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HYPERSURFACES CUTTING OUT A PROJECTIVE VARIETY

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ABSTRACT. Let X be a nondegenerate projective variety of degree d and codimension e in a projective space $\mathbb{P}^N$ defined over an algebraically closed field. We study the following two problems: Is the length of the intersection of X and a line L in $\mathbb{P}^N$ at most $d-e+1$ if $L \not\subseteq X$? Is the scheme-theoretic intersection of all hypersurfaces of degree at most $d-e+1$ containing X equal to X? To study the second problem, we look at the locus of points from which X is projected nonbirationally.

§0. INTRODUCTION

Let $X \subseteq \mathbb{P}^N$ ($N = n+e$) be a projective variety of dimension n, degree d, and codimension e over an algebraically closed field $\Bbbk$. We always assume that X is *nondegenerate* in $\mathbb{P}^N$; i.e., X is not contained in any hyperplane in $\mathbb{P}^N$. Let m be a positive integer. We say that X is *m-regular* if its ideal sheaf $\mathcal{I}_{X/\mathbb{P}^N}$ satisfies $H^i(\mathbb{P}^N, \mathcal{I}_{X/\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(m-i)) = 0$ for all $i > 0$ (see [16], Lecture 14). By $\mathrm{E}_m(X)$, we denote the scheme-theoretic intersection of all hypersurfaces in $\mathbb{P}^N$, containing X, of degree at most m. We consider the following conditions on X:

(A_m) $l(X \cap L) := \mathrm{length}(\mathcal{O}_{X\cap L}) \leq m$ for each line L in $\mathbb{P}^N$ with $L \not\subseteq X$;
(B_m) $X = \mathrm{E}_m(X)$;
(C_m) X is m-regular.

The purpose here is to study (A_m) and (B_m) for $m = d-e+1$. To study (B_m), we also look at the structure of the locus, denoted by $\mathrm{B}(X)$ and $\mathrm{C}(X)$ (see (0.1)), of points from which X is projected nonbirationally.

First we briefly look at the three conditions. It is well-known that (C_m) implies (B_m) since the m-regularity of X implies that the homogeneous ideal of X is generated in degree $\leq m$ ([16], Lecture 14, Proposition). Also it is clear that (B_m) implies (A_m). On the other hand, the condition (C_m) for $m = d-e+1$ is the famous conjecture on Castelnuovo-Mumford regularity, and sometimes the implication (A_m) $\Rightarrow$ (C_m) for m close to $d-e$, with few trivial exceptions, is also included in the conjecture (see [6]; [9], §4). The conjecture is true for $n = 1$ ([9], Theorems 1.1 and 3.1), and the first part of it is true for a smooth surface X of $\mathrm{char}(\Bbbk) = 0$ ([15]). For $n \geq 3$, there are nice approaches to the conjecture but it is still open (see [5] and [13] for information). As evidence of the regularity conjecture, it is natural to expect (A_m) and (B_m) for $m = d-e+1$.

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One of the key ideas to study (A_m) and (B_m) for $m = d - e + 1$ is based on [17], Theorem 1, where it was shown that $X = \mathrm{E}_d(X)$ as a set, and $X = \mathrm{E}_d(X)$ as a scheme if X is smooth. To obtain these results, Mumford considered the image $\pi_\Lambda(X)$ by the linear projection $\pi_\Lambda : \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{n+1}$ from a general $(e-2)$-plane $\Lambda \subseteq \mathbb{P}^N$ and observed that the pull-back of the hypersurface $\pi_\Lambda(X)$ separates a point $v \in \mathbb{P}^N \setminus X$ from X if Λ is suitably chosen according to v. In this paper, we will also consider linear projections π_Λ with $\Lambda \cap X \neq \emptyset$.

Now we will state our results. First we deal with (A_{d-e+1}). We say that a line L is *secant* to X if $X \cap L$ is finite of length > 0. We say that a secant line L to X is *standard* if $\dim \pi_L(X \setminus L) = \dim X$ for the linear projection π_L of $\mathbb{P}^N \setminus L$ to $\mathbb{P}^{N-2}$ with center L.

Theorem 1 ($\mathrm{char}\, \Bbbk \geq 0$)**.** *For a standard secant line L to X, we have $l(X \cap L) \leq d - e + 1$.*

For a nonstandard secant line, the same result is expected. But at this stage, no proof about it would be known for $\mathrm{char}\, \Bbbk \geq 0$. The result here is a generalization of results in $\mathrm{char}\, \Bbbk = 0$, [2], [3], [14], and [18]. When X is smooth, we have a sharp bound including another invariant of X ([19]).

Next we deal with (B_{d-e+1}). To state our result, we introduce some notation:

$$\text{(0.1)} \qquad \begin{aligned} \mathrm{B}(X) &:= \{v \in \mathbb{P}^N \setminus X \mid l(X \cap \langle v, x\rangle) \geq 2 \text{ for general } x \in X\}, \\ \mathrm{C}(X) &:= \{u \in X \setminus \mathrm{Sing}\, X \mid l(X \cap \langle u, x\rangle) \geq 3 \text{ for general } x \in X\}, \end{aligned}$$

where $\mathrm{Sing}\, X$ denotes the singular locus of X. In other words, these are the loci of points from which X is projected nonbirationally onto its image: in the former, points off X, and in the latter, points on $X \setminus \mathrm{Sing}\, X$. When $e = 1$, it is clear that (B_d) holds and $\mathrm{B}(X) = \mathbb{P}^N \setminus X$ and $\mathrm{C}(X) = X \setminus \mathrm{Sing}\, X$ if $d \geq 3$. Thus we consider (B_{d-e+1}) for $e \geq 2$.

Theorem 2 ($\mathrm{char}\, \Bbbk = 0$)**.** *Assume $e \geq 2$.*

1. (Calabri and Ciliberto ([4], Corollary 2); and Sommese, Verschelde, and Wampler ([22])) *As sets, $X \subseteq \mathrm{E}_{d-e+1}(X) \subseteq X \cup \mathrm{B}(X)$.*
2. *As schemes, X and $\mathrm{E}_{d-e+1}(X)$ are equal outside* $\mathrm{B}(X)$*,* $\mathrm{C}(X)$*, and* $\mathrm{Sing}\, X$*.*

As an application of Theorem 2, we have another proof of Theorem 1 in $\mathrm{char}(\Bbbk) = 0$, since a standard secant line L to X is not contained in $\mathrm{B}(X)$ (see Remark 3.6).

As another application, we will prove (B_{d-e+1}) for a special case:

Corollary 3 ($\mathrm{char}\, \Bbbk = 0$)**.** *Suppose that X $(\subseteq \mathbb{P}^N)$ with $e \geq 2$ is contained in the image $v_l(\mathbb{P}^m) \subseteq \mathbb{P}^M$ $(M = \binom{m+l}{l} - 1)$ of an lth $(l \geq 2)$ Veronese embedding v_l of a projective space $\mathbb{P}^m$ in $\mathbb{P}^M (\supseteq \mathbb{P}^N)$ for some $m > 0$. Then* $\mathrm{B}(X)$ *and* $\mathrm{C}(X)$ *are empty. Consequently $X = \mathrm{E}_{d-e+1}(X)$ as sets; and $X = \mathrm{E}_{d-e+1}(X)$ as schemes if X is smooth.*

Finally we will study the structure of $\mathrm{B}(X)$ and $\mathrm{C}(X)$. We show that $\mathrm{B}(X)$ is a closed subset of $\mathbb{P}^N \setminus X$ and that $\mathrm{C}(X)$ is a closed subset of $X \setminus \mathrm{Sing}\, X$ in (4.1) and (4.2). The set $\mathrm{B}(X)$ was first studied by Beniamino Segre [20] and [21], and later by Calabri and Ciliberto [4]. Segre [20] proved that the closure of $\mathrm{B}(X)$ is the union of a finite number of linear subspaces of dimension at most $n-1$ (see Theorem 4.3). Based on Segre's result, we say more about $\mathrm{B}(X)$ and $\mathrm{C}(X)$. Conventionally, we mean $\dim \emptyset = -1$.

Theorem 4 ($\operatorname{char} \Bbbk = 0$)**.** *If $e \geq 2$, then $\dim \mathrm{B}(X) \leq \min\{n-1, \dim \operatorname{Sing} X + 1\}$. In particular, if X is smooth (i.e., $\dim \operatorname{Sing} X = -1$) and $e \geq 2$, then $\mathrm{B}(X)$ is a finite set.*

Theorem 5 ($\operatorname{char} \Bbbk = 0$)**.** *Assume $e \geq 2$. Let Z be an irreducible component of $\mathrm{C}(X)$. Then the closure of Z is a linear subspace of dimension $l \leq \min\{n-1, \dim \operatorname{Sing} X + 2\}$.*

As a consequence of Theorems 2, 3, 4 and 5, we have the following.

Corollary 6 ($\operatorname{char} \Bbbk = 0$)**.** *Suppose that X is smooth, $n \geq 2$ and $e \geq 2$. Then $\mathrm{B}(X)$ is a finite set and $\mathrm{C}(X)$ is the union of a finite number of linear subspaces of dimension ≤ 1. Consequently the intersection of all hypersurfaces containing X, of degree $\leq d-e+1$, is equal to X as a scheme, except for a finite union of linear subspaces of dimension ≤ 1.*

Moreover we will show that the inequality in Theorem 5 is sharp by giving an example in (4.10). Also, in Theorem 4.11, we study the singular locus of X contained in the boundary of $\mathrm{C}(X)$.

