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CAUCHY PROBLEM OF NONLINEAR SCHRÖDINGER EQUATION WITH INITIAL DATA IN SOBOLEV SPACE $W^{s,p}$ FOR $p < 2$

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ABSTRACT. In this paper, we consider in R^n the Cauchy problem for the nonlinear Schrödinger equation with initial data in the Sobolev space $W^{s,p}$ for $p < 2$. It is well known that this problem is ill posed. However, we show that after a linear transformation by the linear semigroup the problem becomes locally well posed in $W^{s,p}$ for $\frac{2n}{n+1} < p < 2$ and $s > n(1 - \frac{1}{p})$. Moreover, we show that in one space dimension, the problem is locally well posed in L^p for any $1 < p < 2$.

1. INTRODUCTION

Consider the Cauchy problem for the linear Schrödinger equation

$$iu_t(t,x) - \triangle u(t,x) = 0, \tag{1.1}$$

$$u(0,x) = u_0(x), \tag{1.2}$$

where $\triangle$ is the Laplace operator in R^n for $n \geq 1$. It is well known that this problem is well posed for initial data $u_0 \in L^p(R^n)$ if and only if $p = 2$. For this reason, it is believed that the initial value problem for the nonlinear Schrödinger equation is not well posed for initial data in the Sobolev space $W^{s,p}$ for $p \neq 2$. However, this is not quite right.

Notice that the solution of the Cauchy problem for (1.1), (1.2) can be written as

$$u(t) = S(t)u_0 = E(t) * u_0, \tag{1.3}$$

where

$$E(t,x) = \frac{1}{(-4\pi it)^{\frac{n}{2}}} e^{-i\frac{|x|^2}{4t}} \tag{1.4}$$

is the fundamental solution and $S(t)$ defines a semigroup. Thus

$$S(-t)u(t) \equiv u_0, \tag{1.5}$$

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and for any norm X we have

$$\|S(-t)u(t)\|_X = \|u_0\|_X. \tag{1.6}$$

There are some examples in the literature in which the nonlinear Schrödinger equation is studied by using the norm $\|u\|_Y$, which is defined by

$$\|S(-t)u(t)\|_X, \tag{1.7}$$

where X is the usual Sobolev or weighted Sobolev norm. Of course, we have the trivial example that when $X = H^s$, we have $X = Y$. The first nontrivial example is to take X to be the weighted L^2-norm. Thus, we take

$$\|w\|_X = \sum_{|\alpha|\leq s} \|x^\alpha w\|_{L^2(R^n)}, \tag{1.8}$$

where α is a multi-index. Then

$$\|u(t)\|_Y = \sum_{|\alpha|\leq s} \|x^\alpha S(-t)u(t)\|_{L^2(R^n)} = \sum_{|\alpha|\leq s} \|S(t)x^\alpha S(-t)u(t)\|_{L^2(R^n)}. \tag{1.9}$$

Noting that

$$S(t)x_k S(-t) = x_k - 2it\partial_{x_k} = L_k, \tag{1.10}$$

we obtain

$$\|u(t)\|_Y = \sum_{|\alpha|\leq s} \|L^\alpha u(t)\|_{L^2(R^n)}. \tag{1.11}$$

This norm was first used by McKean and Shatah [9] and it was proved that one has the following global Sobolev inequality:

$$\|u(t)\|_{L^\infty} \leq C(1+t)^{-\frac{n}{2}}(\sum_{|\alpha|\leq s} \|L^\alpha u(t)\|_{L^2(R^n)} + \|u(t)\|_{H^s(R^n)}), \quad s > \frac{n}{2}. \tag{1.12}$$

This inequalty is similar to the global Sobolev inequality for the wave equation obtained earlier by Klainerman (see [7]) and is very important in studying the nonlinear problem in their paper.

Another more recent example is to take $X = H_t^b H_x^s$. Then Y is the so-called Bourgain space (see [2]). This space plays a very important role in the recent study of low regularity solutions of nonlinear Schrödinger equations.

Therefore, why not take $X = L^p$ (or $W^{s,p}$)? It is our aim to investigate this problem in this paper.

