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CONFORMAL SPECTRAL THEORY FOR THE MONODROMY MATRIX

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Abstract. For any $N \times N$ monodromy matrix we define the Lyapunov function which is analytic on an associated N-sheeted Riemann surface. On each sheet the Lyapunov function has the standard properties of the Lyapunov function for the Hill operator. The Lyapunov function has (real or complex) branch points, which we call resonances. We determine the asymptotics of the periodic, anti-periodic spectrum and of the resonances at high energy. We show that the endpoints of each gap are periodic (anti-periodic) eigenvalues or resonances (real branch points). Moreover, the following results are obtained: 1) We define the quasimomentum as an analytic function on the Riemann surface of the Lyapunov function; various properties and estimates of the quasimomentum are obtained. 2) We construct the conformal mapping with imaginary part given by the Lyapunov exponent, and we obtain various properties of this conformal mapping, which are similar to the case of the Hill operator. 3) We determine various new trace formulae for potentials and the Lyapunov exponent. 4) We obtain a priori estimates of gap lengths in terms of the Dirichlet integral. We apply these results to the Schrödinger operators and to first order periodic systems on the real line with a matrix-valued complex self-adjoint periodic potential.

1. Introduction

There exist many results about the periodic systems; see the books [A], [YS] and the interesting papers of Krein [Kr], Gel'fand and Lidskii [GL], etc. The basic results for spectral theory for the matrix case were obtained by Lyapunov and Poincaré [YS]. We also mention the new papers [BBK], [C1], [C2], [CK], [CHGL], [CG], [CG1], [GKM], [GS], [Sh] and the references therein.

Various properties of the Lyapunov function and quasimomentum for the scalar Hill operator and the 2×2 periodic Zakharov-Shabat systems are well understood [A], [M]. Recall the well-known results for the Hill operator $\mathcal{S}_1 y = -y'' + V(t)y$ in $L^2(\mathbb{R})$ with a periodic potential $V(t+1) = V(t), t \in \mathbb{R}$, and $V \in L^2(0,1)$. The spectrum of $\mathcal{S}_1$ is purely absolutely continuous and consists of intervals $\widetilde{\sigma}_n = [\lambda_{n-1}^+, \lambda_n^-], n \geqslant 1$. These intervals are separated by the gaps $\gamma_n = (\lambda_n^-, \lambda_n^+)$ of length $|\gamma_n| \geqslant 0$. If a gap γ_n is degenerate, i.e. $|\gamma_n| = 0$, then the corresponding segments $\widetilde{\sigma}_n, \widetilde{\sigma}_{n+1}$ merge. The sequence $\lambda_0^+ < \lambda_1^- \leqslant \lambda_1^+ < ...$ is the spectrum of the equation $-y'' + Vy = \lambda y$ with 2-periodic boundary conditions, that is, $y(t+2) = y(t), t \in \mathbb{R}$. Here equality $\lambda_n^- = \lambda_n^+$ means that $\lambda_n^\pm$ is an eigenvalue of multiplicity 2. For

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the equation $-y'' + Vy = \lambda y$ on the real line we define the fundamental solutions $\vartheta(t,\lambda)$ and $\varphi(t,\lambda), t \in \mathbb{R}$, satisfying $\vartheta(0,\lambda) = \varphi'(0,\lambda) = 1, \vartheta'(0,\lambda) = \varphi(0,\lambda) = 0$. The corresponding monodromy matrix M and the Lyapunov function Δ are given by

$$M(\lambda) = \begin{pmatrix} \vartheta(1,\lambda) & \varphi(1,\lambda) \\ \vartheta'(1,\lambda) & \varphi'(1,\lambda) \end{pmatrix}, \qquad \Delta(\lambda) = \frac{\operatorname{Tr} M(\lambda)}{2}, \quad \lambda \in \mathbb{C}. \tag{1.1}$$

Note that $\Delta(\lambda_n^{\pm}) = (-1)^n,\ n \geqslant 1$. The derivative of the Lyapunov function has a zero $\lambda_n \in [\lambda_n^-, \lambda_n^+]$, that is $\Delta'(\lambda_n) = 0$ for each $n \geqslant 1$. We introduce a conformal mapping (the quasimomentum, see [MO]) $k(\cdot) : \mathcal{Z} \to \mathbb{K}$ given by the formula $k(z) = \arccos \Delta(z^2), \quad z \in \mathcal{Z} = \mathbb{C} \setminus g,\ g = \bigcup_{n \neq 0} g_n$, where $g_n = (z_n^-, z_n^+) = -g_{-n}$ and $z_n^{\pm} = \sqrt{\lambda_n^{\pm}} > 0, n \geqslant 1$, and let $\lambda_0^+ = 0$. The quasimomentum domain $\mathbb{K} \equiv \mathbb{C} \setminus \bigcup_{n \neq 0} c_n$, and the vertical slits $c_n = [\pi n + ih_n, \pi n - ih_n] = -c_{-n}$, where a height $h_n \geqslant 0$ is defined by the equation $\cosh h_n = (-1)^n \Delta(\lambda_n) \geqslant 1$. Note that if $V = 0$, then $k(z) = z$. The following asymptotics hold:

$$k(z) = z - \frac{Q_0}{z} - \frac{Q_2 + o(1)}{z^3} \qquad \text{as } y > r_0|x|, \quad y \to \infty, \quad \text{for any} \quad r_0 > 0, \tag{1.2}$$

where $Q_0 = \frac{1}{2}\int_0^1 V(t)dt = \frac{1}{\pi}\int_g q(x)dx$ and $Q_2 = \frac{\|V\|^2}{8} = \frac{1}{\pi}\int_g x^2 q(x)dx$, and $q = \operatorname{Im} k \geqslant 0$ on $\mathbb{R}$. In particular, this implies that if all gaps $\gamma_n = \emptyset, n \geqslant 1$ are empty, then $V = 0$.

Using the quasimomentum as conformal mapping a priori estimates for various parameters of the Hill operator and for the Dirac operator (gap lengths, effective masses, etc., in terms of potentials) were obtained in [GT], [KK1], [KK2], [K2], [K10], [M], [MO]. Conversely (it is significantly more complicated), a priori estimates of potential in terms of spectral data (gap lengths, effective masses, etc.), were obtained in [K2], [K5], [K10], [M]. For example, in [K2] there are estimates

$$\|V_1\| \leqslant 2\|G\|(1 + \|G\|)^{\frac{1}{3}}, \qquad \|G\| \leqslant 2\|V_1\|(1 + \|V_1\|)^{\frac{1}{3}}, \quad \text{where} \quad V_1 = V - 2Q_0,$$

where $\|G\|^2 = \sum_{n \geqslant 1} |\gamma_n|^2$ and $|\gamma_n| \geqslant 0$ is the gap length. Such a priori estimates simplify the proof in the inverse spectral theory for the scalar Hill operator [GT], [KK], [K1] and for the periodic Zakharov-Shabat systems [K4], [K5]; see also [K6], where the author solved the inverse problem for the operator $-y'' + u'y$ on $L^2(\mathbb{R})$, where periodic $u \in L^2_{loc}(\mathbb{R})$.

The corresponding theory for the vector case is still modest. It is well known that the spectrum of the Schrödinger operator on the real line with an $N \times N$ matrix-valued real periodic potential, $N > 1$, is absolutely continuous and consists of intervals separated by gaps [DS]. We recall results from [CK] about this operator with real potentials: the Lyapunov function, which is analytic on an associated N-sheeted Riemann surface, is determined. Moreover, the conformal mapping with imaginary part given by the Lyapunov exponent is constructed, and a priori estimates of gap lengths in terms of potentials are obtained. Some properties of the monodromy matrices and the corresponding Lyapunov functions for periodic nanotubes were obtained in [KL1], [KL2].

Introduce the class $\mathscr{M}_N, N \geqslant 2$, of monodromy matrices given by

Definition M. I) An entire $N \times N$ matrix-valued function $M \in \mathscr{M}_N$ if M satisfies

i) for some unitary matrix J the following identity holds:

$$M(z)JM^*(\overline{z}) = J, \qquad \text{for all } z \in \mathbb{C}. \tag{1.3}$$

$M(z)$ has the eigenvalues $\tau_j(z), j \in \mathbb{N}_N = \{1, ..., N\}$, such that:

ii) If $|\tau_j(z)| = 1$ for some $(j, z) \in \mathbb{N}_N \times \mathbb{C}$, then $z \in \mathbb{R}$.

II) $M \in \mathscr{M}_N$ belongs to $\mathscr{M}_N^0$ if each $\Delta_j(z) = \frac{1}{2}(\tau_j(z) + \tau_j^{-1}(z)), j \in \overline{N}$, satisfies

$$\Delta_j(z) = \frac{1}{2}(\tau_j(z) + \tau_j^{-1}(z)) = \cos z + o(e^{|\operatorname{Im} z|}) \quad \text{as} \quad |z| \to \infty. \tag{1.4}$$

III) A matrix-valued function $M \in \mathscr{M}_N^0$ belongs to $\mathscr{M}_N^{0,r}, r \geqslant 0$, if for some constants $C_0, ..., C_r$ the following asymptotics hold:

(1.5)

$$\det(M(z) + M^{-1}(z)) = \exp -iN\left(z - \sum_0^{2r} \frac{C_s}{z^{s+1}} + \frac{o(1)}{z^{2r+1}}\right) \quad \text{as } z = iy, \quad y \to \infty.$$

Note that monodromy matrices for Schrödinger operators (or canonical systems) with periodic matrix-valued potentials and for Schrödinger operators on periodic nanotubes belong to this class; see [A], [CK], [K3], [KL1].

The main goal of our paper is to obtain new results about the Lyapunov functions, the quasimomentum and a priori estimates of gap lengths in terms of potentials for a class of monodromy matrices $\mathscr{M}_N$. First, we construct Lyapunov functions and the conformal mapping (averaged quasimomentum) $k(\cdot)$, with the imaginary part given by the Lyapunov exponent. In fact, we reformulate some spectral problem for the differential operator with periodic matrix coefficients as problems of conformal mapping theory. Second, we obtain various results from the conformal mapping theory. For solving these "new" problems we use some techniques from [KK1], [KK2], [K2], [K6], [K7], [K8] and [CK], [K3]. In particular, we use the Poisson integral for the domain $\mathbb{C}_+ \cup (-1, 1) \cup \mathbb{C}_-$. We apply these results to the Schrödinger operator and to first order periodic systems on the real line with an $N \times N$ matrix-valued **complex self-adjoint** periodic potential for any $N > 1$. We plan to apply these results to study integrable systems [CD1], [CD2], [Ma] and the integrated density of states for periodic nanotubes. Recall that the Lyapunov functions for the periodic matrix-valued Jacobi operators were studied in [KKu].

An eigenvalue $\tau(z)$ of $M(z)$ is called a multiplier. Note that (1.3) yields that if some $\tau(z), z \in \mathbb{C}$, is a multiplier of multiplicity $d \geqslant 1$, then $1/\overline{\tau}(\overline{z})$ is a multiplier of multiplicity d.

Let $L = \frac{M+M^{-1}}{2}$ and $\Phi(\nu, z) = \det(L(z) - \nu I_N)$. Let $\Delta_j(z), j \in \mathbb{N}_N$, be the zeros of $\Phi(\nu, z) = 0$, where $\mathbb{N}_N = \{1, ..., N\}$. This is an algebraic equation in ν of degree N. The coefficients of $\Phi(\nu, z)$ are entire in $z \in \mathbb{C}$. It is well known (see e.g. [Fo]) that the roots $\Delta_j(z), j \in \mathbb{N}_N$, constitute one or several branches of one or several analytic functions that have only algebraic singularities in $\mathbb{C}$. Thus the number of zeros of $\Phi(\nu, z) = 0$ is a constant N_e with the exception of some special values of z (see below the definition of a resonance). In general, there is an infinite number of such points on the plane. If all functions $\Delta_j(z), j \in \mathbb{N}_N$, are distinct, then $N_e = N$. If some of them are identical, then $N_e < N$, and $\Phi(z, \nu) = 0$ is permanently degenerate.

By definition, the number z_0 is a periodic eigenvalue if z_0 is a zero of the function $\det(M(z) - I_N)$. The number z_1 is an anti-periodic eigenvalue if z_1 is a zero of the function $\det(M(z) + I_N)$. We need the following preliminary results.

Theorem 1.1. *Let $M \in \mathscr{M}_N$. Then there exist analytic functions $\widetilde{\Delta}_s, s = 1, ..., N_0 \leqslant N$, on some N_s-sheeted Riemann surface $\mathscr{R}_s, N_s \geqslant 1$ having the following properties:*

i) There exist disjoint subsets $\omega_s \subset \mathbb{N}_N, s \in \mathbb{N}_{N_0}, \bigcup \omega_s = \mathbb{N}_N$ such that all branches of $\widetilde{\Delta}_s, s \in \mathbb{N}_{N_0}$ are given by $\Delta_j(z) = \frac{1}{2}(\tau_j(z) + \tau_j^{-1}(z))$, $j \in \omega_s$, and satisfy

(1.6)
$$\Phi(\nu, z) = \det\left(\frac{M(z) + M^{-1}(z)}{2} - \nu I_N\right) = \prod_1^{N_0} \Phi_s(\nu, z), \quad \Phi_s(\nu, z) = \prod_{j \in \omega_s} (\nu - \Delta_j(z)),$$

for any $z, \nu \in \mathbb{C}$. Here the functions $\Phi_s(\nu, z)$ are entire with respect to $\nu, z \in \mathbb{C}$. Moreover, if $\Delta_i = \Delta_j$ for some $i \in \omega_k, j \in \omega_s$, then $\Phi_k = \Phi_s$ and $\widetilde{\Delta}_k = \widetilde{\Delta}_s$.

ii) Let some branch $\Delta_j, j \in \mathbb{N}_N$, be real analytic on some interval $Y = (\alpha, \beta) \subset \mathbb{R}$ and $-1 < \Delta_j(z) < 1$ for any $z \in Y$. Then $\Delta_j'(z) \neq 0$ for each $z \in Y$.

iii) Each function $\rho_s, s = 1, ..., N_0$, given by (1.7) is entire and real on the real line,

$$\rho = \prod_1^{N_0} \rho_s, \quad \rho_s(\cdot) = \prod_{i<j, i,j \in \omega_s} (\Delta_i(\cdot) - \Delta_j(\cdot))^2. \tag{1.7}$$

iv) The following identity holds true:

$$\begin{aligned} &\bigcup_{j=1}^{N} \{z \in \mathbb{C} : \Delta_j(z) \in [-1, 1]\} \\ &\qquad = \mathbb{R} \setminus \bigcup_{N_- < n < N_+} (z_n^-, z_n^+), \quad ... < z_n^- < z_n^+ z_{n+1} < ..., \end{aligned} \tag{1.8}$$

where $z_n^\pm$ are either periodic (anti-periodic) eigenvalues or real branch points of Δ_j (for some $j \in \mathbb{N}_N$), which are zeros of ρ (below we call such points resonances).

