

## THE SECOND CLOSED GEODESIC ON FINSLER SPHERES OF DIMENSION $n > 2$

HANS-BERT RADEMACHER

ABSTRACT. We show the existence of at least two geometrically distinct closed geodesics on an  $n$ -dimensional sphere with a bumpy and non-reversible Finsler metric for  $n > 2$ .

### 1. INTRODUCTION

Estimates for the number of closed extremals for a one-dimensional variational problem on  $n$ -spheres have been studied extensively in the calculus of variations. From the geometric point of view one is looking for lower bounds for the number of geometrically distinct closed geodesics on  $n$ -dimensional spheres  $S^n$  carrying a Finsler metric; cf. [An75]. In [LF51] Lyusternik and Fet prove the existence of one closed geodesic for a Finsler metric on a compact and simply connected manifold based on ideas of Birkhoff. For further existence results a fundamental difference between Riemannian metrics, resp. reversible Finsler metrics, and non-reversible Finsler metrics occurs. On one hand there exist non-reversible Finsler metrics on  $S^{2n}$ , resp.  $S^{2n-1}$ , with only  $2n$  geometrically distinct closed geodesics. These metrics occur in the work of Katok, and their geometry is studied by Ziller in [Zi82]. On the other hand there is no example of a reversible Finsler metric, resp. a Riemannian metric, on  $S^n$  with only finitely many closed geodesics. In fact on the two-sphere there are infinitely many geometrically distinct closed geodesics for every Riemannian metric due to the work of Bangert [Ba93] and Franks [Fr92], as well as Hingston [Hi93].

In this paper we consider the following non-degeneracy assumption: We call a Finsler metric *bumpy* if all closed geodesics are non-degenerate, i.e., if there are no non-trivial periodic Jacobi fields orthogonal to the closed geodesic. For a bumpy metric the energy functional on the space of free loops is a Morse function with only non-degenerate critical  $S^1$ -orbits. This is a generic assumption, and the above-mentioned Katok examples with only finitely many closed geodesics are bumpy.

The author showed in [Ra89, ch. 4] that a bumpy Finsler metric on  $S^2$  has at least two geometrically distinct closed geodesics. Instead of the non-degeneracy assumption one can prove the existence of two closed geodesics on  $S^2$ , resp. several geodesics on  $S^n$ , with length estimates also for an open set of Finsler metrics defined, for example, in terms of the flag curvature and the reversibility, cf. [Ra07]. In their recent work [BL07] Bangert and Long prove that for *every* non-reversible Finsler metric on  $S^2$  there are two closed geodesics. From results by Hofer, Wysocki,

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and Zehnder in [HWZ03] one obtains the following statement: A bumpy metric on  $S^2$  for which the stable and unstable manifolds of all closed geodesics intersect transversally carries either two or infinitely many geometrically distinct closed geodesics.

For the  $n$ -sphere of dimension  $n \geq 3$  there are only results for generic metrics or metrics with curvature restrictions. Fet proves in [Fe65] that every bumpy and reversible Finsler metric on a compact manifold carries at least two geometrically distinct closed geodesics. For results on the existence of infinitely many closed geodesics for generic metrics on spheres, resp. compact and simply connected manifolds, we refer to [Hi84], [Ra94], and [Zi82, p. 141]. For further details and references we refer to the surveys [Ba85], [Ta92], and [Lo06].

In this paper we prove for bumpy metrics the following extension of the result by Bangert and Long to all dimensions:

**Theorem.** *Let a compact and simply connected manifold of the rational homotopy type of an  $n$ -dimensional sphere  $S^n$ ,  $n \geq 3$ , carry a bumpy and non-reversible Finsler metric. Then there are at least two geometrically distinct closed geodesics.*

