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THE LEECH LATTICE Λ AND THE CONWAY GROUP ·O REVISITED

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Dedicated to John Horton Conway as he approaches his seventieth birthday.

Abstract. We give a new, concise definition of the Conway group ·O as follows. The Mathieu group M_{24} acts quintuply transitively on 24 letters and so acts transitively (but imprimitively) on the set of $\binom{24}{4}$ tetrads. We use this action to define a progenitor P of shape $2^{\star\binom{24}{4}} : M_{24}$; that is, a free product of cyclic groups of order 2 extended by a group of permutations of the involutory generators. A simple lemma leads us directly to an easily described, short relator, and factoring P by this relator results in ·O. Consideration of the lowest dimension in which ·O can act faithfully produces Conway's elements ξ_T and the 24–dimensional real, orthogonal representation. The Leech lattice is obtained as the set of images under ·O of the integral vectors in $\mathbb{R}_{24}$.

1. Introduction

The Leech lattice Λ was discovered by Leech [14] in 1965 in connection with the packing of non-overlapping identical spheres into the 24–dimensional space $\mathbb{R}^{24}$ so that their centres lie at the lattice points; see Conway and Sloane [9]. Its construction relies heavily on the rich combinatorial structure underlying the Mathieu group M_{24}. The full group of symmeries of Λ is, of course, infinite, as it contains all translations by a lattice vector. Leech himself considered the subgroup consisting of all symmetries fixing the origin O. He had enough geometric evidence to predict the order of this group to within a factor of 2 but could not prove the existence of all the symmetries he anticipated. It was John McKay who told John Conway about Λ – and the rest, as they say, is history. In two elegant papers, see [4, 5], Conway produced a beautifully simple additional symmetry of Λ and found the order of the group it generates together with the monomial group of permutations and sign changes used in the construction of Λ. He proved that this is the full group of symmetries of Λ (fixing O), showed that it is perfect with centre of order 2, and that the quotient by its centre is simple. He called the group ·O to signify that it was the stabiliser of O in the full group of symmetries of Λ and extended the notation to ·2, and ·3, the stabilisers of vectors of type 2 and type 3 respectively. The symbol ·1 was then used to denote the quotient ·O/⟨±1⟩.

In this paper we use the methods of symmetric generation to define ·O directly from M_{24} by considering a homomorphic image of an infinite group which we denote

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by

$$P = 2^{\star\binom{24}{4}} : \mathrm{M}_{24}.$$

Here $2^{\star\binom{24}{4}}$ denotes a free product of $\binom{24}{4}$ cyclic groups of order 2, corresponding to the tetrads of the 24–point set on which M_{24} acts. This free product is extended by M_{24} itself to form a semi-direct product in which M_{24} acts in the natural manner on tetrads. The lemma which first appeared in Curtis [11] is then used to yield a relation by which we can factor P without leading to total collapse. Since P is a semi-direct product every element of it can be written as πw, where π is an element of M_{24} and w is a word in the $\binom{24}{4}$ involutory generators of the free product. The relator by which we factor takes a particularly simple form with the length of w being just 3; thus the corresponding relation has the form $\nu = t_T t_U t_V$, where $\nu \in \mathrm{M}_{24}$ and T, U, V are tetrads.

Having defined this quotient G we use the double coset enumerator of Bray and Curtis [2] to demonstrate that it is indeed a group of the required order. We then seek a faithful representation of minimal degree and, unsurprisingly, come up with dimension 24. Embedding G in $O_{24}(\mathbb{R})$ is readily accomplished, and it turns out that the involutory generators corresponding to the tetrads are simply the negatives of Conway's original elements. The Leech lattice follows, of course, by letting this orthogonal group act on the vectors in the standard basis of $\mathbb{R}^{24}$.

2. Preliminaries

In order to make this paper as self-contained as possible we include here, albeit in a concise form, information about the Mathieu groups which is needed to follow the construction of $\cdot$O. We also give a brief description of what we mean by symmetric generation of groups.

2.1. **The Mathieu group M_{24}.** A Steiner system $\mathrm{S}(5,8,24)$ is a collection of 759 8-element subsets known as *octads* of a 24-element set, Ω say, such that any 5-element subset of Ω is contained in precisely one octad. It turns out that such a system is unique up to relabelling, and the group of permutations of Ω which preserves such a system is a copy of the Mathieu group M_{24} which acts 5-transitively on the points of Ω. We let $\Omega = \mathrm{P}_1(23)$, the 24–point projective line, and choose the Steiner system so that it is preserved by the projective special linear group $\mathrm{L}_2(23) = \langle \alpha : s \mapsto s+1, \gamma : s \mapsto -1/s \rangle$. Let $P(\Omega)$ denote the power set of Ω regarded as a 24–dimensional vector space over the field GF_2. Then the 759 octads span a 12–dimensional subspace, the *binary Golay code* $\mathcal{C}$, which contains the empty set ϕ, 759 octads, 2576 12-element subsets called *dodecads*, 759 16-*ads* which are the complements of octads, and the whole set Ω. The stabiliser in M_{24} of an octad (and its complementary 16-ad) is a subgroup of shape $2^4 : \mathrm{A}_8$, in which the elementary abelian normal subgroup of order 16 fixes every point of the octad. The stabiliser of a dodecad is the smaller Mathieu group M_{12}.

Now a 16-ad can be partitioned into two disjoint octads in 15 ways, and so Ω can be partitioned into 3 mutually disjoint octads in $759 \times 15/3 = 3795$ ways. Such a partition is known as a *trio* of octads and is denoted by $U \cdot V \cdot W$, where U, V and W are the octads. M_{24} acts transitively on such partitions, and the stabiliser in M_{24} of such a trio is a maximal subgroup of shape $2^6{:}(\mathrm{L}_3(2) \times \mathrm{S}_3)$ which, of course, has index 3795 in M_{24}. From the above assertions about $\mathcal{C}$ it is clear that the symmetric difference of two octads which intersect in four points must be another octad, and so we see that the 24 points of Ω can be partitioned into 6 complementary *tetrads* (4-element subsets) such that the union of any two of them is an octad. Such a partition is called a *sextet*, and, since a sextet is determined by any one of its 6 tetrads, 5-transitivity on Ω ensures that M_{24} acts transitively on the $\binom{24}{4}/6 = 1771$ sextets. The stabiliser of one such sextet is a maximal subgroup of shape $2^6 : 3^{\cdot}S_6$. A sextet with tetrads a, b, c, d, e, f is denoted $abcdef$; the visible $2^6{:}3$ of the sextet group stabilises all the tetrads in the sextet, and the quotient S_6 acts naturally upon them.

If the vector space $P(\Omega)$ is factored by the subspace $\mathcal{C}$ we obtain the 12–dimensional *cocode* $\mathcal{C}^\star$, whose elements (modulo $\mathcal{C}$) are the empty set ϕ, 24 *monads* (1-element subsets), 276 *duads* (2-element subsets), 2024 *triads* (3-element subsets), and the 1771 sextets. Since the symmetric difference of two even subsets necessarily has even cardinality itself, the duads and sextets, together with the empty set, form an 11–dimensional subspace which is clearly an irreducible 2-modular module for M_{24}. A duad $\{a, b\}$ is usually denoted ab.

A sextet $abcdef$ is said to *refine* a trio $A \cdot B \cdot C$ if (after suitable reordering) we have $A = a \mathbin{\dot\cup} b$, $B = c \mathbin{\dot\cup} d$ and $C = e \mathbin{\dot\cup} f$. A trio is said to *coarsen* a sextet if the sextet refines the trio. A given sextet refines just 15 trios; conversely, a trio has just 7 refinements to a sextet. These 7 sextets, together with the empty set, form a 3–dimensional subspace of $\mathcal{C}^\star$.