Consider the Cauchy problem for the nonlinear Schrödinger equation

$$iu_t(t,x) - \triangle u(t,x) = \pm|u(t,x)|^2 u(t,x), \tag{1.13}$$

$$u(0,x) = u_0(x). \tag{1.14}$$

This problem can be reformulated as

$$u(t) = S(t)u_0 \pm \int_0^t S(t-\tau)(|u(\tau)|^2 u(\tau))d\tau. \tag{1.15}$$

Motivated by our above discussions, we make a linear transformation

$$v(t) = S(-t)u(t); \tag{1.16}$$

then

$$u(t) = S(t)v(t). \tag{1.17}$$

Therefore, we get

$$v(t) = u_0 \pm \int_0^t S(-\tau)[S(-\tau)\bar{v}(\tau)(S(\tau)v(\tau))^2]d\tau, \tag{1.18}$$

where we use the fact that $\bar{S}(\tau) = S(-\tau)$.

Our main result in this paper is that (1.18) is locally well posed in the Sobolev space $W^{s,p}$ for certain $p < 2$. More precisely, we have the following:

Theorem 1.1. *Consider the nonlinear integral equation* (1.18). *Suppose that*

$$u_0 \in W^{s,p}(R^n) \tag{1.19}$$

for $s > n(1 - \frac{1}{p})$ and $\frac{2n}{n+1} < p < 2$, where $W^{s,p}(R^n)$ is understood as $B^s_{p,p}(R^n)$ and $B^s_{p,q}(R^n)$ is the Besov space. Then there exists a time T which only depends on $\|u_0\|_{W^{s,p}(R^n)}$ such that the integral equation has a unique solution $v \in C([0,T], W^{s,p}(R^n))$ satisfying

$$\|v(t)\|_{W^{s,p}(R^n)} \leq 2\|u_0\|_{W^{s,p}(R^n)}, \quad \forall t \in [0,T]. \tag{1.20}$$

Moreover, suppose that v_1, v_2 are two solutions with initial data u_{01}, u_{02}. It then follows that

$$\|v_1(t) - v_2(t)\|_{W^{s,p}(R^n)} \leq 2\|u_{01} - u_{02}\|_{W^{s,p}(R^n)}, \quad \forall t \in [0,T]. \tag{1.21}$$

Remark 1.2. Our proof relies on a subtle cancellation in the nonlinearity, and thus our result is not valid for the general nonlinearity $F(u,\bar{u})$. However, for a nonlinear term of the form $\pm|u|^{2m}u$, where m is an integer, it is not difficult to generalize our result to this case.

Remark 1.3. By the well-known $L^p - L^{p'}$ estimate, we obtain that for the original solution $u(t) = S(t)v(t)$,

$$\|u(t)\|_{W^{s,p'}} \leq Ct^{-n(\frac{1}{2}-\frac{1}{p'})}\|v(t)\|_{W^{s,p}}, \tag{1.22}$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1. \tag{1.23}$$

Therefore

$$\|u(t)\|_{W^{s,p'}} \leq Ct^{-n(\frac{1}{2}-\frac{1}{p'})}\|u_0\|_{W^{s,p}}, \quad \forall 0 < t < T. \tag{1.24}$$

Remark 1.4. Similar results are expected for other nonlinear dispersive equations and nonlinear wave equations. However, no such result is presently known.

We point out that Theorem 1.1 only shows that one can solve the Cauchy problem in $W^{s,p}$ for $p < 2$; the regularity assumption in Theorem 1.1 need not be optimal and can be improved. As an example, we will show that the problem is locally well posed in L^p for any $1 < p < 2$ in one space dimension. It is proved by Y. Tsutsumi [10] that the problem is locally well posed in L^2. Then it is proved by Grünrock [6] that the problem is locally well posed in $\hat{L}^p$, for any $1 < p < \infty$ (see also Cazenave et al. [3] and Vargas and Vega [11]). Here

$$\|f\|_{\hat{L}^p} = \|\hat{f}\|_{L^{p'}}, \tag{1.25}$$

where $\hat{f}$ is the Fourier transform of f and p' is defined by (1.23). Noting that

$$\|\hat{f}\|_{L^{p'}} \leq C\|f\|_{L^p}, \quad 1 \leq p \leq 2, \tag{1.26}$$

the $\hat{L}^p$ space is slightly larger than the L^p space. However, L^p is the more commonly used space. More recently, there are even some local existence results in H^s for some $s<0$; see Christ et al. [5] as well as Koch and Tataru [8].

