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A NEW APPROACH TO CLASSIFICATION OF INTEGRAL QUADRATIC FORMS OVER DYADIC LOCAL FIELDS

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Abstract. In 1963, O'Meara solved the classification problem for lattices over dyadic local fields in terms of Jordan decompositions. In this paper we translate his result in terms of good BONGs. BONGs (bases of norm generators) were introduced in 2003 as a new way of describing lattices over dyadic local fields. This result and the notions we introduce here are a first step towards a solution of the more difficult problem of representations of lattices over dyadic fields.

1. Introduction

Since the main result of this paper is given in terms of BONGs, which were introduced in [1], we now give a reminder of some of the definitions and results in that paper which we will use here.

Throughout this paper F is a dyadic local field, $\mathcal{O}$ the ring of integers, $\mathfrak{p}$ the prime ideal, $\mathcal{O}^\times := \mathcal{O} \setminus \mathfrak{p}$ the group of units, $e := \operatorname{ord} 2$ and π is a fixed prime element. For $a \in \dot{F}$ we denote its quadratic defect by $\mathfrak{d}(a)$ and let $\Delta = 1 - 4\rho$ be a fixed unit with $\mathfrak{d}(\Delta) = 4\mathcal{O}$.

We denote by $d : \dot{F}/\dot{F}^2 \longrightarrow \mathbb{N} \cup \{\infty\}$ the order of the "*relative* quadratic defect" $d(a) = \operatorname{ord} a^{-1}\mathfrak{d}(a)$. If $a = \pi^R\varepsilon$, with $\varepsilon \in \mathcal{O}^\times$, then $d(a) = 0$ if R is odd and $d(a) = d(\varepsilon) = \operatorname{ord} \mathfrak{d}(\varepsilon)$ if R is even. Thus $d(\dot{F}) = \{0, 1, 3, \ldots, 2e-1, 2e, \infty\}$. This function satisfies the domination principle $d(ab) \geq \min\{d(a), d(b)\}$.

If α is a positive integer then $(1+\mathfrak{p}^\alpha)\dot{F}^2 = \{a \in \dot{F} \mid d(a) \geq \alpha\}$ and $(1+\mathfrak{p}^\alpha)\mathcal{O}^{\times 2} = \{a \in \mathcal{O}^\times \mid d(a) \geq \alpha\}$. For convenience we set $(1+\mathfrak{p}^\alpha)\dot{F}^2 := \{a \in \dot{F} \mid d(a) \geq \alpha\}$ and $(1+\mathfrak{p}^\alpha)\mathcal{O}^{\times 2} := \{a \in \mathcal{O}^\times \mid d(a) \geq \alpha\}$ for any $\alpha \in \mathbb{R} \cup \{\infty\}$. Thus $(1+\mathfrak{p}^\alpha)\dot{F}^2 = \dot{F}^2$ for $\alpha > 2e$ and $(1+\mathfrak{p}^\alpha)\dot{F}^2 = \dot{F}$ for $\alpha \leq 0$. If d is the smallest element in $d(\dot{F})$ s.t. $\alpha \leq d$ then $(1+\mathfrak{p}^\alpha)\dot{F}^2 = (1+\mathfrak{p}^d)\dot{F}^2$.

We denote by $(\cdot,\cdot)_\mathfrak{p} : \dot{F}/\dot{F}^2 \times \dot{F}/\dot{F}^2 \longrightarrow \{\pm 1\}$ the Hilbert symbol, which is a non-degenerate bilinear symmetric form.

If $a \in \dot{F}$, we denote by $\mathrm{N}(a)$ the norm group $\mathrm{N}(F(\sqrt{a})/F) = \{b \in \dot{F} \mid (a,b)_\mathfrak{p} = 1\}$. If $b \in \dot{F}$ and $d(a) + d(b) > 2e$ then $(a,b)_\mathfrak{p} = 1$. However if $\alpha \notin \dot{F}^2$ then there is $b \in \dot{F}$ with $d(b) = 2e - d(a)$ s.t. $(a,b)_\mathfrak{p} = -1$. (For $d(a)$ odd this is just [3, Lemma 3]. If $d(a) = 2e$ and $b \in \dot{F}$ is arbitrary with $d(b) = 0$ then $a \in \Delta\dot{F}^2$ and $\operatorname{ord} b$ is odd

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so $(a,b)_{\mathfrak{p}}=-1$. Similarly if $d(a)=0$ and $d(b)=2e$ we have $(a,b)_{\mathfrak{p}}=-1$.) Thus $(1+\mathfrak{p}^{\alpha})\dot{F}^2\subseteq \mathrm{N}(a)$ iff $\alpha+d(a)>2e$.

An element x of a lattice L is called a *norm generator* of L if $\mathfrak{n}L=Q(x)\mathcal{O}$. A sequence $x_1,\dots,x_n$ of vectors in FL is called a *basis of norm generators* (BONG) for L if x_1 is a norm generator for L and $x_2,\dots,x_n$ is a BONG for $pr_{x_1^\perp}L$. A BONG uniquely determines a lattice, so if $x_1,\dots,x_n$ is a BONG for L, we will write $L=\prec x_1,\dots,x_n\succ$. If moreover $Q(x_i)=a_i$ we say that $L\cong\prec a_1,\dots,a_n\succ$ relative to the BONG $x_1,\dots,x_n$. If $L\cong\prec a_1,\dots,a_n\succ$ then $\det L=a_1\cdots a_n$.

If $x_1,...,x_n$ are mutually orthogonal vectors with $Q(x_i)=a_i$, $L=\mathcal{O}x_1\perp\cdots\perp\mathcal{O}x_n$ and $V=Fx_1\perp\cdots\perp Fx_n$ then we say that $L\cong\langle a_1,\dots,a_n\rangle$ and $V\cong[a_1,\dots,a_n]$ relative to the basis $x_1,\dots,x_n$.

If L is binary with $\mathfrak{n}L=\alpha\mathcal{O}$, we denote by $a(L):=\det L\,\alpha^{-2}$ and by $R(L):=\operatorname{ord}\operatorname{vol}L-2\operatorname{ord}\mathfrak{n}L=\operatorname{ord}a(L)$. $a(L)\in\dot{F}/\mathcal{O}^{\times 2}$ is an invariant of L and it determines the class of L up to scaling. If $L\cong\prec\alpha,\beta\succ$ then $a(L)=\frac{\beta}{\alpha}$.

We denote by $\mathcal{A}=\mathcal{A}_F\subset\dot{F}/\mathcal{O}^{\times 2}$ the set of all possible values of $a(L)$, where L is an arbitrary binary lattice. We have $\mathcal{A}=\{a\in\frac14\mathcal{O}\mid a\neq 0,\mathfrak{d}(-a)\subseteq\mathcal{O}\}$. If $\operatorname{ord}a=R$ and $d(-a)=d$, then $a\in\frac14\mathcal{O}$ means $R\geq -2e$, while $\mathfrak{d}(-a)\subseteq\mathcal{O}$ means $R+d=\operatorname{ord}\mathfrak{d}(-a)\geq 0$.

If $a(L)=a=\pi^R\varepsilon$ with $d(-a)=d$ then:

L is nonmodular, proper modular or improper modular iff $R>0$, $R=0$, resp. $R<0$.

If R is odd then $R>0$.

The inequality $R+2e\geq 0$ becomes equality iff $a\in-\frac14\mathcal{O}^{\times 2}$ or $a\in-\frac{\Delta}{4}\mathcal{O}^{\times 2}$. We have $a(L)=-\frac14$, resp. $a(L)=-\frac{\Delta}{4}$, when $L\cong\pi^rA(0,0)$, resp. $\pi^rA(2,2\rho)$, for some integer r.

The inequality $R+d\geq 0$ becomes equality iff $a\in-\frac{\Delta}{4}\mathcal{O}^{\times 2}$.

A special type of BONG is the so-called "good BONG". If $L\cong\prec a_1,\dots,a_n\succ$ relative to some BONG $x_1,\dots,x_n$ and $\operatorname{ord}a_i=R_i$ we say that the BONG $x_1,\dots,x_n$ is good if $R_i\leq R_{i+2}$ for any $1\leq i\leq n-2$.

Remark. The condition $R_i\leq R_{i+2}$ for $1\leq i\leq n-2$ is equivalent to the condition that the sequence (R_i+R_{i+1}) is increasing.

A set $x_1,\dots,x_n$ of orthogonal vectors with $Q(x_i)=a_i$ and $\operatorname{ord}a_i=R_i$ is a good BONG for some lattice iff $R_i\leq R_{i+2}$ for all $1\leq i\leq n-2$ and $a_{i+1}/a_i\in\mathcal{A}$ for all $1\leq i\leq n-1$. The condition $a_{i+1}/a_i\in\mathcal{A}$ is equivalent to $R_{i+1}-R_i+2e\geq 0$ and $R_{i+1}-R_i+d(-a_ia_{i+1})\geq 0$. As consequences of $a_{i+1}/a_i\in\mathcal{A}$, if $R_{i+1}-R_i$ is odd then it is positive, if $R_{i+1}-R_i=-2e$ then $a_{i+1}/a_i\in-\frac14\mathcal{O}^{\times 2}$ or $-\frac{\Delta}{4}\mathcal{O}^{\times 2}$ and if $R_{i+1}-R_i+d(-a_ia_{i+1})=0$ then $a_{i+1}/a_i\in-\frac{\Delta}{4}\mathcal{O}^{\times 2}$.

The good BONGs enjoy some properties similar to those of orthogonal bases. If $L\cong\prec a_1,\dots,a_n\succ$ relative to some good BONG $x_1,\dots,x_n$ and $\operatorname{ord}a_i=R_i$ then $L^\sharp\cong\prec a_1^{-1},\dots,a_n^{-1}\succ$ relative to the good BONG $x_n^\sharp,\dots,x_1^\sharp$, where $x_i^\sharp=Q(x)^{-1}x_i$. Also if for some $1\leq i\leq j\leq n$ we have $\prec x_i,\dots,x_j\succ\cong\prec b_i,\dots,b_j\succ$ relative to some other good BONG $y_i,\dots,y_j$ then $L\cong\prec a_1,\dots,a_{i-1},b_i,\dots,b_j,a_{i+1},\dots,a_n\succ$ relative to the good BONG $x_1,\dots,x_{i-1},y_i,\dots,y_j,x_{i+1},\dots,x_n$. There are some differences though from the orthogonal bases. E.g. the relation $L=\prec x_1,\dots,x_i\succ\perp\prec x_{i+1},\dots,x_n\succ$ holds iff $R_i\leq R_{i+1}$.

The orders $R_i=\operatorname{ord}a_i$ are independent of the choice of the good BONGs and they are in 1-1 correspondence with the invariants $t,\dim L_k,\mathfrak{s}_k:=\mathfrak{s}L_k$ and $\mathfrak{n}L^{\mathfrak{s}_k}$, where $L = L_1 \perp \ldots \perp L_t$ is a Jordan splitting. More precisely, if $\mathfrak{s}_k = \mathfrak{p}^{r_k}$, $\mathfrak{n}L^{\mathfrak{s}_k} = \mathfrak{p}^{u_k}$ and $n_k = \sum_{l\leq k} \dim L_l$, then the sequence $R_{n_{k-1}+1}, \ldots, R_{n_k}$ is $r_k, \ldots, r_k$ if L_k is proper (i.e. if $r_k = u_k$), and it is $u_k, 2r_k - u_k, \ldots, u_k, 2r_k - u_k$ otherwise; see [1, Lemma 4.7].

The good BONGs are closely connected with the *maximal norm splittings.* A splitting $L = L_1 \perp \ldots \perp L_t$ is called a maximal norm splitting if $\mathfrak{s}L_1 \supseteq \ldots \supseteq \mathfrak{s}L_t$ and $\dim L_i \leq 2$, L_i is modular and $\mathfrak{n}L_i = \mathfrak{n}L^{\mathfrak{s}L_i}$ for all $1 \leq i \leq t$. Condition $\mathfrak{n}L_i = \mathfrak{n}L^{\mathfrak{s}L_i}$ is equivalent to $\mathfrak{n}L_1 \supseteq \ldots \supseteq \mathfrak{n}L_t$ and $\mathfrak{n}L_1^\sharp \subseteq \ldots \subseteq \mathfrak{n}L_t^\sharp$. If we put together the BONGs of the components $L_1, \ldots, L_t$ of a maximal norm splitting we get a good BONG for L. Conversely any good BONG of a lattice can be obtained by putting together some BONGs of the components of some maximal norm splitting. Moreover, the splitting can be chosen such that all binary components are improper modular. An explicit algorithm for finding a maximal norm splitting and, hence, a good BONG of a lattice is provided in [2, Section 7].

2. The invariants α_i

Let L be a lattice over the dyadic field F. Let $L \cong \prec a_1, \ldots, a_n \succ$ relative to a good BONG and let $R_i := \operatorname{ord} a_i$. Also let $L = L_1 \perp \ldots \perp L_t$ be a Jordan decomposition. We keep the notation of [4], $\mathfrak{s}_k := \mathfrak{s}L_k$, $\mathfrak{g}_k := \mathfrak{g}L^{\mathfrak{s}_k}$, $\mathfrak{w}_k := \mathfrak{w}L^{\mathfrak{s}_k}$ but, in order to avoid confusion, we write a_k for O'Meara's a_k. Also we denote $r_k = \operatorname{ord} \mathfrak{s}_k$, $u_k = \operatorname{ord} \mathsf{a}_k = \operatorname{ord} \mathfrak{n}L^{\mathfrak{s}_k}$. Associated to our splitting we have the Jordan chain $L_{(1)} \subset \ldots \subset L_{(t)}$ and the inverse Jordan chain $L^*_{(1)} \supset \ldots \supset L^*_{(t)}$, where $L_{(k)} := L_1 \perp \ldots \perp L_k$ and $L^*_{(k)} := L_k \perp \ldots \perp L_t$.

Since the R_i's are invariants of L we will write $R_i = R_i(L)$.

Definition 1. For any $1 \leq i \leq n-1$ we define $\alpha_i = \alpha_i(L)$ by:

$$\alpha_i := \min(\{(R_{i+1} - R_i)/2 + e\} \cup \{R_{i+1} - R_j + d(-a_j a_{j+1}) \mid 1 \leq j \leq i\} \\ \cup \{R_{j+1} - R_i + d(-a_j a_{j+1}) \mid i \leq j < n\}).$$

Apparently $\alpha_i(L)$ defined this way depends on the choice of the good BONG. We will show later that, in fact, it depends only on L. For the time being we will mean $\alpha_i(L)$ with respect to a given good BONG. We now give some properties of the α_i's.

Lemma 2.1. *If $k \leq i < l$ then, in the set defining α_i, we can replace $(R_{i+1}-R_i)/2+e$ and all the terms corresponding to indices $k \leq j < l$ by $\alpha_{i-k+1}(\prec a_k, \ldots, a_l \succ)$. In particular, $\alpha_i \leq \alpha_{i-k+1}(\prec a_k, \ldots, a_l \succ)$.*

Proof. By definition $\alpha_{i-k+1}(\prec a_k, \ldots, a_l \succ) = \min(\{(R_{i+1} - R_i)/2 + e\} \cup \{R_{i+1} - R_j + d(-a_j a_{j+1}) \mid k \leq j \leq i\} \cup \{R_{j+1} - R_i + d(-a_j a_{j+1}) \mid i \leq j < l\})$, hence the conclusion. □

Lemma 2.2. *The sequence $(R_i + \alpha_i)$ is increasing and the sequence $(-R_{i+1} + \alpha_i)$ is decreasing.*

Proof. Let $1 \leq i \leq h \leq n-1$. We have $R_i + R_{i+1} \leq R_h + R_{h+1}$. From Definition 1 we get $R_i + \alpha_i = \min(\{(R_i + R_{i+1})/2 + e\} \cup \{R_i + R_{i+1} - R_j + d(-a_j a_{j+1}) \mid 1 \leq j \leq i\} \cup \{R_{j+1} + d(-a_j a_{j+1}) \mid i \leq j < n\})$ and $-R_{i+1} + \alpha_i = \min(\{-(R_i + R_{i+1})/2 + e\} \cup \{-R_j + d(-a_j a_{j+1}) \mid 1 \leq j \leq i\} \cup \{R_{j+1} - R_i - R_{i+1} + d(-a_j a_{j+1}) \mid i \leq j < n\})$, and similarly for $R_h + \alpha_h$ and $-R_{h+1} + \alpha_h$. In order to prove that $R_i + \alpha_i \leq R_h + \alpha_h$

we show that the elements in the set that has $R_i + \alpha_i$ as its minimum are less than or equal to the corresponding elements for $R_h + \alpha_h$. The same holds for $-R_{i+1} + \alpha_i \geq -R_{h+1} + \alpha_h$.

The proof is straightforward and uses the fact that $R_l + R_{l+1}$ is an increasing sequence. For terms involving $d(-a_j a_{j+1})$ we consider the cases $j \leq i$, $i \leq j \leq h$ and $h \leq j$ and use the inequalities among $R_i + R_{i+1}$, $R_j + R_{j+1}$ and $R_h + R_{h+1}$ that occur in each case. $\square$

Corollary 2.3. *Suppose that $1 \leq i \leq j \leq n-1$ and $R_i + R_{i+1} = R_j + R_{j+1}$. Then:*

(i) $R_i + \alpha_i = \ldots = R_j + \alpha_j$ and $-R_{i+1} + \alpha_i = \ldots = -R_{j+1} + \alpha_j$.

