^2 - \| u_1 \|_{L^{p+1}(\Omega')}^p + (1 - r \lambda) \| u_2 \|_{L^p(\Omega')}^2 - \| u_2 \|_{L^{p+1}(\Omega')}^p \leq I_{2, \lambda}(u_1, u_2)
\]

\[
\leq (1 + r \lambda) \| u_1 \|_{L^p(\Omega')}^2 - \| u_1 \|_{L^{p+1}(\Omega')}^p + (1 + r \lambda) \| u_2 \|_{L^p(\Omega')}^2 - \| u_2 \|_{L^{p+1}(\Omega')}^p.
\]

We also remark that \( I_{1, \lambda}(u) \ (i = 1, 2) \) has the following properties:

**Lemma 2.12.** For \( i = 1, 2 \) we have

(i) \( I_{i, \lambda}(u_i) : H^1(\Omega') \to \mathbb{R} \) has a mountain pass geometry. That is, there are constants \( r_0, \delta_0 > 0 \) and \( c_i \in H^1_0(\Omega') \) independent of \( \lambda \geq 1 \) such that

\[
I_{i, \lambda}(u_i) \geq \delta_0 \quad \text{for all } u_i \in H^1(\Omega') \quad \text{with } \| u_i \|_{L^p(\Omega')} = r_0,
\]

\[
\| c_i \|_{L^p(\Omega')} > r_0 \quad \text{and} \quad I_{i, \lambda}(c_i) < 0.
\]

(ii) \( I_{i, \lambda}(u_i) \) satisfies the \((PS)_c\)-condition for all \( c \in \mathbb{R} \).

**Proof of Lemma 2.12.** For \( i = 1, 2 \), we remark that \( I_{i, \lambda}(u) \) can be written as

\[
I_{i, \lambda}(u) = \frac{1}{2} \| u \|_{L^p(\Omega')}^2 - \frac{1}{p + 1} \| u \|_{L^{p+1}(\Omega')}^p,
\]

and we can check (i), (ii) in a rather standard way. \( \square \)

**d) Reduction to a problem on** \( \Sigma_{1, \lambda} \times \Sigma_{2, \lambda} \). Next we reduce our problem on an infinite dimensional torus \( \Sigma_{1, \lambda} \times \Sigma_{2, \lambda} \), where

\[
\Sigma_{1, \lambda} = \{ v \in H^1(\Omega') : \| v \|_{L^p(\Omega')} = 1 \} \quad \text{for } i = 1, 2.
\]

We define \( J_{i, \lambda} : \Sigma_{1, \lambda} \to (0, \infty) \) \((i = 1, 2)\), \( J_{\lambda} : \Sigma_{1, \lambda} \times \Sigma_{2, \lambda} \to (0, \infty) \) by

\[
J_{i, \lambda}(v_i) = \sup_{s \geq 0} I_{i, \lambda}(sv_i), \quad J_{\lambda}(v_1, v_2) = \sup_{s, t \geq 0} I_{\lambda}(sv_1, tv_2).
\]
For \( J_{i,\lambda}(v_i) \) \((i = 1, 2)\), we can easily see that \( I_{i,\lambda}(sv_i) : [0, \infty) \to \mathbb{R} \) takes a global maximum at
\[
(2.27) \quad s = s_{i,\lambda}(v_i) = \|v_i\|^{\frac{2}{p+1}}_{\lambda,\Omega_i'} \|v_i\|^{\frac{p+1}{p-1}}_{L^{p+1}(\Omega_i')}
\]
and
\[
(2.28) \quad J_{i,\lambda}(v_i) = I_{i,\lambda}(s_{i,\lambda}(v_i)v_i) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \inf_{v_i \in H^1(\Omega_i')} \|v_i\|^{\frac{2}{p+1}}_{\lambda,\Omega_i'} \|v_i\|^{\frac{p+1}{p-1}}_{L^{p+1}(\Omega_i')} \right)^{\frac{2(p+1)}{p-1}}.
\]
Recalling \( \|v_i\|_{H^1(\Omega_i')} \leq \|v_i\|_{\lambda,\Omega_i'} \leq \|v_i\|_{\lambda,\Omega_i'} \) for \( v_i \in \Sigma_{i,\lambda} \), we also have
\[
(2.29) \quad \inf_{v_i \in \Sigma_{i,\lambda}} J_{i,\lambda}(v_i) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \inf_{v_i \in H^1(\Omega_i')} \|v_i\|^{\frac{2}{p+1}}_{\lambda,\Omega_i'} \|v_i\|^{\frac{p+1}{p-1}}_{L^{p+1}(\Omega_i')} \right)^{\frac{2(p+1)}{p-1}}.
\]

**Remark 2.13.** In a similar way to (2.28), we have for \( J_{i,D}(v_i) \) defined in (1.7)
\[
J_{i,D}(v_i) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{1}{\|v_i\|_{L^{p+1}(\Omega_i)}} \right)^{\frac{2(p+1)}{p-1}} \text{ for } v_i \in \Sigma_{i,D}.
\]
For \( J_{\lambda}(v_1, v_2) \) we have

**Lemma 2.14.** (i) There exists \( r_\lambda > 0 \) satisfying \( r_\lambda \to 0 \) as \( \lambda \to \infty \) such that
\[
(1 - r_\lambda) \frac{2(p+1)}{p-1} (J_{1,\lambda}(v_1) + J_{2,\lambda}(v_2)) \leq J_{\lambda}(v_1, v_2) \leq (1 + r_\lambda) \frac{2(p+1)}{p-1} (J_{1,\lambda}(v_1) + J_{2,\lambda}(v_2))
\]
for all \((v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}\).

(ii) \( \inf_{(v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} J_{\lambda}(v_1, v_2) \geq m_1 + m_2 \), where \( m_1, m_2 > 0 \) are defined in (2.29).

**Proof:** (i) easily follows from (ii) of Corollary 2.11. For (ii), since \( \|u_i\|_{\lambda,\Omega_i'} \geq \|u_i\|_{H^1(\Omega_i')} \) we have
\[
I_{\lambda}(u_1, u_2) \geq \frac{1}{2} \left( \|u_1\|^2_{H^1(\Omega_1')} + \|u_2\|^2_{H^1(\Omega_2')} \right) - \frac{1}{p+1} \left( \|u_1\|^{p+1}_{L^{p+1}(\Omega_1')} + \|u_2\|^{p+1}_{L^{p+1}(\Omega_2')} \right).
\]
Thus (ii) follows.

In what follows, we use the following notation: for \( M \in \mathbb{R} \)
\[
[J_\lambda \leq M |_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} = \{(v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}; J_\lambda(v_1, v_2) \leq M \},
\]
\[
[J_\lambda < M |_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} = \{(v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}; J_\lambda(v_1, v_2) < M \}.
\]
The following propositions state fundamental properties of \( J_{\lambda}(v_1, v_2) \), which we will use repeatedly in the rest of the paper.

**Proposition 2.15.** For any \( M > 0 \) there exist constants \( \lambda(M), \delta_M > 0 \) such that for \( \lambda \geq \lambda(M) \)
\[
(i) \quad \|v_1\|^{p+1}_{L^{p+1}(\Omega_1')} \geq \delta_M, \|v_2\|^{p+1}_{L^{p+1}(\Omega_2')} \geq \delta_M \text{ for all } (v_1, v_2) \in [J_\lambda \leq M |_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}}.
\]

(ii) For any \((v_1, v_2) \in [J_\lambda \leq M |_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}}, (s, t) \mapsto J_\lambda(sv_1, tv_2) \) has a unique maximizer — we denote \((s_\lambda(v_1, v_2), t_\lambda(v_1, v_2)) \) — and it is nondegenerate.
There exist constants $R_0$, $R_1(M) > 0 - R_1(M) > 0$ depends on $M > 0$ but is independent of $(v_1, v_2)$, and $R_0$ is independent of $M$, $(v_1, v_2)$ — such that

\begin{equation}
(2.30) \quad s_\lambda(v_1, v_2)^2 + t_\lambda(v_1, v_2)^2 \leq R_1(M)^2,
\end{equation}

and

\begin{equation}
(2.31) \quad s_\lambda(v_1, v_2), t_\lambda(v_1, v_2) \geq R_0.
\end{equation}

(iv) $(s_\lambda(v_1, v_2), t_\lambda(v_1, v_2)) : [J_\lambda < M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} \to \mathbb{R}^2$ is of class $C^1$ and $J_\lambda : [J_\lambda < M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} \to \mathbb{R}$ is of class $C^1$.

(v) There exists $r_\lambda > 0$ such that

\begin{align}
& r_\lambda \to 0 \quad \text{as} \quad \lambda \to \infty, \\
& (2.32) \quad |J_\lambda(v_1, v_2) - J_{1,\lambda}(v_1) - J_{2,\lambda}(v_2)| < r_\lambda, \\
& (2.33) \quad |J'_\lambda(v_1, v_2)(h_1, h_2) - J'_{1,\lambda}(v_1)h_1 - J'_{2,\lambda}(v_2)h_2| \\
& \quad < r_\lambda(\|h_1\|_{L^1, \Omega_1^0} + \|h_2\|_{L^1, \Omega_2^0})
\end{align}

for all $(v_1, v_2) \in [J_\lambda < M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}}$ and $(h_1, h_2) \in H^1(\Omega_1^0) \oplus H^1(\Omega_2^0)$ satisfying $\langle v_1, h_1 \rangle_{L^1, \Omega_1^0} = \langle v_2, h_2 \rangle_{L^1, \Omega_2^0} = 0$.

**Proposition 2.16.** For $M > 0$ suppose $\lambda \geq \lambda(M)$. Then $J_\lambda(v_1, v_2)$ satisfies the $(PS)_c$-condition for $c \in (-\infty, M)$.

Propositions 2.15 and 2.10 enable us to construct deformation flows to ensure the existence of critical points $J_\lambda(v_1, v_2)$. In Section 9 we will construct deformation flows in various settings.

(e) **Convergence and positivity of critical points.** The following proposition describes the behavior of critical points $v_\lambda = (v_{1,\lambda}, v_{2,\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ as $\lambda \to \infty$.

**Proposition 2.17.** Suppose that $v_\lambda = (v_{1,\lambda}, v_{2,\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ satisfies for some $c > 0$

\begin{equation}
(2.34) \quad \|J'_\lambda(v_\lambda)\|_{\left(\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}\right)} \to 0, \quad J_\lambda(v_\lambda) \to c \quad \text{as} \quad \lambda \to \infty.
\end{equation}

Then after extracting a subsequence $\lambda_n \to \infty$ we have

(i) There exists $v_\infty = (v_{1,\infty}, v_{2,\infty}) \in \Sigma_{1,D} \times \Sigma_{2,D}$ such that for $i = 1, 2$

\begin{align}
& (2.35) \quad \|v_{1,\lambda_n} - v_{1,\infty}\|_{H^1(\Omega_i)} \to 0, \quad \|v_{1,\lambda_n}\|_{L^1(\Omega_1)} \to 0 \quad \text{as} \quad n \to \infty, \\
& (2.36) \quad J'_{1,D}(v_{1,\infty}) = 0, \quad J'_{2,D}(v_{2,\infty}) = 0,
\end{align}

(ii) Define $u_\lambda = (u_{1,\lambda}, u_{2,\lambda}) \in H^1(\Omega_1^0 \cup \Omega_2^0) = H^1(\Omega_1^0) \oplus H^1(\Omega_2^0)$

\begin{equation}
(2.38) \quad u_\lambda = (u_{1,\lambda}, u_{2,\lambda}) = (s_\lambda(v_\lambda)v_{1,\lambda}, t_\lambda(v_\lambda)v_{2,\lambda}), \quad \tilde{u}_\lambda = w_\lambda(u_\lambda).
\end{equation}

Then

\begin{align}
\Phi_\lambda(\tilde{u}_\lambda) &= I_\lambda(v_\lambda) = J_\lambda(v_\lambda) \to c, \quad \Phi'_\lambda(\tilde{u}_\lambda) \to 0, \quad I'_\lambda(u_\lambda) \to 0 \quad \text{as} \quad \lambda \to \infty.
\end{align}

After extracting a subsequence $\lambda_n \to \infty$, there exists $u_\infty = (u_{1,\infty}, u_{2,\infty}) \in H^1_0(\Omega_1 \cup \Omega_2) = H^1_0(\Omega_1) \oplus H^1_0(\Omega_2)$ such that

\begin{enumerate}
\item $\|\tilde{u}_\lambda - u_\infty\|_{L^\infty(\Omega \cup \Omega)} \to 0$ as $n \to \infty$.
\item $u_{1,\infty}$ solves (2.4) - (2.3).
\item $\Psi_{1,D}(u_{1,\infty}) + \Psi_{2,D}(u_{2,\infty}) = c$.
\item $u_{1,\infty} \neq 0$ in $\Omega_1$, $u_{2,\infty} \neq 0$ in $\Omega_2$ and $v_{1,\infty} = \frac{u_{1,\infty}}{\|u_{1,\infty}\|_{H^1(\Omega_1)}}$ for $i = 1, 2$.
\end{enumerate}
Proof. First we show 1°–4° of the second statement (ii). Suppose that \( v_{\lambda} = (v_{1,\lambda}, v_{2,\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda} \) satisfies (2.31). Defining \( u_{\lambda}, \tilde{u}_{\lambda} \) by (2.33), we can see that there exist \( R_0, R_1 > 0 \) such that

\[
(2.39) \quad s_{\lambda}(v_{\lambda}), t_{\lambda}(v_{\lambda}) \in [R_0, R_1] \quad \text{for large } \lambda.
\]

By Proposition 2.23 there exists a subsequence \( \lambda_n \to \infty \) and \( u_{\infty} = (u_{1,\infty}, u_{2,\infty}) \in H_0^1(\Omega_1) \oplus H_0^1(\Omega_2) \) such that 1°–2° hold. By (2.39), we have \( u_{1,\infty} \neq 0, u_{2,\infty} \neq 0 \) and 3°.

Setting \( v_{\infty} = u_{\infty}/\|u_{\infty}\|_{H^1(\Omega_i)} \in \Sigma_{i,D} \) for \( i = 1, 2 \), we can deduce (2.35)–(2.37) from (2.39) and (ii).

In a way similar to Proposition 2.17, we have

**Lemma 2.18.** Let \( i \in \{1, 2\} \) and \( v_{\lambda} \in \Sigma_{i,\lambda} \) satisfies for some \( c > 0 \)

\[
\|J_{i,\lambda}(v_{\lambda})\|_{(T_{\lambda}(\Sigma_{i,\lambda}))^*} \to 0, \quad J_{i,\lambda}(v_{\lambda}) \to c \quad \text{as } \lambda \to \infty.
\]

Then after extracting a subsequence \( \lambda_n \to \infty \), there exists \( v_{\infty} \in \Sigma_{i,D} \) such that

\[
(2.40) \quad \|v_{\lambda_n} - v_{\infty}\|_{H^1(\Omega_i)} \to 0, \quad \|v_{\lambda_n}\|_{L^2(\Omega_i), \Omega_i} \to 0,
\]

\[
(2.41) \quad J_{i,D}(v_{\infty}) = 0, \quad J_{i,D}(v_{\infty}) = c.
\]

To see the positivity of the limit \( v_{\infty} \) in \( \Omega_1 \) (equivalently \( u_{1,\infty} \)), the following proposition is useful.

**Proposition 2.19.** For any \( M > 0 \) there exists \( \delta_0(M) > 0 \) such that for \( i = 1, 2 \)

(i) If \( v \in \Sigma_{i,D} \) satisfies

\[
J_{i,D}(v) = 0, \quad J_{i,D}(v) \leq M, \quad \|v_{\prec}\|_{L_{p+1}(\Omega_i)}^{p+1} \leq \delta_0(M),
\]

then \( v > 0 \) in \( \Omega_1 \).

(ii) If \( v_{\lambda} = (v_{1,\lambda}, v_{2,\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda} \) satisfies

\[
J_{\lambda}(v_{\lambda}) = 0, \quad J_{\lambda}(v_{\lambda}) \leq M, \quad \|(v_{1,\lambda})_{\prec}\|_{L_{p+1}(\Omega_i)}^{p+1} \leq \delta_0(M),
\]

then, after extracting a subsequence \( \lambda_n \to \infty \) there exists \( v_{\infty} = (v_{1,\infty}, v_{2,\infty}) \in \Sigma_{1,D} \times \Sigma_{2,D} \) such that the conclusion of Proposition 2.17 holds, and moreover \( v_{\infty} > 0 \) in \( \Omega_1 \).

(iii) If \( v_{\lambda} = (v_{1,\lambda}, v_{2,\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda} \) satisfies

\[
J_{\lambda}(v_{\lambda}) = 0, \quad J_{\lambda}(v_{\lambda}) \leq M, \quad \|(v_{1,\lambda})_{\prec}\|_{L_{p+1}(\Omega_i)}^{p+1} \leq \delta_0(M), \quad \|(v_{2,\lambda})_{\prec}\|_{L_{p+1}(\Omega_i)}^{p+1} \leq \delta_0(M),
\]

then there exists \( \lambda_M \geq 1 \) such that for large \( \lambda \geq \lambda_M, \tilde{u}_{\lambda} > 0 \) in \( \mathbb{R}^N \). Here \( \tilde{u}_{\lambda} \) is defined in (2.33).

Proof. First we remark that there exists \( \delta_1 > 0 \) such that for \( u \in H_0^1(\Omega_i) \)

\[
(2.42) \quad -\Delta u + u = |u|^{p-1}u \quad \text{in } \Omega_i, \quad \|u_{\prec}\|_{L^p(\Omega_i)} \leq \delta_1
\]

implies \( u_{\prec} \equiv 0 \). In fact, multiplying \( u_{\prec}(x) \) to (2.42) and integrating over \( \Omega_i \), we have

\[
\|u_{\prec}\|_{L^p(\Omega_i)}^{p+1} \leq \delta_1^{p-1}\|u_{\prec}\|_{L^p(\Omega_i)}^{p+1} \leq C\delta_1^{p-1}\|u_{\prec}\|_{L^p(\Omega_i)}^{p+1}.
\]

Thus, choosing \( \delta_1 > 0 \) small so that \( C\delta_1^{p-1} < 1 \), we have \( u_{\prec} \equiv 0 \).
The statements (i), (ii) easily follow from the above property and Proposition 2.17. In a similar way, (iii) follows from the following fact: there exists $\delta_2 > 0$ independent of $\lambda$ such that for $u \in H^1(\mathbb{R}^N)$

\[-\Delta u + (\lambda^2 u(x) + 1)u = g(x,u) \quad \text{in} \quad \mathbb{R}^N, \quad \|u_\pm\|_{L^p(R^N)} \leq \delta_2\]

implies $u_\pm \equiv 0$. \hfill $\square$

3. Sign-changing solutions

We devote Sections 3–4 for a proof of Theorem 1.1. We develop a perturbation theory, and we study $J_{1,\lambda}(v_1, v_2)$ as a perturbation from $J_{1,\lambda}(v_1) + J_{2,\lambda}(v_2)$. First we study basic properties of our “unperturbed” functionals $J_{i,\lambda}(v_i)$.

(a) Minimax methods for $J_{1,\lambda}(v_i)$. As we observed in Section 2, $I_{1,\lambda}(u_1)$ and $I_{2,\lambda}(u_2)$ have a mountain pass geometry and we can apply the mountain pass theorem. Moreover $I_{2,\lambda}(u_2)$ possesses $\mathbb{Z}_2$-symmetry: $I_{2,\lambda}(-u_2) = I_{2,\lambda}(u_2)$, and it enables us to apply symmetric mountain pass theorems. Here we introduce related minimax methods for $J_{1,\lambda}(v_1)$ and $J_{2,\lambda}(v_2)$.