Remark. 1) If $M \in \mathscr{M}_N^0$, then ρ is not a polynomial, since ρ is bounded on $\mathbb{R}$.

2) Let the surface $\mathscr{R} = \bigcup_1^{N_0} \mathscr{R}_s$ be the union of the disjoint Riemann surfaces $\mathscr{R}_s$ and let $\widetilde{\Delta} = \{\widetilde{\Delta}_s, s = 1, ..., N_0\}$ be the corresponding analytic function on $\mathscr{R}$. Let $\zeta \to z$ be the standard projection from the surface $\mathscr{R}$ into the complex plane $\mathbb{C}$. **We set** $\zeta \in \mathscr{R}$ and $z = \phi(\zeta) \in \mathbb{C}$. The surface $\mathscr{R}$ is an N-sheeted branch covering of the complex plane, equipped with the natural projection $\zeta \to z$. Below we will identify (locally) the point $\zeta \in \mathscr{R}$ and the point $z = \phi(\zeta) \in \mathbb{C}$ (see [Fo], Chapter 4). In this case we set $\mathrm{Im}\zeta = \mathrm{Im}\phi(\zeta)$.

3) In the case $M \in \mathscr{M}_4$ and $\tau_1 = \tau_3, \tau_2 = \tau_4$ the function $D(\tau, \cdot) = \det(M - \tau I_4)$ has the form $D(\tau, \cdot) = \tau^4 - T_1\tau^3 + \frac{1}{2}(T_1^2 - T_2)\tau^2 - T_1\tau + 1$, and then

(1.9)
$$D(\tau, \cdot) = \left(\tau^2 - 2\Delta_1\tau + 1\right)\left(\tau^2 - 2\Delta_2\tau + 1\right), \quad \Delta_1 = \frac{T_1}{2} + \frac{\sqrt{\rho}}{2}, \qquad \Delta_2 = \frac{T_1}{2} - \frac{\sqrt{\rho}}{2}.$$

See [BBK], where $\rho = \frac{T_2+4}{2} - \frac{T_1^2}{4}$ and $T_j = \mathrm{Tr}\, M^j(z)$, $j = 1, 2$.

Definition. The number $z_0 \in \mathbb{C}$ is a **resonance** of M if z_0 is a zero of ρ given by (1.7).

Theorem 1.2. *Let $M \in \mathscr{M}_N$ and let $\Delta_j, j = \mathbb{N}_\varkappa$, have a branch point $z_0 \in \mathbb{R}$ for some $\varkappa \in \mathbb{N}$. Assume that each $\Delta_j(z) \in \mathbb{R}, j \in \mathbb{N}_\varkappa$, for all $z \in (z_0, z_0 + \varepsilon)$ (or all*

for $z \in (z_0 - \varepsilon, z_0)$*). Then* $\varkappa = 2$ *and* z_0 *is a branch point of order* $\frac{1}{2}$ *for* Δ_1, Δ_2*, and the function* $(\Delta_1 - \Delta_2)^2$ *is analytic in the disk* $\{|z - z_0| < r\}$ *for some small* $r > 0$ *and* $(\Delta_1 - \Delta_2)^2$ *has a zero* z_0 *of multiplicity* $2m + 1 \geqslant 1$*. If in addition* $\Delta_1(z_0) \in (-1, 1)$*, then* $m = 0$*.*

Remark. 1) This result is important to describe the spectrum of Schrödinger operators with periodic potentials on armchair nanotubes [BBKL].

2) It is very difficult to determine the positions of resonances. We can use only the Levinson Theorem (see Section 2) and Theorem 1.2. It is similar to another case of poles (other resonances) for scattering Schrödinger operators with compactly supported potentials on the real line; see [K11], [Z].

We consider the conformal mapping associated with $M \in \mathscr{M}_N$. We need functions from the subharmonic counterpart of the Cartwright class of the entire functions given by

(1.10)
$$\mathcal{SC} = \left\{ q : \mathbb{C} \to \mathbb{R}, \quad \begin{array}{c} q \text{ is subharmonic in } \mathbb{C} \text{ and harmonic outside } \mathbb{R}, \\ q(\overline{z}) \equiv q(z), z \in \mathbb{C}, \ \int_{\mathbb{R}} \frac{q_+(t)dt}{1+t^2} < \infty, \ \limsup_{z\to\infty} \frac{q(z)}{|z|} < \infty \end{array} \right\}.$$

We recall the class of functions from [KK1] given by

$$\mathcal{SK}_m^+ = \left\{ q \in \mathcal{SC} : q \geqslant 0, \ \lim_{y\to\infty} \frac{q(iy)}{y} = 1, \quad \int_{\mathbb{R}} (1 + t^{2m}) q(t) dt < \infty \right\}, \quad m \geqslant 0.$$

We note that $\mathcal{SK}_{m+1}^+ \subset \mathcal{SK}_m^+, m \geqslant 0$.

We introduce the simple conformal mapping $\eta : \mathbb{C} \setminus [-1, 1] \to \{z \in \mathbb{C} : |z| > 1\}$ by

$$\eta(z) = z + \sqrt{z^2 - 1}, \quad z \in \mathbb{C} \setminus [-1, 1], \ \text{ and } \quad \eta(z) = 2z + o(1) \quad \text{as } |z| \to \infty. \tag{1.11}$$

Note that $\eta(z) = \overline{\eta}(\overline{z}), z \in \mathbb{C} \setminus [-1, 1]$, since $\eta(z) > 1$ for any $z > 1$. The properties of the function Δ_j imply $|\eta(\widetilde{\Delta}_s(\zeta))| > 1$ and $\zeta \in \mathscr{R}_s^+ = \{\zeta \in \mathscr{R}_s : \operatorname{Im} \zeta > 0\}$. Thus we can define the quasimomentum k_j (we fix some branch of arccos and $\Delta_j(z)$) and the function q_j by

(1.12)
$$k_j(z) = \arccos \Delta_j(z) = i \log \eta(\Delta_j(z)), \quad q_j(z) = \operatorname{Im} k_j(z) = \log |\eta(\Delta_j(z))|, \quad k \in \mathbb{N}_N,$$

and $z \in \mathscr{R}_0^+ = \mathbb{C}_+ \setminus \beta_+, \beta_+ = \bigcup_{\beta \in \mathcal{B}(\widetilde{\Delta}) \cap \mathbb{C}_+} [\beta, \beta + i\infty)$, where $\mathcal{B}(f)$ is the set of all branch points of the function f. The branch points of k_j in $\mathbb{C}_+$ belong to $\mathcal{B}(\widetilde{\Delta})$. Define the **averaged quasimomentum** k, the **density** p and the **Lyapunov exponent** q by

$$k(z) = p(z) + iq(z) = \frac{1}{N} \sum_1^N k_j(z), \quad q(z) = \operatorname{Im} k(z), \quad z \in \mathscr{R}_0^+. \tag{1.13}$$

Define the sets $\sigma_{(N)} = \{z \in \mathbb{R} : \Delta_1(z), ..., \Delta_N(z) \in [-1, 1]\}$ and

$$\sigma_{(1)} = \{z \in \mathbb{R} : \Delta_j(z) \in (-1, 1), \ \Delta_s(z) \notin [-1, 1] \ \text{ for some } j, s \in \mathbb{N}_N\}.$$

For the function $k(z) = p(z) + iq(z), z = x + iy \in \overline{\mathbb{C}}_+$ we formally introduce the integrals

(1.14)
$$Q_n = \frac{1}{\pi}\int_{\mathbb{R}} x^n q(x)dx, \qquad I_n^S = \frac{1}{\pi}\int_{\mathbb{R}} x^n q(x)dp(x), \quad I_n^D = \frac{1}{\pi}\iint_{\mathbb{C}_+} |\widetilde{k}'_{(n)}(z)|^2 dxdy,$$
$$\widetilde{k}_{(n)}(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{t^n q(t)}{t-z}dt = z^n\Big(k(z) - z + \sum_{j=0}^{n-1} Q_s z^{-s-1}\Big), \quad z \in \mathbb{C}_+.$$

Let C_{us} denote the class of all real upper semi-continuous functions $h : \mathbb{R} \to \mathbb{R}$. With any $h \in C_{us}$ we associate the "upper" domain $\mathbb{K}(h) = \{k = p + iq \in \mathbb{C} : q > h(p), p \in \mathbb{R}\}$. We formulate our first main result.

Theorem 1.3. *i) Let $M \in \mathscr{M}_N^0$. Then the averaged quasimomentum $k = \frac{1}{N}\sum_1^N k_j$ is analytic in $\mathbb{C}_+$ and $k : \mathbb{C}_+ \to k(\mathbb{C}_+) = \mathbb{K}(h)$ is a conformal mapping for some $h \in C_{us}$. Moreover, $q = \operatorname{Im} k$ has a harmonic extension from $\mathbb{C}_+$ into $\Omega = \mathbb{C}_+ \cup \mathbb{C}_- \cup g$ given by $q(z) = q(\overline{z}), z \in \mathbb{C}_-$ and $q(z) > 0$ for any $z \in \Omega$ and $q \in \mathcal{SC} \cap C(\mathbb{C})$. Furthermore,*

$$q(z) = y + o(1) \quad as \quad |z| \to \infty, y = \operatorname{Im} z \geqslant 0. \tag{1.15}$$

Let, in addition, each $\Delta_j(z) = \cos z + O(|z|^{-1}e^{|\operatorname{Im} z|}), j \in \mathbb{N}_N$ as $|z| \to \infty$. Then

$$q(z) = y + O(|z|^{-\frac{1}{2}}) \quad as \quad |z| \to \infty, y = \operatorname{Im} z \geqslant 0. \tag{1.16}$$

ii) Let $M \in \mathscr{M}_N^{0,r}, r \geqslant 0$. Then $q \in \mathcal{SK}_{2r}^+$, and there exist branches $k_j, j \in \mathbb{N}_N$, such that the following asymptotics, identities and estimates hold true:

(1.17)
$$k(z) = z - \sum_0^{2r} \frac{Q_s}{z^{s+1}} + \frac{o(1)}{z^{2r+1}} \quad as\ y > r_0|x|, \quad y \to \infty, \qquad for\ any\ \ r_0 > 0,$$

(1.18)
$$C_s = Q_s, \qquad I_s^D + I_{2s}^S = Q_{2s} + \frac{sQ_{s-1}^2}{2} + \sum_{n=0}^{s-2}(n+1)Q_nQ_{2s-2-n}, \quad s = 0, ..., r.$$

In particular,

$$I_0^D + I_0^S = Q_0 \quad if \quad r = 0, \quad and \qquad I_1^D + I_2^S = Q_2 + \frac{Q_0^2}{2} \quad if \quad r = 1, \tag{1.19}$$

$$q|_{\sigma_{(N)}} = 0, \quad 0 < q|_{\sigma_{(1)} \cup g} \leqslant \sqrt{2Q_0}. \tag{1.20}$$

Remark. 1) The integral I_0^S is the area between the boundary of $\mathbb{K}(h)$ and the real line.

2) In the case $M \in \mathscr{M}_4$ and $\tau_1 = \tau_3, \tau_2 = \tau_4$ the Lyapunov functions are given by (1.9): $\Delta_1 = \frac{T_1}{2} + \frac{\sqrt{\rho}}{2}$, $\Delta_2 = \frac{T_1}{2} - \frac{\sqrt{\rho}}{2}$. The mapping $k : \mathbb{C}_+ \to \mathbb{K}(h)$ is illustrated in Figure 1. We have $\Delta_1^2 > 1$ on intervals $(A, C), (E, J)$ and $\Delta_2^2 > 1$ on intervals $(B, D), (F, G), (H, I)$, and Δ_1, Δ_2 are not real on (K, L). We also have $\widetilde{A} = k(A), \widetilde{B} = k(B), ..., \widetilde{L} = k(L)$, and, in particular, $k((K, L)) = (3\pi, 3\pi + ih_0]$ is a vertical slit for some $h_0 > 0$.

We describe the properties of the conformal mapping $k(\cdot)$.

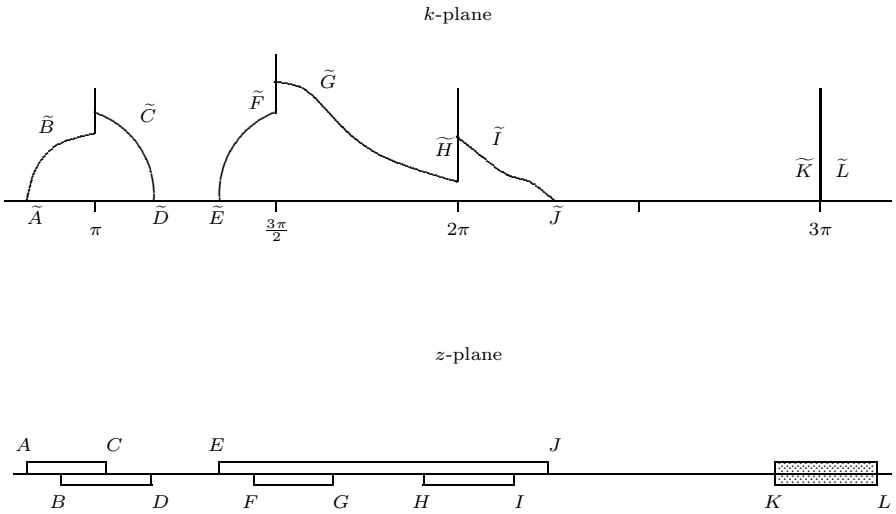

FIGURE 1. $N = 2$. The domain $\mathbb{K}(h) = k(\mathbb{C}_+)$ and gaps in the spectrum.