(ii) $R_k = R_l$ for any $k, l \in [i, j+1]$ of the same parity and $\alpha_k = \alpha_l$ for any $k, l \in [i, j]$ of the same parity.

(iii) If $\alpha_k = (R_{k+1} - R_k)/2 + e$ for some $i \leq k \leq j$ then $\alpha_k = (R_{k+1} - R_k)/2 + e$ for all $i \leq k \leq j$.

In the particular case when $j = i+1$ we get the following statement:

If $1 \leq i \leq n-2$ and $R_i = R_{i+2}$ then $R_i + \alpha_i = R_{i+1} + \alpha_{i+1}$, $-R_{i+1} + \alpha_i = -R_{i+2} + \alpha_{i+1}$ and $\alpha_i = (R_{i+1} - R_i)/2 + e$ is equivalent to $\alpha_{i+1} = (R_{i+2} - R_{i+1})/2 + e$.

Proof. For (i) we note that $R_i + R_{i+1} = (R_i + \alpha_i) - (-R_{i+1} + \alpha_i)$ and $R_j + R_{j+1} = (R_j + \alpha_j) - (-R_{j+1} + \alpha_j)$ and use Lemma 2.2. By using the fact that $R_k + R_{k+1}$ is an increasing sequence we get $R_i + R_{i+1} = R_{i+1} + R_{i+2} = \ldots = R_j + R_{j+1}$, which is equivalent to the first part of (ii). For the second part of (ii) use (i). Finally (iii) follows from $R_i + \alpha_i = \ldots = R_j + \alpha_j$, $R_i + R_{i+1} = \ldots = R_j + R_{j+1}$ and the fact that $\alpha_k = (R_{k+1} - R_k)/2 + e$ is equivalent to $R_k + \alpha_k = (R_k + R_{k+1})/2 + e$. $\square$

Lemma 2.4. *Suppose that $1 \leq i < n$ and $1 \leq k \leq h < l \leq n$. Then:*

(i) If $h \leq i$ then all terms in the definition of α_i corresponding to indices $k \leq j \leq h$ can be replaced by $R_{i+1} - R_{h+1} + \alpha_{h-k+1}(\prec a_k, \ldots, a_l \succ)$. In particular, all terms with $1 \leq j \leq h$ can be replaced by $R_{i+1} - R_{h+1} + \alpha_h$.

(ii) If $i \leq h$ then all terms in the definition of α_i corresponding to indices $h \leq j < l$ can be replaced by $R_h - R_i + \alpha_{h-k+1}(\prec a_k, \ldots, a_l \succ)$. In particular, all terms with $h \leq j < n$ can be replaced by $R_h - R_i + \alpha_h$.

Proof. By Lemma 2.1 we have $\alpha_{h-k+1}(\prec a_k, \ldots, a_l \succ) \geq \alpha_h$.

(i) By Lemma 2.2 we have $\alpha_i \leq R_{i+1} - R_{h+1} + \alpha_h \leq R_{i+1} - R_{h+1} + \alpha_{h-k+1}(\prec a_k, \ldots, a_l \succ)$. If $k \leq j \leq h$ then $\alpha_{h-k+1}(\prec a_k, \ldots, a_l \succ) \leq R_{h+1} - R_j + d(-a_j a_{j+1})$ so $R_{i+1} - R_{h+1} + \alpha_{h-k+1}(\prec a_k, \ldots, a_l \succ) \leq R_{i+1} - R_j + d(-a_j a_{j+1})$. Therefore if we add $R_{i+1} - R_{h+1} + \alpha_{h-k+1}(\prec a_k, \ldots, a_l \succ)$ to the set that defines α_i and remove any one of $R_{i+1} - R_j + d(-a_j a_{j+1})$ with $k \leq j \leq h$ then α_i does not change.

(ii) By Lemma 2.2 we have $\alpha_i \leq R_h - R_i + \alpha_h \leq R_h - R_i + \alpha_{h-k+1}(\prec a_k, \ldots, a_l \succ)$. If $h \leq j < l$ then $\alpha_{h-k+1}(\prec a_k, \ldots, a_l \succ) \leq R_{j+1} - R_h + d(-a_j a_{j+1})$ so $R_h - R_i + \alpha_{h-k+1}(\prec a_k, \ldots, a_l \succ) \leq R_{j+1} - R_i + d(-a_j a_{j+1})$. Thus if we add $R_h - R_i + \alpha_{h-k+1}(\prec a_k, \ldots, a_l \succ)$ to the set that defines α_i and remove any one of $R_{j+1} - R_i + d(-a_j a_{j+1})$ with $h \leq j < l$ then α_i does not change.

If we take $k = 1$ and $l = n$ then $\alpha_{h-k+1}(\prec a_k, \ldots, a_l \succ)$ becomes $\alpha_h(\prec a_1, \ldots, a_n \succ) = \alpha_h(L) = \alpha_h$ so we get the second claims of (i) and (ii). $\square$

Corollary 2.5. *For any $1 \leq i \leq n-1$ we have:*

(i) $\alpha_i = \min\{(R_{i+1} - R_i)/2 + e, R_{i+1} - R_i + d(-a_i a_{i+1}), R_{i+1} - R_i + \alpha_{i-1}, R_{i+1} - R_i + \alpha_{i+1}\}$.

(ii) $\alpha_i = \min\{(R_{i+1}-R_i)/2+e, R_{i+1}-R_i+d(-a_ia_{i+1}), R_{i+1}-R_i+\alpha_{i-1}(\prec a_1,\ldots,a_i\succ), R_{i+1}-R_i+\alpha_1(\prec a_{i+1},\ldots,a_n\succ)\}$.

(The terms that do not make sense, i.e. $R_{i+1}-R_i+\alpha_{i-1}$ *and* $R_{i+1}-R_i+\alpha_{i-1}(\prec a_1,\ldots,a_i\succ)$ *when* $i=1$*, or* $R_{i+1}-R_i+\alpha_{i+1}$ *and* $R_{i+1}-R_i+\alpha_1(\prec a_{i+1},\ldots,a_n\succ)$ *when* $i=n-1$*, are ignored.)*

Proof. (i) By Lemma 2.4 (i), resp. (ii), in the set defining α_i, $R_{i+1}-R_i+\alpha_{i-1}$ can replace all the terms $R_{i+1}-R_j+d(-a_ja_{j+1})$ with $1\leq j\leq i-1$, while $R_{i+1}-R_i+\alpha_{i+1}$ replaces all $R_{j+1}-R_i+d(-a_ja_{j+1})$ with $i+1\leq j<n$. Therefore $\alpha_i=\min\{(R_{i+1}-R_i)/2+e, R_{i+1}-R_i+d(-a_ia_{i+1}), R_{i+1}-R_i+\alpha_{i-1}, R_{i+1}-R_i+\alpha_{i+1}\}$.

(ii) Same as (i) but this time the terms corresponding to $1\leq j\leq i-1$ are replaced by $R_{i+1}-R_i+\alpha_{i-1}(\prec a_1,\ldots,a_i\succ)$ and those corresponding to $i+1\leq j<n$ by $R_{i+1}-R_i+\alpha_1(\prec a_{i+1},\ldots,a_n\succ)$. ☐

Remark 2.6. We have $L^\sharp\cong\prec a_1^\sharp,\ldots,a_n^\sharp\succ$ with $a_i^\sharp=a_{n+1-i}^{-1}$ and $R_i^\sharp:=\operatorname{ord}a_i^\sharp=-R_{n+1-i}$. One can easily see that $\alpha_i^\sharp:=\alpha_i(L^\sharp)=\alpha_{n-i}$. Also, the α_i's are invariant to scaling.

Lemma 2.7. *If* $1\leq i\leq n-1$ *then:*

(i) $\alpha_i\geq 0$ *with equality iff* $R_{i+1}-R_i=-2e$.

(ii) If $R_{i+1}-R_i\geq 2e$ *then* $\alpha_i=(R_{i+1}-R_i)/2+e$.

(iii) If $R_{i+1}-R_i\leq 2e$ *then* $\alpha_i\geq R_{i+1}-R_i$ *with equality iff* $R_{i+1}-R_i=2e$ *or it is odd.*

(iv) α_i *is an odd integer unless* $\alpha_i=(R_{i+1}-R_i)/2+e$.

Proof. We use induction on n. For $n=1$ our lemma is vacuous.

For the induction step let $1\leq i\leq n-1$ and let $L'=\prec a_1,\ldots,a_i\succ$ and $L''=\prec a_{i+1},\ldots,a_n\succ$. By Corollary 2.5(ii) we have $\alpha_i=\min\{(R_{i+1}-R_i)/2+e, R_{i+1}-R_i+d(-a_ia_{i+1}), R_{i+1}-R_i+\alpha, R_{i+1}-R_i+\beta\}$, where $\alpha=\alpha_{i-1}(L')$ and $\beta=\alpha_1(L'')$. (We ignore α and β whenever they are not defined.) By the induction hypothesis α,β satisfy (i)-(iv) of the lemma.

We have $(R_{i+1}-R_i)/2+e\geq 0$ with equality iff $R_{i+1}-R_i=-2e$ and $R_{i+1}-R_i+d(-a_ia_{i+1})\geq 0$ with equality iff $a_{i+1}/a_i\in-\frac{\Delta}{4}\mathcal{O}^{\times 2}$, which implies $R_{i+1}-R_i=-2e$. If $R_{i+2}-R_{i+1}>2e$ then $\beta=(R_{i+2}-R_{i+1})/2+e>2e$ so $R_{i+1}-R_i+\beta>R_{i+1}-R_i+2e\geq 0$. Similarly with $R_{i+1}-R_i+\alpha$ if $R_i-R_{i-1}>2e$. If $R_{i+2}-R_{i+1}\leq 2e$ then, by the induction hypothesis, $\beta\geq R_{i+2}-R_{i+1}$ with equality iff $R_{i+2}-R_{i+1}$ is odd or it is $2e$. Thus $R_{i+1}-R_i+\beta\geq R_{i+2}-R_i\geq 0$ with equality iff $R_i=R_{i+2}$ and $R_{i+2}-R_{i+1}$ is odd or $2e$. Suppose this happens. If $R_{i+2}-R_{i+1}=2e$ then $R_{i+1}-R_i=R_{i+1}-R_{i+2}=-2e$. If $R_{i+2}-R_{i+1}$ is odd then so is $R_{i+1}-R_i=R_{i+1}-R_{i+2}$ so both must be positive. But this is impossible. Similar results hold for $R_{i+1}-R_i+\alpha$ when $R_i-R_{i-1}\leq 2e$. Thus we have (i).

If $R_{i+1}-R_i\geq 2e$ then $\alpha,\beta\geq 0$ so $R_{i+1}-R_i+d(-a_ia_{i+1}), R_{i+1}-R_i+\alpha, R_{i+1}-R_i+\beta\geq R_{i+1}-R_i\geq(R_{i+1}-R_i)/2+e$. Hence $\alpha_i=(R_{i+1}-R_i)/2+e$ and we have (ii).

We now prove (iii). If $R_{i+1}-R_i=2e$ then (ii) implies that $\alpha_i=(R_{i+1}-R_i)/2+e=2e=R_{i+1}-R_i$ so we are done. If $R_{i+1}-R_i<2e$ is odd then $d(-a_ia_{i+1})=0$ and $\alpha,\beta\geq 0$ so $\alpha_i=\min\{(R_{i+1}-R_i)/2+e, R_{i+1}-R_i\}=R_{i+1}-R_i$. Finally if $R_{i+1}-R_i<2e$ is even then $\operatorname{ord}a_ia_{i+1}=R_i+R_{i+1}$ is even so $d(-a_ia_{i+1})>0$. Also $R_i-R_{i-1}, R_{i+2}-R_{i+1}\geq R_i-R_{i+1}>-2e$ ($R_{i-1}\leq R_{i+1}$ and $R_i\leq R_{i+2}$) so by (i)

$\alpha, \beta > 0$. We have $R_{i+1} - R_i + d(-a_1 a_2), R_{i+1} - R_i + \alpha, R_{i+1} - R_i + \beta > R_{i+1} - R_i$. Since also $(R_{i+1} - R_i)/2 + e > R_{i+1} - R_i$ (we have $R_{i+1} - R_i < 2e$) we get $\alpha_i > R_{i+1} - R_i$.

We now prove (iv). If $R_{i+1} - R_i \geq 2e$ then (ii) implies $\alpha_i = (R_{i+1} - R_i)/2 + e$ so (iv) is vacuous. If $R_{i+1} - R_i < 2e$ is odd then (iii) implies $\alpha_i = R_{i+1} - R_i$ so α_i is odd. If $R_{i+1} - R_i < 2e$ is even then again $\operatorname{ord} a_i a_{i+1}$ is even so $d(-a_i a_{i+1}) > 0$. Suppose $\alpha_i < (R_{i+1} - R_i)/2 + e$. If $\alpha_i = R_{i+1} - R_i + d(-a_i a_{i+1})$ then if $d(-a_i a_{i+1})$ is odd α_i will also be odd so we are done. Otherwise $d(-a_i a_{i+1}) = 2e$ or ∞ so $\alpha_i = R_{i+1} - R_i + d(-a_i a_{i+1}) \geq R_{i+1} - R_i + 2e \geq (R_{i+1} - R_i)/2 + e > \alpha_i$. (We have $R_{i+1} - R_i + 2e \geq 0$.) Contradiction. If $\alpha_i = R_{i+1} - R_i + \alpha$ then α_i is odd unless α is not odd, which would imply $\alpha = (R_i - R_{i-1})/2 + e$. So $\alpha_i = R_{i+1} - R_i + (R_i - R_{i-1})/2 + e \geq (R_{i+1} - R_i)/2 + e > \alpha_i$. (We have $R_{i+1} \geq R_{i-1}$.) Contradiction. Similar results hold if $\alpha_i = R_{i+1} - R_i + \beta$ since $R_{i+1} - R_i + (R_{i+2} - R_{i+1})/2 + e \geq (R_{i+1} - R_i)/2 + e > \alpha_i$. (We have $R_{i+2} \geq R_i$.) □

Corollary 2.8. *(i)* $\alpha_i \in \mathbb{Z}$ *except when* $R_{i+1} - R_i$ *is odd and* $> 2e$.

(ii) α_i *is* $< 2e$, $= 2e$ *or* $> 2e$ *if* $R_{i+1} - R_i$ *is* $< 2e$, $= 2e$ *or* $> 2e$ *accordingly.*

(iii) $\alpha_i \in ([0, 2e] \cap \mathbb{Z}) \cup ((2e, \infty) \cap \frac{1}{2}\mathbb{Z})$.

Proof. (i) If $R_{i+1} - R_i > 2e$ then $\alpha_i = (R_{i+1} - R_i)/2 + e$. If $R_{i+1} - R_i$ is even then $\alpha_i \in \mathbb{Z}$, while if it is odd then $\alpha_i \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. Suppose now that $R_{i+1} - R_i \leq 2e$. If $R_{i+1} - R_i$ is odd then $\alpha_i = R_{i+1} - R_i \in \mathbb{Z}$. If $R_{i+1} - R_i$ is even then either α_i is an odd integer or $\alpha_i = (R_{i+1} - R_i)/2 + e \in \mathbb{Z}$.

(ii) If $R_{i+1} - R_i < 2e$ then $\alpha_i \leq (R_{i+1} - R_i)/2 + e < 2e$. If $R_{i+1} - R_i = 2e$ then $\alpha_i = (R_{i+1} - R_i)/2 + e = 2e$. If $R_{i+1} - R_i > 2e$ then $\alpha_i = (R_{i+1} - R_i)/2 + e > 2e$.

(iii) We have $\alpha_i \geq 0$. If $\alpha_i \leq 2e$ then $R_{i+1} - R_i \leq 2e$ so $\alpha_i \in \mathbb{Z}$. If $\alpha_i > 2e$ then $R_{i+1} - R_i > 2e$ so $\alpha_i = (R_{i+1} - R_i)/2 + e \in (2e, \infty) \cap \frac{1}{2}\mathbb{Z}$. □

Corollary 2.9. *In each of the following cases,* α_i *depends only on* $R_{i+1} - R_i$*:*

(i) If $R_{i+1} - R_i \geq 2e$ *or* $R_{i+1} - R_i \in \{-2e, 2 - 2e, 2e - 2\}$ *then* $\alpha_i = (R_{i+1} - R_i)/2 + e$.

(ii) If $R_{i+1} - R_i$ *is odd, then* $\alpha_i = \min\{(R_{i+1} - R_i)/2 + e, R_{i+1} - R_i\}$.

Proof. (i) If $R_{i+1} - R_i \geq 2e$ then $\alpha_i = (R_{i+1} - R_i)/2 + e$ by Lemma 2.7(ii). If $R_{i+1} - R_i = -2e$ then $\alpha_i = 0 = (R_{i+1} - R_i)/2 + e$. If $R_{i+1} - R_i = 2 - 2e$ then $\alpha_i \in \mathbb{Z}$ and $0 < \alpha_i \leq (R_{i+1} - R_i)/2 + e = 1$ so $\alpha_i = 1 = (R_{i+1} - R_i)/2 + e$. If $R_{i+1} - R_i = 2e - 2$ then $\alpha_i \in \mathbb{Z}$ and $2e - 2 = R_{i+1} - R_i < \alpha_i \leq (R_{i+1} - R_i)/2 + e = 2e - 1$ so $\alpha_i = 2e - 1 = (R_{i+1} - R_i)/2 + e$.