2.2. **The Miracle Octad Generator and the hexacode.** The Miracle Octad Generator (MOG) (see [10]) is an arrangement of the 24 points of Ω into a 4×6 array in which the octads assume a particularly recognisable form so that it is easy to read them. Naturally the 6 columns of the MOG will form a sextet, and the pairing of columns $12 \cdot 34 \cdot 56$ will form a trio. The three octads in this trio are known as the *bricks* of the MOG. The hexacode $\mathcal{H}$, see Conway [7], is a 3–dimensional quaternary code of length six whose codewords give an algebraic notation for the binary codewords of $\mathcal{C}$ as given in the MOG. Explicitly, if $\{0, 1, \omega, \bar\omega\} = K \cong \mathrm{GF}_4$, then

$$\begin{aligned}\mathcal{H} &= \langle (1,1,1,1,0,0,), (0,0,1,1,1,1), (\bar\omega,\omega,\bar\omega,\omega,\bar\omega,\omega)\rangle \\ &= \{(0,0,0,0,0,0),\ (0,0,1,1,1,1)(9\text{ such}),\ (\bar\omega,\omega,\bar\omega,\omega,\bar\omega,\omega)(12\text{ such}), \\ &\quad (\bar\omega,\omega,0,1,0,1)(36\text{ such}),\ (1,1,\omega,\omega,\bar\omega,\bar\omega)(6\text{ such})\},\end{aligned}$$

where multiplication by powers of ω are of course allowed, as is an S_4 of permutations of the coordinates corresponding to

$$S_4 \cong \langle (1\ 3\ 5)(2\ 4\ 6), (1\ 2)(3\ 4), (1\ 3)(2\ 4)\rangle$$

(the even permutations in the wreath product of shape $2 \wr S_3$ fixing the pairing $12 \cdot 34 \cdot 56$). Each hexacodeword has an *odd* and an *even* interpretation and each interpretation corresponds to 2^5 binary codewords in $\mathcal{C}$, giving the $2^6 \times 2 \times 2^5 = 2^{12}$ binary codewords of $\mathcal{C}$. The rows of the MOG are labelled in descending order with the elements of K as shown in Figure 1, thus the top row is labelled 0. Let $h = (h_1, h_2, \ldots, h_6) \in \mathcal{H}$. Then in the odd interpretation if $h_i = \lambda \in K$ we place 1 in the λ position in the ith column and 0's in the other three positions, or we may complement this and place 0 in the λth position and 1's in the other three positions. We do this for each of the 6 values of i and may complement freely as long as *the number of* 1*'s in the top row is odd.* So there are 2^5 choices.

In the even interpretation if $h_i = \lambda \neq 0$ we place 1 in the 0th and λth positions and 0's in the other two, or as before we may complement. If $h_i = 0$, then we place

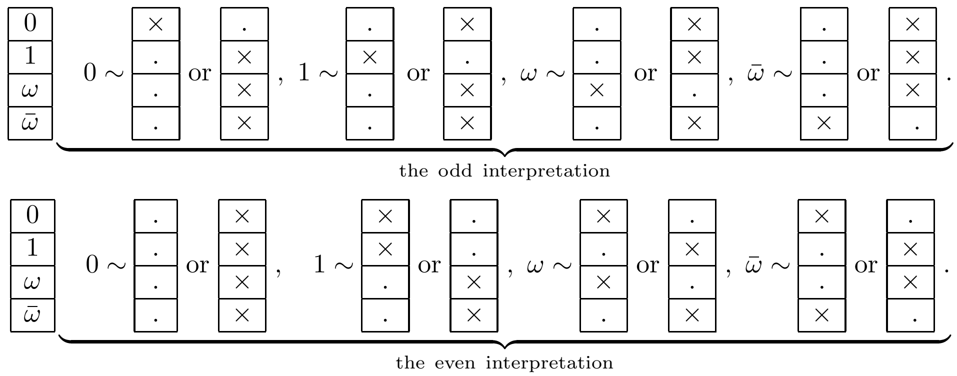


FIGURE 1. The odd and even interpretations of hexacodewords.

0 in all four positions or 1 in all four positions. This time we may complement freely so long as *the number of* 1*'s in the top row is even.* Thus for instance

$(0, 1, \bar\omega, \omega, 0, 1) \sim$

×	.	×	.	×	.
.	×	×	.	.	×
.	.	×	×	.	.
.	.	.	.	.	.

or

.	×	×	×	.	×
.	×	.	.	.	×
.	.	.	×	.	.
.	.	×	.	.	.

,

∞	0	11	1	22	2
3	19	4	20	18	10
6	15	16	14	8	17
9	5	13	21	12	7

,

in the odd and even interpretations respectively, where evenly many complementations are allowed in each case. The last figure shows the standard labelling of the 24 points of Ω with the projective line $P_1(23)$ such that all permutations of $\mathrm{L}_2(23)$ are in M_{24}.

The hexacode also provides a very useful notation for the elements of the elementary abelian normal subgroup of order 2^6 in the sextet group, consisting of those elements which fix every tetrad (in this case the columns of the MOG). An entry $h_i = \lambda$ means that the affine element $x \mapsto x + \lambda$ should be placed in the ith column. Thus 0 means all points in that column are fixed; 1 means 'interchange 1st and 2nd, 3rd and 4th entries in that column'; ω means 'interchange 1st and 3rd, 2nd and 4th entries'; and $\bar\omega$ means 'interchange 1st and 4th, 2nd and 3rd entries'. Note that this implies that 45 of the non-trivial elements are of shape $1^8.2^8$ whilst 18 are fixed-point free involutions. These 18 form the involutions of 6 disjoint Klein fourgroups, such as $\{(\bar\omega, \omega, \bar\omega, \omega, \bar\omega, \omega), (1, \bar\omega, 1, \bar\omega, 1, \bar\omega), (\omega, 1, \omega, 1, \omega, 1)\}$, which are of course permuted by the factor group S_6 of the sextet group. The actions on the 6 fourgroups and the 6 tetrads are non-permutation identical.

2.3. **Finding elements of** $\mathbf{M}_{24}$**.** In the sequel we shall on occasion have to find elements of M_{24} of a certain type. Explicitly we shall find that the subgroup of M_{24} fixing four pairs of points whose union is an octad is of order 2^7 and that this subgroup has a centre of order 2. It is the non-trivial element in this centre which we need to be able to write, given the four pairs. In fact this is easily accomplished as follows: first note that the four pairs define three sextets. These three sextets partition the 16-ad complementary to the original octad into 8 pairs. The element we require has cycle shape $1^8.2^8$, fixes every point of the octad and flips each of the pairs. Thus, obtaining these elements is simply a matter of writing down three sextets, given three defining tetrads.

2.4. **Symmetric generation.** We denote a free product of n copies of the cyclic group of order 2 by $2^{\star n}$, and we let $\{t_1, t_2, \ldots, t_n\}$ be a set of involutory generators of such a free product. A permutation $\pi \in \mathrm{S}_n$ induces an automorphism $\hat{\pi}$ of this free product by permuting these generators; thus $t_i^{\hat{\pi}} = t_{\pi(i)}$. Given N a subgroup of S_n this enables us to form a semi-direct product of form $P = 2^{\star n} : N$ where, for $\pi \in N$,

$$\pi^{-1} t_i \pi = t_{\pi(i)}.$$

In the case when N acts transitively, we call such a semi-direct product a *progenitor*. The t_i are the *symmetric generators*, and N is the *control subgroup*. Note that each of the progenitor's elements can be written πw, where $\pi \in N$ and w is a word in the t_i, and so any homomorphic image of P is obtained by factoring out relations of the form $\pi w = 1$. It turns out that a consideration of the permutation group N can often lead us to suitable relations by which to factor P without leading to total collapse. In particular we include without proof the obvious but surprisingly effective

Lemma 2.1.

$$\langle t_{k_1}, t_{k_2}, \ldots, t_{k_r} \rangle \cap N \leq C_N(N_{k_1 k_2 \ldots k_r}).$$

In other words an element of N that can be written in terms of a given set of symmetric generators without causing collapse must centralise the pointwise stabiliser in N of those generators.

