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ON THE QUASI-HEREDITARY PROPERTY FOR STAGGERED SHEAVES

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Abstract. Let G be an algebraic group over an algebraically closed field, acting on a variety X with finitely many orbits. *Staggered sheaves* are certain complexes of G-equivariant coherent sheaves on X that seem to possess many remarkable properties. In this paper, we construct "standard" and "costandard" objects in the category of staggered sheaves, and we prove that that category has enough projectives and injectives.

1. Introduction

Let X be a variety over an algebraically closed field $\Bbbk$, and let G be a linear algebraic group over $\Bbbk$ acting on X with finitely many orbits. *Staggered sheaves* [A, AT1] are the objects in the heart of certain t-structures on the bounded derived category $\mathcal{D}^{\mathrm{b}}_G(X)$ of G-equivariant coherent sheaves on X. The category of staggered sheaves, denoted $\mathcal{M}(X)$, enjoys a growing list of remarkable properties, analogous in many ways to properties of ℓ-adic mixed perverse sheaves [AT1, AT2]:

- Every object has finite length. Simple objects arise via an "$\mathcal{IC}$" functor and are parametrized by irreducible vector bundles on G-orbits.
- There is a well-behaved notion of "purity" in $\mathcal{D}^{\mathrm{b}}_G(X)$, and every simple staggered sheaf is pure.
- Every pure object in $\mathcal{D}^{\mathrm{b}}_G(X)$ is semisimple, *i.e.*, a direct sum of shifts of simple staggered sheaves.

In this paper, we add to this list as follows. First, we prove that $\mathcal{M}(X)$ is *quasi-hereditary*, meaning that every simple object is a quotient of some "standard" object and is contained in some "costandard" object. (See Section 2 for the definitions.) This answers a question the author was asked by David Vogan. Second, we prove that $\mathcal{M}(X)$ has enough projectives and injectives. These are analogues of results on perverse sheaves due to Mirollo–Vilonen [MV].

The paper is organized as follows. Section 2 contains some generalities on quasi-hereditary abelian categories, and Section 3 is a review of relevant facts about staggered sheaves. In Section 4, we prove that the functor of restriction to an open subscheme has both left and right adjoints in $\mathcal{M}(X)$. (In general, there are no such adjoints in $\mathcal{D}^{\mathrm{b}}_G(X)$, of course.) We use those adjoints to construct standard and costandard objects in Section 5.

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The next two sections contain useful auxiliary results. In Section 6, we show that the subcategory of staggered sheaves supported on some closed subscheme is Serre. (A corollary is that staggered sheaves do not "see" nilpotent thickenings of schemes.) Next, Section 7 gives an explicit description of the structure of standard and costandard objects. We obtain several Ext^1-vanishing results as corollaries. The main theorem, on projectives and injectives in $\mathcal{M}(X)$, is proved in Section 8. Finally, Section 9 presents a brief example.

2. Preliminaries on quasi-hereditary categories

Let $\mathfrak{A}$ be a $\Bbbk$-linear abelian category. Assume that $\mathfrak{A}$ has a small skeleton; *i.e.*, that the class S of isomorphism classes of simple objects forms a set. For each $s \in S$, fix a representative $L(s)$. We also assume that $\mathfrak{A}$ is of *finite type*, meaning that every object has finite length, and we write $[X : L(s)]$ for the multiplicity of $L(s)$ in any composition series for X. We also assume that

$$\mathrm{Hom}(L(s), L(s)) \simeq \Bbbk \qquad \text{for all } s \in S.$$

These assumptions imply that $\mathfrak{A}$ is also *Hom-finite*: the vector space $\mathrm{Hom}(X, Y)$ is finite-dimensional for any two objects $X, Y \in \mathfrak{A}$.

Recall that if a simple object $L(s)$ admits a projective cover (P, ϕ) (where $\phi : P \to L(s)$ is a surjective morphism), then it is unique up to isomorphism, but in general not canonically so. The same holds for injective hulls (I, ψ). (See Lemma 2.2 below, however.)

The next two elementary lemmas will be useful in the sequel. Their proofs are routine and will be omitted.

Lemma 2.1. *Let X be an object of $\mathfrak{A}$. If $L(s)$ has a projective cover (P, ϕ), then $[X : L(s)] = \dim \mathrm{Hom}(P, X)$. If $L(s)$ has an injective hull (I, ψ), then $[X : L(s)] = \dim \mathrm{Hom}(X, I)$.* □

Lemma 2.2. *Assume that for some (and hence any) projective cover (P, ϕ) of $L(s)$, we have $[P : L(s)] = 1$. Then any two projective covers of $L(s)$ are canonically isomorphic.* □

Assume henceforth that the set S is equipped with a fixed order $\preceq$. For $s \in S$, let $\mathfrak{A}_{\preceq s}$ (resp. $\mathfrak{A}_{\prec s}$) denote the Serre subcategory generated by $\{L(t) \mid t \preceq s\}$ (resp. $\{L(t) \mid t \prec s\}$).

In this paper, we work with the following definition of "quasi-hereditary category", taken from a paper of Bezrukavnikov [B2]. Note that this notion is somewhat more general than the older one due to Cline–Parshall–Scott [CPS] (see also [BGS]).

Definition 2.3. A *standard cover* of $L(s)$ is a projective cover $(M(s), \phi(s))$ of $L(s)$ within the category $\mathfrak{A}_{\preceq s}$ with the property that $[M(s) : L(s)] = 1$. Similarly, a *costandard hull* of $L(s)$ is an injective hull $(N(s), \psi(s))$ within $\mathfrak{A}_{\preceq s}$ such that $[N(s) : L(s)] = 1$.

$\mathfrak{A}$ is said to be *quasi-hereditary* if every simple object has a standard cover and a costandard hull.

By Lemma 2.2, standard covers and costandard hulls are unique (when they exist) up to canonical isomorphism.

Lemma 2.4. *The following conditions on an object $X \in \mathfrak{A}$ are equivalent:*

(1) $X \in \mathfrak{A}_{\preceq s}$.
(2) $\mathrm{Hom}(M(t), X) = 0$ *for all* $t \succ s$.
(3) $\mathrm{Hom}(X, N(t)) = 0$ *for all* $t \succ s$.

Proof. If $X \in \mathfrak{A}_{\preceq s}$, it is clear that conditions (2) and (3) hold. Now, suppose $X \notin \mathfrak{A}_{\preceq s}$. We will prove by induction on the length of X that condition (2) also fails; the proof that (3) fails is similar. Let X' be a simple subobject of X, and let $X'' = X/X'$. Then either $X' \notin \mathfrak{A}_{\preceq s}$ or $X'' \notin \mathfrak{A}_{\preceq s}$. If $X' \notin \mathfrak{A}_{\preceq s}$, then $X' \simeq L(t)$ for some $t \succ s$, so there is clearly a nonzero morphism $M(t) \to X'$. Thus, $\mathrm{Hom}(M(t), X) \neq 0$ as well. On the other hand, if $X' \in \mathfrak{A}_{\preceq s}$ but $X'' \notin \mathfrak{A}_{\preceq s}$, then, by assumption, there is some $t \succ s$ such that $\mathrm{Hom}(M(t), X'') \neq 0$. We also have $\mathrm{Hom}(M(t), X') = \mathrm{Ext}^1(M(t), X') = 0$, so the natural morphism $\mathrm{Hom}(M(t), X) \to \mathrm{Hom}(M(t), X'')$ is an isomorphism. In particular, $\mathrm{Hom}(M(t), X) \neq 0$, as desired. □

Assume $\mathfrak{A}$ is quasi-hereditary. For any object $X \in \mathfrak{A}$ and any $s \in S$, we define

$$\langle X : M(s) \rangle = \dim \mathrm{Hom}(X, N(s)) \qquad \text{and} \qquad \langle X : N(s) \rangle = \dim \mathrm{Hom}(M(s), X).$$

Note that if X is a projective cover $P(t)$ of a simple object $L(t)$, Lemma 2.1 gives us an alternate interpretation of $\dim \mathrm{Hom}(X, N(s))$. We see then that

$$\langle P(t) : M(s) \rangle = [N(s) : L(t)]. \tag{2.1}$$

This is, of course, the famous "Brauer–Humphreys reciprocity" formula for highest-weight categories [CPS]. In such a category, the projective cover of a simple object admits a *standard filtration*, *i.e.*, a filtration whose subquotients are standard objects, and the number $\langle P(t) : M(s) \rangle$ is precisely the multiplicity with which $M(s)$ occurs in any standard filtration of $P(t)$. (This follows from the fact that $\mathrm{Hom}(M(t), N(s)) = 0$ for $s \neq t$.)

It is not true that projectives in an arbitrary quasi-hereditary category necessarily admit standard filtrations, and the numbers $\langle P(t) : M(s) \rangle$ cannot always be interpreted as multiplicities. Nevertheless, a weak form of these ideas holds in great generality: the proposition below tells us that for any object $X \in \mathfrak{A}$, the numbers $\langle X : M(s) \rangle$ give information about the subquotients of a certain canonical filtration of X.

Given a finite-dimensional $\Bbbk$-vector space V, consider the object $V \otimes M(s)$. Let us say that a quotient Y of $V \otimes M(s)$ is *essential* if $[Y : L(s)] = \dim V$. Equivalently, Y is an essential quotient if the kernel of the morphism $V \otimes M(s) \to Y$ contains no subobject isomorphic to $M(s)$. Note that if $\mathrm{Hom}(X, N(s)) \neq 0$, it must be the case that $[X : L(s)] \neq 0$, so $\mathrm{Hom}(X, N(s))$ must vanish for all but finitely many s.

Proposition 2.5. *Assume $\mathfrak{A}$ is quasi-hereditary. Given $X \in \mathfrak{A}$, let $s_1 \prec s_2 \prec \cdots \prec s_k$ be the elements of S such that* $\mathrm{Hom}(X, N(s)) \neq 0$. *There is a canonical decreasing filtration*

$$X = X_1 \supset X_2 \supset \cdots \supset X_k \supset X_{k+1} = 0$$

such that X_i/X_{i+1} is an essential quotient of $\mathrm{Hom}(X, N(s_i))^* \otimes M(s_i)$.