Our main result in one space dimension is as follows:

Theorem 1.5. *Consider the nonlinear integral equation* (1.18) *in one space dimension. Suppose that*

$$u_0 \in L^p(R) \tag{1.27}$$

for $1<p<2$. *Then there exists a time* T *which only depends on* $\|u_0\|_{L^p(R)}$ *such that the integral equation has a unique solution* $v \in C([0,T], L^p(R))$ *satisfying*

$$\|v(t)\|_{L^p(R)} \le C_0 \|u_0\|_{L^p(R)}, \quad \forall t \in [0,T], \tag{1.28}$$

and

$$\left\{ \int_0^T \tau^{\theta p'} \|\partial_\tau v(\tau)\|^{p'}_{L^p(R)} d\tau \right\}^{\frac{1}{p'}} \le C_1 \|u_0\|^3_{L^p(R)}, \tag{1.29}$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad \theta = \frac{2}{p} - 1. \tag{1.30}$$

Moreover, suppose that v_1, v_2 *are two solutions with initial data* u_{01}, u_{02}. *Then it follows that*

$$\|v_1(t) - v_2(t)\|_{L^p(R)} \le C_0 \|u_{01} - u_{02}\|_{L^p(R)}, \quad \forall t \in [0,T]. \tag{1.31}$$

Here C_0 *and* C_1 *are positive constants independent of the initial data.*

Remark 1.6. Let $u(t,x)$ be a solution to the nonlinear Schrödinger equation (1.13) with initial data (1.14). Then $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$ is also a solution with initial data $u_{0\lambda} = \lambda u_0(\lambda x)$. If

$$\|u_{0\lambda}\|_{L^p(R^n)} \equiv \|u_0\|_{L^p(R^n)}, \tag{1.32}$$

then p is called a scaling limit. It is easy to see that p is a scaling limit in one space dimension if and only if $p=1$. Thus, as p close to 1, we can go arbitrarily close to the scaling limit.

Remark 1.7. By the well-known $L^p - L^{p'}$ estimate, we obtain that for the original solution $u(t) = S(t)v(t)$,

$$\|u(t)\|_{L^{p'}} \le C t^{-(\frac{1}{2} - \frac{1}{p'})} \|v(t)\|_{L^p}, \tag{1.33}$$

where p' is defined by (1.23). Therefore

$$\|u(t)\|_{L^{p'}} \le C t^{-(\frac{1}{2} - \frac{1}{p'})} \|u_0\|_{L^p}, \quad \forall 0 < t < T.$$

Both Theorem 1.1 and Theorem 1.5 are proved by some trilinear L^p estimates. These kinds of estimates are obtained by interpolation between various well-known L^2 estimates and our new trilinear L^1 estimate (see Lemma 2.1).

In the following, C will denote a positive constant independent of the initial data and its meaning may change from line to line.

Finally, we refer to [1] for the definition of Besov spaces.

2. A key lemma

A key lemma leading to our local well posedness is the following:

Lemma 2.1. *We consider the trilinear form*

$$v_0(\tau) = T(v_1(\tau), v_2(\tau), v_3(\tau)) = S(-\tau)[S(-\tau)v_1(\tau)S(\tau)v_2(\tau)S(\tau)v_3(\tau)]. \tag{2.1}$$

Then,

$$\|v_0(\tau)\|_{L^1(R^n)} \le C\tau^{-n}\|v_1(\tau)\|_{L^1(R^n)}\|v_2(\tau)\|_{L^1(R^n)}\|v_3(\tau)\|_{L^1(R^n)}. \tag{2.2}$$