We organize this paper as follows. In §1, we summarize some results of inner projections which we will use later. In §2, we prove Theorem 1. In §3, we prove Theorem 2 and Corollary 3. In §4, we look at the structure of $\mathrm{B}(X)$ and $\mathrm{C}(X)$ and prove Theorems 4.4, which is a strong form of Theorem 4, and Theorem 5.

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Notation. We use standard terminology from algebraic geometry, e.g, [11]. By a point, we always mean a closed point. A general point means a point off a finite union of suitable proper closed subvarieties. For a point x of X, by $T_x(X)$, we denote the embedded tangent space to X at x in $\mathbb{P}^N$. By $\langle Y, Z \rangle$, we denote the linear span of subschemes Y and Z of $\mathbb{P}^N$, the smallest linear subspace containing both Y and Z. By $\operatorname{Sing} X$ (resp. $\operatorname{Sm} X$), we denote the singular locus (resp. smooth locus) of X.

§1. Inner projection

In this section, we summarize some results of inner projections.

1.1. Let $X \subseteq \mathbb{P}^N$ be as in §0. Let $\Lambda \subseteq \mathbb{P}^N$ be a linear subspace of dimension l with $\Lambda \cap X \neq \emptyset$ but $\Lambda \not\supseteq X$. The linear projection $\pi_\Lambda : \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{N-l-1}$ from Λ induces a morphism $\pi_{\Lambda,X} : X \setminus \Lambda \to \mathbb{P}^{N-l-1}$. Let V be the linear space of linear forms on $\mathbb{P}^N$, and let $W \subseteq V$ be the subspace of linear forms vanishing on Λ, i.e., $\mathbb{P}^N = \mathbb{P}(V)$ and $\mathbb{P}^{N-l-1} = \mathbb{P}(W)$. Then $\pi_{\Lambda,X}$ is defined by the surjection $\varepsilon : W \otimes \mathcal{O}_X \to \mathcal{I}_{X\cap\Lambda/X} \otimes \mathcal{O}_X(1)$ induced from $V \otimes \mathcal{O}_X \to \mathcal{O}_X(1)$. We can identify $\mathbb{P}^{N-l-1}$ to be a subspace of $\mathbb{P}^N$ disjoint from Λ. Let $\sigma : \hat{X} \to X$ be the blowing up of X by the ideal sheaf $\mathcal{I}_{X\cap\Lambda/X}$ of $\mathcal{O}_X$, and let E be the exceptional Cartier divisor. Then we have the morphism $\hat{\pi}_{\Lambda,X} : \hat{X} \to \mathbb{P}^{N-l-1}$ with $\hat{\pi}_{\Lambda,X} \circ \sigma = \pi_{\Lambda,X}$ as a rational map, since ε induces a surjection $\hat{\varepsilon} : W \otimes \mathcal{O}_{\hat{X}} \to \mathcal{O}_{\hat{X}}(-E) \otimes \sigma^*\mathcal{O}_X(1)$. Thus the closure $\bar{X}$ of $\pi_\Lambda(X \setminus \Lambda)$ is exactly $\hat{\pi}_{\Lambda,X}(\hat{X})$ and hence

$$\bar{X} = \pi_\Lambda(X \setminus \Lambda) \cup \hat{\pi}_{\Lambda,X}(E), \quad \text{as a set.} \tag{1.1.1}$$

1.1.2. Assume $\dim \bar{X} = \dim X$. Then $Y := \{\bar{x} \in \bar{X} \mid \dim \hat{\pi}^{-1}_{\Lambda,X}(\bar{x}) \geq 1\} \cup \hat{\pi}_{\Lambda,X}(E)$ is a proper closed subset of $\bar{X}$. Thus the induced morphism $\hat{X} \setminus \hat{\pi}^{-1}_{\Lambda,X}(Y) \to \bar{X} \setminus Y$ from $\hat{\pi}_{\Lambda,X}$ is finite, and hence for the open set $U := \sigma(\hat{X} \setminus \hat{\pi}^{-1}_{\Lambda,X}(Y))$ of X, $\pi_{\Lambda,X}|U$ is finite.

1.1.3. On the other hand, assume that $e \geq 2$ and $\Lambda = \{z\}$ for a smooth point z of X. Then (1) $\dim \bar{X} = \dim X$, since X is not a cone with vertex z nor linear; (2) $\hat{\pi}_{z,X}(E) = \pi_z(T_z(X) \setminus \{z\})$, since the map $H^0(\hat{\varepsilon} \otimes \mathcal{O}_E)$ is $\varepsilon \otimes \Bbbk(z) : W \to m_z/m_z^2$ for the maximal ideal m_z of $\mathcal{O}_{X,z}$; (3) $0 \leq \deg \bar{X} \leq d-1$ by Bézout's theorem or by computing the degree of $\mathcal{O}_{\hat{X}}(-E) \otimes \sigma^*\mathcal{O}_X(1)$; (4) For a point $x\ (\neq z) \in X$, $l(X \cap \langle z, x\rangle) = l(\hat{\pi}^{-1}_{z,X}(\pi_{z,X}(x))) + 1$ by a local computation (see for example, [7], p.269, (8.4.3)). In particular, if $l(X \cap \langle z, x\rangle) = 2$ for a point $x\ (\neq z) \in X$, then $\pi_{z,X}$ is an embedding at x by (1.1.2).

1.2. Returning to the original situation in (1.1), we look at the behavior of tangent spaces under the projection. By $\pi : X \setminus \Lambda \to \bar{X}$ we denote the induced morphism from $\pi_{\Lambda,X}$. Let x be a smooth point of X with $x \notin \Lambda$. Assume $\bar{x} := \pi_\Lambda(x) \in \operatorname{Sm} \bar{X}$. By comparing the bundles of the principal part with respect to $\mathcal{O}_X(1)$ and $\mathcal{O}_{\bar{X}}(1)$ (see [12], (IV.A)), we have $\pi_\Lambda(T_x(X) \setminus \Lambda) \subseteq T_{\bar{x}}(\bar{X})$, and the equality holds if and only if π is smooth at x. If the equality holds, then $\langle T_{\bar{x}}(\bar{X}), \Lambda\rangle = \langle T_x(X), \Lambda\rangle$, and hence

(1.2.1) $$\dim \bar{X} = \dim \pi_\Lambda(T_x(X) \setminus \Lambda) = n - \dim(T_x(X) \cap \Lambda) - 1$$

and

(1.2.2) $$T_{x'}(X) \subseteq \langle T_x(X), \Lambda\rangle \quad \text{for each} \quad x' \in X_{\bar{x}} \cap \operatorname{Sm} X,$$

where $X_{\bar{x}}$ is the closure of $\pi_{\Lambda,X}(\bar{x})$. In particular, if $\operatorname{char}(\Bbbk) = 0$, by the generic smoothness of $\pi|_{\operatorname{Sm} X \setminus \Lambda}$ (e.g. [11], III.10.7), the above holds for general $x \in \operatorname{Sm} X$ and its image $\bar{x} = \pi_\Lambda(x)$.

§2. Multisecant lines: Proof of Theorem 1

In this section we will prove Theorem 1. The key is to find a hypersurface F of degree $\leq d$, containing X, and meeting L in at least $e-1$ distinct points off $X \cap L$.

2.1. Proof of Theorem 1. First we claim that if $M \subseteq \mathbb{P}^N$ is a general e-dimensional linear subspace containing L, then $M \cap X$ is finite and containing at least $e-1$ distinct points off L. To prove this, consider the linear projection $\pi_L : \mathbb{P}^N \setminus L \to \mathbb{P}^{N-2}$ from L. The closure $\bar{X}$ of $\pi_L(X \setminus L)$ has dimension n by assumption. Since X is nondegenerate, then so is $\bar{X} \subseteq \mathbb{P}^{N-2}$. Thus $\bar{d} := \deg \bar{X} \geq e-1$ (see for example [10], (18.12); [8], (I.4.2)). Since $\bar{M} := \pi_L(M \setminus L)$ is a general $(e-2)$-dimensional linear subspace of $\mathbb{P}^{N-2}$, by Bézout's and Bertini's Theorems, $\bar{X} \cap \bar{M}$ is $\bar{d}$ distinct points, contained in a nonempty open subset U of $\bar{X}$ over which the induced morphism $\pi_{L,X}$ is a finite morphism (see (1.1.2) for U). This implies the claim.

Let $M \subseteq \mathbb{P}^N$ be a general e-dimensional linear subspace containing L. By the first part, as sets, $(M \setminus L) \cap X = \{x_1, \ldots, x_m\}$ and $L \cap X = \{y_1, \ldots, y_t\}$ for some distinct points x_j and y_i with

(2.1.1) $$m\ (\geq \bar{d}) \geq e-1.$$

Take a general $(e-2)$-dimensional linear subspace $\Lambda \subseteq M$. We may assume that Λ is disjoint from X, L, and lines $\langle y_i, x_j \rangle$ and $\langle x_j, x_k \rangle$ $(i = 1, \dots, t;\ j \neq k = 1, \dots, m)$. Consequently, $\langle \Lambda, x_j \rangle \cap L$ is a point, say z_j, and $\{z_j\}$ are m distinct points off $X \cap L$. Now consider the linear projection $\pi_\Lambda : \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{n+1}$ from Λ. The image $X' := \pi_\Lambda(X) \subseteq \mathbb{P}^{n+1}$ is a hypersurface of degree d' $(\leq d)$, since the projection $\pi_{\Lambda,X} : X \to \mathbb{P}^{n+1}$ is finite. Let F be the hypersurface obtained by the pulling-back of $X' \subseteq \mathbb{P}^{n+1}$ by π_Λ. In other words, F is the cone over X' with vertex Λ if we consider $\mathbb{P}^{n+1}$ to be a subspace of $\mathbb{P}^N$, disjoint from Λ. Thus $\deg F = d' \leq d$, $F \supseteq X$ and $F \cap L \supseteq (X \cap L) \cup \{z_1, \dots, z_m\}$. Moreover $F \not\supseteq L$, since the closure of $\pi_\Lambda^{-1}(\pi_\Lambda(L))$ is $\langle L, \Lambda \rangle = M$ and $M \cap X$ is finite. Consequently $d' = l(F \cap L) \geq l(X \cap L) + m$, and hence

$$l(X \cap L) \leq d' - m \leq d - m \leq d - e + 1. \qquad (2.1.2) \qquad \square$$

Remark 2.2. Under the same notation and assumptions as in Theorem 1, if $l(X \cap L) = d - e + 1$, then the closure $\bar{X}$ of $\pi_L(X \setminus L)$ is a variety of minimal degree, i.e., $\bar{d} = \bar{e} + 1$ (see [10], (19.9); [8], (I.2.2), (I.5.10)). Furthermore, if $\mathrm{char}(\Bbbk) = 0$, then $\pi : X \setminus L \to \bar{X}$ is separable and hence $\bar{X}$ is birational to X. Indeed, if $l(X \cap L) = d - e + 1$, then it follows from (2.1.2) that $m = e - 1$ and $d' = d$. Hence $m = \bar{d} = e - 1$ by (2.1.1), as required.

