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WEIGHTED AVERAGES OF MODULAR L-VALUES

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Abstract. Using an explicit relative trace formula on GL(2), we derive a formula for averages of modular L-values in the critical strip, weighting by Fourier coefficients, Hecke eigenvalues, and Petersson norms. As an application we show that a GRH holds for these averages as the weight or the level goes to ∞. We also use the formula to give explicit zero-free regions of the form $|\operatorname{Im}(s)| \le \tau_0$ for some particular modular L-functions.

1. Introduction

Let $S_{\mathtt{k}}(N,\omega')$ denote the space of cusp forms h on $\Gamma_0(N)$ satisfying

$$h(\frac{az+b}{cz+d}) = \omega'(d)^{-1}(cz+d)^{\mathtt{k}}\, h(z) \qquad (\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(N)).$$

The Mellin transform of h is the analytic function

$$\Lambda(s,h) = \int_0^\infty h(iy)y^{s-1}dy,$$

which converges absolutely for all $s \in \mathbf{C}$ ([Sh], p. 94). Write $h(z) = \sum_{r>0} a_r(h)e^{2\pi i r z}$. When $\operatorname{Re}(s) > 1 + \mathtt{k}/2$, we have additionally

$$\int_0^\infty \sum_{r>0} |a_r(h)e^{-2\pi r y}y^{s-1}|dy < \infty.$$

Therefore for such s,

$$\Lambda(s,h) = \sum_{r>0} a_r(h)\int_0^\infty e^{-2\pi r y}y^{s-1}dy = \sum_{r>0} a_r(h)\int_0^\infty e^{-t}t^{s-1}(2\pi r)^{-s}dt$$
$$= (2\pi)^{-s}\Gamma(s)\sum_{r>0}\frac{a_r(h)}{r^s} = (2\pi)^{-s}\Gamma(s)L(s,h),$$

where $L(s,h)$ is the Dirichlet series attached to h. The completed L-function $\Lambda(s,h)$ satisfies a functional equation relating values at s and $\mathtt{k}-s$, which in the case of $N=1$ is simply

$$\Lambda(s,h) = i^{\mathtt{k}}\Lambda(\mathtt{k}-s,h). \tag{1}$$

Hence the critical line of the L-function is $\operatorname{Re}(s) = \mathtt{k}/2$. If h is a newform determining the cuspidal representation π, then $\Lambda(s,\pi) = \Lambda(s+\frac{\mathtt{k}-1}{2},h)$, and $\Lambda(s,\pi)$ satisfies a functional equation relating its values at s and $1-s$.

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The central values of L-functions have deep arithmetic significance. If the Hecke eigenvalues are known, one can compute the central values of a particular L-function using the approximate functional equation (see e.g. [Mi], §1.3.2). We can also use the trace formula to get information about averages of L-values as h ranges through an orthogonal Hecke eigenbasis $\mathcal{F}$ for $S_{\mathtt{k}}(N,\omega')$. In this paper, we will explicitly compute such an average, with the L-values weighted by Hecke eigenvalues, Fourier coefficients and Petersson norms.

The asymptotics of such averages have been studied widely. Duke showed that when $\mathtt{k}=2$, N is prime, ω' is trivial, and χ is a Dirichlet character unramified at N,

$$\frac{1}{\psi(N)}\sum_{h\in\mathcal{F}}\frac{a_1(h)L(1,h\otimes\chi)}{\|h\|^2}=4\pi+O(N^{-1/2}\log N),$$

where $\psi(N)=[\mathrm{SL}_2(\mathbf{Z}):\Gamma_0(N)]$, [Du]. Here we have normalized the Petersson norm as in (2) below. With a more careful estimation, Ellenberg improved Duke's error term to $O(N^{-1+\varepsilon})$, while at the same time allowing $a_r(h)$ in place of $a_1(h)$, [El]. Of the many other generalizations of Duke's work, we mention two: Akbary extended it to weight $\mathtt{k}>2$ with an error term of $O_{\mathtt{k}}(N^{-1/2}(\log N)^{\mathtt{k}-1})$ [Ak], and Kamiya further allowed composite N and $L(1+it,h\otimes\chi)$ with an error term of $O_{t,\mathtt{k}}(N^{-\mathtt{k}/4})$ [Ka]. The method of Duke uses the Petersson trace formula. Another approach, based on the Eichler-Selberg trace formula, was found by Royer (see §4.3 of [Ro]).

Here we consider the case $\mathtt{k}>2$. For the weighted averages we obtain an error term of $O(N^{-\mathtt{k}/2})$ on the critical line. In fact, we give an explicit formula for the average (Theorem 1.1). At the same time, we allow s to vary through the whole critical strip. We will also give the asymptotic behavior of the average as $\mathtt{k}\to\infty$.

To state the main theorem, for $h\in S_{\mathtt{k}}(N,\omega')$, let $h^-\in S_{\mathtt{k}}(N,\omega'^{-1})$ denote the "complex conjugate" of h, given by $h^-(z)=\sum\overline{a_n(h)}q^n$. If ω' is trivial, then $h^-=h$, and in general $\Lambda(s,h^-)=\overline{\Lambda(\bar{s},h)}$.

Theorem 1.1. *Let $r,N,\mathtt{n},\mathtt{k}\in\mathbf{Z}^+$ with $(\mathtt{n},N)=1$ and $\mathtt{k}>2$. Fix a Dirichlet character ω' of conductor dividing N, and suppose $S_{\mathtt{k}}(N,\omega')\neq\{0\}$. Let $\mathcal{F}$ be an orthogonal basis for $S_{\mathtt{k}}(N,\omega')$ consisting of eigenfunctions for the Hecke operator $T_{\mathtt{n}}$. Then for any $s\in\mathbf{C}$ with $1<\mathrm{Re}(s)<\mathtt{k}-1$,*

$$\begin{aligned}\sum_{h\in\mathcal{F}}&\frac{\lambda_{\mathtt{n}}(h)a_r(h)\Lambda(s,h^-)}{\|h\|^2}\\ =&\frac{\psi(N)2^{\mathtt{k}-1}\Gamma(s)(2\pi r\mathtt{n})^{\mathtt{k}-s-1}}{(\mathtt{k}-2)!}\sum_{m\mid\gcd(\mathtt{n},r)}\frac{m^{2s-\mathtt{k}+1}}{\omega'(m)}\\ &+\delta_{N,1}\frac{2^{\mathtt{k}-1}\Gamma(\mathtt{k}-s)(2\pi r\mathtt{n})^{s-1}}{(\mathtt{k}-2)!\,i^{\mathtt{k}}}\sum_{m\mid\gcd(\mathtt{n},r)}m^{\mathtt{k}-2s+1}\\ &+\frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}}{N^s(\mathtt{k}-2)!\,e^{i\pi s/2}}\sum_{\substack{a\neq0,d>0\\ \gcd(a,Nd)\mid\gcd(r,\mathtt{n})}}\frac{a^{-(\mathtt{k}-s)}d^{-s}\gcd(a,Nd)}{\omega'(a)\,e^{2\pi ir\ell_0/a}}{}_1f_1(s;\mathtt{k};\tfrac{2\pi ir\mathtt{n}}{Nad}),\end{aligned}$$

where $T_{\mathtt{n}}h=\lambda_{\mathtt{n}}(h)h$, ℓ_0 is any integer satisfying $\ell_0Nd\equiv\mathtt{n}\bmod a$, and

$${}_1f_1(s;\mathtt{k};w)=\frac{\Gamma(s)\Gamma(\mathtt{k}-s)}{\Gamma(\mathtt{k})}{}_1\mathrm{F}_1(s;\mathtt{k};w)$$

for the confluent hypergeometric function ${}_1\mathrm{F}_1(s;\mathtt{k};w) = 1 + \frac{s}{\mathtt{k}}w + \frac{s(s+1)}{\mathtt{k}(\mathtt{k}+1)}\frac{w^2}{2!} + \cdots$. *When* $a < 0$, *we take* $a^s = e^{i\pi s}|a|^s$. *Throughout we use the convention that* $\sum_{m|n}$ *is a sum over* positive *divisors of* n.

This theorem generalizes a result of Kohnen, who derived the special case $\mathtt{n} = N = 1$ using a Poincaré series-type argument ([Ko], p. 188). Our approach here is quite different.

From its integral representation (cf. (17) on page 1439), it follows that

$$|{}_1f_1(s;\mathtt{k};2\pi ir\mathtt{n}/Nad)| \leq 1.$$

Thus the sum over a, d is bounded independently of N (see Proposition 4.2 for a precise bound), and we have the following.

Corollary 1.2. *With notation as above and* $1 < \mathrm{Re}(s) < \mathtt{k} - 1$,

$$\frac{1}{\psi(N)} \sum_{h\in\mathcal{F}} \frac{\lambda_{\mathtt{n}}(h)a_r(h)\Lambda(s,h^-)}{\|h\|^2} = \frac{2^{\mathtt{k}-1}\Gamma(s)(2\pi r\mathtt{n})^{\mathtt{k}-s-1}}{(\mathtt{k}-2)!} \sum_{m|\gcd(\mathtt{n},r)} \frac{m^{2s-\mathtt{k}+1}}{\omega'(m)} + O(N^{-\mathrm{Re}(s)}).$$

The implied constant is effective and depends only on $\mathtt{k}, \mathtt{n}, r$ *and* s, *uniformly for* s *in compact subsets of the given strip.*

According to the Grand Riemann Hypothesis, when h is a Hecke eigenform all zeros of $\Lambda(s,h)$ inside the critical strip $\frac{\mathtt{k}-1}{2} < \mathrm{Re}(s) < \frac{\mathtt{k}+1}{2}$ lie on the critical line $\mathrm{Re}(s) = \mathtt{k}/2$. Using Theorem 1.1, we will show that a GRH holds for averages (see also [Ko] for the $N = 1$ case). Note that Corollary 1.2 implies nonvanishing of the average when N is large, at least when $\gcd(\mathtt{n}, r) = 1$. By the results of Section 4.1 in which we determine the asymptotic behavior as $\mathtt{k} \to \infty$, the average is also nonzero when $\mathtt{k}$ is large. To state the result, we shift the L-functions so that the critical strip becomes $0 \leq \mathrm{Re}(s) \leq 1$, independent of $\mathtt{k}$.

Corollary 1.3. *Assume* $N > 1$, $\mathtt{k} > 3$, $\gcd(\mathtt{n}, r) = 1$, *and that* $S_{\mathtt{k}}(N, \omega') \neq \{0\}$. *For* $\tau_0 > 0$, *let* R *be the rectangle consisting of* s *with* $0 \leq \mathrm{Re}(s) \leq 1$ *and* $|\mathrm{Im}(s)| \leq \tau_0$. *Then there exist constants* $C_{\mathtt{k}}, C_N > 0$ *depending only on* R, $\mathtt{n}$ *and* r, *such that if either* $\mathtt{k} > C_{\mathtt{k}}$ *or* $N > C_N$, *the sum*

$$\sum_{h\in\mathcal{F}} \frac{\lambda_{\mathtt{n}}(h)a_r(h)\Lambda(s+\frac{\mathtt{k}-1}{2},h^-)}{\|h\|^2}$$

is nonzero for every $s \in R$. *In particular, for any* $s \in R$ *there exists an eigenform* $h \in S_{\mathtt{k}}(N, \omega')$ *such that* $\lambda_{\mathtt{n}}(h), a_r(h)$ *and* $\Lambda(s + \frac{\mathtt{k}-1}{2}, h)$ *are all nonzero.*

Some of the hypotheses of Corollary 1.3 can be weakened with minor modifications. To allow $\gcd(\mathtt{n}, r) > 1$, we simply need to exclude the left edge of the strip. Thus the boundary of R should be shrunk to $\delta \leq \mathrm{Re}(s) \leq 1$ for any $0 < \delta < 1/2$. If in addition we exclude the right edge by considering $\delta \leq \mathrm{Re}(s) \leq 1 - \delta$ for such δ, then the statement is also valid for $\mathtt{k} = 3$. When $N = 1$, the situation is a little more delicate because, if s lies on the critical line, the first two terms in the formula for the average may cancel each other out and we cannot say anything. Indeed if $\mathtt{k} \equiv 2 \bmod 4$, the L-values themselves vanish at $s = \mathtt{k}/2$ because of the functional equation (1). So when $N = 1$ we must assume that R is a compact region which does not meet the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

Suppose it happens that $\dim S_{\mathtt{k}}(N, \omega') = 1$. Then the theorem gives a computable formula for the values of the L-function of the cusp form. Using an effective version of Corollary 1.3, we obtain zero-free regions for several such L-functions in Section 4.2. As a final illustration, we show how to use the formula to compute some familiar data, namely values of Ramanujan's τ-function. This is achieved by taking a quotient of two different averages. The resulting expression can be estimated to any desired precision using partial sum approximations, and since $\tau(r)$ is known to be an integer, we can pinpoint its value with just a few terms.

