Abstract. Assume that $\mathbb{K}$ is a complete non-Archimedean valued field. We prove that every infinite-dimensional Fréchet-Montel space over $\mathbb{K}$ which is not isomorphic to $\mathbb{K}^\mathbb{N}$ has a nuclear Köthe quotient. If the field $\mathbb{K}$ is non-spherically complete, we show that every infinite-dimensional Fréchet space of countable type over $\mathbb{K}$ which is not isomorphic to the strong dual of a strict $LB$-space has a nuclear Köthe quotient.

1. Introduction

In this paper all linear spaces are over a non-Archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0, \infty)$. For fundamentals of normed spaces and Hausdorff locally convex spaces (lcs) we refer to [6] and [8, 9].

Any infinite-dimensional Banach space of countable type is isomorphic to the Banach space $c_0$ of all sequences in $\mathbb{K}$ converging to zero with the sup-norm, and any closed subspace of $c_0$ is complemented ([6], Th. 3.16). Any infinite-dimensional Fréchet space of finite type is isomorphic to the Fréchet space $\mathbb{K}^\mathbb{N}$ of all sequences in $\mathbb{K}$ with the product topology.

By a Köthe space we mean an infinite-dimensional Fréchet space with a Schauder basis and with a continuous norm.

We investigated quotients of Fréchet spaces in [10, 11, 12].

If the field $\mathbb{K}$ is spherically complete, then there exist non-normable Fréchet spaces (over $\mathbb{K}$) of countable type with a continuous norm and without a nuclear Köthe quotient ([12], Theorem 10).

In this paper we study when a Fréchet space of countable type has a nuclear Köthe quotient.

We show that for every Fréchet space $E$ with a continuous norm and for every biorthogonal sequence $((x_n, f_n)) \subset E \times E'$ such that $(x_n)$ is linearly dense in $E$ and $(f_n)$ is equicontinuous, there exists an infinite subset $J$ of $\mathbb{N}$ such that the quotient $(E/\bigcap_{n \in J} \ker f_n)$ of $E$ is a Köthe space (Corollary 3.7). It follows that an infinite-dimensional Fréchet space of countable type has a Köthe quotient if and only if it is not isomorphic to $\mathbb{K}^\mathbb{N}$ (Corollary 3.10).

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Next we prove that a Fréchet space $E$ of countable type has a nuclear Köthe quotient if and only if it has a non-decreasing base $(\| \cdot \|_k)$ of continuous seminorms such that the dual norms $\| \cdot \|_k$, $k \in \mathbb{N}$, are pairwise non-equivalent on the subspace $E'_1 = \{ f \in E' : \|f\|_1 < \infty \}$ of $E'$ (Theorem 3.11).

Using this theorem we show that every infinite-dimensional Fréchet-Montel space $E$ which is not isomorphic to $\mathbb{K}^\mathbb{N}$ has a nuclear Köthe quotient (Theorem 3.12).

If $\mathbb{K}$ is non-spherically complete, then every infinite-dimensional Fréchet space $E$ of countable type which is not isomorphic to the strong dual of a strict $LB$-space has a nuclear Köthe quotient (Theorem 3.13).

In our paper we use and develop some ideas of [1].

2. Preliminaries

The field $\mathbb{K}$ is spherically complete if any decreasing sequence of closed balls in $\mathbb{K}$ has a non-empty intersection. Let $B_\mathbb{K}$ denote the set $\{ \alpha \in \mathbb{K} : |\alpha| \leq 1 \}$.

For $S \subset \mathbb{N}$ we put $c_{00}(S) = \{(x_n) \in c_{00} : x_n = 0$ for any $n \in (\mathbb{N} \setminus S)\}$, where $c_{00} = \{(x_n) \in \mathbb{K}^\mathbb{N} : x_n = 0$ for almost all $n \in \mathbb{N}\}$.

Let $E$ be a linear space.

The linear span of a subset $A$ of $E$ is denoted by lin$A$.

A set $A \subset E$ is absolutely convex if for any $\alpha, \beta \in B_\mathbb{K}$ and any $x, y \in A$ we have $\alpha x + \beta y \in A$. Let $A$ be an absolutely convex set in $E$. We put $A^e = A$ if the valuation of $\mathbb{K}$ is discrete and $A^e = \bigcap \{ \alpha A : \alpha \in \mathbb{K}$ with $|\alpha| > 1 \}$ otherwise.

If $A \subset E$, then the set co$A = \{ \sum_{i=1}^n \alpha_i a_i : n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in B_\mathbb{K}, a_1, \ldots, a_n \in A \}$ is the smallest absolutely convex subset of $E$ that contains $A$.

A set $A \subset E$ is $\mathbb{K}$-polar if for each $x \in (E \setminus A)$ there exists a linear functional $f$ on $E$ such that $|f(x)| > 1$ and $|f(a)| \leq 1$ for any $a \in A$.

A seminorm on $E$ is a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}$, $x \in E$ and $p(x + y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$.

Let $t \in [0, 1]$ and let $p$ be a seminorm on a linear space $E$. A sequence $(x_n) \subset E$ is $t$-orthogonal with respect to $p$ if $p(\sum_{i=1}^n \alpha_i x_i) \geq t \max\{p(\alpha_i x_i) : 1 \leq i \leq n\}$ for all $n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \mathbb{K}$. (In [3], a sequence $(x_n)$ in a normed space $(E, \| \cdot \|)$ is called orthogonal if it is 1-orthogonal with respect to the norm $\| \cdot \|$.)

A seminorm $p$ on $E$ is a norm if $\ker p = \{0\}$.

For any seminorm $p$ on $E$ the map $\overline{p} : E_p \to [0, \infty), x + \ker p \to p(x)$ is a norm on $E_p = (E/\ker p)$.

Norms $p, q$ on $E$ are equivalent if there exist positive numbers $a, b$ such that $a p(x) \leq q(x) \leq b p(x)$ for any $x \in E$; then we write $p \approx q$.

Any two norms on a finite-dimensional linear space are equivalent.

In this paper by an lcs we mean a Hausdorff locally convex space.

Let $E$ be an lcs.

The set of all continuous seminorms on $E$ is denoted by $\mathcal{P}(E)$.

$E$ is of finite type if for any $p \in \mathcal{P}(E)$ the space $E_p$ is finite-dimensional.

$E$ is of countable type if for any $p \in \mathcal{P}(E)$ the normed space $(E_p, \overline{p})$ contains a linearly dense countable subset.

The topological dual of $E$ we denote by $E'$. If $A \subset E$ and $M$ is a subspace of $E$, we set $A^0 = \{ f \in E' : |f(x)| \leq 1$ for $x \in A \}$ and $M^\perp = \{ f \in E' : f(x) = 0$ for $x \in M \}$. If $B \subset E'$ and $W$ is a subspace of $E'$, we put $^\circ B = \{ x \in E : |f(x)| \leq 1$ for $f \in B \}$ and $^\perp W = \{ x \in E : f(x) = 0$ for $f \in W \}$. It is easy to see that $M^\perp = M^0$ and
If $A$ is an absolutely convex subset of $E$, then $\circ(A^\circ) = B^\circ$, where $B$ is
the closure of $A$ in $(E, \sigma(E, E'))$ ([S], Proposition 4.10).

A subset $A$ of $E$ is polar if $\circ(A^\circ) = A$. $E$ is polar if for any $p \in P(E)$ there
exists $q \in P(E)$ with $q \geq p$ such that the set $\{x \in E : q(x) \leq 1\}$ is polar.

A set $A \subset E$ is bornivorous if it absorbs any bounded subset of $E$. $E$ is bornological
if any absolutely convex bornivorous subset of $E$ is a neighbourhood of zero. $E$
is polarly barrelled if any polar barrel in $E$ is a neighbourhood of zero. $E$
is polarly bornological if any $K$-polar bornivorous subset of $E$ is a neighbourhood of zero.

A subset $B$ of an lcs $E$ is compactoid if for any neighbourhood $U$ of $0$ in $E$ there
exists a finite subset $S$ of $E$ such that $B \subset U + cS$.

A subspace $D$ of $E$ has the weak extension property in $E$ if for any $g \in D'$ there
exists an $f \in E'$ with $f|_D = g$.

Let $B(E)$ denote the family of all bounded subsets of $E$. The strong dual of
$E$, that is, the topological dual $E'$ of $E$ with the topology $b(E', E)$ of uniform
convergence on bounded subsets of $E$, is denoted by $E_b$.

$E$ is reflexive if the canonical map $j : E \to (E_b')' = \mathbb{K}^E_b$ is an isomorphism.

Let $E$ and $F$ be an lcs. The space of all linear continuous maps from $E$ to $F$ is
denoted by $L(E, F)$. An operator $T \in L(E, F)$ is an isomorphism if $T$ is injective,
surjective and the inverse map $T^{-1}$ is continuous. $E$ is isomorphic to $F$ if $E \cong F$
if there exists an isomorphism $T : E \to F$. A linear map $T : E \to F$ is compact if
there exists a neighbourhood $U$ of $0$ in $E$ such that $T(U)$ is compactoid in $F$.

An lcs $E$ is nuclear if for any $p \in P(E)$ there exists $q \in P(E)$ with $q \geq p$ such
that the map $\varphi_{p, q} : (E_q, \overline{q}) \to (E_p, \overline{p})$, $x + \ker q \to x + \ker p$
is compact.