Proof. From Lemma 2.4, we see that $X \in \mathfrak{A}_{\preceq s_k}$, and that $\mathrm{Hom}(M(t), X) = 0$ for $t \succ s_k$ but $\mathrm{Hom}(M(s_k), X) \neq 0$. Consider the canonical morphism

$$e : \mathrm{Hom}(M(s_k), X) \otimes M(s_k) \to X.$$

Let X_k denote the image of this morphism, and let $X' = X/X_k$. Note that X_k is an essential quotient of $\mathrm{Hom}(M(s_k), X) \otimes M(s_k)$; otherwise, there would be some nonzero $f \in \mathrm{Hom}(M(s_k), X)$ with $\Bbbk f \otimes M(s_k) \subset \ker e$, but that is absurd:

$$e(\Bbbk f \otimes M(s_k)) = \mathrm{im}\, f.$$

Next, consider the exact sequence

$$0 \to \mathrm{Hom}(X_k, N(s)) \to \mathrm{Hom}(\mathrm{Hom}(M(s_k), X) \otimes M(s_k), N(s)) \to \mathrm{Hom}(\ker e, N(s)).$$

When $s \prec s_k$, the middle term vanishes, and therefore the first term does as well. When $s = s_k$, the last term vanishes, so the first two are isomorphic to one another. Note that since $M(s_k)$ is a projective object in $\mathfrak{A}_{\preceq s_k}$, and $N(s_k)$ is injective, there is a nondegenerate pairing

$$\mathrm{Hom}(M(s_k), X) \otimes \mathrm{Hom}(X, N(s_k)) \to \mathrm{Hom}(M(s_k), N(s_k)) \simeq \Bbbk.$$

Thus, we have a sequence of isomorphisms

$$\begin{aligned} \mathrm{Hom}(\mathrm{Hom}(M(s_k), X), \mathrm{Hom}(M(s_k), N(s_k))) &\simeq \mathrm{Hom}(\mathrm{Hom}(M(s_k), X), \Bbbk) \\ &\simeq \mathrm{Hom}(M(s_k), X)^* \simeq \mathrm{Hom}(X, N(s_k)). \end{aligned}$$

Next, consider the sequence

$$0 \to \mathrm{Hom}(X', N(s)) \to \mathrm{Hom}(X, N(s)) \to \mathrm{Hom}(X_k, N(s)).$$

Because $\mathfrak{A}_{\preceq s_k}$ is a Serre subcategory, both X_k and X' belong to it. Thus, all three terms above vanish when $s \succ s_k$. We saw above that the last term vanishes when $s \prec s_k$, and that the map $\mathrm{Hom}(X, N(s)) \to \mathrm{Hom}(X_k, N(s))$ is an isomorphism when $s = s_k$. Combining these observations, we find that

$$\mathrm{Hom}(X', N(s)) \simeq \begin{cases} \mathrm{Hom}(X, N(s)) & \text{if } s \neq s_k, \\ 0 & \text{if } s = s_k. \end{cases}$$

We have shown that X_k is an essential quotient of $\mathrm{Hom}(X, N(s_k))^* \otimes M(s_k)$, so the result follows by induction. □

3. Review of staggered sheaves

In this section, we fix the notation and briefly review relevant facts about staggered sheaves. Let X denote a (not necessarily reduced) scheme of finite type over $\Bbbk$, and let G denote a linear algebraic group over $\Bbbk$, acting on X with finitely many orbits. Here, and throughout the paper, an *orbit* will mean a reduced, locally closed G-invariant subscheme containing no smaller nonempty G-invariant subscheme. Let $\mathcal{C}_G(X)$ denote the category of G-equivariant coherent sheaves on X, and let $\mathcal{D}^{\mathrm{b}}_G(X)$ denote its bounded derived category. We assume throughout that $\mathcal{C}_G(X)$ has enough locally free objects.

Let $\mathbb{O}(X)$ denote the set of G-orbits on X, and let $\Omega(X)$ denote the set of isomorphism classes of pairs $(C, \mathcal{L})$, where $C \in \mathbb{O}(X)$ and $\mathcal{L} \in \mathcal{C}_G(C)$ is an irreducible G-equivariant vector bundle on C. The category of staggered sheaves $\mathcal{M}(X)$ depends on two choices: a *perversity*, which is simply a function $r : \mathbb{O}(X) \to \mathbb{Z}$, and an *s-structure*, which is a certain kind of increasing filtration of $\mathcal{C}_G(X)$ (see [A]). We will not review the rather lengthy and complicated definition of an s-structure here. Instead, we recall only that an s-structure allows us to assign to each pair $(C, \mathcal{L}) \in \Omega(X)$ a certain integer, denoted $\mathrm{step}\, \mathcal{L}$.

We regard both the perversity and the s-structure as fixed, once and for all. Moreover, we assume that the s-structure is "recessed" and "split", so that the results of [AT1, AT2] are available. In particular, the results of [AT1, Section 8] allow us to define $\mathcal{M}(X)$ with no assumption on r. (In contrast, the original construction in [A] required r to obey stringent inequalities.) For examples of s-structures, see [AS, T].

Staggered sheaves on a single orbit $C \subset X$ are easy to describe. Given an irreducible vector bundle $\mathcal{L} \in \mathcal{C}_G(C)$, let $d_{\mathcal{L}} = \operatorname{step} \mathcal{L} - r(C)$. The *staggered t-structure* on $\mathcal{D}^{\mathrm{b}}_G(C)$, denoted $({}^r\mathcal{D}^{\mathrm{b}}_G(C)^{\leq 0}, {}^r\mathcal{D}^{\mathrm{b}}_G(C)^{\geq 0})$, is the unique t-structure on $\mathcal{D}^{\mathrm{b}}_G(C)$ whose heart contains all objects of the form $\mathcal{L}[d_{\mathcal{L}}]$. By definition, $\mathcal{M}(C) = {}^r\mathcal{D}^{\mathrm{b}}_G(C)^{\leq 0} \cap {}^r\mathcal{D}^{\mathrm{b}}_G(C)^{\geq 0}$.

Next, let $\mathcal{D}^{-}_G(C)$ denote the bounded-above derived category of $\mathcal{C}_G(C)$, and let ${}^r\mathcal{D}^{-}_G(C)^{\leq 0}$ denote the full subcategory consisting of objects $\mathcal{F}$ such that $h^i(\mathcal{F})[-i] \in {}^r\mathcal{D}^{\mathrm{b}}_G(C)^{\leq 0}$ for all i. Let $i_C : \overline{C} \hookrightarrow X$ denote the inclusion of the closure of C. The *staggered t-structure* $({}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\leq 0}, {}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\geq 0})$ on $\mathcal{D}^{\mathrm{b}}_G(X)$ is characterized by the property that $\mathcal{F} \in {}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\leq 0}$ if and only if $Li^*_C\mathcal{F}|_C \in {}^r\mathcal{D}^{-}_G(C)^{\leq 0}$ for all orbits C. (This condition must be stated in terms of ${}^r\mathcal{D}^{-}_G(C)^{\leq 0}$ because Li^*_C does not, in general, take values in the bounded derived category.)

A dual version of the description above is as follows. Let $\mathcal{Q}_G(X)$ denote the category of G-equivariant quasi-coherent sheaves, and let $\mathcal{D}^{+}_G(X)$ denote the full subcategory of the bounded-below derived category of $\mathcal{Q}_G(X)$ consisting of objects with coherent cohomology. The full subcategory ${}^r\mathcal{D}^{+}_G(X)^{\geq 0} \subset \mathcal{D}^{+}_G(X)$ consists of objects $\mathcal{G}$ such that $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{F} \in {}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\leq -1}$. It turns out that $\mathcal{G} \in {}^r\mathcal{D}^{+}_G(X)^{\geq 0}$ if and only if $Ri^!_C\mathcal{G}|_C \in {}^r\mathcal{D}^{+}_G(C)^{\geq 0}$ for all orbits C.

Let $j : U \hookrightarrow X$ be the inclusion of a G-invariant open subscheme, and let $i : Z \hookrightarrow X$ be the inclusion of a G-invariant closed subscheme. The restriction functor $j^* : \mathcal{D}^{\mathrm{b}}_G(X) \to \mathcal{D}^{\mathrm{b}}_G(U)$ and the push-forward functor $i_* : \mathcal{D}^{\mathrm{b}}_G(Z) \to \mathcal{D}^{\mathrm{b}}_G(X)$ are t-exact for the staggered t-structure.

We now turn to the construction of simple objects in $\mathcal{M}(X)$. Let $j : U \hookrightarrow X$ be as above. Define two new functions ${}^\flat r, {}^\sharp r : \mathbb{O}(X) \to \mathbb{Z}$ by

$$(3.1) \qquad {}^\flat r(C) = \begin{cases} r(C) & \text{if } C \subset U, \\ r(C) - 1 & \text{if } C \not\subset U, \end{cases} \qquad {}^\sharp r(x) = \begin{cases} r(C) & \text{if } C \subset U, \\ r(C) + 1 & \text{if } C \not\subset U. \end{cases}$$

By [AT1, Proposition 8.7], for any object $\mathcal{F} \in \mathcal{M}(U)$, there is (up to isomorphism) a unique object in $\mathcal{D}^{\mathrm{b}}_G(X)$, denoted ${}^r j_{!*}\mathcal{F}$, such that

$$(3.2) \qquad {}^r j_{!*}\mathcal{F}|_U \simeq \mathcal{F} \qquad \text{and} \qquad {}^r j_{!*}\mathcal{F} \in {}^{\flat r}\mathcal{D}^{\mathrm{b}}_G(X)^{\leq 0} \cap {}^{\sharp r}\mathcal{D}^{\mathrm{b}}_G(X)^{\geq 0}.$$

The assignment $\mathcal{F} \mapsto {}^r j_{!*}\mathcal{F}$ defines a faithfully full functor ${}^r j_{!*} : \mathcal{M}(U) \to \mathcal{M}(X)$, and its essential image, denoted $\mathcal{M}^\natural(X)$, is a Serre subcategory of $\mathcal{M}(X)$. In particular, if $\mathcal{F} \in \mathcal{M}(U)$ is a simple object, then ${}^r j_{!*}\mathcal{F}$ is a simple object of $\mathcal{M}(X)$.

Remark 3.1. The fact that $\mathcal{M}^\natural(X)$ is a Serre subcategory of $\mathcal{M}(X)$ is closely related to the fact that staggered sheaves on a single orbit form a semisimple category (*cf.* Proposition 5.1). The analogous statement in the setting of ℓ-adic mixed perverse sheaves is false in general: the essential image of the intermediate-extension functor for perverse sheaves need not be a Serre subcategory if the category of local systems on some stratum is not semisimple. For instance, consider the intermediate extension to $\mathbb{A}^1$ of a nontrivial 2-dimensional local system on $\mathbb{A}^1 \smallsetminus \{0\}$ with unipotent monodromy. The constant sheaf on the point 0 appears in the composition series for this perverse sheaf.

More generally, given an orbit $C \subset X$, let ∂C denote the closed set $\overline{C} \smallsetminus C$ (not regarded as having a fixed scheme structure), and let $U_C = X \smallsetminus \partial C$. This is an open subscheme of X containing C as a closed orbit. Form the diagram of inclusions

$$\begin{tikzcd} C \arrow[r, "t_C"] \arrow[d, "j_C"'] & U_C \arrow[d, "h_C"] \\ \overline{C} \arrow[r, "i_C"'] & X. \end{tikzcd}$$

If $\mathcal{L} \in \mathcal{C}_G(C)$ is an irreducible vector bundle, then $\mathcal{L}[d_{\mathcal{L}}]$ is a simple object of $\mathcal{M}(C)$. We associate to $(C, \mathcal{L})$ a simple object of $\mathcal{M}(X)$, denoted $\mathcal{IC}(C, \mathcal{L})$, by

$$\mathcal{IC}(C, \mathcal{L}) = i_{C*}({}^{r}j_{C!*}\mathcal{L}[d_{\mathcal{L}}]) \simeq h_{C!*}(t_{C*}\mathcal{L}[d_{\mathcal{L}}]).$$

(The isomorphism $i_{C*} \circ {}^{r}j_{C!*} \simeq h_{C!*} \circ t_{C*}$ follows from the t-exactness of i_{C*} and the uniqueness property (3.2) for $h_{C!*}$.) All simple objects of $\mathcal{M}(X)$ arise in this way. Thus, the set of isomorphism classes of simple objects is in bijection with $\Omega(X)$.