The main ingredients of the proof are the relation between the average indices of closed geodesics for metrics with only finitely many closed geodesics shown in [Ra89, Thm. 3.1] and a detailed analysis of the sequence of Morse indices  $\text{ind}(c^m)$  of the coverings  $c^m$  of a prime closed geodesic  $c$  using a formula due to Bott [Bo56] as well as a careful discussion of the Morse inequalities. For  $n \geq 4$  we use in addition the following indirect argument: If there is only one geometric closed geodesic represented by the prime closed geodesic  $c$ , then it is crucial that we are able to show in Proposition 2 that the sequence  $\text{ind}(c^m)$ ,  $m \geq 1$ , of Morse indices is monotone increasing. Using the Morse inequalities, we conclude that the difference  $\text{ind}(c^{m+1}) - \text{ind}(c^m) \leq 2$  for all  $m \geq 1$ . But since  $\text{ind}(c) = n - 1$ , the *common index jump theorem* due to Long and Zhu (cf. [LZ02, Thm. 4.3]) implies that there are integers  $k$  with  $\text{ind}(c^{2k+1}) - \text{ind}(c^{2k-1}) = 2 \text{ind}(c) = 2n - 2 \geq 6$ .

*Note added in proof:* The same result with a different proof was obtained independently in H. Duan and Y. Long: *Multiple closed geodesics on bumpy Finsler spheres*. J. Diff. Eq. **233** (2007), 221–240.

## 2. CRITICAL POINT THEORY

Closed geodesics on a compact manifold with a Finsler metric  $F$  can be characterized as the critical points of the energy functional

$$E : \Lambda M \rightarrow \mathbb{R}, \quad E(\gamma) = \frac{1}{2} \int_0^1 F^2(\gamma'(t)) dt.$$

Here  $\Lambda M$  is the *free loop space* consisting of closed  $H^1$ -curves  $\gamma : S^1 := [0, 1]/\{0, 1\} \rightarrow M$  on the manifold  $M$ . For the case of a Riemannian manifold cf. [Kl78] for the case of a Finsler metric cf. [Ra92, ch. 5], [Ra04, ch. 2]. References for facts from Finsler geometry are [BCS], [CS05]. On  $\Lambda M$  there is an  $S^1$ -action  $(u, \gamma) \in S^1 \times \Lambda M \mapsto u \cdot \gamma \in \Lambda M$ ,  $u \cdot \gamma(t) = \gamma(t+u)$ ,  $t \in S^1$ , leaving the energy functional invariant. In addition there is the mapping  $^m : \gamma \in \Lambda M \mapsto \gamma^m \in \Lambda M$ ,  $\gamma^m(t) = \gamma(mt)$ ,  $t \in S^1$ , and  $E(\gamma^m) = m^2 E(\gamma)$ . Here  $\gamma^m$  is the  $m$ -fold cover of  $\gamma$ . A closed geodesic  $c$  is called *prime* if there is no closed geodesic  $c_1$  and no integer  $m > 1$  with  $c = c_1^m$ .

We call a Finsler metric  $F$  *reversible* if for all tangent vectors  $X$  we have  $F(-X) = F(X)$ . Otherwise we call the metric *non-reversible*. We call two closed

geodesics  $c_1, c_2 : S^1 \rightarrow M$  of a non-reversible Finsler metric on a differentiable manifold  $M$  *geometrically equivalent* if their traces  $c_1(S^1) = c_2(S^1)$  coincide and if their orientations coincide. The equivalence class is also called a *geometric closed geodesic*. For a closed geodesic  $c_1$  there is a prime closed geodesic  $c$  such that the set of all geometrically equivalent closed geodesics consists of  $u.c^m, m \geq 1, u \in S^1$ . Let  $\theta : \Lambda M \rightarrow \Lambda M$  be orientation reversing; i.e.,  $(\theta c)(t) = c(1 - t)$ . For a non-reversible Finsler metric this mapping in general does not leave the energy functional invariant, and in general for a closed geodesic  $c$  the curve  $\theta c$  is not a geodesic.

The second order behaviour of the energy functional in a neighborhood of a closed geodesic is determined by its *index form*  $\mathcal{H}_c$  which equals the hessian  $d^2E(c)$  of the energy functional by the second variational formula; cf. [Ra04, ch. 2]. The *index*  $\text{ind}(c)$  of the closed geodesic  $c$  is the index of the index form  $\mathcal{H}_c$ , i.e., it is the maximal dimension of a subspace on which  $\mathcal{H}_c$  is negative definite. The nullity  $\text{null}(c)$  is the nullity of the index form  $\mathcal{H}_c$  minus 1. This convention is used since due to the  $S^1$ -action the nullity of the index form  $\mathcal{H}_c$  is at least 1. We call a Finsler metric *bumpy* if all closed geodesics are *non-degenerate*; i.e., for all closed geodesics  $c$  the nullity  $\text{null}(c) = 0$  vanishes.

