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ON SINGULAR INTEGRAL AND MARTINGALE TRANSFORMS

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ABSTRACT. Linear equivalences of norms of vector-valued singular integral operators and vector-valued martingale transforms are studied. In particular, it is shown that the UMD-constant of a Banach space X equals the norm of the real (or the imaginary) part of the Beurling-Ahlfors singular integral operator, acting on $L_X^p(\mathbf{R}^2)$ with $p \in (1,\infty)$. Moreover, replacing equality by a linear equivalence, this is found to be a typical property of even multipliers. A corresponding result for odd multipliers and the Hilbert transform is given. As a corollary we obtain that the norm of the real part of the Beurling-Ahlfors operator equals $p^* - 1$ with $p^* := \max\{p, (p/(p-1))\}$, where the novelty is the lower bound.

1. INTRODUCTION

A Banach space X is said to be a UMD-space[1] if for all (equivalently, for some) $p \in (1,\infty)$ there is a constant $c_p > 0$ such that

$$\sup_{\alpha_k \in \{\pm 1\}} \left\| \sum_{k=1}^n \alpha_k D_k \right\|_{L_X^p} \le c_p \left\| \sum_{k=1}^n D_k \right\|_{L_X^p}$$

for all $n \ge 1$ and all X-valued martingale difference sequences $(D_k)_{k=1}^n$. As the UMD-constant one usually takes $\mathrm{UMD}_p(X) := \inf c_p$. We refer to [11] and the references therein for an overview of the UMD-property. It is known that in the above definition the arbitrary martingale differences can be replaced by Walsh-Paley martingale differences and one gets the same constant (see e.g. [10, p. 12] and [20]; the definition of Walsh-Paley martingales is recalled in Section 2 below). The UMD-property was first investigated by Burkholder [7], who gave a geometric characterization of Banach spaces with the UMD-property. Together with McConnell, Burkholder established in [8] that the Hilbert transform

$$\mathcal{H} f(x) := \frac{1}{\pi} \int_{\mathbf{R}} \frac{f(y)}{x-y}\, dy \tag{1}$$

is bounded on $L_X^p(\mathbf{R})$, $p \in (1,\infty)$, provided that X has the UMD-property. A converse result was proved soon after by Bourgain [5], who showed that boundedness of the Hilbert transform on L_X^p for some $p \in (1,\infty)$ implies the UMD-property for X. It is also known from [6] that the UMD-property implies boundedness of all

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[1]UMD stands for *unconditional martingale differences*.

invariant singular integrals or (more generally) standard multiplier operators under some regularity assumptions. The importance of the UMD-property, especially in connection with PDEs, is further evidenced by recent new results on operator-valued singular integrals (R-boundedness) [26] and other developments.

It is natural to ask for the quantitative nature of the equivalence between the UMD-property and boundedness of the vector-valued Hilbert transform. The proofs in [8] and [5] yield that there is a constant $C > 0$ such that

$$\frac{1}{C}(\mathrm{UMD}_2(X))^{1/2} \leq \| \mathcal{H} \|_{L^2_X(\mathbf{R}) \to L^2_X(\mathbf{R})} \leq C(\mathrm{UMD}_2(X))^2. \tag{2}$$

The curious feature above is the *quadratic equivalence* of the norms, in contrast to the linear dependence one would expect. A better-than-quadratic equivalence obtained from alternative proofs is not known to the authors.

The previous discussion raises the question of whether there is a linear equivalence in (2). We do not know whether this is true or not. However, in this paper we establish that the answer is positive if $\mathcal{H}$ is replaced by the Beurling-Ahlfors transform $\mathcal{BA}$:

Theorem 1.1. *For $p \in (1, \infty)$ and a real Banach space X one has that*

$$\mathrm{UMD}_p(X) = \|\mathrm{Re}(\mathcal{BA})\|_{L^p_X(\mathbf{R}^2) \to L^p_X(\mathbf{R}^2)} = \|\mathrm{Im}(\mathcal{BA})\|_{L^p_X(\mathbf{R}^2) \to L^p_X(\mathbf{R}^2)} \tag{3}$$

with

$$\mathcal{BA}\, f(z) := -\frac{1}{\pi} \int_{\mathbf{C}} \frac{f(w)}{(z-w)^2}\, dm_2(w), \tag{4}$$

where m_2 is the two-dimensional Lebesgue measure on the complex plane $\mathbf{C}$, which is identified with $\mathbf{R}^2$.

Note that $\mathrm{Re}(\mathcal{BA}) = -Id - 2R_1^2$ and $\mathrm{Im}(\mathcal{BA}) = 2R_1R_2$ where R_1 and R_2 are the first and second Riesz transforms, respectively. Equality (3) carries some new information for norm estimates even in the scalar case; see Corollary 3.2 below and the remark after it.

The operator $\mathcal{BA}$ is sometimes called the two-dimensional Hilbert transform. It plays a fundamental role in the theory of quasi-conformal maps and in the theory of elliptic equations in the plane. Quite recently, in connection with the well-known Iwaniec conjecture, there has been much work devoted to the probabilistic approach to estimating the (scalar) L^p-norm of the Beurling-Ahlfors operator; see e.g. [2, 25, 3].

If we replace the equality in (3) by a linear equivalence with multiplicative constants, then the resulting property is shared by an extensive class of operators corresponding to homogeneous multipliers. In this context the crucial difference between the Hilbert transform and the Beurling-Ahlfors operator is the parity of their integral kernels: our Theorem 3.1 shows that *one may replace the real or imaginary part of $\mathcal{BA}$ in* (3) *by any smooth, homogeneous of order zero, and even Fourier multiplier operator* if we allow multiplicative constants. Theorem 4.1 in turn relates odd multipliers to the Hilbert transform.

The proof of Theorem 1.1 is based on a modification of an argument of Bourgain and the already standard representation of suitable combinations of Riesz singular integrals as certain transforms of Ito-integrals; see [16, 2]. In order to define a (wider) class of such transforms, let $W = ((W_t^1, ..., W_t^d))_{t\geq 0}$ be a standard d-dimensional Brownian motion. For a Banach space X, a $p \in (1, \infty)$, and a real

$d \times d$ matrix A we let $\|Id_X|(A,W)\|_p := \inf c$, where the infimum is taken over all $c > 0$ such that

$$\Big\| \int_0^T U \cdot d(AW_t) \Big\|_{L^p_X} \le c \Big\| \int_0^T U \cdot d(W_t) \Big\|_{L^p_X}$$

for all $T \ge 0$ and certain d-tuples U of X-valued processes that take values in a finite-dimensional subspace of X (see Section 2). As a byproduct of our proofs we obtain in Theorem 3.1 that if the matrix A is symmetric and not a constant multiple of the identity matrix, then there is a constant $C = C(A) > 0$, independent of the Banach space X, such that

$$\frac{1}{C}\,\mathrm{UMD}_p(X) \le \|Id_X|(A,W)\|_p \le C\,\mathrm{UMD}_p(X). \tag{5}$$

In the proof of Theorem 1.1 (in fact, Theorem 3.1) we do not employ harmonic extensions of functions to the upper half plane as was done originally by Gundy and Varopoulos [16] in their stochastic proof of the L^p-boundedness of the scalar Riesz transforms. Instead, we follow Bañuelos and Mendez-Hernandez [3] and use the space-time Brownian motion, which corresponds to the heat extension of functions. Earlier the use of heat extensions in norm estimates for singular integrals was initiated by Nazarov, Petermichl and Volberg in [21] and [25]. The use of the space-time Brownian motion in combination with our modified version of Bourgain techniques is one reason that enables us to obtain the equalities (3). In particular, for all $d \ge 2$,

$$\mathrm{UMD}_p(X) = \big\|Id + 2R_1^2\big\|_{L^p_X(\mathbf{R}^d)\to L^p_X(\mathbf{R}^d)} = \|Id_X|(A_s,W)\|_p \tag{6}$$

holds with

$$A_s := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

When one specializes to the case $X = \mathbf{R}$ in Theorem 1.1 one obtains the following:

Corollary 1.2. *For $p \in (1,\infty)$ one has that*

$$\|\mathrm{Re}(\mathcal{BA})\|_{L^p(\mathbf{R}^2)\to L^p(\mathbf{R}^2)} = p^* - 1 \qquad \textit{where } \; p^* := \max\left(p, \frac{p}{p-1}\right).$$

This follows because Burkholder (see [9], [11, Theorem 14], [10, p.12]) established that $\mathrm{UMD}_p(\mathbf{R}) = p^* - 1$. One should also observe here that it is well-known that the L^p-norms of scalar multipliers that preserve real-valued functions do not depend on whether one looks at real or complex L^p-spaces. The upper bound in Corollary 1.2 is due to Volberg and Nazarov [25, Theorem 3.1] who applied Bellman function and heat kernel techniques in their proof (a space-time Brownian motion approach to [25, Theorem 3.1] was subsequently given in [3]). The real novelty in our Corollary 1.2 is the lower bound. As far as we know, this is the first instance where the exact value of the norm of a 'second order' Riesz transform (a non-trivial expression involving products of Riesz transforms) has been determined. Moreover, Corollary 1.2 is of obvious interest in connection with the famous Iwaniec conjecture, which claims that the norm of the *full* complex Beurling-Ahlfors transform equals $p^* - 1$.

The paper is organized as follows: Section 2 recalls necessary notation and contains preparatory material. Our main result, Theorem 3.1, which contains Theorem

1.1, estimate (5), and equality (6) as special cases, is formulated and proved in Section 3. Section 4 treats the case of odd kernels and antisymmetric matrix transforms of stochastic integrals.

The results in Sections 3 and 4 are formulated for tensor product operators $T_m \otimes S$, where $S : X \to Y$ is an arbitrary operator between two Banach spaces X and Y, instead of for the setting of identities Id_X that we have used so far in the introduction. The motivation for doing this comes from certain connections to the geometry of Banach spaces that are explained in Section 5.

We would like to thank Tuomas Hytönen for his careful reading of the manuscript. In his recent paper [17] he obtained results about the linear equivalence of norms of vector-valued spectral multipliers and the UMD-constant, which are in the spirit of the results in this paper (but, as pointed out in [17], the basic singular integral operators such as $\mathcal{BA}$ or $\mathcal{H}$ do not fall into the setting and scope of [17]).