Proof. By scaling invariance, it suffices to prove (2.2) for a fixed value of τ, say $\tau = \frac{1}{2}$. Then, let $M(x) = e^{i|x|^2/2}$, $\overline{M}(x) = e^{-i|x|^2/2}$ and the trilinear form

$$T(f, g, h) = \overline{M} * (\overline{M} * f \cdot M * g \cdot M * h), \tag{2.3}$$

where $*$ denotes the convolution product and $\cdot$ denotes the pointwise multiplication. We only need to prove

$$\|T(f, g, h)\|_{L^1} \le C\|f\|_{L^1}\|g\|_{L^1}\|h\|_{L^1}. \tag{2.4}$$

To see that (2.4) is true, we make use of the identities

$$M * f = M \cdot F(M \cdot f), \quad \overline{M} * f = \overline{M} \cdot \overline{F}(\overline{M} \cdot f), \tag{2.5}$$

where F and $\overline{F}$ denote the Fourier and anti-Fourier transforms

$$F(f)(\xi) = \int e^{-ix\xi} f(x)dx = \hat{f}(\xi), \quad \overline{F}(f)(\xi) = \int e^{ix\xi} f(x)dx = \hat{f}(-\xi). \tag{2.6}$$

Then, the trilinear form becomes

$$T(f, g, h) = \overline{M} \cdot \overline{F}(\overline{M} \cdot \overline{M} \cdot \overline{F}(\overline{M} \cdot f) \cdot M \cdot F(M \cdot g) \cdot M \cdot F(M \cdot h)). \tag{2.7}$$

Now, the key step is to notice that $\overline{M} \cdot \overline{M} \cdot M \cdot M \equiv 1$. Hence

$$|T(f, g, h)| = |\overline{F}(\hat{F} \cdot \hat{G} \cdot \hat{H})|, \tag{2.8}$$

where

$$F(x) = \overline{M}(x) \cdot f(-x), \quad G(x) = M(x) \cdot g(x), \quad H(x) = M(x) \cdot h(x). \tag{2.9}$$

The Fourier transform maps pointwise multiplication of functions into convolution products of their Fourier transforms and in particular we have

$$\overline{F}(\hat{F} \cdot \hat{G} \cdot \hat{H}) = C_n F * G * H. \tag{2.10}$$

We use now the L^1 inequality for convolutions and we obtain

$$\begin{aligned} \|T(f, g, h)\|_{L^1} &= C\|F * G * H\|_{L^1} \\ &\le C\|F\|_{L^1}\|G\|_{L^1}\|H\|_{L^1} = C\|f\|_{L^1}\|g\|_{L^1}\|h\|_{L^1}. \end{aligned} \tag{2.11}$$

□

We also have the following trivial L^2 estimate:

Lemma 2.2. *Let $v_l, l = 0, 1, 2, 3$, satisfy (2.1). Suppose that $2^{j-2} \le |\xi| \le 2^{j+2}$ in the support of $\hat{v}_2(\tau, \xi)$ and $2^{k-2} \le |\xi| \le 2^{k+2}$ in the support of $\hat{v}_3(\tau, \xi)$, where $\hat{v}_2$, $\hat{v}_3$ denote the space Fourier transforms of v_2,v_3. Then*

$$\|v_0(\tau)\|_{L^2} \le C2^{\frac{n}{2}(j+k)}\|v_1(\tau)\|_{L^2}\|v_2(\tau)\|_{L^2}\|v_3(\tau)\|_{L^2}. \tag{2.12}$$

Proof. Let $u_l(\tau) = S(\tau)v_l(\tau), l = 0, 2, 3$ and $u_1(\tau) = S(-\tau)v_1(\tau)$. Then $\hat{u}_0(\tau, \xi) = e^{-i|\xi|^2\tau}\hat{v}_0(\tau, \xi)$, etc. We have

$$u_0(\tau) = u_1(\tau)u_2(\tau)u_3(\tau). \tag{2.13}$$

Therefore

$$\begin{aligned} \|v_0(\tau)\|_{L^2(R^n)} &= \|u_0(\tau)\|_{L^2(R^n)} \\ &\le \|u_1(\tau)\|_{L^2(R^n)}\|u_2(\tau)\|_{L^\infty(R^n)}\|u_3(\tau)\|_{L^\infty(R^n)} \\ &\le C\|u_1(\tau)\|_{L^2(R^n)}\|\hat{u}_2(\tau)\|_{L^1(R^n)}\|\hat{u}_3(\tau)\|_{L^1(R^n)} \\ &= C\|v_1(\tau)\|_{L^2(R^n)}\|\hat{v}_2(\tau)\|_{L^1(R^n)}\|\hat{v}_3(\tau)\|_{L^1(R^n)}. \end{aligned} \tag{2.14}$$

Noting the support property of $\hat{v}_2(\tau)$ and $\hat{v}_3(\tau)$, the desired conclusion follows from the Schwarz inequality. □

We point out that the result of Lemma 2.2 does not depend on the special structure of the trilinear form; it applies to any product of three functions.