For a closed geodesic  $c$  let  $\Lambda(c) := \{\gamma \in \Lambda M \mid E(\gamma) < E(c)\}$ . We call

$$\overline{C}_*(c) := H_*((\Lambda(c) \cup S^1.c) / S^1, \Lambda(c) / S^1; \mathbb{Q})$$

the  $S^1$ -critical group of  $c$ ; cf. [Ra92, ch. 6.3]. Here  $H_*$  is the singular homology; as coefficient field we use in this paper the rationals  $\mathbb{Q}$ . Let  $w_k(c) := \dim \overline{C}_k(c)$ .

For the Morse theory of the energy functional it is important to describe the  $S^1$ -critical groups  $\overline{C}_*(c^m)$  of the infinitely many critical points  $S^1.c^m, m \geq 1$ , produced by a single prime closed geodesic. Therefore the sequence  $\text{ind}(c^m), m \geq 1$ , of the sequence of iterates  $c^m$  is of great importance.

**Definition 1** (Cf. [Ra89, Def. 1.6]). For a closed geodesic  $c$  let  $\gamma_c \in \{\pm 1/2, \pm 1\}$  be the invariant defined by  $\gamma_c > 0$  if and only if  $\text{ind}(c)$  is even and  $|\gamma_c| = 1$  if and only if  $\text{ind}(c^2) - \text{ind}(c)$  is even.

Then we obtain the following crucial

**Lemma 1** (Cf. [Ra89, Prop. 2.2]). *Let  $c$  be a prime and non-degenerate closed geodesic,  $m \geq 1$ . Then for all  $k \geq 0$ ,*

$$w_k(c^m) = \begin{cases} 1, & k = \text{ind}(c^m), m \text{ even or } \gamma_c = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from Bott's iteration formula [Bo56, Thms. A, C], resp. equation (9), that the sequence  $\text{ind}(c^m), m \geq 1$ , grows almost linearly. Therefore the *average index*

$$\alpha_c := \lim_{m \rightarrow \infty} \frac{\text{ind}(c^m)}{m}$$

is well defined and one can show that

$$(1) \quad |\text{ind}(c^m) - m\alpha_c| \leq n - 1$$

for all  $m \geq 1$ , where  $n$  is the dimension of the manifold; cf. [Ra89, (1.4)].

Now we consider bumpy metrics with only a single geometric closed geodesic. Hence there is a prime closed geodesic  $c$  such that any closed geodesic on  $M$  is of

the form  $u.c^m$  for some  $u \in S^1$  and  $m \geq 1$ . Then we define for every  $k \geq 0$ ,

$$(2) \quad w_k := \sum_{m=1}^{\infty} w_k(c^m).$$

This number gives the number of critical orbits  $S^1.c^m$  whose  $S^1$ -critical group is non-trivial in dimension  $k$ . The Morse inequalities relate these local invariants of the critical points to the Betti numbers

$$b_k := b_k(\Lambda M/S^1, \Lambda^0 M/S^1; \mathbb{Q})$$

of the pair  $(\Lambda M/S^1; \Lambda^0 M/S^1)$  of the quotient space  $\Lambda M/S^1$  of the free loop space  $\Lambda M$  divided by the  $S^1$ -action and the space  $\Lambda M^0 := \{c \in \Lambda M; E(c) = 0\}$  of point curves. This space can be identified with the manifold  $M$ ; it is the fixed point set of the  $S^1$ -action. In particular we will use the following Betti numbers of the quotient space  $\Lambda S^n/S^1$ :

**Proposition 1** (Cf. [Ra89, Thm. 2.4]). *If  $M$  is a simply connected compact manifold rationally homotopy equivalent to the  $n$ -dimensional sphere  $S^n$ , then the Betti numbers  $b_k := b_k(\Lambda M/S^1, \Lambda^0 M/S^1; \mathbb{Q})$  of the quotient  $\Lambda M/S^1$  of the free loop space by the  $S^1$ -action satisfy:*

