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TELESCOPE CONJECTURE, IDEMPOTENT IDEALS, AND THE TRANSFINITE RADICAL

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ABSTRACT. We show that for an artin algebra Λ, the telescope conjecture for module categories is equivalent to certain idempotent ideals of $\mathrm{mod}\,\Lambda$ being generated by identity morphisms. As a consequence, we prove the conjecture for domestic standard selfinjective algebras and domestic special biserial algebras. We achieve this by showing that in any Krull-Schmidt category with local d.c.c. on ideals, any idempotent ideal is generated by identity maps and maps from the transfinite radical.

INTRODUCTION

The aim of this paper is to further develop and apply connections between seemingly rather different topics in algebra:

(1) localizations of triangulated compactly generated categories;
(2) theory of cotorsion pairs and induced approximations;
(3) the structure of idempotent ideals in a module category;
(4) representation type of a finite dimensional algebra.

The main motivation for this paper was point (1), the study of so called smashing localizations in triangulated compactly generated categories. There is an important conjecture, the telescope conjecture, which roughly says that any smashing localization of a compactly generated triangulated category comes from a set of compact objects. For an extensive study of this problem and explanation of the terminology we refer to work by Krause [18, 16]. Even though the conjecture is known to be false in this generality—see [14] for a simple algebraic counterexample—it has not been resolved for many important particular settings. Such special solutions would still have significant consequences. In the case of unbounded derived categories of rings, this is discussed in [16].

In this paper, we will focus on another setting. Let R be a quasi-Frobenius ring (that is, the projective and injective left modules coincide), and let $\underline{\mathrm{Mod}}\,R$ be the stable module category of left R-modules. Then $\underline{\mathrm{Mod}}\,R$ is a triangulated compactly generated category in the sense of [18, 16]. If, moreover, R is a self-injective artin algebra, the telescope conjecture has been translated by Krause and Solberg [20] to a statement about modules, or more precisely about certain cotorsion pairs of

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modules. The precise statements and explanation of terminology are given below. Recently, a positive solution to the telescope conjecture for stable module categories over finite group algebras was announced by the authors of [4]. Their methods are, however, closely tied to group algebras and do not allow direct generalization to other self-injective artin algebras. We will develop an alternative approach.

The aforementioned version of the telescope conjecture for cotorsion pairs of modules from [20, §7] makes sense not only for self-injective artin algebras but in fact for any associative ring with unit, leading to a problem in homological algebra which is of interest in itself (see [2, 25]). Even though one loses the translation to triangulated categories, similarities between the new and the original settings are striking and have been analyzed in more detail in [25].

In the present paper, we further develop the approach from [25] and show that the telescope conjecture for module categories depends on the structure of certain idempotent ideals of the category of finitely presented modules. This is another analogy to so called exact ideals from [16]. Further, we prove that the structure of idempotent ideals in the category of finitely presented modules over an artin algebra, as well as in many other categories studied by representation theory, heavily depends on idempotent ideals inside the radical. In particular, if there are no non-zero idempotent ideals in the radical, we get a positive answer to the telescope conjecture.

The condition of no non-zero idempotent ideals in the radical of the module category seems to be closely related to the domestic representation type. These notions were proved to coincide for special biserial algebras by Schröer [27, 24]. A stronger but closely related condition where the infinite radical is nilpotent was studied by several authors; see for example [15, 28, 5, 6]. Our main interest in the existing results stems from the fact that they provide us with non-trivial examples of artin algebras over which the telescope conjecture for module categories holds. Some of these, coming from a paper by Skowroński and Kerner [15], are self-injective, thus allowing us to go all the way back and get a statement about smashing localizations of their stable module categories.

Another condition which seems to be closely related to both the domestic representation type and vanishing of the transfinite radical is that of the Krull-Gabriel dimension of an artin algebra being an ordinal number. The concept of the Krull-Gabriel dimension of a ring R can be interpreted as a measure of complexity for both the category $\mathrm{fp}(\mathrm{mod}\, R, \mathrm{Ab})$ of finitely presented additive functors $\mathrm{mod}\, R \to \mathrm{Ab}$ and the lattice of primitive positive formulas over R. Using a result from [19], we prove that the telescope conjecture for module categories holds true if the Krull-Gabriel dimension of the artin algebra in question is an ordinal number.

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1. Preliminaries

In this text, Λ will always be an artin algebra and all modules will be left Λ-modules. Let us denote by $\mathrm{Mod}\,\Lambda$ the category of all modules and by $\mathrm{mod}\,\Lambda$ the full subcategory of finitely generated modules. Some results in this paper will be proved for more general categories: Krull-Schmidt categories with local d.c.c.

on ideals as defined in Section 3. This setting includes $\operatorname{mod}\Lambda$, derived bounded categories, categories of coherent sheaves, and other categories of representation theoretic significance. A reader who is not interested in the full generality can, nevertheless, read the corresponding statements as if they were stated for $\operatorname{mod}\Lambda$.

A *cotorsion pair* in $\operatorname{Mod}\Lambda$ is a pair $(\mathcal{A},\mathcal{B})$ of full subcategories of $\operatorname{Mod}\Lambda$ such that $\mathcal{A} = \operatorname{Ker}\operatorname{Ext}^1_\Lambda(-,\mathcal{B})$ and $\mathcal{B} = \operatorname{Ker}\operatorname{Ext}^1_\Lambda(\mathcal{A},-)$. A cotorsion pair is called *hereditary* if in addition $\operatorname{Ext}^i_\Lambda(\mathcal{A},\mathcal{B}) = 0$ for all $i \geq 2$. This paper deals with the telescope conjecture for module categories (TCMC) as formulated in [20, Conjecture 7.9]. Actually, we slightly alter the assumptions—we require the cotorsion pair in question to be hereditary (since the cotorsion pairs of interest in [20] always are) and relax the condition that [20] imposes on the class $\mathcal{A}$ of the cotorsion pair. We state the conjecture as follows:

Conjecture (A)**.** *Let Λ be an artin algebra and let $(\mathcal{A},\mathcal{B})$ be a hereditary cotorsion pair in $\operatorname{Mod}\Lambda$ such that $\mathcal{B}$ is closed under taking filtered colimits. Then every module in $\mathcal{A}$ is a colimit of a filtered system of finitely generated modules from $\mathcal{A}$.*

Note that, in view of [1, Theorem 1.5], we can equivalently replace filtered colimits by direct limits in the statement above. We say that a cotorsion pair $(\mathcal{A},\mathcal{B})$ in $\operatorname{Mod}\Lambda$ is of *finite type* if $\mathcal{B} = \operatorname{Ker}\operatorname{Ext}^1_\Lambda(\mathcal{S},-)$ for a set $\mathcal{S}$ of finitely generated modules. Similarly, we define $(\mathcal{A},\mathcal{B})$ to be of *countable type* if we can take $\mathcal{S}$ to be a set of countably generated modules. With this definition we can for any particular algebra Λ equivalently restate Conjecture (A) as follows; see [2, Corollary 4.6]:

Conjecture (B)**.** *Let Λ be an artin algebra and let $(\mathcal{A},\mathcal{B})$ be a hereditary cotorsion pair in $\operatorname{Mod}\Lambda$ such that $\mathcal{B}$ is closed under taking direct limits. Then $(\mathcal{A},\mathcal{B})$ is of finite type.*

As a tool for handling the conjectures, we will need the notion of an ideal of an additive category. Let $\mathcal{C}$ be a skeletally small additive category. A class $\mathfrak{I}$ of morphisms in $\mathcal{C}$ is called a (2-sided) *ideal* of $\mathcal{C}$ if $\mathfrak{I}$ contains all zero morphisms and is closed under addition and under composition with arbitrary morphisms from left and right, whenever the operations are defined. Let us write $\mathfrak{I}(X,Y) = \mathfrak{I} \cap \operatorname{Hom}_\mathcal{C}(X,Y)$. Note that if $\mathcal{C} = \operatorname{mod}\Lambda$, then $\mathfrak{I}(X,Y)$ is always a k-submodule of $\operatorname{Hom}_\Lambda(X,Y)$ where k is the centre of Λ. Since $\mathcal{C}$ was assumed to be skeletally small, ideals of $\mathcal{C}$ form a set.

We say that an additive category $\mathcal{C}$ is a *Krull-Schmidt category* if it is skeletally small, every indecomposable object of $\mathcal{C}$ has a local endomorphism ring, and every object of $\mathcal{C}$ (uniquely) decomposes as a finite coproduct of indecomposables. As an example to keep in mind, we can put $\mathcal{C} = \operatorname{mod}\Lambda$. For Krull-Schmidt categories there is a prominent ideal called the *radical*—it is the ideal generated by all non-invertible morphisms between indecomposable objects. We denote this ideal by $\operatorname{rad}_\mathcal{C}$ and if $\mathcal{C} = \operatorname{mod}\Lambda$ we use the abbreviated notation $\operatorname{rad}_\Lambda$. Let us recall the well known fact that $\operatorname{rad}_\mathcal{C}$ contains no identity morphisms and, clearly, it is the maximal ideal with this property. Here and also later in this paper we, of course, mean no identity morphisms of non-zero objects since zero morphisms are in any ideal by definition.

Following an idea in [23], we can inductively define transfinite powers $\mathfrak{I}^\alpha$ for any ideal $\mathfrak{I}$ and any ordinal number α. Let $\mathfrak{I}^0$ be the ideal of all morphisms in $\mathcal{C}$ and let $\mathfrak{I}^1 = \mathfrak{I}$. For a natural number $n \geq 1$, we define $\mathfrak{I}^n$ as usual to be the ideal generated by all compositions of n-tuples of morphisms from $\mathfrak{I}$. If α is a limit ordinal, we define $\mathfrak{I}^\alpha = \bigcap_{\beta<\alpha} \mathfrak{I}^\beta$. If α is infinite non-limit, then uniquely $\alpha = \beta + n$ for some limit ordinal β and natural number $n \geq 1$, and we set $\mathfrak{I}^\alpha = (\mathfrak{I}^\beta)^{n+1}$. Note that since we assume that $\mathcal{C}$ is skeletally small, the decreasing chain

$$\mathfrak{I}^0 \supseteq \mathfrak{I}^1 \supseteq \mathfrak{I}^2 \supseteq \cdots \supseteq \mathfrak{I}^\alpha \supseteq \mathfrak{I}^{\alpha+1} \supseteq \ldots$$

stabilizes for cardinality reasons. Let us define $\mathfrak{I}^* = \bigcap_\alpha \mathfrak{I}^\alpha$, the minimum of the chain.

We will focus mostly on the case where $\mathfrak{I} = \mathrm{rad}_\mathcal{C}$. In this case we denote by $\mathrm{rad}^*_\mathcal{C}$ the *transfinite radical* of $\mathcal{C}$. Notice that it is not necessarily true that $\mathrm{rad}^*_\mathcal{C} = 0$, even when $\mathcal{C} = \mathrm{mod}\,\Lambda$ for an artin algebra Λ—see the next section or [23, 27]. The main goal of this paper is to prove that the TCMC formulated as Conjecture (B) holds true over those artin algebras for which $\mathrm{rad}^*_\Lambda = 0$. This applies in particular to:

- [15] standard selfinjective algebras of domestic representation type;
- [27] special biserial algebras of domestic representation type.

Recall that a finite dimensional algebra over an algebraically closed field is of *domestic representation type* if there is a natural number N such that for each dimension d, all but finitely many indecomposable modules of dimension d belong to at most N one-parameter families.

2. Transfinite radical

Let $\mathcal{C}$ be an additive category. We call an ideal $\mathfrak{I}$ of $\mathcal{C}$ *idempotent* if $\mathfrak{I} = \mathfrak{I}^2$. Equivalently, $\mathfrak{I}$ is idempotent if and only if for each $f \in \mathfrak{I}$ there are $g, h \in \mathfrak{I}$ such that $f = gh$. Using idempotency, we can give the following characterization of the transfinite radical:

Lemma 1. *Let $\mathcal{C}$ be a Krull-Schmidt category. Then* $\mathrm{rad}^*_\mathcal{C}$ *is the unique maximal idempotent ideal of $\mathcal{C}$ which does not contain any identity morphisms.*

Proof. We use the same (just more verbose) proof as the one given for [19, 8.10] for module categories. Clearly, $\mathrm{rad}^*_\mathcal{C}$ contains no identity morphisms since $\mathrm{rad}_\mathcal{C}$ does not. It is easy to check that $\mathrm{rad}^*_\mathcal{C}$ is idempotent [23, Proposition 0.6]. On the other hand, if $\mathfrak{I}$ is idempotent without identity maps, then $\mathfrak{I} = \mathfrak{I}^* \subseteq \mathrm{rad}^*_\mathcal{C}$ (since $\mathfrak{I} = \mathfrak{I}^\alpha$ for any ordinal α by idempotency). Hence $\mathrm{rad}_\mathcal{C}$ is maximal with respect to these two properties. □

There is also a useful characterization of the morphisms in $\mathrm{rad}^*_\mathcal{C}$ "from inside", shedding more light on the concept than a slightly cryptic definition such as the intersection of a series of transfinite powers. The following statement has been proved in [23] for $\mathcal{C} = \mathrm{mod}\,\Lambda$ using standard means similar to those employed when one deals with Krull dimension of a poset, and the proof reads equally well for any skeletally small Krull-Schmidt category:

Lemma 2. [23, Proposition 0.6] *Let $\mathcal{C}$ be a Krull-Schmidt category and f a morphism in $\mathcal{C}$. Then $f \in$* $\mathrm{rad}^*_\mathcal{C}$ *if and only if there exists a collection of morphisms $f_{pr} : X_r \to X_p$ in* $\mathrm{rad}_\mathcal{C}$*, one for each pair of rational numbers p, r with $0 \leq p < r \leq 1$, such that*