2. Definitions and preliminary results

We shall denote by $\{e_1, \ldots, e_d\}$ the unit vectors of $\mathbf{R}^d$, by $|x|$ the euclidean norm of $x \in \mathbf{R}^d$, and by $B(x, \delta)$ the ball $\left\{y \in \mathbf{R}^d : |x - y| \le \delta\right\}$ for $\delta \ge 0$. Moreover, the set of real $m \times n$ matrices is denoted by $M(m, n)$.

Vector-valued operators. In this paper $L(X, Y)$ stands for the collection of linear and bounded operators between two real Banach spaces X and Y, and we write $L(X) := L(X, X)$. Given a σ-finite measure space (M, μ) and $p \in [1, \infty)$, the space of Bochner integrable random variables $L_X^p(M) = L_X^p(M, \mu)$ consists of all strongly measurable functions $f : M \to X$ such that there is a separable subspace $X_0 \subseteq X$ with $f(M) \subseteq X_0$ and $\|f\|_{L_X^p} := \left(\int_M \|f\|^p d\mu\right)^{1/p} < \infty$. Given operators $S \in L(X, Y)$ and $T \in L(L^p(M, \mu))$ the tensor product $T \otimes S$ can be defined through its action on simple functions, i.e.

$$(T \otimes S)\left(\sum_{k=1}^n x_k \chi_{E_k}\right) := \sum_{k=1}^n S x_k T(\chi_{E_k}).$$

In the case that the quantity

$$\begin{aligned} &\|T \otimes S : L_X^p(M, \mu) \to L_Y^p(M, \mu)\| \\ &\qquad := \sup\left\{\|(T \otimes S)(F)\|_{L_Y^p} : \|F\|_{L_X^p} \le 1, F \text{ a simple function}\right\} \end{aligned}$$

is finite, the operator $T \otimes S$ extends to a bounded linear operator from $L_X^p(M, \mu)$ into $L_Y^p(M, \mu)$. As particular operators T we use multipliers. The usage of operators S instead of identities $\mathrm{Id} : X \to X$ of Banach spaces X might be seen as somewhat artificial at this point. As already mentioned, the motivation for this slightly more general setting can be found in Section 5, but the reader may, if she or he so wishes, replace the operator S by an identity in what follows.

Multipliers. A bounded complex-valued function $m \in C^\infty(\mathbf{R}^d \setminus \{0\})$, $d \ge 1$, is called a (smooth) multiplier. A multiplier m is homogeneous (of order zero) if $m(\lambda\xi) = m(\xi)$ for $\xi \in \mathbf{R}^d \setminus \{0\}$ and $\lambda > 0$. In this paper the term multiplier always refers to smooth and homogeneous multipliers. The multiplier m is called even if $m(\xi) = m(-\xi)$ for $\xi \ne 0$ and odd if $m(\xi) = -m(-\xi)$. The operator $T_m : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$ associated to m is given by

$$T_m f := \mathcal{F}^{-1}(m \mathcal{F} f),$$

where $\mathcal{F}$ stands for the Fourier transform

$$(\mathcal{F}f)(\xi) := \int_{\mathbf{R}^d} \exp(-i\langle \xi, x\rangle) f(x)dx.$$

It is easy to check that an even and real multiplier maps real-valued functions to real-valued functions, and the same is true for odd and purely imaginary multipliers. Consequently, to formulate our results for real Banach spaces, from now on we use the following standing assumption:

Assumption 2.1. All multipliers are even and real, or odd and purely imaginary.

Let $A \in M(d,d)$ be invertible. By applying the simple identity

$$T_{m\circ A} f(x) = (T_m(f \circ A^T))\left((A^T)^{-1}x\right)$$

we deduce that composing a multiplier with a linear invertible map does not change its norm, i.e.

$$\|\mathrm{T}_{m\circ A} \otimes S : L^p_X(\mathbf{R}^d) \to L^p_Y(\mathbf{R}^d)\| = \|\mathrm{T}_m \otimes S : L^p_X(\mathbf{R}^d) \to L^p_Y(\mathbf{R}^d)\|. \tag{7}$$

For a multiplier m on $\mathbf{R}^d$ as above there is a corresponding discrete multiplier $\widetilde{m}$ that acts on functions defined on the d-dimensional torus $\mathbf{T}^d := (-\pi,\pi]^d$: for a finite trigonometric polynomial f we let

$$(T_{\widetilde{m}} f)(\theta) := \sum_{k\in\mathbf{Z}^d} \hat{f}(k) e^{i\langle k,\theta\rangle} m(k),$$

where $\hat{f}(k) := (1/2\pi)^d \int_{\mathbf{T}^d} e^{-i\langle k,\theta\rangle} f(\theta)d\theta$ and $m(0) := \omega_{d-1}^{-1} \int_{S^{d-1}} m(x)dx$ is the average over the boundary of the euclidean unit ball (remember that m is homogeneous of order zero). It follows from Assumption 2.1 that in the above definition $T_{\widetilde{m}} f$ is real whenever f is real. In what follows it will be important that the norms of the corresponding multipliers are equal, which is stated in Lemma 2.2 below.

Lemma 2.2. *Let m be a smooth and homogeneous multiplier on $\mathbf{R}^d$, $d \geq 1$. Then, for any $p \in (1,\infty)$ and $S \in L(X,Y)$,*

$$\begin{aligned} \|T_m \otimes S : L^p_X(\mathbf{R}^d) \to L^p_Y(\mathbf{R}^d)\| &= \|T_{\widetilde{m}} \otimes S : L^p_X(\mathbf{T}^d) \to L^p_Y(\mathbf{T}^d)\| \\ &= \|T_{\widetilde{m}} \otimes S : L^p_{X,0}(\mathbf{T}^d) \to L^p_Y(\mathbf{T}^d)\|, \end{aligned}$$

where $L^p_{X,0}(\mathbf{T}^d)$ is the subset of functions in $L^p_X(\mathbf{T}^d)$ of mean zero.

Proof. The first equality in the scalar case is essentially due to K. de Leeuw [19]; see also [12]. The proof in the monograph [15, pp. 221–223] can easily be seen to carry over to the case considered here to yield the estimate

$$\|T_{\widetilde{m}} \otimes S : L^p_X(\mathbf{T}^d) \to L^p_Y(\mathbf{T}^d)\| \leq \|T_m \otimes S : L^p_X(\mathbf{R}^d) \to L^p_Y(\mathbf{R}^d)\|.$$

The lemma follows as soon as one shows that

$$\|T_m \otimes S : L^p_X(\mathbf{R}^d) \to L^p_Y(\mathbf{R}^d)\| \leq \|T_{\widetilde{m}} \otimes S : L^p_{X,0}(\mathbf{T}^d) \to L^p_Y(\mathbf{T}^d)\|.$$

Observe that when computing the norm of $T_m \otimes S$ we may restrict ourselves to functions of type $f = \sum_{k=1}^n f_k x_k$ where $x_k \in X$ and each $f_k \in C_0^\infty(\mathbf{R}^d)$ has integral zero, because such functions f are dense in $L^p_X(\mathbf{R}^d)$. Then we can also follow the second part of the proof of the transference principle in [15] on pp. 223–225. One also verifies that in the above proofs one may restrict to real-valued scalar functions f_k, and as our multipliers preserve real-valued functions, the result remains true for real Banach spaces X, Y. □

Hilbert transform. The Hilbert transform was defined in (1). It corresponds to the multiplier $m(\xi) = -i \operatorname{sgn}(\xi)$ and maps, by definition, real-valued functions to real-valued functions. The corresponding discrete multiplier operator $T_{\widetilde{m}}$ is the well-known conjugation operator $\widetilde{\mathcal{H}} : L_p(\mathbf{T}) \to L_p(\mathbf{T})$, $p \in (1,\infty)$, which can also be defined through its action on the trigonometric polynomials:

$$(\widetilde{\mathcal{H}} \sin(k\cdot))(\theta) := -\cos(k\theta) \quad \text{and} \quad (\mathcal{H} \cos(k\cdot))(\theta) := \sin(k\theta)$$

for $k = 1, 2, ...$ and $\widetilde{\mathcal{H}}1 :\equiv 0$.

Beurling-Ahlfors transform and Riesz transforms. The k-th Riesz transform R_k, $k = 1, \ldots, d$, is the multiplier operator on $\mathbf{R}^d$ corresponding to the multiplier $\xi_k/(i|\xi|)$. The Beurling-Ahlfors operator, defined through (4) in the introduction, corresponds to the multiplier $m_{BA}(\xi) := \dfrac{\xi_1 - i\xi_2}{\xi_1 + i\xi_2}$. It follows that

$$\mathcal{BA} = R_2^2 - R_1^2 + 2iR_1R_2 =: \operatorname{Re}(\mathcal{BA}) + i\operatorname{Im}(\mathcal{BA}).$$

For our real-valued setting we consider $\operatorname{Re}(\mathcal{BA})$ and $\operatorname{Im}(\mathcal{BA})$ separately and check by (7) that a rotation of the coordinates by angle $\pi/4$ transforms $\operatorname{Re}(\mathcal{BA})$ into $\operatorname{Im}(\mathcal{BA})$ and that their norms coincide.

Martingale transforms. Given independent Bernoulli random variables $\varepsilon_1, \varepsilon_2, \ldots$, i.e. $\mathbf{P}(\varepsilon_k = \pm 1) = 1/2$, and maps $d_k : \mathbf{R}^{k-1} \to X$ where X is a Banach space and d_1 is constant, a sequence $(\varepsilon_k d_k(\varepsilon_1, ..., \varepsilon_{k-1}))_{k \in I}$ with $I = \{1, ..., n\}$ or $I = \{1, 2, ...\}$ is called a Walsh-Paley martingale difference sequence. Given $S \in L(X, Y)$ and $p \in (1, \infty)$, we define $\mathrm{UMD}_p(S) := \inf c$, where the infimum is taken over all $c > 0$ such that

$$\left\| \sum_{k=1}^n \alpha_k S \varepsilon_k d_k \right\|_{L_Y^p} \leq c \left\| \sum_{k=1}^n \varepsilon_k d_k \right\|_{L_X^p} \tag{8}$$

for all X-valued Walsh-Paley martingale difference sequences $(\varepsilon_k d_k)_{k=1}^n$, all real numbers $\alpha_1, ..., \alpha_n$ with $|\alpha_k| \leq 1$, and all $n \geq 1$. It is well-known that by an easy extreme point argument the condition $|\alpha_k| \leq 1$ can be replaced by $\alpha_k \in \{-1, 1\}$ and one gets the same constant $\mathrm{UMD}_p(S)$. The reader is also referred to [11] and the references therein for a more general overview of UMD-spaces.