We define minimax values $c_{min}^{1,\lambda}, b_n^{2,\lambda} \ (n \in \mathbb{N})$ by

\[
c_{min}^{1,\lambda} = \inf_{v_1 \in \Sigma_{1,\lambda}} J_{1,\lambda}(v_1), \quad b_n^{2,\lambda} = \inf_{\gamma \in S_n} \max_{\theta \in S_n} J_{2,\lambda}(\gamma(\theta)),
\]

where $S^n = \{\theta = (\theta_1, \ldots, \theta_{n+1}); |\theta| = 1\}$ and

\[
\Gamma_n^{2,\lambda} = \{\gamma \in C(S^n, \Sigma_{2,\lambda}); \gamma(-\theta) = -\gamma(\theta) \text{ for all } \theta \in S^n\}.
\]

Since $J_{1,\lambda}(v_1), J_{2,\lambda}(v_2)$ satisfy the $(PS)_c$-condition for all $c \in \mathbb{R}$, we have

**Lemma 3.1.**  
(i) $c_{min}^{1,\lambda}$ is a critical value of $J_{1,\lambda}(v_1)$.
(ii) $b_n^{2,\lambda}$ is a critical value of $J_{2,\lambda}(v_2)$.
(iii) $b_1^{2,\lambda} \leq b_2^{2,\lambda} \leq \ldots \leq b_n^{2,\lambda} \leq b_{n+1}^{2,\lambda} \leq \ldots$.
(iv) $b_n^{2,\lambda} \to \infty$ as $n \to \infty$.

The main result of this subsection is the following

**Proposition 3.2.** Let $\frac{1}{c_{min}^{1,\lambda}} (b_n^{2,\lambda}$, respectively) be a critical value of $J_{1,\lambda}(v_1) \in C^1(\Sigma_{1,\lambda}, \mathbb{R})$ $(J_{2,\lambda}(v_2) \in C^1(\Sigma_{2,\lambda}, \mathbb{R})$, respectively) defined in (1.8) (1.9), respectively). Then we have

(i) $c_{min}^{1,\lambda} \to c_{min}^{1,\lambda} \text{ as } \lambda \to \infty$.
(ii) $b_n^{2,\lambda} \to b_n^{2,\lambda} \text{ as } \lambda \to \infty$.

**Proof.** We deal with (ii). (i) can be proved in a similar way. We regard $H^1_0(\Omega_2) \subset H^1(\Omega_2)$, and thus we have $\Sigma_2, D \subset \Sigma_{2,\lambda}$ and $J_{2,\lambda}(v) = J_{2,\lambda}(v)$ for all $v \in \Sigma_{2,\lambda}$. We also remark that $\Gamma_n^{2,\lambda} \subset \Gamma_n^{1,\lambda}$. Thus we have

\[(3.1) \quad b_n^{2,\lambda} \leq b_{n+1}^{2,\lambda} \quad \text{for all } n \in \mathbb{N} \text{ and } \lambda \geq 1.\]

We will show that

\[(3.2) \quad \liminf_{\lambda \to \infty} b_n^{2,\lambda} \geq b_n^{2,\lambda}.\]

To show (3.2), we will show the following.
Lemma 3.3. For any \( m > 0, \delta > 0 \) and any open set \( O \subset \subset \Omega_2 \), there exists \( \lambda_{m,\delta,O} \geq 1 \) with the following property: for \( \lambda \geq \lambda_{m,\delta,O} \) there exists an odd continuous operator \( R_\lambda : \{ v \in \Sigma_{2,\lambda}; J_{2,\lambda}(v) \leq m \} \to \Sigma_{2,D} \) such that

\[
J_{2,D}(R_\lambda v) \leq (1 + \delta) J_{2,\lambda}(v) \quad \text{for all} \ v \in \Sigma_{2,\lambda} \ \text{with} \ J_{2,\lambda}(v) \leq m
\]

and

\[
R_\lambda v = v \quad \text{for all} \ v \in \Sigma_{2,\lambda} \cap H^1_0(O) \ \text{with} \ J_{2,\lambda}(v) \leq m.
\]

We postpone a proof of Lemma 3.3 and we give a proof of Proposition 3.2.

Proof of Proposition 3.2. We show (3.2). Set \( m = b_{n,D}^2 + 1 \) and choose \( \delta \in (0,1) \) arbitrary. (Property (3.4) is not used here and \( O \subset \subset \Omega_2 \) can be fixed arbitrarily. (3.4) will be used later in the proof of Lemma 3.6.) By Lemma 3.3, there exists an odd continuous mapping \( R_\lambda \) satisfying (3.3). Choose \( \gamma(\theta) \in \Gamma_n^2 \) satisfying \( \max_{0 \leq \theta \leq \gamma} J_{2,\lambda}(\gamma(\theta)) \leq b_{n,D}^2 + \delta \). Recalling (3.1), we have \( J_{2,\lambda}(\gamma(\theta)) \leq m \) for all \( \theta \in S^n \). We set \( \tilde{\gamma}(\theta) = R_\lambda(\gamma(\theta)) \in \Gamma_n^2 \). By (3.3),

\[
J_{2,D}(\tilde{\gamma}(\theta)) = J_{2,D}(R_\lambda(\gamma(\theta))) \leq (1 + \delta) J_{2,\lambda}(\gamma(\theta)) \leq (1 + \delta)(b_{n,D}^2 + \delta)
\]

for all \( \theta \in S^n \).

Thus

\[
b_{n,D}^2 \leq \max_{\theta \in S^n} J_{2,D}(\tilde{\gamma}(\theta)) \leq (1 + \delta)(b_{n,D}^2 + \delta) \quad \text{for} \ \lambda \geq \lambda_{m,\delta}.
\]

Therefore \( b_{n,D}^2 \leq (1 + \delta)(\liminf_{\lambda \to \infty} b_{n,D}^2 + \delta) \). Since \( \delta \in (0,1) \) is arbitrary, (3.2) holds.

Next we prove Lemma 3.3.

Proof of Lemma 3.3. We use the following notation: for an open set \( \tilde{\Omega} \subset \mathbb{R}^N \) we write

\[
\Sigma_{\tilde{\Omega}} = \{ v \in H^1_0(\tilde{\Omega}); \| v \|_{H^1(\tilde{\Omega})} = 1 \},
\]

\[
J_{\tilde{\Omega}}(v) = \left( \frac{1}{2} - \frac{1}{p + 1} \right) \left( \| v \|_{L^{p+1}(\tilde{\Omega})} \right)^{\frac{2(p+1)}{p+1}} \in C^1(\Sigma_{\tilde{\Omega}}, \mathbb{R}).
\]

Proof of Lemma 3.3 consists of 3 steps.

Step 1: For any \( \tilde{\Omega} \) satisfying \( \Omega_2 \subset \subset \tilde{\Omega} \subset \subset \Omega'_2 \) and \( m > 0, \epsilon > 0 \), there exists \( \lambda_{\tilde{\Omega},\epsilon,m} \geq 1 \) and an odd continuous operator

\[
R_{\lambda}^1 : \{ v \in \Sigma_{2,\lambda}; J_{2,\lambda}(v) \leq m \} \to \Sigma_{\tilde{\Omega}} \quad (\lambda \geq \lambda_{\tilde{\Omega},\epsilon,m}),
\]

such that

\[
J_{\tilde{\Omega}}(R_{\lambda}^1 v) \leq (1 + \epsilon) J_{2,\lambda}(v) \quad \text{for all} \ v \in \Sigma_{2,\lambda} \ \text{with} \ J_{2,\lambda}(v) \leq m,
\]

\[
R_{\lambda}^1 v = v \quad \text{for all} \ v \in \Sigma_{2,\lambda} \cap H^1_0(\Omega_2) \ \text{with} \ J_{2,\lambda}(v) \leq m.
\]

Step 2: For any \( \epsilon' > 0 \) and any open set \( O \subset \subset \Omega_2 \) there exist a smooth open set \( \tilde{\Omega} \) satisfying \( \Omega_2 \subset \subset \tilde{\Omega} \subset \subset \Omega'_2 \) and an odd continuous operator \( R^2 : \Sigma_{\tilde{\Omega}} \to \Sigma_{2,D} \) such that

\[
J_{2,D}(R^2 v) \leq (1 + \epsilon') J_{\tilde{\Omega}}(v) \quad \text{for all} \ v \in \Sigma_{\tilde{\Omega}},
\]

\[
R^2 v = v \quad \text{for all} \ v \in \Sigma_{\tilde{\Omega}} \cap H^1_0(O).
\]

Step 3: Conclusion.
Step 1: Existence of $R_\lambda^1$ satisfying (3.5), (3.6).
First we remark that $J_{2,\lambda}(v) \leq m$ implies
\[
\left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{1}{\|v\|_{L_{p+1}(\Omega_2')}}\right)^{\frac{2(p+1)}{p}} \leq m.
\]
Recalling $\|v\|_{\lambda,\Omega_2'} \geq \|v\|_{\lambda,\Omega_2} = 1$, we have
\[
\|v\|_{L_{p+1}(\Omega_2')} \geq \tilde{m} = \left(\frac{1}{2} - \frac{1}{p+1}\right)^{\frac{p-1}{p+1}}.
\]

The following proposition will be proved in Section 8, in which the subsolution estimate (cf. Simon [22]) plays an important role.

**Proposition 3.4.** For any $\tilde{m} > 0$, $\tilde{\varepsilon} > 0$ and any open sets $\tilde{\Omega}$, $\tilde{\Omega}$ with smooth boundaries and $\Omega_2 \subset \subset \tilde{\Omega} \subset \subset \tilde{\Omega}$, there exists $\lambda(\tilde{m}, \tilde{\varepsilon}, \tilde{\Omega}, \tilde{\Omega}) \geq 1$ with the following properties: For any $\lambda \geq \lambda(\tilde{m}, \tilde{\varepsilon}, \tilde{\Omega}, \tilde{\Omega})$ there exists an odd continuous mapping
\[
F_\lambda : \{v \in \Sigma_{2,\lambda}; \|v\|_{L_{p+1}(\Omega_2')} \geq \tilde{m}\} \to H_0^1(\tilde{\Omega})
\]
such that for any $v \in \{v \in \Sigma_{2,\lambda}; \|v\|_{L_{p+1}(\Omega_2')} \geq \tilde{m}\}$,

1° $(F_\lambda v)(x) = v(x)$ for all $x \in \tilde{\Omega}$.
2° $\|F_\lambda v\|_{L_{p+1}(\Omega_2')} \leq 1 + \tilde{\varepsilon}$.
3° $\|F_\lambda v\|_{L_{p+1}(\Omega_2')} \geq (1 - \tilde{\varepsilon})\|v\|_{L_{p+1}(\Omega_2')}$. 

Now we can show Step 1. Let $F_\lambda$ be an operator given in the above proposition. We define $R_\lambda^1$ by
\[
R_\lambda^1 v = \frac{F_\lambda v}{\|F_\lambda v\|_{H_0^1(\tilde{\Omega})}}.
\]

By 1°–3° of Proposition 3.4 and (2.28),
\[
J_{\tilde{\Omega}}(R_\lambda^1 v) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{1}{\|R_\lambda^1 v\|_{L_{p+1}(\Omega_2')}}\right)^{\frac{2(p+1)}{p}}
\]
\[
\leq \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{\|R_\lambda^1 v\|_{\lambda,\Omega_2'}}{\|R_\lambda^1 v\|_{L_{p+1}(\Omega_2')}}\right)^{\frac{2(p+1)}{p}}
\]
\[
= \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{\|F_\lambda v\|_{\lambda,\Omega_2'}}{\|F_\lambda v\|_{L_{p+1}(\Omega_2')}}\right)^{\frac{2(p+1)}{p}}
\]
\[
\leq \left(\frac{1 + \tilde{\varepsilon}}{1 - \tilde{\varepsilon}}\right)^{\frac{2(p+1)}{p-1}} \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{1}{\|v\|_{L_{p+1}(\Omega_2')}}\right)^{\frac{2(p+1)}{p}}
\]
\[
\leq \left(\frac{1 + \tilde{\varepsilon}}{1 - \tilde{\varepsilon}}\right)^{\frac{2(p+1)}{p-1}} J_{2,\lambda}(v) \quad \text{for all } v \in \Sigma_{2,\lambda}.
\]

Thus we have (3.5) for sufficiently small $\tilde{\varepsilon} > 0$. (3.6) also clearly holds. \hfill \Box
Step 2: Existence of $\tilde{\Omega}$ and $R^2$ satisfying (3.7), (3.8).

Since $\partial \Omega_2$ is smooth, for any $\nu > 0$ and $O \subset \subset \Omega_2$ we can find a smooth open set $\Omega_2^{(\nu)}$ satisfying $\Omega_2 \subset \subset \Omega_2^{(\nu)} \subset \subset \Omega_2$ and a diffeomorphism $\varphi^{(\nu)} = (\varphi_1^{(\nu)}, \ldots, \varphi_N^{(\nu)}) : \Omega_2 \to \Omega_2^{(\nu)}$ such that

$$
\varphi^{(\nu)}(x) = x \quad \text{for all } x \in O,
$$

$$
\max_{x \in \Omega_2} \left| \frac{\partial \varphi^{(\nu)}(x)}{\partial x_j} - \delta_{ij} \right| < \nu \quad \text{for } i, j \in \{1, 2, \ldots, N\}.
$$

We remark that $\Omega_2^{(\nu)}$ approaches $\Omega_2$ as $\nu \to 0$.

We define $\tilde{\varphi}^{(\nu)} : H^1_0(\Omega_2^{(\nu)}) \to H^1_0(\Omega_2)$ by $(\tilde{\varphi}^{(\nu)}u)(x) = u(\varphi^{(\nu)}(x))$. We can observe that there exists a constant $c_\nu > 0$ such that for $u \in H^1_0(\Omega_2^{(\nu)})$

$$
c_\nu \to 0 \quad \text{as } \nu \to 0,
$$

$$
(1 - c_\nu)\|u\|_{L^{p+1}(\Omega_2^{(\nu)})} \leq \|\tilde{\varphi}^{(\nu)}u\|_{L^{p+1}(\Omega_2)} \leq (1 + c_\nu)\|u\|_{L^{p+1}(\Omega_2^{(\nu)})},
$$

$$
(1 - c_\nu)\|u\|_{H^1(\Omega_2^{(\nu)})} \leq \|\tilde{\varphi}^{(\nu)}u\|_{H^1(\Omega_2)} \leq (1 + c_\nu)\|u\|_{H^1(\Omega_2^{(\nu)})}.
$$

Now we can prove (3.7). For a given $\epsilon > 0$, we choose $\nu \in (0, 1)$ such that

$$
\left(\frac{1+c_\nu}{1-c_\nu}\right)^{2(p+1)/p} < 1 + \epsilon. \quad \text{We will show that (3.7) holds for } \hat{\Omega} = \Omega_2^{(\nu)}.
$$

For $v \in \Sigma_{\hat{\Omega}}$ we define $R^2v \in \Sigma_{2,D}$ by $R^2v = \frac{\tilde{\varphi}^{(\nu)}v}{\|\tilde{\varphi}^{(\nu)}v\|_{H^1(\Omega_2)}}$. Then by (3.10)–(3.11)

$$
J_{2,D}(R^2v) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{\|\tilde{\varphi}^{(\nu)}v\|_{H^1(\Omega_2)}}{\|\tilde{\varphi}^{(\nu)}v\|_{L^{p+1}(\Omega_2)}}\right)^{2(p+1)/(p+1)}
$$

$$
\leq \left(\frac{1+c_\nu}{1-c_\nu}\right)^{2(p+1)/p} \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{\|v\|_{H^1(\Omega_2)}}{\|v\|_{L^{p+1}(\Omega_2)}}\right)^{2(p+1)/(p+1)}
$$

$$
= \left(\frac{1+c_\nu}{1-c_\nu}\right)^{2(p+1)/p} J_{\Omega}(v).
$$

By the choice of $\nu$, we have (3.7). Equation (3.8) also follows from (3.9). \hfill \Box

Step 3: Conclusion.

For a given $\delta > 0$ we choose $\epsilon, \epsilon' > 0$ such that $(1+\epsilon)(1+\epsilon') < 1+\delta$. By Step 2, there exist $\hat{\Omega}$ and $R^2$ such that (3.7) holds. Next, using Step 1, we find $\lambda_{\Omega,\epsilon,m} \geq 1$ and $R^1_{\lambda}$ ($\lambda \geq \lambda_{\Omega,\epsilon,m}$) such that (3.7) holds. Defining $R_{\lambda} : \{v \in \Sigma_{2,\lambda}; J_{2,\lambda}(v) \leq m\} \to \Sigma_{2,D}$ by $R_{\lambda} = R^2 \circ R^1_{\lambda}$, we have (3.3)–(3.4). \hfill \Box

As stated in the Introduction, since $b^{2,D}_n \to \infty$ as $n \to \infty$, there exists a sequence

$$
n(1) < n(2) < \cdots < n(k) < n(k+1) < \cdots \quad \text{such that}
$$

$$
b^{2,D}_{n(1)} < b^{2,D}_{n(2)} < \cdots < b^{2,D}_{n(k)} < b^{2,D}_{n(k+1)}.
$$

We set

$$
\delta_k = \frac{1}{4}(b^{2,D}_{n(k+1)} - b^{2,D}_{n(k)}) > 0
$$

(3.12)
and choose $\gamma_{0k} \in \Gamma^{2,D}_{n(k)}$ such that

$$(3.14) \quad \max_{\theta \in S^{n(k)}} J_{2,D}(\gamma_{0k}(\theta)) < b^{2,D}_{n(k)} + \delta_k,$$

$$(3.15) \quad \text{supp} \gamma_{0k} \subset \Omega_2 \text{ is compact in } \Omega_2 \text{ for all } \theta \in S^{n(k)}.$$  

We define a minimax value $c^{2,D}_k$ by

$$(3.16) \quad c^{2,D}_k = \inf_{\sigma \in \Lambda^{2,D}_k} \max_{\theta \in S^{n(k)}_{+} + 1} J_{2,D}(\sigma(\theta)),$$

where

$$(3.17) \quad \Lambda^{2,D}_k = \{ \sigma \in C(S^{n(k)}_{+} + 1, \Sigma_{2,D}); \sigma(\theta) = \gamma_{0k}(\theta) \text{ for all } \theta \in S^{n(k)} \}.$$  

Here we regard $S^{n(k)} = \{ (\theta_1, \cdots, \theta_{n(k)+1}, 0); \theta_1^2 + \cdots + \theta_{n(k)+1}^2 = 1 \} = \partial S^{n(k)+1}_+ \subset S^{n(k)+1}_+.$

This type of minimax method is introduced in Ambrosetti [1], Bahri-Berestycki [8], Struwe [23] and Rabinowitz [20]. In the definition of $c^{2,D}_k$ we don’t require evenness, and it enables us to detect critical points of perturbed functionals.

As a fundamental property of $c^{2,D}_k$ we have

Lemma 3.5.  

(i) $c^{2,D}_k \geq b^{2,D}_{n(k)+1} (\geq b^{D}_{n(k)})$.

(ii) $c^{2,D}_k$ is a critical value of $J_{2,D}(v_2).$

Proof. (i) For a given $\sigma \in \Lambda^{2,D}_k$ we define $\gamma_{\sigma} \in \Gamma^{2,D}_{n(k)+1}$ by

$$\gamma_{\sigma}(\theta) = \begin{cases} \sigma(\theta) & \text{if } \theta \in S^{n(k)+1}_+ \\ -\sigma(-\theta) & \text{if } \theta \in S^{n(k)+1}_+ \setminus S^{n(k)+1}_+ \end{cases}.$$  

Since $J_{2,D}$ is even, we have $\max_{\theta \in S^{n(k)+1}_+} J_{2,D}(\gamma_{\sigma}(\theta)) = \max_{\theta \in S^{n(k)+1}_+} J_{2,D}(\sigma(\theta)).$

Thus we have $c^{2,D}_k \geq b^{2,D}_{n(k)+1}.$

(ii) By (i), we can see that $\Lambda^{2,D}_k$ is stable under a suitable class of deformations. Thus we can observe that $c^{2,D}_k$ is a critical point value of $J_{2,D}(v_2).$  \qed

Next we introduce related minimax methods to $J_{2,\lambda}(v_2).$ By Proposition 3.2 (3.1), and (3.2), for any $k \in \mathbf{N}$ there exists $\lambda(k)$ such that for $\lambda \geq \lambda(k)$

$$(3.18) \quad b^{2,\lambda}_{n(k)} \leq b^{2,D}_{n(k)} \leq b^{2,\lambda}_{n(k)} + \delta_k, \quad b^{2,\lambda}_{n(k)} \leq b^{2,\lambda}_{n(k) + 1} - 3\delta_k.$$  

We define for $\lambda \geq \lambda(k)$

$$c^{2,\lambda}_k = \inf_{\sigma \in \Lambda^{2,\lambda}_k} \max_{\theta \in S^{n(k)}_{+} + 1} J_{2,\lambda}(\sigma(\theta)),$$

where

$$(3.19) \quad \Lambda^{2,\lambda}_k = \{ \sigma \in C(S^{n(k)}_{+} + 1, \Sigma_{2,\lambda}); \sigma(\theta) = \gamma_{0k}(\theta) \text{ for all } \theta \in S^{n(k)} \}.$$  

Here $\gamma_{0k} \in \Gamma^{2,\lambda}_{n(k)} \subset \Gamma^{2,\lambda}_{n(k)}$ is chosen in (3.14)–(3.15).