Remark 2.3. If a secant line L meets X only at $\mathrm{Sm}\, X$ and $e \geq 2$, by a Bertini-type theorem, $X \cap H$ is irreducible and reduced for a general hyperplane $H \supseteq L$, and hence L is standard (see [19], Lemma for the proof; see also [14] and [18], Lemma 2.1). On the other hand, if X has bad singularity at $X \cap L$, for a general hyperplane $H \supseteq L$ of $\mathbb{P}^N$, $X \cap H$ is not necessarily irreducible (see [18], Example 2.2(2)). Moreover, if X is integral and regular in codimension 1, but not normal at $x \in X \cap L$, then X has Serre's condition S_1 but not S_2 at x (see [1], VII (2.2) and (2.13)), and hence $X \cap H$ is not S_1 at x, i.e., $X \cap H$ is not reduced. Thus to obtain Theorem 1, the method of taking hyperplane sections used in [2], [14], and [19] does not work in case $X \cap L \not\subseteq \mathrm{Sm}\, X$.

§3. Hypersurfaces containing projective variety: Proof of Theorem 2

Let $X \subseteq \mathbb{P}^N$ be as in §0. In this section, we always assume $\mathrm{char}(\Bbbk) = 0$ and will prove Theorem 2. This assumption is necessary only to apply the trisecant lemma, which asserts for general points x, y of X, $l(X \cap \langle x, y \rangle) = 2$ if $e \geq 2$ and $\mathrm{char}(\Bbbk) = 0$: In fact, the set of points $x \neq y \in X$ with $l(X \cap \langle x, y \rangle) \geq 3$ is a closed subset of $X \times X \setminus \Delta_X$, and also a proper subset by the trisecant lemma (see for example [11], IV 3.8) for curves which are obtained by hyperplane sections of X.

Theorem 2 is reduced to proving Propositions 3.1 and 3.2. For scheme-theoretic part (2), see [17], p.34, Lemma.

Proposition 3.1 ($\mathrm{char}\, \Bbbk = 0$)**.** *Let v be a point of $\mathbb{P}^N \setminus X$. Suppose $v \notin \mathrm{B}(X)$ if $e \geq 2$. Then there exists a hypersurface F of degree $\leq d - e + 1$ such that $F \supseteq X$ but $v \notin F$.*

Proof. When $e = 1$, this is clear. By induction on e, suppose $e \geq 2$. Let x be a general point of X, so that $x \in \mathrm{Sm}\, X$ and $l(X \cap \langle v, x \rangle) = 1$ by $v \notin \mathrm{B}(X)$. Consider the projection $\pi_x : \mathbb{P}^N \setminus \{x\} \to \mathbb{P}^{N-1}$ from x and let $\bar{X}$ be the closure of $\pi_x(X \setminus \{x\})$. By (1.1.1) and (1.1.3), $\bar{X}$ is a nondegenerate projective variety of degree $\bar{d} \leq d - 1$, and codimension $\bar{e} = e - 1$, and $\bar{X} = \pi_x(X \setminus \{x\}) \cup \pi_x(T_x(X) \setminus \{x\})$. Thus $\bar{v} := \pi_x(v) \notin \bar{X}$, since $l(X \cap \langle v, x\rangle) = 1$. We assume, for a moment, that $\bar{v} \notin \mathrm{B}(\bar{X})$ for $\bar{e} = e - 1 \geq 2$, and will complete the induction. By the induction, we have a hypersurface $\bar{F} \subseteq \mathbb{P}^{N-1}$, of degree $\leq \bar{d} - \bar{e} + 1 (\leq d - e + 1)$ such that $\bar{F} \supseteq \bar{X}$ but $\bar{v} \notin \bar{F}$. Let F be the hypersurface obtained by the pulling-back of $\bar{F}$ by π_x. Then $\deg F = \deg \bar{F} \leq d - e + 1$, $F \supseteq X$ and $v \notin F$, since F is the cone over $\bar{F}$ with vertex x, as required.

To conclude the proof, we will show that $\bar{v} \notin \mathrm{B}(\bar{X})$ for $e \geq 3$. By contradiction, we assume that for a general point $y \in X$ with $\bar{y} := \pi_x(y) \in \operatorname{Sm} \bar{X}$,

$$l(\langle \bar{v}, \bar{y}\rangle \cap \bar{X}) \geq 2. \tag{3.1.1}$$

By the generality of x and y and the trisecant lemma, the projection $\pi_{x,X} : X \setminus \{x\} \to \mathbb{P}^{N-1}$ is an embedding at y (see (1.1.3)(4)), and consequently $T_{\bar{y}}(\bar{X}) = \pi_x(T_y(X))$ (see (1.2)). Moreover, $v \notin T_x(X)$, $v \notin T_y(X)$, $x \notin \langle T_y(X), v\rangle$, and $y \notin \langle T_x(X), v\rangle$ since $v \notin \mathrm{B}(X)$ and x and y are general in nondegenerate X, and consequently

$$\bar{v} \notin T_{\bar{y}}(\bar{X}) = \pi_x(T_y(X)) \quad \text{and} \quad \langle \bar{y}, \bar{v}\rangle \cap \pi_x(T_x(X) \setminus \{x\}) = \emptyset. \tag{3.1.2}$$

By (3.1.1) and (3.1.2), there is a point $\bar{z} \in \langle \bar{v}, \bar{y}\rangle \cap \bar{X}$ distinct from $\bar{y}$ which is an image $\pi_x(z)$ of some $z \in X$. This means that $\langle v, x, y\rangle$ contains $z \in X$ off $\langle v, x\rangle$ and $\langle v, y\rangle$, since x and y are general and $v \notin \mathrm{B}(X)$. Thus for the linear projection $\pi_v : \mathbb{P}^N \setminus \{v\} \to \mathbb{P}^{N-1}$ from v, the image $\pi_v(X)$ has a secant line meeting at the three distinct points $\pi_v(x)$, $\pi_v(y)$ and $\pi_v(z)$. This contradicts the trisecant lemma, since $\pi_v(x)$ and $\pi_v(y)$ are general points of $\pi_v(X)$ because of the generality of x and y in X. □

Proposition 3.2 (char $\Bbbk = 0$). *Let u be a smooth point of X with embedded tangent space $T_u(X) \subseteq \mathbb{P}^N$ and let w be a point of $\mathbb{P}^N \setminus T_u(X)$. Suppose $u \notin \mathrm{C}(X)$ if $e \geq 2$. Then there exists a hypersurface F containing X, of degree $\leq d - e + 1$, such that F is smooth at u with $w \notin T_u(F) \subseteq \mathbb{P}^N$.*

Proof. When $e = 1$, this is clear. By induction, suppose $e \geq 2$. Let x be a general point of X. Then $x \in \operatorname{Sm} X$ with $l(X \cap \langle x, u\rangle) = 2$ and $x \notin \langle T_u(X), w\rangle$. Consider the projection $\pi_x : \mathbb{P}^N \setminus \{x\} \to \mathbb{P}^{N-1}$ from x, and let $\bar{X}$ be the closure of $\pi_x(X \setminus \{x\})$. The projection $\pi_{x,X} : X \setminus \{x\} \to \mathbb{P}^{N-1}$ is an embedding at u (see (1.1.3)), and hence $\bar{u} := \pi_x(u) \in \operatorname{Sm} \bar{X}$, and $\bar{w} := \pi_x(w) \notin T_{\bar{u}}(\bar{X}) = \pi_x(T_u(X))$ (see (1.2)). Moreover $\bar{X}$ is a nondegenerate projective variety of degree $\bar{d} \leq d - 1$, dimension $\bar{n} = n$, and codimension $\bar{e} = e - 1$, with $\bar{X} = \pi_x(X \setminus \{x\}) \cup \pi_x(T_x(X) \setminus \{x\})$ (see (1.1.1) and (1.1.3)). We assume, for a moment, that $\bar{u} \notin \mathrm{C}(\bar{X})$ for $e \geq 3$, and will complete the induction. By induction, we have a hypersurface $\bar{F} \subseteq \mathbb{P}^{n+e-1}$ containing $\bar{X}$, of degree $\leq \bar{d} - \bar{e} + 1 (\leq d - e + 1)$, smooth at $\bar{u}$ with $\bar{w} \notin T_{\bar{u}}(\bar{F})$. The hypersurfaces F obtained by the pulling-back of $\bar{F}$ by π_x is of degree $\leq \bar{d} - \bar{e} + 1$, smooth at u with $w \notin T_u(F)$, as required.