By the interpolation theorem on the multi-linear functionals (see [1], page 96, Theorem 4.4.1), we can interpolate the inequality in Lemma 2.1 and Lemma 2.2 to obtain the following:

Lemma 2.3. *Let $v_l, l = 0, 1, 2, 3$, satisfy (2.1). Suppose that $2^{j-2} \le |\xi| \le 2^{j+2}$ in the support of $\hat{v}_2(\tau, \xi)$ and $2^{k-2} \le |\xi| \le 2^{k+2}$ in the support of $\hat{v}_3(\tau, \xi)$, where $\hat{v}_2$, $\hat{v}_3$ denote the space Fourier transforms of v_2,v_3. Then*
(2.15)

$$\|v_0(\tau)\|_{L^p} \le C\tau^{-n(\frac{2}{p}-1)}2^{n(1-\frac{1}{p})(j+k)}\|v_1(\tau)\|_{L^p}\|v_2(\tau)\|_{L^p}\|v_3(\tau)\|_{L^p}, \quad 1 \le p \le 2.$$

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by a contraction mapping principle. We point out that we can also slightly improve our result by using Besov spaces.

Theorem 3.1. *Consider the nonlinear integral equation (1.18). Suppose that*

$$u_0 \in \dot{B}^s_{p,1}(R^n) \tag{3.1}$$

for $s = n(1 - \frac{1}{p})$ and $\frac{2n}{n+1} < p < 2$, where $\dot{B}^s_{p,1}(R^n)$ is the homogeneous Besov space. Then there exists a time T which only depends on $\|u_0\|_{\dot{B}^s_{p,1}(R^n)}$ such that the integral equation has a unique solution $v \in C([0, T], \dot{B}^s_{p,1}(R^n))$ satisfying

$$\|v(t)\|_{\dot{B}^s_{p,1}(R^n)} \le 2\|u_0\|_{\dot{B}^s_{p,1}(R^n)}, \quad \forall t \in [0, T]. \tag{3.2}$$

Moreover, suppose that v_1, v_2 are two solutions with initial data u_{01}, u_{02}. Then

$$\|v_1(t) - v_2(t)\|_{\dot{B}^s_{p,1}(R^n)} \le 2\|u_{01} - u_{02}\|_{\dot{B}^s_{p,1}(R^n)}, \quad \forall t \in [0, T]. \tag{3.3}$$

In the following, we will only prove Theorem 3.1 since the proof of Theorem 1.1 is similar.

Let us define the space

$$X = \{w \in C([0, T], \dot{B}^s_{p,1}(R^n)) | \sup_{0\le t\le T} \|w(t)\|_{\dot{B}^s_{p,1}(R^n)} \le 2\|u_0\|_{\dot{B}^s_{p,1}(R^n)}\}, \tag{3.4}$$

where $s = n(1 - \frac{1}{p})$ and $\frac{2n}{n+1} < p < 2$. For any $w \in X$, define a map M by

$$(Mw)(t) = u_0 \pm \int_0^t S(-\tau)[S(-\tau)\bar{w}(\tau)(S(\tau)w(\tau))^2]d\tau. \tag{3.5}$$

We want to show that M maps X into itself and is a contraction provided that T is sufficiently small.

First let us recall the definition of homogeneous Besov spaces. Let $\psi \in C_0^\infty(R^n)$ such that

$$\operatorname{supp} \psi \subset \{\xi | |\xi| \leq 1\} \tag{3.6}$$

and

$$\psi(\xi) \equiv 1, \quad |\xi| \leq \frac{1}{2}. \tag{3.7}$$

Let

$$\phi(\xi) = \psi(2^{-1}\xi) - \psi(\xi). \tag{3.8}$$

Then

$$\sum_{j=-\infty}^{+\infty} \phi(2^{-j}\xi) \equiv 1, \tag{3.9}$$

and we have the following dyadic decomposition:

$$w = \sum_{j=-\infty}^{+\infty} w_j, \tag{3.10}$$

where

$$\hat{w}_j(\xi) = \phi(2^{-j}\xi)\hat{w}(\xi). \tag{3.11}$$

The Besov norm $\dot{B}^s_{p,1}(R^n)$ is defined by

$$\|w\|_{\dot{B}^s_{p,1}(R^n)} = \sum_{j=-\infty}^{+\infty} 2^{js}\|w_j\|_{L^p(R^n)}. \tag{3.12}$$

