EQUIVARIANT LITTLEWOOD-RICHARDSON SKEW TABLEAUX

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Abstract. We give a positive equivariant Littlewood-Richardson rule also discovered independently by Molev. Our proof generalizes a proof by Stembridge of the classical Littlewood-Richardson rule. We describe a weight-preserving bijection between our indexing tableaux and trapezoid puzzles which restricts to a bijection between positive indexing tableaux and Knutson-Tao puzzles.

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1. Introduction

In [MS], Molev and Sagan introduced a rule in terms of barred tableaux for computing the structure constants $c^\nu_{\lambda,\mu}$ for products of two factorial Schur functions. Knutson and Tao [KT] realized that under a suitable specialization these are the structure constants $C^\nu_{\lambda,\mu}$ for products of two Schubert classes in the equivariant cohomology ring of the Grassmannian. Knutson and Tao [KT] also gave a new rule for computing $C^\nu_{\lambda,\mu}$, i.e., an equivariant Littlewood-Richardson rule, which is manifestly positive in the sense of Graham [Gr]. Their rule was expressed in terms of puzzles, generalizations of combinatorial objects first introduced by Knutson, Tao, and Woodward [KTW].
We describe a new nonnegative equivariant Littlewood-Richardson rule, expressed in terms of skew barred tableaux, which was also discovered independently by Molev \cite{Mo1}. By nonnegative we mean that all of the coefficients are either positive or zero; restricting to the positive coefficients then yields a positive rule. The rule includes several equivalent combinatorial tests for determining in advance which skew barred tableaux result in positive coefficients. Although our rule is similar to the Molev-Sagan rule \cite{MS}, it produces a different expression for $c_{\lambda,\mu}^{\nu}$ (see Examples 2.12 and 10.3). For example, the Molev-Sagan rule is not manifestly positive. We remark that unlike the Knutson-Tao rule, whose positivity is obvious from its statement, the nonnegativity and positivity of our rule require proof.

In this paper, we compute the structure constants $c_{\lambda,\mu}^{\nu}$ (as do both \cite{MS} and \cite{Mo1}) and then determine the structure constants $C_{\lambda,\mu}^{\nu}$ by specialization (as does \cite{Mo1}). Our strategy for proving our rule for the structure constants $c_{\lambda,\mu}^{\nu}$ is to generalize a concise proof by Stembridge \cite{St} of a standard Littlewood-Richardson rule from Schur functions to factorial Schur functions; a similar method is used by \cite{Kr2}. This method in fact yields a more general result, namely, a generalization of Zelevinsky’s extension of the Littlewood-Richardson rule \cite{Z}.

We illustrate a weight-preserving bijection $\Phi$ between skew barred tableaux and trapezoid puzzles, combinatorial objects generalizing Knutson-Tao puzzles. The bijection $\Phi$ restricts to a bijection between the skew barred tableaux indexing positive coefficients and Knutson-Tao puzzles. This gives a new proof of Knutson and Tao’s equivariant Littlewood-Richardson rule, and also demonstrates that our positive rule is really the same rule as Knutson and Tao’s, just expressed in terms of different combinatorial indexing sets. Our representation of the bijections generalizes Tao’s ‘proof without words’ \cite{V}, Figure 11, which gives a bijection between tableaux and puzzles in the nonequivariant setting.

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2. Statement of results

Let $\mathbb{N}$ denote the set of nonnegative integers, and let $n \geq d$ be fixed positive integers. For $m \in \mathbb{N}$, define $m' := d + 1 - m$. For $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{N}^d$, define $|\lambda| = \lambda_1 + \cdots + \lambda_d$. Denote by $P_d$ the set of all such $\lambda$ which are partitions, i.e., such that $\lambda_1 \geq \cdots \geq \lambda_d$, and by $P_{d,n}$ the set of all such partitions for which $\lambda_1 \leq n - d$.

Let $\lambda = (\lambda_1, \ldots, \lambda_d), \mu = (\mu_1, \ldots, \mu_d), \rho = (d - 1, d - 2, \ldots, 0)$, and $1 = (1, \ldots, 1)$ be fixed elements of $P_d$. For any sequence $i = i_1, i_2, \ldots, i_t$, $i_j \in \{1, \ldots, d\}$, define the content of $i$ to be $\omega(i) = (\xi_1, \ldots, \xi_d) \in \mathbb{N}^d$, where $\xi_k$ is the number of $k$’s in the sequence.

2.1. Defining the structure constants $c_{\lambda,\mu}^{\nu}$ for products of factorial Schur functions. A reverse Young diagram is a right and bottom justified array of boxes. To $\mu$ we associate the reverse Young diagram whose bottom row has length $\mu_1$, next to bottom row has length $\mu_2$, etc. We also denote this reverse Young diagram by $\mu$. The columns of a reverse Young diagram are numbered from right to left and the rows from bottom to top.
A reverse tableau of shape $\mu$ is a filling of each box of $\mu$ with an integer in \{1, \ldots, d\} in such a way that the entries weakly increase along any row from left to right and strictly increase along any column from top to bottom. Let $R(\mu)$ denote the set of all reverse tableaux of shape $\mu$. Let $x_1, \ldots, x_d$ be a finite set of variables and $(y_i)_{i \in \mathbb{N} > 0}$ an infinite set of variables. For $R \in R(\mu)$, define
\[(x \mid y)^R = \prod_{a \in R} (x_a - y_{a'} + c(a) - r(a)),\]
where for entry $a \in R$, $c(a)$ and $r(a)$ are the column and row numbers of $a$ respectively. The **factorial Schur polynomial** is defined to be
\[s_\mu(x \mid y) = \sum_{R \in R(\mu)} (x \mid y)^R.\]

The factorial Schur function is usually expressed in the literature in terms of Young tableaux rather than reverse tableaux; we show the equivalence of the two formulations below. Factorial Schur functions are special cases of Lascoux and Schützenberger’s double Schubert polynomials [LS1, LS2]. The factorial Schur function $s_\mu(x \mid y)$, under a certain specialization of the $y$ variables, was first defined by Biedenharn and Louck [BL1, BL2] and further studied by Chen and Louck [CL]. The more general factorial Schur function $s_\mu(x \mid y)$ is due to Macdonald [Ma2] and Goulden and Greene [GG]. Factorial Schur functions appear in the study of the center of the enveloping algebra $U(gl_n)$ (see Okounkov [OK], Okounkov and Olshanski [OO], Nazarov [Na], Molev [Mo2, Mo1], and Molev and Sagan [MS]).

We check that our definition of factorial Schur function agrees with the version appearing in [Ma2] and [MS], which is expressed in terms of Young tableaux with entries in \{1, \ldots, d\}. Replacing each entry $a$ in a reverse tableau $R$ by $a'$ and rotating the resulting tableau by 180 degrees, one obtains a Young tableau $T$. This operation defines a bijection between reverse tableaux of shape $\mu$ and Young tableaux of shape $\mu$ with entries in \{1, \ldots, d\}. The polynomials $(x \mid y)^T$, as defined in [MS], and $(x \mid y)^R$, as defined above, are related by a fixed permutation on the indices of the $x_i$’s, namely the involution $i \mapsto i'$. Thus the equivalence of the two definitions follows from the fact that factorial Schur functions are symmetric in the $x_i$’s. (Corollary 5.4 also establishes the equivalence of the two definitions.)

From the definition of $s_\mu(x \mid y)$, one sees that
\[s_\mu(x \mid y) = s_\mu(x) + \text{terms of lower degree in the } x_i \text{'s},\]
where $s_\mu(x)$ is the Schur function in $x_1, \ldots, x_d$. Since the Schur functions form a $\mathbb{Z}$-basis for $\mathbb{Z}[x_1, \ldots, x_d]^{S_d}$, the factorial Schur functions must form a $\mathbb{Z}[y]$-basis for...
\[ \mathbb{Z}[[x_1, \ldots, x_d]]^{S_d}. \quad \text{Thus} \]

\[ s_\lambda(x \mid y)s_\mu(x \mid y) = \sum c^\lambda_{\lambda, \mu} s_\nu(x \mid y), \]

for some polynomials \( c^\lambda_{\lambda, \mu} \in \mathbb{Z}[y] \), where the summation is over all \( \nu \in \mathcal{P}_d \).

We write \( \mu \subseteq \nu \) if \( \mu_i \leq \nu_i, \quad i = 1, \ldots, d \). Using a vanishing theorem of Okounkov [Ok], Molev and Sagan prove [MS, Theorem 3.1]

\[ \mu \not\subseteq \nu \implies c^\nu_{\lambda, \mu} = 0. \]

From the definition one sees that \( s_\mu(x \mid y) \) is a homogeneous polynomial of degree \( |\mu| \). Therefore if \( c^\lambda_{\lambda, \mu} \neq 0 \), then \( |\lambda| + |\mu| - |\nu| = \deg(c^\lambda_{\lambda, \mu}) \). If \( |\lambda| + |\mu| - |\nu| = 0 \), then \( c^\lambda_{\lambda, \mu} \in \mathbb{Z} \) is the classical Littlewood-Richardson coefficient (see [F1], [LR], [Sa]).

2.2. Computing the structure constants \( c^\nu_{\lambda, \mu} \). The skew diagram \( \lambda * \mu \) is obtained by placing the Young diagram \( \lambda \) above and to the right of the reverse Young diagram \( \mu \) (see Figure 2). A skew barred tableau of shape \( \lambda * \mu \) is a filling of each box of the subdiagram \( \lambda \) of \( \lambda * \mu \) with an element of \( \{1, \ldots, d\} \) and each box of the subdiagram \( \mu \) of \( \lambda * \mu \) with an element of \( \{1, \ldots, d\} \cup \{\overline{1}, \ldots, \overline{d}\} \), in such a way that the values of the entries, without regard to whether or not they are barred, weakly increase along any row from left to right and strictly increase along any column from top to bottom. The unbarred column word of \( L \), denoted by \( L^u \), is the sequence of unbarred entries of \( L \) beginning at the top of the rightmost column, reading down, then moving to the top of the next to rightmost column and reading down, etc. (the barred entries are just skipped over in this process).

We say that the unbarred column word of \( L \) is Yamanouchi if, when one writes down the word and stops at any point, one will have written at least as many ones as twos, at least as many twos as threes, \ldots , at least as many \((d - 1)\)'s as \(d\)'s. The unbarred content of \( L \) is \( \omega(L^u) \), the content of the unbarred column word.

Definition 2.3. An equivariant Littlewood-Richardson skew tableau is a skew barred tableau whose unbarred column word is Yamanouchi. We denote the set of all equivariant Littlewood-Richardson skew tableaux of shape \( \lambda * \mu \) and unbarred content \( \nu \) by \( \mathcal{LR}_{\lambda, \mu}^\nu \).

We remark that this definition forces the \( i \)-th row of \( \lambda \) to consist of \( \lambda_i \) unbarred \( i \)'s.