As in Lemma 3.5 and Proposition 3.2 we have

Lemma 3.6.  

(i) $c^{2,\lambda}_k \geq b^{2,\lambda}_{n(k)+1}$ for $\lambda \geq \lambda(k)$.

(ii) $c^{2,\lambda}_k$ is a critical value of $J_{2,\lambda}(v_2)$ for $\lambda \geq \lambda(k)$.

(iii) $c^{2,\lambda}_k \to c^{2,D}_k$ as $\lambda \to \infty.$
Proof. (i) As in (i) of Lemma 3.5 we have (i).
(ii) By (3.14) and (3.18),
\[
\max_{\theta \in S^{n(k)}} J_{2,\lambda}(\gamma_{0k}(\theta)) = \max_{\theta \in S^{n(k)}} J_{2,D}(\gamma_{0k}(\theta)) < b_{n(k)}^{2,D} + \delta_k \\
\leq b_{n(k)}^{2,\lambda} + 2\delta_k \leq b_{n(k)+1}^{2,\lambda} - \delta_k < c_{k}^{2,\lambda}.
\]
Thus we can see that \(c_{k}^{2,\lambda}\) is a critical value of \(J_{2,\lambda}(v_2)\).
(iii) We recall (3.15) and choose \(O \subset \Omega_2\) such that \(\bigcup_{\theta \in S^{n(k)}} \text{supp} \gamma_{0k}(\theta) \subset O\). Applying Lemma 3.3, we can show (iii) as in Proposition 3.2.

(b) Minimax methods for \(J_{\lambda}(v_1, v_2)\). Let \((n(k))_{k=1}^{\infty}, (\gamma_{0k})_{k=1}^{\infty}\) be sequences obtained in (3.12), (3.14) and let \(v_{10} \in \Sigma_{1,D}\) be a minimizer of \(J_{1,D}(v_1)\), that is,
\[
J_{1,D}(v_{10}) = c_{1,D}^{\min}.
\]
We remark that we may assume \(v_{10} > 0\) in \(\Omega_1\). Using \((n(k))_{k=1}^{\infty}, (\gamma_{0k})_{k=1}^{\infty}\) and \(v_{10}\), we introduce minimax values \(c_{k}^{\lambda}\) for \(J_{\lambda}(v_1, v_2)\) by
\[
c_{k}^{\lambda} = \inf_{\tilde{\gamma} \in \Lambda_k^{\lambda}} \max_{\gamma \in S^{n(k)}(\lambda)} J_{\lambda}(\tilde{\gamma}(\theta)),
\]
where
\[
\Lambda_k^{\lambda} = \{ \tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2) \in C(S_{\gamma}^{n(k)}+1, \Sigma_{1,\lambda} \times \Sigma_{2,\lambda});
\]
\[
\tilde{\gamma}(\theta) = (v_{10}, \gamma_{0k}(\theta)) \text{ for all } \theta \in S^{n(k)},
\]
\[
\|\tilde{\gamma}_1(\theta) - p|_{L^{k+1}((\Omega_1')\setminus K)} \leq \nu_{M_k} \text{ for all } \theta \in S^{n(k)+1} \}.
\]
Here \(M_k = c_{1,D}^{\min} + c_{k}^{2,D} + 1 \geq 0\) and \(\nu_{M_k} > 0\) is given in Proposition 9.1 and Corollary 9.3 in Section 9. First we remark
\[
(3.19) \quad c_{k}^{\lambda} \leq c_{1,D}^{\min} + c_{k}^{2,D} \quad \text{for all } \lambda \geq 1.
\]
In fact, for any \(\epsilon > 0\) there exists \(\gamma_{\epsilon} \in \Lambda_k^{2,D}\) such that \(\max_{\theta \in S^{n(k)}(\lambda)} J_{2,D}(\gamma_{\epsilon}(\theta)) < c_{k}^{2,D} + \epsilon\). Setting \(\tilde{\gamma}_{\epsilon}(\theta) = (v_{10}, \gamma_{\epsilon}(\theta)) \in \Lambda_k^{\lambda}\) and noting \(J_{\lambda}(\tilde{\gamma}_{\epsilon}(\theta)) = J_{1,D}(v_{10}, \gamma_{\epsilon}(\theta)) = J_{1,D}(v_{10}) + J_{2,D}(\gamma_{\epsilon}(\theta))\), we have \(c_{k}^{\lambda} \leq c_{1,D}^{\min} + c_{k}^{2,D} + \epsilon\) for all \(\lambda \geq 1\). Since \(\epsilon > 0\) is arbitrary, we have (3.19). We choose \(\lambda_{M_k} \geq 1\) such that (2.32) holds for \(\lambda \geq \lambda_{M_k}\) and \(v \in [J_{\lambda} \leq M_k \mid \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}]\).

Now we can prove

Proposition 3.7. (i) For \(\lambda \geq \lambda_{M_k}\),
\[
(3.20) \quad c_{k}^{\lambda} \geq c_{1,D}^{\min} + c_{k}^{2,D} - r_{\lambda}.
\]
(ii) \(\lambda \to \infty\)
\[
(3.21) \quad \lim_{\lambda \to \infty} c_{k}^{\lambda} = c_{1,D}^{\min} + c_{k}^{2,D}.
\]
(iii) For sufficiently large \(\lambda\), \(c_{k}^{\lambda}\) is a critical value of \(J_{\lambda}(v_1, v_2)\).

Proof. (i) For any \(\tilde{\gamma}(\theta) = (p(\theta), \gamma(\theta)) \in \Lambda_k^{\lambda}\), we observe \(\gamma(\theta) \in \Lambda_k^{2,\lambda}\). Thus for any \(\tilde{\gamma} \in \Lambda_k^{\lambda}\) satisfying \(\max_{\theta \in S^{n(k)}(\lambda)} J_{2,D}(\tilde{\gamma}(\theta)) = M_k\), we have by (2.32) that
\[
J_{\lambda}(\tilde{\gamma}(\theta)) \geq J_{1,\lambda}(p(\theta)) + J_{2,\lambda}(\gamma(\theta)) - r_{\lambda} \geq c_{1,D}^{\min} + J_{2,\lambda}(\gamma(\theta)) - r_{\lambda} \quad \text{for all } \theta \in S^{n(k)+1}.
\]
Thus we get (3.20).
(i) (3.21) follows easily from (3.19), (3.20), (i) of Proposition 3.2 and (iii) of Lemma 3.6.

(ii) For \( \theta \in S^{n(k)} \), we have
\[
J_\lambda(v_{10}, \gamma_{0k}(\theta)) = J_{1,\lambda}(v_{10}) + J_{2,\lambda}(\gamma_{0k}(\theta)) \leq c_{\min}^{1,D} + b_{n(k)}^2 + \delta_k.
\]
Thus we have from (3.21), (3.13) and (i) of Lemma 3.5 that
\[
\max_{\theta \in S^{n(k)}} J_\lambda(v_{10}, \gamma_{0k}(\theta)) \leq c^{1,D}_k - 2\delta_k
\]
for sufficiently large \( \lambda \), and we can show \( \Lambda_\lambda^k \) is stable under a suitable class of deformations. Thus \( c^{1,D}_k \) is a critical value of \( J_\lambda(v_1, v_2) \) for sufficiently large \( \lambda \).

At this stage we remark that if critical point \( v_\lambda = (v_{1\lambda}, v_{2\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda} \) satisfies
\[
J_\lambda(v_\lambda) \to c^{1,D}_\min + c^{2,D}_k,
\]
\[
\|v_{1\lambda}\|_{p+1} < \delta_0(M_k),
\]
then by Proposition 2.17 and Proposition 2.19, a sequence of solutions \( u_\lambda(x) \) corresponding to \( v_\lambda \) satisfies (1.10)–(1.13). We remark that by Corollary 9.3 in Section 9 there exists a critical point \( v_\lambda \) corresponding to the critical value \( c^{1,D}_k \) and \( v_\lambda \) satisfies (3.22)–(3.23).

In the next section we show (1.14).

4. Proof of Theorem 1.1

(a) Construction of a pseudo-gradient vector field. By the argument in the previous sections, to prove Theorem 1.1 we need to show (1.14). For that purpose, we need to construct a deformation flow \( \eta(t) \) under which the level set \( \{v_\in \Sigma_{1,\lambda} \cap \Sigma_{2,\lambda} : J_\lambda(v_\lambda) \leq c^{1,D}_\min + \delta \} \) is invariant for small \( \delta > 0 \). The following isolatedness of critical values is important for the construction of such a flow.

Proposition 4.1 ([Dancer [12, 13]; cf. Cao-Noussair [10]]). Let
\[
\tilde{\Psi}_{1,D}(u) = \int_{\Omega_1} \frac{1}{2}(|\nabla u|^2 + u^2) - \frac{1}{p+1} u^{p+1} dx : H^1_0(\Omega_1) \to \mathbb{R}
\]
and suppose that \( u(x) \) is a positive solution of (1.14) in \( \Omega_1 \). Then \( \tilde{\Psi}_{1,D}(u) \) is an isolated critical value of \( \tilde{\Psi}_{1,D} \).

Proof. See the Appendix of [10] and [12, 13]. Here the analyticity of the map \( u \mapsto (-\Delta)^{-1}(u^p) \) in \( C^1(\Omega) \) near a positive solution is essential.

By Proposition 4.1, we can see \( c^{1,D}_\min \) is an isolated critical value of \( \tilde{\Psi}_{1,D}(u) \) (equivalently of \( J_{1,D}(v) \)).

Remark 4.2. For the following arguments, we need only nowhere denseness of critical values in a neighborhood of \( c^{1,D}_\min \) or \( c^{2,D}_k \). We remark that isolatedness implies nowhere denseness. We refer to Fučík-Kučera-Nečas-Souček-Souček [17] for the Morse-Sard theorem in infinite dimensional setting. We also remark that Sard’s property is also used effectively in the study of the Hamiltonian system in Cieliebak-Séré [11].
First we have

**Lemma 4.3.** Suppose that for \( v \in \Sigma_{1,D} \)

\[
J_{1,D}(v) \in [\alpha, \beta] \quad \text{implies} \quad J'_{1,D}(v) \neq 0
\]

for some \( 0 < \alpha < \beta < \infty \). Then there exist \( \rho_0 > 0 \) and \( \lambda_0 \geq 1 \) such that for \( \lambda \geq \lambda_0 \), \( v \in \Sigma_{1,\lambda} \)

\[
J_{1,\lambda}(v) \in [\alpha, \beta] \quad \text{implies} \quad \| J'_{1,\lambda}(v) \|_{T_v(\Sigma_{1,\lambda})^*} \geq \rho_0.
\]

**Proof.** We argue indirectly, and we assume that there exist \( \lambda_n \geq 1 \), \( v_n \in \Sigma_{1,\lambda_n} \)

(\( n = 1, 2, \ldots \)) satisfying

\[
\lambda_n \to \infty, \quad J_{1,\lambda_n}(v_n) \to c \in [\alpha, \beta], \quad \| J'_{1,\lambda_n}(v_n) \|_{T_v(\Sigma_{1,\lambda_n})^*} \to 0 \quad \text{as} \quad n \to \infty.
\]

By Lemma 2.18 after extracting a subsequence — still denoted by \( n \) —, there exists \( v_\infty \in \Sigma_{1,D} \) such that \((4.10) - (4.11)\) hold. In particular, we have that \( c \) is a critical value of \( J_{1,D}(v) \), which is in contradiction to assumption (1.1). \( \square \)

We also have

**Lemma 4.4.** Suppose that (1.1) holds. Then for any \( M > 0 \) there exists \( \lambda_1(M) \geq \lambda_0 \) such that for any \( \lambda \geq \lambda_1(M) \) and \( v = (v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda} \) satisfying \( J_\lambda(v_1, v_2) \leq M \) and \( J_{1,\lambda}(v_1) \in [\alpha, \beta] \), there exists

\[
X = (X_1, X_2) \in T_v(\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}) = (T_{v_1}(\Sigma_{1,\lambda})) \oplus (T_{v_2}(\Sigma_{2,\lambda}))
\]

such that

\[
\| X_1 \|_{T_{v_1}(\Sigma_{1,\lambda})} = 1, \quad X_2 = 0,
\]

\[
J'_{1,\lambda}(v_1)X_1 \geq \frac{\rho_0}{2}, \quad J_\lambda(v_1, v_2)(X_1, X_2) \geq \frac{\rho_0}{2},
\]

where \( \rho_0 > 0 \) is a constant appearing in (4.12).

**Proof.** This is a consequence from (1.1) and (2.32) - (2.33). \( \square \)

To prove Theorem 1.1 it suffices to show for any \( \delta > 0 \) there exists a critical point \( v_\lambda = (v_{1\lambda}, v_{2\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda} \) of \( J_\lambda(v) \) for sufficiently large \( \lambda \) with the following properties:

\[
| J_{1,\lambda}(v_{1\lambda}) - c_{1,\min}^{1,D} | < \delta, \quad | J_{2,\lambda}(v_{2\lambda}) - c_k^{2,D} | < \delta.
\]

In fact, if the above statement is true, then there exists a sequence \( v_\lambda = (v_{1\lambda}, v_{2\lambda}) \) such that

\[
J_{1,\lambda}(v_{1\lambda}) \to c_{1,\min}^{1,D}, \quad J_{2,\lambda}(v_{2\lambda}) \to c_k^{2,D}.
\]

Clearly it follows from (4.3) that

\[
J_\lambda(v_\lambda) \to c_{1,\min}^{1,D} + c_k^{2,D} \quad \text{as} \quad \lambda \to \infty.
\]

Thus by Proposition 2.17 after extracting a subsequence, we have \( v_{1\lambda_n} \to v_{1\infty} \)

(\( i = 1, 2 \)) and

\[
J_{1,D}(v_{1\infty}) = \lim_{n \to \infty} J_{1,\lambda_n}(v_{1\lambda_n}) = c_{1,\min}^{1,D}.
\]

Thus \( v_{1\infty}(x) \) has a constant sign, and we may assume \( v_{1\infty}(x) > 0 \) and Theorem 1.1 follows. We remark here that critical point \( v_\lambda(x) \) obtained in this section may not satisfy \( J_\lambda(v_\lambda) = c_k^\lambda \).
In what follows, we argue indirectly and assume for some \( \delta_0 > 0 \) that

\[
(4.5) \quad \text{there are no critical points } v = (v_1, v_2) \in \Sigma_{1, \lambda} \times \Sigma_{2, \lambda} \text{ of } J_\lambda(v) \text{ satisfying (4.3) with } \delta = \delta_0.
\]

For \( \epsilon_0 \in (0, \delta_0) \) we choose a neighborhood \( N_{\epsilon_0, \lambda} \) of \((c_{\min}^{1,D}, c_k^{2,D})\) in \( \mathbb{R}^2 \) in the following way: By the isolatedness of \( c_{\min}^{1,D} \), we can choose an interval \([\alpha, \beta]\) such that

\[
[\alpha, \beta] \subset \left( c_{\min}^{1,D} + \frac{1}{2} \delta_0, c_{\min}^{1,D} + \delta_0 \right)
\]

and \( J_{1,D}(v_1) \) no has critical values in \([\alpha, \beta]\). We set

\[
N_{\epsilon_0, \lambda} = \{(j_1, j_2) \in \mathbb{R}^2; \quad c_{\min}^{1,D} + \frac{1}{2} \delta_0 \leq j_1 \leq \beta, \quad (1 + r_\lambda)^{-\frac{2(p+1)}{p-1}} (c_{\min}^{1,D} + c_k^{2,D} - \epsilon_0) \leq j_1 + j_2 \leq (1 - r_\lambda)^{-\frac{2(p+1)}{p-1}} (c_{\min}^{1,D} + c_k^{2,D} + \epsilon_0) \}.
\]

Since \( c_{\min}^{1,D} \rightarrow c_{\min}^{1,D}, c_k^{2,D} \rightarrow c_k^{2,D} \) as \( \lambda \rightarrow \infty \), we can see for \( \epsilon_0 > 0 \) small and \( \lambda > 0 \) large

\[
(4.6) \quad (c_{\min}^{1,D}, c_k^{2,D}) \in N_{\epsilon_0, \lambda},
\]

\[
(4.7) \quad N_{\epsilon_0, \lambda} \subset (c_{\min}^{1,D} - \delta_0, c_{\min}^{1,D} + \delta_0) \times (c_k^{2,D} - \delta_0, c_k^{2,D} + \delta_0),
\]

\[
(4.8) \quad 0 < \epsilon_0 < \delta_k = \frac{1}{4}(b_{n(k)+1} - b_{n(k)}).
\]

By assumption (4.3), we have the following.

**Lemma 4.5.** Let

\[
A_\lambda = \left\{ (v_1, v_2) \in \Sigma_{1, \lambda} \times \Sigma_{2, \lambda}; \quad (J_{1,\lambda}(v_1), J_{2,\lambda}(v_2)) \in N_{\epsilon_0, \lambda} \right\},
\]

where \( \epsilon_0 > 0 \) is fixed so that (4.6)–(4.8) hold. Then for sufficiently large \( \lambda \) there exists a vector field \( X \) on \( A_\lambda \) such that

(i) \( X(v) = (X_1(v), X_2(v)) \in T_v(\Sigma_{1, \lambda} \times \Sigma_{2, \lambda}), \) \( \|X(v)\|_{T_v(\Sigma_{1, \lambda} \times \Sigma_{2, \lambda})} \in [\frac{1}{2}, 1] \) for all \( v \in A_\lambda \) and \( v \rightarrow X(v) \) is locally Lipschitz continuous.

(ii) There is a constant \( \mu_\lambda > 0 \) such that \( J'_\lambda(v)X(v) \geq \mu_\lambda \) for all \( v \in A_\lambda \).

(iii) If \( v = (v_1, v_2) \in A_\lambda \) satisfies \( J_{1,\lambda}(v_1) \in [\alpha, \beta], \) then \( X(v) = (X_1(v), 0) \) and it satisfies \( J'_{1,\lambda}(v_1)X_1(v) \geq \frac{\mu_\lambda}{2}, \) and \( J'_{1,\lambda}(v_1)X(v) \geq \frac{\mu_\lambda}{2}. \)

**Proof.** First we remark that \( v = (v_1, v_2) \in A_\lambda \) implies \( J_{1,\lambda}(v_1) \leq c_{\min}^{1,D} + \delta_0, J_{2,\lambda}(v_2) \leq c_k^{2,D} + \delta_0. \) Thus by (i) of Lemma 2.4 we have

\[
J_\lambda(v) \leq (1 + r_\lambda)^{-\frac{2(p+1)}{p-1}} (c_{\min}^{1,D} + c_k^{2,D} + 2\delta_0).
\]

Choosing \( M = 2^{\frac{2(p+1)}{p-1}} (c_{\min}^{1,D} + c_k^{2,D} + 2\delta_0) \), we can apply Lemma 2.3 in the setting (iii). We also remark that \( J_\lambda(v) \) satisfies the Palais-Smale condition. Thus under assumption (4.3), we have inf \( v \in A_\lambda, \|J'_\lambda(v)\|_{T_v(\Sigma_{1, \lambda} \times \Sigma_{2, \lambda})} > 0. \) In a standard way, we can construct a locally Lipschitz continuous pseudo-gradient vector field satisfying (i)–(iii). \( \square \)
Remark 4.6. It is easily seen that there exists a constant $\nu_0 > 0$ such that for large $\lambda$

\[(4.9) \quad v = (v_1, v_2) \in A_\lambda \quad \text{implies} \quad \|J'_1(v)\|_{T_v((\Sigma_1, \lambda) \times \Sigma_2, \lambda)^*}, \quad \|J'_2(v)(1)\|_{T_v((\Sigma_1, \lambda)^*}, \quad \|J'_2(v_2)\|_{T_v((\Sigma_2, \lambda)^*} \leq \nu_0, \quad J'_1(v_1) \geq c_{1,\lambda}^1 \geq c_{1,\min}^1 - \delta_0.
\]

We also have the following.

**Lemma 4.7.** For sufficiently large $\lambda$, we have for all $v = (v_1, v_2) \in \Sigma_1, \lambda \times \Sigma_2, \lambda$

(i) $J'_1(v_1) \geq c_{1,\min}^1$.