- (1)  $b_k \in \{0, 1, 2\}$ , and  $b_k \geq 1$  if and only if  $k \equiv n - 1 \pmod{2}$  and  $k \geq n - 1$ .
- (2) If  $n \equiv 0 \pmod{2}$ , then  $b_k = 2$  if and only if  $k = (2j + 1)(n - 1)$ ,  $j \geq 1$ .
- (3) If  $n \equiv 1 \pmod{2}$ , then  $b_k = 2$  if and only if  $k = j(n - 1)$ ,  $j \geq 2$ .

This proposition follows from the form of the Poincaré polynomials  $P(t) = \sum_k b_k t^k$  determined in [Ra89, (2.4),(2.5)]:

$$P(\Lambda S^n/S^1, \Lambda^0 S^n; \mathbb{Q})(t) = \begin{cases} t^{n-1} \left\{ \frac{1}{1-t^2} + \frac{t^{2n-2}}{1-t^{2n-2}} \right\}, & n \equiv 0 \pmod{2}, \\ t^{n-1} \left\{ \frac{1}{1-t^2} + \frac{t^{n-1}}{1-t^{n-1}} \right\}, & n \equiv 1 \pmod{2}. \end{cases}$$

It follows that  $B(n, 1) := \lim_{N \rightarrow \infty} \sum_{j=0}^N (-1)^j b_j$  satisfies

$$(3) \quad B(n, 1) = \begin{cases} -\frac{n}{2n-2}, & n \equiv 0 \pmod{2}, \\ \frac{n+1}{2n-2}, & n \equiv 1 \pmod{2}. \end{cases}$$

Then we collect the following conclusions from the results in [Ra89, (2.3),(2.6), Thm. 3.1(a)], resp. [Ra92]:

**Proposition 2.** *Let  $F$  be a bumpy Finsler metric on a compact and simply connected manifold of the rational homotopy type of an  $n$ -sphere with only one geometric closed geodesic represented by the prime closed geodesic  $c$ .*

(a) *The average index  $\alpha_c$  and the invariant  $\gamma_c$  satisfy*

$$(4) \quad (-1)^{n-1} \frac{\alpha_c}{\gamma_c} = \begin{cases} 2 - \frac{2}{n}, & n \equiv 0 \pmod{2}, \\ 2 - \frac{4}{n+1}, & n \equiv 1 \pmod{2}. \end{cases}$$

(b) *The numbers  $w_k$ ,  $k \geq 0$ , of critical  $S^1$ -orbits whose  $S^1$ -critical group is non-trivial in dimension  $k$  (cf. equation (2)) are bounded and there is a*

sequence  $(q_k)_{k \geq 0}$  of non-negative integers satisfying

$$(5) \quad w_k = b_k + q_k + q_{k-1}$$

for all  $k \geq 0$ .

### 3. BOTT'S ITERATION FORMULA AND THE INDEX GROWTH

For a closed geodesic  $c : \mathbb{R} \rightarrow M$  with  $c(t + 1) = c(t)$  for all  $t \in \mathbb{R}$  we define the *linearized Poincaré mapping*  $P_c$  : For  $p = c(0)$  let  $V$  be the  $(n - 1)$ -dimensional orthogonal complement in the tangent space  $T_p M$  to the 1-dimensional subspace generated by  $c'(0)$  (with respect to the osculating Riemannian metric defined by the velocity field  $c'$ ). Then let

$$P_c : V \oplus V \rightarrow V \oplus V, \quad P_c(X, Y) = \left( J(1), \frac{\nabla}{dt} J(1) \right)$$

where  $J$  is the Jacobi field determined uniquely by the initial conditions  $J(0) = X, \frac{\nabla}{dt} J(0) = Y$  along  $c|_{[0, 1]}$ . Here  $\frac{\nabla}{dt}$  is the covariant derivative along  $c$ . The canonical symplectic structure on  $V \oplus V$  is preserved by  $P_c$ ; hence by choosing an orthonormal basis in  $V$ , we consider  $P_c$  as an element of the group  $\text{Sp}(n - 1)$  of linear symplectic maps of  $\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}$ , which is well defined up to conjugation, i.e., independent of the choice of  $p = c(0)$  and the choice of an orthonormal basis of  $V$ .