For \( L \) a skew barred tableau and \( a \in L \), denote by \( L_{<a}^u \) the portion of the unbarred column word of \( L \) which comes before reaching \( a \) when reading entries from \( L \). Define

\[ c_L = \prod_{a \in L \text{ barred}} \left( y_{|a|'} + \omega(L_{<a}^u)|a| - y_{|a|'} + c(a) - r(a) \right), \]

where \( r(a) \) and \( c(a) \) are the row and column numbers of \( a \) considered as entries of \( \mu \) (see Figure 1), and \( |a|' = d + 1 - |a| \) (we use the absolute value symbol, \( |a| \), to stress that we are interested in the integer value of the barred entry \( a \)). As usual, the trivial product is defined to be 1. The main result of this paper, which is proven in Sections 5, 7 and 8, is the following.

Theorem 2.4. \[ c^\nu_{\lambda, \mu} = \sum_{L \in \mathcal{LR}_{\lambda, \mu}^\nu} c_L. \]
Figure 2. An equivariant Littlewood-Richardson skew tableau of shape $\lambda \ast \mu$ and unbarred content $\mu$, where $\lambda = (2, 1, 1)$, $\mu = (4, 3, 1)$, and $\nu = (3, 3, 2, 1)$. The unbarred column word, $1, 1, 2, 3, 2, 4, 3, 1, 2$, is Yamanouchi, as required.

Example 2.5. Let $L$ be the equivariant Littlewood-Richardson skew tableau of Figure 2.

Suppose that $d = 4$. Consider the entry $a = \mathbf{T}$ in row 2, column 2 of $\mu$. We have $L^u_{<a} = 1, 1, 2, 3, 2, 4$, so $\omega(L^u_{<a}) = (2, 2, 1, 1)$. Thus $|a|' + \omega(L^u_{<a})|a| = (d+1-(1))+(2, 2, 1, 1) = 4+2 = 6$. Also, $|a|'+c(a)-r(a) = (d+1-(1))+2-2 = 4$. Therefore the contribution of this entry to $c_L$ is $y_6 - y_4$.

Similarly, one computes the contribution of the entry $2$ in row 1, column 3 to be $y_5 - y_5$ and the contribution of the entry $3$ in row 2, column 1 to be $y_3 - y_1$. Therefore $c_L = (y_5 - y_5)(y_6 - y_4)(y_3 - y_1)$, which equals 0.

2.6. Nonnegativity and positivity. If $L \in LR^\nu_{\lambda, \mu}$, then write $c_L > 0$ if each factor in (3) is of the form $y_i - y_j$ with $i > j$. We write $c_L \geq 0$ if either $c_L > 0$ or $c_L = 0$. Note that the definition of $c_L \geq 0$ does not preclude the possibility that some factor of (3) is of the form $y_i - y_j$ with $i < j$; however, in this case some other factor must be of the form $y_i - y_i$ with $i = j$, thus forcing $c_L = 0$. The following proposition is proven in Section 4.

Proposition 2.7. If $L \in LR^\nu_{\lambda, \mu}$, then $c_L \geq 0$.

By Theorem 2.4 if $c_L = 0$ for all $L \in LR^\nu_{\lambda, \mu}$, then $c^\nu_{\lambda, \mu} = 0$. Proposition 2.7 implies that the converse is true as well:

Corollary 2.8. If $c^\nu_{\lambda, \mu} = 0$, then $c_L = 0$ for all $L \in LR^\nu_{\lambda, \mu}$.

Proof. Denote by $c_L|_{y_i = i}$ the integer obtained by specializing each $y_i$ to $i$ in $c_L$. By Proposition 2.7, $c_L|_{y_i = i} \geq 0$, and $c_L|_{y_i = i} = 0$ if and only if $c_L = 0$. Now assume that $c^\nu_{\lambda, \mu} = 0$. By Theorem 2.4, $\sum_{L \in LR^\nu_{\lambda, \mu}} c_L = c^\nu_{\lambda, \mu} = 0$. Thus $\sum_{L \in LR^\nu_{\lambda, \mu}} (c_L|_{y_i = i}) = 0$, which implies that $c_L|_{y_i = i} = 0$ for all $L \in LR^\nu_{\lambda, \mu}$, which in turn implies $c_L = 0$ for all $L \in LR^\nu_{\lambda, \mu}$. \qed
Remark 2.9. In Example 2.5, \(\mu \not\subseteq \nu\). Thus by (2), \(c_{\lambda, \mu}^\nu = 0\). Hence Corollary 2.8 verifies that \(c_L = 0\).

Let \(LR_{\lambda, \mu}^\nu\) be the set of \(L \in LR_{\lambda, \mu}^\nu\) for which \(c_L > 0\). By Proposition 2.7, we can restrict the summation in Theorem 2.4 to such \(L\):

**Corollary 2.10.** \(c_{\lambda, \mu}^\nu = \sum_{L \in LR_{\lambda, \mu}^\nu} c_L\).

One could, of course, use (3), the definition of \(c_L\), to distinguish between \(c_L > 0\) and \(c_L = 0\): \(c_L > 0\) if and only if \(\omega(L^u_{\leq a})_{[a]} > c(a) - r(a)\) for all barred \(a \in L\). The following proposition gives a number of other tests for more efficiently making this determination.

**Proposition 2.11.** If \(L \in LR_{\lambda, \mu}^\nu\), then the following are equivalent:

1. \(c_L > 0\).
2. \(\omega(L^u_{\leq a})_{[a]} > c(a) - r(a)\) for all barred \(a \in L\).
3. \(\omega(L^u_{\leq a})_{[a]} > c(a) - r(a)\) for all barred \(a \in L\) with \(r(a) = 1\).
4. \(\omega(L^u_{< a})_{[a]} \geq c(a)\) for all barred \(a \in L\).
5. \(\omega(L^u_{< a})_{[a]} \geq c(a)\) for all barred \(a \in L\) with \(r(a) = 1\).

If \(L \in LR_{\lambda, \mu}^\nu\) satisfies any of these equivalent conditions, then we say that \(L\) is **positive**. It is obvious that \(4 \implies 2 \implies 3 \iff 5\). In Section 4 we prove \(3 \implies 4\). Condition 3 states that it suffices to check barred entries on the bottom row of \(L\) for positivity. Condition 4 has the following interpretation: for any barred entry \(a \in L\), the corresponding factor \(y_i - y_j\) in \(c_L\) satisfies \(i - j \geq r(a)\) (which of course implies \(i - j > 0\), the condition required for positivity).

**Example 2.12.** Let \(d = 3\), \(\lambda = (1,1)\), \(\mu = (3,2)\), and \(\nu = (3,2,1)\). We list all \(L \in LR_{\lambda, \mu}^\nu\), and for each \(L\) we give \(c_L\):

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 3 \\
\hline
1 & 1 & 1 \\
2 & 2 & 3 \\
\end{array}
\]

\(c_L = y_6 - y_5\)

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 3 \\
\hline
1 & 1 & 1 \\
2 & 2 & 3 \\
\end{array}
\]

\(c_L = y_6 - y_2\)

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 3 \\
\hline
1 & 1 & 1 \\
2 & 2 & 3 \\
\end{array}
\]

\(c_L = y_5 - y_3\)

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 3 \\
\hline
1 & 1 & 1 \\
2 & 2 & 3 \\
\end{array}
\]

\(c_L = y_6 - y_1\)

Note that if \(L\) has an unbarred 2 in the upper right box of \(\mu\), then the unbarred column word of \(L\) is not Yamanouchi, and if \(L\) has two unbarred 1’s on the top row of \(\mu\) and is not the leftmost diagram, then \(c_L = 0\); thus we do not include such \(L\) among \(LR_{\lambda, \mu}^\nu\). By Corollary 2.10, \(c_{\lambda, \mu}^\nu = (y_6 - y_5) + (y_5 - y_3) + (y_4 - y_2) + (y_3 - y_1)\).

We list all \(L \in LR_{\mu, \lambda}^\nu\), and for each \(L\) we give \(c_L\):

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 3 \\
\hline
1 & 1 & 1 \\
2 & 2 & 3 \\
\end{array}
\]

\(c_L = y_6 - y_2\)

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 3 \\
\hline
1 & 1 & 1 \\
2 & 2 & 3 \\
\end{array}
\]

\(c_L = y_4 - y_1\)
By Corollary 2.10 \( c^\nu_{\lambda,\mu} = (y_6 - y_2) + (y_4 - y_1) \). We see that \( c^\nu_{\lambda,\mu} = c^\nu_{\lambda,\mu} \). This is a general fact ensured by (1); however, it is not apparent from the statement of Corollary 2.10.

See also Example 10.3, where these same coefficients \( c^\nu_{\lambda,\mu} \) are computed using the Molev-Sagan rule.

**Example 2.13.** For cases where \( \mu = \nu \), a formula for \( c^\nu_{\lambda,\mu} \) which produces a different positive expression than Corollary 2.10 appears in [Bi], [IN], and [Kr1]. For example, using this formula, for \( d = 3 \), \( \lambda = (2,1) \), and \( \mu = \nu = (3,3,1) \), one computes:

\[
c^\nu_{\lambda,\mu} = (y_6 - y_1)(y_6 - y_3)(y_5 - y_1) + (y_6 - y_1)(y_5 - y_4)(y_5 - y_1).
\]

Using Corollary 2.10:

\[
c^\nu_{\lambda,\mu} = (y_5 - y_3)(y_5 - y_3)(y_5 - y_1) + (y_6 - y_4)(y_5 - y_1)(y_5 - y_1)
+ (y_6 - y_4)(y_5 - y_3)(y_5 - y_1) + (y_6 - y_4)(y_5 - y_3)(y_5 - y_3)
+ (y_6 - y_4)(y_5 - y_4)(y_5 - y_1) + (y_6 - y_4)(y_5 - y_4)(y_5 - y_3).
\]

\[
c^\nu_{\mu,\lambda} = (y_6 - y_4)(y_6 - y_2)(y_5 - y_1) + (y_5 - y_3)(y_6 - y_2)(y_5 - y_1)
+ (y_6 - y_4)(y_6 - y_2)(y_5 - y_1) + (y_5 - y_3)(y_5 - y_2)(y_5 - y_1)
+ (y_6 - y_4)(y_5 - y_1)(y_2 - y_1) + (y_5 - y_3)(y_5 - y_1)(y_2 - y_1).
\]

These three polynomials are, of course, equal.

For \( L \in \mathcal{LR}^\nu_{\lambda,\mu} \), \( |\lambda| + |\mu| - |\nu| = \#(\text{entries of } L) - \#(\text{unbarred entries of } L) = \#(\text{barred entries of } L) \) which equals \( \deg(c^\nu_{\lambda,\mu}) \) if \( c^\nu_{\lambda,\mu} \neq 0 \). If \( |\lambda| + |\mu| - |\nu| = 0 \), then \( L \) has no barred entries, and Theorem 2.14 reduces to a version of the classical Littlewood-Richardson rule (see [FL], [LR], [Sa]).

