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SIEGEL METRIC AND CURVATURE OF THE MODULI SPACE OF CURVES

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Abstract. We study the curvature of the moduli space M_g of curves of genus g with the Siegel metric induced by the period map $j : M_g \to A_g$. We give an explicit formula for the holomorphic sectional curvature of M_g along a Schiffer variation ξ_P, for P a point on the curve X, in terms of the holomorphic sectional curvature of A_g and the second Gaussian map. Finally we extend the Kähler form of the Siegel metric as a closed current on $\overline{M}_g$ and we determine its cohomology class as a multiple of λ.

1. Introduction

During the last thirty years, some natural metrics on the moduli space of genus g curves M_g have been extensively studied. Many of these metrics come from metrics on the Teichmüller space of which the moduli space of curves is the quotient by the mapping class group. One of these is the Weil-Petersson metric ω_{WP}. It was introduced by Weil and is known to be Kähler, to have nonpositive curvature operator and negative Ricci curvature, and to be geodesically convex. S. A. Wolpert showed that both its holomorphic sectional curvature and Ricci curvature have negative (genus dependent) upper bounds (but no lower bounds do exist). Moreover it is not complete ([23]). The other canonical metrics, namely the Teichmüller metric (or the Kobayashi metric), the Carathéodory metric, the Kähler-Einstein metric, the induced Bergman metric, the McMullen metric, are complete. Recently Liu, Sun and Yau ([10], [11]) showed their equivalence on M_g and the equivalence with the Ricci metric and the perturbed Ricci metric introduced by them. The Kähler form of the WP metric has been extended by Masur ([12]) as a closed current on the Deligne-Mumford compactification $\overline{M}_g$ of M_g. Wolpert ([22]) determined its cohomology class in terms of the first Chern class of the Hodge bundle λ and the classes of the boundary.

Let A_g be the moduli space of principally polarized Abelian varieties of dimension g and let $j : M_g \to A_g$ be the period map sending a curve to its Jacobian. It is an interesting and classical problem to understand the geometry of the image of M_g in A_g.

On A_g there is a natural metric coming from the unique $Sp(2g, \mathbb{R})$ invariant metric on the Siegel space $H_g \simeq Sp(2g, \mathbb{R})/U(g)$ of which A_g is the quotient by

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$Sp(2g, \mathbb{Z})$. The purpose of this paper is to study the metric on M_g induced by this metric through the period map, which we call the Siegel metric. In [4] an explicit expression for the second fundamental form of the immersion j is given and it is proven that the second fundamental form lifts the second Gaussian map $\mu_2 : I_2(K_X) \to H^0(X, 4K_X)$, as stated in an unpublished paper of Green and Griffiths (cf. [7]).

Here we use it to compute the curvature of the Siegel metric. In particular we give an explicit formula for the holomorphic sectional curvature of M_g along a Schiffer variation ξ_P, for P a point on the curve X, in terms of the holomorphic sectional curvature of A_g and the second Gaussian map $\mu_2 : I_2(K_X) \to H^0(X, 4K_X)$.

Finally we give some properties of the holomorphic sectional curvature of M_g, using results of [3]. In particular along a Schiffer variation ξ_P the holomorphic sectional curvature $H(\xi_P)$ of M_g is strictly smaller than the holomorphic sectional curvature of A_g unless P is either a Weierstrass point of a hyperelliptic curve or a ramification point of the g^1_3 on a trigonal curve. In these last cases $H(\xi_P) = -1$.

Furthermore we study the asymptotic behaviour of the Kähler form of the Siegel metric on M_g showing that it extends as a closed current to $\overline{M}_g$; hence it defines a cohomology class in $H^2(\overline{M}_g, \mathbb{C})$, which we compute to be $\pi\lambda$.

In all that we have stated, we have considered M_g as an orbifold. In the paper we make all computations using the covering of M_g given by the moduli space of curves with level $n \geq 3$ structures $M_g^{(n)}$ and the moduli space $A_g^{(n)}$ of principally polarized abelian varieties with level n structures.

In fact $M_g^{(n)}$ and $A_g^{(n)}$ are smooth and by the local Torelli theorem proven in [18] we know that the period map $j^{(n)} : M_g^{(n)} \to A_g^{(n)}$ is a two-to-one immersion outside the hyperelliptic locus and it is an injective immersion if we restrict to the hyperelliptic locus.

The paper is organized as follows: in Section 2 we define the Siegel metric and compute it on the tangent directions given by the Schiffer variations (Lemma 2.2). In Section 3 we give the expression of the curvature of the Siegel metric on $A_g^{(n)}$ restricted to the Schiffer variations. Then, we show that the second fundamental form of the immersion of $M_g^{(n)}$ in $A_g^{(n)}$ is nonzero at any nonhyperelliptic curve, and we exhibit a formula for the curvature of $M_g^{(n)}$ (Thm. 3.7). Finally we write the holomorphic sectional curvature of $M_g^{(n)}$ along a Schiffer variation ξ_P, using the second Gaussian map. In Section 4 we give some applications of results of [3] to the holomorphic sectional curvature of $M_g^{(n)}$. In Section 5 we study in particular the hyperelliptic locus HE_g and we show that the second fundamental form of HE_g in $A_g^{(n)}$ is nonzero at any point. In Section 6 we extend the Kähler form of the Siegel metric as a closed current on $\overline{M}_g$ and we determine its cohomology class (6.1).

2. The Siegel metric

We introduce some notation. Let M_g, resp. $M_g^{(n)}$, be the moduli space of smooth genus g curves, resp. of smooth genus g curves with a fixed n-level structure. Denote by T_g the Teichmüller space and by Γ_g the mapping class group acting on T_g with quotient M_g. Let $K(n) := \ker(\Gamma_g \to Sp(2g, \mathbb{Z}/n\mathbb{Z}))$ and recall that $M_g^{(n)}$ is the quotient of T_g by the action of $K(n)$. Moreover, let $K := \ker(\Gamma_g \to Sp(2g, \mathbb{Z}))$ be the Torelli group and define the Torelli space Tor_g as the quotient of T_g by the action of the Torelli group.

Let A_g, resp. $A_g^{(n)}$, be the moduli space of g-dimensional principally polarized Abelian varieties, resp. of g-dimensional principally polarized Abelian varieties with an n-level structure. Denote by $H_g := \{Z \in M(g,\mathbb{C}) \mid Z = {}^tZ, \operatorname{Im} Z > 0\}$ the Siegel space so that A_g is the quotient of H_g by the action of $Sp(2g,\mathbb{Z})$ and $A_g^{(n)}$ is the quotient of H_g by $\ker(Sp(2g,\mathbb{Z}) \to Sp(2g,\mathbb{Z}/n\mathbb{Z}))$. Denote by j^{Tor}, j and $j^{(n)}$ the period maps which send a curve to its Jacobian. We have the following diagram:

$$\begin{array}{ccc} T_g & & \\ \downarrow & & \\ Tor_g & \stackrel{j^{Tor}}{\to} & H_g \\ \downarrow & & \downarrow \\ M_g^{(n)} & \stackrel{j^{(n)}}{\to} & A_g^{(n)} \\ \downarrow & & \downarrow \\ M_g & \stackrel{j}{\to} & A_g \end{array}$$

The Torelli theorem states that j is injective, while j^{tor} and $j^{(n)}$ are two-to-one on the image and ramified over the hyperelliptic locus. In fact multiplication by -1 in $H^1(X,\mathbb{Z}) = H^1(JX,\mathbb{Z})$, where JX is the Jacobian of the curve X, is induced by an automorphism of Abelian varieties but not by an automorphism of nonhyperelliptic curves. The local Torelli theorem says that outside the hyperelliptic locus and restricted to the hyperelliptic locus the period map is an immersion (cf. [18]). From now on we shall work on $M_g^{(n)}$ and $A_g^{(n)}$, with $n \geq 3$, since they are smooth, everything works in the same way on M_g and A_g but in the orbifold context.