(ii) We use Lemma 2.7(ii) and (iii). If $R_{i+1} - R_i > 2e$ then $\alpha_i = (R_{i+1} - R_i)/2 + e < R_{i+1} - R_i$. If $R_{i+1} - R_i < 2e$ then $\alpha_i = R_{i+1} - R_i < (R_{i+1} - R_i)/2 + e$. In both cases $\alpha_i = \min\{(R_{i+1} - R_i)/2 + e, R_{i+1} - R_i\}$. □

Lemma 2.10. *Let* a *be a norm generator of a lattice* L *and let* $\mathfrak{w} \supseteq 2\mathfrak{s}L$ *be a fractional ideal. Then* $\mathfrak{w} = \mathfrak{w}L$ *iff* $\mathfrak{g}L = \mathsf{a}\mathcal{O}^2 + \mathfrak{w}$ *and we have either* $\mathfrak{w} = 2\mathfrak{s}L$ *or* $\operatorname{ord} \mathsf{a} + \operatorname{ord} \mathfrak{w}$ *is odd.*

Proof. For the necessity, see [4, 93A]. For the sufficiency it is enough to prove that, given another fractional ideal $\mathfrak{w}'$ satisfying the hypothesis of the lemma, we have $\mathfrak{w} = \mathfrak{w}'$. Suppose that $\mathfrak{w} \neq \mathfrak{w}'$. We may assume that $\mathfrak{w} \supset \mathfrak{w}'$. Since $\mathfrak{w} \supset \mathfrak{w}' \supseteq 2\mathfrak{s}L$ we must have that $\operatorname{ord} \mathsf{a} + \operatorname{ord} \mathfrak{w}$ is odd. Let $\mathfrak{w} = \mathsf{b}\mathcal{O}$. Then $\mathsf{a} + \mathsf{b} \in \mathsf{a}\mathcal{O}^2 + \mathfrak{w} = \mathfrak{g}L = \mathsf{a}\mathcal{O}^2 + \mathfrak{w}'$. So $\mathsf{a} + \mathsf{b} = \mathsf{a}\alpha^2 + \mathsf{b}'$ for some $\alpha \in \mathcal{O}$ and $\mathsf{b}' \in \mathfrak{w}' \subset \mathfrak{w}$. It follows that $1 + \mathsf{b}/\mathsf{a} = \alpha^2 + \mathsf{b}'/\mathsf{a}$, which implies that $\mathfrak{d}(1 + \mathsf{b}/\mathsf{a}) \subseteq \mathsf{b}'/\mathsf{a}\mathcal{O} \subset \mathsf{a}^{-1}\mathfrak{w}$.

On the other hand $\operatorname{ord} \mathsf{b}/\mathsf{a} = \operatorname{ord} \mathsf{a}^{-1}\mathfrak{w}$ is odd and, since $\mathsf{b}\mathcal{O} = \mathfrak{w} \subseteq \mathfrak{g}L \subseteq \mathsf{a}\mathcal{O}$ and $\mathsf{b}\mathcal{O} = \mathfrak{w} \supset 2\mathfrak{s}L \supseteq 4\mathsf{a}\mathcal{O}$, we have $4\mathcal{O} \subset \mathsf{b}/\mathsf{a}\mathcal{O} \subseteq \mathcal{O}$. By [4, 63:5] we get $\mathfrak{d}(1 + \mathsf{b}/\mathsf{a}) = \mathsf{b}/\mathsf{a}\mathcal{O} = \mathsf{a}^{-1}\mathfrak{w}$, a contradiction. □

Lemma 2.11. *Let $J_1, \dots, J_s$ be lattices in the same quadratic space and let $J = \sum J_k$. If a_k and a are norm generators for J_k and a and J, respectively, then:*

$$\mathfrak{g}J = \sum \mathfrak{g}J_k + 2\mathfrak{s}J \text{ and } \mathfrak{w}J = \sum \mathfrak{w}J_k + \sum \mathsf{a}^{-1}\mathfrak{d}(\mathsf{a}\mathsf{a}_k) + 2\mathfrak{s}J.$$

Proof. We have $\mathfrak{g}J_k \subseteq \mathfrak{g}J$ and $2\mathfrak{s}J \subseteq \mathfrak{g}J$ so $\mathfrak{g}J \supseteq \sum \mathfrak{g}J_k + 2\mathfrak{s}J$. For the reverse inclusion note that $Q(J) \subseteq \sum Q(J_k) + 2\mathfrak{s}J$. Thus $\mathfrak{g}J = Q(J) + 2\mathfrak{s}J \subseteq \sum(Q(J_k) + 2\mathfrak{s}J_k) + 2\mathfrak{s}J = \sum \mathfrak{g}J_k + 2\mathfrak{s}J$.

We have $\mathsf{a}\mathcal{O}^2 \subseteq \mathfrak{g}J$ and $2\mathsf{a}\mathcal{O} = 2\mathfrak{n}J \subseteq 2\mathfrak{s}J \subseteq \mathfrak{g}J$ so $\mathfrak{g}J = \mathsf{a}\mathcal{O}^2 + 2\mathsf{a}\mathcal{O} + \mathfrak{g}J = \mathsf{a}\mathcal{O}^2 + 2\mathsf{a}\mathcal{O} + \sum \mathfrak{g}J_k + 2\mathfrak{s}J = \mathsf{a}\mathcal{O}^2 + 2\mathsf{a}\mathcal{O} + \sum \mathsf{a}_k\mathcal{O}^2 + \sum \mathfrak{w}J_k + 2\mathfrak{s}J$. But $\mathsf{a}\mathcal{O}^2 + \sum \mathsf{a}_k\mathcal{O}^2 + 2\mathsf{a}\mathcal{O} = \mathfrak{g}(\langle \mathsf{a}, \mathsf{a}_1, \dots, \mathsf{a}_s\rangle)$. (We have $\mathfrak{s}(\langle \mathsf{a}, \mathsf{a}_1, \dots, \mathsf{a}_s\rangle) = \mathfrak{n}(\langle \mathsf{a}, \mathsf{a}_1, \dots, \mathsf{a}_s\rangle) = \mathsf{a}\mathcal{O}$.) But $\mathfrak{w}(\langle \mathsf{a}, \mathsf{a}_1, \dots, \mathsf{a}_s\rangle) = \sum \mathsf{a}^{-1}\mathfrak{d}(\mathsf{a}\mathsf{a}_k) + 2\mathsf{a}\mathcal{O}$. (See [4, p. 280]. We have $\mathsf{a}\mathfrak{d}(\mathsf{a}_k/\mathsf{a}) = \mathsf{a}^{-1}\mathfrak{d}(\mathsf{a}\mathsf{a}_k)$.) So $\mathfrak{g}J = \mathfrak{g}(\langle \mathsf{a}, \mathsf{a}_1, \dots, \mathsf{a}_s\rangle) + \sum \mathfrak{w}J_k + 2\mathfrak{s}J = \mathsf{a}\mathcal{O}^2 + \sum \mathsf{a}^{-1}\mathfrak{d}(\mathsf{a}\mathsf{a}_k) + 2\mathsf{a}\mathcal{O} + \sum \mathfrak{w}J_k + 2\mathfrak{s}J = \mathsf{a}\mathcal{O}^2 + \sum \mathfrak{w}J_k + \sum \mathsf{a}^{-1}\mathfrak{d}(\mathsf{a}\mathsf{a}_k) + 2\mathfrak{s}J$. (Recall, $2\mathsf{a}\mathcal{O} \subseteq 2\mathfrak{s}J$.) Let $\mathfrak{w} = \sum \mathfrak{w}J_k + \sum \mathsf{a}^{-1}\mathfrak{d}(\mathsf{a}\mathsf{a}_k) + 2\mathfrak{s}J$. We have $\mathfrak{g}J = \mathsf{a}\mathcal{O}^2 + \mathfrak{w}$ and $2\mathfrak{s}J \subseteq \mathfrak{w}$. By Lemma 2.10 in order to prove that $\mathfrak{w} = \mathfrak{w}J$ we still need to prove that $\mathfrak{w} = 2\mathfrak{s}J$ or $\operatorname{ord} \mathsf{a} + \operatorname{ord} \mathfrak{w}$ is odd. If $\mathfrak{w} \neq 2\mathfrak{s}J$, i.e. $\mathfrak{w} \supset 2\mathfrak{s}J$, then $\mathfrak{w} = \mathsf{a}^{-1}\mathfrak{d}(\mathsf{a}\mathsf{a}_k)$ or $\mathfrak{w} = \mathfrak{w}J_k$ for some k. Suppose that $\mathfrak{w} = \mathfrak{w}J_k$. We cannot have $\mathfrak{w}J_k = 2\mathfrak{s}J_k \subseteq 2\mathfrak{s}J$. So $\operatorname{ord} \mathsf{a}_k + \operatorname{ord} \mathfrak{w}J_k$ is odd, which implies that $\operatorname{ord} \mathsf{a} + \operatorname{ord} \mathfrak{w}J_k$ is odd unless $\operatorname{ord} (\mathsf{a}\mathsf{a}_k)$ is odd. But this would imply that $\mathsf{a}_k\mathcal{O} = \mathsf{a}^{-1}\mathfrak{d}(\mathsf{a}\mathsf{a}_k) \subseteq \mathfrak{w} = \mathfrak{w}J_k$, so $\mathfrak{w}J_k = \mathsf{a}_k\mathcal{O}$, which contradicts the fact that $\operatorname{ord} \mathsf{a}_k + \operatorname{ord} \mathfrak{w}J_k$ is odd. Finally if $\mathfrak{w} = \mathsf{a}^{-1}\mathfrak{d}(\mathsf{a}\mathsf{a}_k)$ then $\operatorname{ord} \mathsf{a} + \operatorname{ord} \mathfrak{w} = \operatorname{ord} \mathfrak{d}(\mathsf{a}\mathsf{a}_k)$ is odd unless $\mathsf{a}\mathsf{a}_k \in \Delta\dot{F}^2$. (If $\alpha \in \dot{F}$ has odd order then $\mathfrak{d}(\alpha) = \alpha\mathcal{O}$ has odd order. If $\operatorname{ord} \alpha$ is even then $\operatorname{ord} \mathfrak{d}(\alpha) = \operatorname{ord} \alpha + d(\alpha) \equiv d(\alpha) \pmod 2$ is even iff $d(\alpha) = 2e$, i.e. iff $\alpha \in \Delta\dot{F}^2$.) But this implies that $\mathfrak{d}(\mathsf{a}\mathsf{a}_k) = 4\mathsf{a}\mathsf{a}_k\mathcal{O}$, i.e. $\mathfrak{w} = \mathsf{a}^{-1}\mathfrak{d}(\mathsf{a}\mathsf{a}_k) = 4\mathsf{a}_k\mathcal{O} \subset 2\mathfrak{s}J$, a contradiction. □

Lemma 2.12. *Suppose that $\mathfrak{n}L_k = \mathfrak{n}L^{\mathfrak{s}_k}$, $\mathfrak{n}L_{k+1} = \mathfrak{n}L^{\mathfrak{s}_{k+1}}$ and a_k and a_{k+1} are norm generators for L_k and L_{k+1}, respectively. If $u_k + u_{k+1}$ is even, then*

$$\mathfrak{f}_k = \mathfrak{s}_k^{-2}\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) + \mathsf{a}_k\mathfrak{s}_k^{-2}\mathfrak{w}L^*_{(k+1)} + \mathsf{a}_{k+1}\mathfrak{w}L^\sharp_{(k)} + 2\mathfrak{p}^{(u_k+u_{k+1})/2-r_k}.$$

Proof. We have $L^{\mathfrak{s}_k} = \mathfrak{s}_kL^\sharp_{(k)} \perp L^*_{(k+1)}$ and $L^{\mathfrak{s}_{k+1}} = \mathfrak{s}_{k+1}L^\sharp_{(k)} \perp L^*_{(k+1)}$. Now $L_{k+1} \subseteq L^*_{k+1} \subseteq L^{\mathfrak{s}_{k+1}}$ and $L_k \subseteq \mathfrak{s}_kL^\sharp_{(k)} \subseteq L^{\mathfrak{s}_k}$. Thus a_{k+1} is a norm generator for $L^*_{(k+1)}$ and for $L^{\mathfrak{s}_{k+1}}$ and a_k is a norm generator for $\mathfrak{s}_kL^\sharp_{(k)}$ and for $L^{\mathfrak{s}_k}$. Also $\pi^{2(r_{k+1}-r_k)}\mathsf{a}_k$ is a norm generator for $\mathfrak{s}_{k+1}L^\sharp_{(k)}$. By Lemma 2.11 we get $\mathfrak{w}_k = \mathsf{a}_k^{-1}\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) + \mathfrak{w}(\mathfrak{s}_kL^\sharp_{(k)}) + \mathfrak{w}L^*_{(k+1)} + 2\mathfrak{s}_k = \mathsf{a}_k^{-1}\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) + \mathfrak{s}_k^2\mathfrak{w}L^\sharp_{(k)} + \mathfrak{w}L^*_{(k+1)} + 2\mathfrak{s}_k$ and $\mathfrak{w}_{k+1} = \mathsf{a}_{k+1}^{-1}\mathfrak{d}(\pi^{2(r_{k+1}-r_k)}\mathsf{a}_k\mathsf{a}_{k+1}) + \mathfrak{w}(\mathfrak{s}_{k+1}L^\sharp_{(k)}) + \mathfrak{w}L^*_{(k+1)} + 2\mathfrak{s}_{k+1} = \mathsf{a}_{k+1}^{-1}\mathfrak{s}_{k+1}^2\mathfrak{s}_k^{-2}\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) + \mathfrak{s}_{k+1}^2\mathfrak{w}L^\sharp_{(k)} + \mathfrak{w}L^*_{(k+1)} + 2\mathfrak{s}_{k+1}$.

By [4, 93:26] we have $\mathfrak{s}_k^2\mathfrak{f}_k = \mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) + \mathsf{a}_{k+1}\mathfrak{w}_k + \mathsf{a}_k\mathfrak{w}_{k+1} + 2\mathfrak{p}^{(u_k+u_{k+1})/2+r_k} = \mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) + \mathsf{a}_k^{-1}\mathsf{a}_{k+1}\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) + \mathsf{a}_{k+1}\mathfrak{s}_k^2\mathfrak{w}L^\sharp_{(k)} + \mathsf{a}_{k+1}\mathfrak{w}L^*_{(k+1)} + 2\mathsf{a}_{k+1}\mathfrak{s}_k + \mathsf{a}_k\mathsf{a}_{k+1}^{-1}\mathfrak{s}_{k+1}^2\mathfrak{s}_k^{-2}\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) + \mathsf{a}_k\mathfrak{s}_{k+1}^2\mathfrak{w}L^\sharp_{(k)} + \mathsf{a}_k\mathfrak{w}L^*_{(k+1)} + 2\mathsf{a}_k\mathfrak{s}_{k+1} + 2\mathfrak{p}^{(u_k+u_{k+1})/2+r_k}$.

But $\mathsf{a}_k\mathcal{O} \supseteq \mathsf{a}_{k+1}\mathcal{O}$ and $\mathsf{a}_k\mathfrak{s}_k^{-2} \subseteq \mathsf{a}_{k+1}\mathfrak{s}_{k+1}^{-2}$ ([4, 93:25]), so $u_k \le u_{k+1}$ and $u_k - 2r_k \ge u_{k+1} - 2r_{k+1}$. Thus $\mathsf{a}_k^{-1}\mathsf{a}_{k+1}\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}), \mathsf{a}_k\mathsf{a}_{k+1}^{-1}\mathfrak{s}_{k+1}^2\mathfrak{s}_k^{-2}\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) \subseteq \mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1})$. Also $\mathsf{a}_k\mathfrak{s}_{k+1}^2\mathfrak{w}L^\sharp_{(k)} \subseteq \mathsf{a}_{k+1}\mathfrak{s}_k^2\mathfrak{w}L^\sharp_{(k)}$ and $\mathsf{a}_{k+1}\mathfrak{w}L^*_{(k+1)} \subseteq \mathsf{a}_k\mathfrak{w}L^*_{(k+1)}$.

Also $\operatorname{ord} \mathsf{a}_{k+1}\mathfrak{s}_k = u_{k+1} + r_k \geq (u_k + u_{k+1})/2 + r_k$ (we have $u_{k+1} \geq u_k$) and $\operatorname{ord} \mathsf{a}_k\mathfrak{s}_{k+1} = u_k + r_{k+1} \geq (u_k + u_{k+1})/2 + r_k$ (we have $u_k - 2r_k \geq u_{k+1} - 2r_{k+1}$). Hence $2\mathsf{a}_{k+1}\mathfrak{s}_k, 2\mathsf{a}_k\mathfrak{s}_{k+1} \subseteq 2\mathfrak{p}^{(u_k+u_{k+1})/2+r_k}$.