### 2.14. Defining the structure constants \( C^\nu_{\lambda,\mu} \) for products of two Schubert classes in \( H^*_T(Gr_{d,n}) \).

The Grassmannian \( Gr_{d,n} \) is the set of \( d \)-dimensional complex subspaces of \( \mathbb{C}^n \). Let \( \{e_1, \ldots, e_n\} \) be the standard basis for \( \mathbb{C}^n \). Consider the opposite standard flag, whose \( i \)-th space is \( \text{Span}(e_{n,i+1}, \ldots, e_n) \). For \( \lambda \in \mathcal{P}_d \), the (opposite) Schubert variety \( X_\lambda \) of \( Gr_{d,n} \) is defined by the incident relations:

\[
X_\lambda = \{ V \in Gr_{d,n} \mid \dim(V \cap F_i) \geq \dim(\mathcal{C}_\lambda \cap F_i), \; i = 1, \ldots, n \},
\]

where \( \mathcal{C}_\lambda = \text{Span}(e_{\lambda_1}, \ldots, e_{\lambda_1}) \). The Schubert variety \( X_\lambda \) is invariant under the action of the group \( T = (\mathbb{C}^*)^n \) on \( Gr_{d,n} \). Thus it determines a class \( S_\lambda \) in the equivariant cohomology ring \( H^*_T(Gr_{d,n}) \).

Let \( V = Gr_{d,n} \times \mathbb{C}^n \) be the trivial vector bundle on \( Gr_{d,n} \), with diagonal \( T \)-action, where \( T \) acts naturally on \( Gr_{d,n} \) and on \( \mathbb{C}^n \) (thus \( V \) is not equivariantly trivial). Let \( Y_1, \ldots, Y_n \) be the equivariant Chern roots of \( V^* \). Then \( Y_1, \ldots, Y_n \in H^*_T(Gr_{d,n}) \) are algebraically independent, and \( H^*_T(Gr_{d,n}) \) is a free \( \mathbb{Z}[Y_1, \ldots, Y_n] \)-module, with the Schubert classes forming a \( \mathbb{Z}[Y_1, \ldots, Y_n] \)-basis. Thus for \( \lambda, \mu \in \mathcal{P}_d \),

\[
S_\lambda S_\mu = \sum_{\nu \in \mathcal{P}_d} C^\nu_{\lambda,\mu} S_\nu, \text{ for some } C^\nu_{\lambda,\mu} \in \mathbb{Z}[Y_1, \ldots, Y_n].
\]
Let $S = \{(w, v) \in V \mid v \in w\}$ be the tautological vector bundle on $Gr_{d,n}$, a $T$-invariant sub-bundle of $V$, and let $X_1, \ldots, X_d$ be the equivariant Chern roots of $S$. We have (see [F2], [KT], [M])

**Proposition 2.15.** For $\lambda \in \mathcal{P}_d$, $S_\lambda = s_\lambda(X_1, \ldots, X_d, -Y_n, \ldots, -Y_1, 0, 0, \ldots)$. Thus by specializing (1), we can determine the structure constants $C_{\lambda, \mu}^\nu$.

**Corollary 2.16.** For $\lambda, \mu, \nu \in \mathcal{P}_{d,n}$, $C_{\lambda, \mu}^\nu = c_{\lambda, \mu}^\nu(-Y_n, \ldots, -Y_1, 0, 0, \ldots)$.

2.17. **Computing the structure constants** $C_{\lambda, \mu}^\nu$. Let $\lambda, \mu, \nu \in \mathcal{P}_{d,n}$. By Corollary 2.16, $C_{\lambda, \mu}^\nu$ can be computed using the formula for $c_{\lambda, \mu}^\nu$. Letting $L \in LR_{\lambda, \mu}^\nu$, we have:

$$\text{(4)} \quad \text{Both subscripts in equation } (3) \text{ for } c_L \text{ lie between 1 and } n.$$

Indeed,

(i) $|a| + \omega(L_{\lambda, \mu}^\nu)_{|a|} = d + 1 - |a| + \omega(L_{\lambda, \mu}^\nu)_{|a|} \leq d + \omega(L_{\lambda, \mu}^\nu)_{|a|} \leq d + \mu_1 \leq n$.

The last two inequalities are due to $\omega(L^\nu) = \nu$ and $\nu \in \mathcal{P}_{d,n}$ respectively.

(ii) $|a| + |c(a) - r(a)| = d + 1 - |a| + c(a) - r(a) < d + c(a) \leq d + \mu_1 \leq n$. The last two inequalities are due to the facts that the reverse Young diagram $\mu$ has $\mu_1$ columns and $\mu \in \mathcal{P}_{d,n}$ respectively.

(iii) One checks that $|a| \leq d + 1 - r(a)$. Thus $|a| + c(a) - r(a) = d + 1 - |a| + c(a) - r(a) \geq c(a) \geq 1$.

Define

$$\text{(5)} \quad C_L = c_L(-Y_n, \ldots, -Y_1, 0, 0, \ldots)$$

$$\text{(6)} \quad = \prod_{a \in L \text{ barred}} \left( Y_{(n-d)+|a|-|c(a)-r(a)|} - Y_{(n-d)+|a|-\omega(L_{\lambda, \mu}^\nu)_{|a|}} \right).$$

We write $C_L > 0$ if each factor in (6) is of the form $Y_i - Y_j$ with $i > j$, and we write $C_L \geq 0$ if either $C_L > 0$ or $C_L = 0$. By (4), (5), and the algebraic independence of the $Y_i$’s, $c_L = 0 \iff C_L = 0$, and $c_L > 0 \iff C_L > 0$. Thus Propositions 2.7 and 2.11 imply

**Corollary 2.18.** $C_L \geq 0$, and $C_L > 0 \iff L$ satisfies any of the equivalent conditions of Proposition 2.11.

By Theorem 2.4 Corollary 2.16 and Corollary 2.18 we have

**Corollary 2.19.** $C_{\lambda, \mu}^\nu = \sum_{L \in LR_{\lambda, \mu}^\nu} C_L = \sum_{L \in LR_{\lambda, \mu}^\nu}^+ C_L$.

**Example 2.20.** We continue Example 2.12. For $n > 6$, $\lambda, \mu \in \mathcal{P}_{d,n}$. Thus for $\nu \in \mathcal{P}_{d,n}$, $C_{\lambda, \mu}^\nu = (Y_{n+1-2} - Y_{n+1-6}) + (Y_{n+1-1} - Y_{n+1-4}) = (Y_{n-1} - Y_{n-5}) + (Y_n - Y_{n-3})$.

2.21. **Equivalence of Molev’s results.** Our equivariant Littlewood-Richardson skew tableaux are in bijection with Molev’s indexing tableaux [M1]. To determine the tableau in [M1] which corresponds to our $L \in LR_{\lambda, \mu}^\nu$, replace all barred entries and vice versa, and then rotate the resulting object by 180 degrees. If one makes this modification, then Corollary 2.10 is equivalent to [M1] Theorem 2.1] after accounting for the relationship between double Schur functions and factorial Schur functions (see [M1] (1.9)), and Corollary 2.19 is identical to [M1] Corollary 3.1].
In our notation, Molev’s positivity criterion states that for $L \in L^v_{\lambda, \mu}$, $c_L > 0$ if and only if

$$\omega(L^a)^{\prime} \geq |a|$$

for all $a \in L$ with $r(a) = 1$, where $\omega(L^a)^{\prime}$ is the conjugate partition to $\omega(L^a)$ (in this case Molev calls $L|_{\mu, \nu}$-bounded). One can re-express (7) as follows:

$$\omega(L^a) |_{a} \geq c(a)$$

for all $a \in L$ with $r(a) = 1$.

It is not difficult to see that this condition is equivalent to Proposition 2.11.5.

Related and more general results have been achieved in several directions. Robinson [R] has given a Pieri rule in the equivariant cohomology of the flag variety. McNamara [Mc] introduced factorial Grothendieck polynomials, generalizations of factorial Schur functions, and has given a rule for computing the structure constants for various of their products.

This paper is organized as follows. In Section 3 we introduce various types of tableaux which will appear throughout the paper. In Section 4 we prove Propositions 2.7 and 2.11, the nonnegativity property and positivity criteria of $c_L$. In Section 5 we outline the main steps in our proof of Theorem 2.4 whose two difficult technical lemmas are proved in Sections 7 and 8. In Section 6 we define a set of involutions required for the proofs of these two lemmas. In Section 9 we describe a weight preserving bijection between equivariant Littlewood-Richardson skew tableaux and trapezoid puzzles, which restricts to a bijection between positive equivariant Littlewood-Richardson skew tableaux and Knutson-Tao puzzles. In Section 10 we recall the Molev-Sagan rule.

3. Several Types of Tableaux

In this section we collect the definitions of the several types of tableaux which we will encounter in the remainder of the paper: reverse barred tableaux, reverse barred subtableaux, and reverse hatted tableaux. The latter two are refinements of the first.

A reverse barred tableau of shape $\mu$ is a skew barred tableau of shape $\emptyset \ast \mu$; alternatively, it can be defined as a reverse Young diagram of shape $\mu$, each of whose boxes is filled with either an integer $k$ or a barred integer $\overline{k}$, $k \in \{1, \ldots, d\}$, in such a way that the values of the entries, without regard to whether or not they are barred, weakly increase along any row from left to right and strictly increase along any column from top to bottom. We denote the set of all reverse barred tableaux of shape $\mu$ by $B(\mu)$. If $B \in B(\mu)$, then define $\lambda \ast B$ to be the skew barred tableau obtained by placing the Young tableau whose $i$-th row consists of $\lambda_i$'s above and to the right of $B$. Then $B \mapsto \lambda \ast B$ defines a bijection from $\{B \in B(\mu) \mid (\lambda \ast B)^u$ is Yamanouchi$\}$ to the equivariant Littlewood-Richardson skew tableaux of shape $\lambda \ast \mu$, whose inverse map is $L \mapsto L|_{\mu}$. Any $a \in B$ also corresponds to an entry $a \in \lambda \ast B$. Define $B^u$ and $B^u_{<a}$ to be $(\emptyset \ast B)^u$ and $(\emptyset \ast B)_{<a}$ respectively.