(ii) $J'_1(v_1) + J'_2(v_2) \geq (1 - r_\lambda) - \frac{2c_{\min}^2 + c_k^2}{c_{\min}^1 + c_k^2 - \frac{2}{3} \epsilon_0}$ implies $J(v_1, v_2) \geq c_{1,\min}^1 + c_k^2$.

(iii) $J'_1(v_1) + J'_2(v_2) \leq (1 + r_\lambda) - \frac{2c_{\min}^2 + c_k^2 - \epsilon_0}{c_{\min}^1 + c_k^2}$. implies $J(v_1, v_2) \leq c_{1,\min}^1 + c_k^2$.

**Proof.** (i) is obvious. (ii), (iii) follow from (i) of Lemma 2.14 since $c_{\min}^1 \to c_{1,\min}^1$, $c_k^2 \to c_k^2$ as $\lambda \to \infty$.

We choose a function $\psi \in C^\infty(R, [0, 1])$ such that

\[\psi(s) = \begin{cases} 1 & \text{for } s \geq \frac{c_{1,\min}^1 + c_k^2}{c_{\min}^1 + c_k^2 - \frac{1}{3} \epsilon_0}, \\ 0 & \text{for } s \leq \frac{c_{1,\min}^1 + c_k^2 - \frac{2}{3} \epsilon_0}{c_{\min}^1 + c_k^2}.
\]

We define a vector field $\vec{X}$ on $A_\lambda$:

\[\vec{X}(v_1, v_2) = \psi(J(v_1, v_2))X(v_1, v_2)
\]

and consider the following ODE in $A_\lambda$:

\[(4.10) \quad \frac{d\eta}{dt} = -\vec{X}(\eta), \quad \eta(0, u_0) = u_0.
\]

(b) End of the proof of Theorem 1.1. To prove Theorem 1.1, we argue as in Section 3 and choose $\gamma_\epsilon \in \Lambda_{\min}^k$ (defined in (3.17)) such that

\[
\max_{\theta \in S^{n(k)}+1} J_{2.\lambda}(\gamma_\epsilon(\theta)) < c_{1,\min}^1 + c_k^2 + \epsilon.
\]

We consider a deformation of $\tilde{\gamma}_\epsilon(\theta) = (v_{01}, \gamma_\epsilon(\theta)) \in \Lambda_{\min}^k$ through the deformation flow (4.10). In this section we write

\[\Lambda_{\min}^k = \{ \tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2) \in C(S^{n(k)}+1, \Sigma_1, \lambda \times \Sigma_2, \lambda); \tilde{\gamma}(\theta) = (v_{01}, \gamma_0k(\theta)) \text{ for all } \theta \in S^{n(k)} \}.
\]

Choosing $\epsilon \in (0, \epsilon_0)$ small, we have

\[(4.11) \quad J_\lambda(v_{01}, \gamma_\epsilon(\theta)) = J_1(v_{01}) + J_2(\gamma_\epsilon(\theta)) \leq c_{1,\min}^1 + c_k^2 + \epsilon \leq \left(1 - r_\lambda\right) - \frac{2c_{\min}^2 + c_k^2}{c_{\min}^1 + c_k^2 + \epsilon_0}
\]

for all $\theta \in S^{n(k)}+1$.

We will observe that

(i) $\tilde{\gamma}_\epsilon(\theta) \notin A_\lambda$ implies $J_\lambda(\tilde{\gamma}_\epsilon(\theta)) < c_{1,\min}^1 + c_k^2 - \frac{2}{3} \epsilon_0$ (Lemma 4.8).
(ii) For \( \tilde{\gamma}_e(\theta) \in A_\lambda \), we consider \( \eta(t) = \eta(t, \tilde{\gamma}_e(\theta)) \), where \( \eta(t, u_0) \) is a solution of (4.10). Then

(a) \( \frac{d}{dt} J_\lambda(\eta(t)) \leq 0 \) for all \( t \geq 0 \).

(b) \( J_\lambda(\eta(t)) \geq c_{min}^{1,D} + c_k^{2,D} - \frac{1}{3} \epsilon_0 \) implies \( J_{1,\lambda}(\eta(t)) \in [c_{min}^{1,D} - \delta_0, \beta] \) and \( \frac{d}{dt} J_\lambda(\eta(t)) \leq - \mu_\lambda \) (Lemma 4.9).

First we have

**Lemma 4.8.** For sufficiently large \( \lambda \)

\[
J_\lambda(v_{01}, \gamma_e(\theta)) \geq c_{min}^{1,D} + c_k^{2,D} - \frac{2}{3} \epsilon_0
\]

implies

\[
(v_{01}, \gamma_e(\theta)) \in A_\lambda.
\]

**Proof.** First we recall \( J_{1,\lambda}(v_{01}) = c_{min}^{1,D} < \alpha < \beta \). Suppose that \( (v_{01}, \gamma_e(\theta)) (\theta \in S_{n(k)}^{+}) \) satisfies (4.12). By (iii) of Lemma 4.7, we have

\[
J_{1,\lambda}(v_{01}) + J_{2,\lambda}(\gamma_e(\theta)) \geq (1 + r_\lambda)^{-\frac{2r_{e+1}}{r_\lambda^2}} (c_{min}^{1,D} + c_k^{2,D} - \epsilon_0).
\]

Thus recalling (4.11), we can see (4.13) holds for all \( \theta \in S_{n(k)}^{+} \) satisfying (4.12).

Now we define for \( \theta \in S_{n(k)}^{+} \) and \( L > 0 \)

\[
\tilde{\gamma}_L(\theta) = \begin{cases} (v_{01}, \gamma_e(\theta)) & \text{if } (v_{01}, \gamma_e(\theta)) \notin A_\lambda, \\ \eta(L, (v_{01}, \gamma_e(\theta))) & \text{if } (v_{01}, \gamma_e(\theta)) \in A_\lambda, \end{cases}
\]

where \( \eta(t, u_0) \) is a solution of (4.10).

The following lemma is the key of the proof; in particular it ensures \( \tilde{\gamma}_L(\theta) \) is well-defined for all \( L > 0 \) provided \( \epsilon > 0 \) is sufficiently small.

**Lemma 4.9.** Suppose that \( (v_{01}, \gamma_e(\theta)) \) satisfies \( (v_{01}, \gamma_e(\theta)) \in A_\lambda \). Let \( \eta(t) = \eta(t, (v_{01}, \gamma_e(\theta))) \) be the maximal solution of (4.10) and \( [0, \ell_0] \) be its maximal interval of existence satisfying \( \eta(t) \in A_\lambda \) for all \( t \in [0, \ell_0] \). Then for \( \epsilon \in (0, \epsilon_0) \) small and \( \lambda \) large we have

(i) \( \| \eta(t, x) \|_{T_{\theta(t)}(\Sigma_{1,\lambda} \times \Sigma_{2,\lambda})} \leq 1 \) for all \( t \in [0, \ell_0] \).

(ii) \( \frac{d}{dt} J_\lambda(\eta(t)) \leq 0 \) for all \( t \in [0, \ell_0] \).

(iii) \( J_\lambda(\eta(t)) \leq c_{min}^{1,D} + c_k^{2,D} + \epsilon \) for all \( t \in [0, \ell_0] \).

(iv) \( J_{1,\lambda}(\eta(t)) > (1 - r_\lambda)^{-\frac{2r_{e+1}}{r_\lambda^2}} (c_{min}^{1,D} + c_k^{2,D} + \epsilon_0) \) for all \( t \in [0, \ell_0] \).

(v) \( \text{If } \eta(t) = \eta(t, (v_{01}, \gamma_e(\theta))) \text{ satisfies } \eta(t_0) \in \partial A_\lambda \text{ for some } t_0 \in [0, \ell_0], \text{ then} \)

\[
\eta(t) \equiv (v_{01}, \gamma_e(\theta)) \text{ for all } t \in [0, \ell_0],
\]

\[
J_\lambda(v_{01}, \gamma_e(\theta)) \leq c_{min}^{1,D} + c_k^{2,D} - \frac{2}{3} \epsilon_0.
\]

(vi) \( \ell_0 = \infty \) for all \( (v_{01}, \gamma_e(\theta)) \in A_\lambda \). Moreover \( (v_{01}, \gamma_e(\theta)) \in \text{int} A_\lambda \) implies \( \eta(t) \in \text{int} A_\lambda \) for all \( t \geq 0 \) and \( (v_{01}, \gamma_e(\theta)) \notin \text{int} A_\lambda \) implies \( \eta(t) = (v_{01}, \gamma_e(\theta)) \) for all \( t \geq 0 \).

(vii) \( \text{If } \eta(t) = \eta(t, (v_{01}, \gamma_e(\theta))) \text{ satisfies} \)

\[
J_\lambda(\eta(t)) \geq c_{min}^{1,D} + c_k^{2,D} - \frac{1}{3} \epsilon_0,
\]

then \( \eta(t) \in A_\lambda \) and \( \frac{d}{dt} J_\lambda(\eta(t)) \leq - \mu_\lambda. \)
Proof: (i)–(ii) follow from the definition of $\widetilde{X}(v)$.

(iii) Since $J_2(v_01, \gamma(\theta)) = J_1, \lambda(v_01) + J_2, \lambda(\gamma(\theta)) \leq c_{\min} + c_k^2 + \epsilon$, (iii) follows from (ii).

(iv) By (i) of Lemma 4.11 it follows from (iii) that

$$(J_1, \lambda + J_2, \lambda)(\eta(t)) \leq (1 - r_\lambda)^{-\frac{2(\mu + 1)}{\mu}}(c_{\min}^1 + c_k^2 + \epsilon)$$

$$\leq (1 - r_\lambda)^{-\frac{2(\mu + 1)}{\mu}}(c_{\min}^1 + c_k^2 + \epsilon_0).$$

Thus for sufficiently large $\lambda$, we have (iv).

(v) By (iv) and Remark 4.6, $\eta(t) = (\eta_1(t), \eta_2(t))$ never touches a boundary $\{(v_1, v_2); J_1, \lambda(v_1) + J_2, \lambda(v_2) = (1 - r_\lambda)^{-\frac{2(\mu + 1)}{\mu}}(c_{\min}^1 + c_k^2 + \epsilon_0)\} \cup \{(v_1, v_2); J_1, \lambda(v_1) = c_{\min}^1 - \beta\delta_0\}$. We also remark that $\eta(t)$ never touches a boundary $\{(v_1, v_2); J_1, \lambda(v_1) = \beta\}$ by Lemma 4.7 (iii) and (4.11).

Thus $\eta(t) \in \partial A_\lambda$ implies

$$(4.15) \quad (J_1, \lambda + J_2, \lambda)(\eta(t)) = (1 + r_\lambda)^{-\frac{2(\mu + 1)}{\mu}}(c_{\min}^1 + c_k^2 - \epsilon_0).$$

We remark that (4.15) implies

$$(4.16) \quad J_\lambda(\eta(t)) \leq c_{\min}^1 + c_k^2 - \epsilon_0 \leq c_{\min}^1 + c_k^2 - \frac{2}{3}\epsilon_0.$$ 

Since $\widetilde{X}(\eta(t)) = 0$ follows from (4.16), we have (v) from the uniqueness of solutions of (4.10). (vi)–(vii) also follow from the above arguments.

Proof of Theorem 1.1. We choose $\epsilon \in (0, \epsilon_0)$ and let $L_\lambda = \frac{\epsilon_0}{\mu_\lambda} > 0$. We consider $\tilde{\gamma}(\theta) = \eta(L_\lambda, (v_01, \gamma(\theta))) : S_+^{n(k) + 1} \to \Sigma_1, \lambda \times \Sigma_2, \lambda$. Then we can easily see $\tilde{\gamma} \in A_\lambda$ and

$$(4.17) \quad \sup_{\theta \in S_+^{n(k) + 1}} \int \lambda(\tilde{\gamma}(\theta)) \leq c_{\min}^1 + c_k^2 - \frac{1}{3}\epsilon_0.$$ 

In fact, if $(v_01, \gamma(\theta)) \notin A_\lambda$, then by Lemma 4.8 we have

$$J_\lambda(\tilde{\gamma}(\theta)) \leq J_\lambda(v_01, \gamma(\theta)) \leq c_{\min}^1 + c_k^2 - \frac{2}{3}\epsilon_0.$$ 

So we consider the case where $(v_01, \gamma(\theta)) \in A_\lambda$. If $J_\lambda(\eta(L_\lambda, (v_01, \gamma(\theta)))) \leq c_{\min}^1 + c_k^2 - \frac{1}{3}\epsilon_0$, then (4.17) clearly holds. On the other hand, if $J_\lambda(\eta(L_\lambda, (v_01, \gamma(\theta)))) \geq c_{\min}^1 + c_k^2 - \frac{1}{3}\epsilon_0$, then by (vii) of Lemma 4.9,

$$J_\lambda(\eta(L_\lambda, (v_01, \gamma(\theta)))) = J_\lambda(v_01, \gamma(\theta)) + \int_0^{L_\lambda} \frac{d}{dt} J_\lambda(\eta(t)) dt$$

$$\leq c_{\min}^1 + c_k^2 + \epsilon - \mu_\lambda L_\lambda$$

$$\leq c_{\min}^1 + c_k^2 - \epsilon_0,$$

which is a contradiction.

Now we remark that (4.17) implies

$$c_k^1 = \inf_{\gamma \in A_\lambda} \sup_{\theta \in S_+^{n(k) + 1}} J_\lambda(\gamma(\theta)) \leq \sup_{\theta \in S_+^{n(k) + 1}} J_\lambda(\tilde{\gamma}(\theta)) \leq c_{\min}^1 + c_k^2 - \frac{1}{3}\epsilon_0.$$ 

However, by Proposition 8.7, we have $c_k^1 \to c_{\min}^1 + c_k^2$. This is a contradiction and $J_\lambda(v)$ has a critical point in $A_\lambda$. Thus Theorem 1.1 is proved. \qed
5. One dimensional case

In this section we consider 1 dimensional case:

\[ (P_\lambda) \quad -u'' + (\lambda^2 a(x) + 1)u = |u|^{p-1}u \quad \text{in} \ R, \quad u(x) \in H^1(R). \]

For \( a(x) \), we assume

(a1) \( a(x) \in C(R, R) \), \( a(x) \geq 0 \) for all \( x \in R \) and the potential well consists of 2 intervals, that is, \( \text{int} \ a^{-1}(0) = \Omega_1 \cup \Omega_2 \), \( \Omega_1 = (a_1, b_1) \) and \( \Omega_2 = (a_2, b_2) \)

(a2) \( a_1 < b_1 < a_2 < b_2 \).

We remark that the set of all solutions of \( (5.1) \) can be described as \( \{ \pm \bar{w}_n(a, b; x) \mid n \in N \} \cup \{ 0 \} \). In what follows, we define

\[ w_n(a, b; x) = \begin{cases} \bar{w}_n(a, b; x) & \text{in} \ (a, b), \\ 0 & \text{otherwise}. \end{cases} \]

(b) The proof of Theorem 1.2

We have two limit problems for \( (P_\lambda) \) for \( i = 1, 2 \)

\[ -u'' + u = |u|^{p-1}u \quad \text{in} \ (a_i, b_i), \quad u(a_i) = u(b_i) = 0. \]

In order to prove Theorem 1.2 for any \( n_1, n_2 \in N, s_1, s_2 \in \{-1, +1\} \) we will construct solutions of \( (P_\lambda) \) which converge to \( s_iw_n(a_i, b_i; x) \) in \( (a_i, b_i) \) and \( t \) elsewhere. We use the broken-geodesic method which is used in \[15\]. See also \[19\] and \[26\] for related arguments. We consider the case \( n_1, n_2 \geq 3 \). We can deal with the other cases in a similar way.

We set \( t_0^1 = a_1 + \frac{b_2-a_1}{n_1}, t_2^0 = b_1 - \frac{b_1-a_1}{n_1}, t_3^0 = a_2 + \frac{b_2-a_2}{n_2}, t_0^4 = b_2 - \frac{b_2-a_2}{n_2}, \)

\[ t^0 = (t_0^1, t_2^0, t_3^0, t_0^4). \]

For small \( \epsilon \in (0, \min\{\frac{b_2-a_1}{n_1}, \frac{b_2-a_2}{n_2}\}) \) we set

\[ U = \prod_{i=1}^4 (t_i^0 - \epsilon, t_i^0 + \epsilon). \]

To find solutions of \( (P_\lambda) \), first we fix \( t = (t_1, t_2, t_3, t_4) \in U \), \( t_0 = -\infty \), \( t_5 = \infty \) and we solve \( (P_\lambda) \) in the intervals \( (t_i, t_{i+1}) \) for \( i = 0, 1, \ldots, 4 \):

\[ (P_{\lambda,i}) \quad -u'' + (\lambda^2 a(x) + 1)u = |u|^{p-1}u \quad \text{in} \ (t_i, t_{i+1}), \quad u(x) \in H^1_0(t_i, t_{i+1}). \]

Remark 5.1. We remark that \( (P_{\lambda,1}) \) and \( (P_{\lambda,3}) \) are independent of \( \lambda \). Thus \( (P_{\lambda,1}) \) \( (P_{\lambda,3}) \) resp.) has a unique solution \( w_{n_i-2}(t_1, t_2; x) \) \( w_{n_2-3}(t_3, t_4; x) \) resp.) which has \( n_1 - 3 (n_2 - 3 \text{ resp.}) \) zeros and \( w'_{n_i-2}(t_1, t_2; t_1) > 0 \) \( (w'_{n_2-3}(t_3, t_4; t_3) > 0 \text{ resp.}) \).
The following lemma deals with \((P_{\lambda,i})\) \((i = 0, 2, 4)\), which can be shown just as in [15].

**Lemma 5.2.** There exists \(\delta_0 > 0\) such that for any \(\delta \in (0, \delta_0)\), there exists \(\Lambda_0 = \Lambda_0(\delta) > 0\) such that for \(\lambda > \Lambda_0\) and \((t_1, t_2, t_3, t_4) \in U\),

(i) \((P_{\lambda,0})\) has a unique solution \(u_{0,\lambda}(t_1; x)\) which satisfies

\[
\|u_{0,\lambda}(t_1; x) - w_1(a_1, t_1; x)\|_{L^\infty(-\infty, t_1)} < \delta.
\]

(ii) \((P_{\lambda,2})\) has a unique solution \(u_{2,\lambda}(t_2, t_3; x)\) which satisfies

\[
\|u_{2,\lambda}(t_2, t_3; x) - w_1(t_2, b_1; x) - w_1(a_2, t_3; x)\|_{L^\infty(t_2, t_3)} < \delta.
\]

Similarly, \((P_{\lambda,2})\) has a unique solution \(\hat{u}_{2,\lambda}(t_2, t_3; x)\) which satisfies

\[
\|\hat{u}_{2,\lambda}(t_2, t_3; x) - w_1(t_2, b_1; x) + w_1(a_2, t_3; x)\|_{L^\infty(t_2, t_3)} < \delta.
\]

(iii) \((P_{\lambda,4})\) has a unique solution \(u_{4,\lambda}(t_4; x)\) which satisfies

\[
\|u_{4,\lambda}(t_4; x) - w_1(t_4, b_2; x)\|_{L^\infty(t_4, \infty)} < \delta.
\]

**Remark 5.3.** From Lemma 5.2, \(u_{0,\lambda}(t_1; x)\) satisfies

\[
(5.2) \quad \|u_{0,\lambda}(t_1; x) - w_1(a_1, t_1; x)\|_{L^\infty(-\infty, t_1)} \to 0 \quad \text{as} \ \lambda \to \infty.
\]

Since \((P_{\lambda,0})\) is independent of \(\lambda\) in \([a_1, t_1]\), (5.2) implies

\[
\|u_{0,\lambda}(t_1; x) - w_1(a_1, t_1; x)\|_{C^2([a_1, t_1])} \to 0 \quad \text{as} \ \lambda \to \infty.
\]

By uniqueness of \(u_{0,\lambda}(t_1; x)\), we find \(u_{0,\lambda}(t_1; t_1)\) and \(u_{0,\lambda}'(t_1; t_1)\) are continuous with respect to \(t_1 \in (t_1^0 - \epsilon, t_1^0 + \epsilon)\). Similar results hold for \(u_{2,\lambda}(t_2, t_3; x)\), \(\hat{u}_{2,\lambda}(t_2, t_3; x)\) and \(u_{4,\lambda}(t_4; x)\).