We will now define the Siegel metric.

The Siegel space H_g is a homogeneous space, and it can be seen as the quotient $Sp(2g,\mathbb{R})/U(g)$. We call the unique (up to scalar) invariant metric the Siegel metric.

Let F be the homogeneous vector bundle on H_g associated to the standard g-dimensional representation of $U(g,\mathbb{C})$. The Hodge metric h on F is the only (up to multiplication by scalars) invariant metric on the homogeneous bundle F. Moreover, through the identification

$$\Omega^1_{H_g} \simeq S^2F,$$

the Hodge metric on F defines the Siegel metric on H_g .

The Siegel metric on H_g defines a metric on $A_g^{(n)}$ and A_g and, through the period map, an induced metric on $M_g^{(n)}$ and M_g outside the hyperelliptic locus, and on the hyperelliptic locus itself. We call all these metrics the Siegel metrics.

These metrics can be described in terms of polarized variations of Hodge structures. More precisely, on $A_g^{(n)}$ we have the universal family $\phi : \mathcal{A} \to A_g^{(n)}$, and the polarized variation of Hodge structures associated to the local system $R^1\phi_*\mathbb{Z}$. The associated Hodge bundle $\mathcal{F}^1$ can be identified with $\phi_*(\Omega^1_{\mathcal{A}|A_g^{(n)}})$, where $\Omega^1_{\mathcal{A}|A_g^{(n)}}$ is the sheaf of relative holomorphic one-forms. The polarization induces a Hermitian metric on $R^1\phi_*\mathbb{C}$ and on $\mathcal{F}^1$, which we call the Hodge metric. In fact the pullback of $\mathcal{F}^1$ on H_g is the bundle F and the pullback of the metric is the Hodge metric on F. Hence the Siegel metric is induced by the Hodge metric through the identification $S^2\mathcal{F}^1 \cong \Omega^1_{A_g^{(n)}}$.

On $M_g^{(n)}$ we have the universal family $\psi : \mathcal{C} \to M_g^{(n)}$ with induced relative dualizing sheaf $K_{\mathcal{C}|M_g^{(n)}}$. The local system $R^1\psi_*\mathbb{Z}$ coincides with the pullback of $R^1\phi_*\mathbb{Z}$ through the period map: at a point $[X] \in M_g^{(n)}$, we have $H^1(X,\mathbb{Z}) \cong H^1(JX,\mathbb{Z})$. The nondegenerate Hermitian product on $H^1(X,\mathbb{C})$, defined by the polarization, is the following: for any $[\eta],[\xi] \in H^1(X,\mathbb{C})$, we have

$$\langle [\eta],[\xi]\rangle = i\int_X \eta \wedge \overline{\xi}.$$

The Hodge bundle can be identified with $\psi_*(K_{\mathcal{C}|M_g^{(n)}})$, and the corresponding Hodge metric yields a metric on $S^2\mathcal{F}^1 \cong {j^{(n)}}^*\Omega^1_{A_g^{(n)}}$, hence on ${j^{(n)}}^*\mathcal{T}_{A_g^{(n)}}$, and by restriction the Siegel metric on $\mathcal{T}_{M_g^{(n)}}$.

We finally observe that for the sake of simplicity we defined the Siegel metric on the fine moduli space $M_g^{(n)}$, but we also have a Siegel metric on M_g viewed as an orbifold.

2.1. **An explicit formula.** We shall now give an explicit formula for the Siegel metric on $M_g^{(n)}$ at a point $[X] \in M_g^{(n)}$ in terms of the basis of $H^1(T_X)$ given by Schiffer variations ξ_P, for a set of $3g-3$ general points on X.

Now, we briefly recall the definition of ξ_P. Consider the exact sequence

$$0 \to T_X \to T_X(P) \to T_X(P)_{|P} \to 0.$$

Notice that $H^0(T_X(P)_{|P}) \cong \mathbb{C}$. If we denote the coboundary map by $\delta : H^0(T_X(P)_{|P}) \to H^1(T_X)$, we have $\dim(\mathrm{Im}(\delta)) = 1$. Any nonzero element ξ_P in $\mathrm{Im}(\delta)$ is called a Schiffer variation. Let us a choose a local coordinate z in a neighborhood of P. Under the Dolbeault isomorphism $H^1(T_X) \cong H^{0,1}(T_X)$, it is represented by the form

$$\theta_P = \frac{1}{z}\overline{\partial} b_P \otimes \frac{\partial}{\partial z},$$

where b_P is a bump function around P. Notice that if we choose b_P to be one in a neighborhood of P for this choice of local coordinate z, ξ_P depends only on the choice of z. In what follows, we need to express ξ_P in terms of a basis of $S^2(H^0(K_X)^*)$, through the inclusion of $H^1(T_X)$ in $S^2(H^0(K_X)^*)$.

Fix an orthonormal basis $\{\omega_i\}_{i=1,\dots,g}$ of $H^0(K_X)$. Choose a local coordinate z around a point $P \in X$ and write $\omega_j = f_j(z)\,dz$. Since $H^0(K_X) \cong H^{1,0}(X)$, the set $\{\overline{\omega}_i\}_{i=1,\dots,g}$ is a basis of $H^{0,1}(X)$. This set can be viewed as the dual basis of $\{\omega_i\}$, where the nondegenerate pairing is given by

$$\overline{\omega}_i(\omega_j) = i\int_X \omega_i \wedge \overline{\omega}_j = \langle \omega_i, \omega_j\rangle = \delta_{ij}.$$

Observe that we have

$$\langle \overline{\omega}_i, \overline{\omega}_j\rangle = i\int_X \overline{\omega}_i \wedge \omega_j = -i\int_X \omega_j \wedge \overline{\omega}_i = -\delta_{ij}. \tag{2.1}$$

Lemma 2.1. *For a choice of a local coordinate z at P, we have*

$$\langle \xi_P(\omega_i), \overline{\omega_j}\rangle = -2\pi f_i(P) f_j(P). \tag{2.2}$$

Hence in $S^2(H^0(K_X)^*) \cong S^2(H^{0,1}(X))$ it holds

$$\xi_P = 2\pi \sum_{i,j}^{g} f_i(P) f_j(P) (\overline{\omega}_i \odot \overline{\omega}_j).$$