A family of results shows that such progenitors contain perfect subgroups to low index. For our purposes here we include a weak version of these lemmas.

Lemma 2.2. *Let*

$$G = \frac{2^{\star n} : N}{\pi_1 w_1, \pi_2 w_2, \ldots},$$

where N is a perfect permutation group which acts transitively on n letters, and one of the words $w_1, w_2 \ldots$ is of odd length. Then G is itself perfect.

Proof. Certainly $G' \geq N' = N$. Moreover $[t_i, \pi] = t_i t_{\pi(i)} \in G'$ and, since N is transitive, we see that $t_i t_j \in G'$ for all i and j. Thus $K = \{\pi w \mid \pi \in N, l(w) \text{ even}\} \leq G'$. Suppose wlog that $l(w_1)$ is odd; then for any $k \in \{1, \ldots, n\}$ we have $t_k = \pi_1 w_1 t_k \in K \leq G'$, and so $G' = G$. □

3. The progenitor for ·O and the additional relation

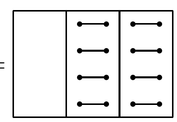
As has been mentioned above, the Mathieu group M_{24} acts quintuply transitively on 24 letters and so permutes the $\binom{24}{4}$ tetrads transitively. This action is not, however, primitive, as the 6 tetrads which together comprise a sextet form a block of imprimitivity. We shall consider the progenitor

$$P = 2^{\star\binom{24}{4}} : \mathrm{M}_{24}.$$

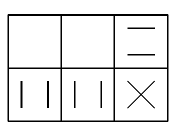
Thus a typical symmetric generator will be denoted by t_T, where T is a tetrad of the 24 points of Ω. Whereas the rank of the symmetric group S_{24} acting on tetrads is just 5, depending only on the number of points in which a tetrad intersects the fixed tetrad T, the tetrad stabilizer in M_{24}, which has shape $2^6 : (3 \times \mathrm{A}_5) : 2$, has 14 orbits on tetrads. We shall be concerned with pairs of tetrads which lie together in a common octad of the system and which intersect one another in 2 points. Having fixed the tetrad T there are clearly $\binom{4}{2}.5.\binom{4}{2} = 180$ possibilities for a tetrad U to intersect it in this manner. The stabilizer in M_{24} of both T and U thus has order $2^6.3.120/180 = 2^7$ and shape $2^4 : 2^3$. Indeed the stabiliser of an octad has shape $2^4 : \mathrm{A}_8$, where the elementary abelian 2^4 fixes every point of the octad, and we have simply fixed a partition of the 8 points of the octad into pairs (and fixed each of those pairs). To be explicit we let T be the tetrad consisting of the top 2 points in each of the first 2 columns of the MOG, and let U be the first and third points in each of the first 2 columns. In what follows we shall often denote the element t_T by $\times$s in the four positions of T as displayed in the MOG diagram. Thus

$$t_T = \begin{array}{|cc|cc|cc|}\hline \times & \times & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}, \quad t_U = \begin{array}{|cc|cc|cc|}\hline \times & \times & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \times & \times & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}, \quad \begin{array}{|cc|cccc|}\hline a & a & & & & \\ \hline b & b & & & & \\ \hline c & c & & & & \\ \hline d & d & & & & \\ \hline \end{array}.$$

Now the tetrads T and U determine a pairing of of the first brick into duads a, b, c and d. The stabilizer of T and U, $\mathrm{Stab}_{\mathrm{M}_{24}}(TU)$, must fix each of these duads and so commute with the symmetric generators $\mathcal{X} = \{t_{ab} = t_T, t_{ac} = t_U, t_{ad}, t_{bc}, t_{bd}, t_{cd}\}$. Now Lemma 2.1 says that the only elements of $N \cong \mathrm{M}_{24}$ which can be written in terms of the elements in $\mathcal{X}$ without causing collapse must lie in the centralizer in M_{24} of $\mathrm{Stab}_{\mathrm{M}_{24}}(TU)$, and we have

$$C_{\mathrm{M}_{24}}(\mathrm{Stab}_{\mathrm{M}_{24}}(TU)) = Z(\mathrm{Stab}_{\mathrm{M}_{24}}(TU)) = \left\langle \nu = \right.$$  $$\left. \right\rangle \cong C_2.$$

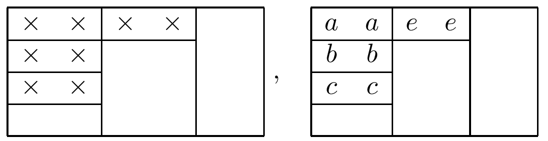
We now seek to write $\nu = w(t_T, t_U, \ldots)$, a word in the elements of $\mathcal{X}$ of shortest possible length without causing collapse. But $l(w) = 1$ would mean that $\nu = t_T$ commutes with the stabilizer of T, which it does not; $l(w) = 2$ would mean that $\nu = t_T t_U$, but this means that $\nu = t_{ab}t_{ac} = t_{ac}t_{ad} = t_{ad}t_{ab}$, and by multiplying these three relations together we obtain $\nu = 1$. So the minimum length for w is 3, and we have a relation $\nu = xyz$. First note that each of x, y and z commutes with ν, and so $xy = z\nu$ is of order 2; thus x and y commute with one another, and similarly all three elements x, y and z commute with one another. In particular the three elements x, y and z must be distinct if we are to avoid collapse. Suppose now that two of the tetrads are complementary within the octad $\{a, b, c, d\}$, so that without loss of generality we have $t_{ab}t_{cd}t_{ac} = \nu$. But an element such as

$\rho =$  $\in \mathrm{M}_{24}$

commutes with ν and acts as $(a)(b)(c\ d)$ on the pairs; conjugating our relation by this element we have $t_{ab}t_{cd}t_{ac} = t_{ab}t_{cd}t_{ad}$, and so $t_{ac} = t_{ad}$, leading to a contradiction. There are thus just two possibilities for a relation of length 3:

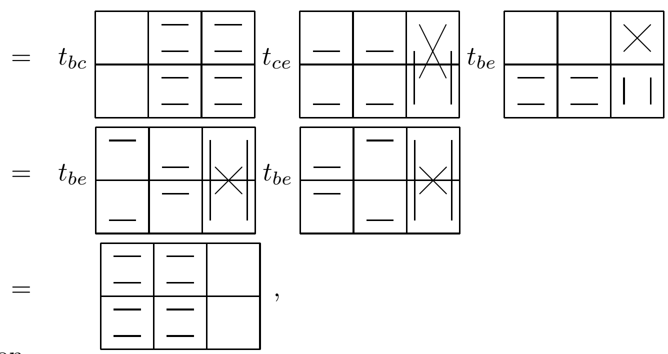
$$\text{(i)} \quad t_{ab}t_{ac}t_{bc} = \nu \text{ and } \quad \text{(ii)} \quad t_{ab}t_{ac}t_{ad} = \nu.$$

In order to see that case (i) fails we proceed as follows. The orbits of M_{24} on the subsets of Ω were calculated by Todd [15] and displayed in a convenient diagram by Conway [8]. The 8–element subsets fall into three orbits which are denoted by S_8, T_8 and U_8 and referred to as *special* (the octads themselves), *transverse*, and *umbral*. Umbral 8-element subsets fall uniquely into four duads in such a way that the removal of any one of the duads leaves a special hexad (that is, a hexad which is contained in an octad). An explicit example is given by the pairs $\{a, b, c, e\}$ below:



We work in the elementary abelian group of order 8 generated by $\{t_{ab}, t_{ac}, t_{ae}\}$. Then

$$1 = \quad t_{ab}^2 t_{ac}^2 t_{ae}^2 = t_{ab}t_{ac}.t_{ac}t_{ae}.t_{ae}t_{ab}$$



a contradiction.