Theorem 1.4. *Let $M \in \mathscr{M}_N^0$. Then the following relations hold true:*

$$p_x'(z) \geqslant 1, \quad z \in \sigma_{(N)} \quad \text{and} \quad p_x'(z) > 0, \qquad z \in \sigma_{(1)}; \tag{1.21}$$

here $p_x'(z) = 1$ for some $z \in \sigma_{(N)}$ iff $\sigma_{(N)} = \sigma(M)$. Moreover,

$$q_{xx}''(z) < 0 < q(z), \qquad p(z) = \text{const} \in \frac{\pi}{N}\mathbb{Z}, \ \text{for all} \ \ z \in g_n = (z_n^-, z_n^+), \tag{1.22}$$

$$q(x) = q_n^0(x)\left(1 + \frac{1}{\pi}\int_{\mathbb{R}\setminus g_n} \frac{q(t)dt}{q_n^0(t)|t-x|}\right), \qquad x \in g_n, \quad q_n^0(z) = |(z - z_n^-)(z_n^+ - z)|^{\frac{1}{2}}, \tag{1.23}$$

$$G^2 = \sum_n |g_n|^2 \leqslant 8Q_0, \quad \text{and} \ \ Q_0 \leqslant C_0 G^2(1 + G^2) \quad \text{if} \quad \sigma_{(N)} = \sigma(M), \tag{1.24}$$

for some absolute constant $C_0 > 0$.

Using this theorem we deduce that the function $h(p) = q(x(p)), p \in \mathbb{R}$, is continuous on $\mathbb{R} \setminus \{p_n, n \in \mathbb{Z}\}$, where $p_n = p(x), x \in g_n$, and $h(p_n \pm 0) \leqslant h(p_n),\ n \in \mathbb{Z}$.

1. The Schrödinger operator. Consider the self-adjoint operator $\mathcal{S}y = -y'' + V(t)y$ acting in $L^2(\mathbb{R})^N, N \geqslant 2$, where V is a 1-periodic $N \times N$ matrix potential, $V(t) = V^*(t), t \in \mathbb{R}/\mathbb{Z}$, and V belongs to the complex Hilbert space $\mathscr{H}$ given by

$$\mathscr{H} = \left\{V(t) = \{V_{jk}(t)\}_{j,k=1}^N, \quad t \in \mathbb{R}/\mathbb{Z}, \quad \|V\|^2 = \int_0^1 \operatorname{Tr} V(t)V^*(t)dt < \infty\right\}.$$

It is well known (see pp. 1486-1494 of [DS], and [Ge]) that the spectrum $\sigma(\mathcal{S})$ of $\mathcal{S}$ is absolutely continuous and consists of non-degenerate intervals $[\lambda_{n-1}^+, \lambda_n^-], n = 1, ..., N_G \leqslant \infty$, and let $\lambda_0^+ = 0$. These intervals are separated by the gaps $\gamma_n = (\lambda_n^-, \lambda_n^+)$ with the length $|\gamma_n| > 0$. We introduce the fundamental $N \times N$-matrix solutions $\varphi(t, z)$, $\vartheta(t, z)$ of the equation

$$-f'' + Vf = z^2 f, \quad \varphi(0, z) = \vartheta'(0, z) = 0, \varphi'(0, z) = \vartheta(0, z) = I_N, \quad z \in \mathbb{C}, \tag{1.25}$$

where $I_N, N \geqslant 1$, is the identity $N \times N$ matrix. Here and below we use the notation $(') = \partial/\partial t$. We define the monodromy $2N \times 2N$-matrix M, the matrix J and the trace $T_m, m \in \mathbb{Z}$, by

$$M(z) = \begin{pmatrix} \vartheta(1,z) & \varphi(1,z) \\ \vartheta'(1,z) & \varphi'(1,z) \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}, \qquad T_m(z) = \operatorname{Tr} M^m(z). \tag{1.26}$$

It is well known that $M(\cdot) \in \mathscr{M}_{2N}$; see [GL], [Kr]. The functions M and $T_m, m \in \mathbb{Z}$, are entire, and $\det M = 1$. Let $\tau_j, j \in \mathbb{N}_{2N}$, be the eigenvalues of M. It is a root of the algebraic equation $D(\tau, z) = 0$, where $D(\tau, z) = \det(M(z) - \tau I_{2N}) = 0, \tau, z \in \mathbb{C}$. Recall that $L = \frac{1}{2}(M + M^{-1})$. Each zero of $\Phi(\nu, z) = \det(L(z) - \nu I_{2N})$ is the Lyapunov function given by $\Delta_j(z) = \frac{1}{2}(\tau_j(z) + \tau_j^{-1}(z)),\ j \in \mathbb{N}_{2N}$. For real V we have exactly $\Delta_j = \Delta_{N+j}, j \in \mathbb{N}_N$, since in this case $\tau_{N+j} = 1/\tau_j$. Then each $\Delta_j(z) \in [-1, 1]$ gives the spectral point $z^2 \in \sigma(\mathcal{S})$ of multiplicity 2; see [CK]. For complex V we have $\Delta_j, j \in \mathbb{N}_{2N}$, where each $\Delta_j(z) \in [-1, 1]$ gives the spectral point $z^2 \in \sigma(\mathcal{S})$ of multiplicity 1. The zeros of $D(1, \sqrt{\lambda})$ (and $D(-1, \sqrt{\lambda})$) (counted with multiplicity) are the periodic (anti-periodic) eigenvalues for the equation $-y'' + Vy = \lambda y$ with periodic (anti-periodic) boundary conditions. Let $g = \bigcup_{n \in \mathbb{Z}} g_n$, where $g_n = (z_n^-, z_n^+), z_n^\pm = \sqrt{\lambda_n^\pm} > 0$ and $g_{-n} = g_n, n \geqslant 1$. We formulate our main result.

Theorem 1.5. *Let $V = V^* \in \mathscr{H}$. Then*

i) M given by (1.26) belongs to $\mathscr{M}_{2N}^{0,1}$, and spectrum $\sigma(\mathcal{S}) = \mathbb{R} \setminus \bigcup \gamma_n$, where the gaps $\gamma_0 = (-\infty, \lambda_0^+), \gamma_n = (\lambda_n^-, \lambda_n^+), 1 \leqslant n < N_g$, and $\lambda_n^\pm$ are either periodic (anti-periodic) eigenvalues or real resonances.

ii) The averaged quasimomentum $k = \frac{1}{2N} \sum_1^{2N} k_j$ is analytic in $\mathbb{C}_+$, and $k : \mathbb{C}_+ \to k(\mathbb{C}_+) = \mathbb{K}(h)$ is a conformal mapping onto $\mathbb{K}(h)$ for some $h \in C_{us}$. Also, $q = \operatorname{Im} k$ has a harmonic extension from $\mathbb{C}_+$ into $\Omega = \mathbb{C}_+ \cup \mathbb{C}_- \cup g$ given by $q(z) = q(\overline{z}), z \in \mathbb{C}_-$, and $q(z) > 0$ for any $z \in \Omega$. Furthermore, $q \in \mathcal{SK}_2^+ \cap C(\mathbb{C})$, and there exist branches $k_j, j \in \mathbb{N}_{2N}$, such that the following asymptotics, identities and estimates hold true:

$$k(z) = z - \frac{Q_0}{z} - \frac{Q_2 + o(1)}{z^3} \qquad \text{as } y > r_0|x|, \quad y \to \infty, \quad \text{for any} \quad r_0 > 0, \tag{1.27}$$

$$Q_0 = I_0^D + I_0^S = \int_0^1 \frac{\operatorname{Tr} V(t)dt}{2N}, \tag{1.28}$$

$$Q_2 = I_1^D + I_2^S - \frac{Q_0^2}{2} = \int_0^1 \frac{\operatorname{Tr} V^2(t)dt}{8N} = \frac{\|V\|^2}{8N}, \tag{1.29}$$

$$q|_{\sigma_{(N)}} = 0, \quad 0 < q|_{\sigma_{(1)} \cup g} \leqslant \sqrt{2Q_0}, \tag{1.30}$$

$$k(z) = -\overline{k(-\overline{z})},\ z \in \overline{\mathbb{C}}_+, \tag{1.31}$$

$$G^2 \equiv \sum |\gamma_n|^2 \leqslant \frac{2\|V\|^2}{N}, \qquad |\gamma_n| = \lambda_n^+ - \lambda_n^-, n \geqslant 1, \tag{1.32}$$

$$\|V\| \leqslant C_0 G(1 + G^{\frac{1}{3}}) \quad \text{if} \quad \sigma_{(N)} = \sigma(\mathcal{S}), \tag{1.33}$$

for some absolute constant C_0.

Remark. 1) Properties of the Lyapunov functions are formulated in Theorems 1.1 and 1.2.

2) Various properties of the quasimomentum k_j are formulated in Theorems 1.3, 1.4, and 3.1.

3) The existence of real and complex resonances was proved in [BBK] for the Schrödinger operator on the real line with a 2×2 matrix real-valued periodic potential $V \in \mathscr{H}$.

4) If the potential $V \in \mathscr{H}$ is real and a matrix $\int_0^1 V(t)dt$ has distinct eigenvalues, then the operator $\mathcal{S}$ has only finite number complex resonances [CK].

5) Let $\sigma(m, A)$ denote the spectrum of a self-adjoint operator A of multiplicity $m, m \geqslant 0$. We have the following simple corollary from Theorem 1.5: *Let* $\sigma(\mathcal{S}) = \sigma(2N, \mathcal{S}) = \mathbb{R}_+$ *for some* $V = V^* \in \mathscr{H}$. *Then* $V = 0$. There are two simple proofs:

a) if $\sigma(\mathcal{S}) = \sigma(N, \mathcal{S}) = \mathbb{R}_+$, then all gaps are close and the identity (1.29) yields $V = 0$;

b) if $\sigma(\mathcal{S}) = \sigma(N, \mathcal{S}) = \mathbb{R}_+$, then all gaps are close and the estimate (1.33) yields $V = 0$.

6) Recall that the so-called Borg Theorem for periodic systems was proved in [CHGL], [GKM] for general cases.

2. The periodic canonical systems. Consider the operator $\mathcal{K}$ given by

$$\mathcal{K}y = -iJy' + V(t)y, \qquad J = I_{N_1} \oplus (-I_{N_2}),$$
$$V = \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix}, \quad N_1 + N_2 = N, \ N_1 \geqslant 1, N_2 \geqslant 1,$$

and acting in the space $L^2(\mathbb{R})^N$, where v is the 1-periodic $N_1 \times N_2$ matrix and $V = V(t) = V(t)^*$ belongs to a subspace $\mathscr{H}_0 \subset \mathscr{H}$ given by

$$\mathscr{H}_0 = \left\{ V = \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix} \in \mathscr{H} : v = \{v_{jk}\}, \ (j,k) \in \mathbb{N}_{N_1} \times \mathbb{N}_{N_2} \right\}.$$

It is well known (see [DS], pp. 1486-1494, and [Ge]) that the spectrum $\sigma(\mathcal{K})$ of $\mathcal{K}$ is absolutely continuous and consists of non-degenerated intervals σ_n. These intervals are separated by the gaps $g_n = (z_n^-, z_n^+)$ with the length $|g_n| > 0, -\infty \leqslant N_g^- < n < N_g^+ \leqslant \infty$, where $N_g = N_g^+ - N_g^- - 1$ is the total number of the gaps. Introduce the fundamental $N \times N$-matrix solutions $\psi(t, z)$ of the canonical periodic system

$$-iJ\psi' + V(t)\psi = z\psi, \qquad z \in \mathbb{C}, \quad \psi(0, z) = I_N. \tag{1.34}$$

It is well known that the monodromy matrix $\psi(1, \cdot) \in \mathscr{M}_N$; see p. 109 of [YS], and [Kr]. The eigenvalues $\tau(z)$ of $\psi(1, z)$ are the *multiplier* of $\mathcal{K}$: to each of them corresponds a solution f of $\mathcal{K}f = zf$ with $f(t+1) = \tau(z)f(t), t \in [0, 1)$. They are roots of the algebraic equation $D(\tau, z) = 0$, where $D(\tau, z) = \det(\psi(1, z) - \tau I_N), \tau, z \in \mathbb{C}$. The zeros of $D(1, z)$ (or $D(-1, z)$) are the eigenvalues of periodic (anti-periodic) problem for the equation $-iJy' + Vy = zy$.

Theorem 1.6. *Let* $V = V^* \in \mathscr{H}_0$. *Then* $\psi(1, z)$ *belongs to* $\mathscr{M}_N^{0,0}$, *the averaged quasimomentum* $k = \frac{1}{N}\sum_1^N k_j$ *is analytic in* $\mathbb{C}_+$, *and* $k : \mathbb{C}_+ \to k(\mathbb{C}_+) = \mathbb{K}(h)$ *is a conformal mapping for some* $h \in C_{us}$. *Furthermore, there exist branches*

$k_j, j \in \mathbb{N}_N$, such that the following asymptotics, identities and estimates hold true:

$$k(z) - z = -\frac{Q_0 + o(1)}{z}, \quad \text{if } y > r|x|, \quad \text{for any } r > 0, \tag{1.35}$$

$$Q_0 = I_0^D + I_0^S = \frac{\|V\|^2}{2N}, \tag{1.36}$$

$$q|_{\sigma_{(N)}} = 0, \quad 0 < q|_{\sigma_{(1)} \cup g} \leqslant \sqrt{2Q_0}, \tag{1.37}$$

$$G^2 = \sum_n |g_n|^2 \leqslant \frac{4\|V\|^2}{N}, \tag{1.38}$$

$$\|V\| \leqslant \sqrt{N} C_0 G(1+G), \quad \text{if} \quad \sigma_{(N)} = \sigma(\mathcal{K}), \tag{1.39}$$

for some absolute constant C_0.

Remark. 1) In the proof we use arguments from [K4], [K5] and [CK]. Theorem 1.6 generalizes the result of [CK] to the case of the canonical systems with periodic matrix potential.

2) We have the following simple corollary from Theorem 1.6: *Let $\sigma(\mathcal{K}) = \sigma(N, \mathcal{K}) = \mathbb{R}$ for some $V = V^* \in \mathscr{H}_0$. Then $V = 0$.* The proof is similar to the case of the Schrödinger operator; see Remark 5) after Theorem 1.5.

3) In [K3] for the case $N_1 = N_2 = N \geqslant 2$ we prove the existence of real or complex resonances. We determine the asymptotics of the periodic, anti-periodic spectrum and of the resonances at high energy (in terms of the Fourier coefficients of the potential). We show that there exist two types of gaps: i) stable gaps, i.e., the endpoints are periodic and anti-periodic eigenvalues, ii) unstable (resonance) gaps, i.e., the endpoints are resonances (real branch points). Moreover, we determine various new trace formulae for potentials and the Lyapunov exponent.

