TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 362, Number 8, August 2010, Pages 4229–4242
S 0002-9947(10)05066-X
Article electronically published on March 26, 2010

ON SOME QUESTIONS RELATED TO THE MAXIMAL OPERATOR ON VARIABLE L^p SPACES

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Abstract. Let $\mathcal{P}(\mathbb{R}^n)$ be the class of all exponents p for which the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. A recent result by T. Kopaliani provides a characterization of $\mathcal{P}$ in terms of the Muckenhoupt-type condition A under some restrictions on the behavior of p at infinity. We give a different proof of a slightly extended version of this result. Then we characterize a weak type $\big(p(\cdot),p(\cdot)\big)$ property of M in terms of A for radially decreasing p. Finally, we construct an example showing that $p\in\mathcal{P}(\mathbb{R}^n)$ does not imply $p(\cdot)-\alpha\in\mathcal{P}(\mathbb{R}^n)$ for all $\alpha<p_--1$. Similarly, $p\in\mathcal{P}(\mathbb{R}^n)$ does not imply $\alpha p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ for all $\alpha>1/p_-$.

1. Introduction

Let $p:\mathbb{R}^n\to[1,\infty)$ be a measurable function. Denote by $L^{p(\cdot)}(\mathbb{R}^n)$ the space of functions f such that for some $\lambda>0$,

$$\int_{\mathbb{R}^n}|f(x)/\lambda|^{p(x)}dx<\infty,$$

with norm

$$\|f\|_{L^{p(\cdot)}}=\inf\left\{\lambda>0:\int_{\mathbb{R}^n}|f(x)/\lambda|^{p(x)}dx\le 1\right\}.$$

Let $\mathcal{P}(\mathbb{R}^n)$ be the class of all functions p for which the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. This class has been a focus of intense study in recent years. By the classical Hardy-Littlewood maximal theorem, any constant function $p\equiv p_0$ with $1<p_0<\infty$ belongs to $\mathcal{P}(\mathbb{R}^n)$. However, it has been observed quite recently that $\mathcal{P}(\mathbb{R}^n)$ consists of many nontrivial, that is, nonconstant functions. We mention briefly the key known results related to $\mathcal{P}(\mathbb{R}^n)$.

Assume that $p_-\equiv \operatorname{ess\,inf}_{x\in\mathbb{R}^n}p(x)>1$ and $p_+\equiv\operatorname{ess\,sup}_{x\in\mathbb{R}^n}p(x)<\infty$. In [5], L. Diening proved that if p satisfies the log-Hölder condition

$$|p(x)-p(y)|\le\frac{c}{\log(e+1/|x-y|)}\tag{1.1}$$

and if p is a constant outside some compact set, then $p\in\mathcal{P}(\mathbb{R}^n)$. The second condition on p, namely the behavior of p at infinity, was improved independently

Received by the editors June 8, 2008.
2000 *Mathematics Subject Classification.* Primary 42B25, 46E30.
Key words and phrases. Maximal operator, variable L^p spaces.
This work was supported by the Spanish Ministry of Education under the program "Programa Ramón y Cajal, 2006".

by D. Cruz-Uribe, A. Fiorenza and C. Neugebauer [4], and A. Nekvinda [16]. It was shown in [4] that if p satisfies (1.1) and

$$|p(x) - p_\infty| \le \frac{c}{\log(e+|x|)} \quad (p_\infty > 1), \tag{1.2}$$

then $p \in \mathcal{P}(\mathbb{R}^n)$. In [16], (1.2) is replaced by a slightly more general integral condition. A new approach to these results, as well as an investigation of the limiting cases when $p_- = 1$ and $p_+ = \infty$, can be found in the very recent works [2, 3, 7].

Conditions (1.1) and (1.2) are optimal in the pointwise sense; the corresponding examples are contained in [19] and [4]. On the other hand, they are not necessary for $p \in \mathcal{P}(\mathbb{R}^n)$. In [17, 18], A. Nekvinda constructed $p \in \mathcal{P}(\mathbb{R}^n)$ satisfying much weaker conditions at infinity than (1.2). In [13], the author established that there exist discontinuous functions $p \in \mathcal{P}(\mathbb{R}^n)$.

In [6], L. Diening showed that $p \in \mathcal{P}(\mathbb{R}^n)$ if and only if there exists $c > 0$ such that for any family of pairwise disjoint cubes π and any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|\sum_{Q\in\pi}(|f|_Q)\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le c\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \tag{1.3}$$

where $f_Q = \frac{1}{|Q|}\int_Q f$. This result implies, for example, that $p \in \mathcal{P}(\mathbb{R}^n)$ if and only if $p' \in \mathcal{P}(\mathbb{R}^n)$, where $p'(x) = \frac{p(x)}{p(x)-1}$.

Note that (1.3) with a single cube on the left-hand side would be a full analogue of the classical Muckenhoupt A_p condition (cf. [14]) in the context of $L^{p(\cdot)}$ spaces. We give a more precise definition.

Definition 1.1. We say that p satisfies condition A[1] ($p \in A$) if $p_- > 1$, $p_+ < \infty$ and there exists $c > 0$ such that for any cube Q and any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$|f|_Q\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le c\|f\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{1.4}$$

It is natural to ask whether (1.3) can be replaced by $p \in A$. In a recent work [11], T. Kopaliani gave the following partial answer: if p is a constant outside some ball, then $p \in \mathcal{P}(\mathbb{R}^n)$ if and only if $p \in A$. Then this was used in [10] in order to give a new sufficient condition for $p \in \mathcal{P}(\mathbb{R}^n)$ in terms of mean oscillations of p.

Observe that the proof in [11] is based essentially on the above mentioned Diening characterization [6], whose proof in turn is long and complicated. In this paper we give a different, self-contained proof of an extended version of Kopaliani's result. Our approach is based on the concept of A_∞ weights and on the standard technique which, for example, can be found in the work of B. Jawerth [9].

Theorem 1.2. *Let $p \in A$, and let $E \subset \mathbb{R}^n$ be a measurable set of positive finite measure. Then there exists a constant $c > 0$ depending on p, n and E such that for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|(Mf)\chi_E\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le c\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

It is still unclear for us whether the class $\mathcal{P}(\mathbb{R}^n)$ can be fully characterized in terms of condition A. However, our next result shows that the weak type $\big(p(\cdot), p(\cdot)\big)$ property of M is equivalent to $p \in A$ for radially decreasing p.