The results of [AT2, Section 9] associate to the perversity r a collection of thick subcategories $(\{\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}\}, \{\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}\})_{w \in \mathbb{Z}}$ that behave much like the weight filtration on ℓ-adic mixed sheaves. These subcategories, together called the *skew co-t-structure* in *loc. cit.*, enjoy the following properties:

$$\begin{aligned} &\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w-1} \subset \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}, && \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w-1}[1] = \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}, \\ &\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w-1} \supset \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}, && \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w-1}[1] = \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}. \end{aligned}$$

An object of $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w} \cap \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}$ is said to be *skew-pure* of skew degree w. The Skew Purity Theorem [AT2, Theorem 10.2] states that every staggered sheaf has a canonical filtration with skew-pure subquotients, and in particular that a simple staggered sheaf is skew-pure. Indeed, the skew degree of such an object is given by

$$\operatorname{sk\,deg} \mathcal{IC}(C, \mathcal{L}) = 2d_{\mathcal{L}} - \dim C.$$

(Skew degrees in [AT2] differ from this formula by the addition of some constant depending on the choice of a dualizing complex, but we may ignore that constant here.) The Skew Decomposition Theorem [AT2, Theorem 11.5] states that every skew-pure object is semisimple.

We conclude with the following useful fact. Here, and throughout the paper, we write $\mathrm{Hom}^n(\mathcal{F}, \mathcal{G})$ for $\mathrm{Hom}(\mathcal{F}, \mathcal{G}[n])$ for any two objects $\mathcal{F}, \mathcal{G} \in \mathcal{D}^{\mathrm{b}}_G(X)$, or even $\mathcal{F} \in \mathcal{D}^{-}_G(X)$ and $\mathcal{G} \in \mathcal{D}^{+}_G(X)$. See [B1, Proposition 2] for the proof.

Lemma 3.2. *Let $j : U \hookrightarrow X$ be the inclusion of a G-invariant open subscheme, and let $\underline{Z} = X \smallsetminus U$ denote the complementary closed subset. For any two objects $\mathcal{F}_1 \in \mathcal{D}^{-}_G(X)$ and $\mathcal{F}_2 \in \mathcal{D}^{+}_G(X)$, there is a long exact sequence*

$$(3.3) \quad \cdots \to \mathrm{Hom}^{-1}(j^*\mathcal{F}_1|_U, j^*\mathcal{F}_2|_U) \to \varinjlim_{Z'} \mathrm{Hom}(Li^*_{Z'}\mathcal{F}_1, Ri^!_{Z'}\mathcal{F}_2) \to$$
$$\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Hom}(j^*\mathcal{F}_1|_U, \mathcal{F}_2|_U) \to \varinjlim_{Z'} \mathrm{Hom}^1(Li^*_{Z'}\mathcal{F}_1, Ri^!_{Z'}\mathcal{F}_2) \to \cdots,$$

where $i_{Z'} : Z' \hookrightarrow X$ ranges over all closed subscheme structures on $\underline{Z}$. □

4. Restriction to an open subscheme

Let $j : U \hookrightarrow X$ be the inclusion of a G-invariant open subscheme. In this section, we construct left and right adjoints to the restriction functor $j^* : \mathcal{M}(X) \to \mathcal{M}(U)$.

The perversities ${}^\flat r$ and ${}^\sharp r$ defined in (3.1) give rise to their own staggered t-structures on $\mathcal{D}^{\mathrm{b}}_G(X)$, and hence their own intermediate extension functors ${}^{\flat r}j_{!*}$ and ${}^{\sharp r}j_{!*}$. Now, ${}^{\flat r}j_{!*}$ takes values in ${}^{\flat(\flat r)}\mathcal{D}^{\mathrm{b}}_G(X)^{\leq 0} \cap {}^{\sharp(\flat r)}\mathcal{D}^{\mathrm{b}}_G(X)^{\geq 0}$. But ${}^\sharp({}^\flat r) = r$, and clearly ${}^{\flat(\flat r)}\mathcal{D}^{\mathrm{b}}_G(X)^{\leq 0} \subset {}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\leq 0}$, so we see that ${}^{\flat r}j_{!*}$ actually takes values in $\mathcal{M}(X)$. The same holds for ${}^{\sharp r}j_{!*}$, by similar reasoning.

We introduce the notation

$$ {}^r j_! = {}^{\flat r}j_{!*} \qquad \text{and} \qquad {}^r j_* = {}^{\sharp r}j_{!*} $$

for these functors regarded as functors $\mathcal{M}(U) \to \mathcal{M}(X)$.

Proposition 4.1. *Let $j : U \hookrightarrow X$ be the inclusion of a G-invariant open subscheme, and let $\underline{Z}$ denote the closed subset complementary to U. For $\mathcal{F} \in \mathcal{M}(U)$ and $\mathcal{G} \in \mathcal{M}(X)$, there are canonical isomorphisms*

$$ \operatorname{Hom}({}^r j_!\mathcal{F}, \mathcal{G}) \simeq \operatorname{Hom}(\mathcal{F}, j^*\mathcal{G}) \qquad \textit{and} \qquad \operatorname{Hom}(\mathcal{G}, {}^r j_*\mathcal{F}) \simeq \operatorname{Hom}(j^*\mathcal{G}, \mathcal{F}). $$

There is a canonical surjective morphism ${}^r j_!\mathcal{F} \to {}^r j_{!*}\mathcal{F}$ whose kernel is supported on $\underline{Z}$, and a canonical injective morphism ${}^r j_{!*}\mathcal{F} \to {}^r j_*\mathcal{F}$ whose cokernel is supported on $\underline{Z}$.

Proof. Let us apply Lemma 3.2 with $\mathcal{F}_1 = {}^r j_!\mathcal{F}$ and $\mathcal{F}_2 = \mathcal{G}$. Since $Li^*_{Z'}({}^r j_!\mathcal{F}) \in {}^{\flat(\flat r)}\mathcal{D}^-_G(Z')^{\leq 0} = {}^r\mathcal{D}^-_G(Z')^{\leq -2}$ and $Ri^!_{Z'}\mathcal{G} \in {}^r\mathcal{D}^+_G(X)^{\geq 0}$ for any closed subscheme structure $i_{Z'} : Z' \hookrightarrow X$ on the complement of U, we see that

$$ \operatorname{Hom}(Li^*_{Z'}({}^r j_!\mathcal{F}), Ri^!_{Z'}\mathcal{G}) = \operatorname{Hom}^1(Li^*_{Z'}({}^r j_!\mathcal{F}), Ri^!_{Z'}\mathcal{G}) = 0. $$

It follows from (3.3) that $\operatorname{Hom}({}^r j_!\mathcal{F}, \mathcal{G}) \simeq \operatorname{Hom}(\mathcal{F}, j^*\mathcal{G})$. The proof of the statement that $\operatorname{Hom}(\mathcal{G}, {}^r j_*\mathcal{F}) \simeq \operatorname{Hom}(j^*\mathcal{G}, \mathcal{F})$ is similar.

The first adjointness statement gives us an isomorphism $\operatorname{Hom}({}^r j_!\mathcal{F}, {}^r j_{!*}\mathcal{F}) \simeq \operatorname{Hom}(\mathcal{F}, \mathcal{F})$, and hence a canonical morphism ${}^r j_!\mathcal{F} \to {}^r j_{!*}\mathcal{F}$. Since the restriction of this map to U is an isomorphism, its kernel and cokernel must both be supported on $\underline{Z}$. But ${}^r j_{!*}\mathcal{F}$, an object of the Serre subcategory $\mathcal{M}^\natural(X)$, has no nonzero quotient supported on $\underline{Z}$, so the morphism ${}^r j_!\mathcal{F} \to {}^r j_{!*}\mathcal{F}$ must be surjective. Similarly, the second adjointness statement gives us a canonical injective morphism ${}^r j_{!*}\mathcal{F} \to {}^r j_*\mathcal{F}$ with cokernel supported on $\underline{Z}$. □

Let us make note of a particular instance of the preceding proposition. For an orbit $C \subset X$ and an irreducible vector bundle $\mathcal{L} \in \mathcal{C}_G(C)$, we put

$$ \begin{aligned} M(C, \mathcal{L}) &= i_{C*}{}^r j_{C!}\mathcal{L}[d_{\mathcal{L}}] \simeq {}^r h_{C!}t_{C*}\mathcal{L}[d_{\mathcal{L}}], \\ N(C, \mathcal{L}) &= i_{C*}{}^r j_{C*}\mathcal{L}[d_{\mathcal{L}}] \simeq {}^r h_{C*}t_{C*}\mathcal{L}[d_{\mathcal{L}}]. \end{aligned} $$

Proposition 4.2. *For any $(C, \mathcal{L}) \in \Omega(X)$, there are canonical nonzero morphisms*

$$ \phi : M(C, \mathcal{L}) \to \mathcal{IC}(C, \mathcal{L}) \qquad \textit{and} \qquad \psi : \mathcal{IC}(C, \mathcal{L}) \to N(C, \mathcal{L}). $$

The kernel of ϕ and cokernel of ψ are both supported on ∂C. □

5. Standard and costandard objects

In this section, we prove that $\mathcal{M}(X)$ is quasi-hereditary. We begin with a result about closed orbits.

Proposition 5.1. *Let $i_C : C \hookrightarrow X$ be the inclusion of a closed orbit. Then $\mathcal{M}(C)$ is a semisimple category, and $i_{C*} : \mathcal{M}(C) \to \mathcal{M}(X)$ is an embedding of it as a Serre subcategory of $\mathcal{M}(X)$.*

Proof. Note that since C is closed, we have $\mathcal{IC}(C, \mathcal{L}) \simeq i_{C*}\mathcal{L}[d_\mathcal{L}]$ for any irreducible vector bundle $\mathcal{L} \in \mathcal{C}_G(C)$. Let $\mathcal{M}'(C) \subset \mathcal{M}(X)$ be the Serre subcategory of $\mathcal{M}(X)$ generated by objects of the form $\mathcal{IC}(C, \mathcal{L})$. This is, of course, the smallest Serre subcategory of $\mathcal{M}(X)$ containing $i_{C*}(\mathcal{M}(C))$. We will show that $\mathcal{M}'(C)$ is semisimple. Since i_{C*} is faithful, that implies that $\mathcal{M}(C)$ is semisimple. Moreover, since $i_{C*}(\mathcal{M}(C))$ is closed under direct sums and contains all simple objects of $\mathcal{M}'(C)$, it also implies that $i_{C*}(\mathcal{M}(C)) = \mathcal{M}'(C)$.