Theorem 1.1 is proven using a relative trace formula on GL(2). We start with a Hecke operator and integrate its associated kernel over the group $N \times M$, where N is unipotent and M is diagonal. This is a hybrid of the techniques of the papers [Li], [KL1] (which used $N \times N$) and [RaRo] (which used $M \times M$). The paper [RaRo] of Ramakrishnan and Rogawski gives an asymptotic formula for certain averages of the form $\sum_{h \in \mathcal{F}} \frac{\lambda_{p^n}(h)\Lambda(\mathtt{k}/2, h\otimes\chi)\Lambda(\mathtt{k}/2,h)}{\|h\|^2}$, which yields a weighted equidistribution result for the Hecke eigenvalues. They use a regularization procedure since they assert that the terms on their geometric side are not absolutely convergent. Thus the replacement here of just one factor of M by the unipotent group N (of compact quotient) is enough to give an absolutely convergent trace formula.

We mention that Feigon and Whitehouse refined the method of [RaRo] in many cases by using the Jacquet-Langlands correspondence to avoid the convergence issues inherent to GL(2), [FW]. They obtain closed formulas for the averages at the central point, over a totally real field.

A version of Theorem 1.1 involving twisted L-functions $\Lambda(s, h \otimes \chi)$ should be obtainable by similar methods, using a test function as in [RaRo]. Of course, the presence of a nontrivial character χ will only help the convergence of the trace formula.

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2. Notation and preliminaries

We briefly recall the notation and test function of [KL2], which contains proofs of the various facts mentioned in this section. Let $\mathbf{A}, \mathbf{A}_{\mathrm{fin}}$ be the adeles and finite adeles of $\mathbf{Q}$, and let $G = \mathrm{GL}(2)$. We write $\overline{G}$ for G/Z, where Z is the center. Fix a level $N \geq 1$ and a Dirichlet character ω' of conductor dividing N. For a weight $\mathtt{k} > 2$, let $S_{\mathtt{k}}(N, \omega')$ denote the space of cusp forms satisfying

$$h(\gamma z) = \omega'(\gamma)^{-1} j(\gamma, z)^{\mathtt{k}} h(z) \qquad (\gamma \in \Gamma_0(N)).$$

Here $\omega'(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \omega'(d)$ and

$$j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z) = (ad - bc)^{-1/2}(cz + d) \qquad (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbf{R})^+).$$

Using $\mathbf{A}^* = \mathbf{Q}^*(\mathbf{R}_+^* \times \widehat{\mathbf{Z}}^*)$, define

$$\omega : \mathbf{A}^* \to \widehat{\mathbf{Z}}^* \to (\mathbf{Z}/N\mathbf{Z})^* \to \mathbf{C}^*,$$

where the last arrow is ω'. For an idele x, let x_N denote the idele which agrees with x at the places $p|N$, and which is 1 at all other places. Then for any integer d prime to N,

$$\omega(d_N) = \omega'(d).$$

To each $h \in S_{\mathtt{k}}(N,\omega')$ we associate $\phi_h \in L_0^2(\omega) = L_0^2(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A}),\omega)$ by using strong approximation:

$$\phi_h(\gamma(g_\infty \times k)) = j(g_\infty, i)^{-\mathtt{k}} h(g_\infty(i))$$

for $\gamma \in G(\mathbf{Q})$, $g_\infty \in G(\mathbf{R})^+$ and $k \in K_1(N) = \{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in G(\widehat{\mathbf{Z}})|\, c, d-1 \in N\widehat{\mathbf{Z}}\}$.

We normalize the Petersson norm by

$$\|h\|^2 = \frac{1}{\psi(N)} \int_{\Gamma_0(N)\backslash\mathbf{H}} |h(z)|^2 y^{\mathtt{k}} \frac{dx\,dy}{y^2}. \tag{2}$$

If we normalize Haar measure on $\overline{G}(\mathbf{A})$ so that $\mathrm{meas}(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})) = \pi/3$, then the Petersson norm corresponds to the L^2-norm and the map $h \mapsto \phi_h$ is an isometry. We normalize Haar measure on $\mathbf{A}$ so that $\mathrm{meas}(\mathbf{Q}\backslash\mathbf{A}) = 1$. We take Lebesgue measure dx on $\mathbf{R}$ and $d^*y = \frac{dy}{|y|}$ on $\mathbf{R}^*$. On $\mathbf{A}_{\mathrm{fin}}^*$ we normalize so that $\mathrm{meas}(\widehat{\mathbf{Z}}^*) = 1$.

Fix $\mathtt{n} \in \mathbf{Z}^+$ with $\gcd(\mathtt{n}, N) = 1$, and define a test function $f = f_\infty \times f^{\mathtt{n}}$ as follows. Define

$$M(\mathtt{n}, N) = \{g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in M_2(\widehat{\mathbf{Z}})|\ \det g \in \mathtt{n}\widehat{\mathbf{Z}}^* \text{ and } c \equiv 0 \bmod N\widehat{\mathbf{Z}}\}.$$

The support of $f_{\mathrm{fin}} = f^{\mathtt{n}}$ is the set $Z(\mathbf{A}_{\mathrm{fin}})M(\mathtt{n},N) = Z(\mathbf{Q}^+)M(\mathtt{n},N)$. By definition,

$$f^{\mathtt{n}}(z_{\mathbf{Q}} m) = \frac{\psi(N)}{\omega(m)} \qquad (z_{\mathbf{Q}} \in Z(\mathbf{Q}^+),\ m \in M(\mathtt{n},N)),$$

where for $m = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in M(\mathtt{n},N)$ we define $\omega(m) = \omega(d_N)$. We take $f_\infty(g) = \frac{1}{d_{\mathtt{k}}}\overline{\langle \pi_{\mathtt{k}}(g)v_0, v_0\rangle}$, where $\pi_{\mathtt{k}}$ is the weight $\mathtt{k}$ discrete series of $\mathrm{GL}_2(\mathbf{R})$ with formal degree $d_{\mathtt{k}} = \frac{4\pi}{\mathtt{k}-1}$ and lowest weight unit vector v_0. Explicitly, if $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, then

$$f_\infty(g) = \begin{cases} \frac{(\mathtt{k}-1)}{4\pi} \frac{\det(g)^{\mathtt{k}/2}(2i)^{\mathtt{k}}}{(-b+c+(a+d)i)^{\mathtt{k}}} & \text{if } \det(g) > 0, \\ 0 & \text{otherwise} \end{cases}$$

(see [KL2], Theorem 14.5). By construction, $f(zg) = \omega(z)^{-1} f(g)$ for $z \in Z(\mathbf{A})$.

This function f is integrable precisely when $\mathtt{k} > 2$. Hence for such $\mathtt{k}$ it defines an operator $R(f)$ on $L^2(\omega)$ by

$$R(f)\phi(x) = \int_{\overline{G}(\mathbf{A})} f(g)\phi(xg)dg.$$

Then as shown in [KL2], we have the following commutative diagram:

$$\begin{array}{ccc} L^2(\omega) & \xrightarrow{\ \mathtt{n}^{\frac{\mathtt{k}}{2}-1}R(f)\ } & L^2(\omega) \\ {\scriptstyle \text{orthog. proj.}}\big\downarrow & & \big\uparrow \\ S_{\mathtt{k}}(N,\omega') & \xrightarrow{\ T_{\mathtt{n}}\ } & S_{\mathtt{k}}(N,\omega') \end{array}$$

where $T_{\mathtt{n}}$ is the classical Hecke operator. Letting $\mathcal{F}$ be any orthogonal basis for $S_{\mathtt{k}}(N,\omega')$, the kernel of $R(f)$ is the function on $G(\mathbf{A})\times G(\mathbf{A})$ given by

$$K(g_1,g_2)=\sum_{\gamma\in\overline{G}(\mathbf{Q})}f(g_1^{-1}\gamma g_2)=\sum_{h\in\mathcal{F}}\frac{R(f)\phi_h(g_1)\overline{\phi_h(g_2)}}{\|\phi_h\|^2}. \tag{3}$$

Lastly, we let $\theta:\mathbf{A}\to\mathbf{C}^*$ denote the standard character of $\mathbf{A}$. It is defined by

$$\theta_\infty(x)=e^{-2\pi ix},\quad x\in\mathbf{R},$$

and

$$\theta_p(x)=e^{2\pi ir_p(x)},\quad x\in\mathbf{Q}_p,$$

where $r_p(x)\in\mathbf{Q}$ is the principal part of x, a number with p-power denominator characterized (up to $\mathbf{Z}_p$) by $x\in r_p(x)+\mathbf{Z}_p$. Then θ is trivial on $\mathbf{Q}$ and $\theta_{\text{fin}}=\prod_p\theta_p$ is trivial precisely on $\widehat{\mathbf{Z}}$. In particular, for any $q\in\mathbf{Q}$, $\theta_{\text{fin}}(q)=\theta_\infty(q)^{-1}=e^{2\pi iq}$. The characters of $\mathbf{Q}\backslash\mathbf{A}$ are parametrized by $r\in\mathbf{Q}$ via

$$\theta_r(x)=\theta(-rx).$$

3. Proof of the theorem

3.1. **Spectral side.** The theorem is proven by computing the following:

$$\int_{\mathbf{Q}^*\backslash\mathbf{A}^*}\int_{\mathbf{Q}\backslash\mathbf{A}}K(\begin{pmatrix}1&x\\0&1\end{pmatrix},\begin{pmatrix}y&0\\0&1\end{pmatrix})\,\overline{\theta_r(x)}|y|^{s-\mathtt{k}/2}dx\,d^*y \tag{4}$$

using the two expressions for the kernel (3). We will see presently that the integral (4) is absolutely convergent for all s.