A reverse barred subtableaux of shape $\mu$ is a reverse Young diagram $\mu$, each of whose boxes contains either an integer $k$, a barred integer $\overline{k}$, or is empty, where $k \in \{1, \ldots, d\}$. A reverse subtableau of shape $\mu$ is a reverse barred tableau of shape $\mu$ which has no barred entries. We do not define any notion of row semistrictness or column strictness for such objects, as no such conditions will be required for our purposes. Denote the set of all reverse subtableaux and reverse
barred subtableaux of shape $\mu$ by $\mathcal{R}_{\text{sub}}(\mu)$ and $\mathcal{B}_{\text{sub}}(\mu)$ respectively. We have the following containments:

$$\mathcal{R}_{\text{sub}}(\mu) \subset \mathcal{B}_{\text{sub}}(\mu) \cup \mathcal{R}(\mu) \subset \mathcal{B}(\mu).$$

For $B \in \mathcal{B}_{\text{sub}}(\mu)$ and $a \in B$, define $B^u$ and $B^u_{\leq a}$ just as for elements of $\mathcal{B}(\mu)$, assuming that when reading the unbarred column word of $B$, both barred entries and empty boxes are skipped over. If $B \in \mathcal{B}_{\text{sub}}(\mu)$, then define $\tilde{B} \in \mathcal{R}_{\text{sub}}(\mu)$ to be the reverse subtableau obtained by removing all bars from entries of $B$, i.e., replacing each barred entry of $B$ by an unbarred entry of the same value.

A reverse hatted tableau of shape $\mu$ is a reverse Young diagram $\mu$, each of whose boxes is filled with either a(n) (un-hatted) integer $k$, a left hatted integer $\hat{k}$, or a right hatted integer $\hat{k}$, $k \in \{1, \ldots, d\}$, such that the values of the entries, without regard to whether or not they are hatted, weakly increase along any row from left to right and strictly increase along any column from top to bottom. Denote the set of all reverse hatted tableaux of shape $\mu$ by $\mathcal{H}(\mu)$. If $H$ is a reverse hatted tableau, then define $\overline{H}$ to be the reverse barred tableau produced by replacing all hats (right and left) by bars. Hence for a reverse barred tableau $B$ with $m$ barred entries, there are $2^m$ reverse hatted tableaux $H$ such that $\overline{H} = B$ (since each $k$ of $B$ can be replaced by either $\hat{k}$ or $\hat{k}$). For $a \in H$, define $H^u$ and $H^u_{\leq a}$ to be $\overline{H^u}$ and $\overline{H^u}_{\leq a}$ respectively. Define $H^l$ (resp. $H^r$) to be the set of left-hatted (resp. right-hatted) entries of $H$.

We next give two different ways to generalize the polynomial $c_L$ defined in Section 2. Let $\xi \in \mathbb{N}^d$. For $B \in \mathcal{B}_{\text{sub}}(\mu)$, define

$$c_{\xi,B} = \prod_{\substack{a \in B \text{ a barred}}} (y_{\xi,B}(a) - y_{f_B}(a)),$$

where $c_{\xi,B}(a) := (\xi + \omega(B^u_{\leq a}))|a|$ and $f_B(a) := |a|' + c(a) - r(a)$, $a \in B$. For $H \in \mathcal{H}(\mu)$, define

$$d_{\xi,H} = \prod_{a \in H^l} y_{\xi,H}(a) \prod_{a \in H^r} (-y_{f_H}(a)),$$

where $c_{\xi,H}(a) := (\xi + \omega(H^u_{\leq a}))|a|$ and $f_H(a) := |a|' + c(a) - r(a)$, $a \in H$. In both $\mathcal{B}$ and $\mathcal{R}$, the empty product is defined to equal 1.

Let $B \in \mathcal{B}(\mu)$. By definition,

$$c_{\lambda+B} = c_{\lambda+\rho+1,B}.$$

In addition, the equation

$$c_{\xi,B} = \sum_{H \in \mathcal{H}(\mu) \atop \overline{H} = B} d_{\xi,H}$$

expresses $c_{\xi,B}$ by expanding (8) in terms of monomials in the $y_i$’s. Combining (11) and (12), we have

$$c_{\lambda+B} = \sum_{H \in \mathcal{H}(\mu) \atop \overline{H} = B} d_{\lambda+\rho+1,H}.$$
If $R \in \mathcal{R}_{\text{sub}}(\mu)$, then define $(x \mid y)^R = \prod_{a \in R} (x_a - y_{f_R(a)})$. This definition is consistent with the definition of $(x \mid y)^R$, $R \in \mathcal{R}(\mu)$, given in Section 2.

4. PROOFS OF NONNEGATIVITY PROPERTY AND POSITIVITY CRITERIA

Let $L \in \mathcal{L}R^u_{\lambda, \mu}$, and let $B = L|_{\mu}$. For $a \in B$, which we also view as an entry of $L$, define $L_u^a$ to be $L^u_{<a}$ if $a$ is barred, or $L^u_{<a}$ appended with $a$ if $a$ is not barred. Define

$$\Delta(a) = \omega(L^u_{<a}|_a) - c(a) + r(a).$$

If $a$ is barred, then $\omega(L^u_{<a}) = \omega(L^u_{\leq a})$; hence $\Delta(a)$ gives the difference between the two indices $i - j$ of the factor $y_i - y_j$ corresponding to $a$ in [3]. Therefore Propositions 2.7 and 2.11 are equivalent to the following two lemmas respectively.

**Lemma 4.1.** If $\Delta(a) < 0$ for some barred $a \in B$, then $\Delta(b) = 0$ for some barred $b \in B$.

**Lemma 4.2.** The following are equivalent:

(i) $\Delta(a) > 0$ for all barred $a \in B$.

(ii) $\Delta(a) > 0$ for all barred $a \in B$ with $r(a) = 1$.

(iii) $\Delta(a) \geq r(a)$ for all barred $a \in B$.

Before proving these two lemmas, we first establish some properties of $\Delta$.

**Lemma 4.3.** The function $\Delta : B \to \mathbb{Z}$ satisfies the following properties:

(i) If $a \in B$ and $c(a) = 1$, then $\Delta(a) \geq 0$, with equality implying that $a$ is barred.

(ii) If one moves left by one box, then $\Delta$ can decrease by at most one. If it does decrease by one, then the left box must be barred.

(iii) If $\Delta(a) \leq 0$ for some $a \in B$, then $\Delta(b) = 0$ for some barred $b \in B$ on the same row as $a$.

(iv) The function $a \mapsto \Delta(a) - r(a)$ is weakly decreasing as one moves down along any column.

**Proof.** (i) Since $r(a) \geq 1$, $\Delta(a) \geq 0$. If $\Delta(a) = 0$, then $r(a) = 1$ and $\omega(L^u_{\leq a}|_a) = 0$. The latter requirement implies that $a$ is barred.

(ii) If entry $m$ lies one box left of $a$, then $-c(m) = -c(a) - 1$, $r(m) = r(a)$, and $\omega(L^u_{\leq m}|_m) \geq \omega(L^u_{\leq a}|_m) \geq \omega(L^u_{\leq a}|_a)$, where the first inequality is an equality if and only if $m$ is barred. The second inequality is a consequence of the fact that the unbarred column word of $L$ is Yamanouchi.

(iii) Let $m$ be the rightmost entry in the same row as $a$. If $\Delta(m) = 0$, then by (i), $m$ is barred, so letting $b = m$ we are done. Otherwise $\Delta(m) > 0$. By (ii), as one moves left from $m$ to $a$ along the row the two entries lie on, one must encounter some barred $b$ for which $\Delta(b) = 0$.

(iv) If entry $m$ lies one box below $a$, then $\omega(L^u_{\leq a}|_a) = \omega(L^u_{\leq m}|_a) \geq \omega(L^u_{\leq m}|_m)$, since the unbarred column word of $L$ is Yamanouchi. \hfill \Box

**Proof of Lemmas 4.1 and 4.2.** Lemma 4.1 is a special case of Lemma 4.3(iii). In Lemma 4.2, implications (iii) $\implies$ (i) $\implies$ (ii) are clear. We prove (ii) $\implies$ (iii). Suppose that $a \in B$ is a barred entry such that $\Delta(a) < r(a)$. Let $m$ be the bottom entry in column $c(a)$. By Lemma 4.3(iv), $\Delta(m) < r(m)$. Since $r(m) = 1$, $\Delta(m) \leq 0$. By Lemma 4.3(iii), $\Delta(b) = 0$ for some barred $b$ on the bottom row of $B$. \hfill \Box
5. Generalization of Stembridge’s proof

In this section we list the main steps in the proof of Theorem 2.4. The bulk of the technical work, however, namely the proofs of Lemmas 5.1 and 5.2, is taken up in the three subsequent sections. The underlying logic and structure of our arguments in this and the following three sections follows Stembridge [St], who works out similar results for ordinary Schur functions.

For $k \in \mathbb{N}$, define the polynomial $(x_j | y)^k = (x_j - y_1) \cdots (x_j - y_k)$. For $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{N}^d$, define $a_\xi(x | y) = \det([x_j | y]_{\xi})_{1 \leq i, j \leq d}$.

Lemma 5.1. $a_{\lambda + \rho}(x | y)s_\mu(x | y) = \sum_{B \in \mathbb{B}(\mu)} c_{\lambda + \rho}(B) a_{\lambda + \rho + \omega(B^\ast)}(x | y)$.

Lemma 5.2. $\sum c_{\lambda + \rho}(B) a_{\lambda + \rho + \omega(B^\ast)}(x | y) = 0$, where the sum is over all $B \in \mathbb{B}(\mu)$ such that the unbarred column word of $\lambda \ast B$ is not Yamanouchi.

The following three corollaries follow easily from these two lemmas.

Corollary 5.3. $a_{\lambda + \rho}(x | y)s_\mu(x | y) = \sum c_{\lambda + \rho}(B) a_{\lambda + \rho + \omega(B^\ast)}(x | y)$, where the sum is over all $B \in \mathbb{B}(\mu)$ such that the unbarred column word of $\lambda \ast B$ is Yamanouchi.

Suppose that $B \in \mathbb{B}(\mu)$ is such that the unbarred column word of $\emptyset \ast B$ is Yamanouchi. If $B$ has barred entries, then by Propositions 2.7 and 2.11.5, $c_{0, B} = 0$. If $B$ has no barred entries, then $B$ must be the unique reverse tableau of shape $\mu$ and content $\mu$: $B$ contains a 1 at the top of each column, and its entries increase by 1 per box as one moves down any column. Thus, by setting $\lambda = \emptyset$ in Corollary 5.3, we arrive at a new proof of the bialternant formula for the factorial Schur function ($\mathbb{G}$, $\mathbb{M}$):  

Corollary 5.4. $s_\mu(x | y) = a_{\rho}(x | y)/a_{\rho}(x | y).

Dividing both sides of the equation in Corollary 5.3 by $a_{\rho}(x | y)$ and applying Corollary 5.4, yields

Corollary 5.5. $s_{\lambda}(x | y)s_\mu(x | y) = \sum c_{\lambda + \rho}(B) s_{\lambda + \rho + \omega(B^\ast)}(x | y)$, where the sum is over all $B \in \mathbb{B}(\mu)$ such that the unbarred column word of $\lambda \ast B$ is Yamanouchi.

Regrouping the terms in this summation:

$$s_{\lambda}(x | y)s_\mu(x | y) = \sum_{\nu} \left( \sum_{B \in \mathbb{B}(\mu) \setminus \text{Yamanouchi}} c_{\lambda + \rho}(B) s_{\mu}(x | y) \right)$$

$$= \sum_{\nu} \left( \sum_{L \in \mathbb{R}_{\lambda, \mu}} c_{L} s_{\mu}(x | y) \right).$$

This proves Theorem 2.4.