Let $E$ be a metrizable lcs. $E$ is of countable type if and only if it contains a
linearly dense countable subset. A sequence $(p_k) \subset P(E)$ is a base in $P(E)$ if for
any $p \in P(E)$ there exists $k \in \mathbb{N}$ such that $p \leq p_k$.

A metrizable complete lcs is a Fréchet space. Let $(x_n)$ be a sequence in a Fréchet
space $E$. The series $\sum_{n=1}^\infty x_n$ is convergent in $E$ if and only if $\lim_{n \to \infty} x_n = 0$.

A normable Fréchet space is a Banach space. Any $n$-dimensional lcs is isomorphic
to the Banach space $\mathbb{K}^n$. A strict LB-space is an lcs $(E, \tau)$ which is the inductive
limit of an inductive sequence $((E_n, \tau_n))$ of Banach spaces such that $\tau_{n+1} = E_n = \tau_n$
for any $n \in \mathbb{N}$; for fundamentals of inductive limits of locally convex spaces we refer
to [S]. A Fréchet space $E$ is a Fréchet-Montel space if any bounded subset of $E$ is
compactoid.

If $(X, \| \cdot \|)$ and $(Y, \| \cdot \|)$ are normed spaces, then the map
$$
\| \cdot \| : L(X, Y) \to [0, \infty), \| T \| = \inf \{ C > 0 : \| Tx \| \leq C \| x \| \text{ for any } x \in X \}
$$
is a norm; the normed space $(L(X, Y), \| \cdot \|)$ is complete if $(Y, \| \cdot \|)$ is complete.

Let $E$ be an lcs. A sequence $(x_n, f_n) \subset E \times E'$ is biorthogonal if $f_n(x_m) = \delta_{n,m}$
for all $n, m \in \mathbb{N}$, where $\delta_{n,m} = 1$ if $n = m$ and $\delta_{n,m} = 0$ otherwise.

A sequence $(x_n)$ in an lcs $E$ is a basis in $E$ if each $x \in E$ can be written uniquely as
$$
x = \sum_{n=1}^\infty \alpha_n x_n \text{ with } (\alpha_n) \subset \mathbb{K}.
$$
If additionally the coefficient functionals $f_n : E \to \mathbb{K}, x \to \alpha_n (n \in \mathbb{N})$ are continuous, then $(x_n)$ is a Schauder basis in $E$.

Let $(t_k) \subset (0, 1]$. A sequence $(x_n)$ in a metrizable lcs $E$ is $t_k$-orthogonal with
respect to $(p_k) \subset P(E)$ if $(x_n)$ is $t_k$-orthogonal with respect to $p_k$ for any $k \in \mathbb{N}$.

A sequence $(x_n)$ in a metrizable lcs $E$ is orthogonal in $E$ if it is $(1)$-orthogonal
with respect to some base $(p_k)$ in $P(E)$.

A linearly dense orthogonal sequence $(x_n)$ of non-zero elements in a metrizable
lcs $E$ is an orthogonal basis in $E$. 
Any orthogonal basis in a metrizable lcs is a Schauder basis, and any Schauder basis in a Fréchet space is an orthogonal basis ([4], Propositions 1.4 and 1.7).

3. Results

We start with the following

**Lemma 3.1.** Let \((X, \| \cdot \|)\) be a normed space and let \(Z\) be a finite-dimensional subspace of \(X'\). Then for any \(e > 0\) there exists a finite-dimensional subspace \(W_e\) of \(X\) with \(\dim W_e = \dim Z\) such that for any \(\phi \in Z'\) there is an \(x \in W_e\) with \(\|\phi\| \leq \|x\| \leq (1 + e)\|\phi\|\) such that \(z(x) = \phi(z)\) for any \(z \in Z\).

**Proof.** Denote by \(Y\) the closed linear subspace \(\perp Z\) of \(X\). Then \(\perp Z = Z\) since \(Z\) is closed in \((X', \sigma(X', X))\). As in the Archimedean case one can show that the linear map

\[ T : Z \to (X/Y)', (Tz)(x + Y) = z(x) \]

is an isometric isomorphism. Finite-dimensional normed spaces are reflexive ([7], Corollary 5.5), so the canonical map \(\pi : (X/Y) \to (X/Y)^{''}\) is an isometric isomorphism. Thus the map \(T'' \circ \pi : (X/Y) \to Z''\) is an isometric isomorphism, too.

Hence for any \(\phi \in Z'\) there is an \(x_0 \in X\) with \(||x_0 + Y|| = \|\phi\|\) such that \(\phi(z) = z(x_0)\) for any \(z \in Z\). Put \(\delta > 0\). For some \(x = x_{\phi, \delta} \in x_0 + Y \subset X\) we have \(\|\phi\| \leq \|x\| \leq (1 + \delta)\|\phi\|\) and \(z(x) = \phi(z)\) for any \(z \in Z\).

Let \(e > 0\), \(t = (2 + e)^{-1}\) and \(\delta = (2 + e)^{-1}\). Let \(\phi_1, \ldots, \phi_n\) be a \(t\)-orthogonal basis in \(Z'\) and let \(x_i = x_{\phi_i, \delta}\) for \(i \leq n\). Put \(W_e = \text{lin}\{x_i : i \leq n\}\). Let \(\phi \in Z'\). Then \(\phi = \sum_{i=1}^{n} \alpha_i \phi_i\) for some \(\alpha_1, \ldots, \alpha_n \in K\). For \(x = \sum_{i=1}^{n} \alpha_i x_i\) and \(z \in Z\) we have

\[ z(x) = \sum_{i=1}^{n} \alpha_i z(x_i) = \sum_{i=1}^{n} \alpha_i \phi_i(z) = \phi(z). \]

Clearly

\[ \|\phi\| = \sup_{z \in Z \setminus \{0\}} \frac{\|\phi(z)\|}{\|z\|} = \sup_{z \in Z \setminus \{0\}} \frac{|z(x)|}{\|z\|} \leq \|x\|. \]

Moreover

\[ \|\phi\| \geq t \max_{i \leq n} \|\alpha_i \phi_i\| \geq t(1 + \delta)^{-1} \max_{i \leq n} \|\alpha_i x_i\| \geq (1 + e)^{-1}\|x\|. \]

Thus for any \(\phi \in Z'\) there exists an \(x \in W_e\) with \(\|\phi\| \leq \|x\| \leq (1 + e)\|\phi\|\) such that \(z(x) = \phi(z)\) for any \(z \in Z\).

We will need the following four results for biorthogonal sequences in normed spaces.

**Proposition 3.2.** Let \((X, \| \cdot \|)\) be a normed space with a biorthogonal sequence \(((x_n, f_n)) \subset X \times X'\) with \(\text{lin}\{x_n : n \in N\} = X\). Let \(L\) be an infinite subset of \(N\) and let \(t \in (0, 1)\). Then there exists an increasing sequence \(\{n_i\} \subset L\) such that \((x_{n_i}, + M)\) is a \(t\)-orthogonal basis in the quotient \(X/M\), where \(M = \bigcap_{i=1}^{\infty} \ker f_{n_i}\).

**Proof.** Put \(c = \sqrt{1/t}\). Since \(c > 1\), there is a sequence \((\epsilon_i)\) of positive numbers with \(\prod_{i=1}^{\infty} (1 + \epsilon_i) < c\). Using Lemma 3.1 we can inductively choose an increasing sequence \(\{n_i\} \subset L\) such that for any \(i \in N\) and any \(\phi \in \text{lin}\{f_{n_j} : j \leq i\}'\) there exists an \(x \in \text{lin}\{x_j : j < n_{i+1}\}\) with \(\|\phi\| \leq \|x\| \leq (1 + \epsilon_i)\|\phi\|\) such that \(f(x) = \phi(f)\) for any \(f \in \text{lin}\{f_{n_j} : j \leq i\}\).
We shall prove that the sequence \((f_n)\) is \(t\)-orthogonal in \(X'\).

Let \((\alpha_j) \subset \mathbb{K}\). Put \(\mu_i = (1 + \epsilon_i)^{-2}\) and \(h_i = \sum_{j=1}^{i} \alpha_j f_n\) for \(i \in \mathbb{N}\). Clearly \(\prod_{j=1}^{i} \mu_j > t\) for any \(i \in \mathbb{N}\).

We shall show that \(\|h_{i+1}\| \geq \mu_i \|h_i\|\) for any \(i \in \mathbb{N}\). Let \(i \in \mathbb{N}\). We can assume that \(h_i \neq 0\). By the Hahn-Banach theorem (\([8, \text{Theorem 4.2}]\)) there exists some \(\phi \in \text{lin}\{f_n : j \leq i\}'\) with \(\|\phi(h_i)\| = 1\) such that \(\|\phi\| \leq (1 + \epsilon_i)\|\phi\| \leq \mu_i^{-1} \|h_i\|^{-1}\) such that \(h_i(x) = \phi(h_i) = 1\). Since \(f_{n_{i+1}}(x) = 0\) we obtain

\[\|h_{i+1}\| \geq \|h_{i+1}(x)\||x|^{-1} = \|h_i(x)\||x|^{-1} \geq \mu_i \|h_i\| .\]

Thus \(\|h_{i+1}\| \geq \mu_i \|h_i\|\) for any \(i \in \mathbb{N}\).