For the spectral side, choose $\mathcal{F}$ in (3) to be an orthogonal basis of eigenvectors of $T_{\mathtt{n}}$. Then $R(f)\phi_h=\mathtt{n}^{1-\mathtt{k}/2}\lambda_{\mathtt{n}}(h)\phi_h$ for $h\in\mathcal{F}$, so (4) is equal to

$$\sum_{h\in\mathcal{F}}\frac{\mathtt{n}^{1-\mathtt{k}/2}\lambda_{\mathtt{n}}(h)}{\|\phi_h\|^2}\int_{\mathbf{Q}\backslash\mathbf{A}}\phi_h(\left(\begin{smallmatrix}1&x\\&1\end{smallmatrix}\right))\overline{\theta_r(x)}dx\int_{\mathbf{Q}^*\backslash\mathbf{A}^*}\overline{\phi_h(\left(\begin{smallmatrix}y&\\&1\end{smallmatrix}\right))}|y|^{s-\mathtt{k}/2}d^*y \tag{5}$$
$$=\frac{\mathtt{n}^{1-\mathtt{k}/2}}{e^{2\pi r}}\sum_{h\in\mathcal{F}}\frac{\lambda_{\mathtt{n}}(h)a_r(h)\Lambda(s,h^-)}{\|h\|^2},$$

by the following lemma.

Lemma 3.1. *For $r\in\mathbf{Q}$,*

$$\int_{\mathbf{Q}\backslash\mathbf{A}}\phi_h(\left(\begin{smallmatrix}1&x\\&1\end{smallmatrix}\right))\,\overline{\theta_r(x)}dx=\begin{cases}e^{-2\pi r}a_r(h)&\textit{if } r\in\mathbf{Z}^+,\\0&\textit{otherwise}\end{cases}$$

and

$$\int_{\mathbf{Q}^*\backslash\mathbf{A}^*}\overline{\phi_h(\left(\begin{smallmatrix}y&\\&1\end{smallmatrix}\right))}\,|y|^{s-\mathtt{k}/2}d^*y=\Lambda(s,h^-).$$

Proof. For a proof of the first statement, see [KL2], Corollary 12.4. For the second, note that $\phi_h(\left(\begin{smallmatrix}y&\\&1\end{smallmatrix}\right))=y^{\mathtt{k}/2}h(iy)$ when $y\in\mathbf{R}_+^*$. Furthermore, $\overline{h(iy)}=\sum\overline{a_r(h)}e^{-2\pi ry}=h^-(iy)$. We can integrate over the fundamental domain $\mathbf{R}_+^*\times\widehat{\mathbf{Z}}^*$. The integrand is invariant under $\widehat{\mathbf{Z}}^*$, which has measure 1. Thus

$$\int_{\mathbf{Q}^*\backslash\mathbf{A}^*}\overline{\phi_h(\left(\begin{smallmatrix}y&\\&1\end{smallmatrix}\right))}|y|^{s-\mathtt{k}/2}d^*y=\int_0^\infty h^-(iy)y^{s-1}dy=\Lambda(s,h^-).$$

□

The two integrals in (5) are absolutely convergent for all s, so we have the following.

Proposition 3.2. *The double integral* (4) *is absolutely convergent for all* $s \in \mathbf{C}$.

3.2. **Geometric side.** On the geometric side, we use the formalism of Jacquet's relative trace formula. Let N be the upper triangular unipotent subgroup of G, and let M be the diagonal subgroup. Let $\overline{M} = M/Z$, where Z is the center. Setting $H = N \times \overline{M}$, the integral (4) is taken over $H(\mathbf{Q})\backslash H(\mathbf{A})$. Using $K(n,m) = \sum_{\gamma\in\overline{G}(\mathbf{Q})} f(n^{-1}\gamma m)$, we would like to pull the sum out of (4); however the individual terms $f(n^{-1}\gamma m)$ are not well-defined modulo $H(\mathbf{Q})$. We have to break $\overline{G}(\mathbf{Q})$ into $H(\mathbf{Q})$-orbits and then sum over these orbits. The action of H is $(n,m)\cdot\gamma = n^{-1}\gamma m$. For $\delta \in \overline{G}(\mathbf{Q})$, its orbit is

$$[\delta] = \{\left(\begin{smallmatrix}1 & -x\\ & 1\end{smallmatrix}\right)\delta\left(\begin{smallmatrix}y & \\ & 1\end{smallmatrix}\right) \mid x \in \mathbf{Q}, y \in \mathbf{Q}^*\} = \{n^{-1}\delta m \mid (n,m) \in H_\delta(\mathbf{Q})\backslash H(\mathbf{Q})\},$$

where H_δ is the stabilizer of δ. It is easy to check that in fact $H_\delta = \{1\}$ for any δ. Thus the geometric expression for (4) is equal to

$$\sum_{[\delta]} \int_{\mathbf{A}^*}\int_{\mathbf{A}} f(\left(\begin{smallmatrix}1 & -x\\ & 1\end{smallmatrix}\right)\delta\left(\begin{smallmatrix}y & \\ & 1\end{smallmatrix}\right))\overline{\theta_r(x)}|y|^{s-\mathtt{k}/2}dx\, d^*y. \tag{6}$$

To justify this manipulation we have to show that (6) converges absolutely.

Proposition 3.3. *Suppose* $1 < \mathrm{Re}(s) < \mathtt{k} - 1$. *Then*

$$\sum_{[\delta]} \int_{\mathbf{A}^*}\int_{\mathbf{A}} \left| f(\left(\begin{smallmatrix}1 & -x\\ & 1\end{smallmatrix}\right)\delta\left(\begin{smallmatrix}y & \\ & 1\end{smallmatrix}\right))\overline{\theta_r(x)}|y|^{s-\mathtt{k}/2}\right| dx\, d^*y < \infty.$$

Thus for such s, *the geometric side* (6) *converges absolutely and equals the spectral side* (5).

We postpone the proof of the proposition until Section 3.3 below. Assuming it for now, let $I_\delta(f)$ denote the double integral attached to δ in (6). By the proposition, $I_\delta(f)$ is absolutely convergent on the given strip. We just need to determine the set of δ and compute each of these geometric integrals. We assume throughout that the hypothesis of the proposition is satisfied.

The set of orbits $[\delta]$ is in one-to-one correspondence with $N(\mathbf{Q})\backslash\overline{G}(\mathbf{Q})/\overline{M}(\mathbf{Q})$. By the Bruhat decomposition

$$G(\mathbf{Q}) = N(\mathbf{Q})M(\mathbf{Q}) \cup N(\mathbf{Q})\left(\begin{smallmatrix} & 1\\ -1 & \end{smallmatrix}\right)N(\mathbf{Q})M(\mathbf{Q}),$$

a set of representatives is given by

$$\{1\} \cup \{\left(\begin{smallmatrix}0 & 1\\ -1 & t\end{smallmatrix}\right) \mid t \in \mathbf{Q}\}.$$

Proposition 3.4. *When* $\delta = 1$, *the integral*

$$I_1(f) = \int_{\mathbf{A}^*}\int_{\mathbf{A}} f(\begin{pmatrix} y & -x\\ 0 & 1\end{pmatrix})\theta(rx)dx|y|^{s-\mathtt{k}/2}d^*y$$

converges absolutely on $0 < \mathrm{Re}(s) < \mathtt{k} - 1$, *and for such* s *it is*

$$= \frac{\mathtt{n}^{1-\mathtt{k}/2}}{e^{2\pi r}} \frac{\psi(N)2^{\mathtt{k}-1}\Gamma(s)(2\pi r\mathtt{n})^{\mathtt{k}-s-1}}{(\mathtt{k}-2)!} \sum_{m\mid \gcd(\mathtt{n},r)} \frac{m^{2s-\mathtt{k}+1}}{\omega'(m)}.$$

Proof. The absolute convergence will be proven in Proposition 3.10 below. For s as given, we factorize the integral as $I_1(f)_\infty I_1(f)_{\mathrm{fin}}$. To start with,

$$I_1(f)_{\mathrm{fin}} = \int_{\mathbf{A}^*_{\mathrm{fin}}} \int_{\mathbf{A}_{\mathrm{fin}}} f^{\mathtt{n}}(\begin{pmatrix} y & -x \\ 0 & 1 \end{pmatrix})\theta_{\mathrm{fin}}(rx)dx|y|_{\mathrm{fin}}^{s-\mathtt{k}/2}d^*y.$$

The value of $f^{\mathtt{n}}$ is nonzero if and only if there exists $m \in \mathbf{Q}^+$ such that $\left(\begin{smallmatrix} my & -mx \\ 0 & m \end{smallmatrix}\right) \in M(\mathtt{n}, N)$. In particular, $m \in \widehat{\mathbf{Z}} \cap \mathbf{Q}^+ = \mathbf{Z}^+$. Furthermore,

(i) $my \in \widehat{\mathbf{Z}}$,
(ii) $m^2 y \in \mathtt{n}\widehat{\mathbf{Z}}^*$,
(iii) $mx \in \widehat{\mathbf{Z}}$.

Together, the first two conditions imply that $m|\mathtt{n}$. Conversely, if $m|\mathtt{n}$, condition (ii) implies condition (i). Assuming that $m|\mathtt{n}$ and y satisfies (ii), we have

$$\int_{\mathbf{A}_{\mathrm{fin}}} f^{\mathtt{n}}(\left(\begin{smallmatrix} y & -x \\ 0 & 1 \end{smallmatrix}\right))\theta_{\mathrm{fin}}(rx)dx = \frac{\psi(N)}{\omega(m_N)} \int_{\frac{1}{m}\widehat{\mathbf{Z}}} \theta_{\mathrm{fin}}(rx)dx.$$

Because $m|\mathtt{n}$, it follows that $(m, N) = 1$, so $\omega(m_N) = \omega'(m)$. Hence the above is

$$= \begin{cases} m\psi(N)/\omega'(m) & \text{if } m|r, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\int_{\mathbf{A}_{\mathrm{fin}}} f^{\mathtt{n}}(\begin{pmatrix} y & -x \\ 0 & 1 \end{pmatrix})\theta_{\mathrm{fin}}(rx)dx = \begin{cases} m\psi(N)/\omega'(m) & \text{if } y \in \frac{\mathtt{n}}{m^2}\widehat{\mathbf{Z}}^* \text{ for} \\ & \quad \text{some } m|\gcd(\mathtt{n}, r), \\ 0 & \text{otherwise.} \end{cases}$$

We note that if such m exists, it is uniquely determined by y. Now

$$I_1(f)_{\mathrm{fin}} = \sum_{m|\gcd(\mathtt{n},r)} \frac{m\psi(N)}{\omega'(m)} \int_{\frac{\mathtt{n}}{m^2}\widehat{\mathbf{Z}}^*} |y|_{\mathrm{fin}}^{s-\mathtt{k}/2} d^*y = \psi(N) \sum_{m|\gcd(\mathtt{n},r)} \frac{m(m^2/\mathtt{n})^{s-\mathtt{k}/2}}{\omega'(m)}.$$

For the infinite part, recall that f_∞ vanishes on matrices with negative determinant. Thus

$$I_1(f)_\infty = \int_0^\infty \int_{\mathbf{R}} f_\infty(\begin{pmatrix} y & -x \\ 0 & 1 \end{pmatrix})\theta_\infty(rx)dx\,|y|^{s-\mathtt{k}/2}d^*y.$$

We have

$$\int_{\mathbf{R}} f_\infty(\left(\begin{smallmatrix} y & -x \\ 0 & 1 \end{smallmatrix}\right))\theta_\infty(rx)dx = \frac{\mathtt{k}-1}{4\pi} y^{\mathtt{k}/2}(2i)^{\mathtt{k}} \int_{-\infty}^{\infty} \frac{e^{-2\pi irx}}{(x+(y+1)i)^{\mathtt{k}}}dx.$$

Use a clockwise semicircular contour integral in the lower complex half-plane. The integrand has a pole at $x = -(y+1)i$ inside the contour. By the residue theorem, the above is

$$= -\frac{\mathtt{k}-1}{4\pi} y^{\mathtt{k}/2}(2i)^{\mathtt{k}} \frac{2\pi i}{(\mathtt{k}-1)!} \left.\frac{d^{\mathtt{k}-1}}{dx^{\mathtt{k}-1}}\right|_{x=-(y+1)i} e^{-2\pi irx}$$

$$= -\frac{\mathtt{k}-1}{4\pi} y^{\mathtt{k}/2}(2i)^{\mathtt{k}} \frac{2\pi i}{(\mathtt{k}-1)!}(-2\pi ir)^{\mathtt{k}-1}e^{-2\pi r(y+1)} = \frac{(4\pi r)^{\mathtt{k}-1}}{(\mathtt{k}-2)!\,e^{2\pi r}}\, y^{\mathtt{k}/2}e^{-2\pi ry}.$$

Therefore using $\mathrm{Re}(s) > 0$,

$$I_1(f)_\infty = \frac{(4\pi r)^{\mathtt{k}-1}}{(\mathtt{k}-2)!e^{2\pi r}} \int_0^\infty y^{s-1}e^{-2\pi ry}dy = \frac{(4\pi r)^{\mathtt{k}-1}}{(\mathtt{k}-2)!e^{2\pi r}}(2\pi r)^{-s}\Gamma(s).$$

All together we have

$$I_1(f) = \frac{\psi(N)2^{\mathtt{k}-1}\Gamma(s)\mathtt{n}^{\mathtt{k}/2-s}(2\pi r)^{\mathtt{k}-s-1}}{(\mathtt{k}-2)!\, e^{2\pi r}} \sum_{m|\,\gcd(\mathtt{n},r)} \frac{m^{2s-\mathtt{k}+1}}{\omega'(m)}.$$

□

Next we need to compute $I_\delta(f)$ for $\delta = \left(\begin{smallmatrix} 0 & 1 \\ -1 & t \end{smallmatrix}\right)$ with $t \in \mathbf{Q}$. We begin with the special case $t = 0$.