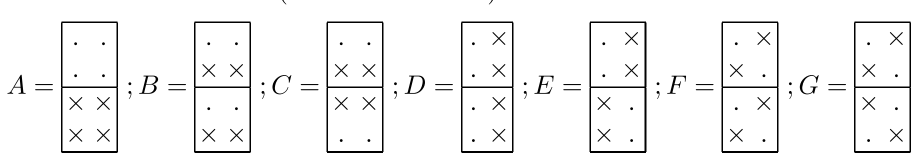
In the next section we shall explore the consequences of the relation in case (ii), namely that $t_{ab}t_{ac}t_{ad} = \nu$.

4. Investigation of the image of our progenitor

The rest of the paper will be devoted to an investigation of the group G defined by

$$G = \frac{2^{\binom{24}{4}} : \mathrm{M}_{24}}{t_{ab}t_{ac}t_{ad} = \nu},$$

where the duads a, b, c, d partition an octad, and $C_{\mathrm{M}_{24}}(\mathrm{Stab}_{\mathrm{M}_{24}}(a, b, c, d)) = \langle \nu \rangle$. From Lemma 2.2 we see that G is perfect. In order to proceed it will be useful to introduce names for certain frequently used tetrads. If T is a tetrad, then the associated symmetric generator will, of course, be denoted by t_T. Thus the following subsets of the first brick (first two columns) of the MOG are labelled as shown:



Mnemonic: Adjacent, Broken, Central, Descending, Echelon, Flagged, Gibbous.

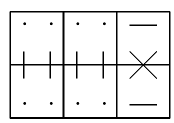
In fact the 7 sextets defined by these 7 tetrads are precisely the set of sextets which refine the MOG trio, the 6 tetrads in each case being the same pattern (and its complement) repeated in each brick of the MOG. If T is one such tetrad, then we denote by T' its complement in the octad shown, which we shall denote by O. Thus we should have $t_A = t_{cd}$ and $t_{A'} = t_{ab}$ in the notation of this section. Our relation then becomes $t_{A'}t_{B'}t_{C'} = \nu$. We have already seen that if T and U are two tetrads which intersect in 2 points and which lie together in the same octad, then t_T and t_U commute; thus, since t_A commutes with $t_{B'}, t_{C'}$ and ν, t_A commutes with $t_{A'}$. So two symmetric generators whose tetrads are disjoint and whose union is an octad also commute. In fact we have

Lemma 4.1. *The involution $t_At_{A'} = t_{A'}t_A = t_Xt_{X'}$ where X is any tetrad in the octad O; and so $t_At_{A'} = \epsilon_O$, say, an involution which commutes with the octad stabilizer of shape $2^4 : \mathrm{A}_8$.*

Proof. The element $t_At_{A'}$ certainly commutes with the subgroup H of the octad stabilizer which fixes a partition of the 8 points into two tetrads, namely a subgroup of M_{24} of shape $2^4 : (\mathrm{A}_4 \times \mathrm{A}_4) : 2$ which is maximal in the octad stabilizer $2^4 : \mathrm{A}_8$. But, using the relation twice, we have

$$t_At_{A'} = t_At_B^2t_{A'} = t_{C'}\ \nu\ t_C\ \nu = t_{C'}t_C,$$

which is fixed by

$\rho =$  .

But ρ does not preserve the partition of O into 2 tetrads, and so $\langle H, \rho\rangle \cong 2^4 : \mathrm{A}_8$. □

In this manner we obtain just 759 *octad-type* involutions of the form ϵ_O for $O \in \mathcal{C}_8$, the set of octads in the Golay code. We shall show that $E = \langle \epsilon_O \mid O \in \mathcal{C}_8\rangle \cong 2^{12}$, an elementary abelian group of order 2^{12}. Before proceeding let us suppose that O and O' are octads which intersect in 4 points. Then $O \setminus O' = T_1, O' \setminus O = T_2$ and $O \cap O' = T_3$ are 3 tetrads of a sextet. By the above we have

$$\epsilon_O\epsilon_{O'} = t_{T_1}t_{T_3}t_{T_2}t_{T_3} = t_{T_1}t_{T_2} = \epsilon_{O+O'},$$

where $O + O'$ denotes the symmetric difference of the octads O and O'. So in this respect ϵ_O and $\epsilon_{O'}$ behave like octads in the Golay code. When we have occasion to display a particular symmetric generator t_T in the MOG, we shall place $\times$ in the positions of the tetrad T. When we wish to display an element ϵ_O we shall insert $\circ$ in the positions of the octad O.

4.1. An elementary abelian subgroup of order 2^{12}.

We may readily convert this problem into the language of progenitors, for we have 759 involutory generators corresponding to the octads, subject to the above relation and permuted by M_{24}. We are attempting to show that

Lemma 4.2.

$$\frac{2^{\star 759} : \mathrm{M}_{24}}{\epsilon_O\epsilon_U\epsilon_{O+U} = 1} = 2^{12} : \mathrm{M}_{24},$$

where O and U are two octads which intersect in 4 points, and $O+U$ denotes their symmetric difference.

This is an ideal opportunity to illustrate the authors' double coset enumerator [2]. We need the action of M_{24} on 759 points, and may obtain this using MAGMA as follows:

```
> g:=Sym(24);
> m24:=sub<g|g!(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23),
> g!(1, 11)(3, 19)(4, 20)(5, 9)(6, 15)(13, 21)(14, 16)(23, 24)>;
> #m24;
244823040
> oct:=Stabilizer(m24,{24,23,3,6,9,19,15,5});
> f,nn,k:=CosetAction(m24,oct);
> Degree(nn);
759
```

We have input two permutations of degree 24 which generate M_{24} and then asked for the action of this group on the cosets of the stabilizer of an octad. We must now feed in our additional relation, and to do this need an octad which intersects the first one in 4 points. This means it must lie in the 280–orbit of the stabilizer of the first octad.

```
> oo:=Orbits(Stabilizer(nn,1));
> [#oo[i]:i in [1..#oo]];
[ 1, 30, 280, 448 ]
> r:=Random(oo[3]);
> r;
52
> Fix(Stabilizer(nn,[1,52]));
{ 1, 52, 367 }
```

The octad labelled 52 is a random member of the 280–orbit. The only other octad fixed by the stabilizer of octads 1 ands 52 is the one labelled 367, and so the relation must be that the product of these three octads is the identity.

```
> RR:=[<[1,52,367],Id(nn)>];
> HH:=[nn];
> CT:=DCEnum(nn,RR,HH:Print:=5, Grain:=100);

Index: 4096 === Rank: 5 === Edges: 13 === Time: 2.169

> CT[4];
[
    [],
    [ 1 ],
    [ 1, 2 ],
    [ 1, 32 ],
    [ 1, 32, 752 ]
]
> CT[7];
[ 1, 759, 2576, 759, 1 ]
>
```

This relation is fed into RR, and the Double Coset Enumerator tells us that there are five double cosets with the expected sizes. With the current labelling of octads they are N, $N\epsilon_1N$, $N\epsilon_1\epsilon_2N$, $N\epsilon_1\epsilon_{32}N$ and $N\epsilon_1\epsilon_{32}\epsilon_{752}N$. The subgroup clearly maps onto the binary Golay code $\mathcal{C}$, and, since it has order 2^{12}, it is isomorphic to $\mathcal{C}$.