By induction we get

\[
\|h_{i+1}\| \geq \prod_{j=1}^{i} \mu_j \max_{j \leq i+1} \|\alpha_j f_n\| \quad \text{for any } i \in \mathbb{N}.
\]

Indeed, for \(i = 1\) we get

\[
\|h_2\| = \max\{\|\alpha_1 f_n\|, \|\alpha_2 f_n\|\} \geq \mu_1 \max\{\|\alpha_1 f_n\|, \|\alpha_2 f_n\|\}
\]

if \(\|\alpha_1 f_n\| \neq \|\alpha_2 f_n\|\), and

\[
\|h_2\| \geq \mu_1 \|\alpha_1 f_n\| = \mu_1 \max\{\|\alpha_1 f_n\|, \|\alpha_2 f_n\|\}
\]

otherwise. Assume that (3.1) is true for some \(i \in \mathbb{N}\). If \(\|\alpha_{i+2} f_{n_{i+2}}\| \leq \|h_{i+1}\|\) we have

\[
\|h_{i+2}\| \geq \|h_{i+1}\| \geq \prod_{j=1}^{i+1} \mu_j \max_{j \leq i+2} \|\alpha_j f_n\| ;
\]

otherwise we get

\[
\|h_{i+2}\| = \|\alpha_{i+2} f_{n_{i+2}}\| = \max\{\|\alpha_{i+2} f_{n_{i+2}}\|, \|h_{i+1}\|\} \geq \prod_{j=1}^{i+1} \mu_j \max_{j \leq i+2} \|\alpha_j f_n\| .
\]

Thus

\[
\|\sum_{j=1}^{i+1} \alpha_j f_n\| \geq t \max_{j \leq i+1} \|\alpha_j f_n\|
\]

for any \((\alpha_j) \subset \mathbb{K}\) and any \(i \in \mathbb{N}\). This means that \((f_n)\) is \(t\)-orthogonal in \(X'\).

Denote by \(F\) the closure of \(\text{lin}\{f_n : i \in \mathbb{N}\}\) in \(X'\). Let \((g_i) \subset F'\) be the sequence of coefficient functionals associated with the basis \((f_n)\) in \(F\). It is easy to check that \((g_i)\) is a \(t\)-orthogonal sequence in \(F'\). Denote by \(G\) the linear span of \((g_i)\) in \(F'\). Put \(M = \bigcap_{i=1}^{\infty} \ker f_n\). Then \(\text{lin}\{x_n : i \in \mathbb{N}\} + M = X\), so \(\text{lin}\{x_n + M : i \in \mathbb{N}\} = X/M\).

The map \(S : X/M \to F'\), \((S(x + M))(f) = f(x)\) is well defined, linear and injective. Moreover \(S(x_n + M) = g_i\) for \(i \in \mathbb{N}\); so \(S(X/M) = G\).

To prove that \((x_n + M)\) is a \(t\)-orthogonal basis in \(X/M\) it is enough to show that \(S\) is an isometry. Let \(x \in X\). For \(y \in M\) we have

\[
\|S(x + M)\| = \sup_{f \in F\setminus\{0\}} \frac{|f(x)|}{\|f\|} = \sup_{f \in F\setminus\{0\}} \frac{|f(x + y)|}{\|f\|} \leq \|x + y\| ;
\]

hence \(\|S(x + M)\| \leq \|x + M\|\). Put \(g = S(x + M)\). For some \(i_0 \in \mathbb{N}\) we have \(g \in \text{lin}\{g_j : j \leq i_0\}\). Let \(i \geq i_0\) and \(F_i = \text{lin}\{f_j : j \leq i\}\). Clearly, \(g|_{F_i} \in F'_i\) and \(\|g|_{F_i}\| \leq \|g\|\). Then there exists some \(y_i \in \text{lin}\{x_j : j < i_{i+1}\}\) with \(\|y_i\| \leq \|x + M\|\).
(1 + \epsilon_i)\|g|_{F_i}\| such that \(f(y_i) = g(f)\) for any \(f \in F_i\). Hence \(f_n_j(y_i) = g(f_n_j)\) for \(j \leq i\); for \(j > i\) we have \(f_n_j(y_i) = 0 = g(f_n_j)\). It follows that \(f(y_i) = g(f)\) for any \(f \in F\). Thus \(S(y_i + M) = g\), so

\[||S^{-1}g|| = ||y_i + M|| \leq ||y_i|| \leq (1 + \epsilon_i)\|g|_{F_i}\| \leq (1 + \epsilon_i)\|g\|.

Since \(\lim_{i \to \infty} \epsilon_i = 0\) we get \(||S^{-1}g|| \leq \|g\|\); so \(||x + M|| \leq ||S(x + M)||\). We have shown that \(||S(x + M)|| = ||x + M||\) for any \(x + M \in X/M\); so \(S\) is an isometry. □

**Lemma 3.3.** Let \((X, \| \cdot \|)\) be a normed space and let \((x_n, f_n) \subset X \times X'\) be a biorthogonal sequence. Then for any finite subset \(A\) of \(\mathbb{N}\) there exists \(d_A \in (0, 1]\) such that

\[
\| \sum_{n=1}^{\infty} \phi_n x_n \| \geq d_A \max_{n \in A} \{ \max \{ \| \phi_n x_n \|, \| \sum_{n \in \mathbb{N} \setminus A} \phi_n x_n \| \} \} \text{ for all } (\phi_n) \in c_{00}.
\]

**Proof.** Put \(d_k = (\|f_k\|\|x_k\|)^{-1}\) for \(k \in \mathbb{N}\); then \((d_k) \subset (0, 1]\). We shall show that

\[
(3.2) \quad \| \sum_{n=1}^{\infty} \phi_n x_n \| \geq d_k \max \{ \| \phi_k x_k \|, \| \sum_{n \neq k} \phi_n x_n \| \} \text{ for } (\phi_n) \in c_{00} \text{ and } k \in \mathbb{N}.
\]

Let \((\phi_n) \in c_{00}\) and let \(k \in \mathbb{N}\). Then

\[
|\phi_k| = |f_k(\sum_{n=1}^{\infty} \phi_n x_n)| \leq \|f_k\| \| \sum_{n=1}^{\infty} \phi_n x_n \|.
\]

Hence

\[
\| \sum_{n=1}^{\infty} \phi_n x_n \| \geq |\phi_k| \|f_k\|^{-1} = d_k \|\phi_k x_k\|.
\]

Using (7), Lemma 3.1, we get

\[
\| \sum_{n=1}^{\infty} \phi_n x_n \| \geq d_k \max \{ \| \phi_k x_k \|, \| \sum_{n \neq k} \phi_n x_n \| \}.
\]

Let \(m \in \mathbb{N}\). Assume that for any \(m\)-element subset \(B\) of \(\mathbb{N}\) we have

\[
\| \sum_{n=1}^{\infty} \phi_n x_n \| \geq \prod_{n \in B} d_n \max \{ \max_{n \in B} \{ \| \phi_n x_n \|, \| \sum_{n \in \mathbb{N} \setminus B} \phi_n x_n \| \} \} \text{ for } (\phi_n) \in c_{00}.
\]

Let \(A \subset \mathbb{N}\) be a set with \(m + 1\) elements and let \((\phi_n) \in c_{00}\). Take an element \(k\) of \(A\) and put \(B = (A \setminus \{k\})\). Let \(\psi_n = 0\) if \(n \in B\) and \(\psi_n = \phi_n\) if \(n \in (\mathbb{N} \setminus B)\); clearly \((\psi_n) \in c_{00}\). Using (3.2) we get

\[
\| \sum_{n \in \mathbb{N} \setminus B} \phi_n x_n \| = \| \sum_{n=1}^{\infty} \psi_n x_n \| \geq d_k \max \{ \| \psi_k x_k \|, \| \sum_{n \neq k} \psi_n x_n \| \}
\]

\[
= d_k \max \{ \| \phi_k x_k \|, \| \sum_{n \in \mathbb{N} \setminus A} \phi_n x_n \| \}.
\]
Hence, by our assumption, we have
\[
\|\sum_{n=1}^{\infty} \phi_n x_n\| \geq \prod_{n \in B} d_n \max_{n \in B} \{\max_{n \in \mathbb{N}\setminus B} \|\phi_n x_n\|, \|\sum_{n \in \mathbb{N}\setminus B} \phi_n x_n\|\}
\]}
\[
\geq \prod_{n \in A} d_n \max_{n \in A} \{\max_{n \in \mathbb{N}\setminus A} \|\phi_n x_n\|, \|\sum_{n \in \mathbb{N}\setminus A} \phi_n x_n\|\}.
\]

Thus, by induction, we have shown our lemma.

**Lemma 3.4.** Let \((X, \| \cdot \|)\) be a normed space and let \(((x_n, f_n)) \subset X \times X'\) be a biorthogonal sequence with \(\text{lin}\{x_n : n \in \mathbb{N}\} = X\). Put \(X_S = \bigcap_{n \in S} \ker f_n\) for \(S \subset \mathbb{N}\). Assume that \(C, B \subset \mathbb{N}\) and \(C \not\subset B\) is finite. If \((x_n + X_B)_{n \in B}\) is \(t\)-orthogonal in \(X/X_B\) for some \(t \in (0, 1]\), then \((x_n + X_C)_{n \in C}\) is \(s\)-orthogonal in \(X/X_C\) for some \(s \in (0, 1]\).