Transforms for stochastic integrals. We recall the definition given in the introduction. Let $W = ((W_t^1, ..., W_t^d))_{t \geq 0}$ be a d-dimensional standard Brownian motion with continuous paths for all $\omega \in \Omega$ and $W_0 \equiv 0$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the completion of $\sigma(W_t : t \geq 0)$ and $(\mathcal{F}_t)_{t \geq 0}$ the augmentation of the natural filtration of W. Let $S \in L(X, Y)$, $p \in (1, \infty)$, and $A = [a_{kl}] \in M(d, d)$. Then $\|S|(A, W)\|_p := \inf c$ such that

$$\Big\| \sum_{k=1}^d \sum_{l=1}^d a_{kl} \int_0^T S U_t^k dW_t^l \Big\|_{L_Y^p} \leq c \Big\| \sum_{k=1}^d \int_0^T U_t^k dW_t^k \Big\|_{L_X^p} \tag{9}$$

for all $T \geq 0$ and $(\mathcal{F}_t)_{t \geq 0}$-adapted left-continuous processes of Radon random variables $U_t^k : \Omega \to X$ which have right-hand limits, take values in a finite-dimensional subspace of X, satisfy

$$\int_0^T \mathbb{E} \|U_t^k\|_X^2 dt < \infty$$

for all $T \geq 0$ and $k = 1, ..., d$, and for which the right-hand side of (9) is finite. To shorten the notation we also write $\int_0^T (SU_t^k)_{k=1}^d \cdot d(AW)_t$ and $\int_0^T (U_t^k)_{k=1}^d \cdot dW_t$, respectively, for the expressions inside the norms.

Some minor notation. Given $A, B \geq 0$ and $c > 0$, the notation $A \sim_c B$ stands for $A/c \leq B \leq cA$. If the dependence of c on the extra quantities involved is clear, we sometimes simply write $A \sim B$.

3. The main result

It will be convenient to have a special notation for particular matrices and multipliers. Thus, we write

$$\mathrm{A_s} := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_{\mathrm{s},d} := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathrm{A_{as}} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{10}$$

We shall also define the special multiplier m_0 on $\mathbf{R}^d$, where $d \geq 2$, by

$$m_0(\xi) := 2\frac{\xi_1^2}{|\xi|^2} - 1, \quad \text{or equivalently} \quad T_{m_0} := -(\mathrm{Id} + 2R_1^2).$$

We recall once more that, for simplicity, all Banach spaces are assumed to be real. Our main result is the following:

Theorem 3.1. *Assume that $m \in C^\infty(\mathbf{R}^d \setminus \{0\})$, $d \geq 2$, is a real, homogeneous, and even multiplier that is not identically constant, and $A \in M(d,d)$ is a real symmetric matrix that is not a multiple of the identity matrix. Let $p \in (1,\infty)$. Then there is a constant $C = C(m, A)$ such that for every pair of Banach spaces X and Y and every operator $S \in L(X,Y)$ one has that*

$$\begin{aligned}\|S|(\mathrm{A_s}, W)\|_p = \mathrm{UMD}_p(S) &= \|T_{m_0} \otimes S : L^p_X(\mathbf{R}^d) \to L^p_Y(\mathbf{R}^d)\| \\ &\sim_C \|T_m \otimes S : L^p_X(\mathbf{R}^d) \to L^p_Y(\mathbf{R}^d)\| \sim_C \|S|(A, W)\|_p.\end{aligned}$$

Exploiting the result that $\mathrm{UMD}_p(\mathbf{R}) = p-1$ for $p \in [2,\infty)$ (see [9], [11, Theorem 14]) and the fact that for $d = 2$ the multiplier m_0 corresponds to $R_2^2 - R_1^2$ and $2R_1R_2$ can be obtained by a rotation of m_0, Theorem 3.1 implies the following result:

Corollary 3.2. *For $d = 2$, $p \in (1,\infty)$, and $S \in L(X,Y)$ one has*

$$\begin{aligned}&\|\mathrm{Re}(\mathcal{BA}) \otimes S : L^p_X(\mathbf{R}^2) \to L^p_Y(\mathbf{R}^2)\| \\ &\qquad = \|\mathrm{Im}(\mathcal{BA}) \otimes S : L^p_X(\mathbf{R}^2) \to L^p_Y(\mathbf{R}^2)\| = \mathrm{UMD}_p(S).\end{aligned}$$

In particular, for $p \in [2,\infty)$,

$$\|\mathrm{Re}(\mathcal{BA}) : L^p(\mathbf{R}^2) \to L^p(\mathbf{R}^2)\| = \|\mathrm{Im}(\mathcal{BA}) : L^p(\mathbf{R}^2) \to L^p(\mathbf{R}^2)\| = p - 1.$$

We break the proof of Theorem 3.1 into a series of auxiliary results, some of which are basically known and some of which may have independent interest. The actual proof of Theorem 3.1 is given at the end of this section. Our first step is to show that non-trivial even multipliers dominate the UMD-constant linearly. To this end we need a generalization of a lemma due to Bourgain [5, Lemma 1]. For the rest of this section we assume that X and Y are Banach spaces and $S \in L(X,Y)$ is a fixed operator.

Lemma 3.3. *Let $p \in (1,\infty)$ and $Q := \mathbf{T}^d$, and assume that the multiplier m satisfies Assumption 2.1. For $k \geq 1$ let E_k be the closure in $L_X^p(Q^k)$ of the finite real trigonometric polynomials*

$$\Phi_k(\theta_1, ..., \theta_k) = \sum_{p=1}^{P} \Phi_k^p(\theta_1, ..., \theta_k) x_p$$

with $x_k \in X$ and

$$\Phi_k^p = \sum_{\ell_1 \in \mathbf{Z}^d} \cdots \sum_{\ell_k \in \mathbf{Z}^d} e^{i\langle \ell_1, \theta_1 \rangle} \cdots e^{i\langle \ell_k, \theta_k \rangle} \alpha^p_{\ell_1,...,\ell_k}, \quad \mathrm{Im}(\Phi_k^p) \equiv 0,$$

where only finitely many of the $\alpha^p_{\ell_1,...,\ell_k} \in \mathbf{C}$ are non-zero and $\alpha^p_{\ell_1,...,\ell_k} = 0$ whenever $\ell_k = 0$ (so that $\int_Q \Phi_k^p(\theta_1, ..., \theta_k) d\theta_k = 0$). Let $T^k_{\widetilde{m}} : L^p(Q^k) \to L^p(Q^k)$ be given by

$$\begin{aligned}(T^k_{\widetilde{m}} \Phi_k)&(\theta_1, ..., \theta_k) \\ &= \sum_{p=1}^{P} \left(\sum_{\ell_1 \in \mathbf{Z}^d} \cdots \sum_{\ell_k \in \mathbf{Z}^d} m(\ell_k) e^{i\langle \ell_1, \theta_1 \rangle} \cdots e^{i\langle \ell_k, \theta_k \rangle} \alpha^p_{\ell_1,...,\ell_k} \right) x_p\end{aligned}$$

for $\ell_1, ..., \ell_k \in \mathbf{Z}^d$. Then one has that

$$\begin{aligned}&\left\| \sum_{k=1}^{n} ((T^k_{\widetilde{m}} \otimes S)\Phi_k)(\theta_1, \ldots, \theta_k) \right\|_{L_Y^p(Q^n)} \\ &\qquad \leq \| T_{\widetilde{m}} \otimes S : L_X^p(\mathbf{T}^d) \to L_Y^p(\mathbf{T}^d) \| \left\| \sum_{k=1}^{n} \Phi_k(\theta_1, \ldots, \theta_k) \right\|_{L_X^p(Q^n)}\end{aligned}$$

for $\Phi_1 \in E_1, ..., \Phi_n \in E_n$.

Proof. It is sufficient to prove the inequality for finite real trigonometric polynomials $\Phi_1(\theta_1), ..., \Phi_n(\theta_1, ..., \theta_n)$. Let $A \geq 1$ be an integer and $\eta \in Q$ an auxiliary variable, denote by $T_{\widetilde{m},\eta}$ the application of $T_{\widetilde{m}}$ with respect to the variable η, and consider the difference

$$\begin{aligned} D_k^A(\theta_1, ..., \theta_k, \eta) \quad := \quad & ((T_{\widetilde{m},\eta} \otimes S)\Phi_k(\theta_1 + A\cdot, \ldots, \theta_k + A^k\cdot))(\eta) \\ & -((T^k_{\widetilde{m}} \otimes S)\Phi_k)(\theta_1 + A\eta, \ldots, \theta_k + A^k\eta). \end{aligned}$$

Note that in the first term on the right-hand side we apply the multiplier $T_{\widetilde{m}}$ to a function, where $\theta_1, ..., \theta_k, A$ act as parameters, whereas in the second term we apply the multiplier $T^k_{\widetilde{m}}$ to Φ_k itself. If we show that for our fixed trigonometric polynomial Φ_k there is the bound

$$\sup_{A \geq 1} A \| D_k^A \|_{L_Y^\infty} < \infty, \tag{11}$$

then the proof is completed by firstly observing that

$$\begin{aligned}&\left\| \sum_{k=1}^{n} ((T_{\widetilde{m},\eta} \otimes S)\Phi_k(\theta_1 + A\cdot, \ldots, \theta_k + A^k\cdot))(\eta) \right\|_{L_Y^p(Q,d\eta)} \\ &\quad \leq \| T_{\widetilde{m}} \otimes S : L_X^p(Q) \to L_Y^p(Q) \| \left\| \sum_{k=1}^{n} \Phi_k(\theta_1 + A\eta, \ldots, \theta_k + A^k\eta) \right\|_{L_X^p(Q,d\eta)}\end{aligned}$$

and then replacing the left-hand side by

$$\left\| \sum_{k=1}^{n} ((T_{\widetilde{m}}^{k} \otimes S)\Phi_k)(\theta_1 + A\eta, \ldots, \theta_k + A^k\eta) \right\|_{L_Y^p(Q,d\eta)}$$

with the corresponding correction terms in A, integrating with respect to the θ's, applying Fubini's theorem so that the variable η is removed, and sending A to infinity.