To conclude our proof, we will show that $\bar{u} \notin \mathrm{C}(\bar{X})$ for $e \geq 3$. By contradiction, assume that for general $y \in X$ with $\bar{y} := \pi_x(y) \in \operatorname{Sm} \bar{X}$, $l(\langle \bar{y}, \bar{u}\rangle \cap \bar{X}) \geq 3$. By the generality of x and y and the trisecant lemma, $\pi_{x,X}$ is an embedding at y, and hence $T_{\bar{y}}(\bar{X}) = \pi_x(T_y(X))$. Moreover, $y \notin \langle T_u(X), x\rangle$, $y \notin \langle T_x(X), u\rangle$, and $x \notin \langle T_y(X), u\rangle$. Hence $\bar{y} \notin T_{\bar{u}}(\bar{X})$, $\bar{y} \notin \langle \bar{u}, \pi_x(T_x(X) \setminus \{x\})\rangle$, and $\bar{u} \notin T_{\bar{y}}(\bar{X})$. Consequently there is a point $\bar{z} \in \bar{X} \cap \langle \bar{y}, \bar{u}\rangle$ distinct from $\bar{y}$ and $\bar{u}$ with $\bar{z} = \pi_x(z)$ for some $z \in X$. This means that z lies on $\langle u, x, y\rangle$, but off $\langle u, x\rangle$ and $\langle u, y\rangle$ by

$u \notin \mathrm{C}(X)$. Thus for the projection $\pi_u : \mathbb{P}^N \setminus \{u\} \to \mathbb{P}^{N-1}$ from u, the closure X' of $\pi_u(X \setminus \{u\})$ has a general secant line $\langle \pi_u(x), \pi_u(y) \rangle$ meeting at the three distinct points, which contradicts the trisecant lemma. □

Remark 3.3. If $v \in \mathrm{B}(X)$, then $\pi_x(v) \in \pi_x(X \setminus \{x\})$ for any $x \in X$. Thus the points of $\mathrm{B}(X)$ cannot separate from X by the hypersurfaces obtained in (3.1). On the other hand, if $u \in \mathrm{C}(X)$, then $\pi_x(u) \in \operatorname{Sing} \pi_x(X \setminus \{x\})$ for any $x \in X$. Thus at the points of $\mathrm{C}(X)$, the hypersurfaces in (3.2) cannot separate the tangent space of X from $\mathbb{P}^N$.

Example 3.4. Let $X \subseteq \mathbb{P}^N$ $(N = n + e)$ be a projective variety of dimension n, degree d, and codimension e over an algebraically closed field $\Bbbk$ of $\operatorname{char} \Bbbk = 0$. Assume X is a variety of minimal degree, or of delta genus $\Delta(X) = 0$, i.e., $d = e+1$ (see [10], (19.9); [8], (I.2.2), (I.5.10)). Assume $e \geq 2$, or equivalently $d \geq 3$. Then $\mathrm{B}(X) = \emptyset$. Moreover, X has no 3-secant line and hence $\mathrm{C}(X) = \emptyset$.

Proof. Suppose $\mathrm{B}(X) \neq \emptyset$, and we will show that $e = 1$. Consider the projection $\pi_v : \mathbb{P}^N \setminus \{v\} \to \mathbb{P}^{N-1}$ from a point $v \in \mathrm{B}(X)$. Set $\bar{X} = \pi_v(X)$ and $\bar{d} = \deg \bar{X}$. Then $\bar{d} \leq d/2$ $(= (e+1)/2)$. Since $\bar{X}$ is nondegenerate in $\mathbb{P}^{N-1}$, we have $\bar{d} \geq N - n = e$ by [10], (18.12) or [8],(I.4.2). From these two inequalities, we obtain $e \leq 1$, and hence $e = 1$, as required. The second part follows from the first part and Theorem 2, noting that $d - e + 1 = 2$. (Or directly, if there is a 3-secant line L, consider the projection from L, and follow the same argument as above. Then we have a contradiction $N = n$.) □

3.5. Proof of Corollary 3. We have only to show that $\mathrm{B}(X)$ and $\mathrm{C}(X)$ are empty. First we will prove $\mathrm{B}(X) = \emptyset$. By contradiction, assume there is a point $v \in \mathrm{B}(X)$. For general $x_1 \neq x_2 \in X$, consider the lines in $\mathbb{P}^M$ joining x_i and v, and observe on the lines that there exist points $y_i \in X$ different from x_i. Let L_i $(i = 1, 2)$ be the line in $\mathbb{P}^m$ joining the preimages of x_i and y_i in $\mathbb{P}^m$. Note that $v \in \langle v_l(L_i) \rangle$. If $L_1 = L_2$, then $X = v_l(L_1)$ with $\mathrm{B}(X) \neq \emptyset$, and hence X is conic by Example 3.4, a contradiction. Thus $L_1 \neq L_2$. On the other hand, it is easy to see for two distinct lines ℓ_1, ℓ_2 in $\mathbb{P}^m$ that

(1) $\langle v_l(\ell_1) \rangle \cap \langle v_l(\ell_2) \rangle = \emptyset$ if ℓ_1 and ℓ_2 are disjoint, and that
(2) $\langle v_l(\ell_1) \rangle \cap \langle v_l(\ell_2) \rangle = v_l(\ell_1 \cap \ell_2)$ if $\ell_1 \cap \ell_2 \neq \emptyset$.

Therefore $v \in v_l(L_1 \cap L_2)$. This implies that $v_l(L_1)$ has a 3-secant line $\langle v, x_i \rangle = \langle x_i, y_i \rangle$, which contradicts Example 3.3, as required. The second part $\mathrm{C}(X) = \emptyset$ follows from the same argument above for $u \in \mathrm{C}(X)$ instead of $v \in \mathrm{B}(X)$. □

Remark 3.6. Proposition 3.1 implies Theorem 1 in case $\operatorname{char} \Bbbk = 0$: Indeed, since L is standard, $L \setminus X \not\subseteq \mathrm{B}(X)$ by the argument as in the proof of (4.4). By (3.1), we obtain a hypersurface F containing X but not L. Consequently $l(X \cap L) \leq l(F \cap L) \leq d - e + 1$.

§4. Study of B(X) and C(X): Proof of Theorems 4 and 5

Let $X \subseteq \mathbb{P}^N$ be as in §0. In this section, we assume $\operatorname{char} \Bbbk = 0$. We will study the structure of $\mathrm{B}(X)$ and $\mathrm{C}(X)$ in (0.1), and prove Theorems 4 and 5. As a consequence, we will obtain Corollary 6.

First we will show that $\mathrm{B}(X)$ and $\mathrm{C}(X)$ are algebraic sets.

Lemma 4.1. $\mathrm{B}(X)$ *is a closed subset of* $\mathbb{P}^N \setminus X$.

Proof. For a hyperplane $H \subseteq \mathbb{P}^N$, we set $U := \mathbb{P}^N \setminus (X \cup H)$. We have only to show that $\mathrm{B}(X) \cap U$ is closed in U. To this purpose, consider the morphism $\varpi : U \times X \to U \times \mathbb{P}^{N-1}$ defined by $(u,x) \mapsto (u, \langle u,x\rangle \cap H)$, that is, the family of the projections $\pi_{u,X} : X \to \mathbb{P}^{N-1}$ from $u \in U$ to $H \cong \mathbb{P}^{N-1}$. Note that ϖ is projective, since $U \times X \to U$ is projective and $U \times \mathbb{P}^{N-1} \to U$ is separated (see [11], Ex.II.4.9). Moreover ϖ is finite, since ϖ is quasi-finite (see [11], Ex.III.11.2). Hence $\mathcal{W} := \{(u,\bar{x}) \in U \times \mathbb{P}^{N-1} | \dim_{\Bbbk(u,\bar{x})} \varpi_* \mathcal{O}_{U\times X} \otimes \Bbbk(u,\bar{x}) \geq 2\}$ is the set of points $(u,\bar{x}) \in U \times \mathbb{P}^{N-1}$ whose fibre $\varpi^{-1}(u,\bar{x})$ $(\cong \pi_u^{-1}(\bar{x}))$ is of length at least 2. Moreover $\mathcal{W}$ is closed in $U \times \mathbb{P}^{N-1}$ by [11], Ex.II.5.8, and hence the first projection $p_1 : \mathcal{W} \to U$ is projective. Therefore a point $u \in U$ is contained in $\mathrm{B}(X) \cap U$ if and only if the fibre $\mathcal{W}_u := p_1^{-1}(u)$ is dense in $\pi_u(X)$, i.e., $\mathcal{W}_u = \pi_u(X)$. Then $\mathcal{W}_u = \pi_u(X)$ if and only if $\dim \mathcal{W}_u = n$, since X is irreducible and π_u is finite. Thus $\mathrm{B}(X) \cap U = \{u \in U | \dim \mathcal{W}_u \geq n\}$, and hence $\mathrm{B}(X) \cap U$ is closed in U, by [11], Ex.II.3.22 (d) and the properness of p_1, as required. □

Lemma 4.2. $\mathrm{C}(X)$ *is a closed subset of the smooth locus* $\mathrm{Sm}\, X$ *of* X.