To show that $\mathcal{M}'(C)$ is semisimple, it suffices to show that

$$\operatorname{Ext}^1(i_{C*}\mathcal{L}[d_\mathcal{L}], i_{C*}\mathcal{L}'[d_{\mathcal{L}'}]) = 0 \tag{5.1}$$

for any two irreducible vector bundles $\mathcal{L}, \mathcal{L}' \in \mathcal{C}_G(C)$. If $d_\mathcal{L} > d_{\mathcal{L}'} + 1$, this vanishing is obvious. Next, if $d_\mathcal{L} = d_{\mathcal{L}'} + 1$, the Ext-group above is isomorphic to $\operatorname{Hom}(\mathcal{L}, \mathcal{L}')$. The two irreducible vector bundles $\mathcal{L}$ and $\mathcal{L}'$ are necessarily nonisomorphic (since $d_\mathcal{L} \neq d_{\mathcal{L}'}$), so $\operatorname{Hom}(\mathcal{L}, \mathcal{L}') = 0$. Finally, if $d_\mathcal{L} \leq d_{\mathcal{L}'}$, then we have $\operatorname{sk\,deg} i_{C*}\mathcal{L}[d_\mathcal{L}] \leq \operatorname{sk\,deg} i_{C*}\mathcal{L}'[d_{\mathcal{L}'}]$. In this case, (5.1) follows from [AT2, Proposition 11.2]. $\square$

Suppose we apply the preceding proposition to the closed embedding $t_C : C \hookrightarrow U_C$. Since $h_{C!*}$ embeds $\mathcal{M}(U_C)$ as a Serre subcategory of $\mathcal{M}(X)$, we obtain the following result.

Corollary 5.2. *For any orbit $C \subset X$, the functor $\mathcal{IC}(C, \cdot) : \mathcal{M}(C) \to \mathcal{M}(X)$ is an embedding of $\mathcal{M}(C)$ as a semisimple Serre subcategory of $\mathcal{M}(X)$.*

Proposition 5.3. *Let $C, C' \subset X$ be orbits, and let $\mathcal{L} \in \mathcal{C}_G(C)$ and $\mathcal{L}' \in \mathcal{C}_G(C')$ be irreducible vector bundles. Assume that either (i) $C \not\subset \overline{C}'$, or (ii) $C = C'$ and $\mathcal{L} \not\simeq \mathcal{L}'$. Then we have*

$$\operatorname{Hom}(M(C, \mathcal{L}), \mathcal{IC}(C', \mathcal{L}')) = \operatorname{Ext}^1(M(C, \mathcal{L}), \mathcal{IC}(C', \mathcal{L}')) = 0,$$

$$\operatorname{Hom}(\mathcal{IC}(C', \mathcal{L}'), N(C, \mathcal{L})) = \operatorname{Ext}^1(\mathcal{IC}(C', \mathcal{L}'), N(C, \mathcal{L})) = 0.$$

Proof. Form the exact sequence (3.3) with $\mathcal{F}_1 = M(C, \mathcal{L})$, $\mathcal{F}_2 = \mathcal{IC}(C', \mathcal{L}')$, and $U = U_C$. For any closed subscheme structure $i_{Z'} : Z' \hookrightarrow X$ on ∂C, we have $Li^*_{Z'} M(C, \mathcal{L}) \in {}_q\mathcal{D}^-_G(Z')^{\leq -2}$, and $Ri^!_{Z'}\mathcal{IC}(C', \mathcal{L}') \in {}_q\mathcal{D}^+_G(Z')^{\geq 0}$. It follows that

$$\operatorname{Hom}(Li^*_{Z'} M(C, \mathcal{L}), Ri^!_{Z'}\mathcal{IC}(C', \mathcal{L}')) = \operatorname{Hom}^1(Li^*_{Z'} M(C, \mathcal{L}), Ri^!_{Z'}\mathcal{IC}(C', \mathcal{L}')) = 0.$$

Note that $h^*_C M(C, \mathcal{L})$ is isomorphic to $t_{C*}\mathcal{L}[d_\mathcal{L}]$; in particular, this is a simple object of $\mathcal{M}(U_C)$. Next, $h^*_C\mathcal{IC}(C', \mathcal{L}')$ is either 0 or a simple object, and in the latter case, it is distinct from $t_{C*}\mathcal{L}[d_\mathcal{L}]$. Therefore, $\operatorname{Hom}(h^*_C M(C, \mathcal{L}), h^*_C\mathcal{IC}(C', \mathcal{L}')) = 0$. Moreover, under the assumptions in the statement of the proposition, the support of $h^*_C\mathcal{IC}(C', \mathcal{L}')$ is either disjoint from C or equal to C. In the first case, it is obvious that $\operatorname{Ext}^1(h^*_C M(C, \mathcal{L}), h^*_C\mathcal{IC}(C', \mathcal{L}')) = 0$, and in the second, this follows from Proposition 5.1. We can then see from the exact sequence (3.3) that $\operatorname{Hom}(M(C, \mathcal{L}), \mathcal{IC}(C', \mathcal{L}')) = \operatorname{Ext}^1(M(C, \mathcal{L}), \mathcal{IC}(C', \mathcal{L}')) = 0$. The proofs of the statements with $N(C, \mathcal{L})$ are similar and will be omitted. $\square$

Together, Propositions 4.2 and 5.3 give us the following result.

Theorem 5.4. *Let $\prec$ be any total order on $\Omega(X)$ such that $C \subsetneq \overline{C}'$ implies $(C, \mathcal{L}) \prec (C', \mathcal{L}')$ for all $\mathcal{L} \in \mathcal{C}_G(C)$ and $\mathcal{L}' \in \mathcal{C}_G(C')$. With respect to this order, $M(C, \mathcal{L})$ is a standard cover of $\mathcal{IC}(C, \mathcal{L})$, and $N(C, \mathcal{L})$ is a costandard hull. Thus, $\mathcal{M}(X)$ is a quasi-hereditary category.* □

6. Staggered sheaves on closed subschemes

We will now make use of standard and costandard objects to show that for any G-invariant closed subscheme $i : Z \hookrightarrow X$, $\mathcal{M}(Z)$ embeds as a Serre subcategory of $\mathcal{M}(X)$. We begin with a result on Hom- and Ext^1-groups.

Proposition 6.1. *Let $i : Z \hookrightarrow X$ be a G-invariant closed subscheme. For any $\mathcal{F}, \mathcal{G} \in \mathcal{M}(Z)$, we have*

$$\mathrm{Hom}_Z(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_X(i_*\mathcal{F}, i_*\mathcal{G}), \tag{6.1}$$

$$\mathrm{Ext}^1_Z(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^1_X(i_*\mathcal{F}, i_*\mathcal{G}). \tag{6.2}$$

(To avoid confusion, in this proposition and in its proof, we explicitly label all Hom- and Ext-groups with the name of the scheme over which that group is to be computed.)

Proof. Assume that $\mathcal{F}$ and $\mathcal{G}$ are both nonzero (otherwise, the statement is trivial). We proceed by Noetherian induction, and assume the statement is already known when either Z or X is replaced by some proper closed subscheme.

We begin by proving the proposition in the case where $\mathcal{F}$ and $\mathcal{G}$, and hence $i_*\mathcal{F}$ and $i_*\mathcal{G}$, are both simple. Suppose $i_*\mathcal{F} \simeq \mathcal{IC}(C, \mathcal{L})$ and $i_*\mathcal{G} \simeq \mathcal{IC}(C', \mathcal{L}')$. In the case where $C = C'$, Corollary 5.2 tells us that both sides of (6.1) are isomorphic to $\mathrm{Hom}_C(\mathcal{L}[d_{\mathcal{L}}], \mathcal{L}'[d_{\mathcal{L}'}])$, and that both sides of (6.2) vanish. Henceforth, assume $C \neq C'$. Both sides of (6.1) automatically vanish, since there are no nonzero morphisms between nonisomorphic simple objects. To prove (6.2), we must consider the various ways in which C and C' may be related.

Suppose first that $C \not\subset \overline{C}'$ and $C' \not\subset \overline{C}$. Let $U = X \smallsetminus (\overline{C} \cap \overline{C}')$. Then $\mathcal{IC}(C, \mathcal{L})|_U$ and $\mathcal{IC}(C', \mathcal{L}')|_U$ have disjoint supports, so $\mathrm{Ext}^1_U(\mathcal{IC}(C, \mathcal{L})|_U, \mathcal{IC}(C', \mathcal{L}')|_U) = 0$. Also, for any closed subscheme structure $i_{Z'} : Z' \hookrightarrow X$ on $\overline{C} \cap \overline{C}'$, we have $Li^*_{Z'}\mathcal{IC}(C, \mathcal{L}) \in {}_q\mathcal{D}^-_G(Z')^{\leq -1}$ and $Ri^!_{Z'}\mathcal{IC}(C', \mathcal{L}') \in {}_q\mathcal{D}^+_G(Z')^{\geq 1}$, so it follows that $\mathrm{Hom}^1_{Z'}(Li^*_{Z'}\mathcal{IC}(C, \mathcal{L}), Ri^!_{Z'}\mathcal{IC}(C', \mathcal{L}')) = 0$, and then

$$\mathrm{Ext}^1_X(\mathcal{IC}(C, \mathcal{L}), \mathcal{IC}(C', \mathcal{L}')) = 0 \tag{6.3}$$

by Lemma 3.2. The same reasoning applies to Z, so both sides of (6.2) above vanish.

Next, suppose that $C \subset \overline{C}'$. Let $J = N(C', \mathcal{L}')/\mathcal{IC}(C', \mathcal{L}')$, and consider the exact sequence

$$\begin{aligned} &\mathrm{Hom}_X(\mathcal{IC}(C, \mathcal{L}), N(C', \mathcal{L}')) \to \mathrm{Hom}_X(\mathcal{IC}(C, \mathcal{L}), J) \to \\ &\qquad \mathrm{Ext}^1_X(\mathcal{IC}(C, \mathcal{L}), \mathcal{IC}(C', \mathcal{L}')) \to \mathrm{Ext}^1_X(\mathcal{IC}(C, \mathcal{L}), N(C', \mathcal{L}')). \end{aligned}$$

The first and last terms vanish by Proposition 5.3, so the middle two terms are isomorphic. By Proposition 4.2, there is some closed subscheme structure $\kappa : Y \hookrightarrow X$ on $\partial C'$ on which J is supported: $J \simeq \kappa_* J'$ for some $J' \in \mathcal{M}(Y)$. Since $C \subsetneq \overline{C}'$, we likewise have $\mathcal{IC}(C, \mathcal{L}) \simeq \kappa_*\mathcal{F}'$ for some simple object $\mathcal{F}' \in \mathcal{M}(Y)$.