In order to prove (11) we observe that $A\|D_k^A(\theta_1, ..., \theta_k, \eta)\|_Y$ is bounded above by a finite number of terms of the form

$$A\Big| T_{\widetilde{m},\eta}\left(e^{i\langle \ell_1, \theta_1 + A\eta\rangle} \cdots e^{i\langle \ell_k, \theta_k + A^k\eta\rangle}\right) \\ - m(\ell_k) e^{i\langle \ell_1, \theta_1 + A\eta\rangle} \cdots e^{i\langle \ell_k, \theta_k + A^k\eta\rangle}\Big| |\alpha^p_{\ell_1,\ldots,\ell_k}| \|Sx_p\|_Y$$

with $\ell_k \neq 0$. Finally we observe that

$$\begin{aligned} & A\Big| T_{\widetilde{m},\eta}\left(e^{i\langle \ell_1, \theta_1 + A\eta\rangle} \cdots e^{i\langle \ell_k, \theta_k + A^k\eta\rangle}\right) \\ & \qquad - m(\ell_k) e^{i\langle \ell_1, \theta_1 + A\eta\rangle} \cdots e^{i\langle \ell_k, \theta_k + A^k\eta\rangle}\Big| \\ = \; & A|m(\ell_1 A + \ldots + \ell_k A^k) - m(\ell_k)| \\ = \; & A|m(\ell_1 A^{-k+1} + \ldots + \ell_{k-1}A^{-1} + \ell_k) - m(\ell_k)|, \end{aligned}$$

which is bounded in A since m is differentiable at $\ell_k \neq 0$. This yields (11). □

Proposition 3.4. *Assume that $m \in C^\infty(\mathbf{R}^d \setminus \{0\})$, $d \geq 2$, is a smooth, homogeneous, even, and non-constant multiplier. Let $\delta^+ := \max_{|\xi|=1} m(\xi)$ and $\delta^- := \min_{|\xi|=1} m(\xi)$ so that $\delta^+ - \delta^- > 0$. Then*

$$\mathrm{UMD}_p(S) \leq \frac{2}{\delta^+ - \delta^-}\left(1 + \frac{|\delta^+ + \delta^-|}{|\delta^+| + |\delta^-|}\right) \|T_m \otimes S : L_X^p(\mathbf{R}^d) \to L_Y^p(\mathbf{R}^d)\|.$$

In particular, in the case where $\max_{|\xi|=1} m(\xi) = -\min_{|\xi|=1} m(\xi) = 1$ we have that

$$\mathrm{UMD}_p(S) \leq \|T_m \otimes S : L_X^p(\mathbf{R}^d) \to L_Y^p(\mathbf{R}^d)\|.$$

Proof. By continuity and compactness there are $\xi^-, \xi^+ \in \mathbf{R}^d$ of length one such that $m(\xi^-) = \delta^-$ and $m(\xi^+) = \delta^+$. Without loss of generality we may assume that $\xi^- = e_1$ and $\xi^+ = e_2$, where e_1 and e_2 are the first two unit vectors in $\mathbf{R}^d$ (otherwise (7) enables us to replace $m(\xi)$ by $m(A\xi)$ with a suitably chosen A). Define the functions $a^-, a^+ \in L^\infty(\mathbf{T}^d)$ by $a^-(\theta) := \mathrm{sgn}\,(\theta_1)$ and $a^+(\theta) := \mathrm{sgn}\,(\theta_2)$ for $\theta \in \mathbf{T}^d$ so that $T_{\widetilde{m}} a^- = \delta^- a^-$ and $T_{\widetilde{m}} a^+ = \delta^+ a^+$. For independent Bernoulli random variables $\varepsilon_1, \varepsilon_2, \ldots$ we consider the X-valued Walsh-Paley martingale difference sequence

$$(\varepsilon_k d_k(\varepsilon_1, \ldots, \varepsilon_{k-1}))_{k=1}^n$$

and a sequence $(\alpha_k)_{k=1}^n$ with $\alpha_k \in \{\delta^-, \delta^+\}$. Define $\psi_k := a^-$ if $\alpha_k = \delta^-$ and $\psi_k := a^+$ if $\alpha_k = \delta^+$, and let

$$\phi_k(\theta_1, \ldots, \theta_{k-1}) := d_k(\psi_1(\theta_1), \ldots, \psi_{k-1}(\theta_{k-1})).$$

Since $(\psi_1(\theta_1), \ldots, \psi_n(\theta_n))$ and $(\varepsilon_1, \ldots, \varepsilon_n)$ have the same distribution (if we normalize the measure on Q^n) and $T_{\widetilde{m}}\psi_k = \alpha_k \psi_k$, Lemma 3.3 implies that

$$\begin{aligned}\left\| \sum_{k=1}^n \alpha_k \varepsilon_k S_k d_k(\varepsilon_1, ..., \varepsilon_{k-1}) \right\|_{L_Y^p} & \\ \leq \| T_{\widetilde{m}} \otimes S : L_X^p(\mathbf{T}^d) \to L_Y^p(\mathbf{T}^d) \| & \left\| \sum_{k=1}^n \varepsilon_k d_k(\varepsilon_1, ..., \varepsilon_{k-1}) \right\|_{L_X^p}.\end{aligned}$$

Let $A := 2/(\delta^+ - \delta^-)$ and $B := (\delta^+ + \delta^-)/(\delta^+ - \delta^-)$ so that the new sequence $\beta_k := A\alpha_k - B$ satisfies $\beta_k = -1$ if $\alpha_k = \delta^-$ and $\beta_k = 1$ if $\alpha_k = \delta^+$. Then

$$\begin{aligned}&\left\| \sum_{k=1}^n \beta_k \varepsilon_k S_k d_k(\varepsilon_1, ..., \varepsilon_{k-1}) \right\|_{L_Y^p} \\ \leq\ & \left[A \| T_{\widetilde{m}} \otimes S : L_X^p(\mathbf{T}^d) \to L_Y^p(\mathbf{T}^d) \| + |B| \|S\| \right] \left\| \sum_{k=1}^n \varepsilon_k d_k(\varepsilon_1, ..., \varepsilon_{k-1}) \right\|_{L_X^p}.\end{aligned}$$

Because $\|m\|_\infty \|S\| = \sup_{\ell \in \mathbf{Z}^d} |m(\ell)| \|S\| \leq \| T_{\widetilde{m}} \otimes S : L_X^p(\mathbf{T}^d) \to L_Y^p(\mathbf{T}^d) \|$ we end up with

$$\begin{aligned}\mathrm{UMD}_p(S) &\leq \left[A + \frac{|B|}{\|m\|_\infty} \right] \| T_{\widetilde{m}} \otimes S : L_X^p(\mathbf{T}^d) \to L_Y^p(\mathbf{T}^d) \| \\ &= \left[A + \frac{|B|}{\|m\|_\infty} \right] \| T_m \otimes S : L_X^p(\mathbf{R}^d) \to L_Y^p(\mathbf{R}^d) \| \\ &\leq \frac{2}{\delta^+ - \delta^-} \left(1 + \frac{|\delta^+ + \delta^-|}{|\delta^+| + |\delta^-|} \right) \| T_m \otimes S : L_X^p(\mathbf{R}^d) \to L_Y^p(\mathbf{R}^d) \|\end{aligned}$$

where the equality follows from Lemma 2.2. □

Proposition 3.5. *For $T_{m_0} = -\mathrm{Id} - 2R_1^2$, $p \in (1, \infty)$, and $d \geq 2$ one has*

$$\| T_{m_0} \otimes S : L_X^p(\mathbf{R}^d) \to L_Y^p(\mathbf{R}^d) \| \leq \left\| S | (A_{\mathrm{s},d}, W) \right\|_p. \tag{12}$$

Proof. We apply the representation of products of Riesz transforms in terms of heat extensions to the upper half space (see Lemma 6.1). Let $f = \sum_{k=1}^m f_k x_k$ and $g = \sum_{l=1}^n g_l b_l$ with $f_k, g_l \in C_0^\infty(\mathbf{R}^d)$ and $x_k \in X$, $b_l \in Y'$. Assume that u_k and v_l are the heat extensions of f_k and g_l, respectively, to the upper half plane and that $(W_t)_{t \geq 0}$ is a standard Brownian motion in $\mathbf{R}^d$ starting at the origin. Let

$u := \sum_{k=1}^{m} u_k x_k$, $v := \sum_{l=1}^{n} v_l b_l$, and $1 = (1/p) + (1/p')$. Lemma 6.1 gives that

$$\begin{aligned}
&\left|\int_{\mathbf{R}^d} \langle ((T_{m_0} \otimes S)f)(x), g(x)\rangle \, dx\right| \\
= \; & \lim_{T\to\infty} (2\pi T)^{d/2} \left| \sum_{k,l} \langle Sx_k, b_l\rangle \times \right. \\
& \left. \times \mathbf{E}\left(\int_0^T \nabla u_k(W_t, T-t)\cdot d(A_{\mathrm{s},d}W)_t \int_0^T \nabla v_l(W_t, T-t)\cdot dW_t \right) \right| \\
= \; & \lim_{T\to\infty} (2\pi T)^{d/2} \\
& \left| \mathbf{E} \left\langle \int_0^T \nabla S u(W_t, T-t)\cdot d(A_{\mathrm{s},d}W)_t, \int_0^T \nabla v(W_t, T-t)\cdot dW_t \right\rangle \right|.
\end{aligned}$$

We continue with

$$\begin{aligned}
&\left| \mathbf{E} \left\langle \int_0^T \nabla S u(W_t, T-t)\cdot d(A_{\mathrm{s},d}W)_t, \int_0^T \nabla v(W_t, T-t)\cdot dW_t \right\rangle \right| \\
\le \; & \left(\mathbf{E} \left\| \int_0^T \nabla S u(W_t, T-t)\cdot d(A_{\mathrm{s},d}W)_t \right\|_Y^p \right)^{1/p} \\
& \times \left(\mathbf{E} \left\| \int_0^T \nabla v(W_t, T-t)\cdot dW_t \right\|_{Y'}^{p'} \right)^{1/p'} \\
\le \; & \|S|(A_{\mathrm{s},d}, W)\|_p \left(\mathbf{E} \left\| \int_0^T \nabla u(W_t, T-t)\cdot dW_t \right\|_Y^p \right)^{1/p} \\
& \times \left(\mathbf{E} \left\| \int_0^T \nabla v(W_t, T-t)\cdot dW_t \right\|_{Y'}^{p'} \right)^{1/p'} \\
= \; & \|S|(A_{\mathrm{s},d}, W)\|_p \|f(W_T) - u(0,T)\|_{L_Y^p} \|g(W_T) - v(0,T)\|_{L_{Y'}^{p'}}
\end{aligned}$$

by Itô's formula, because $(1/2)\Delta u_k = (\partial/\partial t)u_k$ and $(1/2)\Delta v_l = (\partial/\partial t)v_l$. Next, $\sup_{T\ge 0} \|T^{d/2}u(0,T)\|_X < \infty$ gives $\lim_{T\to\infty} T^{d/2}\|u(0,T)\|_X^p = 0$ and hence that $\lim_{T\to\infty}(2\pi T)^{d/2}\mathbf{E}\,\|f(W_T) - u(0,T)\|_X^p = \|f\|_{L_X^p(\mathbf{R}^d)}^p$. The same applies for $g(W_T)$ and we end up with

$$\left|\int_{\mathbf{R}^d} \langle ((T_{m_0} \otimes S)f)(x), g(x)\rangle \, dx\right| \le \|S|(A_{\mathrm{s},d}, W)\|_p \|f\|_{L_X^p(\mathbf{R}^d)} \|g\|_{L_{Y'}^{p'}(\mathbf{R}^d)}.$$