Remark 5.6. Let $\kappa \in \mathbb{P}_d$, $\kappa \leq \mu$, i.e., $\kappa_i \leq \mu_i$, $i = 1, \ldots, d$. One can extend our analysis to factorial skew Schur functions of the form $s_{\mu/\kappa}(x | y)$ (see $\mathbb{M}$). One replaces $\mathbb{B}(\mu)$ with $\mathbb{B}(\mu/\kappa)$, the set of all reverse barred tableaux of shape $\mu/\kappa$. All the above definitions extend naturally. For example, for $B \in \mathbb{B}(\mu/\kappa)$, $c_{\kappa \ast B}$ is computed just as for $B \in \mathbb{B}(\mu)$, but with all boxes of $\kappa \subset \mu$ assumed to be empty.
All proofs are virtually unchanged, modified only by formally replacing \( \mu \) by \( \mu/\kappa \). As a generalization of Corollary 5.5 we obtain
\[
s_{\lambda}(x \mid y)s_{\mu/\kappa}(x \mid y) = \sum c_{\lambda \ast B}s_{\lambda + \omega(B^u)}(x \mid y),
\]
where the sum is over all \( B \in \mathcal{B}(\mu/\kappa) \) such that \((\lambda \ast B)^u\) is Yamanouchi. This generalizes Zelevinsky’s extension of the Littlewood-Richardson rule ([St], [Z]).

6. Involutionson reverse hatted tableaux

In his proof, Stembridge [St] utilizes involutions on Young tableaux introduced by Bender and Knuth [BK]. There is an analogous set of involutions on \( \mathcal{H}(\mu) \) which satisfy properties required for the proofs of Lemmas 5.1 and 5.2 (see Lemma 6.4). We remark that we were unable to find a suitable set of involutions on \( \mathcal{R}(\mu) \), and this is what initially led us to examine \( \mathcal{H}(\mu) \). If the involutions on \( \mathcal{H}(\mu) \) are restricted to \( \mathcal{R}(\mu) \), then the Bender-Knuth involutions are recovered.

6.1. The involutions \( s_1, \ldots, s_{d-1} \) of \( \mathcal{H}(\mu) \). Let \( H \in \mathcal{H}(\mu) \), and let \( i \in \{1, \ldots, d-1\} \) be fixed. Then an entry \( a \) of \( H \) with value \( i \) or \( i+1 \) is

- **free** if there is no entry of value \( i+1 \) or \( i \) respectively in the same column;
- **semi-free** if there is an entry of value \( i+1 \) or \( i \) respectively in the same column, and at least one of the two is hatted; or
- **locked** if there is an entry of value \( i+1 \) or \( i \) respectively in the same column, and both entries are unhatted.

Note that any entry of value \( i \) or \( i+1 \) must be exactly one of these three types, and each hatted entry of value \( i \) or \( i+1 \) must be either free or semi-free. In any row, the free entries are consecutive. Semi-free entries come in pairs, one below the other, as do locked entries.

To define the action of \( s_i \) on \( H \in \mathcal{H}(\mu) \), we first consider how it modifies the free entries of \( H \) (see Example 6.2):

1. Let \( S \) be a maximal string of free entries with values \( i \) and \( i+1 \) on some row of \( H \). Let \( S^0 \), \( S^l \), and \( S^r \) denote the unhatted, left-hatted, and right-hatted entries of \( S \) respectively. Modify \( S^0 \cup S^l \) as follows:
   - A. Change the value of each entry of value \( i \) to \( i+1 \) and each entry of value \( i+1 \) to \( i \), without changing whether or not it has a left hat.
   - B. Swap the entries of value \( i \) with those of value \( i+1 \) as follows: remove all entries of value \( i \); then move each entry of value \( i+1 \), beginning with the rightmost one, into the rightmost available empty box; then put the removed entries of value \( i \) back into the empty boxes of \( B \), preserving the relative order of barred and unbarred entries.

In this step, \( S^0 \cup S^l \) has been modified. No other entries of \( H \), in particular no entries of \( S^r \), have been modified, changed, or moved. Denote the modified string \( S \) by \( S_1 \). A potential problem has been introduced: the values of the entries of \( S_1 \) may not be weakly increasing as one moves from left to right. In step 2 we correct for this.

2. Let \( (S^r_1)_i \) and \( (S^r_1)_{i+1} \) denote the entries of \( S^r_1 \) of values \( i \) and \( i+1 \) respectively. Beginning with the leftmost entry \( a \in (S^r_1)_i \), let \( b \) be the entry of \( S_1 \) to the left of \( a \). If \( b \) has value \( i+1 \), then switch the entries \( b \) and \( a \), and then change the left entry from \( i \) to \( i+1 \). Now move right to the next entry of \( (S^r_1)_i \), and repeat this procedure until it has been performed on
all entries of \((S^r_1)_i\). Next, beginning with the rightmost entry \(a \in (S^r_1)_{i+1}\), let \(b\) be the entry of \(S_1\) to the right of \(a\). If \(b\) has value \(i\), then switch the entries \(b\) and \(a\), and then change the right entry from \(i + 1\) to \(\hat{i}\). Now move left to the next entry of \((S^r_1)_{i+1}\), and repeat this procedure until it has been performed on all entries of \((S^r_1)_{i+1}\).

Upon completion, we denote by \(S_2\) the resulting string obtained by modifying \(S_1\). It is weakly increasing.

We next consider how \(s_i\) modifies the semi-free entries of \(H\):

3. For a semi-free pair consisting of two entries lying in the same column of \(H\), each entry removes its hat (if it has one) and places it on top of the other entry.

The reverse tableau \(s_iH\) is obtained by applying steps 1 and 2 to each maximal string \(S\) of free entries of \(H\) (replacing \(S\) by \(S_2\)) and then applying step 3 to each semi-free pair.

**Example 6.2.** We illustrate steps 1 and 2. Suppose that \(i = 2\) and \(S\) consists of the following maximal string of consecutive free entries lying along some row of \(H\):

\[
S = 2 \hat{2} 2 2 \hat{2} 2 \hat{3} 3 \hat{3} 3 \\
S^o \cup S^l = 2 \hat{2} 2 2 \hat{2} 2 3 \hat{3} 3 3 \\
S_1 = 2 \hat{2} 2 2 \hat{3} 3 2 \hat{3} 3 3 3 \\
S_2 = 2 \hat{2} 2 2 \hat{3} 3 3 \hat{3} 3 3 3
\]

In line 2 we remove the entries of \(S^r\) from the picture for convenience, in order to focus attention on the operations performed in step 1, which only affect \(S^o \cup S^l\). In lines 3 and 4 the results of applying steps 1A and 1B successively to \(S^o \cup S^l\) are shown. In line 5, the removed entries from \(S^r\) are replaced. In line 6, the result of applying step 2 to \(S_1\) is shown. Only two entries are changed in this step.

This algorithm defines maps \(b_l : H^l \to (s_iH)^l\) and \(b_r : H^r \to (s_iH)^r\) as follows. If \(a \in H\) and the value of \(a\) is neither \(i\) nor \(i + 1\), then \(a\) remains unchanged in \(s_iH\). Thus in this case, if \(a \in H^l\) or \(a \in H^r\), then we define \(b_l(a) = a\) or \(b_r(a) = a\) respectively. Assume the value of \(a\) is \(i\) or \(i + 1\). If \(a \in H^l\) is free, then in step 1A, the value of \(a\) is either increased or decreased by 1; in step 1B, it is then moved to a different box; in step 2, this new entry in this new box is moved at most one box and changed by at most one in value, resulting in the entry we denote by \(b_l(a)\). If \(a \in H^r\) is free, then \(a\) is unchanged in step 1 and moved at most one box and changed by at most one in value in step 2. Denote the resulting entry by \(b_r(a)\). If \(a \in H^l\) or \(a \in H^r\) is semi-free, then \(b_l(a)\) or \(b_r(a)\) is the entry in \(s_iH\) to which it gives its hat.

In Example 6.2 if \(a\) is the rightmost entry of \(S\), which is a \(\hat{3}\), then \(b_l(a)\) is the 2 which is the fourth entry of \(S_2\) from the left. These two entries are, of course, entries of \(H\) and \(s_iH\) respectively.

**Lemma 6.3.** \(s_i\) is an involution on \(\mathcal{H}(\mu)\), \(i \in \{1, \ldots, d - 1\}\).
Proof: We begin by showing that \( s_i H \in \mathcal{H}(\mu) \), i.e., \( s_i H \) is row semistrict and column strict. The only nonobvious condition is that if \( S \) is any maximal string of free entries of \( H \) lying along some row, and if \( S_2 \) is the string that replaces it in \( s_i H \), then \( s_i H \) weakly increases along the left and right boundaries of \( S_2 \). To see this, note that if any entry of \( H \) of value \( i + 1 \) is free, then so are all entries of value \( i + 1 \) to the right of it in the same row; and if any entry of \( H \) of value \( i \) is free, then so are all entries of value \( i \) to the left of it in the same row. Thus by the maximality of \( S \), there are no entries of \( H \) of value \( i \) in the same row and to the right of \( S \), and there are no entries of \( H \) of value \( i + 1 \) in the same row and to the left of \( S \). Hence changing values of \( S \) from \( i \) to \( i + 1 \) and vice versa to form \( S_2 \) does not affect the row semistrictness of \( H \) along its boundaries.

We next show that \( s_i^2 = \text{id} \). Since the free entries of \( H \) lie in the same boxes as the free entries of \( s_i H \), it suffices to show that \( s_i^2(S) = S \) for any maximal string \( S \) of free entries of \( H \) (where \( s_i S \) is defined to be \( s_i H \) restricted to \( S \)). If step 1 is applied to \( (s_i S)^o \cup (s_i S)^l \), then one sees that the same entries of \( S^o \cup S^l \) are retrieved, although possibly not in their same boxes. However the relative order of the entries is the same. Now one checks that for \( a \in H^r \), \( b_i^2(a) = a \).

Let \( \sigma_i \) be the simple transposition of the permutation group \( S_d \) which exchanges \( i \) and \( i + 1 \). The involution \( s_i \) satisfies the following properties:

**Lemma 6.4.** Let \( H \in \mathcal{H}(\mu) \), \( a \in H^l \), and \( b \in H^r \). Then

(i) \(|b_i(a)| = \sigma_i|a|\).

(ii) \( \omega((s_i H)^u) = \sigma_i \omega(H^u) \).

(iii) \( \omega((s_i H)^u_{\leq b_i(a)})_{|b_i(a)|} = \omega(H^u_{\leq a})_{|a|} \).

(iv) \( e_{\sigma_i \xi, s_i H}(b_i(a)) = e_{\xi, H}(a) \).

(v) \( f_{s_i H}(b_r(b)) = f_H(b) \).