Proposition 3.5. *If* $\delta = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$, *then*

$$I_\delta(f) = \int_{\mathbf{A}^*}\int_{\mathbf{A}} f(\begin{pmatrix} yx & 1 \\ -y & 0 \end{pmatrix})\, \theta(rx)dx|y|^{s-\mathtt{k}/2}d^*y \tag{7}$$

converges absolutely for $1 < \mathrm{Re}(s) < \mathtt{k}$. *For such* s, $I_\delta(f) = 0$ *unless* $N = 1$. *When* $N = 1$,

$$I_\delta(f) = \frac{\mathtt{n}^{1-\mathtt{k}/2}}{e^{2\pi r}} \frac{2^{\mathtt{k}-1}\Gamma(\mathtt{k}-s)(2\pi r\mathtt{n})^{s-1}}{(\mathtt{k}-2)!\, i^{\mathtt{k}}} \sum_{m|\,\gcd(\mathtt{n},r)} m^{\mathtt{k}-2s+1}.$$

Proof. For the absolute convergence, see Proposition 3.10 below. The value of $f^{\mathtt{n}}$ in $I_\delta(f)_{\mathrm{fin}} = \int_{\mathbf{A}^*_{\mathrm{fin}}}\int_{\mathbf{A}_{\mathrm{fin}}} f^{\mathtt{n}}(\left(\begin{smallmatrix} yx & 1 \\ -y & 0 \end{smallmatrix}\right))\theta_{\mathrm{fin}}(rx)dx|y|_{\mathrm{fin}}^{s-\mathtt{k}/2}d^*y$ is nonzero if and only if there exists $m \in \mathbf{Q}^+$ such that $\left(\begin{smallmatrix} myx & m \\ -my & 0 \end{smallmatrix}\right) \in M(\mathtt{n}, N)$. This means $m \in \mathbf{Z}^+$, $my \in N\widehat{\mathbf{Z}}$ and $m^2y \in \mathtt{n}\widehat{\mathbf{Z}}^*$. It follows that $N|\mathtt{n}$, which is only possible if $N = 1$. Assuming $N = 1$, we have $m|\mathtt{n}$. The last requirement for nonvanishing is $x \in \frac{1}{my}\widehat{\mathbf{Z}} = \frac{m}{\mathtt{n}}\widehat{\mathbf{Z}}$, in which case $f^{\mathtt{n}}(\left(\begin{smallmatrix} yx & 1 \\ -y & 0 \end{smallmatrix}\right)) = 1$. Hence for fixed $m|\mathtt{n}$ and $y \in \frac{\mathtt{n}}{m^2}\widehat{\mathbf{Z}}^*$,

$$\int_{\mathbf{A}_{\mathrm{fin}}} f^{\mathtt{n}}(\begin{pmatrix} yx & 1 \\ -y & 0 \end{pmatrix})\, \theta_{\mathrm{fin}}(rx)dx = \int_{\frac{m}{\mathtt{n}}\widehat{\mathbf{Z}}} \theta_{\mathrm{fin}}(rx)dx = \begin{cases} \mathtt{n}/m & \text{if } \frac{rm}{\mathtt{n}} \in \widehat{\mathbf{Z}}, \\ 0 & \text{otherwise.} \end{cases}$$

Now we have

$$\begin{aligned} I_\delta(f)_{\mathrm{fin}} &= \sum_{\substack{m|\mathtt{n}, \\ \frac{\mathtt{n}}{m}|r}} \frac{\mathtt{n}}{m} \int_{\frac{\mathtt{n}}{m^2}\widehat{\mathbf{Z}}^*} |y|_{\mathrm{fin}}^{s-\mathtt{k}/2} d^*y = \sum_{\substack{m|\mathtt{n}, \\ \frac{\mathtt{n}}{m}|r}} \frac{\mathtt{n}}{m}(m^2/\mathtt{n})^{s-\mathtt{k}/2} \\ &= \mathtt{n}^{\mathtt{k}/2-s+1} \sum_{\substack{m|\mathtt{n}, \\ \frac{\mathtt{n}}{m}|r}} m^{2s-\mathtt{k}-1} = \mathtt{n}^{\mathtt{k}/2-s+1} \sum_{m|\,\gcd(\mathtt{n},r)} (\mathtt{n}/m)^{2s-\mathtt{k}-1} \\ &= \mathtt{n}^{s-\mathtt{k}/2} \sum_{m|\,\gcd(\mathtt{n},r)} m^{\mathtt{k}-2s+1}. \end{aligned}$$

For the infinite part $I_\delta(f)_\infty = \int_{\mathbf{R}^*}\int_{\mathbf{R}} f_\infty(\left(\begin{smallmatrix} yx & 1 \\ -y & 0 \end{smallmatrix}\right))\theta_\infty(rx)dx|y|^{s-\mathtt{k}/2}d^*y$, as before we can assume $y > 0$. We have

$$\begin{aligned} \int_{\mathbf{R}} f_\infty(\begin{pmatrix} yx & 1 \\ -y & 0 \end{pmatrix})e^{-2\pi irx}dx &= \frac{\mathtt{k}-1}{4\pi} y^{\mathtt{k}/2}(2i)^{\mathtt{k}} \int_{-\infty}^{\infty} \frac{e^{-2\pi irx}}{(-1-y+(yx)i)^{\mathtt{k}}}dx \\ &= \frac{(\mathtt{k}-1)(2i)^{\mathtt{k}}}{4\pi} y^{\mathtt{k}/2}(iy)^{-\mathtt{k}} \int_{-\infty}^{\infty} \frac{e^{-2\pi irx}}{(x+(\frac{1+y}{y})i)^{\mathtt{k}}}dx. \end{aligned}$$

Take a clockwise semicircular contour integral in the lower half-plane. The integrand has a pole at $x = -i(1+\frac{1}{y})$. By the residue theorem the above is

$$= -\frac{(\mathtt{k}-1)2^{\mathtt{k}}}{4\pi} y^{-\mathtt{k}/2} \frac{2\pi i}{(\mathtt{k}-1)!} \left.\frac{d^{\mathtt{k}-1}}{dx^{\mathtt{k}-1}}\right|_{x=-i(1+\frac{1}{y})} e^{-2\pi i r x}$$
$$= \frac{-i2^{\mathtt{k}-1}}{(\mathtt{k}-2)!} y^{-\mathtt{k}/2}(-2\pi i r)^{\mathtt{k}-1} e^{-2\pi r(1+1/y)}$$
$$= \frac{(4\pi r)^{\mathtt{k}-1} e^{-2\pi r}}{(\mathtt{k}-2)!\, i^{\mathtt{k}}} y^{-\mathtt{k}/2} e^{-2\pi r/y}.$$

Therefore

$$I_\delta(f)_\infty = \frac{(4\pi r)^{\mathtt{k}-1} e^{-2\pi r}}{(\mathtt{k}-2)!\, i^{\mathtt{k}}} \int_0^\infty y^{s-\mathtt{k}-1} e^{-2\pi r/y} dy.$$

For any $\alpha > 0$, $\int_0^\infty t^{w-1} e^{-\alpha/t} dt = \alpha^w \Gamma(-w)$ when $\mathrm{Re}(w) < 0$, so we get

$$I_\delta(f) = \frac{(4\pi r)^{\mathtt{k}-1} e^{-2\pi r}}{(\mathtt{k}-2)!\, i^{\mathtt{k}}} (2\pi r)^{s-\mathtt{k}} \Gamma(\mathtt{k}-s) \mathtt{n}^{s-\mathtt{k}/2} \sum_{m|\gcd(\mathtt{n},r)} m^{\mathtt{k}-2s+1}.$$

□

For the case of $\delta = \left(\begin{smallmatrix} 0 & 1 \\ -1 & t \end{smallmatrix}\right)$ with $t \in \mathbf{Q}^*$, we use the following lemma, which is very easy to prove.

Lemma 3.6. *For any $n, m, r \in \widehat{\mathbf{Z}}$,*

$$r\widehat{\mathbf{Z}} \cap (n + m\widehat{\mathbf{Z}}) = \begin{cases} rc_0 + \frac{rm}{\gcd(r,m)}\widehat{\mathbf{Z}} & \text{if } \gcd(r,m)|n, \\ \emptyset & \text{if } \gcd(r,m) \nmid n, \end{cases}$$

where $c_0 \in \mathbf{Z}$ is any fixed solution to $rc_0 \equiv n \bmod m\widehat{\mathbf{Z}}$.

We also need to recall the definition of the confluent hypergeometric function

$${}_1\mathrm{F}_1(s;k;w) = \sum_{m=0}^\infty \frac{(s)_m}{(k)_m} \frac{w^m}{m!},$$

where $(s)_0 = 1$ and for $m > 0$, $(s)_m = s(s+1)(s+2)\cdots(s+m-1)$. This is absolutely convergent for all $s, k, w \in \mathbf{C}$, except when k is a nonpositive integer. We have the following useful integral representation:

$$(8)\quad {}_1\mathrm{F}_1(s;k;w) = \frac{\Gamma(k)}{\Gamma(k-s)\Gamma(s)} \int_0^1 e^{wt} t^{s-1}(1-t)^{k-s-1} dt \quad (\mathrm{Re}(k) > \mathrm{Re}(s) > 0)$$

(see [Sl], §3.1).