By removing all unnecessary terms (which are included in others) we get $\mathfrak{s}_k^2\mathfrak{f}_k = \mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) + \mathsf{a}_{k+1}\mathfrak{s}_k^2\mathfrak{w}L^\sharp_{(k)} + \mathsf{a}_k\mathfrak{w}L^*_{(k+1)} + 2\mathfrak{p}^{(u_k+u_{k+1})/2+r_k}$. When we divide by $\mathfrak{s}_k^2$ we get the desired result. □

Suppose $L \cong \prec a_1, \ldots, a_n \succ$ relative to the good BONG $x_1, \ldots, x_n$. Let $L = L^1 \perp \ldots \perp L^m$ be a maximal norm splitting with all the binary components improper such that $x_1, \ldots, x_n$ is obtained by putting together the BONGs of $L^1, \ldots, L^m$. We choose the Jordan decomposition $L = L_1 \perp \ldots \perp L_t$ with components obtained by putting together the L^j's of the same scale (see also the proof of [1, Lemma 4.7]). So the L^j's with $\mathfrak{s}L^j = \mathfrak{s}_k$ make a maximal norm splitting for L_k, those with $\mathfrak{s}L^j \supseteq \mathfrak{s}_k$ a maximal norm splitting for $L_{(k)}$ and those with $\mathfrak{s}L^j \subset \mathfrak{s}_k$ a maximal norm splitting for $L^*_{(k+1)}$. By putting together the BONGs of the components of these maximal norm splittings we get good BONGs for L_k, $L_{(k)}$ and $L^*_{(k+1)}$. It follows that $L_k = \prec x_{n_{k-1}+1}, \ldots, x_{n_k} \succ$, $L_{(k)} = \prec x_1, \ldots, x_{n_k} \succ$ and $L^*_{(k+1)} = \prec x_{n_k+1}, \ldots, x_n \succ$. Also $\mathfrak{n}L_k = \mathfrak{n}L^{\mathfrak{s}_k}$. (For any L^j with $\mathfrak{s}L^j = \mathfrak{s}_k$ we have $L^j \subseteq L_k \subseteq L^{\mathfrak{s}_k}$ and $\mathfrak{n}L^j = \mathfrak{n}L^{\mathfrak{s}L^j} = \mathfrak{n}L^{\mathfrak{s}_k}$.)

Lemma 2.13. *(i) For any* $n_{k-1} + 1 \leq i \leq n_k$ *we have* $R_i = u_k$ *if* $i \equiv n_{k-1} + 1 \pmod 2$ *and* $R_i = 2r_k - u_k$ *if* $i \equiv n_{k-1} \pmod 2$.

(ii) For any $n_{k-1} + 1 \leq i \leq n_k$ *we have* $R_i = u_k$ *if* $i \equiv n_k + 1 \pmod 2$ *and* $R_i = 2r_k - u_k$ *if* $i \equiv n_k \pmod 2$.

(iii) $\pm a_{n_{k-1}+1}$ *and* $\pm\pi^{2u_k-2r_k}a_{n_k}$ *are norm generators for* L_k *and for* $L^{\mathfrak{s}_k}$.

Proof. If L_k is improper then $\dim L_k$ is even so $n_{k-1} \equiv n_k \pmod 2$. Also the sequence $R_{n_{k-1}+1}, \ldots, R_{n_k}$ is $u_k, 2r_k - u_k, \ldots, u_k, 2r_k - u_k$ so we get both (i) and (ii). If L_k is proper then $u_k = r_k$ and the sequence $R_{n_{k-1}+1}, \ldots, R_{n_k}$ is $r_k, \ldots, r_k$. But $u_k = r_k$, so $r_k = u_k = 2r_k - u_k$ and again we get both (i) and (ii).

(iii) We have $L_k \cong \prec a_{n_{k-1}+1}, \ldots, a_{n_k} \succ$ so $a_{n_{k-1}+1}$ is a norm generator for L_k. We have $L_k^\sharp \cong \prec a_{n_k}^{-1}, \ldots, a_{n_{k-1}}^{-1} \succ$ so $a_{n_k}^{-1}$ is a norm generator for $L_k^\sharp = \mathfrak{p}^{-r_k}L_k$. Therefore $\pi^{2r_k}a_{n_k}^{-1}$ is a norm generator for L_k. But $\operatorname{ord} a_{n_k} = 2r_k - u_k$, so $\pi^{2u_k-4r_k}a_{n_k}$ differs from $a_{n_k}^{-1}$ by the square of a unit. Since $\pi^{2r_k}a_{n_k}^{-1}$ is a norm generator for L_k so is $\pi^{2r_k}\pi^{2u_k-4r_k}a_{n_k} = \pi^{2u_k-2r_k}a_{n_k}$. Since $\mathfrak{g}L_k$ is an additive group, $-a_{n_{k-1}+1}$ and $-\pi^{2u_k-2r_k}a_{n_k}$ will also be norm generators for L_k. We have $L_k \subseteq L^{\mathfrak{s}_k}$ and $\mathfrak{n}L_k = \mathfrak{n}L^{\mathfrak{s}_k}$ so $\pm a_{n_{k-1}+1}$ and $\pm\pi^{2u_k-2r_k}a_{n_k}$ are norm generators for $L^{\mathfrak{s}L_k}$ as well. □

We now want to find relations between the α_i's and the O'Meara invariants $\mathfrak{w}_k$ and $\mathfrak{f}_k$. In particular, this will prove that the α_i's are invariants of the lattice L, i.e. they do not depend on the choice of the BONG of L.

Lemma 2.14. $\operatorname{ord} \mathfrak{w}L = \min\{R_1 + \alpha_1, R_1 + e\}$. *(If* $n = 1$ *we ignore* $R_1 + \alpha_1$*.)*

If moreover L_1 *is not unary then* $\operatorname{ord} \mathfrak{w}L = R_1 + \alpha_1$.

Proof. Note that if L_1 is not unary, in particular if L^1 is binary, then $R_1 = u_1 \geq 2r_1 - u_1 = R_2$ so $\alpha_1 \leq (R_2 - R_1)/2 + e \leq e$. Hence $\min\{R_1 + \alpha_1, R_1 + e\} = R_1 + \alpha_1$ and so the two statements of the lemma are equivalent.

We use induction on m, the number of components in the maximal norm splitting we fixed for L. Suppose first that $m = 1$. If $L = L^1$ is unary then $\mathfrak{w}L = 2\mathfrak{s}L = 2\mathfrak{p}^{R_1}$

so $\operatorname{ord}\mathfrak{w}L^1 = R_1 + e$, as claimed. If $L = L^1$ is binary and so improper modular then we may assume that it is unimodular since the statement is invariant upon scaling. Hence $R_1 + R_2 = 0$ and $R_1 = \operatorname{ord}\mathfrak{n}L > \operatorname{ord}\mathfrak{s}L = 0$. Now $a_1 \in Q(L)$ is a norm generator. Thus by [4, 93:10] there is $b \in \mathfrak{w}L$ such that $L \cong A(a_1, b)$. Also if $\mathfrak{w}L \supset 2\mathfrak{s}L = 2\mathcal{O}$ then $\mathfrak{w}L = b\mathcal{O}$. Suppose first that $\mathfrak{w}L = 2\mathcal{O}$. Then $b \in 2\mathcal{O}$ so $\operatorname{ord} b \geq e$. Thus $d(-a_1a_2) = d(-\det L) = d(1 - a_1b) \geq \operatorname{ord} a_1b \geq R_1 + e$ so $R_2 - R_1 + d(-a_1a_2) = -2R_1 + d(-a_1a_2) \geq -R_1 + e$. On the other hand $(R_2 - R_1)/2 + e = -R_1 + e$ so $\alpha_1 = \min\{(R_2 - R_1)/2 + e, R_2 - R_1 + d(-a_1a_2)\} = -R_1 + e$. Thus $\operatorname{ord}\mathfrak{w}L = e = R_1 + \alpha_1$. If $\mathfrak{w}L \supset 2\mathfrak{s}L = 2\mathcal{O}$ then $\mathfrak{w}L = b\mathcal{O}$ and $\operatorname{ord} a_1 + \operatorname{ord} b$ is odd. Also $\operatorname{ord} a_1 = \operatorname{ord}\mathfrak{n}L \leq \operatorname{ord} 2\mathfrak{s}L = e$ and $\operatorname{ord} b = \operatorname{ord}\mathfrak{w}L < \operatorname{ord} 2\mathfrak{s}L = e$. It follows that $\operatorname{ord} a_1b < 2e$ and it is odd. Hence $d(-a_1a_2) = d(1 - a_1b) = \operatorname{ord} a_1b = R_1 + \operatorname{ord} b$ so $R_2 - R_1 + d(-a_1a_2) = -2R_1 + d(-a_1a_2) = -R_1 + \operatorname{ord} b$. Also $(R_2 - R_1)/2 + e = -R_1 + e > -R_1 + \operatorname{ord} b$. It follows that $\alpha_1 = -R_1 + \operatorname{ord} b = -R_1 + \operatorname{ord}\mathfrak{w}L$. So $\operatorname{ord}\mathfrak{w}L = R_1 + \alpha_1$.

We now prove the induction step. We have $L = L^1 \perp L'$, where $L' = L^2 \perp \ldots \perp L^m$. Now let a and a' be norm generators for L^1 and L'. We have $\mathfrak{n}L^1 = \mathfrak{n}L$ so a is also a norm generator for L. By Lemma 2.11 we have $\mathfrak{w}L = \mathfrak{w}L^1 + \mathfrak{w}L' + \mathsf{a}^{-1}\mathfrak{d}(\mathsf{a}\mathsf{a}')$. ($\mathsf{a}^{-1}\mathfrak{d}(\mathsf{a}\mathsf{a}) = 0$ and $2\mathfrak{s}L = 2\mathfrak{s}L^1 \subseteq \mathfrak{w}L^1$ can be ignored.) Since $\operatorname{ord}\mathsf{a}^{-1}\mathfrak{d}(\mathsf{a}\mathsf{a}') = \operatorname{ord}\mathsf{a}' + d(\mathsf{a}\mathsf{a}')$ it follows that $\operatorname{ord}\mathfrak{w}L = \min\{\operatorname{ord}\mathfrak{w}L^1, \operatorname{ord}\mathfrak{w}L', \operatorname{ord}\mathsf{a}' + d(\mathsf{a}\mathsf{a}')\}$.

If L^1 is unary then $R_1 \leq R_2$, $L^1 \cong \prec a_1 \succ$ and $L' \cong \prec a_2, \ldots, a_n \succ$. We take $\mathsf{a} = a_1$ and $\mathsf{a}' = -a_2$. We have $\operatorname{ord}\mathsf{a}' = R_2$, $\mathfrak{w}L^1 = R_1 + e$ and $\operatorname{ord}\mathfrak{w}L' = \min\{R_2 + \alpha_1(L'), R_2 + e\}$. It follows that $\operatorname{ord}\mathfrak{w}L = \min\{R_1 + e, R_2 + \alpha_1(L'), R_2 + e, R_2 + d(-a_1a_2)\}$. Since $R_2 + e \geq R_1 + e$, it can be removed. By Corollary 2.5 (ii) we have $\alpha_1 = \min\{(R_2 - R_1)/2 + e, R_2 - R_1 + d(-a_1a_2), R_2 - R_1 + \alpha_1(L')\}$. It follows that $\min\{R_1 + \alpha_1, R_1 + e\} = \min\{(R_1 + R_2)/2 + e, R_2 + d(-a_1a_2), R_2 + \alpha_1(L'), R_1 + e\}$. But $R_2 \geq R_1$, so $(R_1 + R_2)/2 + e \geq R_1 + e$. Thus $\min\{R_1 + \alpha_1, R_1 + e\} = \min\{R_2 + d(-a_1a_2), R_2 + \alpha_1(L'), R_1 + e\} = \operatorname{ord}\mathfrak{w}L$.

If L^1 is binary then $R_1 \geq R_2$, $L^1 \cong \prec a_1, a_2 \succ$ and $L' \cong \prec a_3, \ldots, a_n \succ$. We prove that $\operatorname{ord}\mathfrak{w}L = R_1 + \alpha_1$. We take $\mathsf{a} = \pi^{2u_1 - 2r_1}a_2$ and $\mathsf{a}' = -a_3$. (See Lemma 2.13(iii).) We have $\mathfrak{w}L^1 = R_1 + \alpha_1(L^1)$, $\mathfrak{w}L' = \min\{R_3 + \alpha_1(L'), R_3 + e\}$ and $\operatorname{ord}\mathsf{a}' + d(\mathsf{a}\mathsf{a}') = R_3 + d(-a_2a_3)$. Thus $\operatorname{ord}\mathfrak{w}L = \min\{R_1 + \alpha_1(L^1), R_3 + \alpha_1(L'), R_3 + d(-a_2a_3), R_3 + e\}$. But $e \geq (R_2 - R_1)/2 + e \geq \alpha_1(L')$, so $R_3 + e \geq R_3 + \alpha_1(L^1)$ and so $R_3 + e$ can be removed. On the other hand $\alpha_1 = \min\{\alpha_1(L^1), R_3 - R_1 + d(-a_2a_3), R_3 - R_1 + \alpha_1(L')\}$. (We have $\alpha_1(L^1) = \min\{(R_2 - R_1)/2 + e, R_2 - R_1 + d(-a_1a_2)\}$ and, by Lemma 2.4(ii), $R_3 - R_1 + \alpha_1(L') = R_3 - R_1 + \alpha_1(\prec a_3, \ldots, a_n \succ)$ can replace all $R_{j+1} - R_1 + d(-a_ja_{j+1})$ with $j \geq 3$.) So $R_1 + \alpha_1 = \min\{R_1 + \alpha_1(L^1), R_3 + d(-a_2a_3), R_3 + \alpha_1(L')\} = \operatorname{ord}\mathfrak{w}L$. □

Lemma 2.15. *If L_k is unary then $\mathfrak{w}_k = \mathfrak{s}_k(\mathfrak{f}_{k-1} + \mathfrak{f}_k + 2\mathcal{O})$. (The term $\mathfrak{f}_{k-1}$ is ignored if $k = 1$ and $\mathfrak{f}_k$ is ignored if $k = t$.)*

Proof. Since L_k is unary we have $\mathfrak{s}_k = \mathsf{a}_k\mathcal{O}$ and $u_k = r_k$. Also $\mathfrak{w}L_k = 2\mathfrak{s}_k$.

We have $L^{\mathfrak{s}_k} = (\perp_{j<k}\ \mathfrak{s}_k\mathfrak{s}_j^{-1}L_j) \perp L_k \perp (\perp_{j>k}\ L_j)$. The first orthogonal sum is included in $\mathfrak{s}_{k-1}^{-1}\mathfrak{s}_kL^{\mathfrak{s}_{k-1}}$, while the last one is included in $L^{\mathfrak{s}_{k+1}}$. Hence $L^{\mathfrak{s}_k} \subseteq L_k + \mathfrak{s}_{k-1}^{-1}\mathfrak{s}_kL^{\mathfrak{s}_{k-1}} + L^{\mathfrak{s}_{k+1}}$. The reverse inclusion follows from [4, 93:24] so $L^{\mathfrak{s}_k} = L_k + \mathfrak{s}_{k-1}^{-1}\mathfrak{s}_kL^{\mathfrak{s}_{k-1}} + L^{\mathfrak{s}_{k+1}}$. Now a_k is a norm generator for both $L^{\mathfrak{s}_k}$ and L_k, $\pi^{2(r_k - r_{k-1})}\mathsf{a}_{k-1}$ for $\mathfrak{s}_{k-1}^{-1}\mathfrak{s}_kL^{\mathfrak{s}_{k-1}}$ and a_{k+1} for $L^{\mathfrak{s}_{k+1}}$. By Lemma 2.11 we have $\mathfrak{w}_k = \mathsf{a}_k^{-1}\mathfrak{d}(\pi^{2(r_k - r_{k-1})}\mathsf{a}_{k-1}\mathsf{a}_k) + \mathsf{a}_k^{-1}\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) + \mathfrak{w}L_k + \mathfrak{w}(\mathfrak{s}_{k-1}^{-1}\mathfrak{s}_kL^{\mathfrak{s}_{k-1}}) + \mathfrak{w}L^{\mathfrak{s}_{k+1}} +$

$2\mathfrak{s}_k = \mathfrak{s}_{k-1}^{-2}\mathfrak{s}_k\mathfrak{d}(\mathsf{a}_{k-1}\mathsf{a}_k) + \mathfrak{s}_k^{-1}\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) + \mathfrak{s}_{k-1}^{-2}\mathfrak{s}_k^2\mathfrak{w}_{k-1} + \mathfrak{w}_{k+1} + 2\mathfrak{s}_k$. (We ignore $\mathsf{a}_k\mathfrak{d}(\mathsf{a}_k\mathsf{a}_k) = 0$.)