We denote by  $\text{Spec}(P) \subset \mathbb{C}$  the set of eigenvalues of the complexification  $P$  of  $P_c$  and denote by  $V(z) = \ker(P - z Id)^{n-1}$  the generalized eigenspace for  $z \in \text{Spec}(P)$ . Then

$$(6) \quad N_c(z) := \dim_{\mathbb{C}}(P_c - z Id) \leq \dim_{\mathbb{C}} V(z)$$

is called the  $z$ -nullity of  $c$ . Since  $P$  is symplectic, with  $z \in \text{Spec}(P)$  also  $z^{-1}, \bar{z}, \bar{z}^{-1} \in \text{Spec}(P)$ . If  $c, c^2$  are non-degenerate (i.e.,  $1, -1 \notin \text{Spec}(P)$ ), it follows:

$$(7) \quad \sum_{z \in \text{Spec}(P), |z|=1, \text{Im}z > 0} N_c(z) \leq n - 1.$$

In addition to the  $z$ -nullity  $N_c(z)$  for a closed geodesic  $c$  Bott defined the  $z$ -index  $I_c : \{z \in \mathbb{C}; |z| = 1\} \rightarrow \mathbb{N}_0$ . Both functions have the following properties (cf. [Bo56, Thms. A, C], [Ra92, ch. 4]): The functions  $I_c, N_c$  are invariant under conjugation; i.e.,  $I_c(\bar{z}) = I_c(z), N_c(\bar{z}) = N_c(z)$  for all  $z$ . The function  $I_c$  is a continuous and constant function outside the set  $\text{Spec}(P_c)$ . The finite jumps define the *splitting numbers*

$$S_c^{\pm}(z) = \lim_{\phi \rightarrow \pm 0} I_c(z \exp(i\phi)) - I_c(z)$$

satisfying

$$(8) \quad 0 \leq S_c^{\pm}(z) \leq N_c(z).$$

The functions  $I_c$  allow the following formula due to Bott [Bo56, Thms. A, C] for the sequence  $\text{ind}(c^m)$  of the indices of the multiples  $c^m$  of a closed geodesic  $c$ :

$$(9) \quad \text{ind}(c^m) = \sum_{z^m=1} I_c(z).$$

As an immediate consequence we obtain that for all  $m \geq 1$ ,

$$(10) \quad \text{ind}(c^m) \geq \text{ind}(c).$$

It was shown by Bott that the splitting numbers  $S_c^\pm(z)$  only depend on the conjugacy class of the linearized Poincaré mapping in the symplectic group.

If  $c$  is a closed geodesic of a bumpy metric and  $z_1 = \exp(2\pi it_1), \dots, z_l = \exp(2\pi it_l)$  are the eigenvalues  $z$  of the linearized Poincaré mapping with  $|z| = 1$ ,  $\text{Im}(z) > 0$ , and  $0 = t_0 < t_1 < t_2 < \dots < t_l < t_{l+1} = 1/2$ , then  $l \leq n - 1$  and the numbers  $t_j$  are irrational since  $\text{null}(c^m) = \sum_{z^m=1} N_c(z)$ .

With the help of the function  $I_c$  we get the following expression for the average index. Let  $I_1 = I_c(0) = I_c(\exp(2\pi it))$ ,  $t \in [0, t_1)$ , and for  $j \in \{1, 2, \dots, l\}$ ,  $I_j := I_c(\exp(2\pi it))$ ,  $t \in (t_{j-1}, t_j)$ , and  $I_{l+1} := I_c(-1) = I_c(\exp(2\pi it))$ ,  $t \in (t_l, 1/2]$ . Hence  $\{I_c(\exp(z)) \mid z \in S^1 - \text{Spec}(P_c)\} = \{I_1, I_2, \dots, I_{l+1}\}$ . Then Bott's formula (9) implies

$$(11) \quad \alpha_c = \int_0^1 I_c(\exp(2\pi it)) dt = I_1 t_1 + \sum_{j=2}^l I_j (t_j - t_{j-1}) + I_{l+1} \left(\frac{1}{2} - t_l\right).$$