Proof. Since $\xi_P = \sum_{i=1}^{g} \xi_P(\omega_i) \odot \overline{\omega}_i$ and

$$\xi_P(\omega_i) = \sum_j -\langle \xi_P(\omega_i), \overline{\omega}_j \rangle \overline{\omega}_j, \tag{2.3}$$

we get

$$\xi_P = \sum_{i,j} -\langle \xi_P(\omega_i), \overline{\omega}_j \rangle \left(\overline{\omega}_j \odot \overline{\omega}_i \right).$$

By definition of ξ_P, the element $\xi_P(\omega_i) \in H^{0,1}(X)$ is represented by the $(0,1)$-form

$$(\frac{1}{z}\overline{\partial} b_P \otimes \frac{\partial}{\partial z}) \rfloor (f_i(z)dz) = \frac{1}{z}\overline{\partial} b_P f_i(z).$$

Let C be a small circle around P such that $b_P \equiv 1$ on C. Then the lemma follows by the Stokes and Cauchy theorems:

$$\langle \xi_P(\omega_i), \overline{\omega_j} \rangle = i \int_X \frac{\overline{\partial} b_P}{z - z(P)} f_i(z) \wedge f_j(z) dz = i \int_X \overline{\partial} \left(\frac{b_P f_i(z) f_j(z)}{z - z(P)} \right) \wedge dz$$
$$= i \int_X d \left(\frac{b_P f_i(z) f_j(z)}{z - z(P)} dz \right) = i \int_C \frac{f_i(z) f_j(z)}{z - z(P)} dz = -2\pi f_i(P) f_j(P).$$

□

Lemma 2.2. *The scalar product of the two Schiffer variations ξ_P, $\xi_{P'}$ has the following form:*

$$\langle \xi_P, \xi_{P'} \rangle = 8\pi^2 (\alpha_{P,P'})^2,$$

where

$$\alpha_{P,P'} = \sum_i f_i(P) \overline{f_i(P')}. \tag{2.4}$$

Proof. Recall that on $S^2 H^{0,1}$ the scalar product is:

$$\langle a \odot b, c \odot d \rangle = \langle a, c \rangle \langle b, d \rangle + \langle a, d \rangle \langle b, c \rangle,$$

induced by the scalar product $\langle a \otimes b, c \otimes d \rangle = 2 \langle a, c \rangle \langle b, d \rangle$ on $H^{0,1} \otimes H^{0,1}$ via the inclusion of $S^2 H^{0,1} \overset{\iota}{\hookrightarrow} H^{0,1} \otimes H^{0,1}$, $\iota(a \odot b) = \frac{1}{2}(a \otimes b + b \otimes a)$.

So, by (2.1) one immediately computes

$$\langle \xi_P, \xi_{P'} \rangle = 8\pi^2 \sum_{i,j} f_i(P) \overline{f_i(P')} f_j(P) \overline{f_j(P')} = 8\pi^2 (\alpha_{P,P'})^2.$$

□

3. Curvature

We would like now to give a formula for the curvature of the Siegel metric on $M_g^{(n)}$. We will do the computation on the tangent vectors given by the ξ_P's. These depend on the choice of the local coordinates, but by linearity one can immediately derive the formulas at the tangent vectors $\frac{\xi_P}{|\xi_P|} = \frac{\xi_P}{2\sqrt{2}\pi \alpha_{P,P}}$, which are intrinsic.

Recall that outside the hyperelliptic locus we have the sequence of tangent bundles:

$$0 \to \mathcal{T}_{M_g^{(n)}} \to j^{(n)*} \mathcal{T}_{A_g^{(n)}} \overset{\pi}{\to} \mathcal{N} \to 0, \tag{3.1}$$

whose dual, under the identifications $j^{(n)*}\Omega^1{}_{A_g^{(n)}} \cong S^2(\psi_* K_{\mathcal{C}|M_g^{(n)}})$, $\Omega^1{}_{M_g^{(n)}} \cong \psi_*(K^2{}_{\mathcal{C}|M_g^{(n)}})$, is

$$0 \to \mathcal{I}_2 \to S^2(\psi_* K_{\mathcal{C}|M_g^{(n)}}) \xrightarrow{m} \psi_*(K^2{}_{\mathcal{C}|M_g^{(n)}}) \to 0, \tag{3.2}$$

where $\mathcal{I}_2 := \mathcal{N}^*$ and m is the multiplication map.

The Hermitian connection of the variation of Hodge structures $\mathcal{R}^1\psi_*\mathbb{C}$, the Gauss-Manin connection, defines a Hermitian connection on $\mathcal{F}^1 = \psi_* K_{\mathcal{C}|M_g^{(n)}}$, thus on ${\mathcal{F}^1}^*$, as well as $S^2\mathcal{F}^1$ and $S^2{\mathcal{F}^1}^* \simeq j^{(n)*}\mathcal{T}_{A_g^{(n)}}$, which we denote by ∇.

The exact sequence (3.1) defines a second fundamental form,

$$\sigma \in Hom(\mathcal{T}_{M_g^{(n)}}, \mathcal{N} \otimes \Omega^1_{M_g^{(n)}}),\ \sigma : s \mapsto \pi(\nabla(s)).$$

Similarly the exact sequence (3.2) defines the second fundamental form $\rho \in Hom\left(\mathcal{I}_2, \psi_*(K^2_{\mathcal{C}|M_g^{(n)}}) \otimes \Omega^1_{M_g^{(n)}}\right)$.

The curvature form R of $\mathcal{T}_{M_g^{(n)}}$ is computed in terms of the curvature form $\tilde{R}$ of ${j^{(n)}}^*(\mathcal{T}_{A_g^{(n)}})$ and the second fundamental form σ. Namely, we have

$$\langle R(s), t\rangle = \langle \tilde{R}(s), t\rangle - \langle \sigma(s), \sigma(t)\rangle, \tag{3.3}$$

where s, t are local sections of $\mathcal{T}_{M_g^{(n)}}$.

At the point $[X] \in M_g^{(n)}$, we need to compute

$$\langle R(\xi_P), \xi_{P'}\rangle(\xi_R, \overline{\xi_T}) = \langle \tilde{R}(\xi_P), \xi_{P'}\rangle(\xi_R, \overline{\xi_T}) - \langle \sigma(\xi_P), \sigma(\xi_{P'})\rangle(\xi_R, \overline{\xi_T}). \tag{3.4}$$

Let us now determine $\langle \tilde{R}(\xi_P), \xi_{P'}\rangle(\xi_R, \overline{\xi_T})$ in terms of the curvature form of the Hodge bundle.