**End of the proof of Theorem 1.2** Let us outline the proof for the case \(n_1, n_2 \geq 3\). We can deal with the other cases in a similar way. We define functions \(u_\lambda(t; x), \hat{u}_\lambda(t; x)\) for \(t = (t_1, \ldots, t_4) \in U\) by

\[
u_\lambda(t; x) = u_{0,\lambda}(t_1; x) - w_{n_1 - 2}(t_1, t_2; x) + (-1)^{n_1 - 1} u_{2,\lambda}(t_2, t_3; x) + (-1)^{n_1 + n_2} u_{4,\lambda}(t_4; x)
\]

or

\[
\hat{u}_\lambda(t; x) = u_{0,\lambda}(t_1; x) - w_{n_1 - 2}(t_1, t_2; x) + (-1)^{n_1 - 1} \hat{u}_{2,\lambda}(t_2, t_3; x) + (-1)^{n_1 + n_2} \hat{u}_{4,\lambda}(t_4; x).
\]

We deal with \(u_\lambda(t; x)\). We can deal with \(\hat{u}_\lambda(t; x)\) in a similar way. We observe that \(u_\lambda(t; x)\) is a solution of \((P_\lambda)\) if and only if \(u_\lambda(t; x)\) is of class \(C^1\) with respect to \(x\).

We define a function \(F_\lambda(t) \in C(U, \mathbb{R}^4)\) by

\[
F_\lambda(t) = \begin{bmatrix} F_{1,\lambda}(t) \\ F_{2,\lambda}(t) \\ F_{3,\lambda}(t) \\ F_{4,\lambda}(t) \end{bmatrix} = \begin{bmatrix} w_{n_1 - 2}(t_1, t_2; t_1) + w_{0,\lambda}(t_1; t_1) \\ u_{2,\lambda}(t_2, t_3; t_2) + (-1)^{n_1 - 1} w_{n_1 - 2}(t_1, t_2; t_2) \\ w_{n_2 - 2}(t_3, t_4; t_3) + u_{2,\lambda}(t_2, t_3; t_3) \\ u_{4,\lambda}(t_4; t_4) + (-1)^{n_2 - 1} w_{n_2 - 2}(t_3, t_4; t_4) \end{bmatrix} : U \to \mathbb{R}^4.
\]

Then \(u_\lambda(t; x)\) is of class \(C^1\) if and only if \(F_\lambda(t) = 0\). We will show \(\deg(F_\lambda, U, 0) = 1\).

We remark that \(\partial U = \bigcup_{i=1}^{4} \{t_i = t_i^0 - \epsilon\} \cup \{t_i = t_i^0 + \epsilon\}\). For \(i = 1, 2, 3, 4\), we have

\[
(5.3) \quad F_{\lambda,i}(t) < 0 \quad \text{if} \quad t_i = t_i^0 - \epsilon, \quad F_{\lambda,i}(t) > 0 \quad \text{if} \quad t_i = t_i^0 + \epsilon
\]
for large $\lambda$. In fact, when $t_1 = t_1^0 - \epsilon$, by the properties of $w_n(a, b; x)$, we find
\[
F_{1, \lambda}(t) = w_n(0, t_1^0 - \epsilon, t_2; t_1^0 - \epsilon) + u_n(t_1^0 - \epsilon; t_1^0 - \epsilon) \\
\leq w_n(0, t_1^0 - \epsilon, t_2^0; t_1^0 - \epsilon) + u_n(t_1^0 - \epsilon; t_1^0 - \epsilon) \\
= w_n'(a_1, t_1^0; a_1) + u_n(t_1^0 - \epsilon; t_1^0 - \epsilon).
\]

By Remark 5.3, we observe
\[
w_n'(a_1, t_1^0; a_1) + u_n(t_1^0 - \epsilon; t_1^0 - \epsilon) \\
\rightarrow w_n'(a_1, t_1^0; a_1) + u_n(t_1^0 - \epsilon; t_1^0 - \epsilon) < 0 \text{ as } \lambda \to \infty.
\]

We can show the other properties in (5.3) in a similar way. By (5.3), $\deg(F_{\lambda}, U, 0)$ is well-defined for large $\lambda$ and $\deg(F_{\lambda}, U, 0) = 1$. Therefore there exists $t_1^\lambda \in U$ such that $u_\lambda(t_1^\lambda; x)$ is a solution of $(P_\lambda)$. Since $\epsilon > 0$ is arbitrary, we can show $t_1^\lambda \to t_0$ as $\lambda \to \infty$; that is, $u_\lambda(t_1^\lambda; x)$ converges to $w_n(a_1, b_1; x)$ in $(a_1, b_1)$, to $(-1)^{n_1-1}w_n(a_2, b_2; x)$ in $(a_2, b_2)$ and to 0 elsewhere. \hfill \Box

6. PROOFS OF THEOREMS 1.3 AND 1.5

In this section we give proofs of Theorem 1.3 and Theorem 1.5. Since the exponent $p$ plays an important role, we write dependence of $J_\lambda, J_{i,D}$ on $p$ explicitly in this section and use the notation
\[
J_\lambda(p; v_1, v_2) = J_\lambda(v_1, v_2) \quad \text{for } (v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda},
\]
\[
J_{i,D}(p; v_i) = \left(1 - \frac{1}{p + 1}\right)\left(\frac{1}{\|v_i\|_{L^{p+1}(\Omega_i)}}\right)^{\frac{2(p+1)}{p}} \quad \text{for } v_i \in \Sigma_{i,D}.
\]

We define
\[
c_p^\lambda = \inf_{(v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} J_\lambda(p; v_1, v_2), \quad c_p(\Omega_i) = \inf_{v_i \in \Sigma_{i,D}} J_{i,D}(p; v_i).
\]

Since $J_\lambda(p; v_1, v_2), J_{i,D}(p; v_i)$ satisfy the Palais-Smale condition, $c_p^\lambda$ and $c_p(\Omega_i)$ are critical values of $J_\lambda(p; v_1, v_2)$ and $J_{i,D}(p; v_i)$. First we observe

**Lemma 6.1.** (i) $c_p^\lambda < c_p(\Omega_1) + c_p(\Omega_2)$ for all $\lambda \geq 1$.

(ii) $c_p^\lambda \to c_p(\Omega_1) + c_p(\Omega_2)$ as $\lambda \to \infty$.

**Proof.** Let $(v_{1,\lambda}, v_{2,\lambda}) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$ be a minimizer of $J_\lambda(p; v_1, v_2)$, that is, $J_\lambda(p; v_1, v_2) = c_p^\lambda$. (i) Since $\Sigma_{1,D} \times \Sigma_{2,D} \subset \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$, we have
\[
c_p^\lambda = \inf_{(v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} J_\lambda(p; v_1, v_2) \leq \inf_{(v_1, v_2) \in \Sigma_{1,D} \times \Sigma_{2,D}} J_\lambda(p; v_1, v_2)
\]
\[
= \inf_{(v_1, v_2) \in \Sigma_{1,D} \times \Sigma_{2,D}} \left(J_{1,D}(p; v_1) + J_{2,D}(p; v_2)\right) = c_p(\Omega_1) + c_p(\Omega_2).
\]

Since $(v_{1,\lambda}, v_{2,\lambda})$ corresponds to a positive solution $u(x)$ of $-\Delta u + (\lambda^2 a(x) + 1)u = g(x, u)$ in $\mathbb{R}^N$, we have $v_{1,\lambda} \in \Sigma_{i,\lambda} \setminus \Sigma_{i,D}$ and we have strict inequality.

(ii) By Proposition 2.17, there exist a sequence $\lambda_n \to \infty$ and critical points $v_i \in \Sigma_{i,D}$ of $J_{i,D}$ $(i = 1, 2)$ such that
\[
(v_{1,\lambda_n}, v_{2,\lambda_n}) \to (v_1, v_2) \quad \text{strongly in } H^1(\Omega_1) \oplus H^1(\Omega_2)
\]
and
\[
J_{\lambda_n}(p; v_{1,\lambda_n}, v_{2,\lambda_n}) \to J_{1,D}(p; v_1) + J_{2,D}(p; v_2) \geq c_p(\Omega_1) + c_p(\Omega_2).
\]

Therefore by (i), $c_p^\lambda_n \to c_p(\Omega_1) + c_p(\Omega_2)$. Since the limit does not depend on the subsequence, conclusion (ii) holds. \hfill \Box
To study the case $p \sim \frac{N+2}{N-2}$, we introduce the notation
\[ c_p(Q) = \inf_{u \in H^1_0(Q), \|u\|_{H^1(Q)}=1} \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{1}{\|u\|_{L^{p+1}(Q)}} \right)^{\frac{2(p+1)}{p-1}} \]
for $Q \subset \mathbb{R}^N$.

$c_p(Q)$ has the following properties:

**Lemma 6.2.**

(i) $\frac{N+2}{N-2}$ implies $c_p(Q') \leq c_p(Q)$.
(ii) $\frac{N+2}{N-2}$ does not depend on $Q$.
(iii) $\frac{N+2}{N-2}$ is never achieved in a proper subset of $\mathbb{R}^N$.

**Proof.** Introducing
\[ d_p(Q) = \inf_{u \in H^1_0(Q), \|u\|_{H^1(Q)}=1} \frac{1}{\|u\|_{L^{p+1}(Q)}} \]
for $Q \subset \mathbb{R}^N$,

(0)–(iii) are equivalent to

(i) $\frac{N+2}{N-2}$ implies $d_p(Q') \leq d_p(Q)$.
(ii) $\frac{N+2}{N-2}$ does not depend on $Q$.
(iii) $\frac{N+2}{N-2}$ is never achieved in a proper subset of $\mathbb{R}^N$.

which can be derived as in [1] (see page 81 and Lemma 4.1). \[ \Box \]

For $r > 0$ we set
\[ \Omega_+ = \{x \in \mathbb{R}^N; dist(x, \Omega_i) < r\}, \quad \Omega_+ = \{x \in \Omega_i; dist(x, \partial \Omega_i) > r\}. \]

We choose $r > 0$ small enough so that

(i) The inclusions $\Omega_+ \subset \Omega_i$ and $\Omega_i \subset \Omega_+$ are homotopy equivalent.
(ii) $B_r(x) \subset \Omega_i$ for all $x \in \Omega_i$.

By the choice of $r > 0$, we have

**Corollary 6.3.**

(i) $c^\lambda \leq c_p(\Omega_1) + c_p(\Omega_2) < 2c_p(B_r)$.
(ii) There exists $p_1 \in (1, \frac{N+2}{N-2})$ such that for any $p \in (p_1, \frac{N+2}{N-2})$, there exists $\Lambda_1(p) > 0$ such that
\[ c_p(B_r) < c^\lambda \quad \text{for all } \lambda \geq \Lambda_1(p). \]

**Proof.** (i) follows from Lemma 6.1 (i) and Lemma 6.2 (0). (ii) By Lemma 6.2 (i),
we have $c_p(\Omega_1) + c_p(\Omega_2) \to 2c_{\frac{N+2}{N-2}}$, $c_p(B_r) \to c_{\frac{N+2}{N-2}}$. Thus (ii) follows from Lemma 6.1 (ii). \[ \Box \]

For $v \in \Sigma_{i,\lambda}$ we set
\[ \beta_i(p; v) = \frac{\int_{\Omega_i'} |v|^{p+1} x \ dx}{\int_{\Omega_i'} |v|^{p+1} \ dx} : \Sigma_{i,\lambda} \to \mathbb{R}^N. \]

We have the following.

**Lemma 6.4.** There exists $\delta > 0$ such that if $p \in \left( \frac{N+2}{N-2}, \delta, \frac{N+2}{N-2} \right)$ and $v \in \Sigma_{i,D}$ satisfies
\[ J_{i,D}(p; v) \leq c_{\frac{N+2}{N-2}} + \delta, \]

then $\beta_i(p; v) \in \Omega_+$. 
We remark that
\[
\inf_{v \in \Sigma_{i,D}} J_{i,D}(p; v) = c_p(\Omega_i) \to c_{N+2 \over N-2} \quad \text{as} \quad p \to {N + 2 \over N - 2} = 2.
\]
Thus \( \{ v \in \Sigma_{i,D}; J_{i,D}(p; v) \leq c_{N+2 \over N-2} + \delta \} \) is not empty for \( p \) close to \( N+2 \over N-2 \).

**Proof.** We argue indirectly and assume that there exist \( p_n, v_n \in \Sigma_{i,D} \) such that
\[
(6.1) \quad p_n \to {N + 2 \over N - 2}, \quad J_{i,D}(p_n; v_n) \to c_{N+2 \over N-2}, \quad \beta_i(p_n; v_n) \not\in \Omega_i^+.
\]

By Hölder’s inequality,
\[
(6.2) \quad J_{i,D}(N+2 \over N-2; v_n) = \frac{1}{N} \int_{\Omega} \frac{1}{L^{2N} \Omega} \leq \frac{1}{N} \|v_n\|_{L^{2N} \Omega}^{N+2 \over p_n} \frac{1}{\|v_n\|_{L^{p_n+1} \Omega}^{N}} \leq \frac{1}{\|v_n\|_{L^{2N} \Omega}}^{N+2 \over p_n} \frac{1}{\|v_n\|_{L^{p_n+1} \Omega}^{N}} \to c_{N+2 \over N-2}.
\]

Since \( J_{i,D}(N+2 \over N-2; v) \geq c_{N+2 \over N-2} \) for all \( v \in \Sigma_{i,D} \), it follows from (6.2) that
\[
\frac{1}{\|v_n\|_{L^{2N} \Omega}^{N+2 \over p_n}} \to d_{N+2 \over N-2},
\]
where \( d_{N+2 \over N-2} \) is defined in the proof of Lemma 6.2. From the well-known compactness result (cf. Struwe [24] and Lions [18]), it follows that there exist \( r_n \to 0 \), \( (x_n)_{n=1}^{\infty} \subset \Omega_i \) and a function \( w_0(x) \in D^{1,2}(\mathbb{R}^N) \) satisfying \( \|\nabla w_0\|_{L^2(\mathbb{R}^N)} = 1 \) and
\[
\|w_0\|_{L^{2N} \mathbb{R}^N}^2 = d_{N+2 \over N-2} \text{ such that } r_n \frac{N+2 \over N-2}{p_n} v_n(r_n x + x_n) \to w_0(x) \text{ strongly in } D^{1,2}(\mathbb{R}^N).
\]
Thus
\[
\beta_i(p_n; v_n) \in \Omega_i^+ \text{ for large } n.
\]
This is a contradiction to assumption (6.1). \( \Box \)

A following proposition is a key of the proof of Theorem 1.3

**Proposition 6.5.** There exists \( p_2 \in (1, N+2 \over N-2) \) such that for any \( p \in (p_2, N+2 \over N-2) \),
there exists \( \Lambda_2(p) \geq 1 \) such that \( (\beta_1(p; v_1), \beta_2(p; v_2)) \in \Omega_i^+ \times \Omega_i^+ \) for all \( \lambda \geq \Lambda_2(p) \) and \((v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda} \) satisfying \( J_\lambda(p; v_1, v_2) \leq 2c_p(B_r) \).

We remark by Corollary 6.3 (i) that the level set
\[
[ J_\lambda(p; v_1, v_2) \leq 2c_p(B_r) ]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} = \{ (v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}; J_\lambda(p; v_1, v_2) \leq 2c_p(B_r) \}
\]
is not empty.

**Proof.** We choose \( p_2 \in (p_1, N+2 \over N-2) \) such that
\[
(6.3) \quad 2c_p(B_r) - c_p(\Omega_i) \leq c_{N+2 \over N-2} + \delta \quad \text{for } i = 1, 2 \text{ and } p \in (p_2, N+2 \over N-2),
\]
where \( \delta > 0 \) is given in Lemma 6.4. We will show the conclusion of Proposition 6.5 holds for \( p_2 \).
Again we argue indirectly, and we assume that there exist \( p \in (p_2, \frac{N+2}{N-2}) \), \( \lambda_n \to \infty \) and \( (v_{1n}, v_{2n}) \in \Sigma_{1, \lambda_n} \times \Sigma_{2, \lambda_n} \) such that

\[
J_{\lambda_n}(p; v_{1n}, v_{2n}) \leq 2c_p(B_r), \quad (\beta_1(p; v_{1n}), \beta_2(p; v_{2n})) \not\in \Omega^+ \times \Omega^+.
\]

By Proposition 2.13 \((v_{1n}, v_{2n})\) satisfies \( \|v_{1n}\|_{L^{p+1}(\Omega')} \geq \delta_M \), where \( \delta_M > 0 \) is given in Proposition 2.13 with \( M = 2e^{\frac{N+2}{N-2}} + 1 \). Extracting a subsequence if necessary, we may assume

\[
v_{1n} \to v_{10} \quad \text{strongly in } L^{p+1}(\Omega') \quad \text{and weakly in } H^1(\Omega').
\]

Thus we have \( \|v_{10}\|_{L^{p+1}(\Omega')} \geq \delta_M \) and

\[
(\beta_1(p; v_{10}), \beta_2(p; v_{20})) \not\in \Omega^+ \times \Omega^+.
\]

We also remark that for any compact set \( D \subset \Omega'_i \backslash \Omega^+_i \)

\[
\|v_{1n}\|_{L^2(D)}^2 \leq \frac{1}{\lambda_n^2} \inf_{x \in D} a(x) \|v_{1n}\|_{\lambda_n \Omega'_i}^2 \to 0.
\]

In particular, we have \( v_{10} \in H^1_0(\Omega_i) \). Thus we have \( \|v_{10}\|_{H^1(\Omega_i)} \leq \lim \inf_{n \to \infty} \|v_{1n}\|_{H^1(\Omega'_i)} \leq \lim \inf_{n \to \infty} \|v_{1n}\|_{\lambda_n \Omega'_i} = 1 \) that

\[
J_{1, D}(\frac{v_{10}}{\|v_{10}\|_{H^1(\Omega_i)}}) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \|v_{10}\|_{H^1(\Omega_i)} \right)^{\frac{2(p+1)}{p+1}} \leq \left( \frac{1}{2} - \frac{1}{p+1} \right) \|v_{10}\|_{L^{p+1}(\Omega'_i)}^{\frac{2(p+1)}{p+1}}.
\]

Thus we have

\[
J_{1, D}(\frac{v_{10}}{\|v_{10}\|_{H^1(\Omega_i)}}) + J_{2, D}(\frac{v_{20}}{\|v_{20}\|_{H^1(\Omega_i)}})
\]

\[
\leq \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \|v_{10}\|_{L^{p+1}(\Omega'_i)}^{\frac{2(p+1)}{p+1}} + \|v_{20}\|_{L^{p+1}(\Omega'_i)}^{\frac{2(p+1)}{p+1}} \right)
\]

\[
= \lim_{n \to \infty} \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \|v_{1n}\|_{L^{p+1}(\Omega'_i)}^{\frac{2(p+1)}{p+1}} + \|v_{2n}\|_{L^{p+1}(\Omega'_i)}^{\frac{2(p+1)}{p+1}} \right)
\]

\[
\leq \lim_{n \to \infty} J_{\lambda_n}(p; v_{1n}, v_{2n}) \leq 2c_p(B_r).
\]

Here we used the fact that

\[
J_{\lambda}(p; v_1, v_2) = \sup_{s, t \geq 0} I_{\lambda}(p; sv_1, tv_2) \geq \sup_{s, t \geq 0} \tilde{I}_{\lambda}(p; sv_1, tv_2)
\]

\[
= \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \|v_1\|_{L^{p+1}(\Omega'_i)}^{\frac{2(p+1)}{p+1}} + \|v_2\|_{L^{p+1}(\Omega'_i)}^{\frac{2(p+1)}{p+1}} \right),
\]

where \( \tilde{I}_{\lambda}(p; v_1, v_2) = \frac{1}{2} \|v_1\|^2_{\lambda \Omega'_i} + \frac{1}{2} \|v_2\|^2_{\lambda \Omega'_i} - \frac{1}{p+1} \|v_1\|_{L^{p+1}(\Omega'_i)}^{p+1} - \frac{1}{p+1} \|v_2\|_{L^{p+1}(\Omega'_i)}^{p+1} \).