Since  $I_j - I_{j+1} = S^-(z_j) - S^+(z_j)$ , we obtain from equation (8)  $|I_j - I_{j+1}| \leq N_c(z_j)$  where  $N_c(z_j) \geq 1$ . Then equation (7) implies

$$(12) \quad \sum_{j=1}^l |I_j - I_{j+1}| \leq \sum_{j=1}^l N_c(z_j) \leq n - 1.$$

The following proposition will be crucial in the proof of our theorem:

**Proposition 3.** *Let  $c$  be a closed geodesic of a bumpy Finsler metric on an  $n$ -dimensional manifold  $M$ . We denote by  $z_j = \exp(2\pi it_j)$ ,  $t_j \in (0, 1/2)$ ,  $j = 1, 2, \dots, l$ ,  $l \leq n - 1$ , the eigenvalues of the linearized Poincaré mapping  $P_c$  whose imaginary part is positive with  $0 < t_1 < t_2 < \dots < t_l < 1/2$ . Let  $t_0 = 0$ ,  $t_{l+1} = 1/2$ , and  $I_j := I_c(\exp(2\pi it))$ ,  $t \in (t_{j-1}, t_j)$ ,  $j = 1, 2, \dots, l + 1$ .*

*Let  $\text{ind}(c) = n - 1$ ,  $\text{ind}(c^2) \geq n$ , and  $\alpha_c < 2|\gamma_c|$ .*

*Then the invariants of the closed geodesic  $c$  satisfy:*

- (a)  $\gamma_c = (-1)^{n-1}$ ;  $\alpha_c > 1$ ;  $\text{ind}(c^2) = n + 1$ .
- (b)  $I_c(1) = I_1 = n - 1 > I_2 > \dots > I_l = 1$ ;  $I_{l+1} = I_c(-1) = 2$ .
- (c) For all integers  $m \geq 1$ ,  $\text{ind}(c^{m+1}) \geq \text{ind}(c^m)$ .

*Remark 1.* Since the number  $\bar{S} := \sum_{j=1}^l |S^+(z_j) - S^-(z_j)| = \sum_{j=1}^l |I_{j+1} - I_j| = n - 1$  is maximal, it follows from the formula for the splitting numbers depending on the symplectic normal form given for example in [BTZ82, 2.13] that the closed geodesic is of elliptic type; i.e., the linearized Poincaré mapping has a symplectic normal form consisting only of 2-dimensional rotations.

*Proof.* We define

$$I_{\min} := \min\{I_1, I_2, \dots, I_l\}, \quad r := \min\{j \in \{1, 2, \dots, l\} \mid I_j = I_{\min}, 1 \leq j \leq l\}.$$

Since  $\text{ind}(c) = n - 1$ , we have  $\gamma_c(-1)^{n-1} \in \{1/2, 1\}$ . If  $\gamma_c = (-1)^{n-1}/2$ , then  $\alpha_c < 1$  by Proposition 2(a); hence  $I_{\min} = 0$  by equation (11). Then equation (12) implies that  $r = l$ ,  $I_{l+1} = 0$ . Bott's formula (9) then implies that  $\text{ind}(c^2) = n - 1 + I_{l+1} = n - 1$ , contradicting the assumption  $\text{ind}(c^2) \geq n$ .

Hence we showed that  $\gamma_c = (-1)^{n-1}$ . By Bott's formula (9) we conclude that  $\text{ind}(c^2) = n - 1 + I_{l+1}$ ; hence by Definition 1 and the assumption  $\text{ind}(c^2) \geq n$ ,  $I_{l+1}$  is an even and positive number. Since  $\alpha_c < 2$ , equation (11) implies that

$I_{\min} = I_r \leq 1$ . From  $I_1 = n - 1$  we conclude  $I_r \in \{0, 1\}$ . If  $I_r = 0$ , then equation (12) implies that  $r = l + 1$  and therefore  $\text{ind}(c^2) = n - 1$ , contradicting our assumption.