Consider the exact sequence

$$0 \to \mathcal{F}^1 \to \mathcal{R}^1\psi_*\mathbb{C} \otimes \mathcal{C}^\infty_{M_g^{(n)}} \to (\mathcal{R}^1\psi_*\mathbb{C} \otimes \mathcal{C}^\infty_{M_g^{(n)}})/\mathcal{F}^1 \to 0. \tag{3.5}$$

At $[X] \in M_g^{(n)}$ we have

$$0 \to H^{1,0}(X) \to H^1(X, \mathbb{C}) \to H^{0,1}(X) \to 0.$$

Lemma 3.1. *The curvature form of the Hodge bundle is given by*

$$\langle R_{\mathcal{F}^1}(\omega_j), \omega_l\rangle(\xi_R, \overline{\xi}_T) = 4\pi^2 \alpha_{R,T} f_j(R)\overline{f_l(T)}.$$

Proof. Since the Gauss-Manin connection on $\mathcal{R}^1\psi_*\mathbb{C} \otimes \mathcal{C}^\infty_{M_g^{(n)}}$ is flat, the following holds:

$$\langle R_{\mathcal{F}^1}(\omega_j), \omega_l\rangle = -\langle \epsilon(\omega_j), \epsilon(\omega_l)\rangle,$$

where $\epsilon \in Hom(H^{1,0}(X), H^{0,1}(X) \otimes H^0(2K_X))$ is the second fundamental form of (3.5) at the point $[X]$. We can also view ϵ as an element in $Hom(H^{1,0}(X) \otimes H^1(T_X), H^{0,1}(X))$, and by a result of Griffiths (cf. e.g. [7], p.32), we have $\epsilon(\omega_i \otimes \zeta) = \zeta(\omega_i)$. Hence, we can write $\epsilon(\omega_j) = \sum_P(\xi_P(\omega_j) \otimes {\xi_P}^*)$. Therefore, we have

$$\langle \epsilon(\omega_j), \epsilon(\omega_l)\rangle = \sum_{P,S} \langle \xi_P(\omega_j), \xi_S(\omega_l)\rangle({\xi_P}^* \otimes \overline{\xi_S}^*).$$

This implies

$$\langle \epsilon(\omega_j), \epsilon(\omega_l)\rangle(\xi_R, \overline{\xi}_T) = \langle \xi_R(\omega_j), \xi_T(\omega_l)\rangle.$$

By (2.3), (2.2), and (2.4) we deduce

$$\begin{aligned}\langle \xi_R(\omega_j), \xi_T(\omega_l)\rangle &= \sum_{k,i} 4\pi^2 f_j(R) f_k(R) \overline{f_l(T) f_i(T)} \langle \overline{\omega}_k, \overline{\omega}_i\rangle \\ &= -4\pi^2 \alpha_{R,T} f_j(R) \overline{f_l(T)}.\end{aligned}$$

□

Finally, we prove the following:

Proposition 3.2. *The curvature* $\tilde{R}$ *of* $j^{(n)^*}(\mathcal{T}_{A_g^{(n)}}) = S^2(\mathcal{F}^{1^*})$ *is given by*

$$\begin{aligned}\langle \tilde{R}(\xi_P), \xi_{P'}\rangle(\xi_S, \overline{\xi_T}) &= -64\pi^4 \alpha_{S,T} \sum_{i,j,l} f_i(P) f_j(P) \overline{f_j(P') f_l(P')} f_l(S) \overline{f_i(T)} \\ &= -64\pi^4 \alpha_{S,T}\alpha_{P,T}\alpha_{P,P'}\alpha_{S,P'}.\end{aligned}$$

Proof. To begin with, by Lemma 2.1 we have

$$\langle \tilde{R}(\xi_P), \xi_{P'}\rangle = 4\pi^2 \sum_{i,j,k,l} f_i(P) f_j(P) \overline{f_k(P') f_l(P')} \langle \tilde{R}(\overline{\omega_j} \odot \overline{\omega_i}), \overline{\omega_l} \odot \overline{\omega_k}\rangle.$$

By standard facts on complex bundles [9], we have

$$\begin{aligned}\langle \tilde{R}(\overline{\omega_j} \odot \overline{\omega_i}), \overline{\omega_l} \odot \overline{\omega_k}\rangle &= \langle R_{\mathcal{F}^{1*}}(\overline{\omega_j}) \odot \overline{\omega_i} + \overline{\omega_j} \odot R_{\mathcal{F}^{1*}}(\overline{\omega_i}), \overline{\omega_l} \odot \overline{\omega_k}\rangle \\ &= -\left(\delta_{ik}\langle R_{\mathcal{F}^{1*}}(\overline{\omega_j}), \overline{\omega_l}\rangle + \delta_{il}\langle R_{\mathcal{F}^{1*}}(\overline{\omega_j}), \overline{\omega_k}\rangle \right. \\ &\quad \left. + \delta_{jl}\langle R_{\mathcal{F}^{1*}}(\overline{\omega_i}), \overline{\omega_k}\rangle + \delta_{jk}\langle R_{\mathcal{F}^{1*}}(\overline{\omega_i}), \overline{\omega_l}\rangle\right).\end{aligned} \tag{3.6}$$

Now, we observe that

$$\langle R_{\mathcal{F}^{1*}}(\overline{\omega_j}), \overline{\omega_l}\rangle = \langle R_{\mathcal{F}^1}(\omega_l), \omega_k\rangle.$$

In fact, set $R_{\mathcal{F}^{1*}}(\overline{\omega_j}) = \sum_i a_{ij}\overline{\omega_i}$, where $a_{ij} \in \Omega^{1,1}_{M_g^{(n)}}$. By duality, we have $R_{\mathcal{F}^1}(\omega_j) = -\sum_i a_{ji}\omega_i$. Hence $\langle R_{\mathcal{F}^{1*}}(\overline{\omega_j}), \overline{\omega_l}\rangle = -a_{lj} = \langle R_{\mathcal{F}^1}(\omega_l), \omega_j\rangle$.

By (3.6) and Lemma 3.1, we deduce

$$\begin{aligned}\langle \tilde{R}(\xi_P), \xi_{P'}\rangle(\xi_S, \overline{\xi_T}) &= -16\pi^4 \alpha_{S,T} \sum_{i,j,k,l} f_i(P) f_j(P) \overline{f_k(P') f_l(P')} \\ &\quad \cdot \left[\delta_{ik} f_l(S)\overline{f_j(T)} + \delta_{il} f_k(S)\overline{f_j(T)} + \delta_{jl} f_k(S)\overline{f_i(T)} + \delta_{jk} f_l(S)\overline{f_i(T)}\right] \\ &= -64\pi^4 \alpha_{S,T} \sum_{i,j,l} f_i(P) f_j(P) \overline{f_j(P') f_l(P')} f_l(S) \overline{f_i(T)} \\ &= -64\pi^4 \alpha_{S,T}\alpha_{P,T}\alpha_{P,P'}\alpha_{S,P'}.\end{aligned}$$