If $u_k + u_{k+1}$ is even then by [4, 93:26] we have $\mathfrak{s}_k^2\mathfrak{f}_k = \mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) + \mathsf{a}_{k+1}\mathfrak{w}_k + \mathsf{a}_k\mathfrak{w}_{k+1} + 2\mathfrak{p}^{(u_k+u_{k+1})/2+r_k})$. This formula also holds in the case when $u_k + u_{k+1}$ is odd if we drop the last term. Indeed, in this case $\mathfrak{s}_k^2\mathfrak{f}_k = \mathsf{a}_k\mathsf{a}_{k+1}\mathcal{O}$ but $\operatorname{ord}\mathsf{a}_k\mathsf{a}_{k+1}$ is odd so $\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) = \mathsf{a}_k\mathsf{a}_{k+1}\mathcal{O}$ and we also have $\mathsf{a}_{k+1}\mathfrak{w}_k, \mathsf{a}_k\mathfrak{w}_{k+1} \subseteq \mathsf{a}_k\mathsf{a}_{k+1}\mathcal{O}$. It follows that $\mathfrak{s}_k(\mathfrak{f}_{k-1} + \mathfrak{f}_k + 2\mathcal{O}) = \mathfrak{s}_k(\mathfrak{s}_{k-1}^{-2}\mathfrak{d}(\mathsf{a}_{k-1}\mathsf{a}_k) + \mathfrak{s}_{k-1}^{-2}\mathsf{a}_k\mathfrak{w}_{k-1} + \mathfrak{s}_{k-1}^{-2}\mathsf{a}_{k-1}\mathfrak{w}_k + 2\mathfrak{p}^{(u_{k-1}+u_k)/2-r_{k-1}} + \mathfrak{s}_k^{-2}\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) + \mathfrak{s}_k^{-2}\mathsf{a}_{k+1}\mathfrak{w}_k + \mathfrak{s}_k^{-2}\mathsf{a}_k\mathfrak{w}_{k+1} + 2\mathfrak{p}^{(u_k+u_{k+1})/2-r_k} + 2\mathcal{O})$. (If $u_k + u_{k+1}$ is odd we ignore $2\mathfrak{p}^{(u_k+u_{k+1})/2-r_k}$. If $u_{k-1} + u_k$ is odd we ignore $2\mathfrak{p}^{(u_{k-1}+u_k)/2-r_{k-1}}$.) But $r_k = u_k$, so $(u_{k-1} + u_k)/2 - r_{k-1} = (u_{k-1} - 2r_{k-1} + 2r_k - u_k)/2 \geq 0$ and $(u_k + u_{k+1})/2 - r_k = (u_{k+1} - u_k)/2 \geq 0$. Hence $2\mathfrak{p}^{(u_{k-1}+u_k)/2-r_{k-1}}, 2\mathfrak{p}^{(u_k+u_{k+1})/2-r_k} \subseteq 2\mathcal{O}$ so these terms can be ignored. Thus $\mathfrak{s}_k(\mathfrak{f}_{k-1}+\mathfrak{f}_k+2\mathcal{O}) = \mathfrak{s}_{k-1}^{-2}\mathfrak{s}_k\mathfrak{d}(\mathsf{a}_{k-1}\mathsf{a}_k)+\mathfrak{s}_{k-1}^{-2}\mathfrak{s}_k^2\mathfrak{w}_{k-1}+\mathfrak{s}_{k-1}^{-2}\mathfrak{s}_k\mathsf{a}_{k-1}\mathfrak{w}_k+\mathfrak{s}_k^{-1}\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1})+ \mathfrak{s}_k^{-1}\mathsf{a}_{k+1}\mathfrak{w}_k + \mathfrak{w}_{k+1} + 2\mathfrak{s}_k = \mathfrak{w}_k + \mathfrak{s}_{k-1}^{-2}\mathfrak{s}_k\mathsf{a}_{k-1}\mathfrak{w}_k + \mathfrak{s}_k^{-1}\mathsf{a}_{k+1}\mathfrak{w}_k = \mathfrak{w}_k$. (We have $\mathfrak{s}_{k-1}^{-2}\mathfrak{s}_k\mathsf{a}_{k-1} = \mathfrak{s}_{k-1}^{-2}\mathsf{a}_{k-1}(\mathfrak{s}_k^{-2}\mathsf{a}_k)^{-1} \subseteq \mathcal{O}$ and $\mathfrak{s}_k^{-1}\mathsf{a}_{k+1} = \mathsf{a}_k^{-1}\mathsf{a}_{k+1}\mathcal{O} \subseteq \mathcal{O}$.) □

Lemma 2.16. *Let $1 \leq i \leq n-1$. Then:*

(i) If $n_{k-1} < i < n_k$ for some $1 \leq k \leq t$, then $R_i+\alpha_i = \operatorname{ord}\mathfrak{w}_k$ and $-R_{i+1}+\alpha_i = \operatorname{ord}\mathfrak{w}_{t-k}^\sharp$.

(ii) Suppose that $i = n_k$ for some $1 \leq k \leq t-1$. If $R_{i+1} - R_i$ is even or $\leq 2e$ then $\alpha_i = \operatorname{ord}\mathfrak{f}_k$; otherwise $\alpha_i = (R_{i+1} - R_i)/2 + e$, $\operatorname{ord}\mathfrak{f}_k = R_{i+1} - R_i = 2\alpha_i - 2e$ and both α_i and $\operatorname{ord}\mathfrak{f}_k$ are $> 2e$.

Proof. (i) Note that $R_i + R_{i+1} = u_k + 2r_k - u_k = 2r_k$. Thus if $R_i + \alpha_i = \operatorname{ord}\mathfrak{w}_k$ then $-R_{i+1} + \alpha_i = \operatorname{ord}\mathfrak{w}_k - 2r_k = \operatorname{ord}\mathfrak{s}_k^{-2}\mathfrak{w}_k = \operatorname{ord}\mathfrak{w}_{t-k}^\sharp$ so it is enough to prove the first part of the statement. Also $R_{n_{k-1}+1} + R_{n_{k-1}+2} = R_{n_k-1} + R_{n_k} = 2r_k$ and so $R_{n_{k-1}+1}+\alpha_{n_{k-1}+1} = \ldots = R_{n_k-1}+\alpha_{n_k-1}$ by Corollary 2.3(i). Thus it is enough to prove our statement for only one value of $n_{k-1} < i < n_k$, say $i = n_{k-1} + 1$.

We use induction on t. Note that if $k = 1$ then $\operatorname{ord}\mathfrak{w}_1 = \operatorname{ord}\mathfrak{w}L = R_1 + \alpha_1$ by Lemma 2.14 so we are done. In particular, (i) is true when $t = 1$. Suppose now that $t \geq 2$. We may assume that $k \geq 2$. We have $L^{\mathfrak{s}_k} = \mathfrak{s}_k L_{(k-1)}^\sharp \perp L_{(k)}^*$. Since $i = n_{k-1} + 1 < n_k$ we have $R_{i-1} = 2r_{k-1} - u_{k-1}$, $R_i = u_k$, $R_{i+1} = 2r_k - u_k$, $L_{(k-1)} \cong \prec a_1, \ldots, a_{i-1} \succ$ and $L_{(k)}^* \cong \prec a_i, \ldots, a_n \succ$. Note that $R_i \geq R_{i+1}$. If a and b are norm generators for $L_{(k)}^*$ and $\mathfrak{s}_k L_{(k-1)}^\sharp$, respectively, then $\mathfrak{n}L_{(k)}^* = \mathfrak{p}^{R_i} = \mathfrak{p}^{u_k} = \mathfrak{n}L^{\mathfrak{s}_k}$. Therefore a is also a norm generator for $L^{\mathfrak{s}_k}$. By Lemma 2.11 we have $\mathfrak{w}_k = \mathfrak{w}L^{\mathfrak{s}_k} = \mathfrak{w}(\mathfrak{s}_k L_{(k-1)}^\sharp) + \mathfrak{w}L_{(k)}^* + \mathsf{a}^{-1}\mathfrak{d}(\mathsf{ab}) + 2\mathfrak{s}_k$, which implies that $\operatorname{ord}\mathfrak{w}_k = \min\{\operatorname{ord}\mathfrak{w}(\mathfrak{s}_k L_{(k-1)}^\sharp), \operatorname{ord}\mathfrak{w}L_{(k)}^*, \operatorname{ord}\mathsf{b} + d(\mathsf{ab}), r_k + e\}$. Now L_k is not unary so $\operatorname{ord}\mathfrak{w}L_{(k)}^* = R_i + \alpha_1(L_{(k)}^*)$ by Lemma 2.14. Also $L_{(k-1)}^\sharp \cong \prec a_{i-1}^{-1}, \ldots, a_1^{-1} \succ$ and $\operatorname{ord} a_{i-1}^{-1} = -R_{i-1}$ so by Lemma 2.14 we have $\operatorname{ord}\mathfrak{w}L_{(k-1)}^\sharp = \min\{-R_{i-1} + \alpha_1(L_{(k-1)}^\sharp), -R_{i-1} + e\} = \min\{-R_{i-1} + \alpha_{i-2}(L_{(k-1)}), -R_{i-1} + e\}$. (We have $\alpha_1(L_{(k-1)}^\sharp) = \alpha_{i-2}(L_{(k-1)})$ by 2.6.) It follows that $\operatorname{ord}\mathfrak{w}(\mathfrak{s}_k L_{(k-1)}^\sharp) = \min\{2r_k - R_{i-1} + \alpha_{i-2}(L_{(k-1)}), 2r_k - R_{i-1} + e\} = \min\{R_i + R_{i+1} - R_{i-1} + \alpha_{i-2}(L_{(k-1)}), R_i + R_{i+1} - R_{i-1} + e\}$. Now a_{i-1}^{-1} is a norm generator for $L_{(k-1)}^\sharp$, so $\mathsf{b} := \pi^{2r_k}a_{i-1}^{-1}$ is a norm generator for $\mathfrak{s}_k L_{(k-1)}^\sharp$, and $\mathsf{a} := -a_i$ is a norm generator for $L_{(k)}$. We get $\operatorname{ord}\mathsf{b} + d(\mathsf{ab}) = 2r_k - R_{i-1} + d(-\pi^{2r_k}a_{i-1}^{-1}a_i) = R_i + R_{i+1} - R_{i-1} + d(-a_{i-1}a_i)$. Also

$r_k + e = (R_i + R_{i+1})/2 + e$. Thus $\operatorname{ord} \mathfrak{w}_k = \min\{R_i + \alpha_1(L^*_{(k)}), R_i + R_{i+1} - R_{i-1} + d(-a_{j-1}a_j), R_i+R_{i+1}-R_{i-1}+\alpha_{i-2}(L_{(k-1)}), R_i+R_{i+1}-R_{i-1}+e, (R_i+R_{i+1})/2+e\}$. But $R_{i-1} \leq R_{i+1} \leq R_i$, so $R_i + R_{i+1} - R_{i-1} + e \geq (R_i + R_{i+1})/2 + e = R_i + (R_{i+1} - R_i)/2 + e \geq R_i + \alpha_1(L^*_{(k)})$, so the last two terms can be removed. By Lemma 2.4(i), $R_{i+1} - R_{i-1} + \alpha_{i-2}(L_{(k-1)}) = R_{i+1} - R_{i-1} + \alpha_{i-2}(\prec a_1, \dots, a_{i-1} \succ)$ replaces all the terms in the definition of α_i with $1 \leq j \leq i-2$, while by Lemma 2.1, $\alpha_1(L^*_{(k)}) = \alpha_1(\prec a_i, \dots, a_n \succ)$ replaces $(R_{i+1}-R_i)/2+e$ and the terms with $i \leq j < n$. Hence $\alpha_i = \min\{R_{i+1}-R_{i-1}+\alpha_{i-2}(L_{(k-1)}), R_{i+1}-R_{i-1}+d(-a_{i-1}a_i), \alpha_1(L^*_{(k)})\}$. It follows that $R_i + \alpha_i = \min\{R_i + R_{i+1} - R_{i-1} + \alpha_{i-2}(L_{(k-1)}), R_i + R_{i+1} - R_{i-1} + d(-a_{i-1}a_i), R_i + \alpha_1(L^*_{(k)})\} = \operatorname{ord} \mathfrak{w}_k$.

(ii) Since $i = n_k$ we have $R_i = 2r_k - u_k$, $R_{i+1} = u_{k+1}$, $L_{(k)} \cong \prec a_1, \dots, a_i \succ$ and $L^*_{(k+1)} \cong \prec a_{i+1}, \dots, a_n \succ$. We have $R_{i+1} - R_i = u_k + u_{k+1} - 2r_k$, so $R_{i+1} - R_i$ is even iff u_k+u_{k+1} is even. If u_k+u_{k+1} and $R_{i+1}-R_i$ are odd then $\mathfrak{f}_k = \mathfrak{s}_k^{-2}\mathsf{a}_k\mathsf{a}_{k+1}$ so $\operatorname{ord} \mathfrak{f}_k = u_k+u_{k+1}-2r_k = R_{i+1}-R_i$. If $R_{i+1}-R_i < 2e$ then $\alpha_i = R_{i+1}-R_i = \operatorname{ord} \mathfrak{f}_k$, while if $R_{i+1} - R_i > 2e$ then $\alpha_i = (R_{i+1} - R_i)/2 + e$. (See Lemma 2.7(ii) and (iii).)

Suppose now that $R_{i+1}-R_i$ is even. By Lemma 2.12 we have $\mathfrak{f}_k = \mathfrak{s}_k^{-2}\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) + \mathsf{a}_k\mathfrak{s}_k^{-2}\mathfrak{w}L^*_{(k+1)} + \mathsf{a}_{k+1}\mathfrak{w}L^\sharp_{(k)} + 2\mathfrak{p}^{(u_k+u_{k+1})/2-r_k}$. We take $\mathsf{a}_k = \pi^{2u_k-2r_k}a_i$ and $\mathsf{a}_{k+1} = -a_{i+1}$. (See Lemma 2.13(iii).) Thus $\operatorname{ord} \mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) = \operatorname{ord}(\mathsf{a}_k\mathsf{a}_{k+1}) + d(\mathsf{a}_k\mathsf{a}_{k+1}) = u_k + u_{k+1} + d(-a_ia_{i+1})$ and so $\operatorname{ord} \mathfrak{s}_k^{-2}\mathfrak{d}(\mathsf{a}_k\mathsf{a}_{k+1}) = -2r_k + u_k + u_{k+1} + d(-a_ia_{i+1}) = R_{i+1} - R_i + d(-a_ia_{i+1})$. By Lemma 2.14 we have $\operatorname{ord} \mathfrak{w}L^*_{(k+1)} = \min\{R_{i+1} + \alpha_1(L^*_{k+1}), R_{i+1} + e\}$. Since $\operatorname{ord} \mathsf{a}_k\mathfrak{s}_k^{-2} = u_k - 2r_k = -R_i$, we get $\operatorname{ord}(\mathsf{a}_k\mathfrak{s}_k^{-2}\mathfrak{w}L^*_{(k+1)}) = \min\{R_{i+1} - R_i + \alpha_1(L^*_{(k+1)}), R_{i+1} - R_i + e\}$. We have $L^\sharp_{(k)} \cong \prec a_i^{-1}, \dots, a_1^{-1} \succ$, so $\operatorname{ord} \mathfrak{w}L^\sharp_{(k)} = \min\{-R_i + \alpha_1(L^\sharp_{(k)}), -R_i + e\}$. Since $\alpha_1(L^\sharp_{(k)}) = \alpha_{i-1}(L_{(k)})$ (see Remark 2.6) and $\operatorname{ord} \mathsf{a}_{k+1} = u_{k+1} = R_{i+1}$ we have $\operatorname{ord}(\mathsf{a}_{k+1}\mathfrak{w}L^\sharp_{(k)}) = \min\{R_{i+1} - R_i + \alpha_{i-1}(L_{(k)}), R_{i+1} - R_i + e\}$. Finally $\operatorname{ord} 2\mathfrak{p}^{(u_k+u_{k+1})/2-r_k} = (u_k + u_{k+1})/2 - r_k + e = (R_{i+1} - R_i)/2 + e$. Thus $\operatorname{ord} \mathfrak{f}_k = \min\{(R_{i+1} - R_i)/2 + e, R_{i+1} - R_i + d(-a_ia_{i+1}), R_{i+1} - R_i + \alpha_{i-1}(L_{(k)}), R_{i+1} - R_i + \alpha_1(L^*_{(k+1)}), R_{i+1} - R_i + e\}$. But $R_i = 2r_k - u_k \leq u_k \leq u_{k+1} = R_{i+1}$ so $R_{i+1} - R_i + e \geq (R_{i+1} - R_i)/2 + e$, so it can be ignored. So $\operatorname{ord} \mathfrak{f}_k = \min\{(R_{i+1} - R_i)/2 + e, R_{i+1} - R_i + d(-a_ia_{i+1}), R_{i+1} - R_i + \alpha_{i-1}(L_{(k)}), R_{i+1} - R_i + \alpha_1(L^*_{(k+1)})\}$, which, by Corollary 2.5(ii), is equal to α_i. (Recall, $L_{(k)} \cong \prec a_1, \dots, a_i \succ$ and $L^*_{(k+1)} \cong \prec a_{i+1}, \dots, a_n \succ$.) □

Corollary 2.17. *(i) If L_k is not unary and $i = n_{k-1}+1$ or n_k-1 then* $\operatorname{ord} \mathsf{a}_k^{-1}\mathfrak{w}_k = \alpha_i$.

(ii) If L_k is unary and $i = n_k$ then $\operatorname{ord} \mathsf{a}_k^{-1}\mathfrak{w}_k = \min\{\alpha_{i-1}, \alpha_i, e\}$. *(We ignore α_{i-1} if $i = 1$, and α_i if $i = n$.)*

Proof. (i) In both cases when $i = n_{k-1} + 1$ or $n_k - 1$ we have $R_i = u_k = \operatorname{ord} \mathsf{a}_k$. Hence $\operatorname{ord} \mathfrak{w}_k = R_i + \alpha_i = \operatorname{ord} \mathsf{a}_k + \alpha_i$ so $\operatorname{ord} \mathsf{a}_k^{-1}\mathfrak{w}_k = \alpha_i$.