The subgroup generated by the 759 involutory generators in the above progenitor is called a *universal representation group*. See for example Ivanov, Pasechnik and Shpectorov [13], for the point-line incidence system $\mathcal{G} = (P, L)$, where P is a set of points and L a set of lines with three points per line. Explicitly we define

$$R(\mathcal{G}) = \langle t_i, i \in P \mid t_i^2 = 1,\ t_i t_j t_k = 1 \text{ for } \{i, j, k\} \in L\rangle,$$

so we are requiring that the three points on a line correspond to the three non-trivial elements of a Klein fourgroup. It is shown in [13] that the universal representation group in our example, where the points are the octads and the lines are sets such as $\{O, U, O + U\}$, where O and U are octads which intersect in four points, is elementary abelian of order 2^{12}. This fact has been proved above by our double coset enumeration, but, since it is of the utmost importance in this paper, we choose to prove it again by hand.

Proof by hand. We are aiming to show that

$$E = \langle \epsilon_O \mid O \in \mathcal{C}_8\rangle \cong 2^{12}$$

is an elementary abelian group of order 2^{12}. We shall show that any two of the generators commute with one another and that the number of elements in the group they generate is at most 2^{12}. Since the octad type vectors in the Golay code certainly satisfy our additional relation, we shall have proved the result. The relation tells us that if two octads O and U intersect in 4 points, then ϵ_O and ϵ_U commute and have product ϵ_{O+U}, another generator.

Suppose now that $O, U \in \mathcal{C}_8$ with $O \cap U = \phi$. Then, if T_1, T_2, T_3, T_4 are tetrads of any one of the 7 sextets refining the trio $O : U : O + U + \Omega$ chosen so that $O = T_1 + T_2, U = T_3 + T_4$, we have

$$\epsilon_O \epsilon_U = t_{T_1} t_{T_2} t_{T_3} t_{T_4} = t_{T_1} t_{T_3} t_{T_2} t_{T_4} = \epsilon_{T_1+T_3} \epsilon_{T_2+T_4}.$$

There are $7 \times 2 \times 2 = 28$ choices for the octad $T_1 + T_3$, which together with O and U gives all 30 octads in the 16–ad $O + U$. So

$$\epsilon_O \epsilon_U = \epsilon_V \epsilon_W$$

for any $V, W \in \mathcal{C}_8$ with $O+U = V+W$, and we may write $\epsilon_O \epsilon_U = \epsilon_{O+U}$, an element which is only dependent on the 16-ad $O + U$. If $U : V : W$ is a trio, then $\epsilon_U \epsilon_V \epsilon_W$ commutes with the whole of the trio stabilizer, since the 3 elements commute with one another. But if $T_1, T_2, T_3, T_4, T_5, T_6$ is any one of the 7 refinements of this trio, then

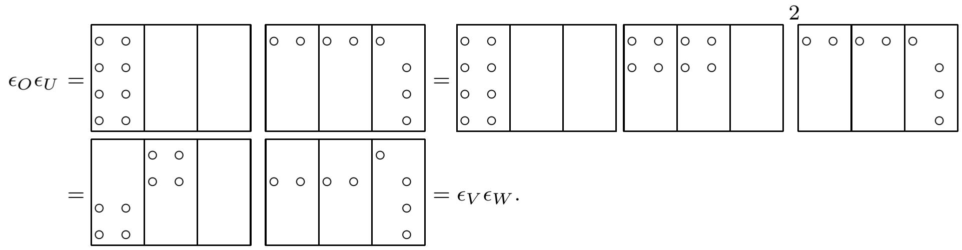
$$\epsilon_U \epsilon_V \epsilon_W = t_{T_1} t_{T_2} t_{T_3} t_{T_4} t_{T_5} t_{T_6},$$

which commutes with the whole of the sextet stabilizer, since these 6 symmetric generators commute with one another. Together these two subgroups generate the whole of M_{24}, and so

$$\epsilon_U \epsilon_V \epsilon_W = \epsilon_{U'} \epsilon_{V'} \epsilon_{W'} = \epsilon_\Omega,$$

say, for any trio $U' : V' : W'$. Clearly $\epsilon_\Omega \epsilon_O = \epsilon_{\Omega+O}$ for any $O \in \mathcal{C}_8$.

We must now consider the case $O, U \in \mathcal{C}_8$ with $|O \cap U| = 2$.



The element $\epsilon_O\epsilon_U$ clearly commutes with a subgroup of M_{24} isomorphic to the symmetric group S_6 preserving the partition of the 24 points into subsets of sizes 6+6+2+10 determined by $O-U, U-O, O\cap U, \Omega-(O\cup U)$. But the foregoing calculation shows that it also commutes with the corresponding copy of S_6 preserving $V - W, W-V, V\cap W, \Omega-(V\cup W)$. Together these two copies of S_6 generate a subgroup of M_{24} isomorphic to the Mathieu group M_{12} acting on $O + U = V + W$. So $\epsilon_O\epsilon_U$ depends only on the dodecad $O + U$, and we may write $\epsilon_O\epsilon_U = \epsilon_{O+U}$. In particular we see that $\epsilon_O\epsilon_U = \epsilon_U\epsilon_O$. We obtain just 2576 new elements of E in this manner. If $O \in \mathcal{C}_8$ and $C \in \mathcal{C}_{16}$, then if $O \cap C = \phi$ we have $\epsilon_O\epsilon_C = \epsilon_\Omega$ as above. Otherwise we have $|O \cap C| = 4$ or 6, and in either case we can find octads V and W so that $C = V \cup W$ and $|O \cap V| = 4$. Then $\epsilon_O\epsilon_C = \epsilon_O\epsilon_V\epsilon_W = \epsilon_{O+V}\epsilon_W$, a case we have already considered. It remains to show that if $O \in \mathcal{C}_8, C \in \mathcal{C}_{12}$, then $\epsilon_O\epsilon_C$ is in the set of elements already produced. Now $|O \cap C| = 6, 4$ or 2. In the first case $\epsilon_O\epsilon_C = \epsilon_{C+O}$, as we have already seen. In the second case we may choose V to be any octad containing the intersection $O \cap C$ and two further points of C; then $\epsilon_O\epsilon_C = \epsilon_O\epsilon_V\epsilon_W = \epsilon_{O+C}\epsilon_W$, which again we have already considered. To deal with the final case when $|O \cap C| = 2$ recall that any pair of points outside a dodecad determine a partition of the dodecad into two hexads such that either hexad together with the pair is an octad. Choose the pair to lie in $O \setminus C$. Then since octads must intersect one another evenly, one of the hexads contains $O \cap C$ and the other is disjoint from O. So we may write $\epsilon_C = \epsilon_V\epsilon_W$, where V and W are octads intersecting in our chosen pair, such that $|O \cap V| = 4$. Then $\epsilon_O\epsilon_C = \epsilon_O\epsilon_V\epsilon_W = \epsilon_{O+V}\epsilon_W$, which we have already considered. Thus our set consisting of the identity, 759 octad-type elements, 759 16–ad type elements, the element ϵ_Ω, and 2576 dodecad type elements is closed under multiplication, and so forms an elementary abelian group of order $4096 = 2^{12}$. □

Note that the element ϵ_Ω defined above commutes with M_{24} and with the symmetric generators t_T. It is thus central in the image group G, and in any faithful irreducible representation of G will be represented by $-I$, a scalar matrix with -1 down the diagonal.