Proof. For a hyperplane $H \subseteq \mathbb{P}^N$, we have only to show that $\mathrm{C}(X) \setminus H$ is closed in $X_0 := \mathrm{Sm}\, X \setminus H$. Recall that the linear projection $\pi_{z,X} : X \setminus \{z\} \to \mathbb{P}^{N-1}$ from $z \in X_0$ to $H \cong \mathbb{P}^{N-1} \subseteq \mathbb{P}^N$ is extendable to the morphism $\hat{\pi}_{z,X} : \hat{X}_z \to \mathbb{P}^{N-1}$ by taking the blowing-up $\sigma_z : \hat{X}_z \to X$ of X at z (see (1.1)). To construct the family of $\hat{\pi}_{z,X}$ over $z \in X_0$, consider the family of linear projections $(\mathbb{P}^N \setminus H) \times \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1} \cong H \subseteq \mathbb{P}^N$, $(z,x) \mapsto \langle z,x\rangle \cap H$ whose base locus is the diagonal of $\mathbb{P}^N \setminus H$. Its restriction $\varpi_2 : \mathcal{X} := X_0 \times X \dashrightarrow \mathbb{P}^{N-1}$ is a rational map whose base locus is the diagonal Δ_{X_0} of X_0. By taking the blowing-up $\sigma : \widehat{\mathcal{X}} \to \mathcal{X}$ by the ideal sheaf of Δ_{X_0} with reduced structure, we have an extension $\hat{\varpi}_2 : \widehat{\mathcal{X}} \to \mathbb{P}^{N-1}$ of ϖ_2. Since $\mathcal{J}_{\Delta_{X_0}/\mathcal{X}} | \{z\} \times X \cong \mathcal{J}_{\{z\}/X}$ by a local computation, σ is the family of blowing-ups σ_z for $z \in X_0$. Thus $\hat{\varpi} := (p_1 \circ \sigma) \times \hat{\varpi}_2 : \widehat{\mathcal{X}} \to X_0 \times \mathbb{P}^{N-1}$ is the required family such that $\hat{\varpi}^{-1}(z,\bar{x}) = \hat{\pi}_{z,X}^{-1}(\bar{x})$ for $(z,\bar{x}) \in X_0 \times \mathbb{P}^{N-1}$. Note that $\hat{\varpi}$ is projective, since $\widehat{\mathcal{X}} \to X_0$ is projective and $X_0 \times \mathbb{P}^{N-1} \to X_0$ is separated.

To prove $\mathrm{C}(X) \setminus H$ is closed in X_0, we consider

$$\mathcal{W} := \{(z,\bar{x}) \in X_0 \times \mathbb{P}^{N-1} | \dim_{\Bbbk(z,\bar{x})} \hat{\varpi}_* \mathcal{O}_{\widehat{\mathcal{X}}} \otimes \Bbbk(z,\bar{x}) \geq 2 \ \text{ or } \ \dim \hat{\pi}_{z,X}^{-1}(\bar{x}) \geq 1\}.$$

By [11], Ex. II.5.8, and Ex. II.3.22 (d), $\mathcal{W}$ is closed in $X_0 \times \mathbb{P}^{N-1}$, and the first projection $p_1 : \mathcal{W} \to X_0$ is projective. Moreover $\mathcal{W}$ is the set of points $(z,\bar{x}) \in X_0 \times \mathbb{P}^{N-1}$ whose fibre $\hat{\varpi}^{-1}(z,\bar{x})$ $(\cong \hat{\pi}_{z,X}^{-1}(\bar{x}))$ is of length at least 2, since $\hat{\varpi}$ is finite around $(z,\bar{x})$ if $\dim \hat{\varpi}^{-1}(z,\bar{x}) = 0$. A point $(z,\bar{x}) \in X_0 \times \mathbb{P}^{N-1}$ belongs to $\mathcal{W}$ if and only if $l(X \cap \langle z,\bar{x}\rangle) \geq 3$, since $l(\hat{\pi}_{z,X}^{-1}(\bar{x})) \geq 2$ if and only if $l(X \cap \langle z,\bar{x}\rangle) \geq 3$ (see (1.1.3)). Thus $z \in X_0$ belongs to $\mathrm{C}(X) \setminus H$ if and only if the fibre $\mathcal{W}_z := p_1^{-1}(z)$ is equal to $\bar{X}$. The last condition is equivalent to $\dim \mathcal{W}_z \geq n$. Therefore $\mathrm{C}(X) \setminus H$ is closed in X_0, as required. □

Next we study the structure of $\mathrm{B}(X)$. Recall the following result.

Theorem 4.3 (Beniamino Segre [20]; see also [4])**.** *Let* $X \subseteq \mathbb{P}^N$ $(N = e + n)$ *be a nondegenerate, projective variety of dimension* n *and codimension* e. *If* $e \geq 2$ *and* $\mathrm{B}(X) \neq \emptyset$, *then every irreducible component of the closure of* $\mathrm{B}(X)$ *is a linear subspace of dimension at most* $n-1$. *Moreover* $\dim \mathrm{B}(X) = n+1$ *if and only if* $e = 1$.

Based on this result and the idea of the proof, we will prove Theorem 4.4 and Theorem 5. Theorem 4 is an immediate consequence of Theorem 4.4.

Theorem 4.4. *Let X be as in §0. Assume $n \geq 2$ and $e \geq 2$. Let Λ be an irreducible component of the closure of* $\mathrm{B}(X)$*, of dimension $l \geq 0$. Then*

(1) $\dim X \cap \Lambda = l - 1$*, and*
(2) $X \cap \Lambda \subseteq \operatorname{Sing} X$.

Consequently $\dim \Lambda \leq \dim \operatorname{Sing} X + 1$*, and in particular,* $\dim \mathrm{B}(X) \leq \dim \operatorname{Sing} X + 1$.

Proof. When $l = 0$, the assertion is trivial, so we assume $l \geq 1$. By (4.3), Λ is linear. First we observe the linear projection $\pi_{\Lambda,X} : X \setminus \Lambda \to \mathbb{P}^{N-l-1}$ from Λ, in particular, its fibre and image. For a general point $x \in X \setminus \Lambda$, let $X_{\bar{x}}$ be the closure of $\pi_{\Lambda,X}^{-1}(\bar{x})$ over $\bar{x} := \pi_\Lambda(x)$. Then $\dim X_{\bar{x}} = l$. Indeed, $\dim X_{\bar{x}} \geq l$, since a line joining x and general $v \in \Lambda \cap \mathrm{B}(X)$ contains a point of X different from x. Hence $\dim X_{\bar{x}} = l$, since $\pi_{\Lambda,X}^{-1}(\bar{x}) \subseteq \langle \Lambda, x \rangle$ and $\Lambda \not\subseteq X$. Consequently the closure $\bar{X}$ of $\pi_{\Lambda,X}(X \setminus \Lambda)$ has dimension $n - l$. Moreover $\bar{d} := \deg \bar{X} \geq 2$: If not, $\bar{X}$ is linear and nondegenerate in $\mathbb{P}^{N-l-1}$, and hence $\bar{X} = \mathbb{P}^{N-l-1}$, which implies $e = 1$, a contradiction.

Now (1) is clear, since $\dim X_{\bar{x}} = l$ and $X_{\bar{x}} \subseteq \langle \Lambda, x \rangle$ with $\Lambda \not\subseteq X$.

Next we will show (2) in case $n = 2$. Then $l = 1$ by assumptions $l \geq 1$ and (4.3). Hence $\dim X_{\bar{x}} = 1$ and $\bar{X} \subseteq \mathbb{P}^{N-2}$ is a nondegenerate curve of degree $\bar{d} \geq 2$. Let H be a general hyperplane containing Λ. The first step is to show that

$$\operatorname{Sing}(X \cap H) \supseteq X \cap \Lambda. \tag{4.4.1}$$

Since $\bar{H} := \pi_\Lambda(H \setminus \Lambda)$ is a general hyperplane in $\mathbb{P}^{N-2}$, by Bézout's theorem, $\bar{X} \cap \bar{H}$ is $\bar{d}$ distinct points, say $\bar{x}_1, \ldots, \bar{x}_{\bar{d}}$, which lie on $\pi_{\Lambda,X}(X \setminus \Lambda)$ (see (1.1.1)). Consequently $X \cap H$ contains $\bar{d}$ (≥ 2) distinct curves $X_{\bar{x}_i}$, each of which contains $X \cap \Lambda$ by Lemma 4.5 below. This implies (4.4.1). Now to obtain (2), we assume, to the contrary, that there is a point $z \in \Lambda \cap \operatorname{Sm} X$. Then $H \not\supseteq T_z(X)$ by the generality of $H \supseteq \Lambda$ and $\Lambda \neq T_z(X)$. Hence $X \cap H$ is smooth at z. This contradicts (4.4.1).