**Proof.** Put \(A = C \setminus B\), \(D = C \cap B\) and \(H = B \setminus C\). Let \(Y_S = \text{lin}\{x_n : n \in S\}\) for \(S \subset \mathbb{N}\); then we have \(Y_S = \{\sum_{n=1}^\infty \varphi_n x_n : (\varphi_n) \in \ell_0(S)\} = X_{\mathbb{N}\setminus S}\). It is easy to see that \(X_D = X_C + Y_A = X_B + Y_H\). Let \(d_A\) be as in Lemma 3.3 and put \(s = td_A^2\).

For \(m \in D\) we have \(\|x_m + X_H\| \geq d_A^2\|x_m + X_C\|\). Indeed, let \(m \in D\) and let \(x \in X_D\). Then \(x = y + \sum_{n=1}^\infty \psi_n x_n\) for some \(y \in X_C\) and some \((\psi_n) \in \ell_0(A)\).

Using Lemma 3.3 we get
\[
\|x_m + x\| = \|x_m + y + \sum_{n=1}^\infty \psi_n x_n\| \geq d_A \max_{n \in N} \{\max_{n \in N} \|\psi_n x_n\|, \|x_m + y\|\} \geq d_A \|\|x_m + X_C\|\|
\]

Hence we obtain
\[
\|\sum_{n=1}^\infty \alpha_n x_n + X_D\| \geq d_A \|\|\alpha_n x_n + X_C\|\| \text{ for } (\alpha_n) \in \ell_0(D).
\]

Indeed, let \((\alpha_n) \in \ell_0(D)\) and let \(x \in X_D\). For some \(y \in X_B\) and some \((\beta_n) \in \ell_0(H)\) we have \(x = y + \sum_{n=1}^\infty \beta_n x_n\). Then
\[
\|\sum_{n=1}^\infty \alpha_n x_n + x\| \geq \|\sum_{n=1}^\infty \alpha_n x_n + \sum_{n=1}^\infty \beta_n x_n + X_B\| \geq t \max_{n \in N} \|\alpha_n x_n + X_B\|
\]
\[
\geq t \max_{n \in N} \|\alpha_n x_n + X_D\| \geq d_A \|\|\alpha_n x_n + X_C\|\|
\]

We shall prove that the sequence \((x_n + X_C)_{n \in C}\) is \(s\)-orthogonal in \(X/X_C\). To show this it is enough to prove that
\[
\|\sum_{n=1}^\infty \phi_n x_n + X_C\| \geq s \max_{n \in N} \|\phi_n x_n + X_C\| \text{ for } (\phi_n) \in \ell_0(C).
\]
Let \((\varphi_n) \in c_{00}(C)\). Then \((\varphi_n) = (\gamma_n) + (\alpha_n)\) for some \((\gamma_n) \in c_{00}(A)\) and some \((\alpha_n) \in c_{00}(D)\). Let \(x \in X_C\). Using Lemma 3.3 we get

\[
\|\sum_{n=1}^{\infty} \varphi_n x_n + x\| = \|\sum_{n=1}^{\infty} \gamma_n x_n + \sum_{n=1}^{\infty} \alpha_n x_n + x\|
\]

\[
\geq d_A \max\{ \max_{n \in \mathbb{N}} \|\gamma_n x_n\|, \|\sum_{n=1}^{\infty} \alpha_n x_n + x\| \}
\]

\[
\geq d_A \max\{ \max_{n \in \mathbb{N}} \|\gamma_n x_n + X_C\|, \|\sum_{n=1}^{\infty} \alpha_n x_n + X_D\| \}
\]

\[
\geq td_A \max_{n \in \mathbb{N}} \|\gamma_n x_n + X_C\|, \max_{n \in \mathbb{N}} \|\alpha_n x_n + X_C\| \} = s \max_{n \in \mathbb{N}} \|\varphi_n x_n + X_C\|.
\]

\[
\square
\]

Lemma 3.5. Let \((X, \| \cdot \|)\) be a normed space and let \(((x_n, f_n)) \subset X \times X'\) be a biorthogonal sequence such that the subspace \(X_0 = \text{lin}\{x_n : n \in \mathbb{N}\}\) is dense in \(X\). Let \(L \subset \mathbb{N}\). Put \(W = \bigcap_{n \in L} \ker f_n\) and \(W_0 = X_0 \cap W\). If the sequence \((x_n + W_0)_{n \in L}\) is \(t\)-orthogonal in \(X_0/W_0\) for some \(t \in (0, 1]\), then \((x_n + W)_{n \in L}\) is a \(t\)-orthogonal basis in \(X/W\) and

\[
\|f_n\|^{-1} \leq \|x_n + W\| \leq (t\|f_n\|)^{-1} \text{ for any } n \in L.
\]

Proof. For any \(x \in X\) we have \(\inf\{\|x + y\| : y \in W_0\} = \inf\{\|x + y\| : y \in \overline{W}_0\}\), where \(\overline{W}_0\) is the closure of \(W_0\) in \(X\). Thus \((x_n + \overline{W}_0)_{n \in L}\) is \(t\)-orthogonal in \((X/\overline{W}_0)\).

Denote by \(\pi\) the quotient map \(X \to X/\overline{W}_0\). We have

\[
\pi(X_0) = \text{lin}\{x_n + \overline{W}_0 : n \in L\},
\]

so \((x_n + \overline{W}_0)\) is linearly dense in \(X/\overline{W}_0\). It follows that \((x_n + \overline{W}_0)_{n \in L}\) is a \(t\)-orthogonal basis in \(X/\overline{W}_0\).

We shall prove that \(\overline{W}_0 = W\). For any \(n \in L\) the functional

\[
\tilde{f}_n : X/\overline{W}_0 \to \mathbb{K}, \quad \tilde{f}_n(x + \overline{W}_0) = f_n(x)
\]

is well defined, linear and continuous. Indeed, for any neighbourhood \(V\) of zero in \(\mathbb{K}\) we have \(f_n^{-1}(V) = \pi^{-1}(\tilde{f}_n^{-1}(V))\), so the set \(\pi^{-1}(V) = \tilde{f}_n^{-1}(V)\) is open in \(X/\overline{W}_0\). Clearly \(\overline{W}_0 \subset W\). Let \(w \in W\). Then for some \((\alpha_m) \subset \mathbb{K}\) we have \(w + \overline{W}_0 = \sum_{m \in L} \alpha_m (x_m + \overline{W}_0)\). Hence for any \(n \in L\) we get

\[
0 = f_n(w) = \tilde{f}_n(w + \overline{W}_0) = \sum_{m \in L} \alpha_m \tilde{f}_n(x_m + \overline{W}_0) = \alpha_n.
\]

Thus \(w + \overline{W}_0 = 0\), so \(w \in \overline{W}_0\). It follows that \(\overline{W}_0 = W\).

We have shown that \((x_n + W)_{n \in L}\) is a \(t\)-orthogonal basis in \(X/W\).

Let \(n \in L\). It is easy to see that

\[
x + W = \sum_{m \in L} f_m(x)(x_m + W) \text{ for } x \in X.
\]

For \(x \in X\) we get

\[
\|x\| \geq \|x + W\| \geq t \max_{m \in L} |f_m(x)| \|x_m + W\| \geq t|f_n(x)| \|x_n + W\|.
\]

Thus

\[
\|f_n\| = \sup_{x \in X\setminus\{0\}} |f_n(x)| \|x\|^{-1} \leq (t\|x_n + W\|)^{-1}.
\]
Hence \( |||x_n + W||| \leq (t||f_n||)^{-1} \). On the other hand we have
\[
1 = |f_n(x_n + w)| \leq ||f_n|| ||x_n + w|| \text{ for } w \in W.
\]
Thus \( ||f_n||^{-1} \leq |||x_n + W||| \). This completes the proof. \( \square \)

Let \( E \) be a metrizable lcs with a non-decreasing base \((\parallel \cdot\parallel_k)\) in \( \mathcal{P}(E) \). For \( k \in \mathbb{N} \) we denote by \( E'_k \) (or \( E'_k^{||\cdot||_k} \)) the linear subspace of \( E' \) consisting of all \( f \in E' \) such that for some \( C > 0 \) we have \( |f(x)| \leq C\parallel x \parallel_k \) for any \( x \in E \). Clearly, the map
\[
\parallel \cdot \parallel'_k : E'_k \to [0, +\infty), \parallel f \parallel'_k = \inf\{C > 0 : |f(x)| \leq C\parallel x \parallel_k \text{ for any } x \in E\}
\]
is a norm on \( E'_k \), and the normed space \((E'_k, \parallel \cdot \parallel'_k)\) is complete. Moreover we have
\[
\bigcup_{k=1}^{\infty} E'_k = E', E'_k \subset E'_{k+1} \text{ and } \parallel f \parallel'_{k+1} \leq \parallel f \parallel'_k \text{ for } f \in E'_k \text{ and } k \in \mathbb{N}.
\]
Clearly, a metrizable lcs has a non-decreasing base of continuous norms if and only if it has a continuous norm.