Since Y is strictly smaller than Z, we know by the inductive assumption that $\mathrm{Hom}_X(\mathcal{IC}(C,\mathcal{L}),J) \simeq \mathrm{Hom}_Y(\mathcal{F}',J')$. The same reasoning also applies to Z, so we have

$$\mathrm{Ext}^1_Z(\mathcal{IC}(C,\mathcal{L}),\mathcal{IC}(C',\mathcal{L}')) \simeq \mathrm{Ext}^1_X(\mathcal{IC}(C,\mathcal{L}),\mathcal{IC}(C',\mathcal{L}')) \simeq \mathrm{Hom}_Y(\mathcal{F}',J'),$$

as desired. A similar argument using $M(C,\mathcal{L})$ applies to the case where $C' \subset \overline{C}$. This completes the proof of the proposition in the case where $\mathcal{F}$ and $\mathcal{G}$ are both simple.

For the general case, we proceed by induction on the sum of the lengths of $\mathcal{F}$ and $\mathcal{G}$. Suppose that $\mathcal{F}$ is not simple and find some short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

with $\mathcal{F}'$ and $\mathcal{F}''$ both nonzero (and therefore of strictly smaller length than $\mathcal{F}$). Consider the eight-term exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(\mathcal{F}'',\mathcal{G}) \to \mathrm{Hom}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}(\mathcal{F}',\mathcal{G}) \to& \\ \mathrm{Ext}^1(\mathcal{F}'',\mathcal{G}) \to \mathrm{Ext}^1(\mathcal{F},\mathcal{G}) \to \mathrm{Ext}^1(\mathcal{F}',\mathcal{G}) &\to \mathrm{Hom}^2(\mathcal{F}'',\mathcal{G}), \end{aligned}$$

together with its analogue obtained by applying i_* to every object. By assumption, the four morphisms

$$\begin{aligned} \mathrm{Hom}(\mathcal{F}',\mathcal{G}) &\to \mathrm{Hom}(i_*\mathcal{F}',i_*\mathcal{G}), & \mathrm{Ext}^1(\mathcal{F}',\mathcal{G}) &\to \mathrm{Ext}^1(i_*\mathcal{F}',i_*\mathcal{G}), \\ \mathrm{Hom}(\mathcal{F}'',\mathcal{G}) &\to \mathrm{Hom}(i_*\mathcal{F}'',i_*\mathcal{G}), & \mathrm{Ext}^1(\mathcal{F}'',\mathcal{G}) &\to \mathrm{Ext}^1(i_*\mathcal{F}'',i_*\mathcal{G}) \end{aligned}$$

are isomorphisms, and $\mathrm{Hom}^2(\mathcal{F}'',\mathcal{G}) \to \mathrm{Hom}^2(i_*\mathcal{F}'',i_*\mathcal{G})$ is injective because i_* is faithful. By two applications of the five lemma, we see that $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}(i_*\mathcal{F},i_*\mathcal{G})$ and $\mathrm{Ext}^1(\mathcal{F},\mathcal{G}) \to \mathrm{Ext}^1(i_*\mathcal{F},i_*\mathcal{G})$ are isomorphisms, as desired. A similar argument establishes (6.1) and (6.2) in the case where $\mathcal{F}$ is simple but $\mathcal{G}$ is not. ☐

Theorem 6.2. *Let $i : Z \hookrightarrow X$ be a G-invariant closed subscheme. Then $i_*(\mathcal{M}(Z))$ is a Serre subcategory of $\mathcal{M}(X)$.*

Proof. The isomorphism (6.1) tells us that the functor $i_* : \mathcal{M}(Z) \to \mathcal{M}(X)$ is full. It remains to show that the essential image of i_* is stable under extensions. Consider the short exact sequence

$$0 \to i_*\mathcal{F}' \to \mathcal{F} \to i_*\mathcal{F}'' \to 0$$

in $\mathcal{M}(X)$, with $\mathcal{F}',\mathcal{F}'' \in \mathcal{M}(Z)$. The object $\mathcal{F}$ is in the essential image of i_* if and only if the corresponding element of $\mathrm{Ext}^1(i_*\mathcal{F}'',i_*\mathcal{F}')$ is in the image of the natural map $\mathrm{Ext}^1(\mathcal{F}'',\mathcal{F}') \to \mathrm{Ext}^1(i_*\mathcal{F}'',i_*\mathcal{F}')$, and the latter is surjective by (6.2). ☐

In particular, since all simple staggered sheaves on a nonreduced scheme are supported on the associated reduced scheme, we obtain the following result.

Corollary 6.3. *Suppose X is not reduced, and let $t : X_{\mathrm{red}} \to X$ be the inclusion of the associated reduced scheme. Then $t_* : \mathcal{M}(X_{\mathrm{red}}) \to \mathcal{M}(X)$ is an equivalence of categories.* ☐

7. Structure of standard and costandard objects

Let $K(C,\mathcal{L})$ denote the kernel of the canonical morphism $M(C,\mathcal{L}) \to \mathcal{IC}(C,\mathcal{L})$, and let $J(C,\mathcal{L})$ denote the cokernel of the canonical morphism $\mathcal{IC}(C,\mathcal{L}) \to N(C,\mathcal{L})$. The goal of this section is to describe the structure of $K(C,\mathcal{L})$ and $J(C,\mathcal{L})$.

Lemma 7.1. *Let $(C,\mathcal{L}),(C',\mathcal{L}') \in \Omega(X)$, and assume that $C \not\subset \partial C'$. There are natural isomorphisms*

$$\mathrm{Hom}(K(C,\mathcal{L}),\mathcal{IC}(C',\mathcal{L}')) \simeq \mathrm{Ext}^1(\mathcal{IC}(C,\mathcal{L}),\mathcal{IC}(C',\mathcal{L}')),$$
$$\mathrm{Hom}(\mathcal{IC}(C',\mathcal{L}'),J(C,\mathcal{L})) \simeq \mathrm{Ext}^1(\mathcal{IC}(C',\mathcal{L}'),\mathcal{IC}(C,\mathcal{L})).$$

Proof. If $(C,\mathcal{L}) \simeq (C',\mathcal{L}')$, then both Hom-groups vanish by Proposition 4.2, and both Ext^1-groups vanish by Corollary 5.2. Suppose now that $(C,\mathcal{L}) \not\simeq (C',\mathcal{L}')$. The first isomorphism above then comes from the long exact sequence associated to $0 \to K(C,\mathcal{L}) \to M(C,\mathcal{L}) \to \mathcal{IC}(C,\mathcal{L}) \to 0$ using Proposition 5.3. The second isomorphism is similar, using $N(C,\mathcal{L})$ instead. □

Lemma 7.2. *For any $(C,\mathcal{L}) \in \Omega(X)$, there are only finitely many pairs $(C',\mathcal{L}') \in \Omega(X)$ with $C \not\subset \partial C'$ such that either*

$$\mathrm{Ext}^1(\mathcal{IC}(C,\mathcal{L}),\mathcal{IC}(C',\mathcal{L}')) \neq 0 \qquad \textit{or} \qquad \mathrm{Ext}^1(\mathcal{IC}(C',\mathcal{L}'),\mathcal{IC}(C,\mathcal{L})) \neq 0.$$

Proof. Clearly, $\mathrm{Hom}(K(C,\mathcal{L}),\mathcal{IC}(C',\mathcal{L}'))$ and $\mathrm{Hom}(\mathcal{IC}(C',\mathcal{L}'),J(C,\mathcal{L}))$ vanish for all but finitely many $(C',\mathcal{L}')$, so this statement follows from the previous lemma. □

Theorem 7.3. *Suppose* $\mathrm{sk}\deg \mathcal{IC}(C,\mathcal{L}) = w$. *Then $K(C,\mathcal{L})$ is skew-pure of degree $w-1$, and there is a natural isomorphism*

$$K(C,\mathcal{L}) = \bigoplus_{\{(C',\mathcal{L}')\in\Omega(X) \mid C'\subset\partial C\}} \mathrm{Ext}^1(\mathcal{IC}(C,\mathcal{L}),\mathcal{IC}(C',\mathcal{L}'))^* \otimes \mathcal{IC}(C',\mathcal{L}').$$

Similarly, $J(C,\mathcal{L})$ is skew-pure of degree $w+1$, and there is a natural isomorphism

$$J(C,\mathcal{L}) = \bigoplus_{\{(C',\mathcal{L}')\in\Omega(X) \mid C'\subset\partial C\}} \mathrm{Ext}^1(\mathcal{IC}(C',\mathcal{L}'),\mathcal{IC}(C',\mathcal{L}')) \otimes \mathcal{IC}(C',\mathcal{L}').$$

Proof. By the Skew Purity Theorem [AT2, Theorem 10.2], $M(C,\mathcal{L})$ admits a canonical filtration

$$0 = M_{\sqsubseteq w-k} \subset \cdots \subset M_{\sqsubseteq w-1} \subset M_{\sqsubseteq w} = M(C,\mathcal{L}),$$

where each $M_{\sqsubseteq w-i} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w-i}$, and $M_{\sqsubseteq w-i}/M_{\sqsubseteq w-i-1}$ is skew-pure of degree $w-i$. (In our case, we know that $M_{\sqsubseteq w}$ must be the last step of the filtration, because the unique simple quotient of $M(C,\mathcal{L})$ is skew-pure of degree w.) Let us begin by showing that in fact $M_{\sqsubseteq w-2} = 0$. It suffices to show that $\mathrm{Hom}(\mathcal{F}, M(C,\mathcal{L})) = 0$ for all $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w-2}$. We employ the exact sequence of Lemma 3.2 with $U = U_C$ and $\underline{Z} = \partial C$. Clearly, $M(C,\mathcal{L})|_{U_C} \simeq \mathcal{IC}(C,\mathcal{L})|_{U_C} \in \mathcal{D}^{\mathrm{b}}_G(U)_{\sqsupseteq w}$, so $\mathrm{Hom}(\mathcal{F}|_{U_C}, M(C,\mathcal{L})|_{U_C}) = 0$.

The Skew Purity Theorem also tells us that any simple staggered sheaf is skew-pure. Since $M(C,\mathcal{L})$ arises by the functor ${}^{r}j_{C!} = {}^{\flat r}j_{C!*}$, we see that $M(C,\mathcal{L})$ is actually a simple object in the category ${}^{\flat r}\mathcal{M}(X)$ of staggered sheaves defined with respect to the perversity ${}^{\flat}r$, and it is skew-pure of degree w with respect to the notion of purity defined with respect to ${}^{\flat}r$. In the notation of [AT2], we have $M(C,\mathcal{L}) \in {}_{\llcorner \flat r \lrcorner}\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w} \cap {}_{\llcorner \flat r \lrcorner}\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}$. It follows that for any subscheme structure $i_{Z'} : Z' \hookrightarrow X$ on ∂C, we have $Ri^!_{Z'}M(C,\mathcal{L}) \in {}_{\llcorner \flat r \lrcorner}\mathcal{D}^{+}_G(Z')_{\sqsupseteq w}$. Since ${}^{\flat}r(C') =$

$r(C')-1$ for all $C' \subset \partial C$, this assertion is equivalent to $Ri^!_{Z'}M(C,\mathcal{L}) \in \mathcal{D}^+_G(Z')_{\sqsupseteq w-1}$, where the last category is defined as usual with respect to the perversity r. It follows that $\mathrm{Hom}(Li^*_{Z'}\mathcal{F}, Ri^!_{Z'}M(C,\mathcal{L})) = 0$, so $\mathrm{Hom}(\mathcal{F}, M(C,\mathcal{L})) = 0$, and $M_{\sqsubseteq w-2} = 0$, as desired.