1. *$f_{ps} = f_{pr} f_{rs}$ whenever $p < r < s$;*
2. *$f_{01} = f$.*

Note that the collection $(f_{pr})_{0\leq p<r\leq 1}$ is none other than an inverse system indexed by $[0,1]\cap\mathbb{Q}$. Using the two lemmas above, we can give some examples of what the transfinite radical can be:

- If Λ is an artin algebra of finite representation type, then $\operatorname{rad}_\Lambda$ is nilpotent. Hence $\operatorname{rad}^*_\Lambda = 0$.
- If Λ is a tame hereditary artin algebra, then $\operatorname{rad}^{\omega+2}_\Lambda = (\operatorname{rad}^{\omega}_\Lambda)^3 = 0$. Hence $\operatorname{rad}^*_\Lambda = 0$.
- If Λ is a standard (that is, having a simply connected Galois covering) selfinjective algebra of domestic representation type, then $\operatorname{rad}^{\omega}_\Lambda$ is nilpotent [15]. Hence $\operatorname{rad}^*_\Lambda = 0$.
- If Λ is a special biserial algebra, then $\operatorname{rad}^*_\Lambda = 0$ if and only if $\operatorname{rad}^{\omega^2}_\Lambda = 0$ if and only if Λ is of domestic representation type. If Λ is not domestic, then there exists an indecomposable Λ-module X such that $0 \neq \operatorname{rad}^*_\Lambda(X,X) \subseteq \operatorname{End}_\Lambda(X)$ (see [27, Theorem 2 and Prop. 6.2]).
- As a special case of the previous point, one may consider the "Gelfand-Ponomarev" algebras $\Lambda_{m,n} = k[x,y]/(xy,yx,x^m,y^n)$; see [11]. The algebra $\Lambda_{2,3}$ is not of domestic representation type and provides a very illustrative example of non-zero maps in the transfinite radical; see [23].
- If Λ is a wild hereditary artin algebra, it is conjectured that $\operatorname{rad}^{\omega}_\Lambda$ is idempotent. In view of Lemma 1, this conjecture can be rephrased as $\operatorname{rad}^*_\Lambda = \operatorname{rad}^{\omega}_\Lambda$.
- It is an unpublished result due to Dieter Vossieck that for the category $\mathcal{C} = \operatorname{mod} k\langle x,y\rangle$ of finite dimensional modules over the free algebra $k\langle x,y\rangle$, the radical $\operatorname{rad}_{\mathcal{C}}$ is idempotent. In particular $\operatorname{rad}^*_{\mathcal{C}} = \operatorname{rad}_{\mathcal{C}}$.

There is an important consequence of some of the examples above for wild artin algebras over an algebraically closed field, namely, they *always* have the transfinite radical non-zero. Let us state this precisely.

Definition 3. Let Λ and Γ be finite dimensional algebras over a field k and let $F : \operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$ be an additive functor. Then F is called a *representation embedding* if F is faithful, exact, preserves indecomposability (i.e. if X is indecomposable, so is FX) and reflects isomorphism classes (i.e. if $FX \cong FY$, then also $X \cong Y$).

A finite dimensional k-algebra is called *wild* if for any other finite dimensional algebra Γ over k, there is a representation embedding $\operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$.

The following statement immediately follows from [27, Proposition 6.2] and [23, Lemma 0.2] (the same idea is also presented in [19, 8.15]):

Proposition 4. *Let Λ be a wild algebra over an algebraically closed field. Then $\operatorname{rad}^*_\Lambda \neq 0$. Moreover, there exists an indecomposable Λ-module X such that $0 \neq \operatorname{rad}^*_\Lambda(X,X) \subseteq \operatorname{End}_\Lambda(X)$.*

3. Idempotent ideals in Krull-Schmidt categories

Let $\mathfrak{I}$ be an ideal of a Krull-Schmidt category. Then clearly, if $\mathfrak{I}$ is generated by a collection of identity morphisms, it is necessarily an idempotent ideal. In what follows we will show that in "nice" categories, any idempotent ideal is generated by a collection of identity morphisms together with some morphisms from the transfinite radical. To make the word *nice* precise, we need the following definition.

Definition 5. A skeletally small additive category $\mathcal{C}$ is said to have *local descending chain condition (d.c.c.) on ideals* if for any decreasing series

$$\mathfrak{I}_0 \supseteq \mathfrak{I}_1 \supseteq \mathfrak{I}_2 \supseteq \dots$$

of ideals of $\mathcal{C}$ and any pair of objects X, Y in $\mathcal{C}$, the decreasing chain

$$\mathfrak{I}_0(X,Y) \supseteq \mathfrak{I}_1(X,Y) \supseteq \mathfrak{I}_2(X,Y) \supseteq \dots$$

stabilizes.

Now, our category is "nice" if it is Krull-Schmidt with local d.c.c. on ideals. In fact, this setting is very common in representation theory. Assume that k is a commutative artinian ring and $\mathcal{C}$ is a skeletally small k-category (i.e. its Hom-spaces are k-modules and its composition is k-linear) which satisfies the following conditions:

- (C1) $\mathcal{C}$ has splitting idempotents (that is, idempotent morphisms have kernels in $\mathcal{C}$);
- (C2) $\mathcal{C}$ is Hom-finite (that is, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a finitely generated k-module for any objects $X, Y \in \mathcal{C}$).

Then $\mathcal{C}$ is "nice":

Lemma 6. *Let k be a commutative artinian ring and $\mathcal{C}$ a skeletally small* Hom-*finite k-category with splitting idempotents. Then $\mathcal{C}$ is Krull-Schmidt with local d.c.c. on ideals.*

Proof. It is a well known fact that $\mathcal{C}$ is Krull-Schmidt under the assumption. It is straightforward to show that $\mathfrak{I}(X,Y)$ is a k-submodule of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ for any ideal $\mathfrak{I}$ and any pair of objects $X, Y \in \mathcal{C}$. Hence $\mathcal{C}$ clearly has local d.c.c. on ideals thanks to (C2). □

As a consequence, we can give plenty of examples of "nice" categories:

- $\mathrm{mod}\,\Lambda$ for an artin algebra Λ;
- $D^b(\Lambda)$, the derived bounded category for an artin algebra Λ;
- the category of finite dimensional modules over any algebra over a field;

and many others.

Let us start with the proof of the aforementioned statement. First we need a technical lemma.