[1]Condition (1.3) is denoted in [6] by $\mathcal{A}$, while condition (1.4) is denoted in [11] by $dx \in A_{p(\cdot)}$. We prefer to denote (1.4) by A, and we hope that this will not mislead the reader.

Given a function p, we say that M is of weak type $\big(p(\cdot),p(\cdot)\big)$ if there exists $c>0$ such that for any $f\in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\sup_{\alpha>0}\alpha\|\chi_{\{x:|Mf(x)|>\alpha\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le c\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

It is easy to see that the weak type $\big(p(\cdot),p(\cdot)\big)$ property of M implies $p\in A$. Using Theorem 1.2, we obtain that the converse is also true for radially decreasing p. Recall that a function p is radially decreasing if $p(x)=\rho(|x|)$, where ρ is a non-increasing function on $(0,\infty)$. The following theorem can be viewed as an analogue of Muckehnoupt's characterization [14] of the weighted weak L^r boundedness of M in terms of the A_r condition.

Theorem 1.3. *Let p be a radially decreasing function with $p_->1$ and $p_+<\infty$. Then M is of weak type $\big(p(\cdot),p(\cdot)\big)$ if and only if $p\in A$.*

Our next result is closely related to the author's work [13]. It was shown there that any pointwise multiplier for $BMO(\mathbb{R}^n)$ generates a function $p\in\mathcal{P}(\mathbb{R}^n)$. A function g is called a pointwise multiplier for BMO if $fg\in BMO$ for any $f\in BMO$. The main result of [13] states the following.

Theorem A. *If p is a pointwise multiplier for $BMO(\mathbb{R}^n)$ with $p_->0$, then there exists a constant $\alpha>0$ such that $p(\cdot)+\alpha\in\mathcal{P}(\mathbb{R}^n)$.*

Observe that conditions (1.1) and (1.2) imply that p is a pointwise multiplier for $BMO(\mathbb{R}^n)$. Therefore, the following was asked in [13] .

Question 1.4. Does any pointwise multiplier for BMO with $p_->1$ belong to $\mathcal{P}(\mathbb{R}^n)$?

Taking into account Theorem A, this question naturally leads to the following one, which is also of some independent interest.

Question 1.5. Let $p\in\mathcal{P}(\mathbb{R}^n)$. Does this imply $p(\cdot)-\alpha\in\mathcal{P}(\mathbb{R}^n)$ for any $\alpha<p_--1$?

The following question, similar to Question 1.5, was asked in [8].

Question 1.6. Let $p\in\mathcal{P}(\mathbb{R}^n)$. Does this imply $\alpha p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ for any $\alpha>1/p_-$?

Our next result shows that the answers to all the above questions are negative.

Theorem 1.7. *Let $n=1$. Also, let $q>1$ and $\delta>0$. There exists a nonnegative function p_0 satisfying the following properties:*

(i) *p_0 is a pointwise multiplier for $BMO(\mathbb{R})$;*
(ii) *if $q(q-1)\le\delta$, then $p_{q,\delta}(x)=q+\delta p_0(x)\notin A$.*

It follows immediately from this theorem that for any $q>1$ and $\delta>0$ such that $q(q-1)\le\delta$, the function $p_{q,\delta}$ yields a counterexample to Question 1.4. This, along with Theorem A, gives a counterexample to Question 1.5. Finally, applying Theorem A to $p_0(\cdot)+\varepsilon_0$, where $\varepsilon_0>0$, we get that there exists $\alpha_0>0$ such that $p_0(\cdot)+\alpha_0\in\mathcal{P}(\mathbb{R}^n)$. Taking this function and $\alpha>1/\alpha_0$ such that $\alpha_0(\alpha\alpha_0-1)\le 1$, by Theorem 1.7 we get that $\alpha(p_0(\cdot)+\alpha_0)\notin\mathcal{P}(\mathbb{R}^n)$, which gives a counterexample to Question 1.6.

The paper is organized as follows. Section 2 contains some preliminaries. The proofs of Theorems 1.2, 1.3 and 1.7 are contained in Sections 3, 4 and 5, respectively.

2. Preliminaries

We recall that the Hardy-Littlewood maximal function is defined for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

where the supremum is taken over all cubes Q containing the point x. We shall use the classical weak type property of M in the following form (see, e.g., [20, p. 7]):

$$|\{x \in \mathbb{R}^n : Mf(x) > \alpha\}| \leq \frac{c}{\alpha} \int_{\{x:|f(x)>\alpha/2\}} |f(x)|dx \quad (\alpha > 0). \tag{2.1}$$

Recall that the conjugate function p' is defined by $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$. The following generalized Hölder inequality and a duality relation can be found in [12]:

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq 2\|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}}, \tag{2.2}$$

$$\|f\|_{L^{p(\cdot)}} \leq \sup_{\|g\|_{L^{p'(\cdot)}} \leq 1} \int_{\mathbb{R}^n} |f(x)g(x)|dx. \tag{2.3}$$

By (2.2) and (2.3) it is easy to see that $p \in A$ if and only if

$$\sup_Q \frac{1}{|Q|} \|\chi_Q\|_{L^{p(\cdot)}} \|\chi_Q\|_{L^{p'(\cdot)}} < \infty.$$

Definition 2.1. Let Q_0 be a cube. We say that a weight w (i.e., a nonnegative locally integrable function) satisfies the $A_\infty(Q_0)$ condition if there exist constants $\alpha, \beta \in (0, 1)$ such that for any cube $Q \subset Q_0$ and for any measurable subset $E \subset Q$, $|E| > \alpha|Q|$ implies $w(E) > \beta w(Q)$.

It is well known that the class A_∞ can be defined in many equivalent ways. In particular, $w \in A_\infty(Q_0)$ if and only if there exist constants $c, \varepsilon > 0$ such that for any cube $Q \subset Q_0$ and for any measurable subset $E \subset Q$,

$$\frac{w(E)}{w(Q)} \leq c \left(\frac{|E|}{|Q|}\right)^\varepsilon$$

(see, e.g., [1] where this result is proved in the case $Q_0 = \mathbb{R}^n$; the local case can be treated exactly in the same way).

3. Proof of Theorem 1.2

We start with the following lemma due to T. Kopaliani [11]. Its proof in [11] is based on some concepts from convex analysis. We give a different and simpler proof here.