Since \( c_p(\Omega_i) \leq J_{1, D}(\frac{v_{10}}{\|v_{10}\|_{H^1(\Omega_i)}}) \), we have

\[
J_{1, D}(\frac{v_{10}}{\|v_{10}\|_{H^1(\Omega_i)}}) \leq 2c_p(B_r) - c_p(\Omega_2), \quad J_{2, D}(\frac{v_{20}}{\|v_{20}\|_{H^1(\Omega_i)}}) \leq 2c_p(B_r) - c_p(\Omega_1).
\]

By (6.3), we have \( J_{1, D}(\frac{v_{10}}{\|v_{10}\|_{H^1(\Omega_i)}}) \leq c_{\frac{N+2}{N-2}} + \delta \). Thus by Lemma 6.4 we have \( (\beta_1(p; v_{10}), \beta_2(p; v_{20})) \in \Omega^+ \times \Omega^+ \), which is in contradiction to (6.4). \( \square \)

In order to prove Theorem 1.3, we need the following lemma.
Let \( A, B, X \) be topological spaces and suppose that there exist maps \( \alpha : A \to X \) and \( \beta : X \to B \) such that \( \beta \circ \alpha : A \to B \) is a homotopy equivalence. Then \( \text{cat}(A) \leq \text{cat}(X) \).

**Proof.** Suppose that \( \text{cat}(X) = k \). Then there exist closed sets \( X_1, \ldots, X_k \subset X \) such that \( X = X_1 \cup \cdots \cup X_k \) and each \( X_i \) are contractible in \( X \). We set \( A_i = \alpha^{-1}(X_i) \subset A \). It follows that

\[
\text{cat}(A) \leq \sum_{i=1}^{k} \text{cat}(A_i).
\]

We claim that, if \( A_i \neq \emptyset \), \( A_i \) is contractible in \( A \); that is, \( \text{cat}(A_i) = 1 \). Since \( X_i \) are contractible in \( X \), there exist \( H_i \in C([0,1] \times X_i, X) \) and \( x_{i0} \in X \) such that \( H_i(0,x) = x, H_i(1,x) = x_{i0} \) for all \( x \in X_i \). Furthermore, since \( \beta \circ \alpha : A \to B \) is a homotopy equivalence, there exist continuous maps \( \varphi : B \to A \) and \( G \in C([0,1] \times A, A) \) such that \( G(0,x) = x, G(1,x) = \varphi(\beta(\alpha(x))) \) for all \( x \in X_1 \). We define \( F_i \in C([0,2] \times A_i, A) \) by

\[
F_i(t,x) = \begin{cases} 
G(t,x) & \text{if } t \in [0,1] \text{ and } x \in A_i, \\
\varphi(H_i(t-1,\alpha(x))) & \text{if } t \in [1,2] \text{ and } x \in A_i.
\end{cases}
\]

Then \( F_i \) satisfies \( F_i(0,x) = x, F_i(2,x) = \varphi(\beta(x_{i0})) \) for all \( x \in A_i \). Therefore, \( A_i \) is contractible in \( A \); that is, \( \text{cat}(A_i) = 1 \). Consequently we get \( \text{cat}(A) \leq k = \text{cat}(X) \). \( \square \)

Set \( M = 2c_{N+2} + 1 \) and let \( \nu_M > 0 \) be a constant given in Proposition 9.1 and Corollary 9.3. We set

\[
X^+_{\lambda} = \{ (v_1, v_2) \in [J_\lambda(p; v_1, v_2) \leq 2c_p(B_r)]_{\Sigma_1,\lambda} \times [\Sigma_2,\lambda]; \varphi_-(v_1) \leq \nu_M, \varphi_-(v_2) \leq \nu_M \},
\]

\[
X^-_{\lambda} = \{ (v_1, v_2) \in [J_\lambda(p; v_1, v_2) \leq 2c_p(B_r)]_{\Sigma_1,\lambda} \times [\Sigma_2,\lambda]; \varphi_+(v_1) \leq \nu_M, \varphi_+(v_2) \leq \nu_M \},
\]

\[
X^0_{\lambda} = \{ (v_1, v_2) \in [J_\lambda(p; v_1, v_2) \leq 2c_p(B_r)]_{\Sigma_1,\lambda} \times [\Sigma_2,\lambda]; \varphi(v_1) \leq \nu_M, \varphi_+(v_2) \leq \nu_M \},
\]

where \( \varphi_\pm(v) = \|v_1\|_{L^{p+1}_{\lambda,\Omega}^p}^{p+1} \) and \( \psi_\pm(v) = \|v_2\|_{L^{p+1}_{\lambda,\Omega}^p}^{p+1} \).

**Proposition 6.7.** Assume (a1)–(a2), (1.5) and \( N \geq 3 \). Then there exist \( p_2 \in (1, \frac{N}{N+2}) \) and \( \lambda \geq \Lambda_3(p) \),

\[
\text{cat}(X_{\lambda}^{\sigma_1,\sigma_2}) \geq \text{cat}(\Omega_1 \times \Omega_2) \quad \text{for all } \sigma_1, \sigma_2 \in \{+, -, \}.
\]

Moreover \( J_\lambda \) has at least \( \text{cat}(\Omega_1 \times \Omega_2) \) critical points in \( X^+_{\lambda} \) (\( X^-_{\lambda} \), \( X^0_{\lambda} \) respectively).

**Proof.** Let \( \tilde{U} \in H^1_0(B_r) \) be a unique solution of

\[
-\Delta u + u = u^p\quad \text{in } B_r, \quad u > 0 \quad \text{in } B_r, \quad u = 0 \quad \text{on } \partial B_r.
\]

We set

\[
U_p(x) = \frac{\tilde{U}(x-y)}{||\tilde{U}||_{L^p_{\lambda,B_r}}} \in H^1_0(B_r(y)).
\]

We note that for all \( (y, z) \in \Omega_1^- \times \Omega_2^- \),

\[
J_\lambda(p; \sigma_1 U_y, \sigma_2 U_z) = 2c_p(B_r) \quad \text{for all } \sigma_1, \sigma_2 \in \{+, -, \},
\]

\[
(\beta_1(p; \sigma_1 U_y), \beta_2(p; \sigma_2 U_z)) = (y, z) \quad \text{for all } \sigma_1, \sigma_2 \in \{+, -, \}.
\]
Let $p_2$ and $\Lambda_2(p)$ be constants given in Proposition 6.5. For any $p \in [p_2, \frac{N+2}{N-2})$ and $\lambda \geq \Lambda_2(p)$, we define two maps by
\[
\alpha^{\sigma_1\sigma_2}(y, z) = (\sigma_1 U_y, \sigma_2 U_z) : \Omega^+ \times \Omega^- \rightarrow X^{\sigma_1\sigma_2},
\beta^{\sigma_1\sigma_2}(v_1, v_2) = (\beta_1(p; v_1), \beta_2(p; v_2)) : X^{\sigma_1\sigma_2} \rightarrow \Omega^+ \times \Omega^-.
\]
By Proposition 6.5, these maps are well defined and $\beta^{\sigma_1\sigma_2} \circ \alpha^{\sigma_1\sigma_2}(y, z) : \Omega^+ \times \Omega^- \rightarrow \Omega^+ \times \Omega^-$ is an identity. Therefore, from Lemma 6.6 we find
\[
\text{cat}(X^{\sigma_1\sigma_2}) \geq \text{cat}(\Omega^+ \times \Omega^-) = \text{cat}(\Omega_1 \times \Omega_2).
\]
Moreover by Corollary 9.3 there exists $\Lambda_3(p) \geq \Lambda_2(p)$ such that for $\lambda \geq \Lambda_3(p)$ there exists a deformation $\eta(t, v)$ in $X^{\sigma_1\sigma_2}$ with the following properties:
\begin{enumerate}
  \item[(i)] $\eta(t, v) \in X^{\sigma_1\sigma_2}$ for all $(t, v) \in [0, 1] \times X^{\sigma_1\sigma_2}$,
  \item[(ii)] $J_\lambda(\eta(t, v)) \leq J_\lambda(v)$ for all $(t, v) \in [0, 1] \times X^{\sigma_1\sigma_2}$,
  \item[(iii)] If $b < M \equiv 2c_{\sigma_1\sigma_2} + 1$ is not a critical value of $J_\lambda$, then there exists $\epsilon > 0$ such that
    \[ J_\lambda(\eta(1, v)) < b - \epsilon \quad \text{for} \quad v \in [J_\lambda < b + \epsilon]. \]
\end{enumerate}
Thus $J_\lambda$ has at least $\text{cat}(\Omega_1 \times \Omega_2)$ critical points in $X^{\sigma_1\sigma_2}$.
\[\Box\]

\textit{End of the proof of Theorem 1.3} By Proposition 6.7, $J_\lambda$ has at least $\text{cat}(\Omega_1 \times \Omega_2)$ critical points in $X^{\sigma_1\sigma_2}$. By Proposition 2.19 (iii) those critical points are positive, and by Corollary 9.3 we have $\inf J_\lambda(v_1, v_2) = c_p^\lambda \geq c_p(B_r)$. In [3], they showed that $(P_\lambda)$ possesses at least $\text{cat}(\Omega_1 \cup \Omega_2) = \text{cat}(\Omega_1) + \text{cat}(\Omega_2)$ positive solutions. We remark that their solutions $u(x)$ satisfy $\Psi_\lambda(u) < c_p(B_r)$. Thus their solutions are different from those obtained in Proposition 6.7. Thus $(P_\lambda)$ has at least $\text{cat}(\Omega_1) + \text{cat}(\Omega_2) + \text{cat}(\Omega_1 \times \Omega_2)$ positive solutions, and the proof of Theorem 1.3 is completed.
\[\Box\]

\textit{End of the proof of Theorem 1.5} By Proposition 6.7, $J_\lambda$ has at least $\text{cat}(\Omega_1 \times \Omega_2)$ critical points $v_\lambda = (v_{1,\lambda}, v_{2,\lambda})$ in $X^{\sigma_1\sigma_2}$. From Propositions 2.17 and 2.19, extracting subsequence $\lambda_n \rightarrow \infty$, $v_{i,\lambda_n}$ converges to a critical point $v_{i,\infty} \in \Sigma_{i,D}$ of $J_{i,D}$ ($i = 1, 2$) and $v_{i,\infty}$ satisfies $v_{i,\infty} > 0$ in $\Omega_1$, $v_{2,\infty} < 0$ in $\Omega_2$. Thus Theorem 1.5 is proved.
\[\Box\]

7. Proofs of Lemmas 2.9, 2.10 and Propositions 2.15, 2.16

In this section we give proofs of various lemmas stated in Section 2 without proofs.

\textit{Proof of Lemma 2.9} Let $V(x)$ be a function satisfying (2.15). (i) is nothing but (2.10) in Lemma 2.6.

(ii) Set $v = Q_{V,\lambda}(u)$, $w = Q_{0,\lambda}(u)$. Since $v - w \in H^1_0(R^N \setminus (\Omega_1 \cup \Omega_2))$, we have from the variational characterization (2.18) of $v = Q_{V,\lambda}(u)$ that
\[
\langle v, v - w \rangle_{\lambda, R^N \setminus (\Omega_1 \cup \Omega_2)} - \int_{R^N \setminus (\Omega_1 \cup \Omega_2)} V(x) v(v - w) \, dx = 0.
\]
Similarly from the variational characterization for $w = Q_{0,\lambda}(u)$, we have
\[
\langle w, v - w \rangle_{\lambda, R^N \setminus (\Omega_1 \cup \Omega_2)} = 0.
\]
Subtracting (7.2) from (7.1),
\[\|v - w\|^2_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')} = \int_{\Omega_1' \cup \Omega_2'} V(x)v(v - w) \, dx\]
\[\leq \left( \int_{\Omega_1' \cup \Omega_2'} |V(x)||v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_1' \cup \Omega_2'} |V(x)||v - w|^2 \, dx \right)^{\frac{1}{2}}.\]

By (2.15)
\[\leq \frac{4}{9}\|v\|_{L^2(\Omega_1' \cup \Omega_2')}\|v - w\|_{L^2(\Omega_1' \cup \Omega_2')}\]
\[\leq r\|v\|_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')}\|v - w\|_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')}\]

Therefore by (i) we have
\[\|v - w\|_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')} \leq r\|v\|_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')} \leq r\|v\|_{\lambda,\Omega_1' \cup \Omega_2'}.\]
Thus we get (ii).

(iii) For (iii) we need the following proposition.

**Proposition 7.1.** For any open sets \(O_1, O_2\) satisfying \(\Omega_1' \subset \subset O_1, \Omega_2' \subset \subset O_2\), there exist \(c_\lambda(O_1), c_\lambda(O_2) > 0\) such that
\[c_\lambda(O_1), c_\lambda(O_2) \to 0 \text{ as } \lambda \to \infty,\]
\[\|Q_0,\lambda(u_1,0)\|_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')} \leq c_\lambda(O_1)\|u_1\|_{\lambda,\Omega_1'} \text{ for all } u_1 \in H^1(\Omega_1'),\]
\[\|Q_0,\lambda(0,u_2)\|_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')} \leq c_\lambda(O_2)\|u_2\|_{\lambda,\Omega_2'} \text{ for all } u_2 \in H^1(\Omega_2').\]

We will give a proof of Proposition 7.1 in Section 8.

For a proof of (iii) we take open sets \(O_1, O_2\) such that \(\Omega_1' \subset \subset O_1, \Omega_2' \subset \subset O_2\) and \(O_1 \cap O_2 = \emptyset\). By Proposition 7.1 there exist constants \(c_\lambda(O_1), c_\lambda(O_2) > 0\) satisfying (7.3) and (7.4).

We have
\[\langle Q_0,\lambda(u_1,0),Q_0,\lambda(0,u_2) \rangle_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')} = \langle Q_0,\lambda(u_1,0),Q_0,\lambda(0,u_2) \rangle_{\lambda,\Omega_1'} + \langle Q_0,\lambda(u_1,0),Q_0,\lambda(0,u_2) \rangle_{\lambda,\Omega_2'} + \langle Q_0,\lambda(u_1,0),Q_0,\lambda(0,u_2) \rangle_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')}.

By (7.3) and (i)
\[\|\langle Q_0,\lambda(u_1,0),Q_0,\lambda(0,u_2) \rangle_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')}\|
\leq \|Q_0,\lambda(u_1,0)\|_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')}\|Q_0,\lambda(0,u_2)\|_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')} + \|Q_0,\lambda(u_1,0)\|_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')}\|Q_0,\lambda(0,u_2)\|_{\lambda,R^N \setminus (\Omega_2' \cup \Omega_2')} + \|Q_0,\lambda(u_1,0)\|_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')}\|Q_0,\lambda(0,u_2)\|_{\lambda,R^N \setminus (\Omega_1' \cup \Omega_2')} \leq (C_3\lambda(O_1) + C_3\lambda(O_2) + c_\lambda(O_1)\lambda(O_2))\|u_1\|_{\lambda,\Omega_1'}\|u_2\|_{\lambda,\Omega_2'}.

Thus we get (iii).

(iv) Recalling Remark 2.8 (i), (iv) follows from (ii).
Next we show

**Proof of Lemma 2.10** (i) We can write

\[
I_\lambda(u_1, u_2) = \frac{1}{2} \|u_1\|_{\lambda, \Omega_1}^2 + \frac{1}{2} \|u_2\|_{\lambda, \Omega_2}^2 - \frac{1}{p+1} \|u_1\|_{L^{p+1}(\Omega_1)}^{p+1} - \frac{1}{p+1} \|u_2\|_{L^{p+1}(\Omega_2)}^{p+1} + \frac{1}{2} \|w_\lambda(u_1, u_2)\|_{\lambda, \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)}^2 - \int_{\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)} G(x, w_\lambda(u_1, u_2)) \, dx.
\]

Recalling (2.24)–(2.25), we have

\[
\begin{align*}
I_\lambda(u_1, u_2) - I_{1, \lambda}(u_1) - I_{2, \lambda}(u_2) &= \frac{1}{2} \|w_\lambda(u_1, u_2)\|_{\lambda, \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)}^2 - \frac{1}{2} \|Q_0,\lambda(u_1, 0)\|_{\lambda, \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)}^2 \quad \text{(i)} \\
&\quad - \frac{1}{2} \|Q_0,\lambda(0, u_2)\|_{\lambda, \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)}^2 - \frac{1}{2} \|w_\lambda(u_1, u_2)\|_{L^2(\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2))}^2 - \frac{1}{2} \|w_\lambda(u_1, u_2)\|_{L^2(\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2))}^2 \quad \text{(II)} \\
&= \frac{1}{2} |(I)| + |(II)|.
\end{align*}
\]

We also recall Remark 2.8 (i), that is,

\[w_\lambda(u) = Q_{V,\lambda}(u) \quad \text{for} \quad V(x) = g(x, w_\lambda(u))/w_\lambda(u),\]

and we have

\[
(I) = \|Q_{V,\lambda}(u) - Q_0,\lambda(u) + Q_0,\lambda(u)\|_{\lambda, \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)}^2 \\
\quad - \|Q_0,\lambda(u_1, 0)\|_{\lambda, \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)}^2 - \|Q_0,\lambda(0, u_2)\|_{\lambda, \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)}^2 \\
\quad + 2\langle Q_{V,\lambda}(u) - Q_0,\lambda(u), Q_0,\lambda(u)\rangle_{\lambda, \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)} \\
\quad + 2\langle Q_0,\lambda(u_1, 0), Q_0,\lambda(0, u_2)\rangle_{\lambda, \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)}.
\]

Thus by Lemma 2.9,

\[
(7.5) \quad |(I)| \leq r_\lambda^2 \|u\|_{\lambda, \Omega_1 \cup \Omega_2}^2 + 2r_\lambda C_3 \|u\|_{\lambda, \Omega_1 \cup \Omega_2}^2 + 2r_\lambda \|u_1\|_{\lambda, \Omega_1} \|u_2\|_{\lambda, \Omega_2}
\]

\[
\leq r_\lambda' \|u\|_{\lambda, \Omega_1 \cup \Omega_2}^2.
\]

On the other hand, using Lemma 2.1

\[
(7.6) \quad |(II)| \leq \frac{2}{3} \|Q_{V,\lambda}(u)\|_{L^2(\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2))}^2 \\
\leq \frac{2}{3} C_2(\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2), \lambda)^2 \|Q_{V,\lambda}(u)\|_{\lambda, \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)}^2 \\
\leq \frac{2}{3} C_2(\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2), \lambda)^2 C_3^2 \|u\|_{\lambda, \Omega_1 \cup \Omega_2}^2.
\]

(i) follows from (7.5)–(7.6).
(ii) For \( \varphi \in H^1_0(\Omega'_1) \), we have

\[
I'_{1,\lambda}(u_1) \varphi = \langle u_1, \varphi \rangle_{\lambda,\Omega'_1} - \int_{\Omega'_1} |u_1|^{p-1} u_1 \varphi \, dx + \langle Q_{0,\lambda}(u_1,0), Q_{0,\lambda}(\varphi,0) \rangle_{\lambda,\mathbb{R}^N \setminus (\Omega'_1 \cup \Omega'_2)},
\]

(7.7)

and

\[
\frac{\partial I_{\lambda}}{\partial u_1}(u_1, u_2) \varphi = \langle u_1, \varphi \rangle_{\lambda,\Omega'_1} - \int_{\Omega'_1} |u_1|^{p-1} u_1 \varphi \, dx + \langle w_\lambda(u_1, u_2), \frac{\partial w_\lambda}{\partial u_1}(u_1, u_2) \varphi \rangle_{\lambda,\mathbb{R}^N \setminus (\Omega'_1 \cup \Omega'_2)} - \int_{\mathbb{R}^N \setminus (\Omega'_1 \cup \Omega'_2)} g(x, w_\lambda(u_1, u_2)) \frac{\partial w_\lambda}{\partial u_1}(u_1, u_2) \varphi \, dx.
\]

(7.8)

By Remark 2.8, we have \( w_\lambda(u) = Q_{f(w_\lambda(u)),\lambda}(\varphi, 0) \) for \( \varphi \in H^1(\Omega'_1) \). Thus from (7.7–7.8),

\[
\frac{\partial I_{\lambda}}{\partial u_1}(u_1, u_2) \varphi - I_{1,\lambda}(u_1) \varphi = -\langle Q_{0,\lambda}(u_1,0), Q_{0,\lambda}(\varphi,0) \rangle_{\lambda,\mathbb{R}^N \setminus (\Omega'_1 \cup \Omega'_2)} + \langle Q_{f(w_\lambda)}(u_1, u_2), Q_{f(w_\lambda)}(\varphi,0) \rangle_{\lambda,\mathbb{R}^N \setminus (\Omega'_1 \cup \Omega'_2)} + \int_{\mathbb{R}^N \setminus (\Omega'_1 \cup \Omega'_2)} g(x, w_\lambda(u_1, u_2)) Q_{f(w_\lambda)}(\varphi,0) \, dx = (I) + (II) + (III).
\]

By Lemma 2.9

\[ |(I) + (II)| \leq r_\lambda \|u\|_{\lambda,\Omega'_1 \cup \Omega'_2} \|\varphi\|_{\lambda,\Omega'_1 \cup \Omega'_2}. \]

We also have from Lemma 2.11

\[
\left| \int_{\mathbb{R}^N \setminus (\Omega'_1 \cup \Omega'_2)} g(x, w_\lambda(u)) Q_{f(w_\lambda)}(\varphi,0) \, dx \right| \leq \frac{2}{3} w_\lambda(u) \|Q_{f(w_\lambda)}(\varphi,0)\|_{\mathbb{R}^N \setminus (\Omega'_1 \cup \Omega'_2)} \leq \frac{2}{3} C_2(\mathbb{R}^N \setminus (\Omega'_1 \cup \Omega'_2), \lambda)^2 \|u\|_{\lambda,\Omega'_1 \cup \Omega'_2} \|\varphi\|_{\lambda,\Omega'_1 \cup \Omega'_2}.
\]

Thus we have for some \( r'_\lambda > 0 \) satisfying \( r'_\lambda \to 0 \) as \( \lambda \to \infty \),

\[
\left| \frac{\partial I_{\lambda}}{\partial u_1}(u_1, u_2) \varphi - I_{1,\lambda}(u_1) \varphi \right| \leq r'_\lambda \|u\|_{\lambda,\Omega'_1 \cup \Omega'_2} \|\varphi\|_{\lambda,\Omega'_1} \quad \text{for all } \varphi \in H^1(\Omega'_1).
\]

Therefore (ii) holds. (iii) can be shown in a similar way.