Lemma 7. *Let $\mathcal{C}$ be a Krull-Schmidt category with local d.c.c. on ideals. Let $X, Y \in \mathcal{C}$ and let α be a limit ordinal. Then there is $\beta < \alpha$ such that $\mathrm{rad}_{\mathcal{C}}^{\beta}(X,Y) = \mathrm{rad}_{\mathcal{C}}^{\alpha}(X,Y)$.*

Proof. Since $\mathcal{C}$ has local d.c.c. on ideals, the decreasing chain $(\mathrm{rad}_{\mathcal{C}}^{\gamma}(X,Y))_{\gamma<\alpha}$ is stationary. Therefore, there is $\beta < \alpha$ such that

$$\mathrm{rad}_{\mathcal{C}}^{\beta}(X,Y) = \bigcap_{\gamma<\alpha} \mathrm{rad}_{\mathcal{C}}^{\gamma}(X,Y) = \mathrm{rad}_{\mathcal{C}}^{\alpha}(X,Y).$$ □

Now, we are in a position to give the structure theorem for idempotent ideals:

Theorem 8. *Let $\mathcal{C}$ be a Krull-Schmidt category with local d.c.c. on ideals. Let $\mathfrak{I}$ be an idempotent ideal of $\mathcal{C}$ and let $f \in \mathfrak{I}$. Then there are $f_1, f_2 \in \mathfrak{I}$ such that $f = f_1 + f_2$, the morphism f_1 is generated by identity morphisms from $\mathfrak{I}$, and $f_2 \in \mathrm{rad}_{\mathcal{C}}^{*}$.*

Proof. We will prove the following statement for all ordinal numbers α by induction:

$(*)$: For every $f \in \mathfrak{I}$ there are $f_{\alpha,1}, f_{\alpha,2} \in \mathfrak{I}$ such that $f = f_{\alpha,1} + f_{\alpha,2}$, the morphism $f_{\alpha,1}$ is generated by identity morphisms from $\mathfrak{I}$, and $f_{\alpha,2} \in \operatorname{rad}_{\mathcal{C}}^{\alpha}$.

Then the theorem will follow if we take α sufficiently big. Let $f : X \to Y$ be a morphism from $\mathfrak{I}$—we can without loss of generality assume that X and Y are indecomposable.

For $\alpha = 0$, we can simply take $f_{0,1} = 0$ and $f_{0,2} = f$. If α is non-zero finite, we can construct by induction morphisms $g^1, g^2, \ldots, g^{\alpha} \in \mathfrak{I}$ such that $f = g^1 g^2 \ldots g^{\alpha}$. The morphisms g^i, $1 \leq i \leq \alpha$, are not necessarily morphisms between indecomposable objects of $\mathcal{C}$, but we can write f as a finite sum of compositions of morphisms between indecomposables, that is:

$$f = \sum_j g^{1j} g^{2j} \ldots g^{\alpha j},$$

where we take g^{ij} as components of g^i so that all g^{ij} are in $\mathfrak{I}$. Finally, we can take $f_{\alpha,1}$ as the sum of those compositions $g^{1j} g^{2j} \ldots g^{\alpha j}$ where at least one of the morphisms in the composition is invertible, and take $f_{\alpha,2}$ to be the sum of the remaining compositions. Then clearly $f_{\alpha,1}$ is generated by identities from $\mathfrak{I}$ and $f_{\alpha,2} \in \operatorname{rad}_{\mathcal{C}}^{\alpha}$.

If α is a limit ordinal, there is an ordinal $\beta < \alpha$ such that $\operatorname{rad}_{\mathcal{C}}^{\beta}(X,Y) = \operatorname{rad}_{\mathcal{C}}^{\alpha}(X,Y)$ by Lemma 7. Of course, β depends on X and Y. Hence we can set $f_{\alpha,1} = f_{\beta,1}$ and $f_{\alpha,2} = f_{\beta,2}$, where the existence of $f_{\beta,1}, f_{\beta,2}$ is given by the inductive hypothesis.

Assume now that α is an infinite non-limit ordinal and $g_{\beta,1}, g_{\beta,2}$ have already been constructed for all $g \in \mathfrak{I}$ and $\beta < \alpha$. We can write $\alpha = \beta + n$ where β is a limit ordinal and $n \geq 1$ is a natural number. Since $\mathfrak{I}$ is idempotent, we can construct as in the finite case $g^1, g^2, \ldots, g^{n+1} \in \mathfrak{I}$ such that $f = g^1 g^2 \ldots g^{n+1}$. By the inductive hypothesis, for each $1 \leq i \leq n+1$ we can write $g^i = g^i_{\beta,1} + g^i_{\beta,2}$ where $g^i_{\beta,1}$ is generated by identity morphisms from $\mathfrak{I}$ and $g^i_{\beta,2} \in \mathfrak{I} \cap \operatorname{rad}_{\mathcal{C}}^{\beta}$. Now,

$$f = \sum g^1_{\beta,k_1} g^2_{\beta,k_2} \cdots g^{n+1}_{\beta,k_{n+1}}$$

where the sum runs through all tuples $(k_1, k_2, \ldots, k_{n+1}) \in \{1,2\}^{n+1}$. Put $f_{\alpha,2} = g^1_{\beta,2} g^2_{\beta,2} \cdots g^{n+1}_{\beta,2}$ and $f_{\alpha,1} = f - f_{\alpha,2}$. Then it immediately follows by the choice of $g^i_{\beta,1}$ and $g^i_{\beta,2}$ that $f_{\alpha,1}$ is generated by identity morphisms from $\mathfrak{I}$ and $f_{\alpha,2} \in (\operatorname{rad}_{\mathcal{C}}^{\beta})^{n+1} = \operatorname{rad}_{\mathcal{C}}^{\alpha}$. □

Just by reformulating Theorem 8, we get the following corollary:

Corollary 9. *Let $\mathcal{C}$ be a Krull-Schmidt category with local d.c.c. on ideals. Let $\mathfrak{I}$ be an idempotent ideal of $\mathcal{C}$, let $\mathfrak{L}$ be a representative set of identity maps contained in $\mathfrak{I}$, and let $\mathfrak{R} = \mathfrak{I} \cap \operatorname{rad}_{\mathcal{C}}^{*}$. Then $\mathfrak{I}$ is generated, as an ideal of $\mathcal{C}$, by $\mathfrak{L} \cup \mathfrak{R}$.*

By combining the above statements, we can also characterize the situation where ideals are idempotent exactly when they are generated by a set of identity maps.

Corollary 10. *Let $\mathcal{C}$ be a Krull-Schmidt category with local d.c.c. on ideals. Then the following are equivalent:*

(1) *Every idempotent ideal of $\mathcal{C}$ is generated by a set of identity maps.*
(2) $\operatorname{rad}_{\mathcal{C}}^{*} = 0$.

Proof. (1) $\implies$ (2). If $\operatorname{rad}^*_{\mathcal{C}} \neq 0$, then by Lemma 1 it is a non-zero idempotent ideal without identity maps, hence (1) does not hold.