We will show (2) in case $n > 2$. By contradiction, assume that there is a point $z \in \Lambda \cap \operatorname{Sm} X$. Take a general line L through z, contained in Λ, not contained in X, so that $L \cap \mathrm{B}(X)$ is dense open in L by (4.1). Let H' be a general hyperplane containing L. We claim that the reduced induced structure $X' := (X \cap H')_{\mathrm{red}}$ is irreducible and nondegenerate in $H' \cong \mathbb{P}^{N-1}$ such that $z \in \operatorname{Sm} X'$ and the closure of $\mathrm{B}(X')$ contains L. If the claim is proved, by induction on n, we have a projective surface X' such that the closure of $\mathrm{B}(X')$ contains L with $L \cap \operatorname{Sm} X' \neq \emptyset$, which contradicts the case $n = 2$. Thus we have only to show the claim. By the same argument as in the first paragraph, a general fibre of the linear projection $\pi_{L,X} : X \setminus L \to \mathbb{P}^{N-2}$ has dimension 1. Hence the image of $\pi_{L,X}$ has dimension $n - 1$ (≥ 2). By Bertini's theorem [24], (I.6.3), $X \cap H'$ is irreducible and generically reduced. (To be precise, apply Bertini's theorem for the normalization $\tilde{X}$ of X and the pull-back $\tilde{H}'$ of H', and take the push-forward of $\tilde{X} \cap \tilde{H}'$.) Moreover, since X is integral, $(X \setminus L) \cap H'$ satisfies Serre's condition S_1 by [7], (3.4.6), and hence $(X \setminus L) \cap H'$ is reduced (see [1], (VII.2.2)). On the other hand, since $X \subseteq \mathbb{P}^N$ is nondegenerate, so is $\pi_{L,X}(X \setminus L) \subseteq \mathbb{P}^{N-2}$, and also so is its hyperplane section (see [7], p.116). Consequently $(X \setminus L) \cap H'$ is nondegenerate. Moreover, $X \cap H'$

is smooth at $z \in \Lambda \cap \operatorname{Sm} X$, since $H' \not\supseteq T_z(X)$ by the generality of $H' \supseteq L$ and $T_z(X) \neq L$. Finally it is clear that $L \not\subseteq X'$ and $L \setminus X' \subseteq \mathrm{B}(X')$. In sum, X' satisfies the property we have claimed. This complete the proof. □

Lemma 4.5. *Let $X \subseteq \mathbb{P}^N$ be a nondegenerate projective variety of dimension n and codimension $e \geq 2$. Let $\Lambda \subseteq \mathbb{P}^N$ be a linear subspace of dimension l $(1 \leq l < n)$, not contained in X. Assume the image of the linear projection $\pi_{\Lambda,X} : X \setminus \Lambda \to \mathbb{P}^{N-l-1}$ has dimension $n - l$. Then for each $x \in X \setminus \Lambda$, the closure $X_{\bar{x}}$ of the fibre of $\pi_{\Lambda,X}$ over $\bar{x} := \pi_{\Lambda,X}(x)$ is a hypersurfaces of $\langle \Lambda, x \rangle$, containing $X \cap \Lambda$.*

Proof. Let $\bar{X}$ be the closure of $\pi_{\Lambda,X}(X \setminus \Lambda)$ in $\mathbb{P}^{N-l-1}$, and let $\nu : Y \to \bar{X}$ be a desingularization of $\bar{X}$. Considering the target $\mathbb{P}^{N-l-1}$ of $\pi_{\Lambda,X}$ to be a subspace of $\mathbb{P}^N$ disjoint from Λ, we have a morphism $\bar{X} \to \mathbb{G} := \operatorname{Grass}(l+1, N)$ from $\bar{X}$ to the Grassmann of the $(l+1)$-plane in $\mathbb{P}^N$, defined by $\bar{x} \mapsto \langle \Lambda, \bar{x} \rangle$. Let $\mathcal{Q}_Y$ be the pull-back of the universal quotient bundle $\mathcal{Q}$ on $\mathbb{G}$ by $Y \to \bar{X} \to \mathbb{G}$, so that $\mathcal{Q}_Y \cong \mathcal{O}_Y^{\oplus l+1} \oplus \mathcal{O}_Y(1)$, where $\mathcal{O}_Y(1) := \nu^*(\mathcal{O}_{\mathbb{P}^{N-l-1}}(1)|\bar{X})$. The projective bundle $\mathbb{P} := \mathbb{P}_Y(\mathcal{Q}_Y)$ with projection $\tau : \mathbb{P} \to Y$ has a natural morphism $\phi : \mathbb{P} \to \mathbb{P}^N$ defined by the tautological line bundle $\mathcal{O}_{\mathbb{P}}(1)$, which is an embedding except on $\mathbb{P}_Y(\mathcal{O}_Y^{\oplus l+1})$ and the fibres of τ over the points of Y at which $Y \to \mathbb{P}^{N-l-1}$ is not an embedding. Thus $\phi(\mathbb{P})$ contains X, and X meets an open subset of $\phi(\mathbb{P})$ where ϕ is an embedding. Hence we can consider a prime divisor $\tilde{X}$ on a smooth variety $\mathbb{P}$, with a birational, surjective, induced morphism $\tilde{X} \to X$. Then $\tilde{X}$ is a member of a linear system $|\mathcal{O}_{\mathbb{P}}(\mu) \otimes \tau^*\mathcal{M}|$ for some positive integer μ and a line bundle $\mathcal{M}$ on Y (see [11], (II.6.11), (II.6.11.1A), (II.6.15), and (Ex.III.12.5)). We write $\mathcal{Q}_Y = \mathcal{O}_Y z_0 \oplus \cdots \oplus \mathcal{O}_Y z_l \oplus \mathcal{O}_Y(1) z_{l+1}$ with formal basis z_i (or homogeneous coordinates of fibres). Then $\tilde{X}$ is the zero of

$$G = \sum_{\mu_0, \cdots, \mu_{l+1} \geq 0, \mu_0 + \cdots + \mu_{l+1} = \mu} g_{\mu_0 \cdots \mu_{l+1}} z_0^{\mu_0} \cdots z_{l+1}^{\mu_{l+1}} \quad \in \quad H^0(\mathcal{O}_{\mathbb{P}}(\mu) \otimes \tau^*\mathcal{M})$$

for some $g_{\mu_0 \cdots \mu_{l+1}} \in H^0(Y, \mathcal{M} \otimes \mathcal{O}_Y(\mu_{l+1}))$. By $\mathbb{P}^l_y$ we denote the fiber of $\mathbb{P}_Y(\mathcal{O}_Y z_0 \oplus \cdots \oplus \mathcal{O}_Y z_l) \to Y$ over $y \in Y$. Then $\tilde{X} \cap \mathbb{P}^l_y$ is the subscheme of $\mathbb{P}^l_y$ $(\subseteq \mathbb{P})$ defined by

$$G|_{\mathbb{P}^l_y} = \sum_{\mu_0, \cdots, \mu_l \geq 0, \mu_0 + \cdots + \mu_l = \mu} g_{\mu_0 \cdots \mu_l 0}(y) z_0^{\mu_0} \cdots z_l^{\mu_l}$$

and $\tilde{X} \cap \mathbb{P}^l_y$ is mapped into $X \cap \Lambda$. If we consider a rational map, defined by $\{g_{\mu_0 \cdots \mu_l 0}\} \subseteq H^0(Y, \mathcal{M})$, from Y to the set $\mathbb{P}^M$ $(M = \binom{l+\mu}{l} - 1)$ of linear forms of degree μ on $\mathbb{P}^l$ with homogeneous coordinates $z_0, \ldots, z_l$, then this map is a morphism from Y whose image is one point, since $\phi(\tilde{X} \cap \mathbb{P}^l_y) \subseteq X \cap \Lambda$ for each $y \in Y$ and since the points of $\mathbb{P}^M$ whose zeros, as μ-forms of $\mathbb{P}^l = \Lambda$, are contained in $X \cap \Lambda$ are finite. Consequently $\mathcal{M} \cong \mathcal{O}_Y$ and $g_{\mu_0 \cdots \mu_l 0} \in H^0(\mathcal{O}_Y) \cong \mathbb{k}$, and moreover $g_{\mu_0 \cdots \mu_l 0} \neq 0$ for some $\mu_0, \ldots, \mu_l$. Therefore set-theoretically, $\phi(\tilde{X} \cap \mathbb{P}^l_y) = X \cap \Lambda$ for each $y \in Y$. This implies that the image $\phi(\tilde{X} \cap \mathbb{P}(\mathcal{Q}_Y \otimes \mathbb{k}(y)))$ contains $X \cap \Lambda$. For $x \in X \setminus \Lambda$, $X_{\bar{x}} = \bigcup_{y \in \nu^{-1}(\bar{x})} \phi(\tilde{X} \cap \mathbb{P}(\mathcal{Q}_Y \otimes \mathbb{k}(y))))$ as sets. Hence $X_{\bar{x}}$ is a hypersurface, containing $X \cap \Lambda$, as required. □

Example 4.6. The induced morphism $\tilde{X} \cap \mathbb{P}^l_y \to X \cap \Lambda$ in the proof of Lemma 4.5 is bijective but not necessarily isomorphic. Let $\mathbb{P}$ be the projective bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{Q})$ over $Y := \mathbb{P}^1$, associated with the vector bundle $\mathcal{Q} = \mathcal{O}_{\mathbb{P}^1} z_0 \oplus \mathcal{O}_{\mathbb{P}^1} z_1 \oplus \mathcal{O}_{\mathbb{P}^1}(4) z_2$

on $\mathbb{P}^1$, with projection $\tau : \mathbb{P} \to \mathbb{P}^1$ and the tautological bundle $\mathcal{O}_{\mathbb{P}}(1)$. Here $\{z_i\}$ is a formal basis. Let s, t be the homogeneous coordinates of $\mathbb{P}^1$. Let $\phi : \mathbb{P} \to \mathbb{P}^5$ be the morphism defined by $z_0, z_1, s^4z_2, s^3tz_2, st^3z_2, t^4z_2 \in H^0(\mathcal{O}_{\mathbb{P}}(1))$. Let $\tilde{X}_i$ $(i = 1, 2)$ be divisors defined by G_i for $G_1 = z_1^2 - s^2t^2z_0z_2$ and $G_2 = z_1^2 - s^3tz_0z_2$ in $H^0(\mathcal{O}_{\mathbb{P}}(2))$. Then the images $X_i = \phi(\tilde{X}_i)$ are nondegenerate projective surfaces of degree 8. The line $L := \phi(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}z_0 \oplus \mathcal{O}_{\mathbb{P}^1}z_1))$ is contained in the closure of $\mathrm{B}(X_i)$. Moreover $X_1 \cap L = X_2 \cap L$ as sets, and $l(X_1 \cap L) = 4$ and $l(X_2 \cap L) = 2$. On the other hand, $l(\tilde{X}_1 \cap \mathbb{P}^1_y) = l(\tilde{X}_2 \cap \mathbb{P}^1_y) = 2$ for each $y \in \mathbb{P}^1$.