Let $w \in X$. To show that M maps X into itself, we need to estimate the nonlinear term:

$$\begin{aligned} F(\tau) &= S(-\tau)[S(-\tau)\bar{w}(\tau)(S(\tau)w(\tau))^2] \\ &= \sum_{j,k,l} S(-\tau)[S(-\tau)\bar{w}_j(\tau)S(\tau)w_k(\tau)S(\tau)w_l(\tau)]. \end{aligned} \tag{3.13}$$

To estimate F, we only need to estimate

$$F_1(\tau) = \sum_{j\geq k\geq l} S(-\tau)[S(-\tau)\bar{w}_j(\tau)S(\tau)w_k(\tau)S(\tau)w_l(\tau)]; \tag{3.14}$$

all the other terms in the summation can be estimated in a similar way.

By Lemma 2.3, we have

$$
\begin{aligned}
&\|F_1(\tau)\|_{\dot{B}^s_{p,1}(R^n)} \\
&\le \sum_{j=-\infty}^{+\infty} \| \sum_{k,l=-\infty}^{j} S(-\tau)[S(-\tau)\bar{w}_j(\tau)S(\tau)w_k(\tau)S(\tau)w_l(\tau)]\|_{\dot{B}^s_{p,1}(R^n)} \\
&\le \sum_{j=-\infty}^{+\infty}\sum_{m=-\infty}^{j+4} 2^{ms}\|\phi(2^{-m}D)\{\sum_{k,l=-\infty}^{j} S(-\tau)[S(-\tau)\bar{w}_j(\tau)S(\tau)w_k(\tau)S(\tau)w_l(\tau)]\}\|_{L^p(R^n)} \\
&\le \sum_{j=-\infty}^{+\infty}\sum_{m=-\infty}^{j+4} 2^{ms}\| \sum_{k,l=-\infty}^{j} S(-\tau)[S(-\tau)\bar{w}_j(\tau)S(\tau)w_k(\tau)S(\tau)w_l(\tau)]\|_{L^p(R^n)} \\
&\le C\sum_{j=-\infty}^{+\infty} 2^{js}\| \sum_{k,l=-\infty}^{j} S(-\tau)[S(-\tau)\bar{w}_j(\tau)S(\tau)w_k(\tau)S(\tau)w_l(\tau)]\|_{L^p(R^n)} \\
&\le C\sum_{j,k,l} 2^{js}\|S(-\tau)[S(-\tau)\bar{w}_j(\tau)S(\tau)w_k(\tau)S(\tau)w_l(\tau)]\|_{L^p(R^n)} \\
&\le C\tau^{-n(\frac{2}{p}-1)}\sum_{j,k,l} 2^{(j+k+l)s}\|w_j(\tau)\|_{L^p(R^n)}\|w_k(\tau)\|_{L^p(R^n)}\|w_l(\tau)\|_{L^p(R^n)} \\
&= C\tau^{-n(\frac{2}{p}-1)}\|w(\tau)\|^3_{\dot{B}^s_{p,1}(R^n)},
\end{aligned}
\tag{3.15}
$$

where $s = n(1-\frac{1}{p})$. Therefore

$$
\|F(\tau)\|_{\dot{B}^s_{p,1}(R^n)} \le C\tau^{-n(\frac{2}{p}-1)}\|w(\tau)\|^3_{\dot{B}^s_{p,1}(R^n)}. \tag{3.16}
$$