Next we give a proof of Proposition 2.13. First we observe some fundamental properties of \( J_{i,\lambda}(v) \): We can find an \( M \)-independent constant \( R_0 \) and constants \( R_i(M), \delta_M > 0 \), which depend on \( M > 0 \) but are independent of \( \lambda \) such that for \( i = 1, 2 \) and \( v_i \in \Sigma_{1,\lambda} \)

(i) \( J_{i,\lambda}(v_i) \leq M \) implies \( \|v_i\|_{L^{p+1}(\Omega'_1)} \geq \delta_M \).

(ii) Suppose that \( J_{i,\lambda}(v_i) \leq M \). Then for \( s \geq 0 \)

\[
\frac{m_i}{2} \leq I_{i,\lambda}(sv_i) \quad \text{implies} \quad R_0 \leq s \leq R_i(M).
\]
Here $m_\iota > 0$ is defined in (2.24). This fact is easily seen from the definition of $J_{i,\iota}(v)$ and (2.26).

For a proof of Proposition 2.15, we first observe that $J_\lambda(v_1, v_2)$ has properties similar to (i)–(ii). In what follows we fix $\lambda_0 \geq 1$ such that

$$r_\lambda \leq \frac{1}{2} \quad \text{for } \lambda \geq \lambda_0.$$ 

We can observe the following lemma easily from Corollary 2.11.

**Lemma 7.2.** There exist $M$-independent constants $R_0, \delta_0 > 0$ and constants $R_1(M), \delta_M > 0$, which depend on $M > 0$ but are independent of $\lambda$ such that for $(v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda}$

(i) $J_\lambda(v_1, v_2) \leq M$ implies $\|v_1\|_{L^{p+1}(\Omega)} \|v_2\|_{L^{p+1}(\Omega)} \in [\delta_M, \delta_0]$.

(ii) Suppose $\lambda \geq \lambda_0$ and $J_\lambda(v_1, v_2) \leq M$. Then for $(s, t) \in [0, \infty)^2$

$$\frac{1}{2}(m_1 + m_2) \leq I_\lambda(sv_1, tv_2) \quad \text{implies} \quad R_0^2 \leq s^2 + t^2 \leq R_1(M)^2.$$

(iii) There exists $\lambda_{1,M} \geq \lambda_0$ such that for $\lambda \geq \lambda_{1,M}$ and $(v_1, v_2) \in [J_\lambda \leq M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}}$

(a) $s^2 + t^2 \leq R_0^2$ implies $I'_{\lambda}(sv_1, tv_2)(sv_1, tv_2) > 0.$

(b) $s^2 + t^2 \geq R_1(M)^2$ implies $I'_{\lambda}(sv_1, tv_2)(sv_1, tv_2) < 0.$

(c) $s = R_0$ and $t \in [0, R_1(M)] \implies I''_{\lambda}(sv_1, tv_2)(v_1, 0) > 0.$

(d) $s \in [0, R_1(M)]$ and $t = R_0 \implies I''_{\lambda}(sv_1, tv_2)(0, v_2) > 0.$

(e) $J_\lambda(v_1, v_2) > \max\{\max\{I_{\lambda}(sv_1, tv_2), \max_{0 \leq \lambda \leq \lambda_0, t \geq 0} I_{\lambda}(sv_1, tv_2)\}\}.$

We remark that (i) of Proposition 2.15 follows from Lemma 7.2. By the above Lemma 7.2 $(s, t) \mapsto I_\lambda(sv_1, tv_2)$ takes its maximum in $\{(s, t) \in \{s, t \geq R_1(M)^2, s, t \geq R_0\}\}$. More precisely we have

**Lemma 7.3.** For any $M > 0$ there exists $\delta_{\lambda,M} > 0$ such that

(i) $\delta_{\lambda,M} \to 0$ as $\lambda \to \infty$.

(ii) For any $(v_1, v_2) \in [J_\lambda \leq M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}},$

$$(7.9)\ I_\lambda(sv_1, tv_2) = J_\lambda(v_1, v_2) \implies |s - s_{1,\lambda}(v_1)| \leq \delta_{\lambda,M}, |t - t_{2,\lambda}(v_2)| \leq \delta_{\lambda,M}.$$ 

Here $s_{1,\lambda}(v_1)$ ($t_{2,\lambda}(v_2)$ respectively) is a unique critical point of $s \mapsto I_{\lambda}(sv_1)$ ($t \mapsto I_{\lambda}(tv_2)$ respectively), which is given in (2.27).

We can show Lemma 7.3 without difficulty, and we omit the proof. We have the following uniqueness and regularity result for the maximizer.

**Lemma 7.4.** For any $M > 0$ there exists $\lambda_M \geq 1$ such that

(i) For any $\lambda \geq \lambda_M$ and $(v_1, v_2) \in [J_\lambda \leq M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}}$

$$\quad (s, t) \mapsto I_\lambda(sv_1, tv_2)$$

has a unique maximum and is nondegenerate.

(ii) We denote the unique maximizer $(s_\lambda(v_1, v_2), t_\lambda(v_1, v_2))$. Then for $\lambda \geq \lambda_M$

$$\quad [J_\lambda \leq M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} \to \mathbb{R}^2; (v_1, v_2) \mapsto (s_\lambda(v_1, v_2), t_\lambda(v_1, v_2))$$

is of class $C^1$.

(iii) $[J_\lambda < M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} \to \mathbb{R}; (v_1, v_2) \mapsto J_\lambda(v_1, v_2) = I_\lambda(s_\lambda(v_1, v_2)v_1, t_\lambda(v_1, v_2)v_2)$

is of class $C^1$ for $\lambda \geq \lambda_M$. 
Proof. For $(v_1, v_2) \in [J, t \leq M]_{\Sigma_1, \Sigma_2}$, we set $f_\lambda(s, t) = I_\lambda(sv_1, tv_2)$. By Lemma 2.23, we have
\[
\partial_s f_\lambda(s, t) = \Phi_\lambda'(sv_1, tv_2)(v_1, 0) = \Phi_\lambda'(w_\lambda(sv_1, tv_2))Q_{0, \lambda}(v_1, 0).
\]
Thus we have
\[
\partial_s^2 f_\lambda(s, t) = \Phi_\lambda''(w_\lambda(sv_1, tv_2))(v_1, 0), Q_{0, \lambda}(v_1, 0)
\]
and
\[
\partial_t f_\lambda(s, t) = \Phi_\lambda'(w_\lambda(sv_1, tv_2))(Q_{f_\lambda(sv_1, tv_2)}(v_1, 0), Q_{0, \lambda}(v_1, 0)).
\]
Analogously,
\[
\partial_t^2 f_\lambda(s, t) = \Phi_\lambda''(w_\lambda(sv_1, tv_2))(Q_{f_\lambda(sv_1, tv_2)}(0, v_2), Q_{0, \lambda}(v_1, 0)).
\]
Since
\[
\Phi_\lambda''(u)(h_1, h_2) = \langle h_1, h_2 \rangle_{\lambda, R}\cdot p \int_{[0, 1]} |u|^{p-1}h_1h_2 dx - \int_{R^N \setminus (0, 1)} f'(u)h_1h_2 dx,
\]
we have by Lemma 2.9
\[
\partial_s^2 f_\lambda(s, t) = \langle Q_{f_\lambda(sv_1, tv_2)}, Q_{0, \lambda}(v_1, 0) \rangle_{\lambda, R^N} - ps^{p-1}||v_1||^{p+1}_{L^{p+1}(\lambda)} - \int_{R^N \setminus (0, 1)} f'(w_\lambda(sv_1, tv_2))Q_{f_\lambda(sv_1, tv_2)}(v_1, 0)Q_{0, \lambda}(v_1, 0) dx
\]
\[
\quad = ||v_1||^2_{\lambda, \Omega_1} - ps^{p-1}||v_1||^{p+1}_{L^{p+1}(\lambda)} + r_\lambda(s, t, v_1, v_2).
\]
Here $r_\lambda(s, t, v_1, v_2)$ satisfies $r_\lambda(s, t, v_1, v_2) \to 0$ as $\lambda \to \infty$ uniformly in $(v_1, v_2) \in [J, t \leq M]_{\Sigma_1, \Sigma_2}$ and $(s, t) \in \{(s, t); s^2 + t^2 \leq R_1(1)^2, s, t \geq R_0\}$.

Now suppose that $(v_1, v_2) \in [J, t \leq M]_{\Sigma_1, \Sigma_2}$ and $(s, t) \in \{(s, t); s^2 + t^2 \leq R_1(1)^2, s, t \geq R_0\}$ is a critical point of $f_\lambda(s, t)$. By Lemma 7.2, for any $\epsilon > 0$ there exists $\lambda_{e, M} \geq 1$ such that
\[
|\partial_s^2 f_\lambda(s, t) - (1-p)||v_1||^{p+1}_{L^{p+1}(\lambda)}| < \epsilon \quad \text{for } \lambda \geq \lambda_{e, M}.
\]
In a similar way, we have
\[
|\partial_t^2 f_\lambda(s, t) - (1-p)||v_2||^{p+1}_{L^{p+1}(\lambda)}| < \epsilon
\]
for $\lambda \geq \lambda_{e, M}$ and for any critical point $(s, t)$ in $\{(s, t); s^2 + t^2 \leq R_1(1)^2, s, t \geq R_0\}$.

Recalling (i) of Lemma 7.2 and choosing $\epsilon > 0$ small, we have
\[
\det \begin{bmatrix} \partial_s^2 f_\lambda(s, t) & \partial_t \partial_s f_\lambda(s, t) \\ \partial_t \partial_s f_\lambda(s, t) & \partial_t^2 f_\lambda(s, t) \end{bmatrix} > 0
\]
for all critical points in $\{(s, t); R_0^2 \leq s^2 + t^2 \leq R_1(1)^2, s, t \geq R_0\}$. Thus uniqueness of critical points follows from (iii) of Lemma 7.2 and we get (i). Since unique maximizer is nondegenerate, the conclusion (ii) follows from the implicit function theorem. (iii) follows from (ii). \qed

Proof of (ii)–(v) of Proposition 2.15: (ii) and (iv) are nothing but (i), (iii) of Lemma 7.4 (iii) follows from Lemma 7.2.
Next we prove (v). For (2.32), we have

\[
|J_\lambda(v_1, v_2) - J_{1,\lambda}(v_1) - J_{2,\lambda}(v_2)|
\]

\[
= |I_\lambda(s_\lambda(v_1, v_2)v_1, t_\lambda(v_1, v_2)v_2) - I_{1,\lambda}(s_\lambda(v_1, v_2)v_1) - I_{2,\lambda}(t_\lambda(v_1, v_2)v_2)|
\]

\[
\leq |I_\lambda(s_\lambda(v_1, v_2)v_1, t_\lambda(v_1, v_2)v_2) - I_{1,\lambda}(s_\lambda(v_1, v_2)v_1) - I_{2,\lambda}(s_\lambda(v_1, v_2)v_2)|
\]

\[
+ |I_{1,\lambda}(s_\lambda(v_1, v_2)v_1) - I_{1,\lambda}(s_\lambda(v_1, v_2)v_1)|
\]

\[
+ |I_{2,\lambda}(s_\lambda(v_1, v_2)v_2) - I_{2,\lambda}(t_\lambda(v_1, v_2)v_2)|.
\]

Thus by Lemma 2.10 and Lemma 7.3, we get (2.32). We can show (2.33) in a similar way.

Finally we give a proof of Proposition 2.16

Proof of Proposition 2.16 Suppose \((v_{1n}, v_{2n}) \in [J_\lambda \leq M]_{\Sigma_1,\lambda \times \Sigma_2,\lambda}\) satisfies

\[J_\lambda(v_{1n}, v_{2n}) \to c \in (-\infty, M), \quad \|I_\lambda'(v_{1n}, v_{2n})\|_{(T(v_{1n}, v_{2n})'(\Sigma_1,\lambda \times \Sigma_2,\lambda))}\to 0,
\]

where

\[\|I_\lambda'(v_{1n}, v_{2n})\|_{(T(v_{1n}, v_{2n})'(\Sigma_1,\lambda \times \Sigma_2,\lambda))} = \text{sup}\{|I_\lambda'(v_{1n}, v_{2n})(h_{1n}, h_{2n})| : (h_{1n}, h_{2n}) \in H^1(\Omega_1') \oplus H^1(\Omega_2'), \langle v_{1n}, h_{1n}\rangle_{\lambda,\partial\Omega_1} = \langle v_{2n}, h_{2n}\rangle_{\lambda,\partial\Omega_2} = 0\}.
\]

We note

\[I_\lambda'(v_{1n}, v_{2n}) = I_\lambda'(s_\lambda(v_{1n}, v_{2n}), t_\lambda(v_{1n}, v_{2n})(s_\lambda(v_{1n}, v_{2n})h_{1n}, v_{1n}, v_{2n})v_{2n}) h_{1n}, t_\lambda(v_{1n}, v_{2n})h_{2n}
\]

and

\[I_\lambda'(s_\lambda(v_{1n}, v_{2n}), t_\lambda(v_{1n}, v_{2n})(\alpha v_1, \beta v_2) = 0 \quad \text{for all} \quad (\alpha, \beta) \in \mathbb{R}^2.
\]

Thus

\[\|I_\lambda'(s_\lambda(v_{1n}, v_{2n}), t_\lambda(v_{1n}, v_{2n})v_{2n})\|_{\lambda,\partial\Omega_1 \cup \partial\Omega_2'}
\]

\[\leq \sqrt{\frac{1}{s_\lambda(v_{1n}, v_{2n})^2 + t_\lambda(v_{1n}, v_{2n})^2} \|J_\lambda'(v_{1n}, v_{2n})\|_{(T(v_{1n}, v_{2n})'(\Sigma_1,\lambda \times \Sigma_2,\lambda))}}.
\]

Recalling (2.31), we have

\[\|I_\lambda'(s_\lambda(v_{1n}, v_{2n}), t_\lambda(v_{1n}, v_{2n})v_{2n})\|_{\lambda,\partial\Omega_1 \cup \partial\Omega_2'} \to 0.
\]

Thus \((s_\lambda(v_{1n}, v_{2n}), t_\lambda(v_{1n}, v_{2n})v_{2n})\) is a \((PS)\)-sequence for \(I_\lambda(u_1, u_2)\). By Proposition 2.14 it has a strongly convergent subsequence in \(H^1(\Omega_1') \oplus H^1(\Omega_2')\). Recalling (2.31) again, we can see that \((v_{1n}, v_{2n})\) has a strongly convergent subsequence in \(\Sigma_1,\lambda \times \Sigma_2,\lambda\), and the \((PS)\)-condition holds for \(J_\lambda(v_{1n}, v_{2n})\).

8. Proofs of Propositions 3.4 and 7.1

The aim of this section is to give proofs to Proposition 3.4 and Proposition 7.1

Suppose that a compact set \(K\) and open sets \(O_1, O_2, O_3\) satisfy the following condition:

\[(8.1) \quad K \subset \subset O_i \subset \subset O_2 \subset \subset O_3\text{ and } \partial K, \partial O_i (i = 1, 2, 3)\text{ are compact and smooth.}
\]

\[(8.2) \quad \inf_{x \in O_3 \setminus K} a(x) > 0.
\]
For a given \( \varphi(x) \in H^1(O_1 \setminus K) \) and \( \lambda \geq 1 \) we consider the following linear elliptic problem:

\[
\begin{aligned}
(8.3) & \quad -\Delta u + (\lambda^2 a(x) + 1)u = 0 \quad \text{in } O_3 \setminus \overline{O_1}, \\
(8.4) & \quad u(x) = \varphi(x) \quad \text{on } \partial O_1, \quad u(x) = 0 \quad \text{on } \partial O_3, \\
(8.5) & \quad u(x) \in H^1(O_3 \setminus \overline{O_1}).
\end{aligned}
\]

Obviously (8.3)–(8.5) has a unique solution — we denote it by \((S_\lambda \varphi)(x)\) — and \((S_\lambda \varphi)(x)\) can be characterized as

\[
\|S_\lambda \varphi\|_{\lambda,O_3 \setminus \overline{O_1}} \leq \|u\|_{\lambda,O_3 \setminus \overline{O_1}}
\]

for all \( u \in H^1(O_3 \setminus \overline{O_1}) \) satisfying (8.4).

The following proposition gives estimates of the solution \( S_\lambda \varphi \):

**Proposition 8.1.** Assume that \( K, O_1, O_2, O_3 \) satisfy (8.1)–(8.2). Then there exist constants \( a, C_4, C_5 > 0 \) such that

\[
\begin{aligned}
(8.7) & \quad \|S_\lambda \varphi\|_{\lambda,O_3 \setminus \overline{O_1}} \leq C_4 \|\varphi\|_{\lambda,O_1 \setminus K}, \\
(8.8) & \quad \|S_\lambda \varphi\|_{\lambda,O_3 \setminus \overline{O_2}} \leq C_5 e^{-a\lambda} \|\varphi\|_{\lambda,O_1 \setminus K},
\end{aligned}
\]

for all \( \lambda \geq 1 \) and \( \varphi \in H^1(O_1 \setminus K) \).

We postpone the proof of Proposition 8.1 until the end of this section. As an immediate corollary to Proposition 8.1 we have

**Corollary 8.2.** Assume (8.1)–(8.2). Then there exists a family of linear bounded operators \( P_\lambda : H^1(O_3 \setminus K) \to H^1(O_3 \setminus K) \) such that

\[
\begin{aligned}
(i) & \quad \text{For all } \lambda \geq 1, \\
(8.9) & \quad (P_\lambda u)(x) = u(x) \quad \text{for all } x \in O_1 \setminus K, \\
(8.10) & \quad (P_\lambda u)(x) = 0 \quad \text{for all } x \in O_3 \setminus \overline{O_2},
\end{aligned}
\]

(ii) There exists a family of positive constants \( \epsilon_\lambda > 0 \) independent of \( u \) such that

\[
(8.11) \quad \epsilon_\lambda \to 0 \quad \text{as } \lambda \to \infty,
\]

\[
(8.12) \quad \|P_\lambda u\|_{\lambda,O_3 \setminus K} \leq (1 + \epsilon_\lambda) \|u\|_{\lambda,O_1 \setminus K} \quad \text{for all } \lambda \geq 1 \text{ and } u \in H^1(O_3 \setminus K).
\]

**Proof.** Choose an open set \( D \) such that \( O_1 \subset D \subset O_2 \subset O_3 \). Applying Proposition 8.1 to \( K, O_1, D, O_3 \), we find a solution operator \( S_\lambda : H^1(O_1 \setminus K) \to H^1(O_3 \setminus K) \) such that

\[
\begin{aligned}
(8.13) & \quad \|S_\lambda u\|_{\lambda,O_3 \setminus K} \leq \|u\|_{\lambda,O_3 \setminus K}, \\
(8.14) & \quad \|S_\lambda u\|_{\lambda,O_3 \setminus \overline{D}} \leq C_5 e^{-a\lambda} \|u\|_{\lambda,O_3 \setminus K},
\end{aligned}
\]

for some constant \( a > 0 \). Here we use the same notation \( S_\lambda \) for an operator \( u \mapsto S_\lambda (u|_{O_1 \setminus K}) : H^1(O_3 \setminus K) \to H^1(O_3 \setminus K) \).