(2) $\implies$ (1). This is immediate by Corollary 9 since, assuming (2), we always get $\mathfrak{R} = 0$. □

4. Telescope conjecture for module categories

The aim of this section is to prove the TCMC for algebras with vanishing transfinite radicals. First, we need to collect some general results about the TCMC from [25]. Even though the results are often proved under weaker assumptions and carry over almost unchanged for left coherent rings, we specialize them to artin algebras since this is our main concern here.

Proposition 11 ([25, Theorems 3.5, 4.8 and 4.9])**.** *Let Λ be an artin algebra, let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in* $\operatorname{Mod}\Lambda$ *such that $\mathcal{B}$ is closed under unions of well ordered chains, and let $\mathfrak{I}$ be the ideal of all morphisms in* $\operatorname{mod}\Lambda$ *which factor through some (infinitely generated) module from $\mathcal{A}$. Then:*

1. *$(\mathcal{A}, \mathcal{B})$ is of countable type.*
2. *$\mathcal{B} = \operatorname{Ker}\operatorname{Ext}^1_\Lambda(\mathfrak{I}, -) = \{X \in \operatorname{Mod}\Lambda \mid \operatorname{Ext}^1(f, X) = 0 \ \ (\forall f \in \mathfrak{I})\}$.*
3. *Every countably generated module in $\mathcal{A}$ is the direct limit of a countable chain*
$$C_1 \stackrel{f_1}{\to} C_2 \stackrel{f_2}{\to} C_3 \stackrel{f_3}{\to} \dots$$
of finitely generated modules such that $f_i \in \mathfrak{I}$ for each $i \geq 1$.

We also need a technical lemma about filtrations which has been studied in [8, 26, 31] and whose origins can be traced back to an ingenious idea of Paul Hill. Let us recall some definitions.

Definition 12. Given a class of modules $\mathcal{S}$, an *$\mathcal{S}$-filtration* of a module M is a well-ordered chain $(M_\alpha \mid \alpha \leq \sigma)$ of submodules of M such that $M_0 = 0$, $M_\sigma = M$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$, and $M_{\alpha+1}/M_\alpha$ is isomorphic to a module from $\mathcal{S}$ for each $\alpha < \sigma$. A module is called *$\mathcal{S}$-filtered* if it possesses at least one $\mathcal{S}$-filtration.

We will use the following specializations of a general statement from [31] for finitely or countably presented modules:

Lemma 13 ([31, Theorem 6])**.** *Let $\mathcal{S}$ be a set of finitely (resp. countably) presented modules over an arbitrary ring and M a module possessing an $\mathcal{S}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$. Then there is a family $\mathcal{F}$ of submodules of M such that:*

1. *$M_\alpha \in \mathcal{F}$ for all $\alpha \leq \sigma$.*
2. *$\mathcal{F}$ is closed under arbitrary sums and intersections.*
3. *For each $N, P \in \mathcal{F}$ such that $N \subseteq P$, the module P/N is $\mathcal{S}$-filtered.*
4. *For each $N \in \mathcal{F}$ and a finite (resp. countable) subset $X \subseteq M$, there is $P \in \mathcal{F}$ such that $N \cup X \subseteq P$ and P/N is finitely (resp. countably) presented.*

Most of what we need to do now before proving the main results is to observe that the ideal $\mathfrak{I}$ from Proposition 11 is always idempotent. We state this statement for artin algebras, but it again admits an almost verbatim generalization to left coherent rings.

Lemma 14. *Let* Λ, $(\mathcal{A}, \mathcal{B})$ *and* $\mathfrak{I}$ *be as in Proposition* 11*. Then* $\mathfrak{I}$ *is an idempotent ideal of* $\operatorname{mod}\Lambda$*.*

Proof. Let $f : X \to Y$ be a morphism from $\mathfrak{I}$. By definition, f factors as $X \stackrel{g}{\to} A \stackrel{h}{\to} Z$ for some $A \in \mathcal{A}$. Since $(\mathcal{A}, \mathcal{B})$ is of countable type, A must be filtered by countably generated modules from $\mathcal{A}$ [31, Theorem 10]. By Lemma 13, we can find a countably generated submodule $A' \subseteq A$ such that $\operatorname{Im} g \subseteq A'$ and $A' \in \mathcal{A}$. More precisely, we use part (4) of the countable version of Lemma 13 with $N = 0$ and X a finite set of generators of $\operatorname{Im} g$. Hence, f factors as $X \stackrel{g'}{\to} A' \stackrel{h'}{\to} Z$, and, by Proposition 11, we can express A' as the direct limit of a system

$$C_1 \stackrel{f_1}{\to} C_2 \stackrel{f_2}{\to} C_3 \stackrel{f_3}{\to} \dots$$

of finitely generated modules such that $f_i \in \mathfrak{I}$ for each $i \geq 1$. Finally, since X is finitely generated, g' factors through C_i for some $i \geq 1$. But then we can write $f = h'vf_{i+1}f_iu$ for some morphisms u and v, and clearly both f_iu and $h'vf_{i+1}$ are in $\mathfrak{I}$. Hence $f \in \mathfrak{I}^2$ and $\mathfrak{I}$ is idempotent. □

Now, we can equivalently rephrase Conjecture (B) in the language of ideals:

Proposition 15. *Let* Λ, $(\mathcal{A}, \mathcal{B})$ *and* $\mathfrak{I}$ *be as in Proposition* 11*. Then the following are equivalent:*

(1) $(\mathcal{A}, \mathcal{B})$ *is of finite type.*
(2) $\mathfrak{I}$ *is generated by a set of identity morphisms from* $\operatorname{mod}\Lambda$*.*

Proof. (1) $\Longrightarrow$ (2). Assume that $(\mathcal{A}, \mathcal{B})$ is of finite type, i.e., $\mathcal{B} = \operatorname{Ker}\operatorname{Ext}^1_\Lambda(\mathcal{S}, -)$ for some set $\mathcal{S}$ of finitely generated modules. We can without loss of generality assume that $\mathcal{S}$ is a representative set of all finitely generated modules in $\mathcal{A}$.

We claim that $\mathfrak{I}$ is then generated by the set $\{1_X \mid X \in \mathcal{S}\}$. To this end we recall that under our assumption, $\mathcal{A}$ consists precisely of direct summands of $\mathcal{S}$-filtered modules (see [32, Theorem 2.2] or [12, Corollary 3.2.3]). Hence, if $f : X \to Y$ is a morphism from $\mathfrak{I}$, then it factors as $X \stackrel{g}{\to} A \stackrel{h}{\to} Z$ for some $\mathcal{S}$-filtered module A. Using part (4) of the finite version of Lemma 13 with $N = 0$ and a finite set X of generators of $\operatorname{Im} g$, we can find a module $A' \subseteq A$ such that A' is isomorphic to some module in $X \in \mathcal{S}$ and $\operatorname{Im} g \subseteq A'$. Thus, f factors through 1_X and since f was chosen arbitrarily, the claim is proved.