5. Mechanical enumeration of the cosets of H in G

The subgroup $N \cong \mathrm{M}_{24}$ is too small for us to enumerate all double cosets of the form NwN. However, we have now constructed, by machine and by hand, a subgroup $H \cong 2^{12} : \mathrm{M}_{24}$, and the first author has modified our double coset enumerator to cope with enumeration of double cosets of the form HwN. Indeed it is particularly adept at performing this kind of enumeration if $N \leq H$, as is the case here. We must first obtain M_{24} as a permutation group of degree $\binom{24}{4} = 10626$ acting on the cosets of the stabilizer of a tetrad, in this case $\{24(=\infty), 3, 6, 9\}$. We now seek a second tetrad which intersects the first tetrad in 2 points and whose union with the first tetrad lies in an octad. Since there are just $\binom{4}{2} \times 5 \times \binom{4}{2} = 180$ such tetrads, we immediately see that we require the fifth orbit of the stabilizer of the first tetrad. We choose a random tetrad in this orbit and are given the tetrad labelled 8696.

```
> s24:=Sym(24);
> m24:=sub<s24|
s24!(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23),
s24!(24,23)(3,19)(6,15)(9,5)(11,1)(4,20)(16,14)(13,21)>;
> #m24;
244823040
> xx:={24,3,6,9};
> sxx:=Stabilizer(m24,xx);
> f,nn,k:=CosetAction(m24,sxx);
> st1:=Stabilizer(nn,1);
> oo:=Orbits(st1);
> [#oo[i]:i in [1..#oo]];
[ 1, 5, 80, 80, 180, 320, 320, 360, 640, 960, 960, 1920, 1920, 2880]
> r:=Random(oo[5]);r;
8696
```

Now the stabilizer of two such tetrads is a subgroup of the octad stabilizer $2^4 : A_8$ of shape $2^4 : 2^3$ with orbits on the 24 points of lengths $2+2+2+2+16$. This clearly fixes 6 tetrads, and we must find out which of the four possibilities completes the word of our relation. Consideration of the orders of the stabilizers of pairs of these six tetrads soon reduces the possibilities to two, one of which leads to collapse as in Section 3 above. It turns out that $[1, 8696, 4203]$ is the word in the symmetric generators we require, and so we set it equal to the unique non-trivial element in the centralizer of the stabilizer of these three tetrads. The union of the tetrads labelled 1 and 325 is an octad, and so the subgroup N together with the element $t_1 t_{325}$ (which is written here as $< [1, 325], Id(N) >$) generate $2^{12} : M_{24}$.

```
> Fix(Stabilizer(nn,[1,8696]));
{ 1, 325, 887, 4203, 5193, 8696 }
> cAB:=Centralizer(nn,Stabilizer(nn,[1,8696]));
> #cAB;
2
> RR:=[<[1,8696,4203],cAB.1>];
> HH:=[*nn,<[1,325],Id(nn)>*];
> CT1:=DCEnum(nn,RR,HH:Print:=5,Grain:=100);
Dealing with subgroup.
Pushing relations at subgroup.
Main part of enumeration.
Index: 8292375 === Rank: 19 === Edges: 1043 === Time: 505.07
> CT1[4];
[
    [],
    [ 1 ],
```

```
    [ 1, 2 ],
    [ 1, 2, 24 ],
    [ 1, 4 ],
    [ 1, 17 ],
    [ 1, 7 ],
    [ 1, 2, 61 ],
    [ 1, 2, 8 ],
    [ 1, 2, 17 ],
    [ 1, 2, 59 ],
    [ 1, 2, 14 ],
    [ 1, 2, 117 ],
    [ 1, 2, 204 ],
    [ 1, 2, 7 ],
    [ 1, 2, 1 ],
    [ 1, 2, 259 ],
    [ 1, 2, 1212 ],
    [ 1, 2, 17, 1642 ]
]
> CT1[7];
[1, 1771, 637560, 2040192, 26565, 637560, 21252, 2266880, 370944,
728640, 91080, 566720, 91080, 425040, 42504, 759, 340032, 1771, 2024]
```

The output shows us that there are 19 double cosets of the form HwN, where $H \cong 2^{12} : \mathrm{M}_{24}$ and $N \cong \mathrm{M}_{24}$; the index of H in G is 8292375 (note the misprint in the first edition of the ATLAS). Canonical double coset representatives are given, and we see that the graph obtained by joining a coset of H to those cosets obtained by multiplication by a symmetric generator has diameter 4.

Alternatively we may enumerate double cosets of the form KwN where $K \cong Co_2$, the stabilizer of a type 2 vector in the Leech lattice. Explicitly we have

$$K = \langle \mathrm{Stab}_N(\infty), t_T \mid \infty \in T \rangle.$$

Note that this is in a sense preferable, as the index is much smaller; however there is more work entailed, as we have not yet identified Co_2.

```
> m23:=Stabilizer(m24,24);
> #m23;
10200960
> HH:=[*f(m23),<[1],Id(nn)>*];
> CT:=DCEnum(nn,RR,HH:Print:=5,Grain:=100);
Dealing with subgroup.
Pushing relations at subgroup.
Main part of enumeration.
Index: 196560 === Rank: 16 === Edges: 178 === Time: 282.661

> CT[4];
[
    [],
    [ 3 ],
    [ 3, 2 ],
    [ 3, 2, 1197 ],
```

```
    [ 3, 26 ],
    [ 3, 5 ],
    [ 3, 2, 4 ],
    [ 3, 2, 1 ],
    [ 3, 53 ],
    [ 3, 2, 14 ],
    [ 3, 2, 90 ],
    [ 3, 2, 9188 ],
    [ 3, 2, 417 ],
    [ 3, 2, 2554 ],
    [ 3, 2, 540 ],
    [ 3, 2, 90, 340 ]
]
>CT[7];
[24, 6072, 53130, 30912, 21252, 12144, 30912, 21252, 759, 12144,
6072, 759,
276, 276, 552, 24]
```

In this case the index is 196560 (note that K does not contain ϵ_Ω, which negates a 2–vector) and the number of double cosets is 16.

6. A representation of the group G

The lowest dimension in which M_{24} can be represented faithfully as matrices over the complex numbers $\mathbb{C}$ is 23, and since G is perfect and the element ϵ_Ω is to be represented by $-I_n$ (which has determinant $(-1)^n$), any faithful representation of G must have even degree. So the lowest dimension in which we could represent G faithfully is 24. Certainly $E = 2^{12} : \mathrm{M}_{24}$ acts monomially in this dimension, with M_{24} acting as permutations and the elements ϵ_C for $C \in \mathcal{C}$ acting as sign changes on the $\mathcal{C}$–set C. Let ρ denote such a representation of G. We are led to seeking elements $\rho(t_T)$, for T a tetrad of the 24 points, which

(1) commute with the tetrad stabilizer of M_{24},
(2) commute with elements ϵ_O, where O is an octad which is the union of 2 tetrads in the sextet defined by T,
(3) have order 2, and
(4) satisfy the additional relation.