For biorthogonal sequences in a metrizable lcs with a continuous norm we get

**Proposition 3.6.** Let \( E \) be a metrizable lcs with a non-decreasing base \((\parallel \cdot\parallel_k)\) of continuous norms and let \((x_n, f_n) \subset E \times E_1'\), where \( E_1 = (E, \parallel \cdot \parallel_1) \), is a biorthogonal sequence such that \((x_n)\) is linearly dense in \( E \). Let \( L \) be an infinite subset of \( \mathbb{N} \). Then there exists an infinite subset \( J \) of \( L \) such that the sequence \((x_n + W)_{n \in J}\), where \( W = \bigcap_{n \in J} \ker f_n \), is an orthogonal basis in the quotient space \( E/W \) with a continuous norm. The space \( E/W \) is nuclear if
\[
\lim_{n \in J} \frac{||f_n||_k}{||f_n||'_{k+1}} = \infty \text{ for any } k \in \mathbb{N}.
\]

**Proof.** Let \( t \in (0, 1) \). Put \( E_0 = \text{lin}\{x_n : n \in \mathbb{N}\}, E_k = (E, \parallel \cdot \parallel_k) \) and \( E_k = (E_0, \parallel \cdot \parallel_k|E_0) \) for \( k \in \mathbb{N} \). Clearly, \( E'_1 \subset E'_k \) and \( E_0 \) is dense in \( E_k \) for any \( k \in \mathbb{N} \). By Proposition 3.2 we can inductively choose a decreasing sequence \((J_k)\) of infinite subsets of \( L \) such that the sequence \((x_n + W_k)_{n \in J_k}\) is \( t\)-orthogonal in \( E_k/W_k \) for \( k \in \mathbb{N} \), where \( W_k = \bigcap_{n \in J_k} \ker f_n \subset E_0 \).

Let \( (m_k) \subset \mathbb{N} \) be an increasing sequence such that \( m_k \in J_k \) for \( k \in \mathbb{N} \). Put \( J = \{m_k : k \in \mathbb{N}\}, W = \bigcap_{n \in J} \ker f_n \) and \( W_0 = E_0 \cap W \). Clearly, the set \((J \setminus J_k)\) is finite for any \( k \in \mathbb{N} \). By Lemma 3.4 there exists \((s_k) \subset (0, 1)\) such that the sequence \((x_n + W_k)_{n \in J}\) is \( s_k\)-orthogonal in \((E_k/W_k)_{n \in J}\) for any \( k \in \mathbb{N} \).

Using Lemma 3.5 we infer that the sequence \((x_n + W)_{n \in J}\) is an \( s_k\)-orthogonal basis in the quotient space \( E_k/W \) for any \( k \in \mathbb{N} \). It follows that \((x_n + W)_{n \in J}\) is an orthogonal basis in \( E/W \) (see [4], Proposition 2.6). \( W \) is closed in \( E_1 \), so the space \( E/W \) has a continuous norm.

If \( \lim_{n \in J} \frac{||f_n||_k}{||f_n||'_{k+1}} = \infty \text{ for any } k \in \mathbb{N} \), then using Lemma 3.5 we get
\[
\lim_{n \in J} \frac{||x_n + W||_k}{||x_n + W||'_{k+1}} = 0 \text{ for any } k \in \mathbb{N}.
\]
It follows that \( E/W \) is nuclear (see [2], Proposition 3.5, and its proof). \( \square \)

**Corollary 3.7.** Let \( E \) be a Fréchet space with a continuous norm. For any biorthogonal sequence \(((x_n, f_n) \subset E \times E'\) such that \((x_n)\) is linearly dense in \( E \) and \((f_n)\) is equicontinuous, there exists an infinite subset \( J \) of \( \mathbb{N} \) such that \((E/\cap_{n \in J} \ker f_n)\) is a Köthe space.
Corollary 3.8. Any infinite-dimensional metrizable lcs $E$ of countable type with a continuous norm has an infinite-dimensional quotient with a continuous norm and with an orthogonal basis.

Proof. Let $(\| \cdot \|_k)$ be a non-decreasing base of continuous norms on $E$ and let $(y_n)$ be a linearly independent sequence in $E$ such that its linear span $E_0$ is dense in $E$. Put $E_1 = (E, \| \cdot \|_1)$.

Using the $p$-adic Hahn-Banach theorem we can inductively choose a sequence $((x_n, f_n)) \subset E_1 \times E_1'$ such that $\text{lin}\{x_i : i \leq n\} = \text{lin}\{y_i : i \leq n\}$, $x_n \in \bigcap_{k < n} \ker f_k$, $f_n \in (\text{lin}\{x_i : i < n\})'$ and $f_n(x_n) = 1$ for any $n \in \mathbb{N}$. Clearly, the sequence $((x_n, f_n))$ is biorthogonal and $\text{lin}\{x_n : n \in \mathbb{N}\} = E_0$. Using Proposition 3.6 completes the proof.

Corollary 3.9. Any infinite-dimensional Fréchet space $E$ of countable type with a continuous norm has a Köthe quotient.

Clearly, any Fréchet space which is not of finite type has an infinite-dimensional quotient with a continuous norm. Hence using the previous corollary we get the following one (see [10], Corollary 13).

Corollary 3.10. An infinite-dimensional Fréchet space $E$ of countable type has a Köthe quotient if and only if it is not isomorphic to $\mathbb{K}^\mathbb{N}$.

Let $E$ be a Fréchet space with a non-decreasing base $(\| \cdot \|_k)$ in $\mathcal{P}(E)$. Let $M$ be a closed subspace of $E$. Let $k \in \mathbb{N}$ and $f \in E_k' \cap M^\perp$. Then the functional $\phi_M(f) : E/M \to \mathbb{K}$, $x + M \to f(x)$ is well defined and linear. Moreover we have

$$|(\phi_M(f))(x + M)| = |f(x)| = |f(x + y)| \leq \|f\|_k \|x + y\|_k$$

for all $x \in E$ and $y \in M$. Hence $|(\phi_M(f))(x + M)| \leq \|f\|_k \|x + M\|_k$ for $x \in E$, so $\phi_M(f) \in (E/M)'_k$ and $\|\phi_M(f)\|_k \leq \|f\|_k$. On the other hand we get $\|f\|_k \leq \|\phi_M(f)\|_k$ since

$$|f(x)| = |(\phi_M(f))(x + M)| \leq \|\phi_M(f)\|_k \|x + M\|_k \leq \|\phi_M(f)\|_k \|x\|_k$$

for $x \in E$. We have shown that $\|\phi_M(f)\|'_k = \|f\|'_k$ for all $f \in E_k' \cap M^\perp$ and $k \in \mathbb{N}$.

If $k \in \mathbb{N}$, $g \in (E/M)'_k$ and $\pi : E \to E/M$ is the quotient map, then $g \circ \pi \in E_k' \cap M^\perp$. It follows that the map

$$\phi_M : M^\perp \to (E/M)'_k, \ f \to \phi_M(f)$$

is well defined and surjective.

Clearly, $\phi_M$ is linear and injective. Thus $\phi_M$ is an isomorphism. Moreover we have

$$\|\phi_M(f)\|'_k = \|f\|'_k$$

for all $f \in E_k' \cap M^\perp$ and $k \in \mathbb{N}$.

Now we can prove our main result.

Theorem 3.11. A Fréchet space $E$ of countable type has a nuclear Köthe quotient if and only if for some non-decreasing base $(\| \cdot \|_k)$ in $\mathcal{P}(E)$ the norms $\| \cdot \|_k'_{E_k'}$ and $\| \cdot \|_{k+1} \big|_{E_{k+1}'}$ are not equivalent for any $k \in \mathbb{N}$.

Proof. (A) Assume that $E$ has a nuclear Köthe quotient $Z = E/M$. Let $(z_n)$ be a Schauder basis in $Z$ and let $(g_n) \subset Z'$ be a sequence of coefficient functionals associated with the basis $(z_n)$. Clearly, $(z_n)$ is $(1)$-orthogonal with respect to some non-decreasing base of norms $(p_k)$ in $\mathcal{P}(Z)$. By the nuclearity of $Z$ we can assume that $\lim_{n \to \infty} [p_{k+1}(z_n)/p_k(z_n)] = \infty$ for $k \in \mathbb{N}$ (see [2], Proposition 3.5).
We shall prove that \( \lim_{n \to \infty} [p_k'(g_n)/p_{k+1}'(g_n)] = \infty \) for \( k \in \mathbb{N} \). We have \( p_k(z) = \max_{m \in \mathbb{N}} |g_m(z)|p_k(z_m) \geq |g_n(z)|p_k(z_n) \) for \( z \in Z \) and \( k, n \in \mathbb{N} \), so \( (g_n) \subset Z_{p_1}' \) and \( p_k'(g_n) \leq [p_k'(z_n)]^{-1} \) for \( k, n \in \mathbb{N} \). On the other hand we have \( 1 = |g_n(z_n)| \leq p_k(g_n)p_k(z_n) \) for \( k, n \in \mathbb{N} \). Thus \( p_k'(g_n) = [p_k'(z_n)]^{-1} \) for all \( k, n \in \mathbb{N} \). Hence we get
\[
\lim_{n \to \infty} [p_k'(g_n)/p_{k+1}'(g_n)] = \infty \quad \text{for} \quad k \in \mathbb{N}.
\]

Let \((\cdot, \cdot)_k\) be a non-decreasing base in \( \mathcal{P}(E) \) and let \((||| \cdot |||_k\)) be the base in \( \mathcal{P}(Z) \) induced by \((\cdot, \cdot)_k\). Passing to subsequences we can assume that \( p_k \leq ||| \cdot |||_k \leq p_{k+1} \) for \( k \in \mathbb{N} \). Then \( p_{k+1}'|_{Z_{p_1}'} \leq &&1||| \cdot |||_{p_1}' \leq p_k|_{Z_{p_1}'} \) for \( k \in \mathbb{N} \).