(2) $\Longrightarrow$ (1). Suppose that $\mathcal{S}$ is a set of finitely generated modules such that $\{1_X \mid X \in \mathcal{S}\}$ generates $\mathfrak{I}$. It is straightforward to deduce from Proposition 11(2) that $\mathcal{B} = \bigcap_{X \in \mathcal{S}} \operatorname{Ker}\operatorname{Ext}^1_\Lambda(1_X, -)$. But this is exactly the same as saying that $\mathcal{B} = \operatorname{Ker}\operatorname{Ext}^1_\Lambda(\mathcal{S}, -)$. Hence, the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is of finite type. □

Finally, we can prove the TCMC formulated as Conjecture (B) for those artin algebras Λ for which $\operatorname{rad}^*_\Lambda = 0$. Note that all we need to do in view of Lemma 14 and Proposition 15 is show that certain idempotent ideals are generated by identities, and this is always the case when $\operatorname{rad}^*_\Lambda = 0$. As mentioned above, $\operatorname{rad}^*_\Lambda = 0$ whenever Λ is a domestic standard selfinjective algebra [15] or a domestic special biserial algebra [27] over an algebraically closed field.

Theorem 16. *Let* Λ *be an artin algebra such that* $\operatorname{rad}^*_\Lambda = 0$*. Then every hereditary cotorsion pair* $(\mathcal{A}, \mathcal{B})$ *in* $\operatorname{Mod}\Lambda$ *such that* $\mathcal{B}$ *is closed under unions of well ordered chains is of finite type.*

Proof. Let $\mathfrak{I}$ be the ideal of all morphisms in $\operatorname{mod}\Lambda$ which factor through some module from $\mathcal{A}$. Then $\mathfrak{I}$ is an idempotent ideal by Lemma 14 and, therefore, is generated by a set of identity maps, by Corollary 10. The latter is equivalent to saying that $(\mathcal{A}, \mathcal{B})$ is of finite type, by Proposition 15. □

Another condition on an artin algebra Λ which seems to be closely related to vanishing of the transfinite radical and the domestic representation type is that of the Krull-Gabriel dimension of Λ being an ordinal number. Let us recall first that the category $\mathcal{C}(\Lambda) = \operatorname{fp}(\operatorname{mod}\Lambda, \mathrm{Ab})$ of finitely presented covariant additive functors $\operatorname{mod}\Lambda \to \mathrm{Ab}$ is an abelian category, and we can inductively define a filtration

$$\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \cdots \subseteq \mathcal{S}_\alpha \subseteq \mathcal{S}_{\alpha+1} \subseteq \dots$$

of Serre subcategories of $\mathcal{C}(\Lambda)$ as follows: Let $\mathcal{S}_0$ be the full subcategory of $\mathcal{C}(\Lambda)$ formed by functors of finite length, and for each ordinal number α, let $\mathcal{S}_{\alpha+1}$ be the full subcategory of all functors whose image under the localization functor $\mathcal{C}(\Lambda) \to \mathcal{C}(\Lambda)/\mathcal{S}_\alpha$ is of finite length. At limit ordinals α, we just take the unions $\mathcal{S}_\beta = \bigcup_{\beta<\alpha} \mathcal{S}_\alpha$. We refer to [19, §7] for more details and further references. The construction leads to the following definition:

Definition 17. The *Krull-Gabriel dimension* of an artin algebra Λ is defined as $\operatorname{KGdim}\Lambda = \alpha$ where α is the least ordinal number such that $\mathcal{S}_\alpha = \mathcal{C}(\Lambda)$. If no such α exists, one puts $\operatorname{KGdim}\Lambda = \infty$.

As a consequence of a deeper and more refined theorem, [19, Corollary 8.14] shows that $\operatorname{rad}^*_\Lambda = 0$ whenever $\operatorname{KGdim}\Lambda < \infty$. In particular, we get as a corollary of Theorem 16 that the TCMC holds for any artin algebra with ordinal Krull-Gabriel dimension:

Corollary 18. *Let Λ be an artin algebra such that* $\operatorname{KGdim}\Lambda < \infty$. *Then every hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in* $\operatorname{Mod}\Lambda$ *such that $\mathcal{B}$ is closed under unions of well ordered chains is of finite type.*

Remark. The concept of the Krull-Gabriel dimension has been nicely illustrated by Geigle for tame hereditary algebras Λ in [9], where he explicitly computed that $\operatorname{KGdim}\Lambda = 2$ and described the localization categories $\mathcal{S}_1/\mathcal{S}_0$ and $\mathcal{S}_2/\mathcal{S}_1$.

The proof in [19] of the fact that $\operatorname{KGdim}\Lambda < \infty$ implies $\operatorname{rad}^*_\Lambda = 0$ goes through a stronger statement and involves many technical arguments. There is, however, a more elementary way to see this, namely, one can define a so called m-dimension of a modular lattice following [22, §10.2]. Then $\operatorname{KGdim}\Lambda$ is equal to the m-dimension of the lattice of subobjects in $\operatorname{fp}(\operatorname{mod}\Lambda, \mathrm{Ab})$ of the forgetful functor $\operatorname{Hom}_\Lambda(\Lambda, -)$; see [19, 7.2]. Such subobjects precisely correspond to pairs (M, m) where $M \in \operatorname{mod}\Lambda$ and $m \in M$, and (M', m') corresponds to a subobject of (M, m) if and only if there is a homomorphism $f : M \to M'$ in $\operatorname{mod}\Lambda$ such that $f(m) = m'$ [19, 7.1]. Now, $\operatorname{KGdim}\Lambda = \infty$ if and only if there is a factorizable system in $\operatorname{mod}\Lambda$ in the sense of [23]. Existence of such a factorizable system is easily implied by Lemma 2 or [23, Proposition 0.6] if $\operatorname{rad}^*_\Lambda \neq 0$.

The Krull-Gabriel dimension of Λ also gives a strong link to model theory of modules, as it is equal to the m-dimension of the lattice of primitive positive formulas in the first order theory of Λ-modules. We refer to [23, Proposition 0.3] and [22, §12] for more details.

5. Telescope conjecture for triangulated categories

We also briefly recall the application of the telescope conjecture to triangulated categories. If Λ is a selfinjective artin algebra, then the stable module category $\underline{\mathrm{Mod}}\,\Lambda$ modulo injective modules is *triangulated* in the sense of [10, IV] or [13, I]. The triangles are, up to isomorphism, of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

where $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a short exact sequence in $\mathrm{Mod}\,\Lambda$ and the suspension functor $\Sigma : \underline{\mathrm{Mod}}\,\Lambda \to \underline{\mathrm{Mod}}\,\Lambda$ corresponds to taking cosyzygies in $\mathrm{Mod}\,\Lambda$. Clearly, Σ is an auto-equivalence of $\underline{\mathrm{Mod}}\,\Lambda$ and the corresponding inverse Σ^{-1} is given by taking syzygies in $\mathrm{Mod}\,\Lambda$.