Noting that when $\frac{2n}{n+1} < p < 2$, we have $0 < n(\frac{2}{p}-1) < 1$, it is easy to see that

$$
\begin{aligned}
\|(Mw)(\tau)\|_{\dot{B}^s_{p,1}(R^n)} &\le \|u_0\|_{\dot{B}^s_{p,1}(R^n)} + \int_0^t \|F(\tau)\|_{\dot{B}^s_{p,1}(R^n)} \\
&\le \|u_0\|_{\dot{B}^s_{p,1}(R^n)} + C\int_0^t \tau^{-n(\frac{2}{p}-1)}\|w(\tau)\|^3_{\dot{B}^s_{p,1}(R^n)}d\tau \\
&\le \|u_0\|_{\dot{B}^s_{p,1}(R^n)} + CT^{1-n(\frac{2}{p}-1)}(\sup_{0\le t\le T}\|w(t)\|_{\dot{B}^s_{p,1}(R^n)})^3 \\
&\le \|u_0\|_{\dot{B}^s_{p,1}(R^n)} + CT^{1-n(\frac{2}{p}-1)}\|u_0\|^3_{\dot{B}^s_{p,1}(R^n)} \\
&\le 2\|u_0\|_{\dot{B}^s_{p,1}(R^n)}
\end{aligned}
\tag{3.17}
$$

provided that T is sufficiently small.

Now we prove that M is a contraction. Let $w^{(1)}, w^{(2)} \in X$. Denote $w^* = w^{(1)} - w^{(2)}$ and $v^* = Mw^{(1)} - Mw^{(2)}$. Then

(3.18)
$$\begin{aligned} v^* &= \pm \int_0^t S(-\tau)[S(-\tau)\bar{w}^{(1)}(\tau)(S(\tau)w^{(1)}(\tau))^2 - S(-\tau)\bar{w}^{(2)}(\tau)(S(\tau)w^{(2)}(\tau))^2]d\tau \\ &= \pm \int_0^t S(-\tau)[S(-\tau)\bar{w}^*(\tau)(S(\tau)w^{(1)}(\tau))^2 \\ &\qquad + S(-\tau)\bar{w}^{(2)}(\tau)S(\tau)(w^{(1)}(\tau) + w^{(2)}(\tau))S(\tau)w^*(\tau)]d\tau. \end{aligned}$$

By a similar argument as before, we can get

(3.19)
$$\begin{aligned} &\|v^*(t)\|_{\dot{B}^s_{p,1}(R^n)} \\ &\leq C\int_0^t \tau^{-n(\frac{2}{p}-1)}(\|w^{(1)}(\tau)\|_{\dot{B}^s_{p,1}(R^n)} + \|w^{(2)}(\tau)\|_{\dot{B}^s_{p,1}(R^n)})^2\|w^*(\tau)\|_{\dot{B}^s_{p,1}(R^n)} \\ &\leq CT^{1-n(\frac{2}{p}-1)}\|u_0\|^2_{\dot{B}^s_{p,1}(R^n)} \sup_{0\leq t\leq T}\|w^*(t)\|_{\dot{B}^s_{p,1}(R^n)} \\ &\leq \frac{1}{2}\sup_{0\leq t\leq T}\|w^*(t)\|_{\dot{B}^s_{p,1}(R^n)}. \end{aligned}$$

Therefore, we have proved the existence and uniqueness of the solution. To prove the stability result, let $v^{(1)}$ and $v^{(2)}$ be two solutions with initial data u_{01} and u_{02}. With a little abuse of notation, we still denote $v^* = v^{(1)} - v^{(2)}$. Then we have

(3.20)
$$\begin{aligned} v^* &= u_{01} - u_{02} \\ &\pm \int_0^t S(-\tau)[S(-\tau)\bar{v}^{(1)}(\tau)(S(\tau)v^{(1)}(\tau))^2 - S(-\tau)\bar{v}^{(2)}(\tau)(S(\tau)v^{(2)}(\tau))^2]d\tau \\ &= u_{01} - u_{02} \\ &\pm \int_0^t S(-\tau)[S(-\tau)\bar{v}^*(\tau)(S(\tau)v^{(1)}(\tau))^2 \\ &\qquad + S(-\tau)\bar{v}^{(2)}(\tau)S(\tau)(v^{(1)}(\tau) + v^{(2)}(\tau))S(\tau)v^*(\tau)]d\tau. \end{aligned}$$