(vi) \( d_{\sigma_i \xi, s_i H} = d_{\xi, H} \).

Proof. If the value of \( a \) is not \( i \) or \( i + 1 \), then (i), (iii), and (iv) are obvious. If the value of \( b \) is not \( i \) or \( i + 1 \), then (v) is obvious. Thus we assume for these parts that the values of \( a \) and \( b \) are either \( i \) or \( i + 1 \). Parts (i), (ii), and (iii) follow from the construction of \( s_i \).

(iv) By parts (i) and (iii),

\[
e_{\sigma_i \xi, s_i H}(b_i(a)) = (\sigma_i \xi + \omega((s_i H)^u_{\leq b_i(a)})_{|b_i(a)|})_{|a|} = (\sigma_i \xi)_{|a|} + \omega((s_i H)^u_{\leq a})_{|b_i(a)|} = (\xi + \omega(H^u_{\leq a}))_{|a|} = e_{\xi, H}(a).
\]

(v) Under \( b_r \), the entry \( b \) is either kept in place, or moved up, down, left, or right by one box. In these cases, its value is either left unchanged, decreased, increased, increased, or decreased by one respectively. The result now follows from the definition of \( f_H \).

(vi) This is a consequence of (iv), (v), and (iii).

Let \( H \in \mathcal{H}(\mu) \) and let \( \sigma \in S_d \). Choose some decomposition of \( \sigma \) into simple transpositions: \( \sigma = \sigma_{i_1} \cdots \sigma_{i_j} \). Define \( \sigma H := s_{i_1} \cdots s_{i_j} H \). Although \( \sigma H \) depends on the decomposition chosen for \( \sigma \), by Lemma 6.4 (ii) and (vi),

\[
\omega((\sigma H)^u) = \sigma \omega(H^u) \quad \text{and} \quad d_{\sigma \xi, \sigma H} = d_{\xi, H}.
\]

In particular, both \( \omega((\sigma H)^u) \) and \( d_{\sigma \xi, \sigma H} \) are independent of the decomposition of \( \sigma \).
7. Proof of Lemma 5.1

Lemma 5.1 is a generalization of [ST, (1)]. In proving [ST, (1)], Stembridge uses the simple fact that if \( S \) is a tableau and \( \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{N}^d \), then \( x^2 x^3 = x^{\xi + \omega(S)} \). The generalization of this fact, which we will need in order to prove Lemma 5.1, is the following lemma. Define \( (x \mid y)^\xi = (x_1 \mid y)^{\xi_1} \cdots (x_d \mid y)^{\xi_d} \).

**Lemma 7.1.** Let \( R \in \mathcal{R}_{\text{sub}}(\mu) \) and let \( \xi \in \mathbb{N}^d \). Then

\[
(x \mid y)^\xi (x \mid y)_R = \sum_{B \in \mathcal{B}_{\text{sub}}(\mu) \atop B \in R} c_{\xi+1,B} \cdot (x \mid y)^{\xi + \omega(B''_n)}.
\]

In fact, we only need this lemma for \( R \in \mathcal{R}(\mu) \). We prove this result more generally for \( R \in \mathcal{R}_{\text{sub}}(\mu) \) only to allow for induction on the number of entries of \( R \) (and thus allow for the possibility that some boxes of \( R \) are empty). We remark that \( \mathcal{R}_{\text{sub}}(\mu) \) and \( \mathcal{B}_{\text{sub}}(\mu) \) were introduced in this paper solely to allow for induction in this proof.

**Proof.** The proof is by induction on the number of entries in \( R \). Let \( a \) be an entry of \( R \) with value \( k \), such that \( R \) has no entry of value \( k \) in any column to the left of \( a \). Let \( \alpha \) be the box containing \( a \). Let \( R' = R \setminus a \) be the reverse subtableau which results from removing \( a \) from \( R \).

If \( B \in \mathcal{B}_{\text{sub}}(\mu) \) is such that \( \bar{B} = R \), then the entry of \( B \) in box \( \alpha \), which we denote by \( B_\alpha \), must be either \( k \) or \( \overline{\mathbb{1}} \). Let \( B' \) denote \( B \setminus B_\alpha \). The following three sets are in bijection with one another:

\[
\{ B \in \mathcal{B}_{\text{sub}}(\mu) \mid \bar{B} = R, B_\alpha = k \} \longleftrightarrow \{ B \in \mathcal{B}_{\text{sub}}(\mu) \mid \bar{B} = R, B_\alpha = \overline{\mathbb{1}} \} \\
\longleftrightarrow \{ D \in \mathcal{B}_{\text{sub}}(\mu) \mid \bar{D} = R' \}.
\]

The first bijection simply adds a bar to \( B_\alpha \), and the second bijection removes \( B_\alpha \) from \( B \), mapping \( B \) to \( B' \). For brevity, we denote \( e_{B, \xi}(B_\alpha) \) and \( f_{B}(B_\alpha) \) by just \( e(B_\alpha) \) and \( f(B_\alpha) \) respectively for the remainder of this proof. If \( B_\alpha \) is unbarred, then

\[
c_{\xi+1,B} = c_{\xi+1,B'} \quad \text{and} \quad (x \mid y)^{\xi + \omega(B''_n)} = (x \mid y)^{\xi + \omega(B''_n)}(x_d - y_{e(B_\alpha)+1}).
\]

On the other hand, if \( B_\alpha \) is barred, then

\[
c_{\xi+1,B} = c_{\xi+1,B'}(y_{e(B_\alpha)+1} - y_{f(B_\alpha)}) \quad \text{and} \quad (x \mid y)^{\xi + \omega(B''_n)} = (x \mid y)^{\xi + \omega((B')_n)}.
\]

Thus,

\[
\sum_{B \in \mathcal{B}_{\text{sub}}(\mu) \atop B \in R} c_{\xi+1,B} (x \mid y)^{\xi + \omega(B''_n)}
\]

\[
= \sum_{B \in \mathcal{B}_{\text{sub}}(\mu) \atop B \in R, B_\alpha = k} c_{\xi+1,B} (x \mid y)^{\xi + \omega(B''_n)} + \sum_{B \in \mathcal{B}_{\text{sub}}(\mu) \atop B_\alpha = \overline{\mathbb{1}}} c_{\xi+1,B} (x \mid y)^{\xi + \omega(B''_n)}
\]

\[
= \sum_{B \in \mathcal{B}_{\text{sub}}(\mu) \atop B \in R, B_\alpha = k} c_{\xi+1,B'} (x \mid y)^{\xi + \omega((B')_n)}(x_{B_\alpha} - y_{e(B_\alpha)+1})
\]

\[
+ \sum_{B \in \mathcal{B}_{\text{sub}}(\mu) \atop B_\alpha = \overline{\mathbb{1}}} c_{\xi+1,B'}(y_{e(B_\alpha)+1} - y_{f(B_\alpha)})(x \mid y)^{\xi + \omega((B')_n)}.
\]
Proof of Lemma 5.1

\[ a_{\lambda+\rho}(x \mid y)s_\mu(x \mid y) = \left( \sum_{\lambda \in S_\lambda} \sum_{\lambda' \in \mathcal{R}(\mu)} c_{\lambda+1,B'}(x \mid y) \xi + \omega((B')^w) \right) (x_{B_{\lambda}} - y_{f(B_{\lambda})} + 1) \]

\[ + \quad \left( \sum_{\lambda \in S_\lambda} \sum_{\lambda' \in \mathcal{R}(\mu)} c_{\lambda+1,B'}(x \mid y) \xi + \omega((B')^w) \right) (x_{B_{\lambda}} - y_{f(B_{\lambda})}) \]

\[ = \left( \sum_{\lambda \in S_\lambda} \sum_{\lambda' \in \mathcal{R}(\mu)} c_{\lambda+1,B'}(x \mid y) \xi + \omega((B')^w) \right) (x_{B_{\lambda}} - y_{f(B_{\lambda})}) \]

\[ = \left( \sum_{\lambda \in S_\lambda} \sum_{\lambda' \in \mathcal{R}(\mu)} c_{\lambda+1,B'}(x \mid y) \xi + \omega((B')^w) \right) (x_{B_{\lambda}} - y_{f(B_{\lambda})}) \]

\[ = (x \mid y)^{\xi}(x \mid y)^{R'} (x_{B_{\lambda}} - y_{f(B_{\lambda})}) \]

\[ = (x \mid y)^{\xi}(x \mid y)^{R'} . \]

Equality (a) follows from the definition of \( a_\mu \), noting that \( \sigma(\lambda+\rho)+1 = \sigma(\lambda+\rho+1) \); (b) follows from Lemma 7.1 setting \( S = R \) and \( \xi = \sigma(\lambda+\rho) \); (c) from (11), with \( \xi = \sigma(\lambda+\rho) \); (d) from (13); and (f) from (10). For (d), we use the fact that for a fixed \( \sigma \) and arbitrary decomposition \( \sigma = \sigma_1 \cdots \sigma_t \), since each \( \sigma_i \) is an involution on \( \mathcal{H}(\mu) \), as \( H \) runs over all elements of \( \mathcal{H}(\mu) \), so does \( \sigma H \).
8. Proof of Lemma 5.2

By (12), Lemma 5.2 is equivalent to the following lemma, whose statement and proof generalize arguments in [51]. For \( H \in \mathcal{H}(\mu) \) and \( j \) a nonnegative integer, define \( H_{\leq j} \) to be the sub-hatted tableau of \( H \) consisting of the portion of \( H \) lying in columns to the right of \( j \), and \( H^u_{\leq j} = (H_{\leq j})^u \) (and similarly for \( H_{\leq j}, H_{> j}, \) etc.).

**Lemma 8.1.** Let \( \lambda \in \mathcal{P}_n \). Then

\[
\sum d_{\lambda+\rho+1, H} a_{\lambda+\rho+\omega(H^u)}(x \mid y) = 0,
\]

the sum being over all \( H \in \mathcal{H}(\mu) \) for which \( \lambda + \omega(H^u_{\leq j}) \notin \mathcal{P}_d \) for some \( j \).