Remark 4.7. In Theorem 4.4, to obtain the inequality $\dim \mathrm{B}(X) \le \dim \operatorname{Sing} X + 1$ for $e \ge 2$, there is an easier argument using Bertini's theorem as follows. Assume to the contrary that $l := \dim \mathrm{B}(X) \ge \dim \operatorname{Sing} X + 2$. By (4.3), $l \le n - 1$. For $m = N - l + 1$, let $M \subseteq \mathbb{P}^N$ be a general m-dimensional linear subspace. By Bertini's theorem, $X' := X \cap M$ is a smooth projective variety of dimension $n-l+1$, nondegenerate in M (see [10], (18.10) or [7], (3.5.8)). Moreover the closure of $\mathrm{B}(X')$ contains a line L with $L \not\subseteq X'$, since $\mathrm{B}(X') \supseteq \mathrm{B}(X) \cap M$. Let $M' \subseteq M$ be a general $(e+1)$-dimensional linear subspace containing L. Since $L \cap X' \subseteq \operatorname{Sm} X'$, by a Bertini-type theorem (see [14], (2.1); [18], (2.1)), $X'' := X' \cap M'$ is a smooth nondegenerate projective curve in M' with $\dim \mathrm{B}(X'') \ge 1$. Since $L \cap \mathrm{B}(X'')$ is dense in L (or apply Theorem 4.3), X'' lies on the 2-plane spanned by L and a general point $x \in X''$, and consequently $e = 1$, a contradiction.

4.8. Proof of Theorem 5. For a general (smooth) point x of X, let Λ_x be the linear span $\langle Z, T_x(X)\rangle$. Now, according to Segre [20], we will show that

$$\dim \Lambda_x = n + 1. \tag{4.8.1}$$

To this purpose, consider the linear projection $\pi_{z,X} : X \setminus \{z\} \to \mathbb{P}^{N-1}$ from a point $z \in Z$. Since $z \in \operatorname{Sm} X$, $\pi_{z,X}$ is generically quasi-finite (see (1.1.3)). By the generic smoothness of $\pi_{z,X}$, the line $\langle x, z\rangle$ meets X at a point $y \in \operatorname{Sm} X$ distinct from x and z. Moreover $T_y(X) \subseteq \langle T_x(X), z\rangle$ (see (1.2.2)). Let Y be an irreducible component of the closure of the set of the points $y \in \langle x, z\rangle \cap X$ for moving $z \in Z$ and fixed $x \in X$. If $y \in Y$ is general, the corresponding point z is also general in Z, and hence,

$$T_y(Y) \subseteq T_y(X) \subseteq \langle T_x(X), z\rangle \supseteq T_x(X).$$

From this, by considering the projection of Y from $T_x(X)$, we observe that $\langle T_x(X), y\rangle$ does not depend on $y \in Y$ (see (1.2.1)). Consequently $\langle T_x(X), y\rangle = \langle T_x(X), Y\rangle = \Lambda_x$, which implies (4.8.1).

Let Λ be the intersection of Λ_x for general points $x \in X$, and set $l := \dim \Lambda$. If $l = n + 1$, then $\Lambda = \Lambda_x$ and hence Λ contains X, which contradicts $e \ge 2$. Thus $l \le n$. Let $\pi_{\Lambda,X} : X \setminus \Lambda \to \mathbb{P}^{N-l-1}$ be the linear projection from Λ to a subspace $\mathbb{P}^{N-l-1}$ of $\mathbb{P}^N$ disjoint from Λ, and let $\bar{X}$ be the closure of $\pi_{\Lambda,X}(X \setminus \Lambda)$. For general $x \in X$, let $X_{\bar{x}}$ be the closure of $\pi_{\Lambda,X}^{-1}(\bar{x})$ over $\bar{x} := \pi_\Lambda(x)$. We claim that $l < n$, $\dim X_{\bar{x}} = l$, and

$$l((X \setminus \Lambda) \cap \langle v, x\rangle) \ge 2 \tag{4.8.2}$$

for general $v \in \Lambda$. If $l = n$, i.e., $\operatorname{codim}(\Lambda, \Lambda_x) = 1$, then $\dim(T_x(X) \cap \Lambda) = n - 1$, and hence $X \subseteq \langle \Lambda, x\rangle$, which contradicts $e \ge 2$. Thus $l < n$. Then $\dim(T_x(X) \cap \Lambda) = l$ or $l - 1$, since $\operatorname{codim}(T_x(X), \Lambda_x) = 1$. If $\dim(T_x(X) \cap \Lambda) = l$, by (1.2.1),

$\dim \bar{X} = n - l - 1$, and hence X is the cone over $\bar{X}$ with vertex Λ, which means $Z \subseteq \Lambda \subseteq \operatorname{Sing} X$, a contradiction. Therefore $\dim(T_x(X) \cap \Lambda) = l - 1$. By (1.2.1), $\dim \bar{X} = n - l$, and $\dim X_{\bar{x}} = l$. The fact that $l((X \setminus \{z\}) \cap \langle z, x\rangle) \geq 2$ for $z \in Z$ implies that the l-dimensional part of $X_{\bar{x}}$ is a hypersurface of $\langle \Lambda, x\rangle$, of degree ≥ 2, not containing Λ. Thus we have (4.8.2).

Next we will show that Λ is the closure of Z. If $\Lambda \not\subseteq X$, by (4.8.2), a general point of Λ lies on $\mathrm{B}(X)$, and hence $Z \subseteq \Lambda \cap X \subseteq \operatorname{Sing} X$ by (4.4), a contradiction. Thus $\Lambda \subseteq X$. Since Z is an irreducible component of $\mathrm{C}(X)$, if $\dim Z < \dim \Lambda$, then $\Lambda \setminus Z \subseteq \operatorname{Sing} X$ by (4.8.2), and hence $Z \subseteq \Lambda \subseteq \operatorname{Sing} X$, a contradiction. Thus $\dim Z = \dim \Lambda$, and hence Λ is the closure of Z.

Finally we look at $\dim Z$. Note that $\dim Z = \dim \Lambda = \dim X_{\bar{x}} = l$. By (1.2.2), $T_{x'}(X) \subseteq \langle T_{\bar{x}}(\bar{X}), \Lambda\rangle (= \langle T_x(X), \Lambda\rangle = \Lambda_x) \subseteq \mathbb{P}^N$ for each $x' \in X_{\bar{x}} \cap \operatorname{Sm} X$. Consequently, by the theorem of tangencies ([23], (I.1.7)),

$$\begin{aligned} l = \dim X_{\bar{x}} &\leq \dim \langle T_{\bar{x}}(\bar{X}), \Lambda\rangle - \dim X + \dim(X_{\bar{x}} \cap \operatorname{Sing} X) + 1 \\ &= 2 + \dim(X_{\bar{x}} \cap \operatorname{Sing} X) \leq 2 + \dim \operatorname{Sing} X. \end{aligned}$$

This completes the proof of Theorem 5. □

4.9. Proof of Corollary 6. If X is smooth, then $\mathrm{B}(X)$ is a finite set and $\mathrm{C}(X)$ is a finite union of linear subspaces of dimension ≤ 1. The rest follows from Theorem 2. □

Here we will show that the inequality of Theorem 5 is sharp, by giving an example of X whose $\mathrm{C}(X)$ contains a linear subspace of dimension $\dim \operatorname{Sing} X + 2$.

Example 4.10. For integers $l \geq 0$, $n \geq l+2$ and $a_n \geq \cdots \geq a_{l+1} > a_l = \cdots = a_0 = 0$, let $\mathbb{P}$ be the projective bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ over $\mathbb{P}^1$, associated with the vector bundle $\mathcal{E} = \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$ on $\mathbb{P}^1$, with projection $\tau : \mathbb{P} \to \mathbb{P}^1$ and the tautological bundle $\mathcal{O}_{\mathbb{P}}(1)$. Assume $a_n \geq 2$ if $n = l+2$. For a general member $\tilde{X} \in |\mathcal{O}_{\mathbb{P}}(\mu) \otimes \tau^*\mathcal{O}_{\mathbb{P}^1}(1)|$ with an integer $\mu \geq 2$, let X be the image of $\tilde{X}$ by the morphism $\phi : \mathbb{P} \to \mathbb{P}^N$ $(N = n + 1 + \sum_{i=0}^n a_i)$ defined by $|\mathcal{O}_{\mathbb{P}}(1)|$. Let $L \subseteq \mathbb{P}^N$ be the l-dimensional linear subspace which is the image $\phi(\tilde{L})$ of the subbundle $\tilde{L} = \mathbb{P}_{\mathbb{P}^1}(\bigoplus_{i=0}^l \mathcal{O}_{\mathbb{P}^1}(a_i)) \subseteq \mathbb{P}$. Then

1. $L \subseteq X$;
2. $\operatorname{Sing} X$ is a subset of L, of codimension 2 if $l \geq 1$, and $\operatorname{Sing} X = \emptyset$ if $l = 0$;
3. $\mathrm{C}(X)$ contains $L \setminus \operatorname{Sing} X$.

Hence $\dim \mathrm{C}(X) \geq \dim L = l \geq \dim \operatorname{Sing} X + 2$ if $l \geq 1$ and, $\dim \mathrm{C}(X) \geq 0$ and $\dim \operatorname{Sing} X = -1$ if $l = 0$. Consequently $\dim \mathrm{C}(X) = \dim \operatorname{Sing} X + 2$ for $l \geq 1$.