□

In order to apply (3.3), we still need to compute

$$\langle \sigma(\xi_P), \sigma(\xi_{P'})\rangle(\xi_S, \overline{\xi_T}).$$

Recall that the exact sequence (3.1) of which σ is the second fundamental form, at $[X] \in M_g^{(n)}$ is

$$0 \to H^1(T_X) \to S^2(H^0(K_X))^* \to I_2(X)^* \to 0; \tag{3.7}$$

thus σ yields a homomorphism

$$\sigma : H^1(T_X) \to Hom(I_2(K_X), H^0(2K_X)). \tag{3.8}$$

Analogously, at $[X] \in M_g^{(n)}$ the exact sequence (3.2) is:

$$0 \to I_2(K_X) \to S^2(H^0(K_X)) \stackrel{m}{\to} H^0(2K_X) \to 0; \tag{3.9}$$

hence the second fundamental form ρ gives a homomorphism

$$\rho : I_2(K_X) \to Hom(H^1(T_X), H^0(2K_X))$$

and for every $v \in H^1(T_X)$, and for every $Q \in I_2(X)$, we have

$$\sigma(v)(Q) = \rho(Q)(v).$$

We recall now some results of [4] on the second fundamental form ρ. In particular we want to use Thm.2.1 and Lemma 3.2 of [4] (cf. also [19](4.8)). Let us fix a point $P \in X$, where $[X] \in M_g^{(n)}$. We have the inclusion $H^0(K_X(2P)) \hookrightarrow H^1(X - \{P\}, \mathbb{C}) \cong H^1(X, \mathbb{C})$. By Riemann-Roch and Hodge decomposition we immediately see that $\dim(H^0(K_X(2P)) \cap H^{0,1}(X)) = 1$, so we define $\eta_P \in H^0(K_X(2P)) \cap H^{0,1}(X)$ as the only generator of $H^0(K_X(2P)) \cap H^{0,1}(X)$ having in a neighborhood of P the following local expression:

$$\eta_P = (-\frac{1}{(z - z(P))^2} + g(z))dz,$$

with $g(z)$ holomorphic.

Lemma 3.3 (cf. [4] (Thm 2.1), (Lemma 3.2)). *Let $Q \in I_2(K_X)$, $Q = \sum_{i,j} a_{ij}\omega_i \otimes \omega_j$. Then*

$$\rho(Q)(\xi_P) = -\eta_P \sum_{i,j} a_{ij} f_i(P)\omega_j \in H^0(2K_X).$$

Corollary 3.4. *If X is any nonhyperelliptic curve, then ρ is injective and σ is nonzero. In particular at any point $[X] \in M_g^{(n)}$ outside the hyperelliptic locus the curvature R of $\mathcal{T}_{M_g^{(n)}}$ and the curvature $\tilde{R}$ of ${j^{(n)}}^*(\mathcal{T}_{A_g^{(n)}})$ are different.*

Proof. By Lemma 3.3, for any $Q \in I_2$, $Q = \sum_{i,j} a_{ij}\omega_i \otimes \omega_j$, $\rho(Q)(\xi_P) = 0$ implies $\sum_{i,j} a_{ij} f_i(P)\omega_j = 0$; hence $\forall j$, $\sum_i a_{ij} f_i(P) = 0$. Then $Q \in \ker(\rho)$ if and only if $\sum_i a_{ij} f_i(P) = 0$ $\forall j$, $\forall P \in X$, so $\sum_i a_{ij}\omega_i = 0$, which implies $Q = 0$.

Since $\sigma(\xi_P)(Q) = \rho(Q)(\xi_P)$ and ρ is injective, there must exist a point $P \in X$ such that $\sigma(\xi_P) \neq 0$. □

Now we compute $\xi_S(\rho(Q)(\xi_P))$, where P and S are two points in X.

Let z be a local coordinate in a neighborhood of S, and consider a local expression of $\rho(Q)(\xi_P) \in H^0(2K_X)$,

$$\rho(Q)(\xi_P) = \Psi_P^Q(z)dz^2.$$

Lemma 3.5. *Let $Q \in I_2(K_X)$. Then*

$$\xi_S(\rho_Q(\xi_P)) = 2\pi i \Psi_P^Q(S).$$

Proof. Recall that ξ_S is represented by a form

$$\theta_S = \frac{1}{z}\overline{\partial} b_S \otimes \frac{\partial}{\partial z},$$

where z is a local coordinate in a neighborhood of S and b_S is a bump function around S which is equal to one in a neighborhood of S.

Let C be a small circle around S such that $b_S \equiv 1$ on C. We have

$$\xi_S(\rho(Q)(\xi_P)) = \int_X \theta_S(\rho(Q)(\xi_P)) = \int_X \overline{\partial}\left(\frac{b_S \Psi_P^Q(z)}{z - z(S)}\right) \wedge dz$$
$$= \int_C \frac{\Psi_P^Q(z)}{z - z(S)} dz = 2\pi i \Psi_P^Q(S).$$

□

We now want to compute $\Psi_P^Q(S)$.

If $P \neq S$ the form η_P has the following local expression in a neighborhood of S:

$$\eta_P(z) = G_P(z) dz,$$

where $G_P(z)$ is holomorphic, so

$$\Psi_P^Q(z) = -G_P(z)\left(\sum_{i,j} a_{ij} f_i(P) f_j(z)\right).$$

If $P = S$, the local expression of η_P in a neighborhood of P is

$$\eta_P = (-\frac{1}{(z - z(P))^2} + g(z)) dz,$$

and we have (cf. also [4], Thm. 3.1)

$$\Psi_P^Q(z) = -\left(-\frac{1}{(z - z(P))^2} + g(z)\right)\left(\sum_{i,j} a_{ij} f_i(P) f_j(z)\right)$$
$$= \frac{\sum_{ij} a_{ij} f_i(P)\left(f_j(P) + f_j'(P)(z - z(P)) + \frac{1}{2} f_j''(P)(z - z(P))^2 + \text{h.o.t.}\right)}{(z - z(P))^2}$$
$$= \frac{1}{2}\sum_{i,j} a_{ij} f_i(P) f_j''(P) + O(1),$$

since $Q \in I_2(K_X)$, so $\sum_{i,j} a_{ij} f_i(P) f_j(P) = 0$, and $\sum_{i,j} a_{ij} f_i(P) f_j'(P) = 0$. Thus we have

$$\Psi_P^Q(P) = \frac{1}{2}\sum_{i,j} a_{ij} f_i(P) f_j''(P) = \frac{1}{2}(\mu_2(Q))(P), \tag{3.10}$$

where $\mu_2(Q)$ is the second Gaussian map of X in Q. For the definition of the second Gaussian map, see Section 4.

Proposition 3.6. *Let ξ_P be a Schiffer variation, let $\{Q_i\}$ be an orthonormal basis of $I_2(K_X)$, and denote by $\Psi_P^i := \Psi_P^{Q_i}$. Then the following holds:*

$$\langle \sigma(\xi_P), \sigma(\xi_{P'})\rangle(\xi_S, \overline{\xi_T}) = 4\pi^2 \sum_i \Psi_P^i(S)\overline{\Psi_{P'}^i(T)}. \tag{3.11}$$