4) Recall the estimates $\frac{1}{\sqrt{2}}\|G\| \leqslant \|V\| \leqslant 2\|G\|(1+\|G\|)$ for the case $N_1 = N_2 = 1$ from [K8].

The plan of our paper is as follows. In Section 2 we obtain the basic properties of the Lyapunov functions. In Section 3 we obtain the main properties of the quasimomentum, and we prove the basic Theorems 1.3 and 1.4, devoted to the conformal mapping theory, and Theorem 1.2 about resonances. In Section 4 we obtain the results for the Schrödinger operator and the first order systems, and Theorems 1.5 and 1.6 will be proved.

2. The Lyapunov functions

Recall that $D(\tau, \cdot) = \det(M - \tau I_N)C$ and $L = \frac{1}{2}(M + M^{-1})$. We need the simple fact

Proposition 2.1. *i) If $M \in \mathscr{M}_N$, then $\Phi(\nu, z) = \det(L(z) - \nu I_N), z, \nu \in \mathbb{C}$, satisfies*

$$\Phi(\nu, z) = \det(L(z) - \nu I_N) = (-1)^N \sum_0^N \phi_j(z) \nu^{N-j}, \quad \phi_0 = 1, \quad \phi_1 = -\mathcal{T}_1, \tag{2.1}$$

$$\phi_2 = -\frac{\mathcal{T}_2 + \mathcal{T}_1 \phi_1}{2}, ..., \phi_j = -\frac{1}{j} \sum_1^j \mathcal{T}_k \phi_{j-k}, ..., \phi_N = \det L, \mathcal{T}_m(z) = \operatorname{Tr} L^m(z),$$

where each ϕ_j is entire and is real on the real line.

ii) Let $\Delta_j(z), j \in \mathbb{N}_N$, be zeros of the equation $\Phi(\nu, z) = 0$. If some $\Delta_j(z) \in [-1, 1], z \in \mathbb{C}$, then $z \in \mathbb{R}$.

Proof. It is well known that the "polynomial" $\Phi(\nu, z), z, \nu \in \mathbb{C}$, is given by (2.1); see pp. 331-333 of [RS]. Using the identity (1.3) we obtain

$$\mathcal{T}_m(z) = \operatorname{Tr} L^m(z) = \frac{1}{2^m} \operatorname{Tr}\Big(M(z) + M(z)^{-1}\Big)^m = \frac{1}{2^m} \operatorname{Tr} \sum_0^m C_p^m M(z)^{2p-m}$$

$$= \frac{1}{2^m} \sum_0^m C_p^m T_{2p-m}(z) = \frac{1}{2^m} \sum_{2p \geqslant m}^m C_p^m \Big(T_{2p-m}(z) + \overline{T}_{2p-m}(\overline{z})\Big),$$

$$C_m^N = \frac{N!}{(N-m)!m!}.$$

This gives that each $\mathcal{T}_m = \operatorname{Tr} L^m, m \in \mathbb{N}$, is the entire and is real on the real line.

ii) Let some some $\Delta_j(z) \in [-1, 1], z \in \mathbb{C}$. Then $\tau_j(z)$ satisfies $|\tau_j(z)| = 1$ and Definition M yields $z \in \mathbb{R}$. □

If $\Phi(\nu, z) = (\cos z - \nu)^N$, then the Lyapunov function $\Delta_j(z) = \cos z, j \in \mathbb{N}_N$ and the zeros of $\Phi(1, z) = 0$ (or $\Phi(-1, z) = 0$) have the form $z_{2n,j} = \pi 2n$ (or $z_{2n+1,j} = \pi(2n+1)$) for $(n, j) \in \mathbb{Z} \times \mathbb{N}_N$. We consider the case $M \in \mathscr{M}_N^0$.

Lemma 2.2. *Let $M \in \mathscr{M}_N^0$. Then the following asymptotics hold:*

$$\Phi(\nu, z) - (\cos z - \nu)^N = o(e^{N|\operatorname{Im} z|}) \quad \text{as} \quad |z| \to \infty, \tag{2.2}$$

where $|\nu| \leqslant A_0$ for some constant $A_0 > 0$. Moreover, there exists an integer n_0 such that:

i) the function $\Phi(1, z)$ has exactly $N(2n_0+1)$ roots, counted with multiplicity, in the disc $\{|z| < \pi(2n_0+1)\}$, and for each $|n| > n_0$, exactly $2N$ roots, counted with multiplicity, in the domain $\{|z - 2\pi n| < \frac{\pi}{2}\}$. There are no other roots.

ii) the function $\Phi(-1, z)$ has exactly $2Nn_0$ roots, counted with multiplicity, in the disc $\{|z| < 2\pi n_0\}$, and for each $|n| > n_0$, exactly $2N$ roots, counted with multiplicity, in the domain $\{|z - \pi(2n+1)| < \frac{\pi}{2}\}$. There are no other roots.

iii) The function $D(z, 1)$ has only real zeros $z_{2n,m}, n \in \mathbb{Z}$, and their labeling is given by

$$\dots \leqslant z_{-2,N} \leqslant \underbrace{z_{0,1} \leqslant z_{0,2} \leqslant \dots \leqslant z_{0,N}}_{n=0} \leqslant \underbrace{z_{2,1} \leqslant \dots \leqslant z_{2,N}}_{n=2} \leqslant z_{4,1} \leqslant \dots, \qquad n \ \text{even}, \tag{2.3}$$

and the function $D(z, -1)$ has only real zeros $z_{2n+1,m}, n \in \mathbb{Z}$, and their labeling is given by

$$\dots \leqslant z_{-1,N} \leqslant \underbrace{z_{1,1} \leqslant \dots \leqslant z_{1,N}}_{n=1} \leqslant \underbrace{z_{3,1} \leqslant \dots \leqslant z_{3,N}}_{n=3} \leqslant z_{5,1} \leqslant \dots, \qquad n \ \text{odd}. \tag{2.4}$$

Moreover, they satisfy

$$z_{n,j} = \pi n + o(1) \quad \text{as} \quad n \to \pm\infty, \quad j \in \mathbb{N}_N. \tag{2.5}$$

Proof. Each function ϕ_n in $\Phi = \sum_0^N \nu^{N-n} \phi_n$ is the symmetric polynomial of $\Delta_m, m \in \mathbb{N}_N$. Then $\Delta_m(z) = \cos z + o(e^{|\operatorname{Im} z|})$ as $|z| \to \infty$ yields (2.2).

Due to Definition Mii), the zeros of the function $D(\pm 1, z)$ (or $\Phi(\pm 1, z)$) are real.

i) Let $n_1 > n_0$ be another integer. Introduce the contour $C_n(r) = \{z : |z - \pi n| = \pi r\}$. Consider the contours $C_0(2n_0+1), C_0(2n_1+1), C_{2n}(\frac{1}{4}), |n| > n_0$. Then (3.3) and the estimate $e^{\frac{1}{2}|\operatorname{Im} z|} < 4|\sin\frac{z}{2}|$ on all contours yield

$$\left|\Phi(1,z) - \left(2\sin\frac{z}{2}\right)^{2N}\right| = o(e^{N|\operatorname{Im} z|}) = o(1)\left|\sin\frac{z}{2}\right|^{2N} < \frac{1}{2}\left|2\sin\frac{z}{2}\right|^{2N} \tag{2.6}$$

for large n_0. Hence, by Rouché's Theorem, $\Phi(1,z)$ has as many roots, counted with multiplicities, as $\sin^{2N}\frac{z}{2}$ in each of the bounded domains and the remaining unbounded domain. Since $\sin^{2N}\frac{z}{2}$ has exactly one root of the multiplicity $2N$ at $2\pi n$, and since $n_1 > n_0$ can be chosen arbitrarily large, the point ii) follows.

ii) The proof for $\Phi(-1,z)$ is similar.

iii) We have $\Delta_j(z) = \cos z + o(1)$ as $z = \pi n + O(1)$. For each $s \in \mathbb{N}_N$ there exists j such that $\Delta_j(z_{n,s}) = (-1)^n$. Thus we have $z_{n,s} = \pi n + o(1)$. □

Proof of Theorem 1.1. The proof repeats the proof of Theorem 1.1 from [CK]. □

We recall some well known facts about entire functions; see [Koo]. An entire function $f(z)$ is said to be of *exponential type* if there is a constant γ such that $|f(z)| \leqslant$ const. $e^{\gamma|z|}$ everywhere. The infimum over the set of γ for which such an inequality holds is called the type of f. The function f is said to belong to the Cartwright class if $f(z)$ is entire, of exponential type, and $\int_{\mathbb{R}} \frac{\log^+|f(x)|dx}{1+x^2} < \infty$. The function f is said to belong to the Cartwright class $\mathcal{E}_C(a_+, a_-)$ if f is entire, of exponential type, and satisfies the following conditions:

$$\int_{\mathbb{R}} \frac{\log^+|f(x)|dx}{1+x^2} < \infty, \quad \rho_\pm(f) = a_\pm, \qquad \text{where} \quad \rho_\pm(f) \equiv \limsup_{y\to\infty} \frac{\log|f(\pm iy)|}{y}.$$

If $f \in \mathcal{E}_C(a_+, a_-)$ and $f \in L^2(\mathbb{R})$, then the Paley-Wiener Theorem gives $f = \int_{-a_-}^{-a_+} \hat{\phi}(t)e^{-itz}dt$ for some $\hat{\phi} \in L^2(-a_-, a_+)$. Denote by $\mathcal{N}(r,f)$ the total number of zeros of f with modulus $\leqslant r$. Recall the following well known result (see p. 69 of [Koo]).

Theorem (Levinson)**.** *Let the function $f \in \mathcal{E}_C(1,1)$. Then $\mathcal{N}(r,f) = \frac{2}{\pi}r + o(r)$ as $r \to \infty$, and for each $\delta > 0$ the number of zeros of f with modulus $\leqslant r$ lying outside both of the two sectors $|\arg z|, |\arg z - \pi| < \delta$ is $o(r)$ for $r \to \infty$.*

We determine the asymptotics of ρ_s, Δ_m and the zeros of $\rho_s, s \in \mathbb{N}_{N_0}$.

Lemma 2.3. *Let $M \in \mathscr{M}_N^0$. If $\Delta_j(z) = \cos z + \frac{b_j \sin z + o(e^{|\operatorname{Im} z|})}{z}$ as $\operatorname{Im} z \to \infty$, and $b_j \neq b_k, j \neq k$ for all $j,k \in \omega_s$ for some $s \in \mathbb{N}_{N_0}$, then $\rho_s \in \mathcal{E}_C(a,a), a = N_s(N_s-1)$ and*

$$\rho_s(z) = c_s\left(\frac{\sin z}{2z}\right)^{N_s(N_s-1)}(1+o(1)) \quad \text{as} \quad \operatorname{Im} z \to \infty, \quad c_s = \prod_{j<k, j,k\in\omega_s}(b_j - b_k)^2. \tag{2.7}$$

Let, in addition, $\Delta_j(z) = \cos z + \frac{b_j \sin z + o(e^{|\operatorname{Im} z|})}{z}$ as $|z| \to \infty$. Then

i) There exists an integer $n_0 \geqslant 1$ such that ρ_s has exactly $2N_s(N_s-1)n_0$ roots, counted with multiplicity, in the disc $\{|z| < \pi(n_0 + \frac{1}{2})\}$, and for each $|n| > n_0$, exactly $N_s(N_s-1)$ roots, counted with multiplicity, in the domain $\{|z - \pi n| < \frac{\pi}{2}\}$.

There are no other roots. Moreover, the following asymptotics hold:

$$\rho_s(z) = c_s\left(\frac{\sin z + o(e^{|\operatorname{Im} z|})}{2z}\right)^{N_s(N_s-1)} \quad as \quad |z| \to \infty. \tag{2.8}$$

ii) The zeros of ρ_s are given by $z_\alpha^{n\pm}, \alpha = (j,k), j < k, j,k \in \omega_s$, and $n \in \mathbb{Z}\setminus\{0\}$. Furthermore, they satisfy $z_\alpha^{n\pm} = \pi n + o(1)$ as $n \to \infty$.

iii) Let, in addition, $\Delta_j(z) = \cos(z - \frac{b_j}{2z}) + o(n^{-2})$ as $|z - \pi n| \leqslant 1, n \to \pm\infty$. Then

$$z_\alpha^{n\pm} = \pi n + \frac{b_j + b_k + o(1)}{2\pi n}, \qquad \alpha = (j,k), \quad n \to \pm\infty. \tag{2.9}$$

Proof. i) Using $\Delta_j(z) - \Delta_k(z) = (b_j - b_k)\frac{\sin z}{2z} + o(z^{-1}e^{|\operatorname{Im} z|})$ as $\operatorname{Im} z \to \infty$, we get

$$\rho_s(z) = \prod_{j<k}(\Delta_j(z) - \Delta_k(z))^2 = c_s\left(\frac{\sin z + o(e^{|\operatorname{Im} z|})}{2z}\right)^{N_s(N_s-1)},$$

which yields (2.7). The proof of (2.8) is similar.

Let $n_1 > n_0$ be another integer. Introduce the contour $C_n(r) = \{z : |z - \pi n| = \pi r\}$.

Consider the case $N_0 = 1$; the proof for $N_0 \geqslant 2$ is similar. Let $n_1 > n_0$ be another integer. Consider the contours $C_0(2n_0+1), C_0(2n_1+1), C_{2n}(\frac{1}{4}), |n| > n_0$. Then (2.8) and the estimate $e^{|\operatorname{Im} z|} < 4|\sin z|$ on all contours (for large n_0) yield $\rho(z) = \rho^0(z)(1+o(1))$, where $\rho^0(z) = c_0(\frac{\sin z}{2z})^{N(N-1)}$. Hence, by Rouché's Theorem, ρ has as many roots, counted with multiplicities, as ρ^0 in each of the bounded domains and the remaining unbounded domain. Since ρ^0 has exactly one root of the multiplicity $N(N-1)$ at $\pi n \neq 0$, and since $n_1 > n_0$ can be chosen arbitrarily large, point i) follows.

ii) Thus the zeros of ρ_s have the form $z_\alpha^{n\pm}, \alpha = (j,j'), j,j' \in \omega_s, j < j', n \in \mathbb{Z}\setminus\{0\}$ and satisfy $|z_\alpha^{n\pm} - \pi n| < \pi/2$.