Proposition 3.7. *If $\delta = \left(\begin{smallmatrix} 0 & 1 \\ -1 & t \end{smallmatrix}\right)$ for $t \in \mathbf{Q}^*$, then $I_\delta(f)$ is absolutely convergent when $0 < \mathrm{Re}(s) < \mathtt{k}$. It vanishes unless $t \in \frac{N}{\mathtt{n}}\mathbf{Z}$. For such t, write $t = \frac{N}{\mathtt{n}}b$. Then*

$$I_\delta(f) = \frac{(4\pi r)^{\mathtt{k}-1}\psi(N)\mathtt{n}^{\mathtt{k}/2}}{(\mathtt{k}-2)!\, e^{i\pi s/2} e^{2\pi r} N^s} b^{s-\mathtt{k}}\, {}_1f_1(s;\mathtt{k};\tfrac{2\pi i r \mathtt{n}}{Nb}) \sum_{\substack{d|b \\ \gcd(b/d, Nd)|\gcd(r,\mathtt{n})}} \frac{\gcd(b/d, Nd)}{d^{2s-\mathtt{k}}\omega'(b/d)} e^{-\frac{2\pi i r \ell_0}{b/d}},$$

where $\ell_0 \in \mathbf{Z}$ is any integer satisfying $\ell_0(Nd) \equiv \mathtt{n} \bmod (b/d)$, and

$${}_1f_1(s;\mathtt{k};w) = \frac{\Gamma(s)\Gamma(\mathtt{k}-s)}{\Gamma(\mathtt{k})} {}_1\mathrm{F}_1(s;\mathtt{k};w).$$

When $b < 0$, we take $b^{s-\mathtt{k}} = |b|^{s-\mathtt{k}} e^{i\pi(s-\mathtt{k})}$.

Proof. The absolute convergence will be proven in Proposition 3.9 below. We can factorize the integral as $I_\delta(f) = I_\delta(f)_\infty I_\delta(f)_{\mathrm{fin}}$. First we compute

$$I_\delta(f)_{\mathrm{fin}} = \int_{\mathbf{A}^*_{\mathrm{fin}}} \int_{\mathbf{A}_{\mathrm{fin}}} f^{\mathtt{n}}(\begin{pmatrix} yx & 1-tx \\ -y & t \end{pmatrix})\theta_{\mathrm{fin}}(rx)dx|y|^{s-\mathtt{k}/2}d^*y.$$

Suppose $f^{\mathtt{n}}(\left(\begin{smallmatrix} yx & 1-tx \\ -y & t \end{smallmatrix}\right)) \neq 0$. Then there exists $m \in \mathbf{Q}^+$ such that

$$\begin{pmatrix} myx & m - mtx \\ -my & mt \end{pmatrix} \in M(\mathtt{n}, N).$$

This means:

$$\begin{array}{llll} \text{(i)} & my \in N\widehat{\mathbf{Z}}, & \text{(iv)} & mxy \in \widehat{\mathbf{Z}}, \\ \text{(ii)} & m^2y \in \mathtt{n}\widehat{\mathbf{Z}}^*, & \text{(v)} & m - mtx \in \widehat{\mathbf{Z}}. \\ \text{(iii)} & mt \in \widehat{\mathbf{Z}}, & & \end{array}$$

The first two conditions imply that $m = \frac{\mathtt{n}}{Nd}$ for some integer $d > 0$ and that $y \in \frac{N^2d^2}{\mathtt{n}}\widehat{\mathbf{Z}}^*$. By the third condition, $t \in \frac{Nd}{\mathtt{n}}\widehat{\mathbf{Z}}$, or equivalently, $t \in \frac{N}{\mathtt{n}}\mathbf{Z}$ and $d|\frac{\mathtt{n}}{N}t$. This proves the first assertion. Condition (iv) is now equivalent to $x \in \frac{1}{Nd}\widehat{\mathbf{Z}}$. Conversely, if m, y, t, x are given in this way, they will satisfy (i)-(iv). Thus we have

$$I_\delta(f)_{\mathrm{fin}} = \sum_{d|\frac{\mathtt{n}}{N}t} \frac{\mathtt{n}^{s-\mathtt{k}/2}}{(Nd)^{2s-\mathtt{k}}} \int_{\frac{N^2d^2}{\mathtt{n}}\widehat{\mathbf{Z}}^*} \int_{\frac{1}{Nd}\widehat{\mathbf{Z}}} f^{\mathtt{n}}(\left(\begin{smallmatrix} yx & 1-tx \\ -y & t \end{smallmatrix}\right))\theta_{\mathrm{fin}}(rx)dx\, d^*y.$$

Write $t = \frac{N}{\mathtt{n}}b$ for nonzero $b \in d\mathbf{Z}$. Then $mt = b/d$, so the fifth condition is equivalent to $x \in \frac{\mathtt{n}}{Nb} + \frac{d}{b}\widehat{\mathbf{Z}}$. Thus the inner integral is taken over

$$x \in \frac{1}{Nd}\widehat{\mathbf{Z}} \cap (\frac{\mathtt{n}}{Nb} + \frac{d}{b}\widehat{\mathbf{Z}}).$$

By Lemma 3.6 (multiply the above through by Nb), this set is nonempty if and only if $\gcd(b/d, Nd)|\mathtt{n}$, in which case it is equal to $\frac{1}{Nd}c_0 + \frac{1}{\gcd(b/d,Nd)}\widehat{\mathbf{Z}}$, where c_0 is any solution to $(b/d)c_0 \equiv \mathtt{n} \bmod Nd$.

Note that $\gcd(b/d, Nd)|\mathtt{n}$ implies that b/d is prime to N. Therefore the value of $f^{\mathtt{n}}$ in the integrand is $\frac{\psi(N)}{\omega'(b/d)}$. Thus

$$\begin{aligned} I_\delta(f)_{\mathrm{fin}} &= \sum_{\substack{d|b \\ \gcd(b/d,Nd)|\mathtt{n}}} \frac{\mathtt{n}^{s-\mathtt{k}/2}}{(Nd)^{2s-\mathtt{k}}} \frac{\psi(N)}{\omega'(b/d)} \int_{\frac{1}{Nd}c_0 + \frac{1}{\gcd(b/d,Nd)}\widehat{\mathbf{Z}}} \theta_{\mathrm{fin}}(rx)dx \\ &= \sum_{\substack{d|b \\ \gcd(b/d,Nd)|\mathtt{n}}} \frac{\mathtt{n}^{s-\mathtt{k}/2}}{(Nd)^{2s-\mathtt{k}}} \frac{\psi(N)}{\omega'(b/d)} \theta_{\mathrm{fin}}(\frac{rc_0}{Nd}) \int_{\frac{1}{\gcd(b/d,Nd)}\widehat{\mathbf{Z}}} \theta_{\mathrm{fin}}(rx)dx \qquad (9) \\ &= \frac{\psi(N)\mathtt{n}^{s-\mathtt{k}/2}}{N^{2s-\mathtt{k}}} \sum_{\substack{d|b \\ \gcd(b/d,Nd)|\gcd(r,\mathtt{n})}} \frac{\gcd(b/d, Nd)}{d^{2s-\mathtt{k}}\omega'(b/d)} e^{2\pi i r c_0/Nd}. \end{aligned}$$

For the archimedean part, the inner integral is

$$\begin{aligned}&\int_{\mathbf{R}} f_\infty\left(\left(\begin{smallmatrix} yx & 1-tx \\ -y & t\end{smallmatrix}\right)\right)e^{-2\pi i r x}dx\\
&= \frac{\mathtt{k}-1}{4\pi}(2i)^{\mathtt{k}}y^{\mathtt{k}/2}\int_{-\infty}^{\infty}\frac{e^{-2\pi i r x}}{(tx-1-y+(yx+t)i)^{\mathtt{k}}}dx\\
&= \frac{\mathtt{k}-1}{4\pi}(2i)^{\mathtt{k}}y^{\mathtt{k}/2}(t+iy)^{-\mathtt{k}}\int_{-\infty}^{\infty}\frac{e^{-2\pi i r x}}{(x-\frac{1+y-it}{i(y-it)})^{\mathtt{k}}}dx.\end{aligned}$$

The integrand has a pole at $x = -i(1+\frac{1}{y-it})$ in the lower half-plane. Using a clockwise lower semicircular contour integral, this is

$$= -\frac{\mathtt{k}-1}{4\pi}(2i)^{\mathtt{k}}\frac{2\pi i}{(\mathtt{k}-1)!}(-2\pi i r)^{\mathtt{k}-1}y^{\mathtt{k}/2}(i)^{-\mathtt{k}}(y-it)^{-\mathtt{k}}e^{-2\pi r(1+\frac{1}{y-it})}.$$

Thus

$$I_\delta(f)_\infty = \frac{(4\pi r)^{\mathtt{k}-1}}{(\mathtt{k}-2)!\, i^{\mathtt{k}}e^{2\pi r}}\int_0^\infty y^{s-1}(y-it)^{-\mathtt{k}}e^{-2\pi r/(y-it)}dy.$$

This has an essential singularity at $y = it$. We define y^{s-1} as a holomorphic function of y by taking the principal value of $\log y$ on the positive real axis and making a branch cut along the positive imaginary axis if $t > 0$ or the negative imaginary axis if $t < 0$. Now pulling out t and making a change of variables, we get

$$I_\delta(f)_\infty = \frac{(4\pi r)^{\mathtt{k}-1}t^{s-\mathtt{k}}}{(\mathtt{k}-2)!\, i^{\mathtt{k}}e^{2\pi r}}\int_0^{\pm\infty} y^{s-1}(y-i)^{-\mathtt{k}}e^{-2\pi r/t(y-i)}dy,$$

where the sign in the upper limit is the sign of t, and, by our choice of branch, $t^{s-\mathtt{k}} = |t|^{s-\mathtt{k}}e^{i\pi(s-\mathtt{k})}$ if $t < 0$. In the notation of the next lemma below, the integral is $\mathtt{G}(s,\mathtt{k},r/t)$. By the result of the lemma and setting $t = Nb/\mathtt{n}$, this gives

$$I_\delta(f)_\infty = \frac{(4\pi r)^{\mathtt{k}-1}N^{s-\mathtt{k}}}{(\mathtt{k}-2)!\, e^{i\pi s/2}e^{2\pi r}\mathtt{n}^{s-\mathtt{k}}}\frac{b^{s-\mathtt{k}}{}_1f_1(s;\mathtt{k};2\pi i r\mathtt{n}/Nb)}{e^{2\pi i r\mathtt{n}/Nb}}.$$

When we multiply this by $I_\delta(f)_{\mathrm{fin}}$, we can combine the terms

$$e^{-2\pi i r\mathtt{n}/Nb}e^{2\pi i r c_0/Nd} = e^{2\pi i r(c_0(b/d)-\mathtt{n})/Nb}.$$

Writing $c_0(b/d) - \mathtt{n} = -Nd\ell_0$ for some $\ell_0 \in \mathbf{Z}$, we have $Nd\ell_0 \equiv \mathtt{n} \bmod (b/d)$, and the above is equal to $e^{-2\pi i r\ell_0/(b/d)}$. The result now follows. $\square$

Lemma 3.8. *For $s, w \in \mathbf{C}$ and $\mathtt{k} \in \mathbf{Z}^+$, define*

$$\mathtt{G}(s,\mathtt{k},w) = \int_0^\infty y^{s-1}(y-i)^{-\mathtt{k}}e^{-2\pi w/(y-i)}dy.$$

This function converges absolutely for $0 < \mathrm{Re}(s) < \mathtt{k}$. *On this strip we can represent* $\mathtt{G}(s, \mathtt{k}, w)$ *in terms of the confluent hypergeometric function:*

$$\mathtt{G}(s, \mathtt{k}, w) = i^{\mathtt{k}} e^{-i\pi s/2} e^{-2\pi i w} \frac{\Gamma(s)\Gamma(\mathtt{k}-s)}{\Gamma(\mathtt{k})} \, {}_1\mathrm{F}_1(s; \mathtt{k}; 2\pi i w).$$

Furthermore, the integral defining $\mathtt{G}$ *is unchanged if we replace* ∞ *by* $-\infty$.