(ii) We have $\mathfrak{s}_k = \mathsf{a}_k\mathcal{O}$ and, by Lemma 2.15, $\mathfrak{w}_k = \mathfrak{s}_k(\mathfrak{f}_{k-1} + \mathfrak{f}_k + 2\mathcal{O})$ so $\mathsf{a}_k^{-1}\mathfrak{w}_k = \mathfrak{f}_{k-1} + \mathfrak{f}_k + 2\mathcal{O}$. Thus $\operatorname{ord} \mathsf{a}_k^{-1}\mathfrak{w}_k = \min\{\operatorname{ord} \mathfrak{f}_{k-1}, \operatorname{ord} \mathfrak{f}_k, e\}$ and we have to prove that it is equal to $\min\{\alpha_{i-1}, \alpha_i, e\}$. Now $i - 1 = n_k - 1 = n_{k-1}$ so, by Lemma 2.16(ii), we have either $\alpha_{i-1} = \operatorname{ord} \mathfrak{f}_{k-1}$ or $\alpha_{i-1}, \operatorname{ord} \mathfrak{f}_{k-1} > 2e$. But if $\alpha_{i-1}, \operatorname{ord} \mathfrak{f}_{k-1} > 2e > e$ then they can be ignored in $\min\{\alpha_{i-1}, \alpha_i, e\}$ and $\min\{\operatorname{ord} \mathfrak{f}_{k-1}, \operatorname{ord} \mathfrak{f}_k, e\}$, respectively. Similarly either $\alpha_i = \operatorname{ord} \mathfrak{f}_k$ or $\alpha_i, \operatorname{ord} \mathfrak{f}_k$ are both $> 2e > e$ so they can be ignored. Thus $\min\{\alpha_{i-1}, \alpha_i, e\} = \min\{\operatorname{ord} \mathfrak{f}_{k-1}, \operatorname{ord} \mathfrak{f}_k, e\}$. □

3. Main theorem

In this section we state and prove the main result of this paper, the classification of integral lattices over dyadic local fields in terms of good BONGs. It is well known that this problem was first solved by O'Meara in [4, Theorem 93:28]. Since our proof uses O'Meara's result we first state Theorem 93:28.

Throughout this section L, K are two lattices with $L \cong \prec a_1, \ldots, a_n \succ$ and $K \cong \prec b_1, \ldots, b_n \succ$ relative to good BONGs. In terms of Jordan decompositions we write $L = L_1 \perp \ldots \perp L_t$ and $K = K_1 \perp \ldots \perp K_{t'}$. Let $\mathfrak{s}_k = \mathfrak{s}L_k$, $\mathfrak{s}'_k = \mathfrak{s}K_k$, $\mathfrak{g}_k = \mathfrak{g}L^{\mathfrak{s}_k}$, $\mathfrak{g}'_k = \mathfrak{g}K^{\mathfrak{s}'_k}$, $\mathfrak{w}_k = \mathfrak{w}L^{\mathfrak{s}_k}$, $\mathfrak{w}'_k = \mathfrak{w}K^{\mathfrak{s}'_k}$, $\mathfrak{f}_k = \mathfrak{f}_k(L)$ and $\mathfrak{f}'_k = \mathfrak{f}_k(K)$. Let a_k and b_k be norm generators for $L^{\mathfrak{s}_k}$ and $K^{\mathfrak{s}'_k}$, respectively. We say that L and K are of the same fundamental type if

$$t = t',\ \dim L_k = \dim K_k,\ \mathfrak{s}_k = \mathfrak{s}'_k,\ \mathfrak{g}_k = \mathfrak{g}'_k$$

for $1 \leq k \leq t$. These conditions are equivalent to

$$t = t',\ \dim L_k = \dim K_k,\ \mathfrak{s}_k = \mathfrak{s}'_k,\ \mathfrak{w}_k = \mathfrak{w}'_k,\ \mathsf{a}_k \cong \mathsf{b}_k \pmod{\mathfrak{w}_k}$$

for $1 \leq k \leq t$. We now state O'Meara's Theorem 93:28.

Theorem 93:28. *Let L, K be lattices with the same fundamental type such that $FL \cong FK$. Let $L_{(1)} \subset \cdots \subset L_{(t)}$ and $K_{(1)} \subset \cdots \subset K_{(t)}$ be Jordan chains for L and K. Then $L \cong K$ if and only if the following conditions hold for $1 \leq i \leq t-1$:*
(i) $\det L_{(k)}/\det K_{(k)} \cong 1 \pmod{\mathfrak{f}_k}$;
(ii) $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{a}_{k+1}]$ *when* $\mathfrak{f}_k \subset 4\mathsf{a}_{k+1}\mathfrak{w}_{k+1}^{-1}$;
(iii) $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{a}_k]$ *when* $\mathfrak{f}_k \subset 4\mathsf{a}_k\mathfrak{w}_k^{-1}$.

We now state our main result.

Theorem 3.1. *Let L, K be two lattices with $FL \cong FK$ and let $L \cong \prec a_1, \ldots, a_n \succ$ and $K \cong \prec b_1, \ldots, b_n \succ$ relative to good BONGs. Let $R_i = R_i(L) = \operatorname{ord} a_i$, $S_i = R_i(K) = \operatorname{ord} b_i$, $\alpha_i = \alpha_i(L)$ and $\beta_i = \alpha_i(K)$. Then $L \cong K$ iff:*
(i) $R_i = S_i$ *for* $1 \leq i \leq n$;
(ii) $\alpha_i = \beta_i$ *for* $1 \leq i \leq n-1$;
(iii) $d(a_1 \cdots a_i b_1 \cdots b_i) \geq \alpha_i$ *for* $1 \leq i \leq n-1$;
(iv) $[b_1, \ldots, b_{i-1}] \rightarrowtail [a_1, \ldots, a_i]$ *for any* $1 < i < n$ *s.t.* $\alpha_{i-1} + \alpha_i > 2e$.

Proof. Condition 3.1(i) is equivalent to $t = t'$, $\dim L_k = \dim K_k$, $\mathfrak{s}_k = \mathfrak{s}'_k$ and $\mathfrak{n}L^{\mathfrak{s}_k} = \mathfrak{n}K^{\mathfrak{s}_k}$, i.e. $\mathsf{a}_k\mathcal{O} = \mathsf{b}_k\mathcal{O}$. (See [1, Lemma 4.7].) Suppose this happens. Denote as before $n_k = \dim L_{(k)} = \dim K_{(k)}$, $\mathfrak{p}^{r_k} = \mathfrak{s}_k$ and $\mathfrak{p}^{u_k} = \mathfrak{n}L^{\mathfrak{s}_k} = \mathsf{a}_k\mathcal{O}$.

As in the previous section, we choose a Jordan splitting of L such that $L_k \cong \prec a_{n_{k-1}+1}, \ldots, a_{n_k} \succ$. Hence for any $1 \leq k \leq n$, a_k can be either $\pm a_{n_{k-1}+1}$ or $\pm \pi^{2u_k - 2r_k} a_{n_k}$. We choose a Jordan splitting for K with the same property.

Assuming that 3.1(i) holds, Lemma 2.16 and Corollary 2.17(ii) imply that 3.1(ii) is equivalent to $\mathfrak{w}_k = \mathfrak{w}'_k$ for $1 \leq k \leq t$ and $\mathfrak{f}_k = \mathfrak{f}'_k$ for $1 \leq k \leq t-1$.

From here the proof of Theorem 3.1 consists of two steps:

1. Assuming that 3.1(i) and (ii) hold, we prove that condition 3.1(iii) is equivalent to $\mathsf{a}_k \cong \mathsf{b}_k \pmod{\mathfrak{w}_k}$ for any $1 \leq k \leq t$ and condition 93:28(i).

2. Assuming that 3.1(i)-(iii) hold, we prove that condition 3.1(iv) is equivalent to conditions 93:28(ii) and (iii). □

Lemma 3.2. *Suppose that L, K satisfy conditions 3.1(i) and 3.1(ii). If $R_{i-1} = R_{i+1}$ for some $1 < i < n$ then:*
(i) If 3.1(iii) holds at $i-2$ or $i-2=0$ then 3.1(iii) holds at i.
(ii) If 3.1(iii) holds at $i+1$ or $i+1=n$ then 3.1(iii) holds at $i-1$.

Proof. (i) We have $d(a_1\cdots a_ib_1\cdots b_i) \geq \min\{d(a_1\cdots a_{i-2}b_1\cdots b_{i-2}), d(-a_{i-1}a_i), d(-b_{i-1}b_i)\}$. (If $i-2=0$ we ignore $d(a_1\cdots a_{i-2}b_1\cdots b_{i-2})$.) But $d(a_1\cdots a_{i-2}b_1\cdots b_{i-2}) \geq \alpha_{i-2} \geq R_{i-1}-R_{i+1}+\alpha_i = \alpha_i$. (We have $-R_{i-1}+\alpha_{i-2} \geq -R_{i+1}+\alpha_i$.) Also $d(-a_{i-1}a_i) = R_{i+1}-R_{i-1}+d(-a_{i-1}a_i) \geq \alpha_i$. Similarly $d(-b_{i-1}b_i) \geq \alpha_i$. Hence $d(a_1\cdots a_ib_1\cdots b_i) \geq \alpha_i$.

(ii) is similar. This time $R_{i+1}+\alpha_{i+1} \geq R_{i-1}+\alpha_{i-1}$ so $d(a_1\cdots a_{i+1}, b_1\cdots b_{i+1}) \geq \alpha_{i+1} \geq R_{i-1}-R_{i+1}+\alpha_{i-1} = \alpha_{i-1}$. (If $i+1=n$ then $d(a_1\cdots a_nb_1\cdots b_n) = \infty > \alpha_{n-2}$.) Also $d(-a_ia_{i+1}) = R_{i+1}-R_{i-1}+d(-a_ia_{i+1}) \geq \alpha_{i-1}$ and similarly $d(-b_ib_{i+1}) \geq \alpha_{i-1}$. □

Lemma 3.3. *Assuming that 3.1(i) and (ii) hold, condition 3.1(iii) is equivalent to $\mathsf{a}_k \cong \mathsf{b}_k \pmod{\mathfrak{w}_k}$ for any $1 \leq k \leq t$ and condition 93:28(i).*

Proof. We have $L_{(k)} \cong \prec a_1,\ldots,a_{n_k} \succ$ and $K_{(k)} \cong \prec b_1,\ldots,b_{n_k} \succ$. Hence $\det L_{(k)} = a_1\cdots a_{n_k}$ and $\det K_{(k)} = b_1\cdots b_{n_k}$. Since the two determinants have the same order, $R_1+\cdots+R_{n_k}$, the condition $\det L_{(k)}/\det K_{(k)} \cong 1 (\mathrm{mod}\ \mathfrak{f}_k)$ is equivalent to $d(a_1\cdots a_{n_k}b_1\cdots b_{n_k}) \geq \mathrm{ord}\,\mathfrak{f}_k$. Let $i = n_k$. We claim that $d(a_1\cdots a_ib_1\cdots b_i) \geq \mathrm{ord}\,\mathfrak{f}_k$ is equivalent to $d(a_1\cdots a_ib_1\cdots b_i) \geq \alpha_i$. By Lemma 2.16(ii) we have either $\alpha_i = \mathrm{ord}\,\mathfrak{f}_k$ or $\alpha_i, \mathrm{ord}\,\mathfrak{f}_k > 2e$. In the first case our claim is obvious and in the second both $d(a_1\cdots a_ib_1\cdots b_i) \geq \mathrm{ord}\,\mathfrak{f}_k$ and $d(a_1\cdots a_ib_1\cdots b_i) \geq \alpha_i$ are equivalent to $a_1\cdots a_ib_1\cdots b_i \in \dot{F}^2$.

Thus condition 3.1(iii) at indices $i = n_k$ with $1 \leq k \leq t-1$ is equivalent to 93:28(i). Assume these equivalent conditions hold. We want to prove that condition $\mathsf{a}_k \cong \mathsf{b}_k \pmod{\mathfrak{w}_k}$ at indices $1 \leq k \leq t$ such that L_k is not unary is equivalent to condition 3.1(iii) at $i = n_{k-1}+1$, while if L_k is unary then it holds unconditionally.

Note that $\mathsf{a}_k \cong \mathsf{b}_k (\mathrm{mod}\ \mathfrak{w}_k)$ is equivalent to $\mathsf{b}_k/\mathsf{a}_k \cong 1 (\mathrm{mod}\ \mathsf{a}_k^{-1}\mathfrak{w}_k)$, i.e. to $d(\mathsf{a}_k\mathsf{b}_k) = d(\mathsf{b}_k/\mathsf{a}_k) \geq \mathrm{ord}\,\mathsf{a}_k^{-1}\mathfrak{w}_k$. We will take $\mathsf{a}_k = a_{n_{k-1}+1} = a_i$ and $\mathsf{b}_k = b_{n_{k-1}+1} = b_i$. So our condition is equivalent to $d(a_ib_i) \geq \mathrm{ord}\,\mathsf{a}_k^{-1}\mathfrak{w}_k$, where $i = n_{k-1}+1$.

If L_k is unary then $\mathrm{ord}\,\mathsf{a}_k^{-1}\mathfrak{w}_k = \min\{\alpha_{i-1}, \alpha_i, e\}$ by Corollary 2.17(ii), where $i = n_{k-1}+1 = n_k$. Since $i-1 = n_{k-1}$ and $i = n_k$, condition 3.1(iii) is satisfied for both. Thus $d(a_1\cdots a_{i-1}b_1\cdots b_{i-1}) \geq \alpha_{i-1} \geq \mathrm{ord}\,\mathsf{a}_k^{-1}\mathfrak{w}_k$ and $d(a_1\cdots a_ib_1\cdots b_i) \geq \alpha_i \geq \mathrm{ord}\,\mathsf{a}_k^{-1}\mathfrak{w}_k$ so $d(a_ib_i) \geq \mathrm{ord}\,\mathsf{a}_k^{-1}\mathfrak{w}_k$. (If $k=1$ so $i = n_0+1 = 1$ we ignore α_{i-1} and we have $d(a_1b_1) \geq \alpha_1 \geq \mathrm{ord}\,\mathsf{a}_1^{-1}\mathfrak{w}_1$. If $k=t$ so $i = n_t = n$ we ignore α_i and, since $a_1\cdots a_n = \det FM = \det FN = b_1\cdots b_n$ in $\dot{F}/\dot{F}^2$, we get $d(a_nb_n) = d(a_1\cdots a_{n-1}b_1\cdots b_{n-1}) \geq \alpha_{n-1} \geq \mathrm{ord}\,\mathsf{a}_t^{-1}\mathfrak{w}_t$.) Thus condition $\mathsf{a}_k \cong \mathsf{b}_k \pmod{\mathfrak{w}_k}$ is superfluous when L_k is unary.

Suppose now that L_k is not unary and let $i = n_{k-1}+1$. By Corollary 2.17(i) we have $\mathrm{ord}\,\mathsf{a}_k^{-1}\mathfrak{w}_k = \alpha_i$. We will prove that $d(a_ib_i) \geq \mathrm{ord}\,\mathsf{a}_k^{-1}\mathfrak{w}_k = \alpha_i$ is equivalent to the condition 3.1(iii) at i i.e. to $d(a_1\cdots a_ib_1\cdots b_i) \geq \alpha_i$. If $k=1$ so $i = n_0+1 = 1$ this is obvious. If $k > 1$ so $i > 1$ note that $-R_i+\alpha_{i-1} \geq -R_{i+1}+\alpha_i$ and $R_i = u_k \geq 2r_k - u_k = R_{i+1}$ so $\alpha_{i-1} \geq \alpha_i$. We have $i-1 = n_{k-1}$ so $d(a_1\cdots a_{i-1}b_1\cdots b_{i-1}) \geq \alpha_{i-1} \geq \alpha_i$ and so $d(a_1b_i) \geq \alpha_i$ is equivalent to $d(a_1\cdots a_ib_1\cdots b_i) \geq \alpha_i$ by the domination principle.

To complete the proof we show that 3.1(iii) is true if it is true for $i = n_k$, where $1 \leq k \leq t-1$, and for $i = n_{k-1}+1$, where $1 \leq k \leq t$ and L_k is not unary. To do this we use Lemma 3.2.

Let $1 \leq k \leq t$. For any $n_{k-1}+1 < i < n_k$ we have $R_{i-1} = R_{i+1}$ (they are both u_k or $2r_k - u_k$) so by Lemma 3.2(i) if 3.1(iii) holds for $i-2$ or $i-2=0$ it will also hold for i. Thus, since 3.1(iii) is true for n_{k-1} (or $n_{k-1}=0$ if $k=1$), it will also be true by induction for any $n_{k-1}+2 \leq i < n_k$ with $i \equiv n_{k-1} \pmod 2$. Similarly since 3.1(iii) is true at $n_{k-1}+1$, it will also be true by induction for any $n_{k-1}+1 \leq i < n_k$ with $i \equiv n_{k-1}+1 \pmod 2$. Hence 3.1(iii) holds for any $n_{k-1} < i < n_k$. Since 3.1(iii) also holds for any $i = n_k$ with $1 \leq k \leq t-1$ it will hold for all $1 \leq i \leq n-1$. □

Lemma 3.4. *If $1 < i < n$ and $R_{i-1} = R_{i+1}$ then $\alpha_{i-1} + \alpha_i \leq 2e$.*

Proof. We have $\alpha_{i-1} + \alpha_i \leq (R_i - R_{i-1})/2 + e + (R_{i+1} - R_i)/2 + e = (R_{i+1} - R_{i-1})/2 + 2e$ so if $R_{i-1} = R_{i+1}$ then $\alpha_{i-1} + \alpha_i \leq 2e$. □

Lemma 3.5. *Let V, W be two quadratic spaces over F. We have:*

(i) If $\dim V - \dim W = 1$ and H is a hyperbolic plane then $W \rightarrowtail V$ iff $V \rightarrowtail W \perp \mathrm{H}$.