An object X in a triangulated category with (set-indexed) coproducts is called *compact* if the representable functor $\mathrm{Hom}(X, -)$ commutes with coproducts. In particular, an object $X \in \underline{\mathrm{Mod}}\,\Lambda$ is compact if and only if it is isomorphic to a finitely generated Λ-module in $\underline{\mathrm{Mod}}\,\Lambda$ (see [18, §1.5] or [17, §6.5]).

A full triangulated subcategory $\mathcal{X}$ of $\underline{\mathrm{Mod}}\,\Lambda$ is called *localizing* if it is closed under forming coproducts in $\underline{\mathrm{Mod}}\,\Lambda$. A localizing subcategory $\mathcal{X}$ is called *smashing* if the inclusion $\mathcal{X} \hookrightarrow \underline{\mathrm{Mod}}\,\Lambda$ has a right adjoint which preserves coproducts. We say that a localizing subcategory $\mathcal{X}$ is *generated* by a class $\mathcal{C}$ of objects if there is no proper localizing subcategory $\mathcal{X}'$ of $\mathcal{X}$ such that $\mathcal{C} \subseteq \mathcal{X}'$. We refer to [18, 16] for a thorough discussion of these concepts. It follows that $\underline{\mathrm{Mod}}\,\Lambda$ is a *compactly generated* triangulated category, that is, $\underline{\mathrm{Mod}}\,\Lambda$ is generated, as a localizing class, by a set of compact objects.

The telescope conjecture studied in [18, 16] asserts that every smashing localizing subcategory of a compactly generated triangulated category is generated by a set of compact objects. Even though it is generally false as mentioned in the introduction, we can give an affirmative answer in a special case. Specifically, Theorem 16 together with results from [20] imply that the conjecture holds for $\underline{\mathrm{Mod}}\,\Lambda$ where Λ is a selfinjective artin algebra with vanishing transfinite radical.

Theorem 19. *Let Λ be a selfinjective artin algebra such that $\mathrm{rad}^*_\Lambda = 0$. Let $\mathcal{X}$ be a smashing localizing subcategory of $\underline{\mathrm{Mod}}\,\Lambda$. Then $\mathcal{X}$ is generated by a set of finitely generated Λ-modules.*

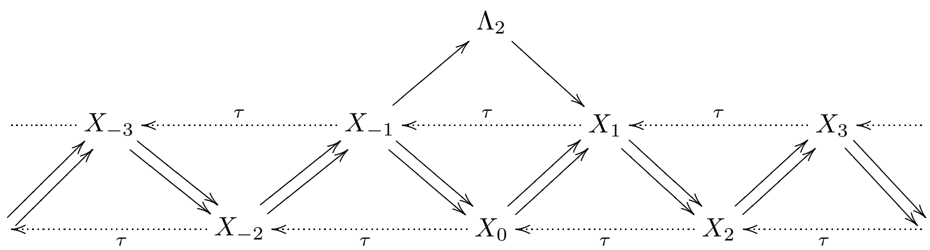
Proof. We know that Conjecture (B) holds for Λ by Theorem 16. Hence Conjecture (A) also holds by the discussion in Section 1. The rest follows immediately from [20, Corollary 7.7]. □

6. Examples

We conclude with some examples of particular representation-infinite selfinjective algebras with vanishing transfinite radical.

Example 20. The simplest example is probably the exterior algebra of a 2-dimensional vector space over an algebraically closed field, i.e. $\Lambda_2 = k\langle x, y\rangle/(x^2, y^2, xy + yx)$. It is a special biserial algebra in the sense of [30] and it has, up to rotation equivalence and inverse, only one band xy^{-1}. In particular, Λ_2 is domestic and we have exactly one one-parameter family of indecomposable modules in each even dimension. For example, we have $M_{(a:b)} = \Lambda_2/\Lambda_2(ax + by)$ for each $(a{:}b) \in \mathbb{P}^1(k)$ in dimension 2. Thus, $\mathrm{rad}^*_{\Lambda_2} = 0$ by [27, Theorem 2].

With a little more effort, we can classify all smashing localizations and all hereditary cotorsion pairs with the right hand class closed under unions of chains. Using the representation theory of special biserial algeras, one can readily compute the Auslander-Reiten quiver of Λ_2. It consists of a family $(\mathcal{T}_{(a:b)} \mid (a{:}b) \in \mathbb{P}^1(k))$ of homogeneous tubes, the corresponding quasi-simples being precisely the modules $M_{(a:b)}$ above. In addition, there is one more component, which we denote by $\mathcal{C}$, of the form



where X_0 is the unique simple module and X_n and X_{-n} are the string modules corresponding to the strings $(yx^{-1})^n$ and $(x^{-1}y)^n$, respectively. In particular, $\dim_k X_n = 2\cdot|n|+1$. It is easy to compute that $\Omega^-(X_n) \cong X_{n+1}$ and $\Omega^-(M) = M$ for each indecomposable finite dimensional module in a tube. This describes the restriction of the suspension functor $\Sigma : \underline{\mathrm{Mod}}\,\Lambda_2 \to \underline{\mathrm{Mod}}\,\Lambda_2$ to $\underline{\mathrm{mod}}\,\Lambda_2$.

We recall that a full triangulated subcategory $\mathcal{X}_0$ of $\underline{\mathrm{mod}}\,\Lambda_2$ is called *thick* if it is closed under direct summands. There is a bijective correspondence between thick subcategories $\mathcal{X}_0$ of $\underline{\mathrm{mod}}\,\Lambda_2$ and localizing subcategories $\mathcal{X}$ of $\underline{\mathrm{Mod}}\,\Lambda_2$ generated by a set of compact objects. More precisely, if $\mathcal{X}$ is generated by $\mathcal{X}_0 \subseteq \underline{\mathrm{mod}}\,\Lambda_2$ and $\mathcal{X}_0$ is thick, then $\mathcal{X} \cap \underline{\mathrm{mod}}\,\Lambda_2 = \mathcal{X}_0$ [21, 2.2]. It is clear that each thick subcategory is uniquely determined by its indecomposable objects.