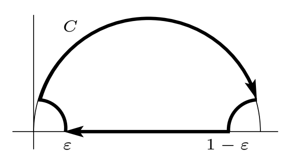
Proof. Let $t = 1 + \frac{i}{y-i}$, so that $y - i = \frac{-i}{1-t}$. This linear fractional transformation takes the positive real axis to the upper semicircle C of radius $1/2$ centered at $z = 1/2$. Then $dy = \frac{-i}{(1-t)^2} dt$ and

$$\mathtt{G}(s, \mathtt{k}, w) = \int_C \left(\frac{-it}{1-t}\right)^{s-1} \left(\frac{-i}{1-t}\right)^{-\mathtt{k}} e^{2\pi i w(t-1)} \frac{-i}{(1-t)^2} dt.$$

We define $y^{s-1} = e^{(s-1)\log y}$ by taking the principal value of $\log y$ for $y > 0$ and making a cut along the positive imaginary axis in the y-plane. This cut corresponds in the t-plane to cuts on the real axis from 0 to $-\infty$ and from 1 to ∞. We choose $\log(-i) = -i\pi/2$ and choose the principal branches of $\log(t)$ and $\log(1-t)$. Then for $t \in (0, 1)$, $-3\pi/2 < \arg(-it/(1-t)) < \pi/2$, and therefore these choices are compatible with the choice of $\log y$. Now

$$\mathtt{G}(s, \mathtt{k}, w) = e^{-is\pi/2} (-i)^{-\mathtt{k}} e^{-2\pi i w} \int_C e^{2\pi i w t} t^{s-1} (1-t)^{\mathtt{k}-s-1} dt.$$

The integrand is holomorphic in t and single-valued in the cut plane, and by Cauchy's theorem, its integral around the following contour vanishes:



Using the fact that $0 < \mathrm{Re}(s) < \mathtt{k}$, it is straightforward to show that the contribution along the small arcs goes to 0 as $\varepsilon \to 0$. It follows that the integral along C can instead be taken along the real axis, so

$$\begin{aligned} \mathtt{G}(s, \mathtt{k}, w) &= i^{\mathtt{k}} e^{-i\pi s/2} e^{-2\pi i w} \int_0^1 e^{2\pi i w t} t^{s-1} (1-t)^{\mathtt{k}-s-1} dt \\ &= i^{\mathtt{k}} e^{-i\pi s/2} e^{-2\pi i w} \frac{\Gamma(\mathtt{k}-s)\Gamma(s)}{\Gamma(\mathtt{k})} \, {}_1\mathrm{F}_1(s; \mathtt{k}; 2\pi i w) \end{aligned}$$

by (8). If the upper limit of $\mathtt{G}$ is replaced by $-\infty$, then t will instead traverse the lower semicircle $\overline{C}$ from 0 to 1, which can likewise be moved to the real axis. In fact a more general path independence property can be proven in a similar way. □

3.3. **Proof of Proposition 3.3.** For each δ, we set

$$I_\delta^{abs}(f) = \int_{\mathbf{A}^*} \int_{\mathbf{A}} \left| f(\left(\begin{smallmatrix} 1 & -x \\ & 1 \end{smallmatrix}\right) \delta \left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)) \overline{\theta_r(x)} |y|^{s-\mathtt{k}/2} \right| dx \, d^*y.$$

Because $f^{\mathtt{n}}$ is compactly supported modulo the center and bounded by $\psi(N)$, the finite part $I_\delta^{abs}(f)_{\mathrm{fin}}$ converges for all s to a value depending on δ. Thus we primarily need to consider the infinite part

$$I_\delta^{abs}(f)_\infty = \int_0^\infty \int_{-\infty}^\infty \left| f_\infty\left(\left(\begin{smallmatrix} 1 & -x \\ & 1 \end{smallmatrix}\right) \delta \left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)\right)\right| dx\, y^{\mathrm{Re}(s)-\mathtt{k}/2-1} dy.$$

We will repeatedly use the fact that for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbf{R})^+$,

$$|f_\infty(g)| = \frac{\mathtt{k}-1}{4\pi} \frac{\det(g)^{\mathtt{k}/2} 2^{\mathtt{k}}}{(a^2+b^2+c^2+d^2+2\det(g))^{\mathtt{k}/2}}. \tag{10}$$

This follows easily from the explicit formula for f_∞.

Proposition 3.9. *Let* $\delta_t = \left(\begin{smallmatrix} 0 & 1 \\ -1 & t \end{smallmatrix}\right)$ *for* $t \in \mathbf{Q}^*$. *Then if* $0 < \mathrm{Re}(s) < \mathtt{k}$,

(a) $I_{\delta_t}^{abs}(f) < \infty$,

(b) $I_{\delta_t}^{abs}(f)_\infty \ll |t|^{\mathrm{Re}(s)-\mathtt{k}}$.

Furthermore, if $1 < \mathrm{Re}(s) < \mathtt{k}-1$, *then*

(c) $\sum_{t \in \mathbf{Q}^*} I_{\delta_t}^{abs}(f) < \infty$.

Proof. We need to estimate the expression

$$\int_0^\infty \int_{-\infty}^\infty \left| f_\infty\left(\begin{pmatrix} yx & 1-tx \\ -y & t \end{pmatrix}\right)\right| dx\, y^{\mathrm{Re}(s)-\mathtt{k}/2-1} dy.$$

By (10), the inner integral is

$$\ll y^{\mathtt{k}/2} \int_{-\infty}^\infty \frac{dx}{(y^2x^2+y^2+t^2+(1-tx)^2+2y)^{\mathtt{k}/2}}$$
$$= y^{\mathtt{k}/2}(t^2+y^2)^{-\mathtt{k}/2} \int_{-\infty}^\infty \frac{dx}{(x^2 - \frac{2t}{t^2+y^2}x + \frac{1+t^2+y^2+2y}{t^2+y^2})^{\mathtt{k}/2}}.$$

We will show that the integral is bounded, independently of y and t. Completing the square, the integral is equal to

$$\int_{-\infty}^\infty \frac{dx}{((x-\frac{t}{t^2+y^2})^2 + \frac{(1+y)^2+t^2}{t^2+y^2} - \frac{t^2}{(t^2+y^2)^2})^{\mathtt{k}/2}} = \int_{-\infty}^\infty \frac{dx}{(x^2 + \frac{(1+2y+y^2+t^2)(t^2+y^2)-t^2}{(t^2+y^2)^2})^{\mathtt{k}/2}}$$
$$= \int_{-\infty}^\infty \frac{dx}{(x^2 + \frac{(t^2+y^2+y)^2}{(t^2+y^2)^2})^{\mathtt{k}/2}} < \int_{-\infty}^\infty \frac{dx}{(x^2+1)^{\mathtt{k}/2}} < \infty.$$

Therefore writing $s = \sigma + i\tau$,

$$I_{\delta_t}^{abs}(f)_\infty \ll \int_0^\infty y^{\sigma-1}(t^2+y^2)^{-\mathtt{k}/2} dy.$$

For convergence as $y \to 0$, we need $\sigma - 1 > -1$, i.e. $\sigma > 0$. For convergence as $y \to \infty$, we need $\sigma - 1 - \mathtt{k} < -1$, i.e. $\sigma < \mathtt{k}$. This proves the absolute convergence of $I_{\delta_t}(f)$ on the given strip.

In order to sum over t, we need to bound the above integral in terms of t. We have

$$I_{\delta_t}^{abs}(f)_\infty \ll \int_0^\infty y^{\sigma-1}|t|^{-\mathtt{k}}(1+\frac{y^2}{t^2})^{-\mathtt{k}/2} dy$$
$$= |t|^{-\mathtt{k}} \int_0^\infty \left(\frac{y^2}{t^2}\right)^{\frac{\sigma}{2}} |t|^\sigma (1+\frac{y^2}{t^2})^{-\mathtt{k}/2} d^*y.$$

Letting $u = (y/t)^2$ so $d^*u = 2d^*y$, the above is

$$= \frac{1}{2}|t|^{\sigma-\mathtt{k}} \int_0^\infty \frac{u^{\frac{\sigma}{2}-1}}{(1+u)^{\mathtt{k}/2}} du,$$

which proves the second assertion since $\mathtt{k} > \sigma > 0$. (As an aside, this last integral equals $B(\frac{\sigma}{2}, \frac{\mathtt{k}-\sigma}{2})$, where $B(n,m) = \Gamma(n)\Gamma(m)/\Gamma(n+m)$ is the Beta function.)

As in the proof of Proposition 3.7, $I_{\delta_t}^{abs}(f)_{\mathrm{fin}}$ vanishes unless $t = \frac{N}{\mathtt{n}}b$ for some $b \in \mathbf{Z} - \{0\}$. By (9), we see that

$$I_{\delta_t}^{abs}(f)_{\mathrm{fin}} \ll \frac{\mathtt{n}^{\sigma-\mathtt{k}/2}\psi(N)}{N^{2\sigma-\mathtt{k}}} \sum_{d|b} d^{-2\sigma+\mathtt{k}}.$$

If $\sigma > \mathtt{k}/2$, then $d^{-2\sigma+\mathtt{k}} \leq 1$. If $\sigma \leq \mathtt{k}/2$, then $d^{-2\sigma+\mathtt{k}} \leq |b|^{-2\sigma+\mathtt{k}}$. The number of divisors of b is $\ll b^\varepsilon$ for any $\varepsilon > 0$. Since $I_{\delta_t}^{abs}(f)_\infty$ contributes $|b|^{\sigma-\mathtt{k}}$, we have

$$\sum_{t\in\mathbf{Q}^*} I_{\delta_t}^{abs}(f) \ll \begin{cases} \sum_{b\in\mathbf{Z}-\{0\}} |b|^{-\sigma+\varepsilon} & \text{if } \sigma \leq \mathtt{k}/2, \\ \sum_{b\in\mathbf{Z}-\{0\}} |b|^{\sigma-\mathtt{k}+\varepsilon} & \text{if } \sigma > \mathtt{k}/2. \end{cases} \tag{11}$$

Hence $\sum_t I_{\delta_t}^{abs}(f) < \infty$ as long as $1 < \sigma < \mathtt{k} - 1$. □

The following will complete the proof of Proposition 3.3.

Proposition 3.10. *For $\delta = 1$, $I_1^{abs}(f) < \infty$, provided*

$$0 < \mathrm{Re}(s) < \mathtt{k} - 1.$$

For $\delta = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$, $I_\delta^{abs}(f) < \infty$, provided

$$1 < \mathrm{Re}(s) < \mathtt{k}.$$

Proof. For any $a > 0$, a change of variables gives

$$\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)^{\mathtt{k}/2}} = a^{-\mathtt{k}+1} \int_{-\infty}^\infty \frac{du}{(u^2+1)^{\mathtt{k}/2}}. \tag{12}$$

We again write $s = \sigma + i\tau$. When $\delta = 1$, using (10) we have

$$I_1^{abs}(f) \ll \int_0^\infty \int_{-\infty}^\infty \frac{y^{\sigma-1}}{(x^2+y^2+2y+1)^{\mathtt{k}/2}} dx\, dy.$$

By (12), this is

$$\ll \int_0^\infty y^{\sigma-1}(y+1)^{-\mathtt{k}+1} dy.$$

This converges precisely when $0 < \sigma < \mathtt{k} - 1$.