**Proof.** We call \( H \in \mathcal{H}(\mu) \) for which \( \lambda + \omega(H^u_{\leq j}) \notin \mathcal{P}_d \) for some \( j \) a Bad Guy. Let \( H \) be a Bad Guy, and let \( j \) be minimal such that \( \lambda + \omega(H^u_{\leq j}) \notin \mathcal{P}_d \). Having selected \( j \), let \( i \) be minimal such that \( (\lambda + \omega(H^u_{\leq j}))_i < (\lambda + \omega(H^u_{\leq j-1}))_{i+1} \). Since \( (\lambda + \omega(H^u_{\leq j-1}))_i \geq (\lambda + \omega(H^u_{\leq j-1}))_{i+1} \) (by the minimality of \( j \)), we must have \((\lambda + \omega(H^u_{\leq j-1}))_i = (\lambda + \omega(H^u_{\leq j-1}))_{i+1} \), and column \( j \) of \( H \) must have an unhatted \( i + 1 \) but not an unhatted \( i \). Thus

\[
(\lambda + \rho + 1 + \omega(H^u_{\leq j}))_i = (\lambda + \rho + 1 + \omega(H^u_{\leq j}))_{i+1}.
\]

Define \( H^* \) to be the reverse tableau of shape \( \mu \) obtained from \( H \) by replacing \( H_{> j} \) by \( s_i(H_{\leq j}) \), and leaving \( H_{\leq j} \) unchanged. Notice first that \( H^* \) is still semistandard. Indeed, since \( s_i \) applied to \( H_{> j} \) can only change the values of its entries from \( i \) to \( i+1 \) and vice versa, the only possible violation of semistandardness of \( H^* \) would occur under the following scenario: (a) \( H \) has an entry \( a \) of value \( i \) in column \( j \) (which has to be either an \( i \) or \( i \), and must lie directly above the entry \( i+1 \)), (b) \( H \) has an entry \( b \) of value \( i \) immediately to the left of \( a \), and (c) \( s_i \) applied to \( H_{> j} \) changes the value of the entry in the position of entry \( b \) to \( i+1 \). However, this scenario is impossible. If (a) and (b) both hold, then since \( H \) is semistandard, the entry of \( H \) immediately below \( b \) must have value \( i+1 \) (we remark that the following property of the shape of a reverse tableau is critical here: if a reverse tableau contains three of the four boxes making up a square, namely the top-left, top-right, and bottom-right boxes, then it must contain the bottom-left box of the square as well). Therefore the entry in box \( b \) is not a free entry of \( H_{> j} \), so \( s_i \) does not change its value, i.e., (c) is violated. Notice second that since \( H^u_{\leq j} = H_{\leq j} \), we have that \( H^* \) is still a Bad Guy, and furthermore the map \( H^* \mapsto H^{**} \) replaces \( (H^*)_{> j} \) by \( s_i((H^*)_{> j}) \) and leaves \( (H^*)_{\leq j} \) unchanged. Therefore \((H^{**})_{> j} = s_i((H^*)_{> j}) = s_i(s_i((H_{> j}))) = H_{> j} \) (since \( s_i \) is an involution on reverse hatted tableaux; see Lemma 6.3), and \((H^{**})_{\leq j} = (H^*)_{\leq j} = H_{\leq j} \). Thus \( H \mapsto H^* \) gives an involution on the set of Bad Guys of \( H(\mu) \).

We define maps \( b_i^r : H^r \rightarrow (H^*)^r \) and \( b_i^r : H^r \rightarrow (H^*)^r \) as follows. If \( a \in (H_{\leq j})^r \), then define \( b_i^r(a) = a \). If \( a \in (H_{> j})^r \), then during the construction of \( H^* \), in the process of applying \( s_i \) to \( H_{> j} \), \( a \) is mapped to \( b_i(a) \in (H_{> j})^r \). This same element \( b_i(a) \), regarded as an element of \( (H^*)^r \), is denoted by \( b_i^r(a) \). The map \( b_i^r \) is defined analogously.

We wish to show that for \( a \in H^r \),

\[
e_{\lambda+\rho+1, H}(a) = e_{\lambda+\rho+1, H^*}(b_i^r(a)),
\]

and for \( a \in H^r \),

\[
f_H(a) = f_{H^*}(b_i^r(a)).
\]

For \( a \in (H_{\leq})^l \) or \( a \in (H_{\leq})^r \), both (16) and (17) are obvious. The proof of (17) for \( a \in (H_{\geq})^r \) follows in much the same manner as the proof of Lemma 6.4.(v).

It remains to prove (16) for \( a \in (H_{\geq})^l \). For such \( a \), by Lemma 6.4.(iii),

\[
\omega(j < (H^u)^{\leq}(b_\nu(a)))|_{\nu} = \omega(j < (H^u)^{\leq}(b_\nu(a)))|_{\nu},
\]

where \( H_{\geq} := H_{\geq}^l \cap H_{\leq} \). By (15),

\[
(\lambda + \rho + 1 + \omega(H^u_{\leq}))|_{\sigma(a)} = (\lambda + \rho + 1 + \omega(H^u_{\leq}))|_{\sigma(a)} = (\lambda + \rho + 1 + \omega((H^u)^{\leq}))|_{\sigma(a)}. \tag{19}
\]

Thus

\[
e_{\lambda + \rho + 1, H^{\ast}}(a) = (\lambda + \rho + 1 + \omega(H^u_{\leq}))|_{\sigma(a)} = (\lambda + \rho + 1 + \omega((H^u)^{\leq}))|_{\sigma(a)} = e_{\lambda + \rho + 1, H^{\ast}}(b_\nu(a)).
\]

Equality (a) follows from (18) and (19); (b) follows from Lemma 6.4.(i). This completes the proofs of (16) and (17).

Now (16) and (17) imply

\[
d_{\lambda + \rho + 1, H} = \prod_{a \in H^r} y_{e_{\lambda + \rho + 1, H}(a)} \prod_{a \in H^r} (1 - y_{f_{H^r}(a)}) = \prod_{a \in H^r} y_{e_{\lambda + \rho + 1, H^r}(b_\nu(a))} \prod_{a \in H^r} (1 - y_{f_{H^r}(b_\nu(a))}) = d_{\lambda + \rho + 1, H^r} \tag{20}
\]

By \( \sigma_i(\omega(H^u_{\leq})) = \omega((H^u)^{\leq}) \) and (19), \( \sigma_i(\lambda + \rho + \omega(H^u)) = \lambda + \rho + \omega((H^u)^{\leq}); \) thus

\[
(\lambda + \rho + \omega(H^u))(x | y) = -a_{\lambda + \rho + \omega((H^u)^{\leq})}(x | y). \tag{21}
\]

By (20) and (21), the contributions to (14) of two Bad Guys paired under the involution \( H \rightarrow H^r \) are negatives, and thus cancel. If a Bad Guy is paired with itself under \( H \rightarrow H^r \), then (21) implies that its contribution to (14) is 0. \( \Box \)

9. Bijection with trapezoid puzzles

In this section we define trapezoid puzzles, which are generalizations of Knutson-Tao puzzles. We give a weight-preserving bijection between equivariant Littlewood-Richardson skew tableaux and trapezoid puzzles which restricts to a bijection between positive equivariant Littlewood-Richardson skew tableaux and Knutson-Tao puzzles. Thus the formulas described in Section 2 for the structure constants \( e_{\lambda, \mu}^{\nu} \) and \( C_{\lambda, \mu}^{\nu} \), \( \lambda, \mu, \nu \in \mathcal{P}_{d,n} \), can be indexed by trapezoid puzzles instead of equivariant Littlewood-Richardson skew tableaux.
9.1. Trapezoid puzzles. A puzzle piece is one of the eight figures shown in Figure 3, each of whose edges has length 1 unit. Each puzzle piece is either an equilateral triangle or a rhombus, together with a fixed orientation, and a labeling of each edge with either a 1 or a 0. The rightmost puzzle piece in Figure 3 is called an equivariant puzzle piece; we color it gray.

![Figure 3. The eight puzzle pieces](image)

Consider the isosceles trapezoid formed by placing an equilateral triangle of side length $n$ on top of a rhombus of side length $n$ and removing the common segment (see Figure 4 in which the common segment is darkened). Consider a partitioning $P$ of this trapezoid into puzzle pieces in such a way that if two puzzle pieces share an edge, then both puzzle pieces must have the same label on that edge. In this partitioning, one assumes that the common segment between the triangle and the rhombus is not present. The boundary of $P$, denoted by $\partial P$, is the set of edges of $P$ lying on the boundary of the trapezoid. It is divided into five parts: northeast, northwest, east, west, and south (denoted by $\partial P_{\text{NE}}$, $\partial P_{\text{NW}}$, $\partial P_{\text{E}}$, $\partial P_{\text{W}}$, and $\partial P_{\text{S}}$). These correspond to the northeast and northwest boundaries of the equilateral triangle, and the east, west, and south boundaries of the rhombus respectively. The partitioning $P$ is called a trapezoid puzzle if $\partial P_{\text{E}}$ and $\partial P_{\text{W}}$ consist entirely of 0 edges. A Knutson-Tao puzzle is a trapezoid puzzle which has no 1-triangle puzzle pieces in the rhombus region.

One forms three $n$-digit binary words by reading the labels along $\partial P_{\text{NE}}$, $\partial P_{\text{NW}}$, and $\partial P_{\text{S}}$: the labels of $\partial P_{\text{NE}}$ are read from top to bottom, the labels of $\partial P_{\text{NW}}$ from bottom to top, and the labels of $\partial P_{\text{S}}$ from left to right. To these three binary words we associate three partitions of $P_{d,n}$ under the map $w \mapsto (\eta_1, \ldots, \eta_d) \in P_{d,n}$, where $\eta_j$ is the number of zeros of $w$ which lie to the right of the $j$-th one of $w$ from the left (for example, 011001010 $\mapsto (5, 5, 2, 1) \in P_{4,10}$). Denote by $P_{\lambda,\mu}$ (resp. $P_{\lambda,\mu}^+$) the set of all trapezoid puzzles (resp. Knutson-Tao puzzles) $P$ for which these three partitions are $\lambda$, $\mu$, and $\nu$, in that order.

Let $D$ denote the common segment forming the south border of the triangle and the north border of the rhombus. For any equivariant puzzle piece of $P$, draw two lines from the center of the puzzle piece to $D$: one line $L_1$ parallel to $\partial P_{\text{NW}}$ and the other $L_2$ parallel to $\partial P_{\text{NE}}$. The lines $L_1$ and $L_2$ cross $D$ at $e-.5$ and $f-.5$ units from its right endpoint, respectively ($e$, $f$ are both integers). If the equivariant puzzle piece lies above $D$, then $e > f$; if it lies below $D$, then $e < f$; if it is bisected by $D$, then $e = f$. The factorial weight of the puzzle piece is $y_e - y_f$, and the equivariant weight of the puzzle piece is $Y_{n+1-f} - Y_{n+1-e}$. Let $c_P$ (resp. $C_P$) denote the product of the factorial weights (resp. equivariant weights) of all the equivariant puzzle pieces of $P$. For example, in Figure 4 $c_P = (y_5 - y_6)(y_5 - y_4)(y_2 - y_1) = 0$ and $C_P = (Y_4 - Y_3)(Y_4 - Y_4)(Y_4 - Y_5)(Y_7 - Y_8) = 0$. 
Figure 4. A trapezoid puzzle $P$, with $n = 8$, $d = 2$. The $n$-digit binary words of the NE, NW, and S sides of the boundary are 00001010, 10000010, and 00100100 respectively. Thus $P \in \mathcal{P}^\nu_{\lambda,\mu}$, where $\lambda = (2, 1)$, $\mu = (6, 1)$, and $\nu = (4, 2)$. The darkened common segment is displayed only to illustrate that the trapezoidal shape is formed from an equilateral triangle and a rhombus; it is not part of the trapezoid puzzle.