Next, according to the Skew Decomposition Theorem [AT2, Theorem 11.5], any skew-pure object is semisimple. In particular, $M_{\sqsubseteq w-1}$ and $M(C,\mathcal{L})/M_{\sqsubseteq w-1}$ are both semisimple. Because $M(C,\mathcal{L})$ has a unique simple quotient, we must in fact have $M(C,\mathcal{L})/M_{\sqsubseteq w-1} \simeq \mathcal{IC}(C,\mathcal{L})$, and then we identify $M_{\sqsubseteq w-1}$ with $K(C,\mathcal{L})$. We may write

$$K(C,\mathcal{L}) \simeq \bigoplus_{\{(C',\mathcal{L}')\in\Omega(X) \mid C'\subset\partial C\}} V_{C',\mathcal{L}'} \otimes \mathcal{IC}(C',\mathcal{L}')$$

for some vector spaces $V_{C',\mathcal{L}'}$, which are zero for all but finitely many pairs $(C',\mathcal{L}')$. By applying the functor $\mathrm{Hom}(\cdot, \mathcal{IC}(C',\mathcal{L}'))$ to both sides, one sees that $V_{C',\mathcal{L}'} \simeq \mathrm{Hom}(K(C,\mathcal{L}), \mathcal{IC}(C',\mathcal{L}'))^*$. The desired formula for $K(C,\mathcal{L})$ then follows from Lemma 7.1. The second part of the theorem is proved similarly. ☐

From this structure theorem, we can deduce the following constraint on when Ext^1-groups may be nonzero. This strengthens the Ext^1-vanishing result contained in [AT2, Proposition 11.2].

Corollary 7.4. *Suppose* $\mathrm{Ext}^1(\mathcal{IC}(C,\mathcal{L}), \mathcal{IC}(C',\mathcal{L}')) \neq 0$. *Then either* $C \subset \partial C'$ *or* $C' \subset \partial C$, *and* $\mathrm{sk\,deg}\,\mathcal{IC}(C',\mathcal{L}') = \mathrm{sk\,deg}\,\mathcal{IC}(C,\mathcal{L}) - 1$.

Proof. By Corollary 5.2, we cannot have $C = C'$. Next, if $C \not\subset \partial C'$ and $C' \not\subset \partial C$, then we have established the vanishing statement (6.3) in the course of the proof of Propostion 6.1. Finally, if $C \subset \partial C'$, then $\mathcal{IC}(C',\mathcal{L}')$ must occur as a direct summand of $K(C,\mathcal{L})$ by Lemma 7.1, and then its skew degree must be $\mathrm{sk\,deg}\,\mathcal{IC}(C,\mathcal{L}) - 1$ according to Theorem 7.3. Similar reasoning applies if $C' \subset \partial C$. ☐

Since the category $\mathcal{M}(C)$ is semisimple, Theorem 7.3 immediately implies the following more general statement about the functors ${}^r\!j_{C!}$ and ${}^r\!j_{C*}$.

Corollary 7.5. *For any object* $\mathcal{F} \in \mathcal{M}(C)$, *the kernel of the natural morphism* ${}^r\!j_{C!}\mathcal{F} \to {}^r\!j_{C!*}\mathcal{F}$ *is naturally isomorphic to*

$$\bigoplus_{\{(C',\mathcal{L}')\in\Omega(X) \mid C'\subset\partial C\}} \mathrm{Ext}^1({}^r\!j_{C!*}\mathcal{F}, \mathcal{IC}(C',\mathcal{L}'))^* \otimes \mathcal{IC}(C',\mathcal{L}').$$

Likewise, the cokernel of the natural morphism ${}^r\!j_{C!*}\mathcal{F} \to {}^r\!j_{C*}\mathcal{F}$ *is naturally isomorphic to*

$$\bigoplus_{\{(C',\mathcal{L}')\in\Omega(X) \mid C'\subset\partial C\}} \mathrm{Ext}^1(\mathcal{IC}(C',\mathcal{L}'), {}^r\!j_{C!*}\mathcal{F}) \otimes \mathcal{IC}(C',\mathcal{L}'). \qquad \square$$

Theorem 7.6. *For any* $(C,\mathcal{L}) \in \Omega(X)$, *there are only finitely many pairs* $(C',\mathcal{L}') \in \Omega(X)$ *such that either*

$$\mathrm{Ext}^1(\mathcal{IC}(C,\mathcal{L}), \mathcal{IC}(C',\mathcal{L}')) \neq 0 \qquad \textit{or} \qquad \mathrm{Ext}^1(\mathcal{IC}(C',\mathcal{L}'), \mathcal{IC}(C,\mathcal{L})) \neq 0.$$

Proof. Since X contains only finitely many orbits, and in view of Lemma 7.2, it suffices to prove that for a fixed orbit C' with $C \subset \partial C'$, there are only finitely many irreducible vector bundles $\mathcal{L}' \in \mathcal{C}_G(C')$ such that one of the Ext^1-groups above is nonzero. Moreover, by Proposition 6.1, those Ext^1-groups may be computed in

$\mathcal{M}(\overline{C}')$ instead. Thus, we henceforth assume without loss of generality that $X = \overline{C}'$. Recall that $\mathcal{C}_G(X)$ is assumed to have enough locally free objects. Let us fix some locally free resolution $\mathcal{P}^\bullet$ of $\mathcal{IC}(C, \mathcal{L})$. ($\mathcal{P}^\bullet$ may be unbounded below, of course.) Let

$$m = d_{\mathcal{L}} - \dim C + \dim C' - 1.$$

Suppose now that $(C', \mathcal{L}')$ is such that

$$\operatorname{Ext}^1(\mathcal{IC}(C, \mathcal{L}), \mathcal{IC}(C', \mathcal{L}')) \neq 0. \tag{7.1}$$

Corollary 7.4 tells us that $\operatorname{sk}\deg \mathcal{IC}(C', \mathcal{L}') = \operatorname{sk}\deg \mathcal{IC}(C, \mathcal{L}) - 1$, and it follows that $d_{\mathcal{L}'} = m$. In particular, $d_{\mathcal{L}'}$ is independent of $\mathcal{L}'$. Now, choose some bounded complex $\mathcal{F}^\bullet$ of G-equivariant coherent sheaves that represents $\mathcal{IC}(C', \mathcal{L}')$. Then any nonzero class $f \in \operatorname{Ext}^1(\mathcal{IC}(C, \mathcal{L}), \mathcal{IC}(C', \mathcal{L}'))$ may be represented by a morphism of chain complexes $\tilde{f} : \mathcal{P}^\bullet[-1] \to \mathcal{F}^\bullet$. Let $\mathcal{G}^\bullet$ denote the image of $\tilde{f}$.

We claim that $h^m(\mathcal{G}^\bullet)|_{C'} \simeq \mathcal{L}'$. Indeed, we clearly have $h^i(\mathcal{G}^\bullet) \subset h^i(\mathcal{IC}(C', \mathcal{L}'))$ for all i, and in particular, $h^i(\mathcal{G}^\bullet)|_{C'} = 0$ for $i \neq m$. From the characterization of ${}^{\flat r}\mathcal{D}^{\mathrm{b}}_G(X)^{\leq 0}$ in terms of cohomology sheaves in [AT1, Section 2], it follows that the complex $\mathcal{G}^\bullet$, regarded as an object of $\mathcal{D}^{\mathrm{b}}_G(X)$, also belongs to ${}^{\flat r}\mathcal{D}^{\mathrm{b}}_G(X)^{\leq 0}$. Moreover, if $h^i(\mathcal{G}^\bullet)$ is supported on $\partial C'$ for all i, then we actually have $\mathcal{G}^\bullet \in {}^{r}\mathcal{D}^{\mathrm{b}}_G(X)^{\leq -1}$. But that is impossible: the natural map $\mathcal{G}^\bullet \to \mathcal{IC}(C', \mathcal{L}')$ is nonzero (because $f \neq 0$), but there is no nonzero morphism from an object of ${}^{r}\mathcal{D}^{\mathrm{b}}_G(X)^{\leq -1}$ to an object of $\mathcal{M}(X)$. We noted earlier that $h^i(\mathcal{G}^\bullet)$ is supported on $\partial C'$ for $i \neq m$, so we must have $h^m(\mathcal{G}^\bullet)|_{C'} \neq 0$. Indeed,

$$h^m(\mathcal{G}^\bullet)|_{C'} \subset h^m(\mathcal{IC}(C', \mathcal{L}'))|_{C'} \simeq \mathcal{L}',$$

and since $\mathcal{L}'$ is irreducible, the containment is an equality. Finally, note that $h^m(\mathcal{G}^\bullet)|_{C'}$ is a subquotient of $\mathcal{P}^m|_{C'}$.

We have shown that the condition (7.1) implies that $\mathcal{L}'$ is a subquotient of $\mathcal{P}^m|_{C'}$. Since $\mathcal{C}_G(C')$ is a finite-type category, there are, up to isomorphism, only finitely many possible irreducible vector bundles $\mathcal{L}'$ for which (7.1) may hold, as desired.

It remains to consider $\mathcal{L}'$ such that $\operatorname{Ext}^1(\mathcal{IC}(C', \mathcal{L}'), \mathcal{IC}(C, \mathcal{L})) \neq 0$. Unusually for this paper, the proof is not parallel to the case considered above. Instead, we use the Serre–Grothendieck duality functor, which we denote by $\mathbb{D} : \mathcal{D}^{\mathrm{b}}_G(X) \to \mathcal{D}^{\mathrm{b}}_G(X)$. By [AT1, Theorem 8.6], $\mathbb{D}$ carries the staggered t-structure to another staggered t-structure (that is, with respect to another perversity). Since

$$\operatorname{Ext}^1(\mathcal{IC}(C', \mathcal{L}'), \mathcal{IC}(C, \mathcal{L})) \simeq \operatorname{Ext}^1(\mathbb{D}\mathcal{IC}(C, \mathcal{L}), \mathbb{D}\mathcal{IC}(C', \mathcal{L}')),$$

the desired result follows from the case (7.1) in the dual category $\mathbb{D}\mathcal{M}(X)$. □

8. Projective and injective objects

In this section, we prove the main result of the paper:

Theorem 8.1. *Every simple object in $\mathcal{M}(X)$ admits a canonical projective cover and a canonical injective hull.*

Remark 8.2. For the reader familiar with [BGS, Theorem 3.2.1], on whose proof the argument below is based, we briefly indicate the relationship between the two. That theorem states that in any quasi-hereditary category with finitely many isomorphism classes of simple objects and a certain Ext^2-vanishing condition on standard and costandard objects, every simple object has a projective cover and an injective hull. Both assumptions are false in general for staggered sheaves, so we cannot

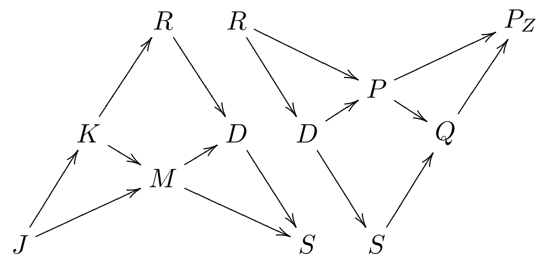

Figure 1. In this diagram, every collinear sequence of three objects is a short exact sequence.

simply invoke that theorem. However, the fact that $\mathcal{M}(X)$ usually has infinitely many isomorphism classes of simple objects is not a major obstacle: we proceed by induction on the number of orbits instead of on the number of simple objects, making use of Theorems 6.2 and 7.6 to handle infinitely many simple objects at each step. The failure of the Ext^2-vanishing hypothesis is more serious: getting around this requires a delicate argument using the explicit structure theorem from Section 7 to gain control over what happens in the relevant Ext^2-groups. It is this aspect of the proof that causes it to be so lengthy.