Hence we conclude  $I_r = 1$ . Equation (12) implies that  $r \in \{l, l + 1\}$  and  $I_{l+1} - I_l \leq 1$  if  $r = l$ . Since  $I_{l+1}$  is even and positive, we have  $r = l$  and therefore  $I_{l+1} = 2$ . Since  $I_1 - I_l = n - 2$  and  $I_{l+1} - I_l = 1$  in equation (12), the inequalities are actually equalities, and hence  $n - 1 = I_1 > I_2 > \dots > I_l = 1$  and  $I_{l+1} = 2$ , which finishes the proof of parts (a) and (b).

It remains to prove part (c): We use the notation  $e(x) := \exp(2\pi ix)$ . Using Bott's formula (9) and  $I_c(-1) = 2$ , we obtain

$$(13) \quad \text{ind}(c^{m+1}) - \text{ind}(c^m) = A_m + B_m$$

with

$$(14) \quad A_m := \begin{cases} 2, & m \equiv 1 \pmod{2}, \\ 2 I_c \left( e \left( \frac{m}{2m+2} \right) \right) - 2, & m \equiv 0 \pmod{2} \end{cases}$$

and

$$(15) \quad B_m := 2 \sum_{1 \leq j < m/2} \left\{ I_c \left( e \left( \frac{j}{m+1} \right) \right) - I_c \left( e \left( \frac{j}{m} \right) \right) \right\}.$$

From part (b) we conclude that  $I_c(e(j/(m+1))) - I_c(e(j/m)) < 0$  for some  $j \in \mathbb{N}$ ,  $j < m/2$ , implies  $j \in J_m := \{p \in \mathbb{N}; p/(m+1) < t_l < p/m, p < m/2\}$  and therefore  $I_c(e(j/(m+1))) - I_c(e(j/m)) = -1$ . On the other hand  $J_m$  is either empty or consists of a single element  $J_m = \{j_m\}$ . Hence  $B_m \geq -2$  and  $B_m = -2$  if and only if  $J_m = \{j_m\}$ . Therefore we obtain immediately that  $A_m + B_m \geq 0$  for odd  $m$ . Let  $m$  be even. Since  $I_{\min} = 1$ , we have  $A_m \geq 0$ . On the other hand  $A_m = 0$  implies that  $m/(2m+2) < t_l < 1/2$ . Hence  $J_m = \emptyset$  and  $A_m + B_m \geq 0$ . If  $A_m > 0$ , then  $A_m \geq 2$ ; hence also in this case  $A_m + B_m \geq 0$ . Therefore we obtain in all cases  $\text{ind}(c^{m+1}) - \text{ind}(c^m) = A_m + B_m \geq 0$ .  $\square$

We use the *common index jump theorem* due to Y. Long and C. Zhu for a single closed geodesic  $c$  of a bumpy metric. For a closed geodesic  $c$  of a bumpy metric the nullities  $\text{null}(c^m)$  for all  $m$  vanish as well as the splitting number  $S_c^+(1)$ . Hence we obtain as a particular case of the common index jump theorem the following proposition. Here we note that the common index jump theorem also applies to the closed geodesic problem on Finsler manifolds; cf. [Lo02, Rem. 12.2.5].

**Proposition 4** ([LZ02, Thm. 4.3] [Lo02, Thm. 11.2.1]). *Let  $c$  be a closed geodesic of a bumpy Finsler metric with positive average index  $\alpha_c$ . Then there are infinitely many  $k$  with*

$$\text{ind}(c^{2k+1}) - \text{ind}(c^{2k-1}) = 2 \text{ind}(c).$$

#### 4. PROOF OF THE THEOREM

We assume that there is a prime closed geodesic  $c$  such that all closed geodesics on  $M$  are geometrically equivalent to  $c$  and  $n = \dim M \geq 3$ . For the average index  $\alpha_c$  and the invariant  $\gamma_c$  we obtain from Proposition 2(a):

$$(16) \quad 1 \leq \frac{\alpha_c}{|\gamma_c|} < 2, \quad \alpha_c = |\gamma_c| \Leftrightarrow n = 3.$$