Lemma 3.1. *Let $p \in A$. Suppose that $|f|_Q \geq c_1$ and $\|f\|_{L^{p(\cdot)}} \leq c_2$, where $c_1, c_2 > 0$. Then*

$$\int_Q (|f|_Q)^{p(x)} dx \leq c \int_Q |f(x)|^{p(x)} dx,$$

where c depends on p, c_1 and c_2.

Proof. We consider the case $c_1 = c_2 = 1$; the same proof with trivial modifications works for general c_1 and c_2.

Let α be a positive constant satisfying $\int_Q \alpha^{p'(y)-1}dy = \int_Q |f|$. Then

$$\int_Q (|f|_Q)^{p(x)}dx = \int_Q \left(\frac{1}{|Q|}\int_Q \alpha^{p'(y)-1}dy\right)^{p(x)}dx \tag{3.1}$$
$$= \left(\frac{1}{|Q|}\int_Q\left(\frac{1}{|Q|}\int_Q \alpha^{p'(y)-p'(x)}dy\right)^{p(x)-1}dx\right)\int_Q \alpha^{p'(y)}dy.$$

Since $|f|_Q \geq 1$, we have $\alpha \geq 1$. On the other hand, since $\|f\|_{L^{p(\cdot)}} \leq 1$, by (2.2) we get $\int_Q \alpha^{p'(y)-1}dy \leq 2\|\chi_Q\|_{L^{p'(\cdot)}}$. Therefore, $\alpha \leq \frac{c}{\|\chi_Q\|_{L^{p'(\cdot)}}}$.

Setting $E_1(x) = \{y \in Q : p'(y) > p'(x)\}$ and $E_2(x) = Q \setminus E_1(x)$, and using the above estimates for α, we obtain

$$\begin{aligned}\int_Q \alpha^{p'(y)-p'(x)}dy &= \int_{E_1(x)} \alpha^{p'(y)-p'(x)}dy + \int_{E_2(x)} \alpha^{p'(y)-p'(x)}dy \\ &\leq c\|\chi_Q\|_{L^{p'(\cdot)}}^{p'(x)} + |Q|.\end{aligned}$$

This, along with $p \in A$, gives

$$\left(\frac{1}{|Q|}\int_Q\left(\frac{1}{|Q|}\int_Q \alpha^{p'(y)-p'(x)}dy\right)^{p(x)-1}dx\right) \tag{3.2}$$
$$\leq c + c\int_Q\left(\frac{\|\chi_Q\|_{L^{p'(\cdot)}}}{|Q|}\right)^{p(x)}dx \leq c + c\int_Q\left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}}\right)^{p(x)}dx \leq c.$$

Further,

$$\begin{aligned}\int_Q \alpha^{p'(y)}dy &= 2\alpha\int_Q |f| - \int_Q \alpha^{p'(y)}dy \\ &\leq 2\alpha\int_{\{y\in Q: 2\alpha|f(y)|>\alpha^{p'(y)}\}} |f(y)|dy \\ &\leq c\int_Q |f(y)|^{p(y)}dy.\end{aligned} \tag{3.3}$$

Combining (3.1) with (3.2) and (3.3) completes the proof. □

Corollary 3.2. *Let $p \in A$. Suppose that $\xi_1 \leq t \leq \frac{\xi_2}{\|\chi_{Q_0}\|_{L^{p(\cdot)}}}$, where $\xi_1, \xi_2 > 0$. Then $t^{p(x)} \in A_\infty(Q_0)$ with the A_∞ constants depending only on p, ξ_1 and ξ_2.*

Proof. Let $E \subset Q' \subset Q_0$, where Q' is a cube and $|E| \geq |Q'|/2$. Then $f = t\chi_E$ satisfies the conditions of Lemma 3.1 with $Q = Q'$, $c_1 = \xi_1/2$ and $c_2 = \xi_2$. Hence,

$$\frac{1}{2^{p_+}}\int_{Q'} t^{p(x)}dx \leq c\int_E t^{p(x)}dx,$$

which proves the $A_\infty(Q_0)$ condition. □

Proof of Theorem 1.2. For each integer k set

$$\Omega_k = \{x \in \mathbb{R}^n : Mf(x) > 3^{nk}\}$$

and $D_k = \Omega_k \setminus \Omega_{k+1}$. Let F_k be an arbitrary compact subset of D_k.

Fix a function $\varphi \geq 0$ supported in E and such that $\|\varphi\|_{L^{p'(\cdot)}} \leq 1$. We are going to show that

$$\int_{\bigcup_{k=-\infty}^{\infty} F_k} (Mf)\varphi\, dx \leq c\|f\|_{L^{p(\cdot)}}, \tag{3.4}$$

where $c = c\big(p, n, E\big)$. By (2.3) and by the standard limiting argument, this inequality readily gives the desired result.

By the Vitali covering lemma, there exists a finite collection of pairwise disjoint cubes $\{Q_j^k\}_{j\geq 1}$ such that $F_k \subset \bigcup_j 3Q_j^k$ and $|f|_{Q_j^k} > 3^{nk}$. Let $E_1^k = 3Q_1^k \cap F_k$ and $E_j^k = (3Q_j^k \setminus \bigcup_{s<j} 3Q_s^k) \cap F_k$, $j > 1$. Note that the sets E_j^k are pairwise disjoint and $\bigcup_j E_j^k = F_k$.

Using the above definitions and (2.2), we get

$$\begin{aligned} \int_{\bigcup_{k=-\infty}^{\infty} F_k} (Mf)\varphi\, dx &\leq 3^n \sum_{k=-\infty}^{\infty} \sum_j |f|_{Q_j^k} \int_{E_j^k} \varphi \\ &= 3^n \int_{\mathbb{R}^n} |f| T\varphi\, dx \leq 2 \cdot 3^n \|f\|_{L^{p(\cdot)}} \|T\varphi\|_{L^{p'(\cdot)}}, \end{aligned}$$

where

$$T\varphi(x) = \sum_{k=-\infty}^{\infty} \sum_j \left(\frac{1}{|Q_j^k|} \int_{E_j^k} \varphi \right) \chi_{Q_j^k}(x).$$