In [BGS], the Ext^2-vanishing condition is also used to show that projective covers admit standard filtrations, and hence to deduce the reciprocity formula (2.1). However, these properties need not hold for $\mathcal{M}(X)$ (except in the weak sense of Proposition 2.5). Indeed, they fail even in the example of the $\mathbb{G}_m$-action on $\mathbb{A}^1$ considered in [A, Section 11.2].

Remark 8.3. In the remarks following [BGS, Theorem 3.2.1], the authors sketch a different and much shorter existence proof for projective covers, due to Ringel. That argument (which also assumes finitely many isomorphism classes of simple objects) could likely also be modified to work for $\mathcal{M}(X)$. However, the proof below has the advantage of giving a rather explicit description of those objects.

To prove Theorem 8.1, we proceed by induction on the number of orbits in X. Choose an open orbit $C_0 \subset X$, and let $Z = X \smallsetminus C_0$. Given a pair $(C, \mathcal{L}) \in \Omega(X)$, we will show that $\mathcal{IC}(C, \mathcal{L})$ admits a projective cover. The construction of injective hulls is parallel and will be omitted. If $C = C_0$, then clearly $M(C, \mathcal{L})$ is the desired projective cover. Assume henceforth that $C \subset Z$. The treatment of this case occupies the remainder of the section.

8.1. **Definition of auxiliary objects.** In this step, we define eight new objects, arranged in seven interlocking short exact sequences. A summary diagram is shown in Figure 1.

By assumption, within the category $\mathcal{M}(Z)$, the object $\mathcal{IC}(C, \mathcal{L})$ has a projective cover, which we denote by P_Z. For each irreducible vector bundle $\mathcal{L} \in \mathcal{C}_G(C_0)$, let

$$B_{\mathcal{L}} = \mathrm{Ext}^1(P_Z, \mathcal{IC}(C_0, \mathcal{L})).$$

It follows from Theorem 7.6 that $B_{\mathcal{L}} = 0$ for all but finitely many $\mathcal{L}$ up to isomorphism. Thus, we may form the direct sum

$$S = \bigoplus B_{\mathcal{L}}^* \otimes \mathcal{IC}(C_0, \mathcal{L}).$$

Note that for any irreducible vector bundle $\mathcal{L} \in \mathcal{C}_G(C_0)$, we have

$$\text{(8.1)} \quad \operatorname{Hom}(S, \mathcal{IC}(C_0, \mathcal{L})) \simeq \operatorname{Hom}(B^*_{\mathcal{L}} \otimes \mathcal{IC}(C_0, \mathcal{L}), \mathcal{IC}(C_0, \mathcal{L})) \simeq \operatorname{Hom}(B^*_{\mathcal{L}}, \Bbbk) \simeq B_{\mathcal{L}}.$$

This observation will be used later. Consider now the sequence of isomorphisms

$$\operatorname{Ext}^1(P_Z, S) \simeq \bigoplus B^*_{\mathcal{L}} \otimes \operatorname{Ext}^1(P_Z, \mathcal{IC}(C, \mathcal{L})) \simeq \bigoplus \operatorname{End}(B_{\mathcal{L}}).$$

Thus, $\operatorname{Ext}^1(P_Z, S)$ contains a canonical element α, corresponding to the identity operator in $\bigoplus \operatorname{End}(B_{\mathcal{L}})$. We define an object Q by forming the short exact sequence corresponding to α:

$$\text{(8.2)} \qquad 0 \to S \to Q \to P_Z \to 0.$$

Since P_Z is projective as an object of the subcategory $\mathcal{M}(Z)$, and since $\mathcal{M}(Z)$ is Serre by Theorem 6.2, it follows that even in the larger category $\mathcal{M}(X)$, we have

$$\text{(8.3)} \qquad \operatorname{Ext}^1(P_Z, \mathcal{IC}(C', \mathcal{L}')) = 0 \qquad \text{when } C' \subset Z.$$

Thus, whenever $C' \subset Z$, we obtain from (8.2) an exact sequence

$$\text{(8.4)} \quad 0 \to \operatorname{Ext}^1(Q, \mathcal{IC}(C', \mathcal{L}')) \to \operatorname{Ext}^1(S, \mathcal{IC}(C', \mathcal{L}')) \to \operatorname{Ext}^2(P_Z, \mathcal{IC}(C', \mathcal{L}')).$$

Let us put

$$E_{C',\mathcal{L}'} = \operatorname{Ext}^1(Q, \mathcal{IC}(C', \mathcal{L}')) \qquad \text{and} \qquad F_{C',\mathcal{L}'} = \operatorname{Ext}^1(S, \mathcal{IC}(C', \mathcal{L}')).$$

Next, we denote the cokernel of the inclusion $E_{C',\mathcal{L}'} \hookrightarrow F_{C',\mathcal{L}'}$ by

$$\text{(8.5)} \qquad \bar{F}_{C',\mathcal{L}'} = F_{C',\mathcal{L}'}/E_{C',\mathcal{L}'} \subset \operatorname{Ext}^2(P_Z, \mathcal{IC}(C', \mathcal{L}')).$$

We now define a new object R by a construction analogous to that of S:

$$R = \bigoplus E^*_{C',\mathcal{L}'} \otimes \mathcal{IC}(C', \mathcal{L}').$$

As we saw with $\operatorname{Ext}^1(P_Z, S)$, the group $\operatorname{Ext}^1(Q, R) \simeq \bigoplus \operatorname{End}(E_{C',\mathcal{L}'})$ contains a canonical element β. Let

$$0 \to R \to P \to Q \to 0$$

be the corresponding short exact sequence. Now, S can be identified with a subobject of Q. Let D be its preimage in P. We thus obtain an additional short exact sequence

$$\text{(8.6)} \qquad 0 \to R \to D \to S \to 0.$$

The class γ of this extension in $\operatorname{Ext}^1(S, R)$ can be described as follows. As in (8.4), the long exact sequence formed from (8.2) gives us an injective map $\operatorname{Ext}^1(Q, R) \hookrightarrow \operatorname{Ext}^1(S, R)$. The element γ is simply the image of β under this inclusion. For an alternate description, note that

$$\operatorname{Ext}^1(S, R) \simeq \bigoplus E^*_{C',\mathcal{L}'} \otimes \operatorname{Ext}^1(S, \mathcal{IC}(C', \mathcal{L}')) \simeq \bigoplus \operatorname{Hom}(E_{C',\mathcal{L}'}, F_{C',\mathcal{L}'}).$$

Then β corresponds to the inclusions $E_{C',\mathcal{L}'} \to F_{C',\mathcal{L}'}$.

It is clear that $P/D \simeq Q/S \simeq P_Z$, so we also have

$$\text{(8.7)} \qquad 0 \to D \to P \to P_Z \to 0.$$

Next, consider the objects

$$M = \bigoplus B^*_{\mathcal{L}} \otimes M(C_0, \mathcal{L}) \qquad \text{and} \qquad K = \bigoplus F^*_{C',\mathcal{L}'} \otimes \mathcal{IC}(C', \mathcal{L}').$$

We of course have a surjective map $M \to S$. According to Corollary 7.5, its kernel is K, so we have a short exact sequence

$$0 \to K \to M \to S \to 0. \tag{8.8}$$

For each pair $(C', \mathcal{L}')$, there is a natural surjective map $F^*_{C',\mathcal{L}'} \to E^*_{C',\mathcal{L}'}$ with kernel $\bar{F}^*_{C',\mathcal{L}'}$. Together, they give rise to a surjective map $K \to R$ with kernel

$$J = \bigoplus \bar{F}^*_{C',\mathcal{L}'} \otimes \mathcal{IC}(C', \mathcal{L}').$$

To complete the picture in Figure 1, it remains to show the existence of a short exact sequence

$$0 \to J \to M \to D \to 0 \tag{8.9}$$

making the diagram commute. There is a natural isomorphism $\mathrm{Ext}^1(S, K) \simeq \mathrm{End}(F_{C',\mathcal{L}'})$, and the canonical element $\delta \in \mathrm{Ext}^1(S, K)$ corresponds to the extension (8.8). Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ext}^1(S,K) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1(S,R) \arrow[d, equal] & \mathrm{Ext}^1(Q,R) \arrow[l] \arrow[d, equal] \\
\mathrm{End}(F_{C',\mathcal{L}'}) \arrow[r] & \mathrm{Hom}(E_{C',\mathcal{L}'}, F_{C',\mathcal{L}'}) & \mathrm{End}(E_{C',\mathcal{L}'}). \arrow[l]
\end{tikzcd}$$

The image of δ in $\mathrm{Ext}^1(S, R)$ is clearly γ. It follows that the sequences (8.6) and (8.8) are related by a commutative diagram as shown in Figure 1, and, moreover, that the kernel of the surjective morphism $M \to D$ coincides with that of $K \to R$, as desired.