The proof is complete because f and g as above are dense in $L_X^p(\mathbf{R}^d)$ and $L_{Y'}^{p'}(\mathbf{R}^d)$, respectively. □

In order to exploit the quantities $\|S|(A, W)\|_p$ in rigorous arguments, one needs in some places approximations of stochastic integrals by discrete martingales. So for the general reader's convenience we switch to the simple discretized version $\|S|(A, g)\|_p$ introduced below. To prove just part of our main result one could proceed more directly (see Remark 3.9). Given $p \in (1,\infty)$, $S \in L(X,Y)$, and $A \in M(d,d)$, we let

$$\|S|(A,g)\|_p := \inf c,$$

where the infimum is taken over all $c > 0$ such that

$$\Big\| \sum_{k=1}^{N} \sum_{l=1}^{d} \left[S d_{(k-1)d+l}(\varphi_1, ..., \varphi_{(k-1)d}) \right] \left\langle A(\varphi_{(k-1)d+1}, ..., \varphi_{kd}), e_l \right\rangle \Big\|_{L_Y^p}$$
$$\leq c \Big\| \sum_{k=1}^{N} \sum_{l=1}^{d} \left[d_{(k-1)d+l}(\varphi_1, ..., \varphi_{(k-1)d}) \right] \varphi_{(k-1)d+l} \Big\|_{L_X^p},$$

where $N = 1, 2, ...,$ $d_j : \mathbf{R}^{\lfloor \frac{j-1}{d} \rfloor d} \to X$ are continuous bounded functions taking values in a finite-dimensional subspace of X and $\varphi_1, \varphi_2, ...$ are independent standard Gaussian random variables. The Nd terms on the right-hand side (in their natural order) we call a Gaussian block martingale difference sequence of order $d \geq 1$; on the left-hand side we have its A-*transform*. Observe that these transforms are not the traditional martingale transforms appearing in the definition of UMD-spaces.

Lemma 3.6. *Let $A = [a_{lk}] \in M(d,d)$, $S \in L(X,Y)$, and $p \in (1,\infty)$. Then the following hold:*

(i) $\|S|(A,g)\|_p = \|S|(A,W)\|_p$.

(ii) *If $U \in M(d,d)$ is real and unitary, then*

$$\|S|(U^T A U, g)\|_p = \|S|(A,g)\|_p.$$

(iii) *If $M \geq 1$ is an integer and the tensor product $\otimes^M A$ is defined to be the block diagonal matrix with A as each diagonal block, then*

$$\|S|(\otimes^M A, g)\|_p = \|S|(A,g)\|_p.$$

(iv) *Consider a sub-matrix B of A obtained from A by choosing indices $I = \{k_1, \ldots, k_{d'}\}$ with $1 \leq k_1 < k_2 < \ldots < k_{d'} \leq d$, $1 \leq d' < d$, and deleting the corresponding rows and columns from A. Then*

$$\|S|(B,g)\|_p \leq \|S|(A,g)\|_p.$$

Proof. (i) It is evident that $\|S|(A,g)\|_p \leq \|S|(A,W)\|_p$. The inequality in the other direction follows by a standard approximation of Ito-integrals by discrete Gaussian martingales.

(ii) Here we observe that

$$\begin{aligned}
& \Big\| \sum_{k=1}^{N} \sum_{l=1}^{d} \left[S d_{(k-1)d+l}(\varphi_1, ..., \varphi_{(k-1)d}) \right] \left\langle U^T A U(\varphi_{(k-1)d+1}, ..., \varphi_{kd}), e_l \right\rangle \Big\|_{L_Y^p} \\
= \; & \Big\| \sum_{k=1}^{N} \Big\langle U\big((S d_{(k-1)d+l}(\varphi_1, ..., \varphi_{(k-1)d}))_{l=1}^d\big), AU(\varphi_{(k-1)d+1}, ..., \varphi_{kd}) \Big\rangle \Big\|_{L_Y^p} \\
\leq \; & \|S|(A,g)\|_p \\
& \cdot \Big\| \sum_{k=1}^{N} \Big\langle U\big((d_{(k-1)d+l}(\varphi_1, ..., \varphi_{(k-1)d}))_{l=1}^d\big), U(\varphi_{(k-1)d+1}, ..., \varphi_{kd}) \Big\rangle \Big\|_{L_Y^p} \\
= \; & \|S|(A,g)\|_p \Big\| \sum_{k=1}^{N} \sum_{l=1}^{d} \left[d_{(k-1)d+l}(\varphi_1, ..., \varphi_{(k-1)d}) \right] \varphi_{(k-1)d+l} \Big\|_{L_X^p}.
\end{aligned}$$

Above we used in the second step the observation that the transformed sequence

$$\langle U(\varphi_1,...,\varphi_d),e_1\rangle,\dots,\langle U(\varphi_1,...,\varphi_d),e_d\rangle,$$
$$\langle U(\varphi_{d+1},...,\varphi_{2d}),e_1\rangle,\dots,\langle U(\varphi_{d+1},...,\varphi_{2d}),e_d\rangle,\dots$$

consists again of independent Gaussian random variables because U is unitary.

(iii) The $\otimes^M A$-transform of an appropriate Gaussian martingale block difference sequence of order dM is obtained by simply performing the A-transform of the same sequence (which is also of order d). This shows that $\|S|(\otimes^M A,g)\|_p \le \|S|(A,g)\|_p$. The converse inequality is a special case of (iv) which we treat now.

(iv) We first perform a unitary permutation of the coordinates, which is justified by part (ii), so that we may assume that $B = (a_{lk})_{1\le l,k\le d'}$. Next we consider the unitary map U which maps the unit vector e_k to e_k if $k = 1,...,d'$ and to $-e_k$ if $k = d'+1,...,d$. From (ii) it follows that for $C := (1/2)(A + U^TAU)$, one has $\|S|(C,g)\|_p \le \|S|(A,g)\|_p$. The entries of C satisfy $c_{lk} = a_{lk}$ if $l,k \in \{1,...,d'\}$ and $c_{lk} = 0$ if $l \in \{1,...,d'\}$ and $k \notin \{1,...,d'\}$. Now the inequality $\|S|(B,g)\|_p \le \|S|(C,g)\|_p$ can be proved by an appropriate augmentation of the Gaussian random variables and of the martingale difference sequence that has to be transformed: we add to each block $\phi_{(k-1)d'+1},\dots,\phi_{kd'}$ of Gaussian random variables $d-d'$ independent Gaussian random variables to obtain a block size d, and add appropriate zero martingale differences to the original martingale difference sequence. Now the transformation with respect to B can be artificially written as a transformation with respect to C. □

The previous lemma will be used to prove the following result:

Proposition 3.7. (i) *Assume that $A \in M(d,d)$ is real and symmetric, and denote by $\lambda_{\max}$ (respectively, $\lambda_{\min}$) the largest (respectively, the smallest) eigenvalue of A. Let $B \in M(d',d')$ be another real and symmetric matrix such that each eigenvalue λ of B satisfies $\lambda_{\min} \le \lambda \le \lambda_{\max}$. Then for any $S \in L(X,Y)$ and $p \in (1,\infty)$,*

$$\|S|(B,W)\|_p \le \|S|(A,W)\|_p.$$

(ii) *Assume that $A \in M(d,d)$ is real and antisymmetric, and denote by $\lambda_{\max}$ (respectively, $\lambda_{\min}$) the largest (respectively, the smallest) eigenvalue of iA. Let $B \in M(d',d')$ be another real and antisymmetric matrix such that each eigenvalue λ of iB satisfies $\lambda_{\min} \le \lambda \le \lambda_{\max}$. Then for any $S \in L(X,Y)$ and $p \in (1,\infty)$,*

$$\|S|(B,W)\|_p \le \|S|(A,W)\|_p.$$

Proof. (i) By Lemma 3.6 (i) it is enough to show that $\|S|(B,g)\|_p \le \|S|(A,g)\|_p$. Moreover, by applying the spectral theorem for symmetric matrices and part (ii) of the same lemma we may assume that both A and B are (real and) diagonal. In addition, by Lemma 3.6 (iii) we may replace A by the tensor product $\otimes^{d'}A$. Observe that the tensor product $\otimes^{d'}A$ is diagonal and at least d' of its diagonal elements have the value $\lambda_{\max}$, and the same holds true for $\lambda_{\min}$. By applying again a unitary permutation of the coordinates and part (iv) of Lemma 3.6, we obtain that for any given sequence $\Lambda := (\lambda_1,\dots,\lambda_{d'})$ with $\lambda_j \in \{\lambda_{\max},\lambda_{\min}\}$ for all $1 \le j \le d'$, the diagonal matrix $A_\Lambda \in M(d',d')$ (with diagonal Λ) satisfies

$$\|S|(A_\Lambda,W)\|_p \le \|S|(A,W)\|_p.$$

By the assumption on the eigenvalues of the matrix B, we may express B as a convex combination of matrices of the form A_Λ. This clearly yields the claim.

(ii) The matrix iA is self-adjoint, so that the eigenvalues are real. By a simple examination of the spectral decomposition of iA (observe that λ and $-\lambda$ are simultaneously eigenvalues for iA) we may write A, after a unitary transformation, in the form

$$A = B_1 \otimes \ldots \otimes B_\ell,$$

where each B_k is of the form $B_k = c_k \mathrm{A}_{\mathrm{as}}$ (A_{as} was defined in (10)), with $\lambda_{\min} \leq c_k \leq \lambda_{\max}$ (in the case where d is odd, the last one, B_ℓ, equals the 1×1 zero matrix and $\lambda_{\min} \leq 0 \leq \lambda_{\max}$). Now the claim follows by an extreme point argument as in (i). $\square$

Proposition 3.8. *Let m be a homogeneous, even, and smooth multiplier on $\mathbf{R}^d$, $d \geq 2$. Then for any $S \in L(X,Y)$,*

$$\left\| T_m \otimes S : L_X^p(\mathbf{R}^d) \to L_Y^p(\mathbf{R}^d) \right\| \leq c \left\| T_{m_0} \otimes S : L_X^p(\mathbf{R}^d) \to L_Y^p(\mathbf{R}^d) \right\|$$

for $S \in L(X,Y)$ and $p \in (1,\infty)$, where $c > 0$ depends at most on m.