Hence, in order to prove (3.4), it suffices to show that

$$\|T\varphi\|_{L^{p'(\cdot)}} \leq c\big(p, n, E\big). \tag{3.5}$$

Let $\alpha_{j,k}(\varphi) = \frac{1}{|Q_j^k|} \int_{E_j^k} \varphi$ and

$$T_l\varphi(x) = \sum_{k=-\infty}^{\infty} \sum_j \alpha_{j,k}(\varphi) \chi_{Q_j^k \cap D_{k+l}}(x) \quad (l = 0, 1, \dots).$$

Note that $Q_j^k \subset \Omega_k = \bigcup_{l=0}^{\infty} D_{k+l}$, and hence $T\varphi(x) = \sum_{l=0}^{\infty} T_l\varphi(x)$. Also, since the sets $Q_j^k \cap D_{k+l}$ are pairwise disjoint, we have

$$\int_{\mathbb{R}^n} (T_l\varphi)^{p'(x)} dx = \sum_{k=-\infty}^{\infty} \sum_j \int_{Q_j^k \cap D_{k+l}} \alpha_{j,k}(\varphi)^{p'(x)} dx.$$

We divide the last sum into two sums corresponding the indices $\mathcal{I}_1 = \{(j,k) : \alpha_{j,k}(\varphi) > 1\}$ and $\mathcal{I}_2 = \{(j,k) : \alpha_{j,k}(\varphi) \leq 1\}$.

Suppose first that $(j,k) \in \mathcal{I}_1$. By (2.2) and by condition A,

$$\begin{aligned} \alpha_{j,k}(\varphi) &\leq \frac{2}{|Q_j^k|} \|\chi_{E_j^k}\|_{L^{p(\cdot)}} \leq \frac{2}{|Q_j^k|} \|\chi_{3Q_j^k}\|_{L^{p(\cdot)}} \\ &\leq \frac{c}{|Q_j^k|} \|\chi_{Q_j^k}\|_{L^{p(\cdot)}} \leq \frac{c}{\|\chi_{Q_j^k}\|_{L^{p'(\cdot)}}} \end{aligned}$$

(we have used the fact that condition A implies the following "doubling" property: there exists $c > 0$ such that $\|\chi_{2Q}\|_{L^{p(\cdot)}} \leq c\|\chi_Q\|_{L^{p(\cdot)}}$ for any cube Q). Hence, by

Corollary 3.2, $\alpha_{j,k}(\varphi)^{p'(x)}\in A_\infty(Q_j^k)$. From this and from Lemma 3.1,

$$\begin{aligned}\int_{Q_j^k\cap D_{k+l}}\alpha_{j,k}(\varphi)^{p'(x)}dx &\le c\Big(\frac{|Q_j^k\cap D_{k+l}|}{|Q_j^k|}\Big)^\varepsilon\int_{Q_j^k}\alpha_{j,k}(\varphi)^{p'(x)}dx\\ &\le c\Big(\frac{|Q_j^k\cap D_{k+l}|}{|Q_j^k|}\Big)^\varepsilon\int_{E_j^k}\varphi(x)^{p'(x)}dx.\end{aligned}\tag{3.6}$$

Assume now that $(j,k)\in\mathcal{I}_2$. Then

$$\begin{aligned}\int_{Q_j^k\cap D_{k+l}}\alpha_{j,k}(\varphi)^{p'(x)}dx &\le \int_{Q_j^k\cap D_{k+l}}\alpha_{j,k}(\varphi)dx\\ &= \frac{|Q_j^k\cap D_{k+l}|}{|Q_j^k|}\int_{E_j^k}\varphi.\end{aligned}\tag{3.7}$$

Let us show now that for each Q_j^k,

$$|Q_j^k\cap D_{k+l}|\le 3^{n(3-l)}|Q_j^k|\quad (l\ge 4).\tag{3.8}$$

Indeed, let $x\in Q_j^k$ and let Q' be an arbitrary cube such that $x\in Q'$. Observe that either $Q'\subset 3Q_j^k$ or $Q_j^k\subset 3Q'$. If the second inclusion holds, then $3Q'\cap D_k\ne\emptyset$, and hence

$$|f|_{Q'}\le 3^n|f|_{3Q'}\le 3^n3^{n(k+1)}\le 3^{n(k+l)}\quad (l\ge 2).$$

Therefore, if $|f|_{Q'}>3^{n(k+l)}$, then $Q'\subset 3Q_j^k$. From this and from the weak type $(1,1)$ property of M, we get

$$\begin{aligned}|Q_j^k\cap D_{k+l}| &\le |\{x\in Q_j^k: M(f\chi_{3Q_j^k})(x)>3^{n(k+l)}\}|\\ &\le \frac{3^n}{3^{n(k+l)}}\int_{3Q_j^k}|f|\le\frac{9^n|Q_j^k|}{3^{n(k+l)}}|f|_{3Q_j^k}\le\frac{9^n}{3^{n(l-1)}}|Q_j^k|,\end{aligned}$$

proving (3.8).

Combining (3.6), (3.7) and (3.8), we get (for $0\le l\le 3$ we use a trivial estimate $|Q_j^k\cap D_{k+l}|\le|Q_j^k|$)

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\sum_j\int_{Q_j^k\cap D_{k+l}}\alpha_{j,k}(\varphi)^{p'(x)}dx\\ &\le\sum_{(j,k)\in\mathcal{I}_1}c3^{n\varepsilon(3-l)}\int_{E_j^k}\varphi(x)^{p'(x)}dx+\sum_{(j,k)\in\mathcal{I}_2}3^{n(3-l)}\int_{E_j^k}\varphi\\ &\le c3^{n\varepsilon(3-l)}\Big(\int_{\mathbb{R}^n}\varphi(x)^{p'(x)}dx+\|\varphi\|_{L^1}\Big).\end{aligned}$$

However, $\int_{\mathbb{R}^n}\varphi(x)^{p'(x)}dx\le 1$, and by (2.2),

$$\|\varphi\|_{L^1}=\int_E\varphi\le 2\|\chi_E\|_{L^{p(\cdot)}}.$$

Therefore,

$$\int_{\mathbb{R}^n}(T_l\varphi)^{p'(x)}dx\le c\|\chi_E\|_{L^{p(\cdot)}}3^{n\varepsilon(3-l)},$$

which easily implies

$$\|T_lf\|_{L^{p'(\cdot)}}\le c\big(p,n,E\big)(3^{n\varepsilon/(p')_+})^{-l}\quad (l=0,1,\dots).$$

This estimate, along with

$$\|Tf\|_{L^{p'(\cdot)}} \le \sum_{l=0}^{\infty} \|T_l f\|_{L^{p'(\cdot)}},$$

proves (3.5), and therefore the proof is complete. □

Remark 3.3. The proof of Theorem 1.2 shows that

$$\|(Mf)\chi_E\|_{L^{p(\cdot)}} \le c\big(p,n\big)c(E)\|f\|_{L^{p(\cdot)}}, \tag{3.9}$$

where $c(E) = \max\big(1, \|\chi_E\|_{L^{p(\cdot)}}^{1/(p')_-}\big)$.