We choose a function \( \zeta(x) \in C^\infty(\mathbb{R}^N, [0,1]) \) such that

\[
\zeta(x) = \begin{cases} 
1 & \text{for all } x \in D, \\
0 & \text{for all } x \in \mathbb{R}^N \setminus \overline{O_2}.
\end{cases}
\]
Setting \((P_{\lambda}u)(x) = \zeta(x)(S_{\lambda}u)(x)\), we have \([8.9] - [8.12]\). In fact, \([8.9] - [8.10]\) are clear. For \([8.11] - [8.12]\), first we observe by \([8.14]\)

\[
\|S_{\lambda}u\|_{\lambda,O_3}^2 \leq 2 \int_{O_3} |\nabla(S_{\lambda}u)|^2 + |S_{\lambda}u|^2 \, dx
\]

Thus by \([8.13]\) and \([8.15]\)

\[
\|P_{\lambda}u\|_{\lambda,O_3}^2 = \|S_{\lambda}u\|_{\lambda,O_3}^2 + \|\zeta S_{\lambda}u\|_{\lambda,O_3}^2
\]

Thus we get \([8.11] - [8.12]\). 

Now we are in a position to prove Proposition 3.4.

**Proof of Proposition 3.4** We choose an open set \(D\) such that

\[
\Omega_3 \subset D \subset \tilde{\Omega} \subset \tilde{\Omega} \subset \Omega_2.
\]

We apply Corollary 8.2 to \((K, O_1, O_2, O_3) = (\overline{D}, \tilde{\Omega}, \tilde{\Omega}, \Omega_2')\), and we find a family of operators \(P_{\lambda} : H^1(\Omega_2' \setminus \overline{D}) \to H^1(\Omega_2' \setminus \overline{D})\) satisfying \([8.9] - [8.12]\). Defining \(F_{\lambda} : \{v \in \Sigma_2, \|v\|_{L^{p+1} (\Omega_2')} \geq \tilde{m}\} \to H^1_0(\tilde{\Omega})\) by

\[
(F_{\lambda}u)(x) = \begin{cases} u(x) & \text{for } x \in \overline{D}, \\ (P_{\lambda}u)(x) & \text{for } x \in \tilde{\Omega} \setminus \overline{D}, \end{cases}
\]

1°, 2° of Proposition 3.4 follow from \([8.9] - [8.12]\).

For 3°, first we remark for \(v \in \Sigma_2, \|v\|_{\lambda,O_2'} \leq \|v\|_{\lambda,O_2'} = 1\). Thus

\[
\|F_{\lambda}v\|_{L^{p+1} (\Omega_2')} \geq \|F_{\lambda}v\|_{L^{p+1} (\Omega_2')} - \|v\|_{L^{p+1} (\Omega_2')} \leq \|v\|_{L^{p+1} (\Omega_2')} - C_{p+1} (\Omega_2' \setminus \tilde{\Omega}, \lambda) \|v\|_{\lambda,O_2'}^2
\]

Therefore, 3° also holds for large \(\lambda\). 

**Proof of Proposition 7.1** We choose an open set \(D\) such that \(\Omega_4 \subset D \subset \Omega_4' \subset O_1\). Applying Proposition 8.1 with \((K, O_1, O_2, O_3) = (\overline{D}, \tilde{\Omega}_4, \tilde{\Omega}_4, \Omega_2')\), we have for some constant \(a > 0\)

\[
\|Q_{0,\lambda}(u_1, 0)\|_{\lambda, R^N \setminus (O_1 \cup O_2')} \leq C_5 e^{-a\lambda}\|u_1\|_{\lambda, O_4'} \leq C_5 e^{-a\lambda}\|u_1\|_{\lambda,O_4}
\]

for all \(\lambda \geq 1\) and \(u_1 \in H^1(\Omega_1)\). Thus we have \([7.3]\). We can show \([7.4]\) analogously.
Proof of Proposition 8.1 We choose open sets $D_1, D_2, E_1, E_2, E_3$ such that

$$K \subset \subset O_1 \subset \subset D_1 \subset \subset D_2 \subset \subset O_2 \subset \subset E_2 \subset \subset E_1 \subset \subset O_3.$$ 

In what follows, for $\varphi \in H^1(O_1 \setminus K)$ we denote $u(x) = (S_\lambda \varphi)(x)$, that is, $u(x)$ solves (8.3)–(8.5). First we remark that using the characterization (8.6) we can prove (8.7) as in Lemma 2.5.

The proof of (8.8) consists of 4 steps. As the first step we show

Step 1: There exists a constant $C_6 > 0$ independent of $\varphi \in H^1(O_1 \setminus K)$ such that

$$\max_{x \in E_1 \setminus D_1} |u(x)| \leq C_6 \|\varphi\|_{\lambda, O_1 \setminus K}. \tag{8.16}$$

We choose $r > 0$ such that $\overline{B(x_0, r)} \subset O_3 \setminus \overline{O_1}$ for all $x_0 \in \overline{E_1 \setminus D_1}$, where $B(x_0, r) = \{y \in \mathbb{R}^N : |y - x_0| < r\}$. By the subsolution estimate (22, Theorem C.1.2), we have for some constant $C_7 > 0$ independent of $\lambda, x_0, \varphi$

$$|u(x_0)| \leq C_7 \int_{B(x_0, r)} |u(x)| dx \leq C_7 \sqrt{\operatorname{meas}(B(0, r))} \|u\|_{L^2(B(x_0, r))} \leq C_7 \sqrt{\operatorname{meas}(B(x_0, r))} \|\varphi\|_{\lambda, O_1 \setminus K}. \tag{8.16}$$

Thus (8.16) holds.

Step 2: There exists $\mu_1 > 0$ independent of $\lambda$ and $\varphi$ such that

$$\max_{x \in E_2 \setminus D_2} |u(x)| \leq e^{-\mu_1 \lambda} \|\varphi\|_{\lambda, O_1 \setminus K}. \tag{8.17}$$

We choose $b \in (0, \inf_{x \in O_3 \setminus K} a(x))$. From (8.3) it holds that

$$(-\Delta + 2\lambda^2 b)u^2 = -2(\lambda^2 a(x) + 1 - \lambda^2 b)u^2 - 2|\nabla u|^2.$$ 

In particular, we have

$$(-\Delta + 2\lambda^2 b)u^2 \leq 0. \tag{8.18}$$

We choose $r > 0$ such that $\overline{B(x_0, r)} \subset \overline{E_1 \setminus D_1}$ for all $x_0 \in \overline{E_2 \setminus D_2}$. Now we consider a solution $w(x)$ of

$$-\Delta w + 2\lambda^2 bw = 0 \quad \text{in} \ B(x_0, r), \quad w = (C_0 \|\varphi\|_{\lambda, O_1 \setminus K})^2 \quad \text{on} \ \partial B(x_0, r). \tag{8.19}$$

Then we have for some $b' > 0$

$$|w(x_0)| \leq C_6^2 e^{-b' \lambda} \|\varphi\|_{\lambda, O_1 \setminus K}^2 \quad \text{for all} \ \lambda \geq 1.$$ 

Comparing (8.18) and (8.19), the conclusion (8.17) follows from Step 1.

Step 3: There exists $\mu_2 > 0$ independent of $\lambda$ and $\varphi$ such that

$$\|u\|_{\lambda, E_3 \setminus \overline{D_2}} \leq e^{-\mu_2 \lambda} \|\varphi\|_{\lambda, O_1 \setminus K}. \tag{8.20}$$

Choose $\psi(x) \in C^\infty_0(\mathbb{R}^N, [0, 1])$ such that

$$\psi(x) = \begin{cases} 1 & \text{for} \ x \in E_3 \setminus \overline{D_2}, \\ 0 & \text{for} \ x \in \mathbb{R}^N \setminus (\overline{E_2 \setminus D_2}). \end{cases}$$

Multiplying $\psi u$ to (8.8) and integrating, we have

$$\int_{O_1 \setminus K} \nabla u \nabla (\psi u) + (\lambda^2 a(x) + 1)\psi u^2 dx = 0.$$
Thus, by (8.17) and (8.7),
\[
\|u\|_{L^2(O_3 \setminus O_2)}^2 \leq \int_{E_3 \setminus D_2} |\nabla \psi||u||\nabla u| \, dx
\]
\[
\leq C_8 \|\nabla \psi\|_{L^1(\mathbb{R}^N)} \left( \max_{x \in E_3 \setminus D_2} |u(x)| \right) \|u\|_{L^2(O_3 \setminus O_2)}
\]
\[
\leq C_8 C_4 e^{-\mu_2 \lambda} \|\nabla \psi\|_{L^1(\mathbb{R}^N)} \|\varphi\|_{L^1(O_1 \setminus K)}^2.
\]
Thus we get (8.20). □

**Step 4: Conclusion.**
From the characterization (8.6), it follows that
\[
\|u\|_{L^2(O_3 \setminus O_2)} \leq \|w\|_{L^2(O_3 \setminus O_2)}
\]
for all \( w \in H^1(O_3 \setminus O_2) \) satisfying \( w = u \) on \( \partial O_2 \) and \( w = 0 \) on \( \partial O_3 \). In particular, setting \( w = \zeta u \), where \( \zeta(x) \in C^\infty(\mathbb{R}^N, [0, 1]) \) is a function such that
\[
\zeta(x) = \begin{cases} 
0 & \text{for } x \in O_3 \setminus E_3, \\
1 & \text{in a neighborhood of } \partial O_2,
\end{cases}
\]
we can deduce as in the proof of Corollary 8.2 that
\[
\|u\|_{L^2(O_3 \setminus O_2)} \leq \|\zeta u\|_{L^2(O_3 \setminus O_2)} \leq C_9 \|\zeta\|_{L^2(O_3 \setminus O_2)} \leq C_9 e^{-\mu_2 \lambda} \|\varphi\|_{L^1(O_1 \setminus K)}.
\]
Thus the proof of Proposition 8.1 is completed. □

**9. Construction of a deformation flow**

The aim of this section is to construct a deformation flow which enables us to find critical points whose limits are positive or negative in \( \Omega_i \).

We define
\[
\varphi_\pm(v_1) = \|v_{1\pm}\|_{L^{p+1}(\Omega_i)} : \Sigma_{1,\lambda} \rightarrow \mathbb{R},
\]
\[
\psi_\pm(v_2) = \|v_{2\pm}\|_{L^{p+1}(\Omega_i)} : \Sigma_{2,\lambda} \rightarrow \mathbb{R},
\]
and for \( a < b \), we use the following notation:
\[
[a \leq \varphi_\pm \leq b]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} = \{(v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda} ; a \leq \varphi_\pm(v_1) \leq b\},
\]
\[
[a \leq \psi_\pm \leq b]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} = \{(v_1, v_2) \in \Sigma_{1,\lambda} \times \Sigma_{2,\lambda} ; a \leq \psi_\pm(v_2) \leq b\}.
\]

First we have

**Proposition 9.1.** For \( M > 0 \) and \( \sigma_1, \sigma_2 \in \{+,-\} \), there exist \( \lambda_M > 1 \), \( \nu_M \in (0, \delta_0(M)] \) such that for any \( \lambda \geq \lambda_M \), if \( b < M \) is not a critical value of \( J_\lambda \), then there exist \( \epsilon, \mu > 0 \) and a vector field \( X(v) \) on \( [J_\lambda \leq M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} \) with the following properties:

(i) \( X(v) = (X_1(v), X_2(v)) \in T_v(\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}) \) for all \( v \in [J_\lambda \leq M] \) and \( v \mapsto X(v) \) is locally Lipschitz continuous.

(ii) \( J'_\lambda(v)X(v) > -\mu \) for all \( v \in [b - \epsilon \leq J_\lambda \leq b + \epsilon]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} \).

(iii) \( X(v) = 0 \) for all \( v \in [J_\lambda \leq b - 2\epsilon]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} \).

(iv) \( J'_\lambda(v)X(v) \leq 0 \) for all \( v \in [J_\lambda \leq M]_{\Sigma_{1,\lambda} \times \Sigma_{2,\lambda}} \).
have way. For standard way, we can find a pseudo-gradient vector field \( \tilde{J} \)

We only prove (i) with

Here we use the inequality

We have the following.

To prove Proposition 9.1 we define

\[ Y_1^\pm(v_1) = -v_1 \pm \|v_1\|^2_{\lambda, \Omega_1}v_1 \in T_{v_1}(\Sigma_1, \lambda), \text{ for } v_1 \in \Sigma_1, \lambda, \]

\[ Y_2^\pm(v_2) = -v_2 \pm \|v_2\|^2_{\lambda, \Omega_2}v_2 \in T_{v_2}(\Sigma_2, \lambda), \text{ for } v_2 \in \Sigma_2, \lambda. \]

We have the following.

**Lemma 9.2.** For \( M > 0 \), there exist \( \lambda_M > 1, \nu_M \in (0, \delta_0(M)) \) and \( \rho_M > 0 \) such that for any \( \lambda > \lambda_M, \sigma_1, \sigma_2 \in \{+, -\}, \)

(i) for \( v \in [J_\lambda \leq M]_{\Sigma_1, \lambda} \cap [\sqrt{2}\nu_M \leq \varphi_{\sigma_1} \leq 2\nu_M]_{\Sigma_1, \lambda} \),

\[ J'_\lambda(v)(Y_{\sigma_1}^1, v) \leq -\rho_M, \quad \varphi'_{\sigma_1}(v_1)Y_{\sigma_1}^1 \leq -\rho_M; \]

(ii) for \( v \in [J_\lambda \leq M]_{\Sigma_1, \lambda} \cap [\sqrt{2}\nu_M \leq \varphi_{\sigma_2} \leq 2\nu_M]_{\Sigma_1, \lambda} \),

\[ J'_\lambda(v)(0, Y_{\sigma_2}^2) \leq -\rho_M, \quad \varphi'_{\sigma_2}(v_2)Y_{\sigma_2}^2 \leq -\rho_M. \]

**Proof.** We only prove (i) with \( \sigma_1 = + \). We can deal with other cases in a similar way. For \( v \in [J_\lambda \leq M]_{\Sigma_1, \lambda} \), by Lemma 2.10 and Proposition 2.15 we have

\[ \varphi'_{\sigma_1}(v_1)Y_{\sigma_1}^1 = (p + 1) \int_{\Omega_1} (v_1 + pY_{\sigma_1}^1) \, dx \]

\[ = -(p + 1)(1 - \|v_1\|^2_{\lambda, \Omega_1})\|v_1\|^p \|\nabla v_1\|_{L^{p+1}(\Omega_1)}^p \]

\[ \leq -(p + 1)(1 - C^{-p}L^{p+1}(\Omega_1))\|v_1\|^p \|\nabla v_1\|_{L^{p+1}(\Omega_1)}^p. \]

Here we use the inequality \( \|v_1\|^p \|\nabla v_1\|_{L^{p+1}(\Omega_1)}^p \leq C \|v_1\|^p \|\lambda, \Omega_1 \| \).

Writing \( s = s_\lambda(v_1, v_2), t = t_\lambda(v_1, v_2), \) by Lemma 2.10 and Proposition 2.15 we have

\[ J'_\lambda(v_1, v_2)(Y_{\sigma_1}^1) \]

\[ = I'_\lambda(sv_1, tv_2)(sv_1)^p \|v_1\|^p \|\nabla v_1\|_{L^{p+1}(\Omega_1)}^p \]

\[ \leq -s^2\|v_1\|^2_{\lambda, \Omega_1} + s^p\|v_1\|^p \|\nabla v_1\|_{L^{p+1}(\Omega_1)}^p + r_\lambda(s + t) \]

\[ \leq -s^2C^{-p}L^{p+1}(\Omega_1) + s^p\|v_1\|^p \|\nabla v_1\|_{L^{p+1}(\Omega_1)}^p + r_\lambda(s + t) \]

\[ \leq -R\lambda s^p\|v_1\|^p \|\nabla v_1\|_{L^{p+1}(\Omega_1)}^p + R_1(M)s^p\|v_1\|^p \|\nabla v_1\|_{L^{p+1}(\Omega_1)}^p + 2\lambda R_1(M)^2. \]

From (9.2) and (9.3), we can find \( \nu_M > 0 \) and \( \rho_M > 0 \) such that if \( \varphi_{\sigma_1}(v_1) = \|v_1\|^p \|\nabla v_1\|_{L^{p+1}(\Omega_1)}^p \in [2\nu_M, 2\nu_M] \), (9.1) holds.

**Proof of Proposition 9.1.** Assume that \( b < M \) is not a critical value. In a standard way, we can find a pseudo-gradient vector field \( \tilde{X}(v) \) satisfying (i)–(iv) of Proposition 9.1.

We choose smooth functions \( \zeta : \mathbb{R} \to [0, 1] \) such that \( \zeta(s) = 0 \) for \( s \leq \nu_M \), \( \zeta(s) = 1 \) for \( s \geq \nu_M \) and \( \xi : \mathbb{R} \to [0, 1] \) such that \( \xi(s) = 0 \) for \( s \leq b - 2\epsilon, \xi(s) = 1 \) for
s ≥ b − ε and set
\[ X(v) = (1 - \zeta(\varphi_{\sigma_1}(v_1))) (1 - \zeta(\psi_{\sigma_2}(v_2))) \tilde{X}(v) + \xi(J_\lambda(v_1, v_2))(\varphi_{\sigma_1}(v_1))(Y^{\sigma_1}_{v_1}(v_1), 0) + \zeta(\psi_{\sigma_2}(v_2))(0, Y^{\sigma_2}_{v_2}(v_2)). \]

Then we can easily observe that \( X(v) \) has the desired properties (i)–(v) of Proposition 9.1. □

As an immediate corollary to Proposition 9.1, we have

**Corollary 9.3.** For \( M > 0 \) and \( \sigma_1, \sigma_2 \in \{+,-\} \), there exist \( \lambda_M > 1 \), \( \nu_M \in (0, \delta_0(M)) \) such that for any \( \lambda \geq \lambda_M \), if \( b < M \) is not a critical value of \( J_\lambda \), then there exist \( \epsilon > 0 \) and a continuous mapping \( \eta(t, v) : [0, 1] \times [J_\lambda \leq M]_{\Sigma_1, \lambda, \Sigma_2, \lambda} \rightarrow [J_\lambda \leq M]_{\Sigma_1, \lambda, \Sigma_2, \lambda} \) such that

1. \( J_\lambda \eta(t, v) \leq J_\lambda(v) \) for all \( (t, v) \in [0, 1] \times [J_\lambda \leq M]_{\Sigma_1, \lambda, \Sigma_2, \lambda} \),
2. \( J_\lambda \eta(1, v) \leq b - \epsilon \) for all \( v \in [J_\lambda \leq b + \epsilon]_{\Sigma_1, \lambda, \Sigma_2, \lambda} \),
3. \( \eta(t, v) \in [J_\lambda \leq M]_{\Sigma_1, \lambda, \Sigma_2, \lambda} \cap \{ \varphi_{\sigma_1} \leq \nu_M \}_{\Sigma_1, \lambda, \Sigma_2, \lambda} \) for all \( (t, v) \in [0, 1] \times ([J_\lambda \leq M]_{\Sigma_1, \lambda, \Sigma_2, \lambda} \cap \{ \varphi_{\sigma_1} \leq \nu_M \}_{\Sigma_1, \lambda, \Sigma_2, \lambda}) \),
4. \( \eta(t, v) \in [J_\lambda \leq M]_{\Sigma_1, \lambda, \Sigma_2, \lambda} \cap \{ \psi_{\sigma_2} \leq \nu_M \}_{\Sigma_1, \lambda, \Sigma_2, \lambda} \) for all \( (t, v) \in [0, 1] \times ([J_\lambda \leq M]_{\Sigma_1, \lambda, \Sigma_2, \lambda} \cap \{ \psi_{\sigma_2} \leq \nu_M \}_{\Sigma_1, \lambda, \Sigma_2, \lambda}) \).

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