(ii) If $\dim V = \dim W$ and $a \in \dot{F}$ then $W \rightarrowtail V \perp [a]$ iff $V \rightarrowtail W \perp [a \det V \det W]$.

(iii) If $\dim V = \dim W$, $a, b \in \dot{F}$ and $(ab, \det V \det W)_{\mathfrak{p}} = 1$ (in particular, if $d(ab) + d(\det V \det W) > 2e$) then $W \rightarrowtail V \perp [a]$ iff $W \rightarrowtail V \perp [b]$.

Proof. This is a direct consequence of [4, 63:21]. For (iii) we also use the fact that if $xy = zt$ then $[x, y] \cong [z, t]$ iff $z \rightarrowtail [x, y]$, which in turn is equivalent to $(xz, yz)_{\mathfrak{p}} = 1$. □

Lemma 3.6. *Suppose that L, K satisfy the conditions 3.1(i)-(iii) (or, equivalently, they have the same fundamental type and they satisfy the condition 93:28(i)). Then:*

(i) If $\mathfrak{f}_k \subset 4\mathsf{a}_k \mathfrak{w}_k^{-1}$ and both a_k and b_k are norm generators for $L^{\mathfrak{s}_k}$ then $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{a}_k]$ is equivalent to $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{b}_k]$, and also to $FK_{(k)} \rightarrowtail FL_{(k)} \perp [\mathsf{b}_k]$.

(ii) If $\mathfrak{f}_k \subset 4\mathsf{a}_{k+1} \mathfrak{w}_{k+1}^{-1}$ and both a_{k+1} and b_{k+1} are norm generators for $L^{\mathfrak{s}_{k+1}}$ then $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{a}_{k+1}]$ is equivalent to $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{b}_{k+1}]$ and also to $FK_{(k)} \rightarrowtail FL_{(k)} \perp [\mathsf{b}_{k+1}]$.

Proof. (i) $\mathfrak{f}_k \subset 4\mathsf{a}_k \mathfrak{w}_k^{-1}$ is equivalent to $\operatorname{ord} \mathfrak{f}_k + \operatorname{ord} \mathsf{a}_k^{-1} \mathfrak{w}_k > 2e$. We have $\mathsf{a}_k^{-1} \mathfrak{w}_k \supseteq 2\mathsf{a}_k^{-1} \mathfrak{s}_k \supseteq 2\mathcal{O}$ so $\operatorname{ord} \mathsf{a}_k^{-1} \mathfrak{w}_k \leq e < \operatorname{ord} \mathfrak{f}_k$. Since $\mathsf{a}_k, \mathsf{b}_k$ are both norm generators for $L^{\mathfrak{s}_k}$ we have $d(\mathsf{a}_k \mathsf{b}_k) \geq \operatorname{ord} \mathsf{a}_k^{-1} \mathfrak{w}_k$. Since also $d(\det L_{(k)} \det K_{(k)}) \geq \operatorname{ord} \mathfrak{f}_k > \operatorname{ord} \mathsf{a}_k^{-1} \mathfrak{w}_k$ we also have $d(\mathsf{a}_k \mathsf{b}_k \det L_{(k)} \det K_{(k)}) \geq \operatorname{ord} \mathsf{a}_k^{-1} \mathfrak{w}_k$. Since $d(\det L_{(k)} \det K_{(k)}) + d(\mathsf{a}_k \mathsf{b}_k) \geq \operatorname{ord} \mathfrak{f}_k + \operatorname{ord} \mathsf{a}_k^{-1} \mathfrak{w}_k > 2e$ we get by Lemma 3.5(iii) that $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{a}_k]$ iff $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{b}_k]$. Similarly, since $d(\det L_{(k)} \det K_{(k)}) + d(\mathsf{a}_k \mathsf{b}_k \det L_{(k)} \det K_{(k)}) \geq \operatorname{ord} \mathfrak{f}_k + \operatorname{ord} \mathsf{a}_k^{-1} \mathfrak{w}_k > 2e$, we have $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{a}_k]$ iff $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{b}_k \det L_{(k)} \det K_{(k)}]$ which, by Lemma 3.5(ii), is equivalent to $FK_{(k)} \rightarrowtail FL_{(k)} \perp [\mathsf{b}_k]$.

(ii) The proof is the same as (i) but with $\mathsf{a}_k, \mathsf{b}_k, \mathfrak{w}_k$ replaced by $\mathsf{a}_{k+1}, \mathsf{b}_{k+1}, \mathfrak{w}_{k+1}$. □

Lemma 3.7. *Suppose that L, K satisfy the conditions 3.1(i) - (iii). If $1 \le k \le t-1$ then:*

(i) If $\mathfrak{f}_k \subset 4\mathsf{a}_k\mathfrak{w}_k^{-1}$ then $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{a}_k]$ iff $[b_1, \dots, b_{i-1}] \rightarrowtail [a_1, \dots, a_i]$, with $i = n_k$.

(ii) If $\mathfrak{f}_k \subset 4\mathsf{a}_{k+1}\mathfrak{w}_{k+1}^{-1}$ then $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{a}_{k+1}]$ iff $[b_1, \dots, b_{i-1}] \rightarrowtail [a_1, \dots, a_i]$, with $i = n_k + 1$.

Proof. (i) We take $\mathsf{b}_k = -\pi^{2u_k - 2r_k} b_i$ as a norm generator for $K^{\mathfrak{s}_k}$, so for $L^{\mathfrak{s}_k}$. (See Lemma 2.13(iii).) By Lemma 3.6(i) $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{a}_k]$ iff $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{b}_k]$, i.e. iff $[a_1, \dots, a_i] \rightarrowtail [b_1, \dots, b_i] \perp [-b_i] \cong [b_1, \dots, b_{i-1}] \perp \mathrm{H}$. By Lemma 3.5(i) this is equivalent to $[b_1, \dots, b_{i-1}] \rightarrowtail [a_1, \dots, a_i]$.

(ii) We take $\mathsf{b}_{k+1} = a_i$ as a norm generator for $L^{\mathfrak{s}_{k+1}}$. By Lemma 3.6(ii) $FL_{(k)} \rightarrowtail FK_{(k)} \perp [\mathsf{a}_{k+1}]$ iff $FK_{(k)} \rightarrowtail FL_{(k)} \perp [\mathsf{b}_{k+1}]$, i.e. iff $[b_1, \dots, b_{i-1}] \rightarrowtail [a_1, \dots, a_{i-1}] \perp [a_i] \cong [a_1, \dots, a_i]$. □

Lemma 3.8. *(i) If $i = n_k > n_{k-1} + 1$ then $\alpha_{i-1} + \alpha_i > 2e$ iff $\mathfrak{f}_k \subset \mathsf{a}_k\mathfrak{w}_k^{-1}$.*

(ii) If $i = n_k + 1 < n_{k+1}$ then $\alpha_{i-1} + \alpha_i > 2e$ iff $\mathfrak{f}_k \subset \mathsf{a}_{k+1}\mathfrak{w}_{k+1}^{-1}$.

(iii) If $i = n_k = n_{k-1} + 1$ then $\alpha_{i-1} + \alpha_i > 2e$ iff $\mathfrak{f}_k \subset \mathsf{a}_k\mathfrak{w}_k^{-1}$ or $\mathfrak{f}_{k-1} \subset \mathsf{a}_k\mathfrak{w}_k^{-1}$.

(In (iii) we ignore the condition $\mathfrak{f}_k \subset \mathsf{a}_k\mathfrak{w}_k^{-1}$ if $k = t$ and we ignore $\mathfrak{f}_{k-1} \subset \mathsf{a}_k\mathfrak{w}_k^{-1}$ if $k = 1$.)

Proof. (i) Condition $\mathfrak{f}_k \subset 4\mathsf{a}_k\mathfrak{w}_k^{-1}$ is equivalent to $\operatorname{ord} \mathsf{a}_k^{-1}\mathfrak{w}_k + \operatorname{ord} \mathfrak{f}_k > 2e$. By Corollary 2.17(i) we have $\operatorname{ord} \mathsf{a}_k^{-1}\mathfrak{w}_k = \alpha_{i-1}$. By Lemma 2.16(ii) we have either $\alpha_i = \operatorname{ord} \mathfrak{f}_k$ or $\alpha_i, \operatorname{ord} \mathfrak{f}_k > 2e$. In the first case $\operatorname{ord} \mathsf{a}_k^{-1}\mathfrak{w}_k + \operatorname{ord} \mathfrak{f}_k = \alpha_{i-1} + \alpha_i$ and in the second both $\operatorname{ord} \mathsf{a}_k^{-1}\mathfrak{w}_k + \operatorname{ord} \mathfrak{f}_k > 2e$ and $\alpha_{i-1} + \alpha_i > 2e$ hold. In both cases $\operatorname{ord} \mathsf{a}_k^{-1}\mathfrak{w}_k + \operatorname{ord} \mathfrak{f}_k > 2e$ iff $\alpha_{i-1} + \alpha_i > 2e$.

(ii) We have $\mathfrak{f}_k \subset 4\mathsf{a}_{k+1}\mathfrak{w}_{k+1}^{-1}$ iff $\operatorname{ord} \mathsf{a}_{k+1}^{-1}\mathfrak{w}_{k+1} + \operatorname{ord} \mathfrak{f}_k > 2e$. By Corollary 2.17(i) $\operatorname{ord} \mathsf{a}_{k+1}^{-1}\mathfrak{w}_{k+1} = \alpha_i$ and by Lemma 2.16(ii) $\operatorname{ord} \mathfrak{f}_k$ and α_{i-1} are either equal or they are both $> 2e$. Thus $\operatorname{ord} \mathsf{a}_{k+1}^{-1}\mathfrak{w}_{k+1} + \operatorname{ord} \mathfrak{f}_k > 2e$ iff $\alpha_{i-1} + \alpha_i > 2e$.

(iii) $\mathfrak{f}_{k-1} \subset 4\mathsf{a}_k\mathfrak{w}_k^{-1}$ and $\mathfrak{f}_k \subset 4\mathsf{a}_k\mathfrak{w}_k^{-1}$ are equivalent to $\operatorname{ord} \mathfrak{f}_{k-1} + \operatorname{ord} \mathsf{a}_k^{-1}\mathfrak{w}_k > 2e$, resp. $\operatorname{ord} \mathfrak{f}_k + \operatorname{ord} \mathsf{a}_k^{-1}\mathfrak{w}_k > 2e$. By Corollary 2.17(ii) we have $\operatorname{ord} \mathsf{a}_k^{-1}\mathfrak{w}_k = \min\{\alpha_{i-1}, \alpha_i, e\} \ge 0$. By Lemma 2.16(ii) we have that $\operatorname{ord} \mathfrak{f}_{k-1} = \alpha_{i-1}$ or $\operatorname{ord} \mathfrak{f}_{k-1}, \alpha_{i-1} > 2e$ and $\operatorname{ord} \mathfrak{f}_k = \alpha_i$ or $\operatorname{ord} \mathfrak{f}_k, \alpha_i > 2e$. Therefore $\mathfrak{f}_{k-1} \subset 4\mathsf{a}_k\mathfrak{w}_k^{-1}$ and $\mathfrak{f}_k \subset 4\mathsf{a}_k\mathfrak{w}_k^{-1}$ are equivalent to $\alpha_{i-1} + \min\{\alpha_{i-1}, \alpha_i, e\} > 2e$, resp. $\alpha_i + \min\{\alpha_{i-1}, \alpha_i, e\} > 2e$. Obviously either of them implies $\alpha_{i-1} + \alpha_i > 2e$. Conversely, suppose that $\alpha_{i-1} + \alpha_i > 2e$. If both α_{i-1} and α_i are $> e$ then we have both $\alpha_{i-1} + \min\{\alpha_{i-1}, \alpha_i, e\} > 2e$ and $\alpha_i + \min\{\alpha_{i-1}, \alpha_i, e\} > 2e$. Otherwise we have $\min\{\alpha_{i-1}, \alpha_i, e\} = \min\{\alpha_{i-1}, \alpha_i\}$ and so $\max\{\alpha_{i-1}, \alpha_i\} + \min\{\alpha_{i-1}, \alpha_i, e\} = \max\{\alpha_{i-1}, \alpha_i\} + \min\{\alpha_{i-1}, \alpha_i\} = \alpha_{i-1} + \alpha_i > 2e$, which implies that either $\alpha_{i-1} + \min\{\alpha_{i-1}, \alpha_i, e\} > 2e$ or $\alpha_i + \min\{\alpha_{i-1}, \alpha_i, e\} > 2e$. □

Lemma 3.9. *Assuming that 3.1(i)-(iii) hold, condition 3.1(iv) is equivalent to 93:28(ii) and (iii).*

Proof. Take $1 < i < n$. If $n_{k-1} + 1 < i < n_k$ for some $1 \le k \le t$ then $R_{i-1} = R_{i+1}$, by Lemma 2.13, so, by Lemma 3.4, $\alpha_{i-1} + \alpha_i \le 2e$, which makes 3.1(iv) vacuous at i. Therefore we can restrict ouselves to $i = n_k$ or $n_k + 1$ for some $1 \le k \le t-1$. We have three cases:

1. $i = n_k$ and $\dim L_k > 1$, i.e. $i = n_k > n_{k-1} + 1$. By Lemma 3.8(i) $\mathfrak{f}_k \subset 4\mathsf{a}_k\mathfrak{w}_k^{-1}$ is equivalent to $\alpha_{i-1} + \alpha_i > 2e$. On the other hand if $\mathfrak{f}_k \subset 4\mathsf{a}_k\mathfrak{w}_k^{-1}$ then

$FL_{(k)} \to\!\!\!\!\!- FK_{(k)} \perp [\mathsf{a}_k]$ is equivalent to $[b_1, \dots, b_{i-1}] \to\!\!\!\!\!- [a_1, \dots, a_i]$ by Lemma 3.7(i). Therefore 3.1(iv) at index i is equivalent to 93:28(iii) at index k.

2. $i = n_k + 1$ and $\dim L_{k+1} > 1$, i.e. $i = n_k + 1 < n_{k+1}$. By Lemma 3.8(ii) $\mathfrak{f}_k \subset 4\mathsf{a}_{k+1}\mathfrak{w}_{k+1}^{-1}$ is equivalent to $\alpha_{i-1} + \alpha_i > 2e$. On the other hand if $\mathfrak{f}_k \subset 4\mathsf{a}_{k+1}\mathfrak{w}_{k+1}^{-1}$ then $FL_{(k)} \to\!\!\!\!\!- FK_{(k)} \perp [\mathsf{a}_{k+1}]$ is equivalent to $[b_1, \dots, b_{i-1}] \to\!\!\!\!\!- [a_1, \dots, a_i]$ by Lemma 3.7(ii). Therefore 3.1(iv) at index i is equivalent to 93:28(ii) at index k.

3. $i = n_k = n_{k-1} + 1$ for some $1 \le k \le t$. In this case L_k is unary. We will prove that the condition 3.1(iv) at index i is equivalent to 93:28(iii) at index k and 93:28(ii) at index $k-1$. First note that if $k = t$ then 3.1(iv) is vacuous at $i = n_t = n$. On the other hand 93:28(iii) is vacuous at index $k = t$. Also if $\mathfrak{f}_{t-1} \subset 4\mathsf{a}_t\mathfrak{w}_t^{-1}$ then, by Lemma 3.7(ii), $FL_{(t-1)} \to\!\!\!\!\!- FK_{(t-1)} \perp [\mathsf{a}_t]$ is equivalent to $[b_1, \dots, b_{n-1}] \to\!\!\!\!\!- [a_1, \dots, a_n]$ (we have $i = n_{t-1} + 1 = n_t = n$). But this follows from $[a_1, \dots, a_n] \cong [b_1, \dots, b_n]$. Thus 93:28(ii) is superfluous at index $k - 1 = t - 1$. Next we note that if $k = 1$ then 3.1(iv) is vacuous at $i = n_0 + 1 = 1$. On the other hand 93:28(ii) is vacuous at index $k - 1 = 0$. Also if $\mathfrak{f}_1 \subset 4\mathsf{a}_1^{-1}\mathfrak{w}_1$ then $FL_{(1)} \to\!\!\!\!\!- FK_{(1)} \perp [\mathsf{a}_1]$ is equivalent, by Lemma 3.7(i), to $0 \to\!\!\!\!\!- [a_1]$ (we have $i = n_1 = 1$). Here 0 is not the scalar zero, but the zero lattice, of dimension 0, so $0 \to\!\!\!\!\!- [a_1]$ holds trivially. Thus 93:28(iii) is superfluous at $k = 1$.