Proof. Fix an orthonormal basis $\{Q_i\}$ of $I_2(K_X) \subset S^2(H^0(K_X))$. Let $\{Q_i^*\}$ be the dual basis of $I_2(K_X)^*$. By (3.8), $\sigma(\xi_P) \in I_2^* \otimes H^0(2K_X)$; hence

$$\sigma(\xi_P) = \sum_i \sigma(\xi_P)(Q_i) \otimes Q_i^*.$$

Then $\sigma(\xi_P)(Q_i) = \rho(Q_i)(\xi_P) =: \rho_{Q_i}(\xi_P) \in H^0(2K_X)$, so

$$\sigma(\xi_P) = \sum_i \rho_{Q_i}(\xi_P) \otimes Q_i^*.$$

On the other hand, a basis of $H^0(2K_X)$ is given by the set $\{\xi_S^*\}$, where S runs in a set of $3g-3$ general points of X. This implies that

$$\rho_{Q_i}(\xi_P) = \sum_S \xi_S(\rho_{Q_i}(\xi_P)){\xi_S}^*.$$

Therefore, the following holds:

$$\begin{aligned}&\langle \sigma(\xi_P), \sigma(\xi_{P'})\rangle(\xi_S, \overline{\xi_T})\\ &\quad= \sum_i \sum_{V,V'} \langle \xi_V(\rho_{Q_i}(\xi_P)){\xi_V}^*, \xi_{V'}(\rho_{Q_i}(\xi_{P'})){\xi_{V'}}^*\rangle(\xi_S, \overline{\xi_T})\\ &\quad= \sum_i \xi_S(\rho_{Q_i}(\xi_P))\overline{\xi_T(\rho_{Q_i}(\xi_{P'}))}.\end{aligned}$$

Using Lemma 3.5 we get

$$\begin{aligned}\langle \sigma(\xi_P), \sigma(\xi_{P'})\rangle(\xi_S, \overline{\xi_T}) &= \sum_i \xi_S(\rho_{Q_i}(\xi_P))\overline{\xi_T(\rho_{Q_i}(\xi_{P'}))}\\ &= 4\pi^2 \sum_i \Psi_P^i(S)\overline{\Psi_{P'}^i(T)}.\end{aligned}$$

□

From Proposition 3.2 and Proposition 3.6 we obtain a closed expression for the curvature form of $\mathcal{T}_{M_g^{(n)}}$ at $[X] \in M_g^{(n)}$. More precisely, the following holds.

Theorem 3.7.

$$\begin{aligned}&\langle R(\xi_P), \xi_{P'}\rangle(\xi_S, \overline{\xi_T})\\ &\quad= -64\pi^4 \alpha_{S,T}\alpha_{P,T}\alpha_{P,P'}\alpha_{S,P'} - 4\pi^2 \sum_i \Psi_P^i(S)\overline{\Psi_{P'}^i(T)}.\end{aligned}$$

Corollary 3.8. *The holomorphic sectional curvature of $\mathcal{T}_{M_g^{(n)}}$ at $[X] \in M_g^{(n)}$ computed at the tangent vector ξ_P is given by*

$$\begin{aligned}H(\xi_P) &= \frac{1}{\langle \xi_P, \xi_P\rangle\langle \xi_P, \xi_P\rangle}\langle R(\xi_P), \xi_P\rangle(\xi_P, \overline{\xi_P})\\ &= -1 - \frac{1}{64\pi^2(\alpha_{P,P})^4}\sum_i |\mu_2(Q_i)(P)|^2.\end{aligned}$$

Proof. The proof immediately follows from (3.7), (3.10) and (2.2). □

By Corollary 3.8 we see that the holomorphic sectional curvature of $A_g^{(n)}$ calculated along the tangent directions at $[X] \in M_g^{(n)}$ given by the Schiffer variations ξ_P is equal to -1, for all $P \in X$.

We shall now give another proof of this, showing in fact that this holds only for the Schiffer variations. We recall that the image of the sectional curvature of H_g is the segment $[-1, -\frac{1}{g}]$ and that the tangent directions V such that $H(V) = -1$ correspond to the symmetric matrices of rank 1.

Let us now consider as usual an element $\xi \in H^1(T_X)$ as a symmetric homomorphism $H^0(K_X) \to H^0(K_X)^*$ through the exact sequence (3.7). Then the above observation shows that $H(\xi) = -1$ if and only if ξ has rank one. We therefore recall the characterisation of the elements $\xi \in H^1(T_X)$ such that ξ has rank 1. Observe that the Schiffer variations are the points of the bicanonical curve $\phi_{2K}(X) \subset \mathbb{P}H^1(X, T_X)$. Then the characterization follows by the following result of Griffiths and by the theorem of Enriques-Babbage and Petri.

Define $\mathcal{X} \subset \mathbb{P}H^1(X, T_X)$,

$$\mathcal{X} = \{\xi \in \mathbb{P}H^1(X, T_X) \mid \operatorname{rank}(\xi) \leq 1\}.$$

Theorem 3.9 ([8])**.** *Assume that $g \geq 3$ and X is not hyperelliptic. Consider the image of the bicanonical map $\phi_{2K}(X) \subset \mathbb{P}H^1(X, T_X)$. Then $\phi_{2K}(X) \subset \mathcal{X}$ with equality holding if and only if the canonical curve $\phi_K(X)$ is cut out by quadrics.*

Corollary 3.10 ([8])**.** *Assume that $g \geq 3$ and X is not hyperelliptic. Then $\phi_{2K}(X) \subset \mathcal{X}$ with equality holding if and only if the canonical curve $\phi_K(X)$ is not trigonal, and it is not isomorphic to a plane quintic.*

4. Second Gaussian map and holomorphic sectional curvature

We first recall the definition of the Gaussian maps (cf. [21]). Let X be a smooth projective curve, $S := X \times X$, $\Delta \subset S$ be the diagonal. Let L be a line bundle on X and $L_S := p_1^*(L) \otimes p_2^*(L)$, where $p_i : S \to X$ are the natural projections. Consider the restriction map

$$\tilde{\mu}_{n,L} : H^0(S, L_S(-n\Delta)) \to H^0(\Delta, L_S(-n\Delta)_{|\Delta}).$$

Notice that since $\mathcal{O}(\Delta)_{|\Delta} \cong T_X$, we have

$$H^0(\Delta, L_S(-n\Delta)_{|\Delta}) \cong H^0(X, 2L \otimes nK_X).$$

In the case $L = K_X$, $I_2(K_X) \subset H^0(S, K_S(-2\Delta))$, so we can define the second Gaussian map

$$\mu_2 : I_2(K_X) \to H^0(X, 4K_X)$$

as the restriction $\tilde{\mu}_{2,K|I_2(K_X)}$.

As above we fix a basis $\{\omega_i\}$ of $H^0(K_X)$. In local coordinates $\omega_i = f_i(z)dz$. Let $Q \in I_2(K_X)$, $Q = \sum_{i,j} a_{ij}\omega_i \otimes \omega_j$, recall that $\sum_{i,j} a_{ij} f_i f_j \equiv 0$, and since $a_{i,j}$ are symmetric, we also have $\sum_{i,j} a_{ij} f_i' f_j \equiv 0$. The local expression of $\mu_2(Q)$ is

$$\mu_2(Q) = \sum_{i,j} a_{ij} f_i'' f_j (dz)^4 = -\sum_{i,j} a_{ij} f_i' f_j' (dz)^4. \tag{4.1}$$

We recall the following results of [3].

Theorem 4.1 ([3] Lem.4.1, Thm.4.3)**.** *For any trigonal nonhyperelliptic curve X of genus $g \geq 4$, the image of μ_2 is contained in $H^0(4K_X - (q_1 + \ldots + q_{2g+4}))$, where $q_1 + \ldots + q_{2g+4}$ is the ramification divisor of the g^1_3.*

If $g \geq 8$, the rank of μ_2 is $4g - 18$.