We have $0 = \Delta_j(z) - \Delta_{j'}(z) = (b_j - b_{j'})\frac{\sin z + o(1)}{2z}$ as $|z_\alpha^{n\pm} - \pi n| < \pi/2, |n| \to \infty$. Then we deduce that $z_\alpha^{n\pm} = \pi n + o(1)$ as $n \to \infty$.

iii) We have the identity $\Delta_j(z) - \Delta_{j'}(z) = 0$ at $z = z_\alpha^{n\pm}$ and the asymptotics

$$\begin{aligned}\cos\left(z_\alpha^{n\pm} - \frac{b_j}{2\pi n}\right) &- \cos\left(z_\alpha^{n\pm} - \frac{b_{j'}}{2\pi n}\right)\\ &= 2(-1)^n \sin\frac{b_{j'} - b_j}{4\pi n}\sin\left(z_\alpha^{n\pm} - \pi n - \frac{b_j + b_{j'}}{4\pi n}\right) = \frac{o(1)}{n^2},\end{aligned}$$

which yields (2.9). □

We remark that by using this lemma and the Levinson Theorem we describe the position of resonances, but unfortunately it is very rough.

3. The conformal mappings

In this section we study properties of the quasimomentum and prove theorems about the conformal mapping. Recall that the Lyapunov function $\widetilde{\Delta}_s(\zeta)$ is analytic on some N_s–sheeted Riemann surface $\mathscr{R}_s$, and $\mathscr{R} = \bigcup_1^{N_0}\mathscr{R}_s$. Let $z = x + iy \in \mathbb{C}$ be the natural projection of $\zeta \in \mathscr{R}$, $\mathcal{B}(\widetilde{\Delta})$ be the set of all branch points of the Lyapunov function and $\mathscr{R}^\pm = \{\zeta \in \mathscr{R} : \pm\operatorname{Im}\zeta > 0\}$. Recall that the domains $\mathscr{R}_0^\pm =$

$\mathbb{C}_\pm \setminus \beta_\pm \subset \mathbb{C}_\pm$ are simply connected, and define a domain $\mathscr{R}_0 = \mathbb{C} \setminus (\beta_+ \cup \beta_- \cup \beta_0)$, where

$$\beta_\pm = \bigcup\nolimits_{\beta \in \mathcal{B}(\widetilde{\Delta}) \cap \mathbb{C}_\pm} [\beta, \beta \pm i\infty), \qquad \beta_0 = \{z \in \mathbb{R} : \Delta_j(z) \notin \mathbb{R} \textit{ for some } j \in \mathbb{N}_N\}.$$

Due to Definition M, $\widetilde{\Delta}(\zeta) \notin [-1,1]$ for $\zeta \in \mathscr{R}^+$. Recall that $q(\zeta) = |\log \eta(\widetilde{\Delta}(\zeta))|$ is the single-valued on $\mathscr{R}^+$ imaginary part of the (in general, many-valued on $\mathscr{R}^+$) quasimomentum $k(\zeta) = p(\zeta) + iq(\zeta) = \arccos \widetilde{\Delta}(\zeta) = i \log \eta(\widetilde{\Delta}(\zeta))$, where

$$\eta(z) \equiv z + \sqrt{z^2 - 1}, \qquad \eta : \mathbb{C} \setminus [-1,1] \to \{z \in \mathbb{C} : |z| > 1\}.$$

We denote by $q_j(z)$, $(z,j) \in \mathbb{C}_+ \times \mathbb{N}_N$, the branches of $q(\zeta)$ and by $p_j(z)$, $k_j(z)$, $z \in \mathscr{R}_0^+$, the single-valued branches of $p(\zeta)$, $k(\zeta)$, respectively.

Theorem 3.1. *Let $M \in \mathscr{M}_N$ and let $s \in \mathbb{N}_{N_0}$ (N_0 is defined in Theorem 1.1). Then the function $\widetilde{q}_s(\zeta) = \log|\eta(\widetilde{\Delta}_s(\zeta))|$ is subharmonic on the Riemann surface $\mathscr{R}_s$. Moreover,*

1) If $M \in \mathscr{M}_N^0$, then the following asymptotics hold true:

$$\widetilde{q}_s(\zeta) = y + o(1), \qquad |\zeta| \to \infty, \qquad \operatorname{Im} z \geqslant 0. \tag{3.1}$$

If $\widetilde{\Delta}_s(z) = \cos z + O(1/z)$ as $|z| \to \infty$, then the following asymptotics hold true:

$$\widetilde{q}_s(\zeta) = y + O(|z|^{-\frac{1}{2}}), \quad |\zeta| \to \infty, \ \zeta \in \mathscr{R}_s, \quad \operatorname{Im} z \geqslant 0. \tag{3.2}$$

2) Let Δ_j be analytic on some bounded interval $Y = (\alpha, \beta) \subset \mathbb{R}$ for some $j \in \omega_s$. Then

i) If $\Delta_j(z) \in \mathbb{R} \setminus [-1,1]$ for all $z \in Y$, then $k_j(\cdot)$ has an analytic extension from $\mathscr{R}_0^+$ into $\mathscr{R}_0^+ \cup \mathscr{R}_0^- \cup Y$ such that

$$\operatorname{Re} k_j(z) = \text{const} \in \pi\mathbb{Z}, \quad z \in Y, \tag{3.3}$$

$$q_j(z) = q_j(\overline{z}) > 0, \qquad z \in \mathscr{R}_0^+ \cup \mathscr{R}_0^- \cup Y. \tag{3.4}$$

ii) If $\Delta_j(z) \notin \mathbb{R}$ for any $z \in Y$, then there exists a branch $\Delta_{j'}, j' \in \omega_s$, such that $\overline{\Delta}_{j'}(z) = \Delta_j(z)$ for any $z \in Y$. The functions $\Delta_j(z)$ and $k_{j'} + k_j$ have analytic extensions from $\mathscr{R}_0^+$ into $\mathscr{R}_0^+ \cup \mathscr{R}_0^- \cup Y$ such that

$$\Delta_j(z) = \begin{cases} \Delta_j(z) & \textit{if} \quad z \in \mathscr{R}_0^+, \\ \overline{\Delta}_{j'}(\overline{z}) & \textit{if} \quad z \in \mathscr{R}_0^-, \end{cases} \tag{3.5}$$

$$p_j(z) + p_{j'}(z) = \text{const} \in 2\pi\mathbb{Z}, \quad z \in Y, \quad q_j(z) = q_{j'}(z), \qquad z \in Y, \tag{3.6}$$

$$q_j(z) + q_{j'}(z) = q_j(\overline{z}) + q_{j'}(\overline{z}) > 0, \quad z \in \mathscr{R}_0^+ \cup \mathscr{R}_0^- \cup Y. \tag{3.7}$$

Proof. The proof repeats the proof of Theorem 4.1 from [CK]. □

Proof of Theorem 1.2. The Puiseux series for $\Delta_j, j \in \mathbb{N}_\varkappa$, are given by

$$\Delta_j(z) = \Delta_1(z_0) + a_1 et + a_2 e^2 t^2 + ..., \quad e = e^{i\frac{2\pi}{p}}, \quad t = (z - z_0)^{\frac{1}{\varkappa}} \in D_r = \{t : |t| < r\}. \tag{3.8}$$

If $j = \varkappa$, then we obtain $a_n \in \mathbb{R}$ for all $n \geqslant 1$. Furthermore, $\varkappa = 2$, since $\Delta_j(z)$ is real for all $j \in \mathbb{N}_\varkappa$ and all $z \in (z_0, z_0 + \varepsilon)$ (or all $z \in (z_0 - \varepsilon, z_0)$). Moreover, these arguments give

$$\Delta_j(z_0 + t^2) = f(t) + (-1)^j t^{1+2m} g(t), \quad j = 1,2, \quad t \in D_r, \ m - 1 \in \mathbb{N}, \ g(0) \neq 0, \tag{3.9}$$

where f, g are analytic function in some disk $D_r, r > 0$, and f, g are real on $(-r, r)$. Thus $F = (\Delta_1 - \Delta_2)^2$ is analytic in D_r and $F(z) = z^{1+2m}(C + O(z))$ as $z \to z_0$ for some $C \neq 0$.

Let $\Delta_1(z_0) \in (-1, 1)$. Then using (3.9), identity $\Delta_j(z) = \cos k_j(z)$ and the Implicit Function Theorem, we obtain

$$k_j(z_0 + t^2) = f_j(t) + t^{1+2m_j} g_j(t), \quad j = 1, 2, \quad t \in D_r,\ m_j - 1 \in \mathbb{N},\ g_j(0) \neq 0,$$

where f_j, g_j are some analytic functions in the disk D_r for some $r > 0$. Then (3.7) yields

$$k_1(z_0 + t^2) + k_2(z_0 + t^2) = 2\pi n_0 + t^s(c + O(t)) \quad \text{as} \quad t \to 0, t \in \overline{\mathbb{C}}_+,$$

for some $(n_0, s) \in \mathbb{Z} \times \mathbb{N}$ and $c \neq 0$. The case $s \geqslant 2$ is absent, since $q_1, q_1 > 0$ in $D_r \cup \mathbb{C}_+$ for some $r > 0$, which yields $s = m_1 = m_2 = 1$. □

Recall the needed properties of the functions $q \in \mathcal{SC}$ defined in Section 1 and $k = p + iq$. It is well known that $p \in C(\overline{\mathbb{C}}_+)$ and $\frac{1}{2\pi}(\partial_x^2 + \partial_y^2)q = \mu_q$ (in the sense of distribution) is a so-called Riesz measure of the function v. Moreover, the following identities are fulfilled:

$$\pi\mu_q((x_1, x_2)) = p(x_2) - p(x_1), \quad \text{for any} \quad x_1 < x_2,\ x_1, x_2 \in \mathbb{R}, \tag{3.10}$$

$$\frac{\partial q(z)}{\partial y} = y \int_{\mathbb{R}} \frac{d\mu_v(t)}{(t - x)^2 + y^2}, \quad z = x + iy \in \mathbb{C}_+, \tag{3.11}$$

which yields $\frac{\partial q(z)}{\partial y} \geqslant 0, z \in \mathbb{C}_+$. Moreover, $q(x) = q(x \pm i0), x \in \mathbb{R}$. It is well known that if $q \in \mathcal{SC}$, then

$$\int_{\mathbb{R}} \frac{d\mu_q(t)}{1 + t^2} < +\infty, \quad \limsup_{z \to \infty} \frac{q(z)}{|z|} = \lim_{y \to +\infty} \frac{k(iy)}{iy} = \lim_{y \to +\infty} \frac{q(iy)}{y} = \lim_{x \to \pm\infty} \frac{p(x)}{x} \geqslant 0. \tag{3.12}$$

Now we recall the well known fact (see [Ah]).

Theorem (Nevanlinna)**.** *i) Let μ be a Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}(1 + x^{2r})d\mu(x) < \infty$ for some integer $r \geqslant 0$. Then for each $s > 0$ the following asymptotics hold:*

$$\int_{\mathbb{R}} \frac{d\mu(t)}{t - z} = -\sum_{k=0}^{2r} \frac{Q_k}{z^{k+1}} + \frac{o(1)}{z^{2p+1}}, \quad |z| \to \infty,\ y > s|x|,$$

$$Q_n = \int_{\mathbb{R}} x^n d\mu(x),\ 0 \leqslant n \leqslant 2r.$$

ii) Let f be an analytic function in $\mathbb{C}_+$ such that $\operatorname{Im} f(z) \geqslant 0$ for all $z \in \mathbb{C}_+$ and

$$\operatorname{Im} f(iy) = c_0 y^{-1} + \ldots + c_{2p-1} y^{-2r} + O(y^{-2r-1}) \ \text{as} \ y \to \infty \tag{3.13}$$

for some $c_0, ..., c_{2r-1} \in \mathbb{R}$ and $r \geqslant 0$. Then $f(z) = C + \int_{\mathbb{R}} \frac{d\mu(t)}{t-z}$, $z \in \mathbb{C}_+$, for some Borel measure μ on ρ such that $\int_{\mathbb{R}}(1 + x^{2r})d\mu(x) < \infty$ and $C \in \mathbb{R}$.

Proof of Theorem 1.3. i) The proof repeats the proof of Theorem 4.2 from [CK].

ii) Below, for each real harmonic function $q(z), z \in \mathbb{C}_+$, we introduce an analytic function $k = p + iq$ in $\mathbb{C}_+$, where $(-p)$ is some harmonic conjugate of q for $\mathbb{C}_+$. If $q \in \mathcal{SK}_0^+$, then the function $k = p + iq$ in $\mathbb{C}_+$ is defined by

$$k(z) = z + \frac{1}{\pi} \int_{\mathbb{R}} \frac{q(t)dt}{t - z}, \quad z \in \mathbb{C}_+. \tag{3.14}$$

Due to i), the function $q = \frac{1}{N}\sum_1^N q_j \in \mathcal{SC} \cap C(\mathbb{C})$ and q is positive in $\mathbb{C}_+$. Let $z = iy, y \to \infty$, and using $q_m(iy) = y + o(1)$ (see (3.1)) we obtain

(3.15)
$$\Phi(z,0) = \prod_1^N \Delta_m(z) = \frac{(1+e^{-2Ny}O(1))}{2^N} e^{-i\sum_1^N k_m(z)} = \frac{(1+e^{-2Ny}O(1))}{2^N} e^{-iNk(z)}.$$

Thus these asymptotics and $\Phi(z,0) = (\cos^N z)\exp i\Big(-\sum_{j=0}^{2r}\frac{C_j}{z^{j+1}} + \frac{o(1)}{z^{2r+1}}\Big)$, see (1.5), give

$$k(z) = z - \sum_{j=0}^{2r}\frac{C_j}{z^{j+1}} + \frac{o(1)}{z^{2r+1}} \qquad \text{as} \quad z = iy,\ y \to \infty. \tag{3.16}$$

Then by i) and the Nevanlinna Theorem, the function $q \in \mathcal{SK}_{2r}^+$. We need the following result from [KK1]: *Let $q \in \mathcal{SK}_m^+$ for some $m \geqslant 0$ and $q \neq const$. Then $k : \mathbb{C}_+ \to k(\mathbb{C}_+) = \mathbb{K}(h)$ is a conformal mapping for some $h \in C_{us}, h \geqslant 0$. Moreover, the following asymptotics, estimates and identities are fulfilled:*

$$k(z) = z - \sum_{k=0}^{2m}\frac{Q_k}{z^{k+1}} + \frac{o(1)}{z^{2m+1}} \quad \text{as} \quad |z| \to \infty, \quad y \geqslant r|x|, \quad \text{for any } r > 0, \tag{3.17}$$

$$I_s^D + I_{2s}^S = Q_{2s} + \frac{sQ_{s-1}^2}{2} + \sum_{n=0}^{s-2}(n+1)Q_nQ_{2s-2-n}, \qquad s = 0, ..., m, \tag{3.18}$$

$$\sup_{x\in\mathbb{R}} q^2(x) \leqslant 2Q_0, \tag{3.19}$$

where $I_n^S, I_n^D, Q_n, n \geqslant 0$, are given by (1.14). Thus the above results give that $k : \mathbb{C}_+ \to k(\mathbb{C}_+) = \mathbb{K}(h)$ is a conformal mapping for some $h \in C_{us}$. Moreover, asymptotics (1.17) and identities (1.18), (1.19) hold true.