Proof. Note that $X \setminus L$ is smooth, since ϕ gives an embedding of $\mathbb{P} \setminus \tilde{L}$ into $\mathbb{P}^N$ and since $\tilde{X}$ is smooth by the generality of $\tilde{X}$. To see (1) and (2), let us look at $\tilde{X} \cap \tilde{L}$ and the induced morphism $\tilde{X} \cap \tilde{L} \to L$. Let s, t be the homogeneous coordinates of $\mathbb{P}^1$ and let z_i be the formal basis of $\mathcal{O}_{\mathbb{P}^1}(a_i)$ in $\mathcal{E}$. Then $\tilde{X}$ is defined as a subscheme of $\mathbb{P}$ by

$$G = \sum_{\mu_0, \cdots, \mu_n \geq 0, \mu_0 + \cdots + \mu_n = \mu} g_{\mu_0 \cdots \mu_n} \cdot z_0^{\mu_0} \cdots z_n^{\mu_n} \in H^0(\mathcal{O}_{\mathbb{P}}(\mu) \otimes \tau^*\mathcal{O}_{\mathbb{P}^1}(1)) \quad (\mu_i \geq 0)$$

for some homogeneous polynomials $g_{\mu_0 \cdots \mu_n} \in \Bbbk[s,t]$ of degree $1 + \sum_{i=l+1}^n a_i \mu_i$. Moreover $\tilde{X} \cap \tilde{L}$ is defined by $G|\tilde{L}$ in $\tilde{L} \cong \mathbb{P}^1 \times L$. Since $\tilde{L}$ is defined by $z_{l+1} = \cdots = z_n = 0$ in $\mathbb{P}$, the degree of $G|\tilde{L}$ with respect to s and t is one, and we may write $G|\tilde{L} = h_1 s + h_2 t$ for some h_1 and $h_2 \in \Bbbk[z_0, \dots, z_l]$. Let W be the subscheme of L defined by $h_1 = h_2 = 0$. Then $\operatorname{codim}(W, L) = 2$ if $l \geq 1$, and $W = \emptyset$ if $l = 0$, since $\tilde{X}$ is general and hence h_1 and h_2 are general. This together with $\phi^{-1}(L) = \tilde{L}$ implies that $\tilde{X} \to X$ is one-to-one and unramified at every point of $L \setminus W$ and also X contains L. Thus $X \setminus W \subseteq \operatorname{Sm} X$.

To show (2), we will prove $W \subseteq \operatorname{Sing} X$. For a point $x \in W$, take general points $\tilde{x}_1 \neq \tilde{x}_2$ of $\phi^{-1}(x) \cong \mathbb{P}^1$ so that $\tau|\tilde{X} : \tilde{X} \to \mathbb{P}^1$ is unramified at $\tilde{x}_i$. Set $\tilde{\mathbb{P}}^n_i = \tau^{-1}(\tau(\tilde{x}_i))$ and $\mathbb{P}^n_i = \phi(\tilde{\mathbb{P}}^n_i)$ $(\cong \tilde{\mathbb{P}}^n_i)$. To look at the dimension of the Zariski tangent space $\Theta_{x,X}$ to X at x, consider $\Theta_{x,X}$ as a subspace of $\Theta_{x,\mathbb{P}^N}$. The space $\Theta_{x,X}$ contains the image of $\Theta_{\tilde{x}_i,\tilde{X}}$ $(i = 1, 2)$ by the tangent map of ϕ. Then $\dim \Theta_{\tilde{x}_i,\tilde{X}} = \dim \Theta_{\tilde{x}_i,\mathbb{P}} - 1$ and $\Theta_{\tilde{x}_i,\tilde{\mathbb{P}}^n_i} \not\subseteq \Theta_{\tilde{x}_i,\tilde{X}}$. Since $\Theta_{\tilde{x}_i,\tilde{\mathbb{P}}^n_i} \cong \Theta_{x,\mathbb{P}^n_i}$ and $\Theta_{x,\mathbb{P}^n_1} \cap \Theta_{x,\mathbb{P}^n_2} = \Theta_{x,L}$ as subspaces of $\Theta_{x,\mathbb{P}^N}$, we have $\dim \Theta_{x,X} \geq 2(n-1) - l = n + (n - l - 2)$. If $n \geq l + 3$, this means that $x \in \operatorname{Sing} X$. When $n = l + 2$, we take another general point $\tilde{x}_3 \in \phi^{-1}(x)$, and the same argument implies that $x \in \operatorname{Sing} X$.

Now (3) is easy. In fact, for $p \in \mathbb{P}^1$, $\tilde{X} \cap \tau^{-1}(p)$ is a hypersurface of degree $\mu \geq 2$ in $\tau^{-1}(p)$ $(\cong \mathbb{P}^n)$, and hence $l(X \cap \langle x, y \rangle) \geq 3$ for every $x \in L \setminus W$ and general $y \in X$. □

Finally we look at the relation between $\operatorname{Sing} X$ and the boundary of $\mathrm{C}(X)$.

Theorem 4.11. *Let $X \subseteq \mathbb{P}^N$ be a nondegenerate, projective variety of dimension n and codimension $e \geq 2$. Let Λ be an irreducible component of the closure of* $\mathrm{C}(X)$*, of dimension l, which is necessarily linear by Theorem 5.*

(1) *Assume $l \geq 2$. (Hence necessarily $n \geq l + 1 > l \geq 2$.) Then* $\dim \operatorname{Sing} X \geq n - 2$*, or* $\dim(\Lambda \cap \operatorname{Sing} X) \geq l - 2$.
(2) *Assume $l \geq 3$. (Hence necessarily $n \geq l + 1 > l \geq 3$.) Then* $\dim(\Lambda \cap \operatorname{Sing} X) \geq l - 3$.

Proof. (1). Assume $\dim(\Lambda \cap \operatorname{Sing} X) \leq l - 3$. We will show $\dim \operatorname{Sing} X \geq n - 2$. By the assumption, there exists a 2-dimensional subspace M of Λ with $M \subseteq \operatorname{Sm} X$, and consequently $M \subseteq \mathrm{C}(X)$ by (4.2). Consider the linear projection $\pi_{M,X} : X \setminus M \to \mathbb{P}^{N-3}$ from M and let $\bar{X}$ be the closure of $\pi_{M,X}(X \setminus M)$. Let x be a general point of X and set $\bar{x} := \pi_{M,X}(x)$. Let $X_{\bar{x}}$ be the closure of $\pi^{-1}_{M,X}(\bar{x}) = \langle M, x \rangle \cap (X \setminus M)$ over $\bar{x}$. Since $M \subseteq \mathrm{C}(X)$, we have $\dim X_{\bar{x}} = 2$ or 3, and hence $\dim \bar{X} = n - 2$ or $n - 3$. In the latter, X is the cone over $\bar{X}$ with vertex Λ and hence $\Lambda \subseteq \operatorname{Sing} X$, a contradiction. Thus $\dim \bar{X} = n - 2$. Hence $\dim \langle T_{\bar{x}}(\bar{X}), M \rangle = n + 1$ and $T_{x'}(X) \subseteq \langle T_{\bar{x}}(\bar{X}), M \rangle$ for each $x' \in X_{\bar{x}} \cap \operatorname{Sm} X$ (see (1.2.2)). By the theorem of tangencies ([23], (I.1.7)), $\dim(X_{\bar{x}} \cap \operatorname{Sing} X) \geq \dim X_{\bar{x}} - \dim \langle T_{\bar{x}}(\bar{X}), M \rangle + n - 1 = 0$. This implies that $\langle M, x \rangle \cap (X \setminus M) \cap \operatorname{Sing} X \neq \emptyset$, since $M \subseteq \operatorname{Sm} X$ and $X_{\bar{x}} \setminus M = \langle M, x \rangle \cap (X \setminus M)$. By the generality of $x \in X$, $\operatorname{Sing} X$ dominates $\bar{X}$, and consequently $\dim \operatorname{Sing} X \geq n - 2$, as required.

(2). Assume $\dim(\Lambda \cap \operatorname{Sing} X) \leq l - 4$ to get a contradiction. There exists a 3-dimensional subspace M' of Λ with $M' \subseteq \operatorname{Sm} X$, and consequently $M' \subseteq \mathrm{C}(X)$. Consider the linear projection $\pi_{M',X} : X \setminus M' \to \mathbb{P}^{N-4}$ from M' and the closure $\bar{X}$ of $\pi_{M',X}(X \setminus M')$. For a general point $x \in X$, let $X_{\bar{x}}$ be the closure of $\pi^{-1}_{M',X}(\bar{x})$ over $\bar{x} := \pi_{M',X}(x)$. By the same argument as in (1), $\dim X_{\bar{x}} = 3$, $\dim \bar{X} = n - 3$, and $\dim(X_{\bar{x}} \cap \operatorname{Sing} X) \geq 1$. Since $X_{\bar{x}} \subseteq \langle M', x \rangle$, we have $X_{\bar{x}} \cap \operatorname{Sing} X \cap M' \neq \emptyset$, which contradicts $M' \subseteq \operatorname{Sm} X$. □

Remark 4.12. In Example 4.10, by (4.11), we can relax the assumption $n \geq l+2$ to $n \geq l+1$ if $l \geq 2$. Indeed, $W \supseteq \operatorname{Sing} X$ without the assumption. Since $\dim W = l-2$, we have $\dim(\Lambda \cap \operatorname{Sing} X) \geq l-2$ by (4.11)(1). Since W is irreducible by the generality of $\tilde{X}$ and hence by the generality of h_1 and h_2, we have $W = \operatorname{Sing} X = \Lambda \cap \operatorname{Sing} X$.

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