8.2. **Properties of D.** Given a pair $(C', \mathcal{L}') \in \Omega(X)$, form the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(D, \mathcal{IC}(C', \mathcal{L}')) &\to \mathrm{Hom}(M, \mathcal{IC}(C', \mathcal{L}')) \\ &\to \mathrm{Hom}(J, \mathcal{IC}(C', \mathcal{L}')) \to \mathrm{Ext}^1(D, \mathcal{IC}(C', \mathcal{L}')) \to 0. \end{aligned} \tag{8.10}$$

Here, we have used the fact that M is projective, and hence $\mathrm{Ext}^1(M, \mathcal{IC}(C', \mathcal{L}')) = 0$. Now, if $C' \subset Z$, then the second term above vanishes, and it follows that

$$\mathrm{Hom}(D, \mathcal{IC}(C', \mathcal{L}')) = 0 \tag{8.11}$$

and that

$$\mathrm{Ext}^1(D, \mathcal{IC}(C', \mathcal{L}')) \simeq \mathrm{Hom}(J, \mathcal{IC}(C', \mathcal{L}')) \simeq \bar{F}_{C',\mathcal{L}'}.$$

In particular, the natural morphism $\mathrm{Ext}^1(D, \mathcal{IC}(C', \mathcal{L}')) \to \mathrm{Ext}^2(P_Z, \mathcal{IC}(C', \mathcal{L}'))$ in the long exact sequence for (8.7) can be identified with the inclusion in (8.5). That is, the morphism

$$\mathrm{Ext}^1(D, \mathcal{IC}(C', \mathcal{L}')) \hookrightarrow \mathrm{Ext}^2(P_Z, \mathcal{IC}(C', \mathcal{L}')) \tag{8.12}$$

is injective. Next, consider the sequence (8.10) in the case where $C' = C_0$. In this case, the third term vanishes, so

$$\mathrm{Ext}^1(D, \mathcal{IC}(C_0, \mathcal{L}')) = 0, \tag{8.13}$$

and, moreover, the first two terms are isomorphic. Since $\mathrm{Hom}(K, \mathcal{IC}(C_0, \mathcal{L}')) = 0$, it follows from the long exact sequence for (8.8) that

$$\begin{aligned}(8.14) \quad \mathrm{Hom}(D, \mathcal{IC}(C_0, \mathcal{L}')) &\simeq \mathrm{Hom}(M, \mathcal{IC}(C_0, \mathcal{L}')) \\ &\simeq \mathrm{Hom}(S, \mathcal{IC}(C_0, \mathcal{L}')) \simeq \mathrm{Ext}^1(P_Z, \mathcal{IC}(C_0, \mathcal{L}')),\end{aligned}$$

where the last isomorphism is from (8.1).

8.3. **Conclusion of the proof.** Given $(C', \mathcal{L}') \in \Omega(X)$, form the sequence

$$\begin{aligned}0 \to \mathrm{Hom}(P_Z, \mathcal{IC}(C', \mathcal{L}')) &\to \mathrm{Hom}(P, \mathcal{IC}(C', \mathcal{L}')) \to \\ \mathrm{Hom}(D, \mathcal{IC}(C', \mathcal{L}')) &\to \mathrm{Ext}^1(P_Z, \mathcal{IC}(C', \mathcal{L}')) \to \mathrm{Ext}^1(P, \mathcal{IC}(C', \mathcal{L}')) \\ &\to \mathrm{Ext}^1(D, \mathcal{IC}(C', \mathcal{L}')) \to \mathrm{Ext}^2(P_Z, \mathcal{IC}(C', \mathcal{L}')) \to \cdots .\end{aligned}$$

Depending on whether $C' \subset Z$ or $C' = C_0$, we use either (8.11) or (8.14) to see that the first two terms are isomorphic:

$$\mathrm{Hom}(P, \mathcal{IC}(C', \mathcal{L}')) \simeq \mathrm{Hom}(P_Z, \mathcal{IC}(C', \mathcal{L}')) \simeq \begin{cases} \Bbbk & \text{if } (C', \mathcal{L}') = (C, \mathcal{L}), \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\mathcal{IC}(C, \mathcal{L})$ is the unique simple quotient of P. A purity argument as in the proof of Theorem 7.3 then shows that $[P : \mathcal{IC}(C, \mathcal{L})] = 1$. Next, we check that P is projective: if $C' \subset Z$, we see from (8.3) and (8.12) that

$$\mathrm{Ext}^1(P, \mathcal{IC}(C', \mathcal{L}')) = 0.$$

If $C' = C_0$, the same result follows from (8.13) and (8.14). Thus, P is a projective cover of $\mathcal{IC}(C, \mathcal{L})$, and it is unique up to canonical isomorphism by Lemma 2.2.

9. An example

Let $G = (\mathbb{G}_m)^2$ act on $\mathbb{A}^2$ with weights $\pi_+ = (2, 1)$ and $\pi_- = (-2, 1)$. That is, G acts on $\mathbb{A}^2$ by

$$(z, t) \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} z^2 tu \\ z^{-2} tv \end{bmatrix}.$$

Let $X = \operatorname{Spec} \Bbbk[x, y]/(xy)$; this is the union of the two coordinate axes in $\mathbb{A}^2$. Clearly, G has three orbits on X, which we denote $C_+ = \operatorname{Spec} \Bbbk[x, x^{-1}]$, $C_- = \operatorname{Spec} \Bbbk[y, y^{-1}]$, and $C_0 = \operatorname{Spec} \Bbbk$. The G-stabilizer of any point in C_+ is $H_+ = \ker \pi_+$, and the G-stabilizer of any point in C_- is $H_- = \ker \pi_-$. Note that H_+ is precisely the image of the cocharacter $\chi_+ = (-1, 2)$ (that is, the map $\chi_+ : \mathbb{G}_m \to G$ given by $s \mapsto (s^{-1}, s^2)$), and H_- is the image of $\chi_- = (1, 2)$. The weight lattice of H_+ or H_- can be identified with $\mathbb{Z}$, and the restriction of any G-weight λ to H_+ or H_- is given by $\langle \chi_+, \lambda \rangle$ or $\langle \chi_-, \lambda \rangle$, respectively.

Remark 9.1. This example is related to the Lie group $SL(2, \mathbb{R})$ in the following way. Let K denote a maximal compact subgroup of $SL(2, \mathbb{R})$, and let $\mathfrak{k} \subset \mathfrak{sl}(2, \mathbb{R})$ denote its Lie algebra. Its complexification $K_{\mathbb{C}} \simeq \mathbb{C}^\times$ acts on the complex vector space $(\mathfrak{sl}(2, \mathbb{R})/\mathfrak{k})^* \otimes \mathbb{C}$ with weights 2 and -2. The variety $\mathcal{N}$ of nilpotent elements in that space is the union of the two weight spaces. Now, let us extend this $K_{\mathbb{C}}$-action to a $(K_C \times \mathbb{C}^\times)$-action by having the second factor act with weight 1 on the whole vector space. Then, the $(K_{\mathbb{C}} \times \mathbb{C}^\times)$-action on $\mathcal{N}$ is isomorphic to the G-action on X described above.

For any G-stable closed subscheme $Y \subset X$ and any G-weight λ, we write $\mathcal{O}_Y(\lambda)$ for the G-equivariant sheaf obtained by twisting the structure sheaf of Y by λ. Next, note that irreducible line bundles on C_+ or on C_- are indexed by $\mathbb{Z}$, *i.e.*, by characters of H_+ or H_-. We denote these line bundles by $\mathcal{L}_+(n)$ and $\mathcal{L}_-(n)$.

Let $i : C_0 \hookrightarrow X$ be the inclusion morphism. It is easy to check that

$$\begin{aligned} &Li^*\mathcal{O}_{\overline{C}_+}(\lambda) \simeq \mathcal{O}_{C_0}(\lambda), \\ &Ri^!\mathcal{O}_{\overline{C}_+}(\lambda) \simeq \mathcal{O}_{C_0}(\lambda+\pi_+)[-1], \end{aligned} \qquad \mathcal{O}_{\overline{C}_+}(\lambda)|_{C_+} \simeq \mathcal{L}_+(\langle\chi_+,\lambda\rangle). \tag{9.1}$$

Analogous results hold for $\mathcal{O}_{\overline{C}_-}(\lambda)$.

Let us endow X with an s-structure. By [AT2, Theorem 7.4], we may specify an s-structure on an orbit by giving a cocharacter of its isotropy group. We give C_+ the s-structure corresponding to χ_+ (which may, of course, be regarded as a cocharacter of the isotropy group H_+), and we give C_- the s-structure corresponding to χ_-. For C_0, we use the cocharacter $\chi_0 = (0, 1)$.

To combine these into an s-structure on all of X, we will use the gluing theorem [AS, Theorem 1.1]. That theorem requires us to check that the conormal bundle of any orbit C lies in $\mathcal{C}_G(C)_{\le -1}$. This condition holds trivially for the open orbits C_+ and C_-. For C_0, the conormal bundle is $\mathcal{O}_{C_0}(-\pi_+) \oplus \mathcal{O}_{C_0}(-\pi_-)$. We have $\langle\chi_0, -\pi_\pm\rangle \le -1$, as required. We thus obtain an s-structure on X. It is automatically recessed, and it is split by [AT2, Theorem 7.6], as required.

Let $r : \mathbb{O}(X) \to \mathbb{Z}$ denote the constant perversity $r(C) = 0$. We now determine the simple staggered sheaves with respect to this perversity. For brevity, we adopt the notation

$$\begin{aligned} L_+(n) &= \mathcal{IC}(\overline{C}_+, \mathcal{L}_+(n)), \\ L_-(n) &= \mathcal{IC}(\overline{C}_-, \mathcal{L}_-(n)), \end{aligned} \qquad L_0(n,k) = \mathcal{IC}(C_0, \mathcal{O}_{C_0}(n,k)).$$

The three families of simple objects can be explicitly described as follows:

$$\begin{aligned} L_+(n) &= \mathcal{O}_{\overline{C}_+}(n-2, n-1)[n], \\ L_-(n) &= \mathcal{O}_{\overline{C}_-}(-n+2, n-1)[n], \end{aligned} \qquad L_0(n,k) = \mathcal{O}_{C_0}(n,k)[k].$$

(These assertions are easily verified using (9.1).) Next, we turn to standard and costandard objects. It turns that every simple object is also standard, and of course the $L_0(n,k)$ are also costandard. The nontrivial costandard objects are

$$N_+(n) = \mathcal{O}_{\overline{C}_+}(n,n)[n] \qquad \text{and} \qquad N_-(n) = \mathcal{O}_{\overline{C}_-}(-n,n)[n].$$

These objects fit into short exact sequences as follows:

$$\begin{gathered} 0 \to L_+(n) \to N_+(n) \to L_0(n,n) \to 0, \\ 0 \to L_-(n) \to N_-(n) \to L_0(-n,n) \to 0. \end{gathered} \tag{9.2}$$

Finally, we turn to projective covers and injective hulls. The objects $L_\pm(n)$ are already projective, and their injective hulls are simply their costandard hulls, the $N_\pm(n)$. On the other hand, the objects $L_0(n,k)$ are already injective, and they are also projective except when $k = |n|$. In that case, the nonsplit sequences (9.2) show that they are not projective. In fact, when $n \ne 0$, we can read off the projective cover $P_0(\pm n, n)$ of $L_0(\pm n, n)$ from (9.2):

$$P_0(n,n) = N_+(n) \qquad \text{and} \qquad P_0(-n,n) = N_-(n) \qquad \text{if } n \ne 0.$$

In the special case $n = 0$, the sequences (9.2) give two distinct nontrivial extensions of $L_0(0, 0)$. The projective cover of this object is $P_0(0, 0) \simeq \mathcal{O}_X$, and we have a short exact sequence

$$0 \to L_+(0) \oplus L_-(0) \to P_0(0, 0) \to L_0(0, 0) \to 0.$$

Because every standard object in this example happens to be simple, it is obviously true that projective covers of simple objects have standard filtrations. As we saw in Section 2, the multiplicities of standard objects in projective covers obey the reciprocity formula (2.1).

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