For \( g \in Z_{p_1}' = (E/M)'_{p_1} \) and \( h = \phi_M^{-1}(g) \) we have
\[
|h(x)| = |(\phi_M(h))(x + M)| = |g(x + M)| \\
\leq p_1'(g)p_1(x + M) \leq p_1'(g)|||x + M|||_1 \leq p_1'(g)|x|_1
\]
for \( x \in E \), so \( \phi_M^{-1}(g) \in E_{11}', \) for \( g \in Z_{p_1}' \).

Put \( ||| \cdot ||| = ||| \cdot |||_{2k-1} \) for \( k \in \mathbb{N} \). Then \((||| \cdot |||_k\)) is a non-decreasing base in \( \mathcal{P}(E) \) and \((\phi_M^{-1}(g_n)) \subset E_{11}', \). Moreover we have
\[
\frac{|||\phi_M^{-1}(g_n)|||_{2k-1}}{|||\phi_M^{-1}(g_n)|||_{2k+1}} = \frac{|||\phi_M^{-1}(g_n)|||_{2k-1}}{|||\phi_M^{-1}(g_n)|||_{2k+1}} = \frac{|||g_n|||_{2k-1}}{|||g_n|||_{2k+1}} \geq \frac{p_{2k+1}(g_n)}{p_{2k+1}(g_n)} \quad \text{for all} \quad k, n \in \mathbb{N}.
\]

It follows that the norms \( ||| \cdot |||_{E_{11}'} \) and \( ||| \cdot |||_{2k+1}' \) are not equivalent for any \( k \in \mathbb{N} \).

(B) Now assume that \( E \) has a non-decreasing base \((||| \cdot |||_k\)) in \( \mathcal{P}(E) \) such that the norms \( \cdot |||_k \) \( E_{1}' \text{ and } ||| \cdot |||_{k+1} \) \( E_{1}' \) are not equivalent for any \( k \in \mathbb{N} \). Without loss of generality we can assume that \( \cdot |||_k, k \in \mathbb{N}, \) are norms. Indeed, put \( M = E_{1}' \text{ and } Z = E/M \). Let \( x \in E \) with \( |||x + M|||_1 = 0 \). Then there exists \( y_n \in M \) such that \( \lim_{n \to \infty} |||x - y_n|||_1 = 0 \). Let \( f \in E_{1}' \). Then \( |f(x)| = \lim_{n \to \infty} |f(x - y_n)| = 0, \) so \( f(x) = 0 \). It follows that \( x \in M \), so \((||| \cdot |||_k\)) is a non-decreasing base of norms in \( \mathcal{P}(Z) \).

Clearly \( E_{1}' \subset M^{\perp} \). Thus \( |||\phi_M(f)|||_k = |||f|||_k \) for all \( f \in E_{1}' \) and \( k \in \mathbb{N} \). Let \( k \in \mathbb{N} \). Then there exists a sequence \((h_n) \subset E_{1}' \) such that \( \lim_{n \to \infty} |||h_n|||_{k}/|||h_n|||_{k+1} = \infty \). Hence \( \phi_M(h_n) \subset Z_{11}' \) and \( \lim_{n \to \infty} |||\phi_M(h_n)|||_{k}/|||\phi_M(h_n)|||_{k+1} = \infty \), so the norms \( \cdot |||_{k} \) \( Z_{11}' \) and \( \cdot |||_{k+1} \) \( Z_{11}' \) are not equivalent for any \( k \in \mathbb{N} \). If \( Z \) has a nuclear Köthe quotient, then \( E \) has one.

Thus from now on we shall assume that \( \cdot |||_k, k \in \mathbb{N}, \) are norms on \( E \). We show that for any linear subspace \( G \) of \( E_{1}' \) with \( \dim(E_{1}'/G) < \infty \) we have
\[
(3.3) \quad \sup \{ |||g|||_{k}/|||g|||_{k+1} : g \in (G \setminus \{0\}) \} = \infty \quad \text{for} \quad k \in \mathbb{N}.
\]

Indeed, suppose by contradiction that for some subspace \( G \) of finite codimension in \( E_{1}' \) there exist some \( k \in \mathbb{N} \) and \( C > 0 \) such that \( |||g|||_{k} \leq C|||g|||_{k+1} \) for any \( g \in G \).

Denote by \( G_k \) the closure of \( G \) in the Banach space \((E_{1}', \cdot |||_k)\). For any \( h \in G_k \) there exists \((g_n) \subset G \) with \( \lim_{n \to \infty} |||h - g_n|||_{k} = 0 \). Since \( |||h - g_n|||_{k+1} \leq |||h - g_n|||_{k} \), then \( \lim_{n \to \infty} |||g_n|||_{k+1} = |||h|||_{k+1} \) and \( \lim_{n \to \infty} |||g_n|||_{k} = |||h|||_{k} \). By our assumption we get \( |||g_n|||_{k} \leq C|||g_n|||_{k+1} = n \in \mathbb{N} \). Thus \( |||h|||_{k+1} \leq |||h|||_{k} \leq C|||h|||_{k+1} \) for any \( h \in G_k \).

It follows that the norms \( ||| \cdot |||_{k+1} \) \( G_k \) and \( \cdot |||_{k} \) \( G_k \) are equivalent, so the normed space \((G_k, \cdot |||_{k+1}) \) \( G_k \) is complete. Hence \( G_k \) is closed in \((E_{1}', \cdot |||_{k+1}) \). For some finite-dimensional subspace \( S \) of \( E_{1}' \) we have \( E_{1}' = G + S \). Put \( H_k = G_k + S \).
Then $E'_1 \subset H_k$; moreover $H_k$ is closed in $E'_k$ and $E'_{k+1}$ (3.14). Thus the norms $\| \cdot \|_{k|H_k}$ and $\| \cdot \|_{k+1|H_k}$ are equivalent, a contradiction.

Using (3.3) we shall construct a biorthogonal sequence $((x_n, f_n)) \subset E \times E'$ such that the sequence $(x_n)$ is linearly dense in $E$ and $\lim_{n \to \infty} (\|f_{n+1}|_{k+1}\|/\|f_n|_{k}) = 0$ for $k \in \mathbb{N}$. Let $E_0$ be an $\mathfrak{N}_0$-dimensional dense subspace of $E$ with a Hamel basis $(y_n)$. Put $x_1 = y_1$. Let $f_1 \in E'_1$ with $f_1(x_1) = 1$. Suppose that for some $n \geq 2$ we have $\{(x_k, f_k) : k < n\} \subset E \times E'_1$ with $\lim \{x_k : k < n\} = \{y_k : k < n\}$ such that $f_k(x_i) = \delta_{k,i}$ for $k < n$, $i < n$, and $\|f_m|_{k+1}/\|f_m|_{k}\| < m^{-1}$ for $m < n$ and $k < m$. Let $g_n \in E'_1 \cap (\{y_k : k < n\})^\perp$ such that $g_n(y_n) = 1$ and $|g_n(y_{n+1})| > \|y_{n+1}\|$; then $\|g_n\|_n > 1$. Using (3.3) we can inductively choose $g_{n-1}, \ldots, g_1 \in E'_1 \cap (\{y_s : s < n\})^\perp$ such that

\[ \|g_k\|_{k+1} < 1 \text{ and } \|g_k\|_k > n \max_{k < i \leq n} \|g_i\|_k \text{ for } k = n - 1, \ldots, 1. \]

Put $x_n = y_n - \sum_{k=1}^{n-1} f_k(y_k)x_k$ and $f_n = \sum_{k=1}^n g_k$. Then $(x_n, f_n) \in E \times E'_1$ and $\lim \{x_k : k \leq n\} = \lim \{g_k : k \leq n\}$. Moreover $f_n(x_n) = 1$ and $f_i(x_n) = 0 = f_n(x_i)$ for $i < n$. We shall show that $\|f_n|_{k+1}/\|f_n|_k\| < n^{-1}$ for $k < n$. Let $k < n$. Then

\[ \max_{i \leq k} \|g_i\|_{k+1} \leq \max_{i \leq k} \|g_i\|_k < 1 < \|g_n\|_n \leq \|g_n\|_{k+1} \leq \max_{k < i \leq n} \|g_i\|_{k+1} \]

and

\[ \max_{k < i \leq n} \|g_i\|_k \leq \max_{k < i \leq n} \|g_i\|_k < n^{-1}\|g_k\|_k. \]

Thus $\|f_n\|_{k+1} < n^{-1}\|g_k\|_k$. On the other hand we have

\[ \|g_k\|_k \leq \|g_k\|_{k+1} < 1 < \|g_n\|_n \leq \|g_n\|_k \leq \|g_k\|_k \text{ for } k < n. \]

Thus $\|g_k\|_k < \|g_k\|_{k+1}$ for $k \leq n$. Hence $\|g_k\|_{k+1} = \|g_k\|_k$. It follows that $\|f_n|_{k+1}/\|f_n|_k\| < n^{-1}$.

Thus we have inductively constructed a biorthogonal sequence $((x_n, f_n)) \subset E \times E'_1$ with $\lim \{x_n : n \in \mathbb{N}\} = E_0$ such that $\|f_n|_{k+1}/\|f_n|_k\| < n^{-1}$ for all $k, n \in \mathbb{N}$ with $k < n$, so $\lim_{n \to \infty} (\|f_n|_{k+1}/\|f_n|_k\|) = 0$ for any $k \in \mathbb{N}$.