We also recall

Theorem 4.2 ([3]Thm.6.1)**.** *Assume that X is smooth curve of genus $g \geq 5$, which is nonhyperelliptic and nontrigonal. Then for any $P \in X$ there exists a quadric $Q \in I_2$ such that $\mu_2(Q)(P) \neq 0$. Equivalently* $\operatorname{Im}(\mu_2) \cap H^0(4K_X - P) \neq \operatorname{Im}(\mu_2)$, $\forall P \in X$.

Assume $[X] \in M_g^{(n)}$, with $g \geq 4$, X nonhyperelliptic. Then Corollary 3.8 allows us to define a function $F : X \to \mathbb{R}$, given by the holomorphic sectional curvature evaluated along the tangent vectors given by the Schiffer variations:

$$F(P) = H(\xi_P) = -1 - \frac{1}{64\pi^2(\alpha_{P,P})^4} \sum_i |\mu_2(Q_i)(P)|^2 \leq -1,$$

where $\{Q_i\}$ is an orthonormal basis of $I_2(K_X)$.

Proposition 4.3. *If $g = 4$, the set of points $P \in X$ such that $F(P) = -1$ is finite, which implies that F is nonconstant.*

If $g \geq 5$, X not hyperelliptic, nor trigonal, then $F(P) < -1$ for all $P \in X$.

If X is a trigonal curve of genus ≥ 4, then $F(P) = H(\xi_P) = -1$ for every $P \in X$ which is a ramification point of the g^1_3.

Proof. Assume X has genus 4. Then the dimension of I_2 is one and I_2 can be generated by a quadric Q of rank 4 which has norm 1. So $\forall P \in X$, $F(P) = -1 - \frac{1}{64\pi^2(\alpha_{P,P})^4}|\mu_2(Q)(P)|^2$. Hence there is a finite number of points P such that $\mu_2(Q)(P) = 0$, so in these points we have $F(P) = -1$, while $F(P) < -1$ elsewhere.

As regards the second statement, we observe that $F(P) = -1$ if and only if $\mu_2(Q_i)(P) = 0$ for all i, where $\{Q_i\}$ is an orthonormal basis of I_2. But then we must have $\mu_2(Q)(P) = 0$ for all $Q \in I_2$. So the proof follows by Theorem 4.2.

The last statement follows from (4.1). □

Remark 4.4. The previous statements imply that for any curve $X \in M_g^{(n)}$, not hyperelliptic, nor trigonal, for every point $P \in X$ the holomorphic sectional curvature of $M_g^{(n)}$ at X, along the tangent directions given by ξ_P, is strictly smaller than the holomorphic sectional curvature of $A_g^{(n)}$. Hence the Schiffer variations are never tangent directions of totally geodesic submanifolds of $A_g^{(n)}$.

On the other hand, in the trigonal case, along the Schiffer variations at the ramification points of the g^1_3 (which are a basis of the tangent space to the trigonal locus), the holomorphic sectional curvature of $M_g^{(n)}$ coincides with the holomorphic sectional curvature of $A_g^{(n)}$.

5. The hyperelliptic locus

We will now study the hyperelliptic locus $HE_g \subset M_g^{(n)}$. Recall that by local Torelli, the restriction of the period map to HE_g is an injective immersion (cf. [18]). Therefore we have the exact sequence

$$0 \to \mathcal{T}_{HE_g} \to \mathcal{T}_{A_g^{(n)}|HE_g} \to \mathcal{N}_{HE_g|A_g^{(n)}} \to 0,$$

and we denote by

$$\sigma_{HE} : \mathcal{T}_{HE_g} \to Hom(\mathcal{T}_{HE_g}, \mathcal{N}_{HE_g|A_g^{(n)}})$$

the associated second fundamental form and by ρ_{HE} the second fundamental form of the dual exact sequence. At the point $[X] \in HE_g$ the dual exact sequence is

$$0 \to I_2 \to S^2(H^0(K_X)) \to H^0(2K_X)^+ \to 0,$$

where $H^0(2K_X)^+$ is the invariant part of $H^0(2K_X)$ under the hyperelliptic involution and I_2 is the vector space of the quadrics containing the rational normal curve, so that

$$\rho_{HE} : I_2 \to Hom(\mathcal{T}_{HE_g,[X]}, H^0(2K_X)^+).$$

We recall that the set of Schiffer variations at the Weierstrass points P_i generates $\mathcal{T}_{HE_g,[X]}$.

Proposition 5.1. *If X is hyperelliptic, then ρ_{HE} is injective and thus σ_{HE} is nonzero. This implies that the curvature R_{HE} of $\mathcal{T}_{HE_g}$ is different from the curvature $\tilde{R}$ of $\mathcal{T}_{A_g^{(n)}|HE_g}$ at any point $[X] \in HE_g$.*

Proof. With the same proof of Thm 2.1, Lemma 3.2 of [4] one can show that

$$\rho_{HE}(Q)(\xi_P) = -\eta_P \sum_{i,j} a_{ij} f_i(P)\omega_j \in H^0(2K_X)^+$$

if P is a Weierstrass point of X and $Q = \sum_{i,j} a_{ij}\omega_i \otimes \omega_j \in I_2$.

So $\rho_{HE}(Q)(\xi_P) = 0$ implies $\sum_{i,j} a_{ij} f_i(P)\omega_j = 0$; hence $\forall j$, $\sum_{i,j} a_{ij} f_i(P) = 0$. Then $Q \in \ker(\rho_{HE})$ if and only if $\sum_{i,j} a_{ij} f_i(P) = 0$ for every Weierstrass point $P \in X$. Since there are $2g+2$ Weierstrass points, this implies that $\sum_{i,j} a_{ij}\omega_i = 0$; hence $Q = 0$.

Since $\sigma_{HE}(\xi_P)(Q) = \rho_{HE}(Q)(\xi_P)$ and ρ_{HE} is injective, there must exist a Weierstrass point $P \in X$ such that $\sigma_{HE}(\xi_P) \neq 0$. □

We also observe that with the same proof as in Lemma 3.5 and formula (3.10) one shows that

$$\xi_P(\rho_{HE}(Q)(\xi_P)) = \mu_2(Q)(P)$$

at a Weierstrass point $P \in X$.

Let us denote by H_{HE} the holomorphic sectional curvature of $\mathcal{T}_{HE_g}$. Then if $[X] \in HE_g$ and $P \in X$ is a Weiestrass point, we have the same expression for $H_{HE}(\xi_P)$ as in (3.8), namely

$$H_{HE}(\xi_P) = -1 - \frac{1}{64\pi^2(\alpha_{P,P})^4} \sum_i |\mu_2(Q_i)(P)|^2, \tag{5.1}$$

where $\{Q_i\}$ is an orthonormal basis of I_2.

We recall now a result on the second Gaussian map proven in [3].