Remark 3.4. Theorem 1.2 easily implies the following result due to T. Kopaliani [11] mentioned in the Introduction: if p is a constant outside some ball and $p \in A$, then M is bounded on $L^{p(\cdot)}$. Indeed, let $p(x) = p_0$ on B^c, and let $\|f\|_{L^{p(\cdot)}} = 1$. Then Theorem 1.2 with $E = 2B$ gives

$$\int_{\mathbb{R}^n} (Mf)^{p(x)} dx \le c + \int_{(2B)^c} (Mf)^{p_0} dx.$$

Next, setting $f_1 = f\chi_B$ and $f_2 = f - f_1$, we get

$$\int_{(2B)^c} (Mf)^{p_0} dx \le 2^{p_0-1}\bigg(\int_{(2B)^c} (Mf_1)^{p_0} dx + \int_{(2B)^c} (Mf_2)^{p_0} dx\bigg).$$

By the Hardy-Littlewood maximal theorem,

$$\int_{(2B)^c} (Mf_2)^{p_0} dx \le c\int_{B^c} |f|^{p_0} dx \le c.$$

Finally, by (2.2), $\int_B |f| \le c_B$, and hence

$$\int_{(2B)^c} (Mf_1)^{p_0} dx \le c\bigg(\int_{(2B)^c} \frac{1}{|x-x_0|^{p_0}} dx\bigg)\bigg(\int_B |f|\bigg)^{p_0} \le c,$$

where x_0 is the center of B. We have proved that $\int (Mf)^{p(x)} dx \le c$ whenever $\|f\|_{L^{p(\cdot)}} = 1$, which means the boundedness of M on $L^{p(\cdot)}$.

4. Proof of Theorem 1.3

The necessity part of Theorem 1.3 follows immediately from the fact that

$$|f|_Q\chi_Q(x) \le Mf(x)$$

for any cube Q. In proving the sufficiency part it will be more convenient to work with the maximal function defined with respect to balls, so we shall assume in this section that

$$Mf(x) = \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)|dy,$$

where the supremum is taken over all balls B containing the point x.

Proof of the sufficiency part of Theorem 1.3. Let p be radially decreasing and $p \in A$. The weak type $\big(p(\cdot), p(\cdot)\big)$ of M means that there exists a constant $c > 0$ such that for any $f \in L^{p(\cdot)}$ with $\|f\|_{L^{p(\cdot)}} = 1$ one has

$$\sup_{\alpha>0} \int_{\{x:Mf(x)>\alpha\}} \alpha^{p(x)} dx \le c. \tag{4.1}$$

Fix an $f \in L^{p(\cdot)}$ with $\|f\|_{L^{p(\cdot)}} = 1$. Observe that the case corresponding to $\alpha \geq 1$ follows easily from Theorem 1.2. Indeed, let $E = \{x : Mf(x) > 1\}$. By (2.1),

$$|E| \leq c \int_{\{x:|f(x)|>1/2\}} |f(x)|dx \leq c \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \leq c.$$

Therefore, $\|\chi_E\|_{L^{p(\cdot)}} \leq c$, and by (3.9) we get

$$\int_{\{x:Mf(x)>\alpha\}} \alpha^{p(x)} dx \leq c \int_E (Mf)^{p(x)} dx \leq c \quad (\alpha \geq 1). \tag{4.2}$$

The case when $\alpha < 1$ is more complicated. We are going to show that

$$\sup_{0<\alpha<1} \int_{\{x:Mf(x)>2^{n+1}\alpha\}} \alpha^{p(x)} dx \leq c. \tag{4.3}$$

Clearly, this estimate, along with (4.2), would imply (4.1). We start by defining several auxiliary functions.

Given a ball B, denote by $\widetilde{B}$ the ball of the same radius centered at the origin. Suppose that $|B| \geq 1$. Then $\|\chi_{\widetilde{B}}\|_{L^{p'(\cdot)}} \geq 1$. Since $p(x)$ is radially decreasing, $p'(x)$ is radially increasing, and thus

$$\int_B \left(\frac{1}{\|\chi_{\widetilde{B}}\|_{L^{p'(\cdot)}}} \right)^{p'(x)} dx \leq \int_{\widetilde{B}} \left(\frac{1}{\|\chi_{\widetilde{B}}\|_{L^{p'(\cdot)}}} \right)^{p'(x)} dx = 1.$$

Hence,

$$\|\chi_B\|_{L^{p'(\cdot)}} \leq \|\chi_{\widetilde{B}}\|_{L^{p'(\cdot)}},$$

and therefore for $h > 1$ we have

$$\psi(h) \equiv \frac{1}{h} \sup_{|B|=h} \|\chi_B\|_{L^{p'(\cdot)}} = \frac{1}{h} \|\chi_{\{|x| \leq (h/v_n)^{1/n}\}}\|_{L^{p'(\cdot)}},$$

where v_n is the volume of the unit ball. Observe that if $|B| \geq 1$, then

$$|B|^{1/(p')_+} \leq \|\chi_B\|_{L^{p'(\cdot)}} \leq |B|^{1/(p')_-}.$$

From this and since $(p')_+ = (p_-)'$ and $(p')_- = (p_+)'$, we get

$$(1/h)^{1/p_-} \leq \psi(h) \leq (1/h)^{1/p_+} \quad (h > 1). \tag{4.4}$$

Now, for $0 < \alpha < 1$ define

$$\varphi(\alpha) = \sup\{h > 1 : \psi(h) > \alpha\}.$$

By (4.4),

$$(1/\alpha)^{p_-} \leq \varphi(\alpha) \leq (1/\alpha)^{p_+} \quad (0 < \alpha < 1). \tag{4.5}$$

Further, setting $B_\alpha = \{x : |x| \leq (\varphi(\alpha)/v_n)^{1/n}\}$, we have

$$\frac{\|\chi_{B_\alpha}\|_{L^{p'(\cdot)}}}{|B_\alpha|} = \psi(\varphi(\alpha)) = \alpha.$$

Hence, since $p \in A$, we obtain $\alpha \|\chi_{B_\alpha}\|_{L^{p(\cdot)}} \leq c$, or equivalently,

$$\int_{B_\alpha} \alpha^{p(x)} dx \leq c \tag{4.6}$$

(we have used an obvious fact that the definitions of class A in terms of cubes and balls are equivalent).