Using Proposition 3.6 we infer that $E$ has a nuclear Köthe quotient; in fact for any infinite subset $L$ of $\mathbb{N}$ there exists an infinite subset $J$ of $L$ such that $E/W$ is a nuclear Köthe quotient of $E$, where $W = \bigcap_{n \in J} \ker f_n$. \hfill $\square$

In [10], Theorem 11, we have shown that any nuclear Fréchet space which is not of finite type has a nuclear Köthe quotient. Now we can generalize that result by proving the following.

**Theorem 3.12.** Any Fréchet-Montel space $E$ which is not of finite type has a nuclear Köthe quotient.

**Proof.** By Theorem 3.11 it is enough to show that for some non-decreasing base $\{k\} \in \mathcal{P}(E)$ the norms $\|\cdot\|_k$ and $\|\cdot\|_{k+1}$ are not equivalent for any $k \in \mathbb{N}$. Suppose, by contradiction, that it is not true. Let $\{k\}$ be a non-decreasing base in $\mathcal{P}(E)$. Then for any $l \in \mathbb{N}$ there is some $k \geq l$ such that $\|\cdot\|_k \approx \|\cdot\|_l$ for
any \( j > k \). Passing to a subsequence we can assume that \(|| \cdot | |_{l+1}^{E'_l} \approx || \cdot | |_{l+2}^{E'_l}| |_{l+1}^{E'_l} \) for all \( l, j \in \mathbb{N} \) with \( l + 1 < j \). Let \( F_l \) denote the closure of \( E'_l \) in \((E_{l+1}^{F'_l} | | \cdot | |_{l+1}^{E'_l})\) for \( l \in \mathbb{N} \).

We have \(| | \cdot | |_{l+1}^{F_l} \approx | | \cdot | |_{l+2}^{F_l}| |_{l+1}^{F_l} \) for \( l \in \mathbb{N} \). Indeed, let \( l \in \mathbb{N} \). For some \( C > 0 \) we have \(| f|_{l+1}^F \leq C| f|_{l+2}^F \) for any \( f \in E'_l \). Let \( g \in F_l \) and let \( (f_n) \subset E'_l \) with \( \lim_{n \to \infty} |f_n - g|_{l+2}^{F_l} = 0 \). Thus \( \lim_{n \to \infty} |f_n - g|_{l+2}^{F_l} = 0 \), so \( \lim_{n \to \infty} |f_n|_{l+1}^{F_l} = |g|_{l+1}^{F_l} \) and \( \lim_{n \to \infty} |f_n|_{l+2}^{F_l} = |g|_{l+2}^{F_l} \). Hence \(| f|_{l+2}^{F'} \leq |g|_{l+1}^{F_l} \leq C|g|_{l+2}^{F_l} \), so \(| | \cdot | |_{l+1}^{F_l} \approx | | \cdot | |_{l+2}^{F_l}| |_{l+1}^{F_l} \).

It follows that the normed space \((F_l, | | \cdot | |_{l+2}^{F_l})\) is complete, so \( F_l \) is a closed subspace of \( F_{l+1} = (F_{l+1}, | | \cdot | |_{l+2}^{F_{l+1}}) \). Thus \((F_l)\) is a strict inductive sequence with \( \bigcup_{l=1}^{\infty} F_l = F' \). Let \( F = \lim F_l \).

It is easy to see that the identity map \( I : F \to E'_b \) is continuous. \( E \) is a Fréchet-Montel space, so its strong dual \( E'_b \) is an LB-space (\([8], \text{ Corollary 2.5.9}\)). Hence, by the open mapping theorem for LF-spaces (\([5], \text{ Theorem 3.1}\)), the map \( F \) is bounded, so continuous, for any \( E_b \subset F \). It follows that \( (F_l) \) is a strict inductive sequence with \( \bigcup_{l=1}^{\infty} F_l = F' \). Let \( F = \lim F_l \).

In the case when \( \mathbb{K} \) is not spherically complete we get

**Theorem 3.13.** Assume that \( \mathbb{K} \) is not spherically complete. Let \( E \) be an infinite-dimensional Fréchet space of countable type which is not isomorphic to the strong dual of a strict LB-space. Then \( E \) has a nuclear Köthe quotient.

**Proof.** Suppose, by contradiction, that \( E \) has no nuclear Köthe quotient. Then, as in the proof of Theorem 3.12, we get a non-decreasing base \(| | \cdot | |_k \) in \( \mathcal{P}(E) \) such that the sequence \((F_l)\), where \( F_l \) is the closure of \( E'_l \) in \((E_{l+1}^{F'_l} | | \cdot | |_{l+1}^{E'_l})\), is a strict inductive sequence with \( \bigcup_{l=1}^{\infty} F_l = F' \). Let \( F = \lim F_l \). We shall prove that \( F'_b = \lim F'_b \).

Let \( V_n = \{ x \in E : |x|_n \leq 1 \} \) and \( B_n = \{ f \in E'_n : |f|_n \leq 1 \} \) for \( n \in \mathbb{N} \). Clearly \( B_n \subset \{ f \in F_n : |f|_{n+1} \leq 1 \} \subset B_{n+1} \) for \( n \in \mathbb{N} \).

We shall prove that \((B_n)\) is a fundamental sequence of bounded subsets of \( E'_b \). Let \( A \) be a closed absolutely convex bounded subset of \( E \) and let \( n \in \mathbb{N} \). Then \( \alpha(A^n) = A^n \) (\([8], \text{ p. 199}\)), so \( \alpha(A^n) \subset \alpha V_n \subset \alpha(A^n) \) for some \( \alpha \in \mathbb{K} \). Hence \( B_n \subset \alpha(A^n) \subset \alpha(A^n) \subset \alpha(A^n) \). Thus \((B_n) \subset \mathcal{B}(E'_b) \). Let \( B \subset \mathcal{B}(E'_b) \) and let \( \beta \in \mathbb{K} \) with \( |\beta| > 1 \). Then \( \beta B \) is a barrel in \( E \), so it is a neighbourhood of zero in \( E \). Therefore \( \beta V_n \subset \beta B \) for some \( n \in \mathbb{N} \). Hence \( B \subset \beta B \subset \beta^{-1} V_n \subset B_k \).

It follows that \( \mathcal{B}(F) = \mathcal{B}(E'_b) \).

Let \( f \) be a linear functional on \( F \) which is bounded on bounded sets. Then \( f|F_n \) is bounded, so continuous, for any \( n \in \mathbb{N} \). Hence \( f \) is continuous on \( F \) (\([3], \text{ Proposition 1.1.6}\)). By Proposition 3.14 (see below) any linear functional on \( E'_b \) which is bounded on bounded sets is continuous. It follows that \( F'_b = \lim F'_b \).

The Fréchet space \( E \) is reflexive since \( \mathbb{K} \) is not spherically complete (\([8], \text{ Theorem 10.3}\)). Thus \( E \) is isomorphic to the strong dual of a strict LB-space, a contradiction.

In the proof of our previous theorem we used the following

**Proposition 3.14.** Assume that \( \mathbb{K} \) is not spherically complete. Let \( E \) be a Fréchet space of countable type. Then any linear functional \( f \) on \( E'_b \) which is bounded on bounded subsets of \( E'_b \) is continuous.
Proof. Let \((p_k)\) be a non-decreasing base in \(\mathcal{P}(E)\). Let \(k \in \mathbb{N}\). Denote by \((G_k, q_k)\) the Banach space \((E'_k, p'_k)\). By our assumption the functional \(f|_{G_k}\) is continuous on \((G_k, q_k)\). Let \(\pi_k : E \to E_{p_k}\) be the quotient map. For any \(h \in G_k\) the functional \(h_k : (E_{p_k}, \overline{\pi_k}) \to \mathbb{K}\), \(\pi_k(x) \mapsto h(x)\) is well defined, linear and continuous and the linear map \((G_k, q_k) \to (E_{p_k}, p_k') : h \mapsto h_k\) is an isomorphism. The space \((E_{p_k}, \overline{\pi_k})\) is of countable type, so its completion \((\tilde{E}_{p_k}, \tilde{\pi}_k)\) is a reflexive Banach space [8], Corollary 9.9.

Thus \((G_k, q_k)'\) is isomorphic to \((\tilde{E}_{p_k}, \tilde{\pi}_k)\), so there exists \(y_k \in \tilde{E}_{p_k}\) such that \(f(h) = h_k(y_k)\) for any \(h \in G_k\), where \(h_k \in (E_{p_k}, \overline{\pi_k})'\) with \(h_k|_{E_{p_k}} = h_k\).

It follows that \(\tilde{h}_k(y_k) = h_{k+1}(y_{k+1})\) for all \(h \in G_k\), \(k \in \mathbb{N}\).