Proposition 5.2 ([3]Lem.4.1, Prop.4.2)**.** *Let X be a hyperelliptic curve of genus $g \geq 3$. Then the rank of μ_2 is $2g-5$ and its image is contained in $H^0(4K_X - (q_1 + ... + q_{2g+2}))$, where $\{q_1, ..., q_{2g+2}\}$ are the Weierstrass points.*

Corollary 5.3. *Let $[X] \in HE_g$. Then $H_{HE}(\xi_P) = -1$, for any Weierstrass point $P \in X$.*

Proof. The proof immediately follows from (5.1) and from (5.2). □

6. The class of the Siegel metric

Let $\overline{M}_g$ ($\overline{M_g^{(n)}}$) be the Deligne-Mumford compactification of M_g ($M_g^{(n)}$). In [15] it is shown that the Hodge bundle extends to $\overline{M}_g$ ($\overline{M_g^{(n)}}$) and its g-th exterior power is ample on M_g ($M_g^{(n)}$).

We denote by λ both the first Chern class of the extension of the Hodge bundle on $\overline{M_g}$ and on $\overline{M_g^{(n)}}$. We will prove that the Kähler form of the Siegel metric on M_g extends as a closed current to $\overline{M}_g$; hence it defines a cohomology class in $H^2(\overline{M}_g, \mathbb{C})$ which is a multiple of λ.

Theorem 6.1. *The Kähler form ω of the Siegel metric on M_g extends as a closed current to $\overline{M}_g$. Its class $[\omega] \in H^2(\overline{M}_g, \mathbb{C})$ satisfies $[\omega] = \pi\lambda$.*

Proof. On H_g the Hodge metric is the only (up to multiplication by scalars) invariant metric on the homogeneous bundle F.

Therefore we have an invariant metric on the line bundle $\Lambda^g F$, and thus its curvature is an invariant $(1,1)$ form β on H_g.

On the other hand, the Siegel metric is the invariant metric obtained by the metric on S^2F^* induced by the Hodge metric, and we denote by $\tilde{\omega}$ its Kähler form.

Since both β and $\tilde{\omega}$ are invariant $(1,1)$ forms and we are on the irreducible symmetric domain H_g, there exists a constant c such that $\tilde{\omega} = c\beta$. This relation still holds on the corresponding forms on $A_g^{(n)}$, which we denote in the same way.

In [1] a compactification $\overline{A_g^{(n)}}$ of $A_g^{(n)}$ is constructed and it has the property that it is nonsingular and that $D_\infty := \overline{A_g^{(n)}} - A_g^{(n)}$ is a divisor with normal crossings.

In [14] it is shown that the Hodge bundle $\mathcal{F}^1$ on $A_g^{(n)}$ extends as a bundle on $\overline{A_g^{(n)}}$, such that the Hodge metric has only logarithmic singularities at D_∞.

Moreover in [14] (see also [5]), it is also proven that the extension of the second symmetric power is isomorphic to the sheaf of differential forms with logarithmic poles at D_∞:

$$S^2(\mathcal{F}^1) \cong \Omega^1_{\overline{A_g^{(n)}}}[D_\infty].$$

Furthermore Mumford proves in ([14], Thm.(3.1), Thm.(1.4)) that the extension of the Hodge metric has "good" singularities and that this implies that its first Chern class yields a closed current on $\overline{A_g^{(n)}}$ and thus a cohomology class $\tilde{\lambda} \in H^2(\overline{A_g^{(n)}}, \mathbb{C})$.

Therefore, since on $A_g^{(n)}$ our Kähler form $\tilde{\omega} = c\beta$, then also $\tilde{\omega}$ can be extended as a closed (1,1) current on $\overline{A_g^{(n)}}$, which we still call $\tilde{\omega}$, and its cohomology class $[\tilde{\omega}] \in H^2(\overline{A_g^{(n)}}, \mathbb{C})$ is given by $[\tilde{\omega}] = c\tilde{\lambda}$.

In ([16] (18.9), see also [17]) it is shown that the period map $j^{(n)} : M_g^{(n)} \to A_g^{(n)}$ extends to a period map $\overline{j} : \overline{M_g^{(n)}} \to \overline{A_g^{(n)}}$, so we can consider the pullback $\overline{j}^*([\tilde{\omega}]) \in H^2(\overline{M_g^{(n)}}, \mathbb{C})$. Moreover, since the image of $\overline{j}$ is not contained in the locus where the current is singular, the pullback $\omega := \overline{j}^*(\tilde{\omega})$ is a well-defined closed current and $[\omega] = \overline{j}^*([\tilde{\omega}])$ (cf. [13]). Moreover it gives a closed current on $\overline{M}_g$, still denoted by ω. Observe that $\overline{j}^*(\tilde{\lambda}) = \lambda$, so $[\omega] = c\lambda$ in $H^2(\overline{M_g^{(n)}}, \mathbb{C})$, hence in $H^2(\overline{M}_g, \mathbb{C})$.

In order to compute the constant c, we use the cycles introduced by Wolpert in [22]. In our case, since $[\omega]$ is a multiple of λ, it is sufficient to compute the value of $[\omega]$ on the 1-dimensional family given by a varying 1-pointed elliptic curve attached to a fixed $g-1$ curve with one marked point $\mathcal{E}_l$ of [22](2.2). More precisely, let us denote by $H := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, by $\Gamma := SL(2, \mathbb{Z})$, and by

$$\Gamma_l = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod l \right\}.$$

Since the $g-1$ curve in $\mathcal{E}_l$ is constant we identify $\mathcal{E}_l$ with the curve $H/\Gamma_l = A_1^{(l)}$. We have then to compute

$$\int_{H/\Gamma_l} \omega = [\Gamma : \Gamma_l] \int_{H/\Gamma} \omega.$$

Set $E_z := \mathbb{C}/(\mathbb{Z} \oplus z\mathbb{Z})$, where $z \in H/\Gamma$, and let ξ_z be a holomorphic coordinate on E_z, so that $H^{1,0}(E_z) = \langle d\xi_z \rangle$. Then a cotangent direction to the curve $\mathcal{E}_l$ can be identified with $d\xi_z \odot d\xi_z$ and we have: $\langle d\xi_z \odot d\xi_z, d\xi_z \odot d\xi_z \rangle = 2\langle d\xi_z, d\xi_z \rangle^2$,

$$\langle d\xi_z, d\xi_z \rangle = i \int_{E_z} d\xi_z \wedge \overline{d\xi_z} = 2\,\mathrm{Im}(z).$$

Then

$$\langle [\omega], \mathcal{E}_l \rangle = \int_{H/\Gamma_l} \omega = i[\Gamma : \Gamma_l] \int_D \frac{1}{8(\mathrm{Im}(z))^2} (dz \wedge \overline{dz}) = [\Gamma : \Gamma_l] \frac{\pi}{12},$$

where D is the fundamental domain of the action of Γ on H and the last equality is a standard integral computation.

Since one has $\langle \lambda, \frac{\mathcal{E}_l}{[\Gamma:\Gamma_l]} \rangle = \frac{1}{12}$, we have

$$\frac{\pi}{12} = \langle \zeta, \frac{\mathcal{E}_l}{[\Gamma : \Gamma_l]} \rangle = \langle c\lambda, \frac{\mathcal{E}_l}{[\Gamma : \Gamma_l]} \rangle = c\frac{1}{12};$$

thus we obtain $c = \pi$, so finally $[\omega] = \pi\lambda$. □

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