Setting $S_k(\alpha)=(k+1)B_\alpha\setminus kB_\alpha$ and using (4.6), we get

$$\int_{\{x:Mf(x)>2^{n+1}\alpha\}}\alpha^{p(x)}dx\le c+\sum_{k=1}^\infty\int_{S_k(\alpha)\cap\{Mf>2^{n+1}\alpha\}}\alpha^{p(x)}dx. \tag{4.7}$$

Now set $\widetilde{S}_k(\alpha)=(k+3/2)B_\alpha\setminus(k-1/2)B_\alpha$. Note that

$$\bigcup_{x\in S_k(\alpha)}\{B:x\in B,|B|\le|B_\alpha/2|\}\subset\widetilde{S}_k(\alpha).$$

Further, if $|B|>|B_\alpha/2|$, then $|2B|>\varphi(\alpha)$, and hence, by the definition of φ and by (2.2) we get

$$\frac{1}{|B|}\int_B|f|\le\frac{2^n}{|2B|}\int_{2B}|f|\le 2^{n+1}\frac{\|\chi_{2B}\|_{L^{p'(\cdot)}}}{|2B|}\le 2^{n+1}\alpha.$$

Therefore,

$$S_k(\alpha)\cap\{Mf>2^{n+1}\alpha\}\subset\{M(f\chi_{\widetilde{S}_k(\alpha)})>2^{n+1}\alpha\}.$$

Hence, setting $\gamma_\alpha=(\varphi(\alpha)/v_n)^{1/n}$ and using the fact that p is radially decreasing along with (2.1), we get (recall that $p(x)=\rho(|x|)$)

$$\begin{aligned}\int_{S_k(\alpha)\cap\{Mf>2^{n+1}\alpha\}}\alpha^{p(x)}dx&\le\alpha^{\rho((k+1)\gamma_\alpha)}|\{M(f\chi_{\widetilde{S}_k(\alpha)})>2^{n+1}\alpha\}|\\&\le c\alpha^{\rho((k+1)\gamma_\alpha)-1}\int_{\widetilde{S}_k(\alpha)\cap\{|f|>2^n\alpha\}}|f|\\&\le c\alpha^{\rho((k+1)\gamma_\alpha)-\rho((k-1/2)\gamma_\alpha)}\int_{\widetilde{S}_k(\alpha)}|f(x)|^{p(x)}dx.\end{aligned}\tag{4.8}$$

It is easy to see that $\sum_{k=1}^\infty\chi_{\widetilde{S}_k(\alpha)}(x)\le 2$, and hence

$$\sum_{k=1}^\infty\int_{\widetilde{S}_k(\alpha)}|f(x)|^{p(x)}dx\le 2.$$

Combining this with (4.7) and (4.8), we have that in order to prove (4.3), it suffices to show that

$$\sup_{0<\alpha<1}\alpha^{\rho((k+1)\gamma_\alpha)-\rho((k-1/2)\gamma_\alpha)}\le c, \tag{4.9}$$

where c does not depend on k.

Let $\xi_1=\big((k+3/2)\gamma_\alpha,0,\dots,0\big)$, $\xi_2=\big((k-1)\gamma_\alpha,0,\dots,0\big)$, and let B_1 and B_2 be the balls of radius $\gamma_\alpha/2$ centered at ξ_1 and ξ_2, respectively. Next, let $\xi_3=(\xi_1+\xi_2)/2$ and let B_3 be the ball centered at ξ_3 of radius $2\gamma_\alpha$. Then the balls B_1, B_2 and B_3 satisfy the following properties:

(i) $\inf_{x\in B_1}|x|=(k+1)\gamma_\alpha$ and $\sup_{x\in B_2}|x|=(k-1/2)\gamma_\alpha$;
(ii) $|B_1|=|B_2|=\varphi(\alpha)/2^n$;
(iii) $B_1,B_2\subset B_3$ and $|B_3|=2^n\varphi(\alpha)$.

If the supremum in (4.9) is taken over $2^{-n/p_-}<\alpha<1$, then the bound is trivial. Hence, one can assume that $\alpha\le 2^{-n/p_-}$. Then, by (4.5), $|B_1|=|B_2|\ge 1$, and therefore,

$$\varphi(\alpha)2^{-n}\left(\frac{1}{\|\chi_{B_1}\|_{L^{p(\cdot)}}}\right)^{\rho((k+1)\gamma_\alpha)}\le\int_{B_1}\left(\frac{1}{\|\chi_{B_1}\|_{L^{p(\cdot)}}}\right)^{p(x)}dx=1$$

and

$$\varphi(\alpha)2^{-n}\left(\frac{1}{\|\chi_{B_2}\|_{L^{p'(\cdot)}}}\right)^{\rho'((k-1/2)\gamma_\alpha)} \leq \int_{B_2}\left(\frac{1}{\|\chi_{B_2}\|_{L^{p'(\cdot)}}}\right)^{p'(x)} dx = 1.$$

Using these estimates and condition A, we get

$$\begin{aligned}\varphi(\alpha)^{1/\rho((k+1)\gamma_\alpha)+1/\rho'((k-1/2)\gamma_\alpha)} &\leq c\|\chi_{B_1}\|_{L^{p(\cdot)}}\|\chi_{B_2}\|_{L^{p'(\cdot)}} \\ &\leq c\|\chi_{B_3}\|_{L^{p(\cdot)}}\|\chi_{B_3}\|_{L^{p'(\cdot)}} \leq c\varphi(\alpha).\end{aligned}$$

Hence

$$\varphi(\alpha)^{1/\rho((k+1)\gamma_\alpha)-1/\rho((k-1/2)\gamma_\alpha)} \leq c,$$

and thus

$$\varphi(\alpha)^{\frac{1}{(\rho_+)^2}\rho((k-1/2)\gamma_\alpha)-\rho((k+1)\gamma_\alpha)} \leq c.$$

Combining this with the left-hand side of (4.5) proves (4.9), and therefore the theorem is proved. □

5. Proof of Theorem 1.7

We start with the following characterization of condition A for big cubes for radially decreasing p.