Condition (1) above requires the matrix representing $\rho(t_T)$ to have form

$$\rho(t_T) = \left(\begin{array}{c|c|c|c|c|c} aI+bJ & eJ & eJ & eJ & eJ & eJ \\ fJ & cI+dJ & gJ & gJ & gJ & gJ \\ fJ & gJ & cI+dJ & gJ & gJ & gJ \\ fJ & gJ & gJ & cI+dJ & gJ & gJ \\ fJ & gJ & gJ & gJ & cI+dJ & gJ \\ fJ & gJ & gJ & gJ & gJ & cI+dJ \end{array}\right),$$

where the 24×24 matrix has been partitioned into blocks corresponding to the sextet defined by the tetrad (which itself corresponds to the block in the first row and the first column); I denotes the 4×4 identity matrix and J denotes the 4×4 all 1's matrix. But this must commute with sign changes on the first two tetrads; that is to say that negating the first two columns must have the same effect as negating the first two rows. Thus $e = f = g = 0$, and the matrix takes the form

$$\rho(t_T) = \mathrm{diag}(aI + bJ, (cI + dJ)^5).$$

Now condition (3) requires

$$(aI + bJ)^2 = (cI + dJ)^2 = I_4 = a^2 I + (2ab + 4b^2)J = c^2 I + (2cd + 4d^2)J.$$

So we have $a = \pm 1$ and either $b = 0$ or $a = -2b$, and similarly $c = \pm 1$ and $d = 0$ or $c = -2d$. But if T_1, T_2 are two tetrads of the same sextet, then $t_{T_1} t_{T_2} = \epsilon_{T_1 + T_2}$, and so

$$(aI + bJ)(cI + dJ) = acI + (ad + bc + 4bd)J = -I.$$

So $ac = -1$, and if either of b or d is 0, then so is the other. Both being 0 would mean that $\rho(t_T)$ is a diagonal matrix, and so the additional relation of condition (4) could not possibly hold. So neither is 0, and $a = \pm 1, c = -a, b = -a/2, d = a/2$, and it remains to determine the sign. In order to do this we first restrict our attention to the action on the octad which is the union of the three tetrads in the additional relation, and we order the rows and columns to correspond to the pairings a, b, c, d. We let

$$L = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

so that $L^2 = -2L, J^2 = 2J, LJ = JL = 0$. Our relation requires

$$\pm\frac{1}{8}\left(\begin{array}{cc|cc} L & J & & \\ J & L & & \\ \hline & & -L & -J \\ & & -J & -L \end{array}\right)\left(\begin{array}{cc|cc} L & & J & \\ & -L & & -J \\ \hline J & & L & \\ & -J & & -L \end{array}\right)\left(\begin{array}{cc|cc} L & & & J \\ & -L & -J & \\ \hline & -J & -L & \\ J & & & L \end{array}\right)$$

$$\pm\frac{1}{8}\left(\begin{array}{cc|cc} L & J & & \\ J & L & & \\ \hline & & -L & -J \\ & & -J & -L \end{array}\right)\left(\begin{array}{cc|cc} -2L & -2J & & \\ -2J & -2L & & \\ \hline & & 2L & 2J \\ & & 2J & 2L \end{array}\right) = \pm\frac{1}{8}\left(\begin{array}{cc|cc} -8I & & & \\ & -8I & & \\ \hline & & -8I & \\ & & & -8I \end{array}\right) = I_8,$$

and so we require the negative sign. To complete the verification that the additional relation holds and that this product does indeed produce the required permutation of M_{24}, we restrict our attention to the second brick of the MOG when we obtain

$$\frac{1}{8}\left(\begin{array}{cc|cc} L & J & & \\ J & L & & \\ \hline & & L & J \\ & & J & L \end{array}\right)\left(\begin{array}{cc|cc} L & & J & \\ & L & & J \\ \hline J & & L & \\ & J & & L \end{array}\right)\left(\begin{array}{cc|cc} L & & & J \\ & L & J & \\ \hline & J & L & \\ J & & & L \end{array}\right)$$

$$= \frac{1}{8}\left(\begin{array}{cc|cc} L & J & & \\ J & L & & \\ \hline & & L & J \\ & & J & L \end{array}\right)\left(\begin{array}{cc|cc} -2L & 2J & & \\ 2J & -2L & & \\ \hline & & -2L & 2J \\ & & 2J & -2L \end{array}\right)$$

$$= \frac{1}{2}\begin{pmatrix} L+J & & & \\ & L+J & & \\ & & L+J & \\ & & & L+J \end{pmatrix} = \left(\begin{array}{cc|cc|cc|cc} & 1 & & & & & & \\ 1 & & & & & & & \\ \hline & & & 1 & & & & \\ & & 1 & & & & & \\ \hline & & & & & 1 & & \\ & & & & 1 & & & \\ \hline & & & & & & & 1 \\ & & & & & & 1 & \end{array}\right),$$

as required. Note that the element t_T is in fact $-\xi_T$, where ξ_T is the element produced by Conway [8, page 237] to show that the Leech lattice is preserved by more than the monomial group $H = 2^{12} : \mathrm{M}_{24}$. Observe moreover that $\rho(t_T)$ is an orthogonal matrix, and so the group $\rho(G)$ preserves lengths of vectors and angles between them. In the notation of [8] we let $\{v_i \mid i \in \Omega\}$ be an orthonormal basis for a 24–dimensional space over $\mathbb{R}$, and for $X \subset \Omega$ we let v_X denote

$$v_X = \sum_{i \in X} v_i.$$

For T a tetrad of points in Ω, we let $\{T = T_0, T_1, \ldots, T_5\}$ be the sextet defined by T. Then the element t_T acts as

$$t_T = -\xi_T : v_i \mapsto \begin{cases} v_i - \frac{1}{2}v_T & \text{for } i \in T = T_0, \\ \frac{1}{2}v_{T_i} - v_i & \text{for } i \in T_i, i \neq 0, \end{cases}$$

so, as described in [8], t_T is best applied to a vector in $\mathbb{R}^{24}$ as follows:

> *for each tetrad T_i work out one half the sum of the entries in T_i and subtract it from each of the four entries; then negate on every entry except those in $T = T_0$.*

7. The Leech lattice Λ

In order to obtain the Leech lattice Λ we simply apply the group we have constructed to the basis vectors and consider the $\mathbb{Z}$–lattice spanned by the set of images. More specifically, in order to avoid fractions, we normalise by applying the group to the vectors $8v_i$. Let Λ denote this lattice. If T denotes the first column of the MOG we have

$$t_T = \begin{array}{|cc|cc|cc|} \hline \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} : \quad \begin{array}{|cc|cc|cc|} \hline 8 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \mapsto \begin{array}{|cc|cc|cc|} \hline 4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline -4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}.$$

So, under the permutations of the quintuply transitive M_{24} and the sign changes of E, every vector of the shape $(\pm 4)^4.0^{20}$ is in Λ. In particular, we have $(4, 4, 4, 4, 0, 0^{19}) + (0, -4, -4, -4, -4, 0^{19}) = (4, 0^3, -4, 0^{19}) \in \Lambda$, and so every vector of the form $((\pm 4)^2.0^{22})$ is in Λ. Moreover we see that

$$t_T : \begin{array}{|cc|cc|cc|} \hline 4 & 4 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \mapsto \begin{array}{|cc|cc|cc|} \hline 2 & -2 & \cdot & \cdot & \cdot & \cdot \\ -2 & 2 & \cdot & \cdot & \cdot & \cdot \\ \hline -2 & 2 & \cdot & \cdot & \cdot & \cdot \\ -2 & 2 & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array};$$

$$\begin{array}{|cc|cc|cc|} \hline 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \mapsto \begin{array}{|cc|cc|cc|} \hline -3 & -1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 & 1 & 1 \\ \hline -1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}.$$

The first image shows that Λ contains every vector of the form $((\pm 2)^8.0^{16})$, where the non-zero entries are in the positions of an octad of the Steiner system preserved by our copy of M_{24} and the number of minus signs is even (since $\mathcal{C}$–sets intersect one another evenly). The second shows that every vector of the form $(-3, 1^{23})$ followed by a sign change on a $\mathcal{C}$–set is also in Λ. We may readily check that this set of vectors, which have normalised length 2×16, is closed under the action of t_T and hence of G; it is normally denoted by Λ_2 and it consists of

shape	calculation	number
$(4^2.0^{22})$	$\binom{24}{2}.2^2$	1,104
$(2^8.0^{16})$	759.2^7	97,152
(-3.1^{23})	24.2^{12}	98,304
	Total	196,560

Clearly G acts as a permutation group on the $196560/2 = 98280$ pairs consisting of a type 2 vector and its negative. The stabiliser of such a pair has just three non-trivial orbits on the other pairs, where the orbit in which a particular pair lies depends only on the angles its vectors make with the fixed vectors. The permutation character of this action is $\chi_1 + \chi_3 + \chi_6 + \chi_{10}$, of degrees 1, 299, 17250, 80730 respectively, as listed in the Atlas [3].