Estimate (3.19) gives $q|_{\mathbb{R}} \leqslant \sqrt{2Q_0}$. If $z \in \sigma_{(N)}$, then $q(z) = 0$, since $\Delta_j(z) \in [-1,1]$ and $q_j(z) = 0$ for all $j \in \mathbb{N}_N$. If $z \in \sigma_{(1)} \cup g$, then $\Delta_j(z) \notin [-1,1]$ for some $j \in \mathbb{N}_N$. Thus $q_j(z) > 0$ and $q(z) > 0$, which gives (1.20). □

Proof of Theorem 1.4. i) We show (1.21). Using $k(z) = z + \frac{1}{\pi}\int_{\mathbb{R}}\frac{q(t)dt}{(t-x)}, z \in \mathbb{C}_+$, and (1.20) we obtain $k'(z) = 1 + \frac{1}{\pi}\int_{\mathbb{R}}\frac{q(t)dt}{(t-x)^2} > 1, \quad z \in \sigma_{(N)}$, which yields $p_x'(z) > 1, z \in \sigma_{(N)}$. Moreover, we have $p_x'(z) = 1$ for some $z \in \sigma_{(N)}$ iff $q|_{\mathbb{R}} = 0$.

The function $\widetilde{q}_s$ is harmonic in $\mathscr{R}_s^+ = \{\zeta \in \mathscr{R}_s : \operatorname{Im}\zeta > 0\}$. The function $f = \widetilde{q}_s - y$ is harmonic in $\mathscr{R}_s^+$ and $f > 0$.

Let $x \in \sigma_{(1)}$. Then some branch $\Delta_m(x) \in (-1,1)$ for all $x \in Y \subset \sigma_{(1)}$ for some small interval $Y = (\alpha,\beta)$. We have $\Delta_m(x) = \cos k_m(x)$. Thus we get $k_m'(x) = -\frac{\Delta_m'(x)}{\sin k_m(x)} \neq 0$. Hence we get $p_m'(x) = k_m'(x) > 0$, since $q_m(x) = 0$ and (3.11) yields $p_x(x) \geqslant 0$ for $x \in Y$, which gives (1.21).

We show (1.22). Note that $q(z) > 0, z \in g$; otherwise we do not have a gap. Using (3.11) we obtain $-\frac{\partial^2 q(z)}{\partial x^2} = \frac{\partial^2 q(z)}{\partial y^2} = \int_{\mathbb{R}}\frac{d\mu_q(t)}{(t-x)^2} > 0, \quad z = x \in g_n$, which yields $q_{xx}''(z) < 0, z \in g_n$. Consider the function $p(z), z \in g_n$. Theorem 3.1 yields $N\operatorname{Re}k(z) = \sum_1^N \operatorname{Re}k_m = \sum_1^N \pi n_m = \pi N_n$, which gives (1.22)

We recall the result from [KK2]. Let a function f be harmonic and positive in the domain $\mathbb{C} \setminus \overline{g}_n, g_n = (z_n^-, z_n^+) \neq \emptyset$ and $f(iy) = y(1+o(1))$ as $y \to \infty$. Assume

$f(z) = f(\overline{z}),\ z \in \mathbb{C} \setminus \overline{g}_n$ and $f \in C(\overline{\mathbb{C}_+})$. Then

$$f(x) = q_n(x)\left(1 + \frac{1}{\pi}\int_{\mathbb{R}\setminus I} \frac{f(t)dt}{|t-x|q_n(t)}\right), \quad q_n(x) = |(z - z_n^-)(z_n^+ - z)|^{\frac{1}{2}}, \quad x \in g_n.$$

Hence the last identity and properties of q yield (1.23) and the estimate $q(z) \geqslant q_n(z),\ z \in g_n = (z_n^-, z_n^+)$. This estimate implies $Q_0 \geqslant \frac{1}{\pi}\sum\int_{g_n} q_n(t)dt = \frac{1}{8}\sum |g_n|^2$, which yields the first estimate in (1.24).

Assume that $\sigma_{(N)} = \sigma(F)$. In this case the function h is given by $h(p) = 0, p \neq \frac{\pi}{N}\mathbb{Z}$ and $(h(\frac{\pi}{N}n))_{n\in\mathbb{Z}} \in \ell^2$. Such conformal mapping were considered in [K9]. In this case for some absolute constant C_0 the following estimate $Q_0 \leqslant C_0 G^2(1+G^2)$ was obtained in [K9]. This gives the second estimate in (1.24). □

4. Proof of Theorems 1.5 and 1.6

We begin with some notational convention. A vector $h = \{h_n\}_1^N \in \mathbb{C}^N$ has the Euclidean norm $|h|^2 = \sum_1^N |h_n|^2$, while an $N \times N$ matrix A has the operator norm given by $|A| = \sup_{|h|=1} |Ah|$. Note that $|A|^2 \leqslant \operatorname{Tr} A^*A$.

1. The first order periodic systems. In this case $J = I_{N_1} \oplus (-I_{N_2})$. Below we use arguments from [K4], [K5]. We need the identities

$$JV = -VJ, \quad e^{zJ}V = Ve^{-zJ}, \quad \text{for all} \quad (z, V) \in \mathbb{C} \times \mathscr{H}_0. \tag{4.1}$$

The solution of the equation $-iJ\psi' + V\psi = z\psi, \psi_0(z) = I_N$ satisfies the integral equation

$$\psi(t,z) = e^{iztJ} - i\int_0^t e^{izJ(t-s)}JV(s)\psi(s,z)ds, \quad t \geqslant 0,\ z \in \mathbb{C}, \tag{4.2}$$

and ψ is given by

(4.3)

$$\psi(t,z) = \sum_{n\geqslant 0}\psi_n(t,z), \quad \psi_n(t,z) = -i\int_0^t e^{izJ(t-s)}JV(s)\psi_{n-1}(t,z)ds, \quad \psi_0(t,z) = e^{iztJ},$$

$n \geqslant 1$. Using (4.1), (4.3) we have

$$\psi_1(t,z) = -i\int_0^t e^{izJ(t-s)}JV(s)e^{izsJ}ds = -i\int_0^t e^{izJ(t-2s)}JV(s)ds, \tag{4.4}$$

$$\begin{aligned}\psi_2(t,z) &= -i\int_0^t e^{izJ(t-t_1)}JV(t_1)\psi_1(t_1,z)dt_1 \\ &= \int_0^t dt_1 \int_0^{t_1} e^{izJ(t-2t_1+2t_2)}V(t_1)V(t_2)dt_2.\end{aligned} \tag{4.5}$$

Proceeding by induction,

$$\psi_{2n}(1,z) = \int_0^1 dt_1 \ldots \int_0^{t_{2n-1}} e^{izJ\varkappa_{2n}}V(t_1)\ldots V(t_{2n})dt_{2n}, \tag{4.6}$$

$$\varkappa_n = 1 - 2t_1 + 2t_2 + \cdots + (-1)^n 2t_n,$$

$$\psi_{2n+1}(1,z) = -iJ\int_0^1 dt_1 \ldots \int_0^{t_{2n}} e^{izJ\varkappa_{2n+1}}V(t_1)\ldots V(t_{2n+1})dt_{2n+1}. \tag{4.7}$$

We need the following estimates.

Lemma 4.1. *Let $V \in \mathscr{H}_0$. For each $z \in \mathbb{C}$ there exists a unique solution ψ of* (4.2) *given by* (4.3)*, and the series* (4.3) *converges uniformly on bounded subsets of $\mathbb{R} \times \mathbb{C} \times \mathscr{H}$. For each $t \geqslant 0$ the function $\psi(t,z)$ is entire on $\mathbb{C}$. Moreover, for any $n \geqslant 0$ and $(t,z) \in [0,\infty) \times \mathbb{C}$ the following estimates and asymptotics hold:*

$$|\psi_n(t,z)| \leqslant \frac{e^{|\operatorname{Im} z|t}}{n!}\left(\int_0^t |V(s)|ds\right)^n, \tag{4.8}$$

$$|\psi(t,z) - \sum_0^{n-1} \psi_j(t,z)| \leqslant \frac{(\sqrt{t}\|V\|)^n}{n!} e^{t|\operatorname{Im} z| + \int_0^t |V(s)|ds}, \tag{4.9}$$

$$\psi(t,z) - e^{iztJ} = o(e^{t|\operatorname{Im} z|}) \quad \text{as} \quad |z| \to \infty, \tag{4.10}$$

$$T_m(z) = N_1 e^{iz} + N_2 e^{-iz} + o(e^{m|\operatorname{Im} z|}) \quad \text{as} \quad |z| \to \infty. \tag{4.11}$$

If $V^\nu, V \in \mathscr{H}_0$ and if the sequence $V^\nu \to V$ weakly in $\mathscr{H}_0$ as $\nu \to \infty$, then $\psi(t,z,V^\nu) \to \psi(t,z,V)$ uniformly on bounded subsets of $\mathbb{R} \times \mathbb{C}$.

Proof. Equations (4.6), (4.7) give

$$\begin{aligned} |\psi_n(t,z)| &\leqslant \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} e^{|\operatorname{Im} z|t} |V(t_1)| \ldots |V(t_n)| dt_n \\ &\leqslant \frac{e^{|\operatorname{Im} z|t}}{n!}\left(\int_0^t |V(s)|ds\right)^n, \end{aligned}$$

since $|t - 2t_1 + 2t_2 + \cdots + (-1)^n 2t_n| \leqslant t$, which yields (4.8). This shows that for each $t > 0$ the series (4.3) converges uniformly on bounded subsets of $\mathbb{C} \times \mathcal{H}_0$. Each term of this series is an entire function. Hence the sum is an entire function. Summing the majorants we obtain estimates (4.9). The proof of asymptotics in (4.10) is standard (see e.g. [K4] or [K5]). Equation (4.10) implies (4.11).

Assume that the sequence $V^\nu \to V$ weakly in $\mathscr{H}$, as $\nu \to \infty$. Then each term $\psi_n(t,z,V^\nu) \to \psi_n(t,z,V)$ uniformly on bounded subsets of $\mathbb{R} \times \mathbb{C}$ and fixed $n \geqslant 1$. Then (4.9) gives that $\psi(t,z,V^\nu) \to \psi(t,z,V)$ uniformly on bounded subsets of $\mathbb{R} \times \mathbb{C}$. □

We need asymptotics of $L(z) = \frac{1}{2}(M(z) + M^{-1}(z))$ as $\operatorname{Im} z \to \infty$. The identities (1.3), (4.3) yield

$$L(z) = \frac{1}{2}(\psi(1,z) + J\psi^*(1,\overline{z})J) = \sum_{n \geqslant 0} L_n(z), \quad L_n(z) = \frac{1}{2}(\psi_n(1,z) + J\psi_n^*(1,\overline{z})J), \tag{4.12}$$

where the series (4.12) converges uniformly on bounded subsets of $\mathbb{C} \times \mathscr{H}_0$. Using (4.1), (4.6), (4.7), we obtain

$$J{\psi_{2n}}^*(1,\overline{z})J = \int_0^1 dt_1 \ldots \int_0^{t_{2n-1}} e^{-izJ\varkappa_{2n}} V(t_{2n}) \ldots V(t_1) dt_{2n}, \tag{4.13}$$

$$J{\psi_{2n+1}}^*(1,\overline{z})J = iJ \int_0^1 dt_1 \ldots \int_0^{t_{2n}} e^{izJ\varkappa_{2n+1}} V(t_{2n+1}) \ldots V(t_1) dt_{2n+1}, \tag{4.14}$$

and in particular, $J\psi_1^*(1,\overline{z})J = -\psi_1(1,z)$. Thus we deduce that

$$L = \cos z + L_2 + \sum_{n\geqslant 3} L_n, \quad L_1 = 0, \tag{4.15}$$

$$L_2(z) = \frac{1}{2}\int_0^1\int_0^{t_1}\Big(e^{izJ(1-2t_1+2t_2)}V(t_1)V(t_2) + e^{-izJ(1-2t_1+2t_2)}V(t_2)V(t_1)\Big)dt, ... \tag{4.16}$$

Lemma 4.2. *Let $V \in \mathscr{H}_0$. Then for $y \geqslant r_0|x|, y \to \infty, r_0 > 0$, the following asymptotics hold:*

$$\int_0^1 dt_1 \dots \int_0^{t_{n-1}} e^{\pm izJ\varkappa_n} V(t_1)\dots V(t_n)dt_n = \frac{o(e^y)}{|z|}, \tag{4.17}$$

$$L_2(z) = \frac{\sin z}{2z}(\mathscr{V} + o(1)), \qquad \textit{where} \quad \mathscr{V} = \int_0^1 V^2(t)dt, \tag{4.18}$$

$$L(z) = \cos\Big(zI_N - \frac{\mathscr{V} + o(1)}{2z}\Big), \tag{4.19}$$

$$\det L(z) = (\cos^N z)\exp\Big(\frac{i\|V\|^2 + o(1)}{2z}\Big). \tag{4.20}$$

Proof. Let $V_t = V(t)$. Asymptotics (5.1), (5.2) give

$$\int_0^1 dt\int_0^t e^{i2z(t-s)}V_tV_s ds = \frac{i}{2z}\Big(\mathscr{V} + o(1)\Big), \quad \int_0^1 dt\int_0^t e^{-i2z(t-s)}V_tV_s ds = \frac{o(e^{2y})}{|z|}. \tag{4.21}$$

Using (4.21) and $\varkappa_2 = 1 - 2t + 2s$ we get

$$\begin{aligned} L_2(z) &= \frac{1}{2}\int_0^1\int_0^t\Big(e^{izJ\varkappa_2}V_tV_s + e^{-izJ\varkappa_2}V_sV_t\Big)dt \\ &= \frac{1}{2}\int_0^1\int_0^{t_1}\Big(\cos z\varkappa_2(V_tV_s + V_sV_t) + iJ\sin z\varkappa_2(V_tV_s - V_sV_t)\Big)dt \\ &= \frac{i\cos}{2z}\Big(\int_0^1 V_t^2 dt + o(1)\Big), \end{aligned} \tag{4.22}$$

which gives (4.18). The similar arguments yield (4.17).