Similarly, for $\delta = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$,

$$\begin{aligned} I_\delta^{abs}(f) &\ll \int_0^\infty \int_{-\infty}^\infty \frac{y^{\sigma-1}}{(x^2y^2+y^2+2y+1)^{\mathtt{k}/2}} dx\, dy \\ &= \int_0^\infty y^{\sigma-1-\mathtt{k}} \int_{-\infty}^\infty \frac{dx}{(x^2+(1+\frac{1}{y})^2)^{\mathtt{k}/2}}\, dy \\ &\ll \int_0^\infty y^{\sigma-\mathtt{k}-1}(1+y^{-1})^{-\mathtt{k}+1} dy. \end{aligned}$$

As $y \to 0$, we need $\sigma - \mathtt{k} - 1 + \mathtt{k} - 1 > -1$, i.e. $\sigma > 1$. As $y \to \infty$, we need $\sigma - \mathtt{k} - 1 < -1$, i.e. $\sigma < \mathtt{k}$. This proves the proposition. □

3.4. **Proof of Theorem 1.1.** We have now proven that the geometric side converges absolutely when $1 < \mathrm{Re}(s) < \mathtt{k}-1$, and therefore it is equal to the spectral side on this strip. When we sum the contribution of Proposition 3.7 over all $b \neq 0$, we set $a = b/d$ so that $b = ad$. Then

$$\sum_{b\neq 0} b^{s-\mathtt{k}} {}_1f_1(s;\mathtt{k};\tfrac{2\pi i r\mathtt{n}}{Nb}) \sum_{\substack{d|b \\ \gcd(b/d,Nd)|\gcd(r,\mathtt{n})}} \frac{\gcd(b/d,Nd)}{d^{2s-\mathtt{k}}\omega'(b/d)} e^{-\frac{2\pi i r\ell_0}{b/d}}$$
$$= \sum_{\substack{a\neq 0, d>0 \\ \gcd(a,Nd)|\gcd(r,\mathtt{n})}} \frac{a^{s-\mathtt{k}} d^{-s}\gcd(a,Nd)}{\omega'(a)e^{2\pi i r\ell_0/a}} {}_1f_1(s;\mathtt{k};\tfrac{2\pi i r\mathtt{n}}{Nad}).$$

The theorem now follows immediately upon equating the two sides of the trace formula and dividing through by $e^{-2\pi r}\mathtt{n}^{1-\mathtt{k}/2}$.

4. Estimates and examples

4.1. **Asymptotic behavior.** For two functions A, B, we write $A \sim B$ to mean that $A/B \to 1$ in a limiting sense which will be clear from the context. For example, by Stirling's approximation we have the following:

$$\Gamma(z+b) \sim \sqrt{2\pi}e^{-z}z^{z+b-1/2} \qquad (z\to\infty, |\arg z| < \pi) \tag{13}$$

([AS], 6.1.39). The $\sim$ notation here depends on b; i.e. given $\varepsilon > 0$ there is a constant $N(b) > 0$ such that the quotient is within ε of 1 whenever $|z| > N(b)$.

We now estimate each term of Theorem 1.1 as $\mathtt{k}\to\infty$. It will turn out that the first two terms are dominant, provided their sum does not vanish. In order to ensure nonvanishing of $\sum_{m|\gcd(\mathtt{n},r)} m^{2s-\mathtt{k}+1}/\omega'(m)$, we will assume for simplicity that $\gcd(\mathtt{n},r) = 1$. However, in general one can prove that this sum can only vanish on the left edge of the critical strip, i.e. on the line $\mathrm{Re}(s) = \frac{\mathtt{k}-1}{2}$.

Proposition 4.1. *Let $s = \mathtt{k}/2 + \alpha + i\tau$, with $1 < \mathtt{k}/2+\alpha < \mathtt{k}-1$. Assume $\gcd(\mathtt{n},r) = 1$. Then as $\mathtt{k}\to\infty$ the identity term in Theorem 1.1 satisfies*

$$\left|\frac{\psi(N)2^{\mathtt{k}-1}\Gamma(s)(2\pi r\mathtt{n})^{\mathtt{k}-s-1}}{(\mathtt{k}-2)!}\right| \sim \frac{2\sqrt{\pi}\,\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}/2-\alpha-1}\mathtt{k}^{\mathtt{k}/2+\alpha-1/2}}{(\mathtt{k}-2)!\,e^{\mathtt{k}/2}} \tag{14}$$
$$\sim \sqrt{2}\psi(N)e^{\alpha+1}\left(\frac{4\pi r\mathtt{n}e}{\mathtt{k}}\right)^{\mathtt{k}/2-\alpha-1}.$$

If $N = 1$, then as $\mathtt{k}\to\infty$ the second term in Theorem 1.1 satisfies

$$\left|\frac{2^{\mathtt{k}-1}\Gamma(\mathtt{k}-s)(2\pi r\mathtt{n})^{s-1}}{(\mathtt{k}-2)!\,i^{\mathtt{k}}}\right| \sim \frac{2\sqrt{\pi}(4\pi r\mathtt{n})^{\mathtt{k}/2+\alpha-1}\mathtt{k}^{\mathtt{k}/2-\alpha-1/2}}{(\mathtt{k}-2)!\,e^{\mathtt{k}/2}} \tag{15}$$
$$\sim \sqrt{2}e^{-\alpha+1}\left(\frac{4\pi r\mathtt{n}e}{\mathtt{k}}\right)^{\mathtt{k}/2+\alpha-1}.$$

Remark. The $\sim$ notation here depends on $\alpha + i\tau$ as discussed after (13).

Proof. Using (13), the left-hand side of (14) is

$$= \frac{\psi(N)2^{\mathtt{k}-1}|\Gamma(\mathtt{k}/2+\alpha+i\tau)|(2\pi r\mathtt{n})^{\mathtt{k}/2-\alpha-1}}{(\mathtt{k}-2)!}$$
$$\sim \frac{\psi(N)2^{-1}2^{\mathtt{k}}(2\pi r\mathtt{n})^{\mathtt{k}/2-\alpha-1}\sqrt{2\pi}e^{-\mathtt{k}/2}(\mathtt{k}/2)^{\mathtt{k}/2+\alpha-1/2}}{(\mathtt{k}-2)!}.$$

For the second line of (14) we substitute $(\mathtt{k}-2)! = \Gamma(\mathtt{k}-1) \sim \sqrt{2\pi}e^{-\mathtt{k}}\mathtt{k}^{\mathtt{k}-1-1/2}$. The second estimate is similar, as the left-hand side of (15) is

$$\sim \frac{2^{-1}2^{\mathtt{k}}(2\pi r\mathtt{n})^{\mathtt{k}/2+\alpha-1}e^{-\mathtt{k}/2}(\mathtt{k}/2)^{\mathtt{k}/2-\alpha-1/2}}{(\mathtt{k}-2)!}.$$

□

We now show that the third term in Theorem 1.1 decays much more rapidly in comparison with the first terms as $\mathtt{k} \to \infty$. We can rewrite it as a sum over $a, d > 0$. Note that $\omega'(-a) = (-1)^{\mathtt{k}}\omega'(a)$. Thus the third term is equal to

$$\frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}}{N^s(\mathtt{k}-2)!\, e^{i\pi s/2}} \sum_{\substack{a,d>0 \\ \gcd(a,Nd)\mid\gcd(r,\mathtt{n})}} \Big[a^{s-\mathtt{k}}d^{-s}\,{}_1f_1(s;\mathtt{k};\tfrac{2\pi ir\mathtt{n}}{Nad})e^{-2\pi ir\ell_0/a} + e^{i\pi s}a^{s-\mathtt{k}}d^{-s}\,{}_1f_1(s;\mathtt{k};-\tfrac{2\pi ir\mathtt{n}}{Nad})e^{2\pi ir\ell_0/a}\Big]\frac{\gcd(a,Nd)}{\omega'(a)}, \tag{16}$$

where ℓ_0 is any integer satisfying $\ell_0 Nd \equiv \mathtt{n} \bmod a$. Write $s = \sigma + i\tau$. If w is real,

$$\begin{aligned} |{}_1f_1(s;\mathtt{k};2\pi iw)| &= \left|\int_0^1 e^{2\pi iwt}t^{s-1}(1-t)^{\mathtt{k}-s-1}dt\right| \\ &\le \int_0^1 t^{\sigma-1}(1-t)^{\mathtt{k}-\sigma-1}dt = B(\sigma,\mathtt{k}-\sigma) \end{aligned} \tag{17}$$

for the Beta function B. Furthermore, $|e^{i\pi s/2}| = e^{-\pi\tau/2}$. Thus the absolute value of (16) is

$$\le \frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}\gcd(r,\mathtt{n})B(\sigma,\mathtt{k}-\sigma)}{N^\sigma(\mathtt{k}-2)!}e^{\pi\tau/2}\sum_{a,d>0}a^{-(\mathtt{k}-\sigma)}d^{-\sigma}\left|1+e^{i\pi s}\right|.$$

Note that $|1+e^{i\pi s}| \le (1+e^{-\pi\tau})$. Pulling this out of the sum, we obtain

$$(e^{\pi\tau/2}+e^{-\pi\tau/2}) = 2\cosh(\tau\pi/2),$$

and we immediately arrive at the following.

Proposition 4.2. *Write $s = \sigma + i\tau$ for $1 < \sigma < \mathtt{k}-1$. Then the absolute value of the last term* (16) *of Theorem* 1.1 *is*

$$\le \frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}\gcd(r,\mathtt{n})B(\sigma,\mathtt{k}-\sigma)}{N^\sigma(\mathtt{k}-2)!}2\cosh(\tau\pi/2)\zeta(\mathtt{k}-\sigma)\zeta(\sigma)$$

for the Beta function B and the Riemann zeta function ζ.

We remark that when $1 < \mathrm{Re}(s) < \mathtt{k}-1$ as is the case here, the integrand in (17) is smaller than 1, so $0 < B(\sigma,\mathtt{k}-\sigma) < 1$.

If we restrict s to the critical strip $\frac{\mathtt{k}-1}{2} < \mathrm{Re}(s) < \frac{\mathtt{k}+1}{2}$, then both zeta values approach 1 as $\mathtt{k} \to \infty$. Therefore we see that if $N > 1$, the identity term is dominant as $\mathtt{k} \to \infty$. If $N = 1$, then $I_1(f)$ is the main term when $\sigma > \mathtt{k}/2$, while $I_{\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)}(f)$ is the main term when $\sigma < \mathtt{k}/2$.