Lemma 5.1. *Let p be a radially decreasing function ($p(x) = \rho(|x|)$) with $p_- > 1$ and $p_+ < \infty$. Then*

$$\sup_{|Q|\geq 1} \frac{\|\chi_Q\|_{L^{p(\cdot)}}\|\chi_Q\|_{L^{p'(\cdot)}}}{|Q|} < \infty \tag{5.1}$$

if and only if

$$\sup_{t\geq 1}\int_0^1 t^{n\left(\frac{\rho(\xi t)-\rho(t)}{\rho(t)(\rho(\xi t)-1)}\right)}\xi^{n-1}d\xi < \infty. \tag{5.2}$$

Proof. Given $t > 0$, let B_t be the ball centered at the origin of radius t. Observe that (5.2) is equivalent to

$$\sup_{t\geq 1}\|\chi_{B_t}\|_{L^{p'(\cdot)}}t^{-n/\rho'(t)} < \infty. \tag{5.3}$$

Indeed, (5.3) holds if and only if

$$\sup_{t\geq 1}\int_{B_t}\left(\frac{1}{t^n}\right)^{\frac{p'(x)}{\rho'(t)}} dx < \infty.$$

However,

$$\begin{aligned}\int_{B_t}\left(\frac{1}{t^n}\right)^{\frac{p'(x)}{\rho'(t)}} dx &= \omega_{n-1}\int_0^t\left(\frac{1}{t^n}\right)^{\frac{\rho'(\xi)}{\rho'(t)}}\xi^{n-1}d\xi \\ &= \omega_{n-1}\int_0^1\left(\frac{1}{t^n}\right)^{\frac{\rho'(\xi t)}{\rho'(t)}-1}\xi^{n-1}d\xi = \omega_{n-1}\int_0^1 t^{n\left(\frac{\rho(\xi t)-\rho(t)}{\rho(t)(\rho(\xi t)-1)}\right)}\xi^{n-1}d\xi,\end{aligned}$$

where ω_{n-1} is the surface area of the unit sphere in $\mathbb{R}^n$.

Assume now that (5.1) holds. By (2.2) and (2.3), this is equivalent to

$$|f|_Q\|\chi_Q\|_{L^{p(\cdot)}} \leq c\|f\chi_Q\|_{L^{p(\cdot)}}$$

for any locally integrable f and any cube with $|Q| \geq 1$. In particular, setting $f = \chi_{Q/2}$, we get

$$\|\chi_Q\|_{L^{p(\cdot)}} \leq c\|\chi_{Q/2}\|_{L^{p(\cdot)}} \quad (|Q| \geq 1). \tag{5.4}$$

Let Q_t be the smallest cube containing B_t. From (5.4), for $t \geq 1$ we get

$$\begin{aligned} \big(2^n(2^n-1)t^n\big)^{1/\rho(t)} &= |Q_{2t} \setminus Q_t|^{\frac{1}{p_+(Q_{2t}\setminus Q_t)}} \\ &\leq \|\chi_{Q_{2t}\setminus Q_t}\|_{L^{p(\cdot)}} \leq \|\chi_{Q_{2t}}\|_{L^{p(\cdot)}} \leq c\|\chi_{Q_t}\|_{L^{p(\cdot)}} \end{aligned}$$

(we use the notion $p_-(E) = \operatorname*{ess\,inf}_{x\in E} p(x)$ and $p_+(E) = \operatorname*{ess\,sup}_{x\in E} p(x)$). Therefore, by (5.1),

$$\|\chi_{B_t}\|_{L^{p'(\cdot)}} \leq \|\chi_{Q_t}\|_{L^{p'(\cdot)}} \leq c\frac{|Q_t|}{\|\chi_{Q_t}\|_{L^{p(\cdot)}}} \leq ct^{n/\rho'(t)},$$

which proves (5.3) and so (5.2).

Suppose now that (5.2) holds. Let $Q = \prod_{i=1}^n (a_i, a_i + h)$, where $h \geq 1$. Denote $\alpha = \max_{1\leq i\leq n} |a_i|$, and assume that $\alpha \leq 2h$. Then it is easy to see that $Q \subset B_{3\sqrt{n}h}$. Next, since p is radially decreasing,

$$\|\chi_{B_{3\sqrt{n}h}}\|_{L^{p(\cdot)}} \leq |B_{3\sqrt{n}h}|^{1/\rho(3\sqrt{n}h)}.$$

From this and from (5.3),

$$\begin{aligned} \frac{\|\chi_Q\|_{L^{p(\cdot)}}\|\chi_Q\|_{L^{p'(\cdot)}}}{|Q|} &\leq \frac{\|\chi_{B_{3\sqrt{n}h}}\|_{L^{p(\cdot)}}\|\chi_{B_{3\sqrt{n}h}}\|_{L^{p'(\cdot)}}}{h^n} \\ &\leq c\|\chi_{B_{3\sqrt{n}h}}\|_{L^{p'(\cdot)}} h^{-n/\rho'(3\sqrt{n}h)} \leq c. \end{aligned}$$

It remains to consider the case when $\alpha > 2h$. In this case,

$$\sup_{x\in Q} |x| \leq \frac{3\sqrt{n}}{2}\alpha \quad \text{and} \quad \inf_{x\in Q} |x| \geq \frac{1}{2}\alpha,$$

and therefore,

$$p_+(Q) \leq \rho(\alpha/2) \quad \text{and} \quad p_-(Q) \geq \rho(3\sqrt{n}\alpha/2).$$

Next, since $|Q| \geq 1$,

$$\|\chi_Q\|_{L^{p(\cdot)}} \leq |Q|^{1/p_-(Q)} \quad \text{and} \quad \|\chi_Q\|_{L^{p'(\cdot)}} \leq |Q|^{1-1/p_+(Q)}.$$

Combining these estimates yields

$$\begin{aligned} \frac{\|\chi_Q\|_{L^{p(\cdot)}}\|\chi_Q\|_{L^{p'(\cdot)}}}{|Q|} &\leq |Q|^{\frac{p_+(Q)-p_-(Q)}{p_+(Q)p_-(Q)}} \\ &\leq c\alpha^{n(\rho(\alpha/2)-\rho(3\sqrt{n}\alpha/2))/p_-^2}. \end{aligned} \tag{5.5}$$