In order to estimate L_n we consider the function $f_n^{\pm} = f_n^{\pm}(y) = \int_{G_n} e^{\pm y\varkappa_n}dt$. Recall $\varkappa_n = 1 - 2t_1 + 2t_2 + \dots + (-1)^n 2t_n$, $t = (t_1, ..., t_n) \in G_n = \{0 < t_n < ... < t_2 < t_1 < 1\} \subset \mathbb{R}^n$. We need the simple estimate

$$\int_0^a e^{2yt}dt \leqslant \frac{e^{2ya}}{2y}, \qquad \int_0^a\int_0^t e^{-2y(t-s)}dtds \leqslant \frac{1}{2y}, \qquad \text{for any} \quad a, y > 0. \tag{4.23}$$

The direct calculations imply $f_p^{\pm} \leqslant e^y y^{-2}, y \geqslant 1,\ p = 2, 3$. Consider f_n^+; the proof for f_n^- is similar. Using the second estimate from (4.23) we have $f_{2n}^+(y) \leqslant \frac{e^y}{(2y)^n}$. This and the first estimate from (4.23) yield $f_{2n+1}^+(y) \leqslant \frac{f_{2n}^+(y)}{2y} \leqslant \frac{e^y}{(2y)^{n+1}}$. Thus we obtain

$$f_n^{\pm}(y) \leqslant \frac{e^y}{(2y)^{\frac{n}{2}}}, \quad n, y \geqslant 1. \tag{4.24}$$

Hence we have

$$\begin{aligned} |L_n(z)| &\leqslant \int_{G_n} (e^{y\varkappa_n} + e^{-y\varkappa_n}) \prod_1^n |V(t_j)| dt \\ &\leqslant \Big(\int_{G_n} (e^{y\varkappa_n} + e^{-y\varkappa_n})^2 dt \Big)^{\frac{1}{2}} \frac{\|V\|^n}{\sqrt{n!}} \leqslant \frac{2e^y}{\sqrt{n!}}\, \varepsilon^n, \end{aligned} \tag{4.25}$$

where $\varepsilon = \frac{\|V\|}{(4y)^{\frac{1}{4}}} \to 0$, since

$$\int_{G_n} \prod_1^n |V_{t_j}|^2 dt = \frac{\|V\|^{2n}}{n!}, \quad \int_{G_n} (e^{y\varkappa_n} + e^{-y\varkappa_n})^2 dt \leqslant 2 \int_{G_n} (e^{2y\varkappa_n} + e^{-2y\varkappa_n})^2 dt \leqslant \frac{4e^{2y}}{(4y)^{\frac{n}{2}}}.$$

Then the last estimate gives

$$|\sum_{n \geqslant 5} L_n(z)| \leqslant e^y |\sum_{n \geqslant 5} \varepsilon^n| = \frac{\varepsilon^5 e^y}{1 - \varepsilon} \leqslant 2\varepsilon^5 e^y. \tag{4.26}$$

Equations (4.12)-(4.14) and (4.17) give $L_s(z) = o(e^{2y})$ as $y \to \infty$. Then combining this and (4.26), (4.18) we obtain (4.19).

By using the asymptotics $L = \cos z (I_N + S + O(|S|^2))$ and $\det(I + S) = \exp(\operatorname{Tr} S + O(|S|^2))$, where $S = \frac{i\mathscr{V} + o(1)}{2z}$ as $\operatorname{Im} z \to \infty$, we obtain (4.20). □

Proof of Theorem 1.6. Recall the simple fact: Let A, B be matrices and and $\sigma(B)$ be a spectra of B. If A is normal, then $\operatorname{dist}\{\sigma(A), \sigma(A + B)\} \leqslant |B|$ (see p. 291 of [Ka]).

From (4.10) we have $L(z) = \cos z I_N + o(e^{|\operatorname{Im} z|}),\ |z| \to \infty$, where the operator $\cos z I_N$ has the eigenvalues $\cos z$ of the multiplicity N. Due to the result from [Ka] and asymptotics above, we deduce that the eigenvalues $\Delta_m(z)$ of the matrix $L(z)$ satisfy the asymptotics $\Delta_j(z) = \cos z + o(e^{|\operatorname{Im} z|}), j = 1, ..., N$, which gives the fact that $M \in \mathscr{M}_N^0$.

Lemma 4.2 implies $M \in \mathscr{M}_N^{0,0}$. Thus using Theorems 1.1-1.3, we obtain the proof of Theorem 1.6. □

2. The Schrödinger operator. In order to prove Theorem 1.5 we determine the asymptotics of M. The fundamental solution φ satisfies the integral equations

$$\varphi(t, z) = \varphi_0(t, z) + \int_0^t \frac{\sin z(t - s)}{z} V(s) \varphi(s, z) ds, \quad \varphi_0(t, z) = \frac{\sin zt}{z} I_N, \tag{4.27}$$

where $(t,z)\in\mathbb{R}\times\mathbb{C}$. The standard iterations in (4.27) yield

$$\varphi(t,z)=\sum\nolimits_{n\geqslant 0}\varphi_n(t,z)\,,\quad \varphi_{n+1}(t,z)=\int_0^t\frac{\sin z(t-s)}{z}V(s)\varphi_n(s,z)ds. \tag{4.28}$$

The similar expansion $\vartheta=\sum_{n\geqslant 0}\vartheta_n$ with $\vartheta_0(t,z)=I_N\cos zt$ holds. We need

Lemma 4.3. *Let $V\in\mathscr{H}$ and let $\varkappa=\frac{\|V\|}{|z|_1}$ and $|z|_1=\max\{1,|z|\}$. Then for any integers $m\geqslant t\geqslant 0, n_0\geqslant -1$ the following estimates hold:*

$$\begin{aligned}\max\Bigg\{&\Bigg|\vartheta(1,z)-\sum_0^{n_0}\vartheta_n(1,z)\Bigg|,\Bigg||z|_1\Bigg(\varphi(1,z)-\sum_0^{n_0}\varphi_n(1,z)\Bigg)\Bigg|,\\ &\Bigg|\frac{1}{|z|_1}\Bigg(\vartheta'(1,z)-\sum_0^{n_0}\vartheta_n'(1,z)\Bigg)\Bigg|,\\ &\Bigg|\varphi'(1,z)-\sum_0^{n_0}\varphi^{(n)\prime}(1,z)\Bigg|\Bigg\}\leqslant\frac{\varkappa^{n_0+1}}{(n_0+1)!}e^{(|\operatorname{Im}z|+\varkappa)},\end{aligned} \tag{4.29}$$

$$|T_m(z)-2N\cos mz-\frac{\sin mz}{z}m\operatorname{Tr}V^0|\leqslant 2Nm\varkappa e^{m(|\operatorname{Im}z|+\varkappa)}. \tag{4.30}$$

Proof. We prove the estimates of φ; the proof for $\varphi',\vartheta,\vartheta'$ is similar. Equation (4.28) gives

$$\varphi_n(t,z)=\int_{D_n}f_n(t,s)V(s_1)\cdot\ldots\cdot V(s_n)ds,\qquad f_n(t,s)=\varphi_0(s_n,z)\prod_1^n\frac{\sin z(s_{k-1}-s_k)}{z},$$

where $s=(s_1,\ldots,s_n)\in\mathbb{R}^n, s_0=t$ and $D_n=\{0<s_n<\ldots<s_2<s_1<t\}$. Substituting the estimate $|\varphi_0(t,z)|=|z^{-1}\sin zt|\leqslant|z|_1^{-1}e^{|\operatorname{Im}z|t}$ into the last integral, we obtain

$$|\varphi_n(t,z)|\leqslant\frac{e^{|\operatorname{Im}z|t}}{|z|_1^{n+1}}\int_{D_n}|V(s_1)|\cdot\ldots\cdot|V(s_n)|ds\leqslant\frac{e^{|\operatorname{Im}z|t}}{|z|_1^{n+1}}\cdot\frac{1}{n!}\Bigg(\int_0^t|V(x)|dx\Bigg)^n.$$

This shows that for each $t\geqslant 0$ the series (4.28) converges uniformly on bounded subsets of $\mathbb{C}$. Each term of this series is an entire function. Hence the sum is an entire function. Summing the majorants we obtain estimates (4.29).

The function $T_m, m\geqslant 1$, is entire, since φ,ϑ are entire. We have $T_m=\operatorname{Tr}M^m(z)=\sum_{n\geqslant 0}T_{m,n}(z)$, where

$$T_{m,0}=2N\cos mz,\qquad T_{m,n}(z)=\operatorname{Tr}\vartheta_n(m,z)+\operatorname{Tr}\varphi_n'(m,z),\quad n\geqslant 1,$$
$$T_{m,1}(z)=\frac{1}{z}\int_0^m(\sin z(m-t)\cos zt+\cos z(m-t)\sin zt)\operatorname{Tr}V(t)dt=\frac{\sin mz}{z}m\operatorname{Tr}V^0.$$

The two estimates $|\operatorname{Tr}\varphi_n'(m,z)|\leqslant\frac{(m\varkappa)^n}{n!}e^{|\operatorname{Im}z|m}$ and $|\operatorname{Tr}\vartheta_n(m,z)|\leqslant\frac{(m\varkappa)^n}{n!}e^{|\operatorname{Im}z|m}$ give $|T_{m,n}(z)|\leqslant(2N)\frac{(m\varkappa)^n}{n!}e^{m|\operatorname{Im}z|},\quad n\geqslant 0$, which yields (4.30). □

We will obtain the simple properties of the monodromy matrix. We introduce the modified monodromy matrix Ψ and the matrix Λ by

$$\Psi = \mathscr{U}^{-1} M \mathscr{U} = \begin{pmatrix} \vartheta(1,z) & z\varphi(1,z) \\ z^{-1}\vartheta'(1,z) & \varphi'(1,z) \end{pmatrix}, \quad \mathscr{U} = I_N \oplus zI_N \quad z \in \mathbb{C}, \tag{4.31}$$

$$\Lambda = \mathscr{U}^{-1} L \mathscr{U} = \frac{\Psi + \Psi^{-1}}{2} = \frac{1}{2} \begin{pmatrix} \vartheta(1,z) + \varphi'(1,\overline{z})^* & z(\varphi(1,z) - \varphi(1,\overline{z})^* \\ z^{-1}(\vartheta'(1,z) - \vartheta'(1,\overline{z})^*) & \vartheta(1,\overline{z})^* + \varphi'(1,z) \end{pmatrix}, \tag{4.32}$$

where we used the identity (1.3). Note that M and Ψ have the same eigenvalues and the same traces. Using (4.29) we obtain

$$\Lambda(z) = \Lambda_1(z) + \frac{\Lambda_2(z)}{2z^2} + \frac{\Lambda_3(z)}{2z^3} + \frac{O(e^{|\operatorname{Im} z|})}{z^4}, \quad \Lambda_1(z) = \cos z + \frac{\sin z}{2z} V^0 \ \text{ as } \ |z| \to \infty, \tag{4.33}$$

where

$$\Lambda_2(z) = \int_0^1 dt \int_0^t \sin z(t-s) \begin{pmatrix} a_{11}(t,s,z) & a_{12}(t,s,z) \\ a_{21}(t,s,z) & a_{22}(t,s,z) \end{pmatrix} ds, \quad z \in \mathbb{C}, \tag{4.34}$$

$$a_{11} = \sin z(1-t) \cos zs V_t V_s + \cos z(1-t) \sin zs V_s V_t, \quad a_{22}(z) = a_{11}(\overline{z})^*, \tag{4.35}$$

$$a_{12} = \sin z(1-t) \sin zs (V_t V_s - V_s V_t), \quad a_{21} = \cos z(1-t) \cos zs (V_t V_s - V_s V_t), \tag{4.36}$$

where $V_t = V(t)$ and

$$\Lambda_3(z) = \int_0^1 dt \int_0^t ds \int_0^s \sin z(t-s) \sin z(s-u) \begin{pmatrix} b_{11}(t,s,u,z) & b_{12}(t,s,u,z) \\ b_{21}(t,s,u,z) & b_{22}(t,s,u,z) \end{pmatrix} du, \tag{4.37}$$

$$b_{11} = \sin z(1-t) \cos zu V_t V_s V_u + \cos z(1-t) \sin zu V_u V_s V_t, \quad b_{22}(z) = b_{11}(\overline{z})^*, \tag{4.38}$$

$$\begin{aligned} b_{12} &= \sin z(1-t) \sin zu (V_t V_s V_u - V_u V_s V_t), \\ b_{21} &= \cos z(1-t) \cos zu (V_t V_s V_u - V_u V_s V_t). \end{aligned} \tag{4.39}$$

Lemma 4.4. *For each $(r, V) \in \mathbb{R}_+ \times \mathscr{H}$ and $V^* = V$ the following asymptotics hold:*

$$2^{2N} \det \Lambda(z) = \exp\Big(-2Niz + i\frac{\operatorname{Tr} V^0}{z} + \frac{i\|V\|^2 + o(1)}{4z^3} \Big), \tag{4.40}$$

$$\operatorname{Tr} \Lambda_2(z) = \Big(-B_2 + \frac{i\|V\|^2 + o(1)}{2z} \Big) \cos z, \tag{4.41}$$

$$\operatorname{Tr} V^0 \Lambda_2(z) = -3B_3 \cos z + O(e^{|\operatorname{Im} z|}/z), \tag{4.42}$$

$$\operatorname{Tr} \Lambda_3(z) = -iB_3 \frac{\cos z}{2} + o(e^{|\operatorname{Im} z|}) \tag{4.43}$$

as $y \geqslant r|x|, y \to \infty$, where $V^0 = \int_0^1 V(t)dt$, $B_n = \operatorname{Tr} \frac{(V^0)^n}{n!}$.