In [6, Theorem 5] Conway gives a beautifully simple characterisation of the vectors of Λ, normalised as above:

Theorem 1 (Conway)**.** *The integral vector $x = (x_1, x_2, \ldots, x_{24})$ is in Λ if, and only if,*

(i) *the x_i all have the same parity,*
(ii) *the set of i where x_i takes any given value (modulo 4) is a $\mathcal{C}$–set, and*
(iii) $\sum x_i \equiv 0$ *or* 4 *(modulo 8) according to whether* $x_i \equiv 0$ *or* 1 *(modulo 2).*

It is readily checked that the above list of 2–vectors contains all vectors of (normalised) length 32 having these properties, and it is clear that these properties are enjoyed by all integral combinations of them.

8. A presentation of the Conway group ·O

It is of interest to deduce an ordinary presentation of ·O from our symmetric presentation. We first need a presentation for the control subgroup M_{24} and choose one based on that given by Curtis [11, page 390], which defines M_{24} as an image of the progenitor

$$2^{\star 7} : \mathrm{L}_3(2).$$

Consider first

$$\langle x, y, t \mid x^7 = t^2 = y^2 = (xy)^3 = [x, y]^4 = [y^x, t] = 1, y = [t, x^2]^3 \rangle.$$

We see that $L = \langle x, y \rangle \cong \mathrm{L}_3(2)$ or the trivial group. If it is the former, which must be the case since we can find permutations of 24 letters satisfying all these relations, then without loss of generality we may calculate within L by letting $x \sim (0\ 1\ 2\ 3\ 4\ 5\ 6), y \sim (3\ 4)(5\ 1)$. Moreover $\langle t, t^{x^2} \rangle \cong \mathrm{D}_{12}$, or an image thereof, and centralises y. Thus t commutes with

$$\langle y, y^{x^{-2}}, y^x \rangle = \langle (3\ 4)(5\ 1), (1\ 2)(3\ 6), (4\ 5)(6\ 2) \rangle \cong \mathrm{S}_4,$$

and so $|t^L| = 7$. Labelling $t^{x^i} = t_i$ for $i = 0, 1, \ldots, 6$, we see that elements of L must permute the t_i by conjugation, just as the above permutations representing x and y permute their subscripts. Thus $\langle x, y, t \rangle = \langle x, t \rangle$ is a homomorphic image of the progenitor $2^{\star 7} : \mathrm{L}_3(2)$. Following Curtis, Hammas and Bray [12, Table 8, page 33] we factor this by the additional relators

$$\begin{array}{ccc} (yt^{x^{-1}}t^x)^4 & \sim & ((3\ 4)(5\ 1)t_6t_1)^4\,, \\ (yxt)^{11} & \sim & ((0\ 1\ 6)(2\ 3\ 5)t_0)^{11}\,. \end{array}$$

To verify the claim made in [12] that this is a presentation for M_{24}, we observe that restricting the given relations to the four symmetric generators $\{t_0, t_3, t_6, t_5\}$ yields $(0\ 3) = (t_5t_6)^3$, and $((0\ 3\ 6)t_0)^{11} = 1$. Thus we have the symmetric presentation

$$J = \frac{2^{\star 4} : \mathrm{S}_4}{(3\ 4) = (s_1s_2)^3, ((1\ 2\ 3)s_1)^{11}},$$

which is equivalent to the presentation

$$\langle u, v, s \mid u^4 = v^2 = (uv)^3 = s^2 = (uvs)^{11} = 1, v = [s, u]^3\rangle.$$

Either the symmetric or the ordinary presentation is readily shown to define the projective special linear group $\mathrm{L}_2(23)$.

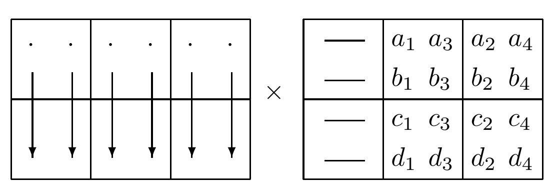
We have now identified a sufficiently large subgroup and can perform a coset enumeration of

$$\begin{array}{rl} M = \langle x, t \mid & x^7 = t^2 = y^2 = (xy)^3 = [x, y]^4 = [y^x, t] = 1, y = [t, x^2]^3, \\ & (yt^{x^{-1}}t^x)^4 = (yxt)^{11} = 1\rangle \end{array}$$

over the subgroup $\langle t, t^{x^3}, t^{x^6}, t^{x^5}\rangle \cong \mathrm{L}_2(23)$ to obtain index $40,320$, and so $|M| = 244,823,040$. In fact the subgroup $\langle x, ytxy\rangle \cong \mathrm{M}_{23}$, and so the group M can be obtained by acting on 24 letters.

In order to extend this to a presentation of $\cdot\mathrm{O}$ we must adjoin a generator s, say, which commutes with the stabilizer in M of a tetrad of the 24 points, and we must then factor out the additional relation. The clue is to consider an element of M_{24} of order 12 and cycle shape 2.6.4.12. Let σ be such an element; then σ^3 has shape $2.2^3.4.4^3$, written so as to reveal the cycles of σ. Now the 2-orbit of σ together with each of the other 2-orbits of σ^3 in turn may be taken to be the three tetrads in our additional relation, when the element ν is simply σ^6. So our additional relation may be taken to be $(\sigma^2 s)^3 = 1$, where s is the symmetric generator corresponding to any one of these three tetrads. The above will be clarified by exhibiting such an element in the MOG diagram. Thus

$\sigma =$



is an element of the required shape. It is expressed as the product of an element of order three which fixes the columns and the top row whilst rotating the other three rows downwards, and an element of order four which fixes the rows and acts on the six columns as the permutation $(1\ 2)(3\ 5\ 4\ 6)$. The three tetrads are those labelled A', B' and C' in Section 4 and the element $\sigma^6 = \nu$ as in Section 3. We find that the element $\sigma = xyt^x$ is in this conjugacy class.

We must now seek elements in M_{24} which generate the stabilizer of a tetrad which contains the 2-cycle of σ and is fixed by σ^3. Noting that any element of

M_{24} can be written in the form πw, where $\pi \in L_3(2)$ and w is a word in the seven symmetric generators, we find computationally that

$$\begin{aligned} u_1 = &\quad (1\ 6\ 5)(2\ 4\ 3)t_5t_1t_2t_0t_6t_5 &= x^3yx^3t^{x^{-2}}t^xt^{x^2}tt^{x^{-1}}t^{x^{-2}}, \\ u_2 = &\quad (2\ 6)(4\ 5)t_4t_1t_0t_1t_5 &= (y^xt)^{t^xt^{x^{-2}}} \end{aligned}$$

will suffice, so we adjoin s such that $s^2 = [s, u_1] = [s, u_2] = 1$ and have now defined the progenitor

$$2^{\star\binom{24}{4}} : M_{24}.$$

It remains to factor this by the additional relation which now takes the simple form $(\sigma^2 s)^3 = 1$. Our double coset enumeration has proved that

$$\begin{aligned} \cdot O = \langle x, y, t, s \mid \quad & x^7 = t^2 = y^2 = (xy)^3 = [x, y]^4 = [y^x, t] = 1, y = [t, x^2]^3, \\ & (yt^{x^{-1}}t^x)^4 = (yxt)^{11} = s^2 = [s, x^3yx^3t^{x^{-2}}t^xt^{x^2}tt^{x^{-1}}t^{x^{-2}}] \\ & = [s, (y^xt)^{t^xt^{x^{-2}}}] = ((xyt^x)^2s)^3 = 1\rangle. \end{aligned}$$

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