But it follows from (5.2) that for $t \geq 1$,

$$t^{\rho(\xi t)-\rho(t)} \leq c \quad (0 < \xi < 1),$$

where c depends only on ξ and p. Indeed, since ρ is nonincreasing, by (5.2) we get

$$t^{\frac{n}{\rho_+(\rho_+-1)}(\rho(\xi t)-\rho(t))}\frac{\xi^n}{n} \leq \int_0^\xi t^{n\left(\frac{\rho(\tau t)-\rho(t)}{\rho(t)(\rho(\tau t)-1)}\right)}\tau^{n-1}d\tau \leq c.$$

Since $\alpha > 2h \geq 2$, we obtain that the right-hand side of (5.5) is bounded, which completes the proof. $\square$

Proof of Theorem 1.7. Let $E = \bigcup_{k=1}^{\infty}(e^{k^3}, e^{k^3 e^{1/k^2}})$ and

$$p_0(x) = \int_{|x|}^{\infty} \frac{1}{\tau \log \tau}\chi_E(\tau)d\tau.$$

Let us show that p_0 is a pointwise multiplier for $BMO(\mathbb{R})$. This is just a combination of several known facts. First, it was proved in [15] that p is a pointwise multiplier for $BMO(\mathbb{R})$ if and only if $p \in L^\infty(\mathbb{R})$ and

$$\sup_I \frac{\ell(I)}{|I|^2}\int_I\int_I |p(x)-p(y)|dxdy < \infty, \tag{5.6}$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$,

$$\ell(I) = \log\left(e + \max(|I|, |I|^{-1}, |\mathrm{cen}_I|)\right)$$

and cen_I denotes the center of I. Next, in proving [13, Proposition 4.2] it contains the proof of the fact that $g(x) = (\log\log|x|)\chi_{\{|x|\ge e\}}(x)$ satisfies (5.6). But $p_0 \in L^\infty$, and it is easy to see that for all $x, y \in \mathbb{R}$,

$$|p_0(x) - p_0(y)| \le |g(x) - g(y)|.$$

Therefore, p_0 satisfies (5.6), which proves that p_0 is a pointwise multiplier for $BMO(\mathbb{R})$.

Let $p_{q,\delta}(x) = q + \delta p_0(x)$, and assume that $q(q-1) \le \delta$. Let us show that $p_{q,\delta} \notin A$. By Lemma 5.1, it suffices to prove that

$$\sup_{t\ge 1}\int_0^1 t^{\frac{\rho(\xi t)-\rho(t)}{\rho(t)(\rho(\xi t)-1)}}d\xi = \infty. \tag{5.7}$$

Denote $\alpha_k = e^{k^3}$ and $\beta_k = e^{k^3 e^{1/k^2}}$. We have

$$\int_0^1 \beta_k^{\frac{\rho(\xi\beta_k)-p(\beta_k)}{\rho(\beta_k)(\rho(\xi\beta_k)-1)}}d\xi \ge \int_{\alpha_k/\beta_k}^1 \beta_k^{\frac{\delta(\log\log\beta_k - \log\log\xi\beta_k)}{\rho(\beta_k)(\rho(\alpha_k)-1)}}d\xi.$$

There exists a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$ and

$$\log\log\beta_k - \log\log\xi\beta_k = \log\left(\frac{\log(1/\xi)}{\log(\xi\beta_k)} + 1\right) \ge (1-\varepsilon_k)\frac{\log(1/\xi)}{\log(\xi\beta_k)}$$

for any $\xi \in (\alpha_k/\beta_k, 1)$. Hence,

$$\beta_k^{\log\log\beta_k - \log\log\xi\beta_k} \ge \beta_k^{(1-\varepsilon_k)\frac{\log(1/\xi)}{\log(\xi\beta_k)}} = (1/\xi)^{(1-\varepsilon_k)\frac{\log\beta_k}{\log\xi\beta_k}} \ge (1/\xi)^{(1-\varepsilon_k)}.$$

From this,

$$\int_{\alpha_k/\beta_k}^1 \beta_k^{\frac{\delta(\log\log\beta_k - \log\log\xi\beta_k)}{\rho(\beta_k)(\rho(\alpha_k)-1)}}d\xi \ge \int_{\alpha_k/\beta_k}^1 \xi^{-\frac{\delta(1-\varepsilon_k)}{\rho(\beta_k)(\rho(\alpha_k)-1)}}d\xi.$$

Since $\alpha_k/\beta_k \to 0$ and $\frac{\delta(1-\varepsilon_k)}{\rho(\beta_k)(\rho(\alpha_k)-1)} \to \frac{\delta}{q(q-1)} \ge 1$, we have

$$\int_{\alpha_k/\beta_k}^1 \xi^{-\frac{\delta(1-\varepsilon_k)}{\rho(\beta_k)(\rho(\alpha_k)-1)}}d\xi \to \infty \quad \text{as} \quad k \to \infty.$$

Indeed, for any $\varepsilon > 0$ there exists K such that $\alpha_k/\beta_k < \varepsilon$ for all $k \ge K$. Hence, for all $k \ge K$ we obtain

$$\int_{\alpha_k/\beta_k}^1 \xi^{-\frac{\delta(1-\varepsilon_k)}{\rho(\beta_k)(\rho(\alpha_k)-1)}}d\xi \ge \int_{\varepsilon}^1 \xi^{-\frac{\delta(1-\varepsilon_k)}{\rho(\beta_k)(\rho(\alpha_k)-1)}}d\xi,$$

and thus,

$$\liminf_{k\to\infty}\int_{\alpha_k/\beta_k}^{1}\xi^{-\frac{\delta(1-\varepsilon_k)}{\rho(\beta_k)(\rho(\alpha_k)-1)}}\,d\xi \geq \int_{\varepsilon}^{1}\xi^{-\frac{\delta}{q(q-1)}}\,d\xi \geq \log\frac{1}{\varepsilon}.$$

This proves (5.7), and therefore the proof is complete. □

Remark 5.2. It is not difficult to show that the restrictions on q and δ in Theorem 1.7 are sharp in the sense that $p_{q,\delta} \in A$ if $q(q-1) > \delta$.

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