Proof. In the following we always assume that $\xi, \theta \in S^{d-1}$, so that (for example) $m_0(\xi) = 2\xi_1^2 - 1$. Let $a \in (0,1)$. By composing m_0 with the linear map $\xi \mapsto B_a(\xi) := (\sqrt{1-a^2}\xi_1, \xi_2, \ldots, \xi_d)$ we infer from (7) that

$$\|T_{m_a} \otimes S : L_X^p(\mathbf{R}^d) \to L_Y^p(\mathbf{R}^d)\| \leq \frac{1}{1+a} \|T_{m_0} \otimes S : L_X^p(\mathbf{R}^d) \to L_Y^p(\mathbf{R}^d)\|,$$

where

$$\begin{aligned} m_a(\xi) &:= \frac{2-a^2}{2(1+a)} + \frac{a^2}{2(1+a)} m_0 \circ B_a(\xi) = (1-a) \frac{1}{1-a^2\xi_1^2} \\ &= (1-a) \sum_{k=0}^{\infty} a^{2k} \xi_1^{2k}, \end{aligned}$$

and we also use the estimate

$$\|Id \otimes S : L_X^p(\mathbf{R}^d) \to L_Y^p(\mathbf{R}^d)\| \leq \|S\| \leq \|T_{m_0} \otimes S : L_X^p(\mathbf{R}^d) \to L_Y^p(\mathbf{R}^d)\|$$

(for example, use Lemma 2.2 and $\|m_0\|_\infty = 1$). Of course, given $\theta \in S^{d-1}$ we obtain the same estimates for

$$m_a^\theta(\xi) := (1-a) \sum_{k=0}^{\infty} a^{2k} \langle \xi, \theta \rangle^{2k}.$$

Given $r \in (0,1)$, the d-dimensional Poisson kernel for $B(0,1)$ has the form

$$\begin{aligned} P(r\xi, \theta) &= \frac{1}{\omega_{d-1}} \frac{(1-r^2)}{(1+r^2-2r\langle \xi, \theta \rangle)^{d/2}} \\ &= \frac{(1-r^2)(1+r^2)^{-d/2}}{\omega_{d-1}} \left(1 - \frac{2r\langle \xi, \theta \rangle}{1+r^2} \right)^{-d/2} \end{aligned}$$

with $\omega_{d-1} = |\partial B(0,1)|$. By the Taylor expansion of $y \mapsto (1-y)^{-d/2}$ we obtain that

$$\begin{aligned} P^s(r\xi, \theta) &:= \frac{1}{2}(P(r\xi, \theta) + P(r\xi, -\theta)) \\ &= \frac{(1-r^2)(1+r^2)^{-d/2}}{\omega_{d-1}} \sum_{k=0}^{\infty} \left(\frac{2r}{1+r^2} \right)^{2k} \langle \xi, \theta \rangle^{2k} \binom{-d/2}{2k}. \end{aligned}$$

Fix $\varepsilon > 0$ and let

$$c_0 := 2\omega_{d-1} \frac{\Gamma(\varepsilon+1)\Gamma(d/2)}{\Gamma((d/2)+\varepsilon)}.$$

Using the substitution $u = 2r/(1+r^2)$ and Euler's β-integral yields that

$$\begin{aligned}
&\int_0^1 (1+r^2)^{-2+d/2} \left(1 - \frac{2r}{1+r^2}\right)^{d/2-1+\varepsilon} P^s(r\xi,\theta)\, dr \\
&= \frac{1}{2\omega_{d-1}} \sum_{k=0}^{\infty} \left(\int_0^1 (1-u)^{d/2-1+\varepsilon} u^{2k}\, du \right) \binom{-d/2}{2k} \langle \xi, \theta \rangle^{2k} \\
&= \frac{1}{2\omega_{d-1}} \frac{\Gamma(d/2+\varepsilon)}{\Gamma(d/2)} \sum_{k=0}^{\infty} \left(\frac{\Gamma(d/2+2k)}{\Gamma(d/2+2k+1+\varepsilon)} \right) \langle \xi, \theta \rangle^{2k} \\
&= \frac{1}{c_0} \int_0^1 (1-a)^{\varepsilon-1} a^{d/2-1} m_a^\theta(\xi)\, da. \qquad (13)
\end{aligned}$$

Let us denote by $\mathbf{sh}(d)$ a complete orthonormal L^2-basis of $L^2(S^{d-1},\lambda)$, where λ is the normalized Haar measure, consisting of spherical harmonics on S^d (we refer to [23, 24] for more details on spherical harmonics). For $\psi \in \mathbf{sh}(d)$ we denote by $\deg(\psi)$ the degree of ψ. Because the function $(r,\xi) \mapsto r^k\psi(\xi)$ is harmonic if $\deg(\psi) = k$, we have in this case that

$$\int_{S^{d-1}} P(r\xi,\theta)\psi(\theta)\, d\theta = r^k \psi(\xi). \qquad (14)$$

Define

$$\lambda_k := c_0 \int_0^1 (1+r^2)^{-2+d/2} \left(1 - \frac{2r}{1+r^2}\right)^{d/2-1+\varepsilon} r^k\, dr \sim_{c(\varepsilon,d)} k^{1-d-2\varepsilon}.$$

Let $a_\psi := \langle m, \psi \rangle / \lambda_k$ if $\psi \in \mathbf{sh}(d)$ and $\deg(\psi) = k$. Applying [23, p. 70] gives

$$\sum_{\psi \in \mathbf{sh}_d} |a_\psi|^2 < \infty.$$

Because m is even, we can find an even $f \in L^2(S^{d-1},\lambda)$ with $\langle f, \psi \rangle = a_\psi$ so that

$$m = \sum_{k=0}^{\infty} \lambda_k \sum_{\psi \in \mathbf{sh}_d,\ \deg(\psi)=k} \langle f, \psi \rangle \psi \qquad (15)$$

in $L^2(S^{d-1},\lambda)$. If we can show that

$$m(\xi) = \int_{S^{d-1}} \int_0^1 (1-a)^{\varepsilon-1} a^{\frac{d}{2}-1} m_a^\theta(\xi) f(\theta)\, da d\theta \qquad (16)$$

for $\xi \in S^{d-1}$, then this would give our assertion since

$$\int_{S^{d-1}} \int_0^1 (1-a)^{\varepsilon-1} a^{\frac{d}{2}-1} |f(\theta)|\, da d\theta < \infty$$

implies that m is a convex combination of the multipliers m_a^θ. In order to verify equality (16) it is sufficient to show that

$$\int_{S^{d-1}} \left[\int_{S^{d-1}} \int_0^1 (1-a)^{\varepsilon-1} a^{\frac{d}{2}-1} m_a^\theta(\xi) f(\theta) da d\theta \right] \overline{\psi}(\xi) d\xi$$
$$= c_0 \int_0^1 (1+r^2)^{-2+d/2} \left(1 - \frac{2r}{1+r^2} \right)^{d/2-1+\varepsilon} r^k \, dr \langle f, \psi \rangle$$

for $\psi \in \mathbf{sh}_d$ with $\deg(\psi) = k$. But this follows from a computation using (13) and (14). □

Remark 3.9. A slight additional argument shows that in the above proposition one may allow any even multiplier m such that $m_{|S^{d-1}} \in W^{s,1}(S^{d-1})$ for some $s > d-1$. In particular, if $d = 2$ this class contains all m such that for some $\varepsilon > 0$ the derivative $D^\varepsilon m_{|S^1}$ is a function of bounded variation.

We are now ready to prove our main theorem.

Proof of Theorem 3.1. We first verify that $\mathrm{UMD}_p(S)$ is comparable to the corresponding norm of the operator $T_{m_0} \otimes S$.

Observe that $\max_{|\xi|=1} m_0(\xi) = 1$ and $\min_{|\xi|=1} m_0(\xi) = -1$. An application of Proposition 3.4 yields that

$$\mathrm{UMD}_p(S) \le \| T_{m_0} \otimes S : L^p_X(\mathbf{R}^d) \to L^p_Y(\mathbf{R}^d) \|. \tag{17}$$

By Proposition 3.5 we have in turn that

$$\| T_{m_0} \otimes S : L^p_X(\mathbf{R}^d) \to L^p_Y(\mathbf{R}^d) \| \le \| S | (A_{\mathrm{s},d}, W) \|_p. \tag{18}$$

The eigenvalues of $A_{\mathrm{s},d}$ and $\mathrm{A_s}$ are ± 1. Hence Proposition 3.7 (i) and Lemma 3.6 (i) yield that

$$\| S | (A_{\mathrm{s},d}, W) \|_p \le \| S | (\mathrm{A_s}, W) \|_p = \| S | (\mathrm{A_s}, g) \|_p. \tag{19}$$

However, the $\mathrm{A_s}$-transform of discrete Gaussian martingales is a special case of a UMD-martingale transform and hence in the case of $S = \mathrm{Id}$ it follows immediately that

$$\| S | (\mathrm{A_s}, g) \|_p \le \mathrm{UMD}_p(S). \tag{20}$$

There are various ways to check this inequality for general S (note that we started with Walsh-Paley martingales in the definition of $\mathrm{UMD}_p(S)$). An easy and self-consistent way would be to apply the central limit theorem argument from Lemma 4.2 to the Bernoulli variables and replace the Bernoulli variables by Gaussian random variables (there are arguments to switch from Walsh-Paley martingale difference sequences to arbitrary martingale difference sequences; see for example [10, p. 12] and [20]). By combining the inequalities (17)–(20) we obtain that

$$\mathrm{UMD}_p(S) = \| T_{m_0} \otimes S : L^p_X(\mathbf{R}^d) \to L^p_Y(\mathbf{R}^d) \| = \| S | (\mathrm{A_s}, W) \|_p. \tag{21}$$

This proves the equalities in the theorem.