Corollary 1.3 now follows easily. In fact we can make it effective. Assume $N > 1$, $\mathtt{k} > 3$ and $\gcd(\mathtt{n}, r) = 1$. Let

$$F(s) = \frac{\psi(N)2^{\mathtt{k}-1}\Gamma(s)(2\pi r\mathtt{n})^{\mathtt{k}-s-1}}{(\mathtt{k}-2)!}$$

denote the first term of the geometric side of Theorem 1.1, and let $T(s)$ denote the other term, given in (16). Clearly the average of L-values is nonzero whenever $|T(s)| < |F(s)|$. By Proposition 4.2, this holds whenever

$$\frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}B(\sigma,\mathtt{k}-\sigma)}{N^\sigma(\mathtt{k}-2)!}2\cosh(\pi\tau/2)\zeta(\mathtt{k}-\sigma)\zeta(\sigma) < \frac{\psi(N)2^{\mathtt{k}-1}|\Gamma(s)|(2\pi r\mathtt{n})^{\mathtt{k}-\sigma-1}}{(\mathtt{k}-2)!}.$$

Using $B(\sigma,\mathtt{k}-\sigma) = \Gamma(\sigma)\Gamma(\mathtt{k}-\sigma)/(\mathtt{k}-1)!$, the above is equivalent to

$$2\cosh(\tau\pi/2) < \left(\frac{N}{2\pi r\mathtt{n}}\right)^\sigma \frac{(\mathtt{k}-1)!\,|\Gamma(s)|}{\zeta(\mathtt{k}-\sigma)\zeta(\sigma)\Gamma(\mathtt{k}-\sigma)\Gamma(\sigma)}. \tag{18}$$

Lemma 4.3. *For any $s = \sigma + i\tau$ with $\sigma > 1$,*

$$\left|\frac{\Gamma(s)}{\Gamma(\sigma)}\right| \geq e^{-\tau\arg(s-1/2)}\left(\frac{\sigma - 1/2 - \frac{\ln(2)}{\pi\sqrt{2\pi e}}}{\sigma - 1/2 + \frac{\ln(2)}{\pi\sqrt{2\pi e}}}\right).$$

Proof. This follows immediately from the following approximation due to Spouge:

$$\Gamma(s) = \sqrt{2\pi}(s-1/2)^{s-1/2}e^{-s+1/2}[1+\varepsilon(s)] \qquad (\sigma > 1), \tag{19}$$

where

$$|\varepsilon(s)| < \frac{\frac{\ln(2)}{\pi\sqrt{2\pi e}}}{\sigma - 1/2}$$

([Sp], Theorem 1.3.2). We apply this to $\Gamma(s)$ and $\Gamma(\sigma)$, and use

$$\left|(s-1/2)^{s-1/2}\right| = |s-1/2|^{\sigma-1/2}e^{-\tau\arg(s-1/2)} \geq (\sigma-1/2)^{\sigma-1/2}e^{-\tau\arg(s-1/2)}.$$

□

By the lemma and (18), we see that the average of Theorem 1.1 is nonzero whenever

$$2\cosh(\tau\pi/2)e^{\tau\arg(s-1/2)} < \frac{\left(\frac{N}{2\pi r\mathtt{n}}\right)^\sigma(\mathtt{k}-1)!}{\zeta(\mathtt{k}-\sigma)\zeta(\sigma)\Gamma(\mathtt{k}-\sigma)}\left(\frac{\sigma - 1/2 - \frac{\ln(2)}{\pi\sqrt{2\pi e}}}{\sigma - 1/2 + \frac{\ln(2)}{\pi\sqrt{2\pi e}}}\right). \tag{20}$$

We remark that since $|\arg(s-1/2)| < \pi/2$, the left-hand side is bounded above by $2\cosh(\tau\pi/2)e^{|\tau|\pi/2} = e^{\pi|\tau|} + 1$, which would simplify but weaken the inequality.

Since the left-hand side of (20) increases with $|\tau|$, we obtain the following.

Proposition 4.4. *Suppose $N > 1$, $\mathtt{k} > 3$, and $\gcd(\mathtt{n}, r) = 1$. Fix $\tau_0 > 0$, and let R denote the set of $s = \sigma + i\tau$ with $|\tau| \leq \tau_0$ and $\frac{\mathtt{k}-1}{2} \leq \sigma \leq \frac{\mathtt{k}+1}{2}$. Then the average in Theorem 1.1 is nonzero at every point of R if*

$$2\cosh(\tau_0\pi/2)e^{\tau_0\tan^{-1}(\frac{\tau_0}{\mathtt{k}/2-1})} < \frac{\left(\frac{N}{2\pi r\mathtt{n}}\right)^{\frac{\mathtt{k}\pm1}{2}}(\mathtt{k}-1)!}{\zeta(\frac{\mathtt{k}-1}{2})^2\,\Gamma(\frac{\mathtt{k}+1}{2})}\left(\frac{\frac{\mathtt{k}}{2} - 1 - \frac{\ln(2)}{\pi\sqrt{2\pi e}}}{\frac{\mathtt{k}}{2} + \frac{\ln(2)}{\pi\sqrt{2\pi e}}}\right). \tag{21}$$

Here we choose $\frac{\mathtt{k}-1}{2}$ if $N > 2\pi r\mathtt{n}$, and $\frac{\mathtt{k}+1}{2}$ otherwise.

Because the right-hand side of (21) tends to ∞ as $N + \mathtt{k} \to \infty$, Corollary 1.3 follows immediately.

4.2. **Zero-free regions.** We can use Proposition 4.4 to find zero-free regions of certain modular L-functions. The idea is to apply the proposition with $\mathtt{n} = r = 1$ when $\dim S_{\mathtt{k}}(N, \omega') = 1$, since the average then gives an actual L-value. The exponent of $\frac{N}{2\pi}$ in (21) is $\frac{\mathtt{k}+1}{2}$ unless $N \geq 7$.

Example 4.5. Let h denote the unique normalized cusp form in $S_{10}(2)$. When $\mathtt{n} = r = 1$, $N = 2$ and $\mathtt{k} = 10$, the right-hand side of (21) is 8.97346 and the inequality holds for $\tau_0 = 1.169259$. Hence the value of $\Lambda(s, h)$ is nonzero for all s in the critical strip with $|\operatorname{Im}(s)| \leq 1.169259$.

Example 4.6. Let h denote the unique normalized cusp form in $S_8(3)$. Then $\Lambda(s, h)$ is nonzero for all s in the critical strip with $|\operatorname{Im}(s)| \leq 1.119308$.

Example 4.7. Let h denote the unique normalized cusp form in $S_6(5)$. Then the value of $\Lambda(s, h)$ is nonzero for all s in the critical strip with $|\operatorname{Im}(s)| \leq 0.852608$.

Example 4.8. According to Stein's Modular Forms Database, there exists a Dirichlet character χ mod 7 (unique up to Galois conjugacy) for which $\dim S_5(7, \chi) = 1$. If h is the normalized cusp form, then $\Lambda(s, h)$ is nonzero for all s in the critical strip with $|\operatorname{Im}(s)| \leq 0.501352$.

4.3. **Approximation by partial sums.** In order to estimate the geometric side, we can truncate the last term (16). Let A, D be positive integers. Define the partial sum

$$S_{A,D} = \frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}}{N^s(\mathtt{k}-2)!\, e^{i\pi s/2}} \sum_{\substack{1\leq a\leq A,\, 1\leq d\leq D \\ \gcd(a,Nd)\mid \gcd(r,\mathtt{n})}} \Big[a^{s-\mathtt{k}} d^{-s}\, {}_1f_1(s;\mathtt{k};\tfrac{2\pi i r\mathtt{n}}{Nad}) e^{-2\pi i r \ell_0/a} + e^{i\pi s} a^{s-\mathtt{k}} d^{-s}\, {}_1f_1(s;\mathtt{k};-\tfrac{2\pi i r\mathtt{n}}{Nad}) e^{2\pi i r\ell_0/a}\Big] \frac{\gcd(a,Nd)}{\omega'(a)},$$

where as usual $\ell_0 Nd \equiv \mathtt{n} \bmod a$. The error is given by the tail of the series

$$\Delta_{A,D} = \frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1}}{N^s(\mathtt{k}-2)!\, e^{i\pi s/2}} \sum_{\substack{a,d>0 \\ a>A \text{ or } d>D \\ \gcd(a,Nd)\mid \gcd(r,\mathtt{n})}} \Big[a^{s-\mathtt{k}} d^{-s}\, {}_1f_1(s;\mathtt{k};\tfrac{2\pi i r\mathtt{n}}{Nad}) e^{-2\pi i r \ell_0/a} + e^{i\pi s} a^{s-\mathtt{k}} d^{-s}\, {}_1f_1(s;\mathtt{k};-\tfrac{2\pi i r\mathtt{n}}{Nad}) e^{2\pi i r\ell_0/a}\Big] \frac{\gcd(a,Nd)}{\omega'(a)}.$$

As in the proof of Proposition 4.2, we have the following bound for the error:

$$|\Delta_{A,D}| \leq \frac{\psi(N)(4\pi r\mathtt{n})^{\mathtt{k}-1} \gcd(r,\mathtt{n}) B(\sigma, \mathtt{k}-\sigma)}{N^\sigma(\mathtt{k}-2)!} 2\cosh(\pi\tau/2) \sum_{\substack{a>A \text{ or } d>D \\ \gcd(a,Nd)\mid \gcd(r,\mathtt{n})}} a^{-(\mathtt{k}-\sigma)} d^{-\sigma}.$$

We can estimate the error using the following easy lemma.

Lemma 4.9. *For* $s = \sigma + i\tau$,

$$\sum_{\substack{a>A \text{ or } d>D \\ \gcd(a,Nd)\mid \gcd(r,\mathtt{n})}} a^{-(\mathtt{k}-\sigma)} d^{-\sigma} \leq \zeta(\mathtt{k}-\sigma)\zeta(\sigma) - \sum_{a=1}^{A} a^{-(\mathtt{k}-\sigma)} \sum_{d=1}^{D} d^{-\sigma}.$$

4.4. **Computing the τ-function.** As a simple example, consider Ramanujan's $\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z} \in S_{12}(1)$. Writing

$$\tau(r) = \frac{\tau(r)\Lambda(6,\Delta)/\|\Delta\|^2}{\tau(1)\Lambda(6,\Delta)/\|\Delta\|^2}, \tag{22}$$

we can use the geometric side of Theorem 1.1 to compute the top and bottom. Taking $\mathtt{n} = 1$, let $F(r)$ denote the sum of the first two terms of the formula for $\frac{\tau(r)\Lambda(6,\Delta)}{\|\Delta\|^2}$. We find that

$$F(r) = \frac{2^{12}(2\pi r)^5 5!}{10!}.$$

Let $S_A(r)$ denote the A^{th} partial sum (taking $A = D$ above) of the last term of the formula. Then $\frac{\tau(r)\Lambda(6,\Delta)}{\|\Delta\|^2} \approx F(r) + S_A(r)$ with an error of

$$\leq \frac{2(4\pi r)^{11} B(6,6)}{10!} \left[\zeta(6)^2 - \left(\sum_{a=1}^{A} \frac{1}{a^6} \right)^2 \right] \tag{23}$$

by Lemma 4.9.

As an illustration, we will compute $\tau(2)$. To estimate the denominator of (22), take $r = 1$ and $A = 1$. This gives

$$\frac{\Lambda(6,\Delta)}{\|\Delta\|^2} \approx \frac{2^{12}(2\pi)^5 5!}{10!} - \frac{(4\pi)^{11}}{10!} \left[{}_1f_1(6;12;2\pi i) + {}_1f_1(6;12;-2\pi i) \right] = 1492.55$$

with an error of ≤ 8.584. So the exact value is in the interval $[1483, 1502]$.

For $r = 2$ we need to use $A = 3$ to get a reasonable approximation. We get

$$\begin{aligned} \frac{\tau(2)\Lambda(6,\Delta)}{\|\Delta\|^2} \approx & \frac{2^{12}(4\pi)^5 5!}{10!} - \frac{(8\pi)^{11}}{10!} \sum_{\substack{a,d \in \{1,2,3\} \\ \gcd(a,d)=1}} (ad)^{-6} \Big[{}_1f_1(6;12;\tfrac{4\pi i}{ad}) e^{-2\pi i r \ell_0 / a} \\ & \qquad + {}_1f_1(6;12;-\tfrac{4\pi i}{ad}) e^{2\pi i r \ell_0 / a} \Big] \\ = & -35769.72. \end{aligned}$$

By (23) the error here is

$$\leq \frac{2(8\pi)^{11} B(6,6)}{(10)!} \left(\zeta(6)^2 - (1 + \frac{1}{2^6} + \frac{1}{3^6})^2 \right) = 354.008.$$

Thus the exact value is in the interval $[-36124, -35415]$.

Taking the quotient of the estimates, we find that

$$\frac{-36124}{1483} \leq \tau(2) \leq \frac{-35415}{1502},$$

i.e.

$$-24.359 \leq \tau(2) \leq -23.578.$$

Because $\tau(2)$ is an integer, it must equal -24.

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