**Proposition 9.2.** There is a weight preserving bijection $\Phi : \mathcal{P}^\nu_{\lambda,\mu} \rightarrow \mathcal{LR}^\nu_{\lambda,\mu}$, which restricts to a weight preserving bijection $\mathcal{P}^\nu_{\lambda,\mu} \rightarrow \mathcal{LR}^\nu_{\lambda,\mu}$. By weight-preserving, we mean that for $P \in \mathcal{P}^\nu_{\lambda,\mu}$, $c_P$ and $c_{\Phi(P)}$ are equal, and moreover are identical expressions, and similarly for $C_P$ and $C_{\Phi(P)}$. 
Figure 5. The three equivariant puzzle pieces have factorial weights $y_6 - y_3$, $y_5 - y_5 (= 0)$, and $y_1 - y_4$, and equivariant weights $Y_5 - Y_2$, $Y_3 - Y_3 (= 0)$, and $Y_7 - Y_4$ respectively.

Proof. The bijection $\Phi$, illustrated in Figure 6, generalizes Tao’s ‘proof without words’ of the bijection between puzzles and tableaux in the nonequivariant setting [V, Figure 11]. The object in the center of Figure 6 represents a truncated generic trapezoid puzzle $P$. The bottom of the rhombus portion of the trapezoid has been removed. To retrieve this portion, one extends the rhombus portion downward, meanwhile extending the white paths to the bottom of the figure. In the diagram, black represents regions of 1-triangles, dark gray represents regions of 0-triangles, white represents regions of nonequivariant rhombi, and light gray represents regions of equivariant rhombi.

From $P$, one may construct the Young tableau $Y$ and reverse barred tableau $B$ appearing in Figure 6. The shape of $Y$ is determined by the lengths indicated on $\partial P_{\text{NE}}$, and the $i$-th row is filled with unbarred $i$’s. The reverse barred tableau $B$ is constructed using the regions of $P$ consisting of rhombus puzzle pieces labelled by $x_{i,j}$ (where $x = a, b, c, \text{ or } d$). To each puzzle piece in such a region there corresponds an entry of value $j$ in row $i$ of $B$. An equivariant puzzle piece corresponds to a barred entry of $B$; a nonequivariant puzzle piece corresponds to an unbarred entry of $B$. The skew barred tableau $\Phi(P)$ is constructed by placing $Y$ above and to the right of $B$.

We list two properties of any $L \in \mathcal{LR}_{\lambda,\mu}$:

(a) $L|_{\mu}$ is column strict; and
(b) the unbarred column word of $L$ is Yamanouchi.

Let $P \in \mathcal{P}_{\lambda,\mu}$. For $i \in \{1, \ldots, d\}$ (where $d = 4$ in Figure 6), there is a path $P_i$ in $P$ consisting of 1-triangles and rhombi which begins on $\partial P_{\text{NE}}$, moves only west
or southwest, and ends on \( \partial P_3 \) (see Figure 7). Each path \( P_i \) has segments \( P_{i,j} \) consisting of the rhombus pieces lying in the regions of Figure 6 labelled by \( x_{i,j} \) (where \( x = a, b, c, \) or \( d \)).

We list two properties of \( P \):

(a)' For \( i = 2, \ldots, d \) and all \( j \), the distance from the leftmost edge of \( P_{i,j} \) to \( \partial P_{\text{NE}} \) is greater than or equal to the distance from the leftmost edge of \( P_{i-1,j} \) to \( \partial P_{\text{NE}} \).

(b)' The interiors of the \( P_i \) do not touch.

Properties (a)' and (b)' of \( P \) imply properties (a) and (b) of \( \Phi(P) \) respectively. Conversely, given any \( L \in \mathcal{LR}_\lambda^\mu \), Figure 6 shows how to construct a puzzle \( \Phi^{-1}(L) \).

Properties (a) and (b) of \( L \) ensure that the puzzle \( \Phi^{-1}(L) \) can be constructed, and imply that it satisfies (a)' and (b)'. Uniqueness is clear.

To each equivariant puzzle piece of \( P \) there corresponds a barred entry of \( \Phi(P) \), and they both determine the same factor \( y_i - y_j \) of \( c_P \) and \( c_{\Phi(P)} \) respectively.
Figure 7. The paths $P_i$, $i = 1, \ldots, 4$, of the trapezoid puzzle $P$ of Figure 6. The segments $P_{i,j}$ of each path are shaded. The segments may contain two types of puzzle pieces: equivariant puzzle pieces and rhombi with horizontal 0-edges.

Therefore $\Phi$ is weight-preserving, and thus restricts to a bijection from $\{ P \in P^\nu_{\lambda,\mu} \mid c_P > 0 \}$ to $\{ L \in LR^\nu_{\lambda,\mu} \mid c_L > 0 \} = LR^\nu_{\lambda,\mu}$. To see that the former set is $P^\nu_{\lambda,\mu}$, observe that $P$ is not a Knutson-Tao puzzle if and only if $P$ contains a 1-triangle lying below $D$ if and only if $P$ contains an equivariant puzzle piece which is bisected by $D$ if and only if $c_P = 0$. □

Using Theorem 2.4, Corollary 2.10, and Proposition 9.2 we obtain a proof of the following theorem; these formulas, expressed in terms of $P^\nu_{\lambda,\mu}$, are due to Knutson and Tao [KT].

**Theorem 9.3.** For $\lambda, \mu, \nu \in P_{d,n}$,

$$c^\nu_{\lambda,\mu} = \sum_{P \in P^\nu_{\lambda,\mu}} c_P = \sum_{P \in P^\nu_{\lambda,\mu}^+} c_P \quad \text{and} \quad C^\nu_{\lambda,\mu} = \sum_{P \in P^\nu_{\lambda,\mu}} C_P = \sum_{P \in P^\nu_{\lambda,\mu}^+} C_P.$$
Figure 8. A trapezoid puzzle $P$ (center), with the color scheme described in Figure 6. Here $d = 3$, $n = 13$, $\lambda = (5, 2, 1)$, $\mu = (8, 5, 1)$, $\nu = (9, 4, 2)$. The corresponding equivariant Littlewood-Richardson skew tableau $\Phi(P)$ appears top right, top left. The common segment $D$ separating the triangle from the rhombus is darkened. The fact that 1-triangles lie below $D$ implies that $c_P = C_P = 0$. Indeed, $c_P = (y_9 - y_4)(y_5 - y_2)(y_2 - y_1)(y_2 - y_2)(y_2 - y_3)(y_3 - y_5)(y_6 - y_8) = 0$, and $C_P = (Y_{10} - Y_5)(Y_{12} - Y_6)(Y_{13} - Y_{12})(Y_{12} - Y_{12})(Y_{11} - Y_{12})(Y_9 - Y_{11})(Y_6 - Y_8) = 0$.

10. The Molev-Sagan rule

In this section we recall the Molev-Sagan rule for computing the coefficients $c^\nu_{\lambda, \mu}$ [MS].

The forward skew diagram $\lambda \star \mu$ is obtained by placing the Young diagram $\lambda$ above and to the right of the Young diagram $\mu$ (see Figure 9). Note the difference between this object and the skew diagram $\lambda \ast \mu$ defined in Section 2 in $\lambda \ast \mu$, $\mu$ is represented by a Young diagram rather than a reverse Young diagram. One defines a forward skew barred tableau $L$, an unbarred column word of $L$, $L_u$, and $L_{u<}^\mu$, in essentially the same way as their counterparts in Section 2.2 with the only difference being that the definitions are applied to the shape $\lambda \star \mu$ rather than $\lambda \ast \mu$.

Definition 10.1. An equivariant Littlewood-Richardson forward skew tableau is a forward skew barred tableau whose unbarred column word is $Y_a$-
manouchi. Denote the set of all equivariant Littlewood-Richardson forward skew tableaux of shape \( \lambda \ast \mu \) and unbarred content \( \nu \) by \( \mathcal{LRF}^\nu_{\lambda,\mu} \).

For \( L \) a forward skew barred tableau, define

\[
e_L = \prod_{a \in L \text{ barred}} \left( y_{|a'| + \omega(L_{\mu,a}|_a)} - y_{|a|} + c(a) - r(a) \right),
\]

where the rows of \( \mu \) are numbered from top to bottom and the columns from left to right, and \( |a'| = d + 1 - |a| \).

**Theorem 10.2** (Molev-Sagan rule). \( c^\nu_{\lambda,\mu} = \sum_{L \in \mathcal{LRF}^\nu_{\lambda,\mu}} e_L \).

We remark that [MS, (8) and Theorem 3.1] is more general than Theorem 10.2, as it allows for the \( y \) variables in \( s_\lambda(x|y) \) and the \( y \) variables in \( s_\mu(x|y) \) of (1) to be two different families of variables.

In the following example, we recompute the coefficient \( c^\nu_{\lambda,\mu} \) of Example 2.12 using the Molev-Sagan rule.

**Example 10.3.** Let \( d = 3 \), \( \lambda = (1,1) \), \( \mu = (3,2) \), and \( \nu = (3,2,1) \). We list all \( L \in \mathcal{LRF}^\nu_{\lambda,\mu} \), and for each \( L \) we give \( e_L \):

![Tableau](image)

**Figure 9.** An equivariant Littlewood-Richardson forward skew tableau \( L \) of shape \( \lambda \ast \mu \) and unbarred content \( \nu \), where \( \lambda = (2,1,1) \), \( \mu = (4,3,2) \), and \( \nu = (3,2,2,1) \). The unbarred column word, 1,1,2,3,2,4,3,5,1,4, is Yamanouchi. If \( d = 5 \), then \( e_L = (y_2 - y_7)(y_6 - y_3)(y_5 - y_2) \).
By Theorem 10.2, \( c_{\lambda, \mu}^\nu = (y_4 - y_3) + (y_3 - y_4) + (y_1 - y_5) + (y_5 - y_2) + (y_6 - y_1) + (y_3 - y_2) + (y_2 - y_3) + (y_4 - y_1) \).

The presence of terms of the form \( y_i - y_j \) with both \( i > j \) and \( i < j \) in this example and the example of Figure 9 illustrate that in general the Molev-Sagan rule does not yield a positive formula for \( c_{\lambda, \mu}^\nu \).

We remark that Corollary 2.8 does not hold if \( c_L \) is replaced by \( e_L \) and \( LR_{\lambda, \mu}^\nu \) by \( LRF_{\lambda, \mu}^\nu \). (In fact, Figure 9 gives a counterexample, since \( \mu \not\subseteq \nu \), implying \( c_{\lambda, \mu}^\nu = 0 \), but \( e_L \neq 0 \).) This is due to the nonpositivity of the Molev-Sagan rule.

The Molev-Sagan rule and Theorem 2.4 of course produce the same coefficients \( c_{\lambda, \mu}^\nu \). It would be interesting to deduce one rule directly from the other.

References


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