Now suppose that $1 < k < t$. By Lemma 3.8(iii) we have $\alpha_{i-1} + \alpha_i > 2e$ iff $\mathfrak{f}_{k-1} \subset 4\mathsf{a}_k\mathfrak{w}_k^{-1}$ or $\mathfrak{f}_k \subset 4\mathsf{a}_k\mathfrak{w}_k^{-1}$. To complete the proof we note that if $\mathfrak{f}_{k-1} \subset 4\mathsf{a}_k\mathfrak{w}_k^{-1}$ then $FL_{(k-1)} \to\!\!\!\!\!- FK_{(k-1)} \perp [\mathsf{a}_k]$ is equivalent to $[b_1, \dots, b_{i-1}] \to\!\!\!\!\!- [a_1, \dots, a_i]$ by Lemma 3.7(ii) (we have $i = n_{k-1} + 1$) and if $\mathfrak{f}_k \subset 4\mathsf{a}_k\mathfrak{w}_k^{-1}$ then $FL_{(k)} \to\!\!\!\!\!- FK_{(k)} \perp [\mathsf{a}_k]$ is equivalent to $[b_1, \dots, b_{i-1}] \to\!\!\!\!\!- [a_1, \dots, a_i]$ by Lemma 3.7(i) (we have $i = n_k$). □

4. The 2-adic case

In this section we will assume that F is 2-adic, i.e. that $e = 1$.

In [4, §93G] O'Meara gives a solution to the classification problem in the 2-adic case which only involves the Jordan invariants t, $\dim L_k$, $\mathfrak{s}_k$ and $\mathfrak{n}_k := \mathfrak{n}L_k$. The invariants $\mathfrak{g}_k$ and $\mathfrak{w}_k$ are no longer necessary since they can be written as $\mathfrak{g}_k = \mathfrak{n}_k$ and $\mathfrak{w}_k = 2\mathfrak{s}_k$. A similar phenomenon occurs when we use good BONGs instead of Jordan decompositions. This time the invariants α_i are no longer necessary.

Lemma 4.1. *If $e = 1$ then $\alpha_i = 1$ if $R_{i+1} - R_i = 1$ and $\alpha_i = (R_{i+1} - R_i)/2 + 1$ otherwise.*

Proof. We have $R_{i+1} - R_i \ge -2e = -2$ and if $R_{i+1} - R_i$ is negative then it is even. Thus $R_{i+1} - R_i$ is either -2 or it is ≥ 0. If $R_{i+1} - R_i = -2e = -2$ or $R_{i+1} - R_i = 2e - 2 = 2 - 2e = 0$ or if $R_{i+1} - R_i \ge 2e = 2$ then $\alpha_i = (R_{i+1} - R_i)/2 + e = (R_{i+1} - R_i)/2 + 1$ by Corollary 2.9(i). If $R_{i+1} - R_i = 1$, which is odd and $< 2e$, we have $\alpha_i = R_{i+1} - R_i = 1$ by Lemma 2.7(iii). □

Since the α_i's are uniquely defined by the R_i's, condition (ii) of the main theorem is superfluous since it follows from (i). Also, $\operatorname{ord} a_1 \cdots a_i = \operatorname{ord} b_1 \cdots b_i$ so $\operatorname{ord} a_1 \cdots a_i b_1 \cdots b_i$ is even. So if $R_{i+1} - R_i \le 1$ we have $d(a_1 \cdots a_i b_1 \cdots b_i) \ge 1 \ge \alpha_i$. So condition (iii) is superfluous if $R_{i+1} - R_i \le 1$. If $R_{i+1} - R_i = 2$ then $\alpha_2 = 2$, while if $R_{i+1} - R_i > 2$ then $\alpha_i > 2$. Thus in these cases (iii) becomes $a_1 \cdots a_i b_1 \cdots b_i \in \dot{F}^2 \cup \Delta\dot{F}^2$ if $R_{i+1} - R_i = 2$ and $a_1 \cdots a_i b_1 \cdots b_i \in \dot{F}^2$ if $R_{i+1} - R_i > 2$. Finally, it is easy to see that the condition $\alpha_{i-1} + \alpha_i > 2$ from

3.1(iv) is satisfied iff $R_{i-1} < R_{i+1}$ and the pair $(R_i - R_{i-1}, R_{i+1} - R_i)$ is different from $(0,1),(1,0),(1,1)$. So we have:

Theorem 4.2. *Suppose that F is 2-adic, $L \cong \prec a_1,\dots,a_n \succ$ and $K \cong \prec b_1,\dots,b_n \succ$ relative to good BONGs, $R_i = R_i(L) = \operatorname{ord} a_i$, $S_i = R_i(K) = \operatorname{ord} b_i$ and $FL \cong FK$. Then $L \cong K$ if and only if the following conditions hold:*

(i) $R_i = S_i$ for any $1 \le i \le n$.

(ii) For any $1 \le i \le n-1$ we have $a_1 \cdots a_i b_1 \cdots b_i \in \dot{F}^2 \cup \Delta \dot{F}^2$ if $R_{i+1} - R_i = 2$, and $a_1 \cdots a_i b_1 \cdots b_i \in \dot{F}^2$ if $R_{i+1} - R_i > 2$.

(iii) $[b_1,\dots,b_{i-1}] \rightarrowtail [a_1,\dots,a_i]$ for any $1 < i < n$ s.t. $R_{i-1} < R_{i+1}$ and $(R_i - R_{i-1}, R_{i+1} - R_i) \ne (0,1),(1,0),(1,1)$.

5. Remarks

1. The binary case

If $L \cong \prec \alpha, \beta \succ$ and $\eta \in \mathcal{O}^\times$ then [1, 3.12] states that $L \cong \prec \eta\alpha, \eta\beta \succ$ iff $\eta \in g(a(L)) = g(\frac{\beta}{\alpha})$.

The function $g : \mathcal{A} \longrightarrow Sgp(\mathcal{O}^\times / \mathcal{O}^{\times 2})$ was introduced in [1, Definition 6]. Here $Sgp\, H$ is the set of all subgroups of a group H. We recall the definition of g.[1]

Definition. If $a = \pi^R \varepsilon \in \mathcal{A}$ and $d(-a) = d$ then:

I. If $R > 2e$ then $g(a) = \mathcal{O}^{\times 2}$.

II. If $R \le 2e$ then:

$$g(a) = \begin{cases} (1+\mathfrak{p}^{R/2+e})\mathcal{O}^{\times 2} & \text{if } d > e - R/2, \\ (1+\mathfrak{p}^{R+d})\mathcal{O}^{\times 2} \cap \mathrm{N}(-a) & \text{if } d \le e - R/2. \end{cases}$$

The following lemma gives a more compact formula for $g(a)$.

Lemma 5.1. *If $a \in \mathcal{A}$ and $\operatorname{ord} a = R$ and $d(-a) = d$ then $g(a) = (1+\mathfrak{p}^{\alpha(a)})\mathcal{O}^{\times 2} \cap \mathrm{N}(-a)$, where $\alpha(a) = \min\{R/2+e, R+d\}$.*

Proof. By [1, 3.16] we have $g(a) \subseteq \mathrm{N}(-a)$. If $\eta \in \mathcal{O}^\times$ then $\eta \in g(a)$ iff $\eta \in \mathrm{N}(-a)$ and (I) If $R > 2e$ then $\eta \in \mathcal{O}^{\times 2}$; (II) If $R \le 2e$ then $d(\eta) \ge R+d$ if $d \le e - R/2$, and $d(\eta) \ge R/2+e$ if $d > e - R/2$. (See [1, Definition 6].)

We have to prove that the conditions from (I) and (II) are equivalent to $d(\eta) \ge \alpha(a)$. If $R > 2e$ then $R+d > 2e$ and $R/2+e > 2e$ so $\alpha(a) > 2e$. Thus $d(\eta) \ge \alpha(a)$ is equivalent to $\eta \in \mathcal{O}^{\times 2}$. If $R \le 2e$ then $d \le e - R/2$ is equivalent to $R+d \le R/2+e$. Hence if $d \le e - R/2$ then $\alpha(a) = R+d$ and if $d > e-R/2$ then $\alpha(a) = R/2+e$. ☐

If $n = 2$ then from [1, 3.12] we have $\prec a_1, a_2 \succ \cong \prec \eta a_1, \eta a_2 \succ$ iff $\eta \in g(a_2/a_1)$. By Lemma 5.1 this is equivalent to $\eta \in \mathrm{N}(-a_1 a_2)$ and $d(\eta) \ge \alpha(a_2/a_1)$. The first condition is equivalent to the isometry of quadratic spaces $[a_1, a_2] \cong [\eta a_1, \eta a_2]$, while the second means $d(\eta) \ge \alpha(a_2/a_1) = \min\{(R_2 - R_1)/2 + e, R_2 - R_1 + d(-a_1 a_2)\} = \alpha_1(\prec a_1, a_2 \succ)$, which is consistent with condition (iii) of the main theorem.

Remark 5.2. Since $\alpha(a_2/a_1) = \alpha_1(\prec a_1, a_2 \succ)$ we have by Lemma 5.1 $g(a_2/a_1) = (1+\mathfrak{p}^{\alpha_1(\prec a_1, a_2 \succ)})\mathcal{O}^{\times 2} \cap \mathrm{N}(-a_1 a_2)$. Equivalently, $g(a(L)) = (1+\mathfrak{p}^{\alpha_1(L)})\mathcal{O}^{\times 2} \cap \mathrm{N}(-\det FL)$.

[1] In [1, Definition 6] there are some mistakes which we corrected here.

2. The formula for α_i

We will now show the heuristical method by which the invariants α_i were found. We want to know, given that $L \cong \prec a_1,\dots,a_n \succ$ relative to a good BONG and $1\leq i\leq n-1$, how much the product $a_1\cdots a_i$ can be altered by a change of good BONGs. That is, if $L\cong\prec b_1,\dots,b_n\succ$ relative to another good BONG we want to know how big the quadratic defect of $(b_1\cdots b_i)/(a_1\cdots a_i)$ can be. So we are looking for a lower bound $\alpha_i=\alpha_i(L)$ for $d(a_1\cdots a_ib_1\cdots b_i)$.

For any $\eta\in g(a_{i+1}/a_i)$ we have $\prec a_i,a_{i+1}\succ\cong\prec \eta a_i,\eta a_{i+1}\succ$ so, by [1, Lemma 4.9(ii)], $L\cong\prec a_1,\dots,a_{i-1},\eta a_i,\eta a_{i+1},a_{i+2},\dots,a_n\succ$. By this change of BONGs, $a_1\cdots a_i$ was changed by the factor η. We have $\eta\in g(a_{i+1}/a_i)$ which, by Lemma 5.1, implies $d(\eta)\geq\alpha(a_{i+1}/a_i)=\min\{(R_{i+1}-R_i)/2+e,R_{i+1}-R_i+d(-a_ia_{i+1})\}$. (See Lemma 5.1.) This lower bound can be further decreased if we decrease $d(-a_ia_{i+1})$. This can be done by changing the good BONGs of $\prec a_1,\dots,a_i\succ$ and $\prec a_{i+1},\dots,a_n\succ$. If $\prec a_1,\dots,a_i\succ\cong\prec a'_1,\dots,a'_i\succ$ and $\prec a_{i+1},\dots,a_n\succ\cong\prec a'_{i+1},\dots,a'_n\succ$ then $d(-a_ia_{i+1})$ is replaced by $d(-a'_ia'_{i+1})$. But $d(a_{i+1}a'_{i+1})\geq\alpha_1(\prec a_{i+1},\dots,a_n\succ)$. Also, by reason of determinant, $a_1\cdots a_ia'_1\cdots a'_i\in\dot F^2$ so $d(a_ia'_i)=d(a_1\cdots a_{i-1}a'_1\cdots a'_{i-1})\geq\alpha_{i-1}(\prec a_1,\dots,a_i\succ)$. It follows that $d(-a'_ia'_{i+1})\geq\min\{d(-a_ia_{i+1}),\alpha_{i-1}(\prec a_1,\dots,a_i\succ),\alpha_1(\prec a_{i+1},\dots,a_n\succ)\}$. Hence the new lower bound for η is $\min\{(R_{i+1}-R_i)/2+e,R_{i+1}-R_i+d(-a_ia_{i+1}),R_{i+1}-R_i+\alpha_{i-1}(\prec a_1,\dots,a_i\succ),R_{i+1}-R_i+\alpha_1(\prec a_{i+1},\dots,a_n\succ)\}$. This leads to the recursive formula $\alpha_i=\min\{(R_{i+1}-R_i)/2+e,R_{i+1}-R_i+d(-a_ia_{i+1}),R_{i+1}-R_i+\alpha_{i-1}(\prec a_1,\dots,a_i\succ),R_{i+1}-R_i+\alpha_1(\prec a_{i+1},\dots,a_n\succ)\}$ from Corollary 2.5(ii).

In the case $i=1$ and $n\geq 3$ the formula becomes $\alpha_1=\min\{(R_2-R_1)/2+e,R_2-R_1+d(-a_1a_2),R_2-R_1+\alpha_1(\prec a_2,\dots,a_n\succ)\}$. In the case $i=n-1$ and $n\geq 3$ we have $\alpha_{n-1}=\min\{(R_n-R_{n-1})/2+e,R_n-R_{n-1}+d(-a_{n-1}a_n),R_n-R_{n-1}+\alpha_{n-2}(\prec a_1,\dots,a_{n-1}\succ)\}$. Finally if $i=1$ and $n=2$ then $\alpha_1=\min\{(R_2-R_1)/2+e,R_2-R_1+d(-a_1a_2)\}$. Starting with the case $n=2$ it is easy to prove by induction that $\alpha_1=\min(\{(R_2-R_1)/2+e\}\cup\{R_{j+1}-R_1+d(-a_ja_{j+1})\mid 1\leq j<n\})$ and $\alpha_{n-1}=\min(\{(R_n-R_{n-1})/2+e\}\cup\{R_n-R_j+d(-a_ja_{j+1})\mid 1\leq j<n\})$. By plugging $\alpha_{i-1}(\prec a_1,\dots,a_i\succ)=\min(\{(R_i-R_{i-1})/2+e\}\cup\{R_i-R_j+d(-a_ja_{j+1})\mid 1\leq j<i\})$ and $\alpha_1(\prec a_{i+1},\dots,a_n\succ)=\min(\{(R_{i+2}-R_{i+1})/2+e\}\cup\{R_{j+1}-R_{i+1}+d(-a_ja_{j+1})\mid i+1\leq j<n\})$ in the recursive formula for α_i we get the formula from Definition 1. (The extra terms $R_{i+1}-R_i+(R_i-R_{i-1})/2+e$ and $R_{i+1}-R_i+(R_{i+2}-R_{i+1})/2+e$ that appear are $\geq(R_{i+1}-R_i)/2+e$ so they can be removed.)

Of course this is only a guess and does not constitute a proof. In fact the relation $d(a_1\cdots a_ib_1\cdots b_i)\geq\alpha_i$ is only proved this way in the particular case when $b_1,\dots,b_n$ are obtained from $a_1,\dots,a_n$ through a succession of "binary transformations" of the type $a_1,\dots,a_n\longrightarrow a_1,\dots,\eta a_j,\eta a_{j+1},\dots,a_n$ with $1\leq j\leq n-1$ and $\eta\in g(a_{j+1}/a_j)$. It is not hard to prove that conditions (i)-(iv) of the main theorem are necessary if $b_1,\dots,b_n$ are obtained this way. However, for the proof of the necessity in the general case and for the proof of sufficiency the use of O'Meara's theorem is necessary.

3. In the view of the previous remark there is the natural question that asks whether, given that $L\cong\prec a_1,\dots,a_n\succ\cong\prec b_1,\dots,b_n\succ$ relative to good BONGs, there is always a succession of binary transformations as defined above from $a_1,\dots,a_n$ to $b_1,\dots,b_n$. The answer to this question is YES but only if we make the assumption that $F/\mathbb{Q}_2$ is not totally ramified, i.e. that the residual field $\mathcal{O}/\mathfrak{p}$ has more than 2 elements.

If $|\mathcal{O}/\mathfrak{p}| = 2$ we have the following counterexample. Let $0 < d < 2e$ be odd and let $R = 2e - 2d$ and $\varepsilon, \eta \in \mathcal{O}^\times$ with $d(\varepsilon) = d$ and $d(\eta) = 2e - d$. It can be proved that $\prec 1, -\pi^R\varepsilon, \varepsilon\eta, -\pi^R\eta \succ \cong \prec \eta, -\pi^R\varepsilon\eta, \varepsilon, -\pi^R \succ$ but one cannot go from $1, -\pi^R\varepsilon, \varepsilon\eta, -\pi^R\eta$ to $\eta, -\pi^R\varepsilon\eta, \varepsilon, -\pi^R$ through binary transformations.

E.g., if $F = \mathbb{Q}_2$ and we take $d = 1$, so $R = 0$, and $\varepsilon = \eta = -1$ then $\prec 1, 1, 1, 1 \succ \cong \prec 7, 7, 7, 7 \succ$. However from $1, 1, 1, 1$ we can go through binary transformations only to a_1, a_2, a_3, a_4, where an even number of a_i's belong to $\mathcal{O}^{\times 2}$ and the rest to $5\mathcal{O}^{\times 2}$. This happens because $g(1) = g(5) = \mathcal{O}^{\times 2} \cup 5\mathcal{O}^{\times 2}$ so the only binary relations involving 1 and 5 are $\prec 1, 1 \succ \cong \prec 5, 5 \succ$ and $\prec 1, 5 \succ \cong \prec 5, 1 \succ$. Similarly from $7, 7, 7, 7$ we can only go to a_1, a_2, a_3, a_4, where an even number of a_i's belong to $7\mathcal{O}^{\times 2}$ and the rest to $3\mathcal{O}^{\times 2}$.

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