Let  $w_k := \sum_{m=1}^{\infty} w_k(c^m)$ ; cf. equation (2). Since the Betti numbers  $b_k = b_k(\Lambda M/S^1, \Lambda^0 M/S^1; \mathbb{Q})$  vanish in dimensions  $k \equiv n \pmod{2}$  (cf. Proposition 1), we conclude from the Morse inequalities (see (5)) that the sequence  $(q_j)_{j \geq 0}$  vanishes identically, i.e.,

$$(17) \quad w_k = \#\{m \in \mathbb{N}; \text{ind}(c^m) = k, m \text{ odd or } \gamma_c = \pm 1\} = b_k \in \{0, 1, 2\}$$

for all  $k \in \mathbb{N}_0$  with  $b_k \in \{1, 2\}$  if and only if  $k \equiv n - 1 \pmod{2}$  and  $k \geq n - 1$ ; cf. Proposition 1 and Lemma 1.

We conclude from equation (10) and from  $b_0 = b_1 = \dots = b_{n-2} = 0, b_{n-1} = 1$  (cf. Proposition 1) that

$$(18) \quad \text{ind}(c) = n - 1.$$

We conclude from equation (1) and equation (11) that  $n - 1 = \text{ind}(c) \leq \text{ind}(c^2) < 2 \cdot 2 + n - 1 = n + 3$ . If  $\text{ind}(c^2) = \text{ind}(c) = n - 1$ , then Definition 1 and Lemma 1 imply  $w_{n-1} \geq w_{n-1}(c) + w_{n-1}(c^2) \geq 2$ , contradicting equation (17). Therefore  $\text{ind}(c^2) \geq n$ , and we conclude from Proposition 3(a) that  $\gamma_c = (-1)^{n-1}$  and  $\alpha_c > 1$ . In dimension  $n = 3$  this yields a contradiction to equation (16) and finishes the proof.

Hence we can now assume  $n \geq 4$ : Since the sequence  $\text{ind}(c^m)$ ,  $m \geq 1$ , is monotone increasing by Proposition 3(c) and  $b_k = w_k \geq 1$  for all  $k \equiv n - 1 \pmod{2}$ ,  $k \geq n - 1$ , we have  $\text{ind}(c^{m+1}) - \text{ind}(c^m) \in \{0, 2\}$  for all  $m \geq 1$ . Therefore for  $n \geq 4$  we obtain for all  $m \geq 1$ ,

$$(19) \quad \text{ind}(c^{m+2}) - \text{ind}(c^m) \leq 4.$$

But we conclude from Proposition 4 that for  $n \geq 4$  there is a  $k_1 \in \mathbb{N}$  with

$$(20) \quad \text{ind}(c^{2k_1+1}) - \text{ind}(c^{2k_1-1}) = 2n - 2 \geq 6,$$

contradicting equation (19). Therefore we obtain a contradiction to our assumption made at the beginning of our proof.  $\square$

*Remark 2.* It is likely that two is not the optimal lower bound for the number of closed geodesics of a non-reversible bumpy metric on  $S^n$ ,  $n \geq 3$ . Katok's examples carry  $2n$  closed geodesics on  $S^{2n}$  and  $S^{2n-1}$ . One can show that any non-reversible and bumpy Finsler metric sufficiently  $C^2$ -close to the standard Riemannian metric of constant curvature 1 on  $S^{2n}$ , resp.  $S^{2n-1}$ , carries at least  $2n$  closed geodesics of length approximately  $2\pi$ ; cf. [Zi82, p. 141]. Estimates on the number of closed geodesics under pinching assumptions for the flag curvature depending on the reversibility are presented for example in [Ra07, Thm. 3, Thm. 8]. Various questions, resp. conjectures, about the optimal lower bound for an arbitrary Finsler metric on  $S^n$  are given in [Zi82, p. 155] and [Lo06, ch. 4].

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MATHEMATISCHES INSTITUT, UNIVERSITÄT LEIPZIG, 04081 LEIPZIG, GERMANY  
*E-mail address:* rademacher@math.uni-leipzig.de