We will now describe thick subcategories of $\underline{\mathrm{mod}}\,\Lambda_2$. It is straightforward to check that if an indecomposable non-injective module $M \in \mathrm{mod}\,\Lambda_2$ is contained in a thick subcategory $\mathcal{X}_0$, then all modules in the same component of the Auslander-Reiten quiver are in $\mathcal{X}_0$, too. On the other hand, if $\mathcal{T}_p$ is a tube for some $p \in \mathbb{P}^1(k)$, then one can check that in $\mathrm{mod}\,\Lambda_2$, the additive closure of $\mathcal{T}_p \cup \{\Lambda_2\}$ equals

$$\{X \in \mathrm{mod}\,\Lambda_2 \mid \underline{\mathrm{Hom}}_{\Lambda_2}(X, \mathcal{T}_q) = 0 = \underline{\mathrm{Hom}}_{\Lambda_2}(\mathcal{T}_q, X) \ \ (\forall q \in \mathbb{P}^1(k) \setminus \{p\})\}.$$

Therefore, $\mathrm{add}(\mathcal{T}_p \cup \{\Lambda_2\})$ is closed under extensions, syzygies and cosyzygies in $\mathrm{mod}\,\Lambda_2$, and consequently $\mathrm{add}\mathcal{T}_p$ is thick in $\underline{\mathrm{mod}}\,\Lambda_2$. It is easy to see that $\underline{\mathrm{Hom}}_{\Lambda_2}(\mathcal{T}_p, \mathcal{T}_q) = 0$ for $p \neq q$, so the additive closure of any set of tubes is thick in $\underline{\mathrm{mod}}\,\Lambda_2$. Finally, there is an exact sequence $0 \to M \to X_m \to X_{m+1} \to 0$ for each $m < 0$ and each quasi-simple module M in a tube; hence a thick subcategory containing the component $\mathcal{C}$ contains all the tubes, too. Upon summarizing all the facts (and using Theorem 19), we obtain the following classification:

Proposition 21. *Let k be an algebraically closed field, let $\Lambda_2 = k\langle x, y\rangle/(x^2, y^2, xy+yx)$, and let $\mathcal{C}$ and $\mathcal{T}_p$, $p \in \mathbb{P}^1(k)$, be the components of the Auslander-Reiten quiver of Λ_2 as above. Then each smashing localizing class $\mathcal{X}$ in $\underline{\mathrm{Mod}}\,\Lambda_2$ is generated by $\mathcal{X}_0 = \mathcal{X} \cap \underline{\mathrm{mod}}\,\Lambda_2$, and the possible intersections $\mathcal{X}_0$ are classified as follows:*

(1) $\mathcal{X}_0 = 0$*; or*
(2) $\mathcal{X}_0$ *is the additive closure of* $\bigcup_{p\in P} \mathcal{T}_p$ *for some* $P \subseteq \mathbb{P}^1(k)$*; or*
(3) $\mathcal{X}_0 = \underline{\mathrm{mod}}\,\Lambda_2$.

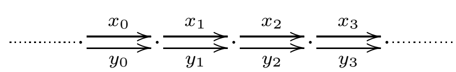
In the same spirit, we can classify the hereditary cotorsion pairs $(\mathcal{A}, \mathcal{B})$ in $\operatorname{Mod}\Lambda_2$ such that $\mathcal{B}$ is closed under unions of chains. Recall that a subcategory $\mathcal{A}_0$ of $\operatorname{mod}\Lambda_2$ is called *resolving* if it contains Λ_2 and is closed under extensions, kernels of epimorphisms and direct summands. There is a bijective correspondence between resolving subcategories $\mathcal{A}_0$ in $\operatorname{mod}\Lambda_2$ and hereditary cotorsion pairs $(\mathcal{A}, \mathcal{B})$ of finite type in $\operatorname{Mod}\Lambda_2$ [3, 2.5]. Note that if $\mathcal{A}_0$ is resolving and contains a module $X_m \in \mathcal{C}$, it must contain all X_z, $z \leq m$, and all tubes. On the other hand, it is not difficult to see that there is an exact sequence $0 \to X_n \to U \to X_{-k} \to 0$ with an indecomposable (string) module U from a tube for each $n, k > 0$. Hence $\mathcal{A}_0$ must contain all of $\mathcal{C}$, too. We will leave details of the following statement (using Theorem 16) for the reader:

Proposition 22. *Let k be an algebraically closed field, let $\Lambda_2 = k\langle x, y\rangle/(x^2, y^2, xy+yx)$, and let $\mathcal{C}$ and $\mathcal{T}_p$, $p \in \mathbb{P}^1(k)$, be the components of the Auslander-Reiten quiver of Λ_2 as above. Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in $\operatorname{Mod}\Lambda_2$ such that $\mathcal{B}$ is closed under unions of chains, and let $\mathcal{A}_0 = \mathcal{A} \cap \operatorname{mod}\Lambda_2$. Then $\mathcal{B} = \operatorname{Ker}\operatorname{Ext}^1_{\Lambda_2}(\mathcal{A}_0, -)$, and the possible classes $\mathcal{A}_0$ are classified as follows:*

(1) $\mathcal{A}_0 = \operatorname{add}\{\Lambda_2\}$*; or*
(2) $\mathcal{A}_0$ *is the additive closure of* $\{\Lambda_2\} \cup \bigcup_{p\in P} \mathcal{T}_p$ *for* $P \subseteq \mathbb{P}^1(k)$*; or*
(3) $\mathcal{A}_0 = \operatorname{mod}\Lambda_2$.

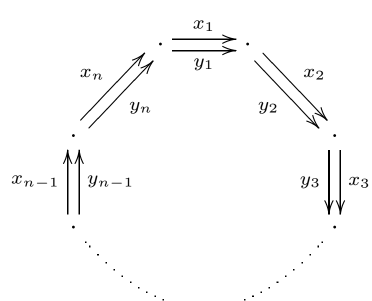
Example 23. A recipe for construction of more complicated examples is given in [15]. Let B be a representation-infinite tilted algebra of Euclidean type over an algebraically closed field and $\hat{B}$ its repetitive algebra. Put $\Lambda = \hat{B}/G$ where G is an admissible infinite cyclic group of k-linear automorphisms of $\hat{B}$ (see [29, §1] for unexplained terminology). Then Λ is selfinjective and $\operatorname{rad}^*_\Lambda = 0$ by the main result of [15].

We illustrate the construction on $B = k(\cdot \rightrightarrows \cdot)$, the Kronecker algebra. The repetitive algebra $\hat{B}$ is then given by the following infinite quiver with relations



$$x_{i+1}x_i - y_{i+1}y_i = 0, \quad x_{i+1}y_i = 0, \quad y_{i+1}x_i = 0 \qquad \text{for each } i \in \mathbb{Z}.$$

Let $n \geq 1$ and let $\bar{q} = (q_1, \ldots, q_n)$ be an n-tuple of non-zero elements of k. It is not difficult to see that we get the algebra $\Lambda_{n,\bar{q}}$, described by the quiver and relations below, as $\hat{B}/G$ for a suitable G:



$$x_{i+1}y_i + q_i y_{i+1}x_i = 0, \;\; x_{i+1}x_i = 0, \;\; y_{i+1}y_i = 0 \quad \text{for each } i \in \{1, 2, \ldots, n\}.$$

The addition in indices of the arrows above is considered modulo n. It is easy to see that $\Lambda_{n,\bar{q}}$ is special biserial and that there are exactly n one-parameter families of indecomposable $\Lambda_{n,\bar{q}}$-modules in each even dimension. They correspond to the bands $x_i y_i^{-1}$. In fact, if $n = 1$ and $q_1 = 1$, we get precisely the exterior algebra on a 2-dimensional space.

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