Now let us assume that $A \in M(d,d)$ is symmetric and non-trivial, and that m is an even, smooth, and non-trivial multiplier. Proposition 3.4 shows that $\mathrm{UMD}_p(S)$ is dominated by a multiple of the norm of $T_m \otimes S$, and Proposition 3.8 verifies that this norm in turn is dominated by a multiple of the norm of $T_{m_0} \otimes S$. The linear equivalence of the norm of $T_m \otimes S$ to the UMD-constant of X follows now from (21).

Finally, the equivalence of $\mathrm{UMD}_p(S)$ to $\|S|(A,W)\|_p$ will be deduced from (21) by an application of Proposition 3.7 (i). Denote by $\lambda_{\min}$ and $\lambda_{\max}$ the minimal and maximal eigenvalues of A. Firstly, we obtain that

$$\|S|(A,W)\|_p \leq c\|S|(\mathrm{A_s},W)\|_p$$

with $c := \max(|\lambda_{\max}|,|\lambda_{\min}|)$. To consider the other direction, let $\alpha := 2/(\lambda_{\max} - \lambda_{\min})$ and $\beta := -(\lambda_{\max}+\lambda_{\min})/(\lambda_{\max}-\lambda_{\min})$, so that $B := \alpha A + \beta$ satisfies $\lambda_{\max}(B) = -\lambda_{\min}(B) = 1$. Consequently,

$$\begin{aligned} \|S|(\mathrm{A_s},W)\|_p &\leq \alpha\|S|(A,W)\|_p + \beta\|S|(Id,W)\|_p \\ &\leq \alpha\|S|(A,W)\|_p + \frac{\beta}{\max\{|\lambda_{\max}|,|\lambda_{\min}|\}}\|S|(A,W)\|_p \end{aligned}$$

and the proof of Theorem 3.1 is complete. □

Remark 3.10. There is a shorter argument for part of Theorem 3.1 which, modulo some 'hand waving', bypasses Lemma 3.6 and Proposition 3.7. Specifically, in order to prove e.g. the statement

$$\mathrm{UMD}_p(X) = \|T_{m_0} : L^p_X(\mathbf{R}^d) \to L^p_Y(\mathbf{R}^d)\| \sim_C \|T_m : L^p_X(\mathbf{R}^d) \to L^p_Y(\mathbf{R}^d)\|$$

one only needs Propositions 3.4, 3.5, and 3.8 and the bound $\|S|(A_{\mathrm{s},d},W)\|_p \leq \mathrm{UMD}_p(S)$. Because $A_{\mathrm{s},d}$ is a diagonal matrix, the latter inequality can be suitably approximated by a UMD-transform inequality.

4. Odd multipliers

A canonical example of an odd multiplier is the Hilbert transform. In this section we verify that all odd multipliers, as well as all antisymmetric transforms for stochastic integrals, are linearly comparable to the Hilbert transform. Observe that an odd multiplier maps real functions to purely imaginary ones, hence we consider below the operator T_{im} in order to be able to allow also real Banach spaces in the statement.

Theorem 4.1. *Assume that $m \in C^\infty(\mathbf{R}^d\backslash\{0\})$, $d \geq 2$, is a real non-zero, homogeneous, and odd multiplier, and that $A \in M(d,d)$ is a non-zero and antisymmetric matrix. Let $p \in (1,\infty)$. Then there is a constant $C = C(A,m)$ such that for every pair of Banach spaces X and Y and every operator $S \in L(X,Y)$ one has*

$$\begin{aligned} \|T_{im} \otimes S : L^p_X(\mathbf{R}^d) \to L^p_Y(\mathbf{R}^d)\| &\sim_C \|S|(A,W)\|_p \sim_C \|S|(\mathrm{A_{as}},W)\|_p \\ &= \|\mathcal{H} \otimes S : L^p_X(\mathbf{R}) \to L^p_Y(\mathbf{R})\|. \end{aligned}$$

We start the proof by showing the following:

Lemma 4.2. *For $p \in (1,\infty)$ and $S \in L(X,Y)$ one has that*

$$\|S|(\mathrm{A_{as}},W)\|_p \leq \|\widetilde{\mathcal{H}} \otimes S : L^p_X(\mathbf{T}) \to L^p_Y(\mathbf{T})\|.$$

Proof. Assume that $M \geq 1$ and that $a_k, b_k : [-1,1]^{2k-2} \to X$, $k = 1, ..., M$, are bounded and continuous. Lemma 3.3 yields that

$$
\begin{aligned}
(22) \qquad & \Big\| \sum_{k=1}^{M} \Big(\widetilde{\mathcal{H}} \sin(\theta_k) S a_k(\sin(\theta_1), ..., \sin(\theta_{k-1}); \cos(\theta_1), ..., \cos(\theta_{k-1})) \\
& + \widetilde{\mathcal{H}} \cos(\theta_k) S b_k(\sin(\theta_1), ..., \sin(\theta_{k-1}); \cos(\theta_1), ..., \cos(\theta_{k-1})) \Big) \Big\|_{L_p^Y} \\
\leq \ & \|\widetilde{\mathcal{H}} \otimes S : L_p^X(\mathbf{T}) \to L_p^Y(\mathbf{T})\| \ \times \\
& \times \Big\| \sum_{k=1}^{M} \Big(\sin(\theta_k) a_k(\sin(\theta_1), ..., \sin(\theta_{k-1}); \cos(\theta_1), ..., \cos(\theta_{k-1})) \\
& + \cos(\theta_k) b_k(\sin(\theta_1), ..., \sin(\theta_{k-1}); \cos(\theta_1), ..., \cos(\theta_{k-1})) \Big) \Big\|_{L_p^X}.
\end{aligned}
$$

Next, we apply a blocking argument. Let $N, L \geq 1$ be integers and assume that $A_k, B_k : [-1,1]^{2k-2} \to X$, $k = 1, ..., N$, are bounded and continuous functions. For $M = NL$ we apply (22) to

$$
\begin{aligned}
& a_{(k-1)L+j}(\sin(\theta_1), ..., \sin(\theta_{(k-1)L+j-1}); \cos(\theta_1), ..., \cos(\theta_{(k-1)L+j-1})) \\
= \ & A_k \Big((L/2)^{-1/2}(\sin(\theta_1) + \ldots + \sin(\theta_L)), \\
& (L/2)^{-1/2}(\sin(\theta_{L+1}) + \ldots + \sin(\theta_{2L})), \\
& \qquad \ldots, (L/2)^{-1/2}(\sin(\theta_{(k-2)L+1}) + \ldots + \sin(\theta_{(k-1)L})); \\
(23) \qquad & (L/2)^{-1/2}(\cos(\theta_1) + \ldots + \cos(\theta_L)), \\
& (L/2)^{-1/2}(\cos(\theta_{L+1}) + \ldots + \cos(\theta_{2L})), \\
& \qquad \ldots, (L/2)^{-1/2}(\cos(\theta_{(k-2)L+1}) + \ldots + \cos(\theta_{(k-1)L})) \Big)
\end{aligned}
$$

and to an analogous choice of the coefficients $b_{(k-1)L+j}$ for $1 \leq k \leq N$ and $1 \leq j \leq L$. By taking into account the action of $\widetilde{\mathcal{H}}$ on the trigonometric polynomials we obtain that

$$
\begin{aligned}
& \Big\| \sum_{k=1}^{N} \Big(- S c_k A_k(s_1, ..., s_{k-1}; c_1, ..., c_{k-1}) \ + \\
& \qquad + S s_k B_k(s_1, ..., s_{k-1}; c_1, ..., c_{k-1}) \Big) \Big\|_{L_p^Y} \\
\leq \ & \|\widetilde{\mathcal{H}} \otimes S : L_p^X(\mathbf{T}) \to L_p^Y(\mathbf{T})\| \ \times \\
& \times \Big\| \sum_{k=1}^{N} \Big(s_k A_k(s_1, ..., s_{k-1}; c_1, ..., c_{k-1}) \ + \\
& \qquad + c_k B_k(s_1, ..., s_{k-1}; c_1, ..., c_{k-1}) \Big) \Big\|_{L_p^X}
\end{aligned}
$$

where we have used the abbreviations $s_k := (L/2)^{-1/2} \sum_{j=1}^{L} \sin(t_{(k-1)L+j})$ and $c_k := (L/2)^{-1/2} \sum_{j=1}^{L} \cos(t_{(k-1)L+j})$ for $k = 1, \ldots, N$. By keeping N fixed, letting $L \to \infty$, applying the central limit theorem (here we may normalize the Lebesgue measure on $\mathbf{T}$) and using the fact that cos and sin are uncorrelated, one gets that

$$\|S|(\mathrm{A}_{\mathrm{as}}, g)\|_p \le \|\widetilde{\mathcal{H}} \otimes S : L^p_X(\mathbf{T}) \to L^p_Y(\mathbf{T})\|.$$

Finally, Lemma 3.6 verifies that $\|S|(\mathrm{A}_{\mathrm{as}}, g)\|_p = \|S|(\mathrm{A}_{\mathrm{as}}, W)\|_p$ and we are done. □

The following lemma is well-known. For the convenience of the reader we recall the idea of its proof.