For any \(k \in \mathbb{N}\) there exists \((y_{k,n}) \in E_{p_k}\) with \(\lim_{n \to \infty} \tilde{\pi}_k(y_{k,n}) = y_{k} \in \tilde{E}_{p_k}\); clearly \(\lim_{n \to \infty} \tilde{\pi}_k(y_{k,n}) = 0\). Moreover \(h_k \circ \phi_k = h_{k+1}\), where

\[
\phi_k : E_{p_{k+1}} \to E_{p_k}, \pi_{k+1}(x) \mapsto \pi_k(x) \in \tilde{E}_{p_k}\text{ for } k \in \mathbb{N}.
\]

Thus \(\lim_{n \to \infty} h_k(y_{k,n}) = \lim_{n \to \infty} h_k(\phi_k(y_{k+1,n})) = 0\) for all \(\phi \in (E_{p_k}, p_k')\), \(k \in \mathbb{N}\). We have shown that \(\lim_{n \to \infty} \tilde{\pi}_k(y_{k,n} - \phi_k(y_{k+1,n})) = 0\) converges weakly to 0 in \((E_{p_k}, \overline{\pi_k})\) for any \(k \in \mathbb{N}\). By [8], Proposition 4.11, we infer that \(\lim_{n \to \infty} \tilde{\pi}_k(y_{k,n} - \phi_k(y_{k+1,n})) = 0\), \(k \in \mathbb{N}\).

For some \((x_{k,n}) \subset E\) we have \(\pi_k(x_{k,n}) = y_{k,n}\) for all \(k, n \in \mathbb{N}\). Then we obtain \(\lim_{n \to \infty} p_k(x_{k,n} - x_{k,m}) = 0\) and \(\lim_{n \to \infty} p_k(x_{k,n} - x_{k+1,n}) = 0\) for any \(k \in \mathbb{N}\).

Let \((\epsilon_k)\) be a decreasing sequence of positive numbers with \(\lim_{k \to \infty} \epsilon_k = 0\). Then for any \(k \in \mathbb{N}\) there exists \(n_k \in \mathbb{N}\) such that

\[
p_k(x_{k,n} - x_{k,m}) < \epsilon_k \quad \text{and} \quad p_k(x_{k,n} - x_{k+1,n}) < \epsilon_k \quad \text{for all } n, m \in \mathbb{N} \text{ with } n, m \geq n_k.
\]

Clearly, we can assume that the sequence \((n_k)\) is increasing.

We shall prove that \(\lim_{m \to \infty} p_k(x_{k,m} - x_{k,m-1}) = 0\) for \(k \in \mathbb{N}\).

Let \(k \in \mathbb{N}\). For \(m \in \mathbb{N}\) with \(m > k\) we have

\[
p_k(x_{k,m} - x_{k,m-1}) \leq \max \{p_k(x_{k,m} - x_{k,m-1}), \max_{0 \leq i < m - k} p_k(x_{i+1,n} - x_{i+1,n})\}.
\]

Let \(\epsilon > 0\). Then there exists \(m_0 > k\) such that \(\epsilon_m \leq \epsilon\) for \(m \geq m_0\).

We have \(\lim_{m \to \infty} p_k(x_{k,m} - x_{k,m-1}) = 0\) and

\[
\lim_{m \to \infty} p_k(x_{i+1,n} - x_{i+1,n}) = 0 \quad \text{for } 0 \leq i < m_0 - k
\]

and

\[
p_k(x_{k+1,n} - x_{k+1,n}) \leq \epsilon_{k+1} \leq \epsilon \quad \text{for } m_0 - k \leq i < m_0 - k.
\]

It follows that \(\lim_{m \to \infty} p_k(x_{k,m} - x_{k,m-1}) = 0\).

Put \(z_m = x_{m,m-1}\) for \(m \in \mathbb{N}\). Then \(\lim_{m \to \infty} \tilde{\pi}_k(z_m) = 0\). Since

\[
p_k(z_m - z_l) \leq \max \{p_k(x_{m,m-1} - x_{k,m}), \max_{0 \leq i < m, l} p_k(x_{i+1,n} - x_{i+1,n})\}
\]

we infer that \((z_m)\) is a Cauchy sequence in \(E\), so it converges to some \(z_0\) in \(E\).

Let \(k \in \mathbb{N}\). For any \(h \in G_k\) we have

\[
f(h) = \tilde{h}_k(y_k) = \lim_{m \to \infty} h_k(y_{k,m}) = \lim_{m \to \infty} h_k(\pi_k(z_m)) = \lim_{m \to \infty} h(z_m) = h(z_0).
\]

Thus \(f(h) = h(z_0)\) for any \(h \in \text{End}\), so \(f\) is continuous on \(E'_k\). \(\Box\)

It is known that the strong dual \(E'_k\) of a reflexive Fréchet space \(E\) over a spherically complete field is bornological [9], Proposition 15.6. For Fréchet spaces over a non-spherically complete field we get the following.
Corollary 3.15. Assume that $\mathbb{K}$ is not spherically complete. If $E$ is a Fréchet space of countable type, then $E_b'$ is polarly bornological.

Proof. By [5], Lemma 9.5 and Corollary 9.9, the strong dual $E_b'$ of $E$ is polarly barreled. Let $A$ be a bornivorous $\mathbb{K}$-polar subset of $E_b'$. Then $A = \bigcap_{f \in P} f^{-1}(B_2)$ for some set $P$ of linear functionals on $E_b'$. Since $A$ absorbs bounded subsets of $E_b'$, each $f \in P$ is bounded on bounded subsets of $E_b'$. Using Proposition 3.14 we get $P \subset (E_b')'$. Hence $A$ is a polar barrel in $E_b'$, so it is a neighbourhood of zero in $E_b'$. Thus $E$ is polarly bornological.

In connection with Theorem 3.13 we show the following

Proposition 3.16. If a Fréchet space $E$ of countable type is isomorphic to the strong dual of a strict LB-space $F = \lim\limits_{\rightarrow} F_n$, such that $F_n$ has the weak extension property in $F_{n+1}$, $n \in \mathbb{N}$, then $E$ is isomorphic to a countable product of Banach spaces.

Proof. Let $i_n : F_n \to F_{n+1}$ be the inclusion map for $n \in \mathbb{N}$. By the weak extension property the adjoint map $i_n^* : F_{n+1}' \to F_n'$, $f \mapsto f|_{F_n}$ is surjective for $n \in \mathbb{N}$. Put $H_1 = F_1'$ and $H_n = \ker i_n^*$ for $n > 1$. The closed subspace

$$H = \{ (f_n) \in \prod_{n=1}^{\infty} F'_n : i_n^*(f_{n+1}) = f_n \text{ for } n \in \mathbb{N} \}$$

of the Fréchet space $\prod_{n=1}^{\infty} F'_n$ is the projective limit of the projective sequence $(F'_n)$ ([3], 1.3.2).

By [3], Theorem 1.3.5, $H$ is isomorphic to $F_b'$, so $H$ is of countable type. Let $n \in \mathbb{N}$. It is easy to check that the linear continuous map $\pi_n : H \to F_n$, $(f_n) \mapsto f_n$ is surjective. Thus $H$ has a quotient isomorphic to $F_n'$, so $F_n'$ is of countable type. Thus $H_n$ is complemented in $F_n'$ ([3], Theorem 3.12), so there exists $T_n \in L(F_n', F_{n+1}')$ such that $i_n^* \circ T_n$ is the identity map on $F_n'$.

Hence the map

$$T : \prod_{n=1}^{\infty} H_n \to H, (f_n) \mapsto (f_1, T_1 f_1 + f_2, \ldots, T_{n-1} f_1 + \cdots + T_1 f_1 + f_n, f_{n+1}, \ldots)$$

is well defined. Clearly $T$ is linear and injective.

We show that $T$ is surjective. Let $(g_n) \in H$. Put $f_1 = g_1$. Then $g_2 - T_1 f_1 \in H_2$, so there exists an $f_2 \in H_2$ such that $g_2 = T_1 f_1 + f_2$. Assume that for some $n \geq 1$ we have a sequence $(f_1, \ldots, f_{n+1}) \in \prod_{i=1}^{n+1} H_i$ with $g_{n+1} = T_n \circ \cdots \circ T_1 f_1 + \cdots + T_n f_n + f_{n+1}$. Then $g_{n+2} = T_{n+1} g_{n+1} + f_{n+2} = T_{n+1} \circ \cdots \circ T_1 f_1 + \cdots + T_{n+1} f_{n+1} + f_{n+2}$. Thus we can inductively construct a sequence $(f_n) \in \prod_{n=1}^{\infty} H_n$ with $T((f_n)) = (g_n)$.

The linear maps $T_n, n \in \mathbb{N}$, are continuous, and the spaces $\prod_{n=1}^{\infty} H_n$ and $H$ have the product topologies. It follows that the maps $T$ and $T^{-1}$ are continuous. Thus $T$ is an isomorphism, so $E$ is isomorphic to a countable product of Banach spaces.

Corollary 3.17. If a Fréchet-Montel space $E$ is isomorphic to the strong dual of a polar strict LB-space $F = \lim\limits_{\rightarrow} F_n$, then it is of finite type.
Proof: $F$ is polar and polarly barreled ([3], Proposition 1.1.10), so it is isomorphic to a subspace of $(F_b')_b$ ([8], Lemmas 9.2 and 9.3). Clearly $(F_b')_b \simeq E'_b$; by [8], Theorem 8.5, $E'_b$ is of countable type. Thus $F$ and $F_n, n \in \mathbb{N}$, are of countable type ([8], Proposition 4.12 and [3], Theorem 1.4.7). It follows that $F_n$ has the weak extension property in $F_{n+1}, n \in \mathbb{N}$. Using Proposition 3.16 we infer that $E$ is isomorphic to a countable product of Banach spaces. Thus $E$ is of finite type since it is a Fréchet-Montel space.

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References


Faculty of Mathematics and Computer Science, Adam Mickiewicz University, ul. Umultowska 87, 61-614 Poznań, Poland
E-mail address: sliwa@amu.edu.pl

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