Proof. Let $V_t = V(t)$. Asymptotics (5.1), (5.2) yield $\int_0^1 \int_0^t \cos z(1-2t+2s) \operatorname{Tr} V_t V_s = \frac{i\|V\|^2 + o(1)}{2z} \cos z$. Then using (4.34) and $\int_0^1 \int_0^t \operatorname{Tr} V_t V_s dt ds = \frac{1}{2} \operatorname{Tr} \Big(\int_0^1 V_t dt \Big)^2 = B_2$,

we obtain

$$\begin{aligned}\operatorname{Tr}\Lambda_2(z) &= 2\int_0^1\int_0^t \sin z(t-s)\sin z(1-t+s)\operatorname{Tr}V_tV_s dtds\\ &= \int_0^1\int_0^t(\cos z(1-2t+2s)-\cos z)\operatorname{Tr}V_tV_s dtds\\ &= \frac{i\|V\|^2+o(1)}{2z}\cos z - B_2\cos z,\end{aligned}$$

which yields (4.41). We show (4.42). Let $G(t,s)=\operatorname{Tr}V_0(V_tV_s+V_sV_t)$. Using (4.34), (5.1), (5.2) we have

$$\begin{aligned}\operatorname{Tr}V^0\Lambda_2(z) &= \int_0^1\int_0^t \sin z(t-s)z(1-t+s)\operatorname{Tr}V^0(V_tV_s+V_sV_t)ds\\ &= \frac{1}{2}\int^1 dt\int_0^t(-\cos z+\cos z(1-2t+2s))G(t,s)ds\\ &= -3B_3\cos z + O(e^{|\operatorname{Im} z|}/z),\end{aligned}$$

since

$$\int_0^1 dt\int_0^t \operatorname{Tr}V_0(V_tV_s+V_sV_t)ds = \operatorname{Tr}V^0\biggl(\int_0^1 V_t dt\biggr)^2 = 6B_3,$$

which yields (4.42). Consider $\operatorname{Tr}\Lambda_3$. The identity (4.37) gives

$$\begin{aligned}\operatorname{Tr}\Lambda_3(z) &= \int_0^1 dt\int_0^t ds\int_0^s \sin z(t-s)\sin z(s-u)\operatorname{Tr}(b_{11}+b_{22})du\\ &= \int_0^1 dt\int_0^t ds\int_0^s \sin z(t-s)\sin z(s-u)\sin z(1-t+u)Rdu,\\ R &= \operatorname{Tr}(V_tV_sV_u+V_uV_sV_t).\end{aligned}$$

Using the identity

$$\begin{gathered}4\sin z(t-s)\sin z(s-u)\sin z(1-t+u) = -\sin z + P,\\ P = \sin z(1-2s+2u)-\sin z(1-2t+2s)-\sin z(1-2t+2u),\end{gathered}$$

we get

$$\Lambda_3 = -\Lambda_3^0\frac{\sin z}{2}+\Lambda_3^1,\quad \Lambda_3^0 = \frac{1}{2}\int_0^1 dt\int_0^t ds\int_0^s Rdu,\quad \Lambda_3^1 = \frac{1}{4}\int_0^1 dt\int_0^t ds\int_0^s PRdu,$$

where

$$\Lambda_3^0 = \frac{\operatorname{Tr}}{2}\int_0^1 V_t dt\int_0^t ds\int_0^s (V_sV_u+V_uV_s)du = \frac{\operatorname{Tr}}{2}\int_0^1 V_t\biggl(\int_0^t V_s ds\biggr)^2 = B_3.$$

Due to (5.3) we obtain $\Lambda_3^1 = o(e^{|\operatorname{Im} z|})$, which yields (4.43).

We will show (4.40). The asymptotics (4.33) yield

$$\frac{\Lambda}{\cos z} = I_{2N}+S,\quad S = i\frac{V^0I_{2N}}{2z}+\frac{\Lambda_2}{2z^2\cos z}+\frac{\Lambda_3}{2z^3\cos z}+O(z^{-4}), \tag{4.44}$$

as $|z|\to\infty, y\geqslant r|x|$, since $\sin z = i\cos z + O(e^{-y})$. In order to use the identity

$$\det(I+S) = \exp\biggl(\operatorname{Tr}S - \operatorname{Tr}\frac{S^2}{2}+\operatorname{Tr}\frac{S^3}{3}+o(z^{-3})\biggr),\quad |S| = O(1/z),$$

we need the traces of $S^m, m = 1, 2, 3$. Due to (4.41)-(4.43) we get

$$\frac{\operatorname{Tr} S^3}{3} = -i \operatorname{Tr} \frac{(V^0)^3 I_{2N}}{3(2z)^3} + O(z^{-4}) = -i \frac{B_3}{2z^3} + O(z^{-4}), \tag{4.45}$$

$$-\operatorname{Tr} \frac{S^2}{2} = \frac{\operatorname{Tr}}{2} \left(\frac{(V^0)^2}{(2z)^2} I_{2N} - 2i \frac{V^0 \Lambda_2}{4z^3 \cos z} + O(z^{-4}) \right) = \frac{B_2}{2z^2} + i \frac{3B_3}{4z^3} + O(z^{-4}), \tag{4.46}$$

and

$$\operatorname{Tr} S = \operatorname{Tr} \left(i \frac{V^0}{2z} + \frac{\Lambda_2}{2z^2 \cos z} + \frac{\Lambda_3 + O(z^{-1})}{2z^3 \cos z} \right) = i \frac{B_1}{z} + \left(-\frac{B_2}{2z^2} + i \frac{\|V\|^2}{4z^3} \right) - i \frac{B_3 + o(1)}{4z^3}, \tag{4.47}$$

and summing (4.45)-(4.47) we get (4.40). □

Proof of Theorem 1.5. Recall the simple fact: Let A, B be matrices and $\sigma(B)$ be spectra of B. If A is normal, then $\operatorname{dist}\{\sigma(A), \sigma(A+B)\} \leqslant |B|$ (see p. 291 of [Ka]).

From (4.29), (4.33) we have $\Lambda(z) = \Lambda_1(z) + O(z^{-2} e^{|\operatorname{Im} z|})$, $|z| \to \infty$, where the diagonal operator $\Lambda_1 = \cos z + \frac{\sin z}{2z} V^0$ has the eigenvalues $\Delta_j^0(z) = \cos z - V_j^0 \frac{\sin z}{2z}, j \in \overline{1, N}$, with multiplicity 2. Using the result from [Ka] and the asymptotics above we deduce that the eigenvalues $\Delta_m(z)$ of matrix $\Lambda(z)$ satisfy the asymptotics $\Delta_j(z) = \Delta_j^0(z) + O(z^{-2} e^{|\operatorname{Im} z|}), j \in \overline{N}$. Then $M \in \mathscr{M}_N^0$, and Lemma 4.4 yields $M \in \mathscr{M}_N^{0,1}$.

The function $\Phi(iy, \nu)$ is real for $y, \nu \in \mathbb{R}$. Then $\phi_j(z) = \overline{\phi}_j(-\overline{z})$ for all $(z, j) \in \overline{\mathbb{C}}_+ \times \overline{1, N}$. This yields that the set $\{\Delta_m(z)\}_1^N = \{\Delta_m(-\overline{z})\}_1^N, z \in \overline{\mathbb{C}}_+$, which gives $q(z) = q(-\overline{z}), z \in \overline{\mathbb{C}}_+$. Thus $q(x) = q(-x)$ for all $x \in \mathbb{R}$, and the identities

$$k(-\overline{z}) = -\overline{z} + \frac{1}{\pi} \int_{\mathbb{R}} \frac{q(t)dt}{t + \overline{z}} = -\overline{z} - \frac{1}{\pi} \int_{\mathbb{R}} \frac{q(s)dt}{s - \overline{z}} = -\overline{k}(z), z \in \mathbb{C}_+$$

give $-k(-\overline{z}) = \overline{k}(z), z \in \mathbb{C}_+$. Thus by Theorems 1.1-1.3, we obtain the proof of Theorem 1.5 with the exception of (1.32), (1.33). Note that similar arguments give

$$Q_4 = I_2^D + I_4^S - Q_0 Q_2 = \int_0^1 \frac{\operatorname{Tr}(V'(t)^2 + 2V^3(t))dt}{2^5 N} \qquad \text{if } V' \in \mathscr{H}.$$

Defining $z_n^0 = \frac{z_n^+ + z_n^-}{2}, r = \frac{|g_n|}{2}$ and using $(z_n^0 + x)^2 + (z_n^0 - x)^2 \geqslant 2(z_n^0)^2$, we have

$$\begin{aligned} \int_{g_n} \frac{t^2 q(t)dt}{\pi} &\geqslant \int_{g_n} \frac{t^2 q_n^0(t)dt}{\pi} \geqslant \int_0^r ((z_n^0 + x)^2 + (z_n^0 - x)^2) \frac{\sqrt{r^2 - x^2}}{\pi} dx \\ &\geqslant \frac{r^2}{2} (z_n^0)^2 = \frac{|\gamma_n|^2}{32}, \\ Q_2 = \frac{1}{\pi} \int_{\mathbb{R}} t^2 q(t)dt &= \sum_{n \in \mathbb{Z}} \frac{1}{\pi} \int_{g_n} t^2 q(t)dt \geqslant \frac{1}{32} \sum_{n \in \mathbb{Z}} |\gamma_n|^2 = \frac{1}{16} \sum_{n \geqslant 1} |\gamma_n|^2, \end{aligned}$$

which yields (1.32).

If $\sigma_{(N)} = \sigma(F)$, then the function h is given by $h(p) = 0, p \neq \frac{\pi}{N}\mathbb{Z}$, and $(h(\frac{\pi}{N} n))_{n \in \mathbb{Z}} \in \ell_1^2$. Such conformal mapping was considered in [K9]. In this case for some absolute constant C_0 the estimate $Q_2 \leqslant C_0 G^2 (1 + G^{\frac{1}{3}})$ was obtained in [K9]. This gives estimates (1.33). □

5. Appendix

Lemma 5.1. *Let functions $h_1, ..., h_n \in L^2(0,1)$ for some $n \geqslant 3$. Then*

$$\int_0^1 dt \int_0^t e^{i2z(t-s)} h_1(t)h_2(s)ds = \frac{i}{2z}\Big(\int_0^1 h_1(t)h_2(t)dt + o(1)\Big), \tag{5.1}$$

$$\int_0^1 dt \int_0^t e^{-i2z(t-s)} h_1(t)h_2(s)ds = \frac{o(e^{2y})}{|z|}, \tag{5.2}$$

$$\int_0^1 dt \int_0^t ds \int_0^s e^{\pm iz(1-2\zeta)} h_1(t)h_2(s)h_3(u)du = o(e^y), \tag{5.3}$$

$$\int_0^1 dt_1 ... \int_0^{t_{n-1}} dt_n e^{\pm iz(1-2t_1+2t_2+\cdots+(-1)^n 2t_n)} \prod_1^n h_j(t_j) = \frac{o(e^y)}{|z|}, \quad \textit{for all} \quad n \geqslant 3, \tag{5.4}$$

as $r|x| < y \to \infty$ for any fixed $r > 0$, where ζ is one of the functions $s-u, t-u$, or $t-s$.

Proof. Let $F(t,s) = h_1(t)h_2(s), t, s \in (0,1)$. We have

$$f_1(z) \equiv \int_0^1 dt \int_0^t e^{i2z(t-s)} F(t,s)ds = \frac{1}{2}\int_0^1 ds \int_0^1 e^{i2z|t-s|} F(t,s)ds. \tag{5.5}$$

Substituting the identities

$$\frac{2z}{\pi i}\int_{\mathbb{R}} \frac{e^{i(t-s)k}dk}{k^2-4z^2} = e^{i2z|t-s|}, \quad z \in \mathbb{C}_+, \qquad \hat{F}(k) \equiv \frac{1}{2\pi}\iint_{[0,1]^2} e^{ik(t-s)} F(t,s)dtds$$

into (5.5) we obtain

$$f_1 = \frac{2z}{i}\int_{\mathbb{R}} \frac{\hat{F}(k)dk}{k^2-4z^2} = \frac{1}{i2z}\int_{\mathbb{R}}\Big(-1+\frac{k^2}{k^2-4z^2}\Big)\hat{F}(k)dk = \frac{i}{2z}\Big(\int_0^1 F(t,t)dt + o(1)\Big),$$

which yields (5.1). Consider $f(z) \equiv \int_0^1 dt \int_0^t e^{i2z(t-s)} F(t,s)ds$. We have

$$|f(z)|^2 \leqslant g(z)\|h_1\|^2\|h_2\|^2, \qquad g(z) = \int_0^1 dt \int_0^t e^{4y(t-s)}ds \leqslant \int_0^1 \frac{e^{4yt}}{4y}dt \leqslant \frac{e^{4y}}{(4y)^2}. \tag{5.6}$$

Let F_0 be a smooth function such that $\|F - F_0\| = \varepsilon$ for some small $\varepsilon > 0$. Define the function $f_0(z) = \int_0^1 dt \int_0^t e^{-i2z(t-s)} F_0(t,s)ds$. Using (5.6) we obtain

$$|f(z)| \leqslant |f_0(z)| + |f(z) - f_0(z)| \leqslant |f_0(z)| + \|F - F_0\|\frac{e^{2y}}{(4y)}, \tag{5.7}$$

and the integration by parts yields $f_0(z) = O(\frac{e^{2y}}{y^2})$. Thus we obtain (5.2), since ε is arbitrarily small. Similar arguments yield (5.3) and (5.4). □

We formulate the following results from [CK].

Lemma 5.2. *The function $f(z) = \log|\xi(z)|, z \in \mathbb{C} \setminus [-1,1]$, is subharmonic and continuous in $\mathbb{C}$. Moreover, for some absolute constant C the following estimate is fulfilled:*

$$c|f(z) - f(z_0)| \leqslant C\varepsilon^{\frac{1}{2}} \quad \textit{if} \quad |z - z_0| \leqslant \varepsilon \max\{2, |z_0|\}, \ \ 0 \leqslant \varepsilon \leqslant \frac{1}{8}, \ \ z, z_0 \in \mathbb{C}. \tag{5.8}$$

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