Lemma 4.3. *For $p \in (1,\infty)$ one has that*

$$\|\,\mathcal{H} \otimes S : L^p_X(\mathbf{R}) \to L^p_Y(\mathbf{R})\| \le \|S|(\mathrm{A}_{\mathrm{as}}, W)\|_p.$$

Proof. We use $\|\,\mathcal{H} \otimes S : L^p_X(\mathbf{R}) \to L^p_Y(\mathbf{R})\| = \|\widetilde{\mathcal{H}} \otimes S : L^p_{X,0}(\mathbf{T}) \to L^p_Y(\mathbf{T})\|$ from Lemma 2.2 and consider $f(\theta) := \sum_{k=1}^n (\sin(k\theta)x_k + \cos(k\theta)y_k)$ with $x_k, y_k \in X$ as a function on the unit circle. Then, a.s.,

$$u(W_\tau) = \int_0^\tau \nabla u(W_t) \cdot dW_t \quad \text{and} \quad v(W_\tau) = -\int_0^\tau \nabla u(W_t) \cdot d(\mathrm{A}_{\mathrm{as}}\, W)_t$$

by Itô's formula, where u and v are the harmonic extensions to the unit disc of f and $\sum_{k=1}^n (-\cos(k\theta)x_k + \sin(k\theta)y_k)$, $(W_t)_{t\ge 0}$ is a standard two-dimensional standard Brownian motion, and $\tau := \inf\{t \ge 0 : |W_t| = 1\}$. □

Proof of Theorem 4.1. The equality

$$\|S|(\mathrm{A}_{\mathrm{as}}, W)\|_p = \|\,\mathcal{H} \otimes S : L^p_X(\mathbf{R}) \to L^p_Y(\mathbf{R})\|$$

follows from Lemmas 4.2, 4.3, and 2.2. Moreover,

$$\|S|(A, W)\|_p \sim_C \|S|(\mathrm{A}_{\mathrm{as}}, W)\|_p$$

is a consequence of Proposition 3.7 (ii) (note that the eigenvalues of $i\,\mathrm{A}_{\mathrm{as}}$ are ± 1 and that iA has at least two symmetric non-zero real eigenvalues). Moreover, by a classical argument called 'the method of rotations' (see e.g. [15, p.271, formula (4.2.20)]) one may express any odd (and smooth) multiplier operator T_{im} as an average of directional Hilbert transforms, which immediately yields that the norm of $T_{im} \otimes S$ is linearly dominated by that of $\mathcal{H} \otimes S$. Finally, if m is an odd and non-zero multiplier, we may assume that $m(e_1) = 1$. The corresponding discrete multiplier $\widetilde{m}$ satisfies $-i\widetilde{m}(k_1 e_1) = -i\mathrm{sgn}\,(k_1)$ for $k_1 \ne 0$ and $-i\widetilde{m}(0e_1) = 0$. By considering $\widetilde{f}(\theta_1, ..., \theta_d) = f(\theta_1) := \sum_{k=1}^n (\sin(k_1\theta_1)x_{k_1} + \cos(k_1\theta_1)y_{k_1})$ with $x_{k_1}, y_{k_1} \in X$ and observing that $((\widetilde{H} \otimes S)f)(\theta_1) = ((-T_{i\widetilde{m}} \otimes S)\widetilde{f})(\theta_1, ..., \theta_d)$, we immediately get that

$$\|\widetilde{H} \otimes S : L^p_{X,0}(\mathbf{T}) \to L^p_Y(\mathbf{T})\| \le \|T_{i\widetilde{m}} \otimes S : L^p_X(\mathbf{T}^d) \to L^p_Y(\mathbf{T}^d)\|$$

so that $\|H \otimes S : L^p_X(\mathbf{R}) \to L^p_Y(\mathbf{R})\| \le \|T_{im} \otimes S : L^p_X(\mathbf{R}^d) \to L^p_Y(\mathbf{R}^d)\|$ by Lemma 2.2. □

5. Additional remarks

The main open problem that remains is whether or not we have linear equivalence of the norms:

$$\mathrm{UMD}_p(S) \sim \|\,\mathcal{H} \otimes S : L^p_X(\mathbf{R}) \to L^p_Y(\mathbf{R})\|. \tag{24}$$

By the results obtained in Theorems 3.1 and 4.1 this problem can be formulated now in various ways, for example in a purely probabilistic way via martingale transforms or via multipliers. To find counterexamples to (24) in the category of Banach spaces (i.e. $S = Id_X$ for a Banach space X) seems to be harder than finding counterexamples for operators $S \in L(X,Y)$, which is one reason for our usage of the operator setting. At the same time the operator setting gives easier control over whether or not estimates are linear (for example, $\mathrm{UMD}_p(Id_X) < \infty$ if and only if $\| \mathcal{H} \otimes Id_X : L^p_X(\mathbf{R}) \to L^p_X(\mathbf{R}) \| < \infty$ because $Id_X^2 = Id_X$, which does not work for operators). Natural candidates to disprove (24) in the setting of operators are the operators of summation $\sigma_n : \ell^1_n \to \ell^\infty_n$. For these it is known that

$$\| \mathcal{H} \otimes \sigma_n : L^2_{\ell^1_n}(\mathbf{R}) \to L^2_{\ell^\infty_n}(\mathbf{R}) \| \sim \log(n+1);$$

see [13] (Section IV, Satz 2.1 (proof) and Korollar 2.4) (cf. also [22] (2.4.4)). So far, the best estimates for the UMD-constants are

$$\frac{1}{c}\sqrt{\log(n+1)} \leq \mathrm{UMD}_2(\sigma_n) \leq c \log(n+1)$$

where the lower estimate follows from [14] and the upper one is a consequence of $\mathrm{UMD}_2(Id_{\ell^\infty_n}) \leq c \log(n+1)$, which is folklore.

A determination of the UMD-constant of σ_n would be of interest for several reasons: in the case of $\mathrm{UMD}_2(\sigma_n) \sim \log(n+1)$ this would imply that each non-superreflexive Banach space X contains n-dimensional subspaces E_n such that the lower bound $\mathrm{UMD}_2(Id_{E_n}) \geq \log(n+1)/c$ holds, because due to R.C. James a Banach space X is non-superreflexive if and only if the operators σ_n can be uniformly factorized through certain n-dimensional subspaces $E_n \subseteq X$ (see [18]). Having the lower estimate, one might ask for more connections between the quantitative behavior of UMD-constants of the finite-dimensional subspaces of a Banach space and the property that the space is non-superreflexive. On the other hand, any estimate of type $\mathrm{UMD}_2(\sigma_n) = o(\log(n+1))$ would offer some new insight into martingale transforms.

6. Appendix

We recall the well-known connection between singular integrals and A-transforms of stochastic integrals corresponding to heat extensions of functions to the upper half space. A simple proof will be sketched below for the reader's convenience. We refer to [3] for the original result, and to [4] and the references therein for corresponding results that use the harmonic extension instead. In order to state the relation let $d \geq 1$ and consider a real matrix $A = [a_{kl}] \in M(d,d)$. We define the operator U_A for smooth elements $f, g \in C_0^\infty(\mathbf{R}^d)$ through the bilinear form

$$\begin{aligned} (25) \qquad & \int_{\mathbf{R}^d} (U_A f)(x) g(x)\, dx \\ := \quad & \lim_{T\to\infty} (2\pi T)^{d/2} \mathbf{E} \left[\left(\int_0^T \nabla u(W_t, T-t) \cdot d(AW)_t \right) \right. \\ & \left. \times \left(\int_0^T \nabla v(W_t, T-t) \cdot dW_t \right) \right], \end{aligned}$$

where u and v are the heat extensions of f and g, respectively, to the upper half space $\mathbf{R}^d \times \mathbf{R}_+$ (i.e. $u(t,x) := \mathbb{E} f(x+W_t)$ and similarly for v), ∇ denotes differentiation with respect to the x-variables, and $(W_t)_{t\geq 0}$ is a standard d-dimensional Brownian motion starting at the origin. Hence, for example, $u_t = \frac{1}{2}\Delta u$ and $u(x,0) = f(x)$ for $x \in \mathbf{R}^d$.

Lemma 6.1. *Let $p \in (1,\infty)$ and $d \geq 2$. The operator U_A is well-defined and extends to a bounded operator on $L^p(\mathbf{R}^d)$ which can be expressed in terms of the Riesz transforms as*

$$U_A = -\sum_{k,l=1}^{d} a_{kl} R_k R_l.$$

Proof. We let $f, g \in C_0^\infty(\mathbf{R}^d)$. From Itô's isometry we obtain that

$$\begin{aligned}
&(2\pi T)^{d/2}\mathbf{E}\left[\left(\int_0^T \nabla u(W_t, T-t)\cdot d(AW)_t\right)\right.\\
&\qquad\qquad\qquad\left.\times\left(\int_0^T \nabla v(W_t,T-t)\cdot dW_t\right)\right]\\
= \;& (2\pi T)^{d/2}\mathbf{E}\int_{[0,T]} \langle A^T \nabla u(W_t,T-t), \nabla v(W_t, T-t)\rangle dt\\
= \;& \int_{\mathbf{R}^d\times[0,T]} \langle A^T\nabla u(x,t), \nabla v(x,t)\rangle (2\pi T)^{d/2} d\mu_{t,T}(x)dt,
\end{aligned}$$

where $\mu_{t,T} = law(W_{T-t})$. Because $\sup_{x\in\mathbf{R}^d, t>0} t^{\frac{d+1}{2}}|\nabla u(x,t)| < \infty$ and similarly for v, upon splitting the integration over $\mathbf{R}^d\times[0,T]$ into integrals over $\mathbf{R}^d\times[0,T/2]$ and $\mathbf{R}^d\times(T/2,T]$ we see by standard arguments that

$$\begin{aligned}
&\lim_{T\to\infty}(2\pi T)^{d/2}\mathbf{E}\left[\left(\int_0^T \nabla u(W_t,T-t)\cdot d(AW)_t\right)\times\right.\\
&\quad\left.\times\left(\int_0^T \nabla v(W_t,T-t)\cdot dW_t\right)\right] = \int_{\mathbf{R}^d\times[0,\infty)} \langle A^T\nabla u(x,t), \nabla v(x,t)\rangle dxdt
\end{aligned}$$

(note that $\int_{\mathbf{R}^d\times[0,\infty)} |\nabla u(x,t)|^2 dxdt < \infty$, and similarly for v, which follows from the argument below; cf. also [21, Lemma 1.1]). If $\mathcal{F} u$ is the Fourier transform of u with respect to the x-variables, then it is well-known that $\mathcal{F} u(\xi,t) = (\mathcal{F} f)(\xi)e^{-t|\xi|^2/2}$ for $\xi\in\mathbf{R}^d$ and $t>0$. Observe also that $\mathcal{F}((d/dx_k)u) = i\xi_k \mathcal{F} u$ for $k=1,...,d$. By Parseval's formula and Fubini's theorem we may compute that

$$\begin{aligned}
&\int_{\mathbf{R}^d} (U_A f)(x) g(x)\, dx\\
= \;& (2\pi)^{-d}\int_{\mathbf{R}^d}\left(\int_{\mathbf{R}^+} e^{-t|\xi|^2}\, dt\right)\mathcal{F} f(\xi)\overline{\mathcal{F} g(\xi)}\langle i\xi, A(i\xi)\rangle\, d\xi\\
= \;& (2\pi)^{-d}\int_{\mathbf{R}^d} \mathcal{F} f(\xi)\overline{\mathcal{F} g(\xi)}|\xi|^{-2}\langle \xi, A\xi\rangle\, d\xi\\
= \;& \int_{\mathbf{R}^d} (T_m f)(x) g(x)\, dx,
\end{aligned}$$

where m is the multiplier $m(\xi) := |\xi|^{-2}\langle \xi, A\xi\rangle$. By recalling that R_j corresponds to the multiplier $\xi_j/(i|\xi|)$ the claim follows immediately. □

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