Thus,

(3.21)
$$\begin{aligned} \|v^*(t)\|_{\dot{B}^s_{p,1}(R^n)} &\leq \|u_{01} - u_{02}\|_{\dot{B}^s_{p,1}(R^n)} \\ &\quad + C\int_0^t \tau^{-n(\frac{2}{p}-1)}(\|v^{(1)}(\tau)\|_{\dot{B}^s_{p,1}(R^n)} + \|v^{(2)}(\tau)\|_{\dot{B}^s_{p,1}(R^n)})^2\|v^*(\tau)\|_{\dot{B}^s_{p,1}(R^n)} \\ &\leq \|u_{01} - u_{02}\|_{\dot{B}^s_{p,1}(R^n)} \\ &\quad + CT^{1-n(\frac{2}{p}-1)}(\|u_{01}\|_{\dot{B}^s_{p,1}(R^n)} + \|u_{02}\|_{\dot{B}^s_{p,1}(R^n)})^2 \sup_{0\leq t\leq T}\|v^*(t)\|_{\dot{B}^s_{p,1}(R^n)} \\ &\leq \|u_{01} - u_{02}\|_{\dot{B}^s_{p,1}(R^n)} + \frac{1}{2}\sup_{0\leq t\leq T}\|v^*(t)\|_{\dot{B}^s_{p,1}(R^n)}. \end{aligned}$$

Therefore

$$\sup_{0\leq t\leq T}\|v^*(t)\|_{\dot{B}^s_{p,1}(R^n)} \leq 2\|u_{01} - u_{02}\|_{\dot{B}^s_{p,1}(R^n)}. \tag{3.22}$$

We have completed the proof of Theorem 3.1.

4. Proof of Theorem 1.5

In this section, we will prove Theorem 1.5.

Lemma 4.1. *Let $n=1$ and v_l, $l=0,1,2,3$, be defined by Lemma 2.1. Then*

$$\sup_{0\le\tau\le T}(\tau\|v_0(\tau)\|_{L^1(R)}) \le C\prod_{i=1}^{3}\{\|v_i(0)\|_{L^1(R)}+\int_0^T\|\partial_\tau v_i(\tau)\|_{L^1(R)}d\tau\}. \tag{4.1}$$

Proof. (4.1) follows from Lemma 2.1 by

$$v_i(t)=v_i(0)+\int_0^t\partial_\tau v_i(\tau)d\tau. \tag{4.2}$$

□

Lemma 4.2. *Let $n=1$ and v_l, $l=0,1,2,3$, be defined by Lemma 2.1. Then*

$$\left\{\int_0^T\|v_0(\tau)\|^2_{L^2(R)}d\tau\right\}^{\frac12} \le C\prod_{i=1}^{3}\{\|v_i(0)\|_{L^2(R)}+\int_0^T\|\partial_\tau v_i(\tau)\|_{L^2(R)}d\tau\}. \tag{4.3}$$

Proof. Let

$$u_1(\tau)=S(\tau)\bar v_1(\tau),\quad u_2(\tau)=S(\tau)v_2(\tau),\quad u_3(\tau)=S(\tau)v_3(\tau). \tag{4.4}$$

Then it follows from Hölder's inequality that

$$\left\{\int_0^T\|v_0(\tau)\|^2_{L^2(R)}d\tau\right\}^{\frac12} \le C\prod_{i=1}^{3}\left\{\int_0^T\|u_i(\tau)\|^6_{L^6(R)}d\tau\right\}^{\frac16}. \tag{4.5}$$

Noting that

$$iu_{1t}(t,x)-\triangle u_1(t,x)=S(t)\partial_t\bar v_1(t), \tag{4.6}$$

$$u_1(0)=\bar v_1(0) \tag{4.7}$$

as well as similar equations for u_2, u_3, the desired conclusion follows from Strichartz' inequality. □

By the interpolation theorem on multi-linear functionals (see [1], page 96, Theorem 4.4.1), we can interpolate the inequality in Lemma 4.1 and Lemma 4.2 to get the following.

Lemma 4.3. *Let $n=1$ and v_l, $l=0,1,2,3$, be defined by Lemma 2.1. Then*

$$\left\{\int_0^T\tau^{\theta p'}\|v_0(\tau)\|^{p'}_{L^p(R)}d\tau\right\}^{\frac{1}{p'}} \le C\prod_{i=1}^{3}\{\|v_i(0)\|_{L^p(R)}+\int_0^T\|\partial_\tau v_i(\tau)\|_{L^p(R)}d\tau\}, \tag{4.8}$$

where $1<p<2$ and p',θ satisfy (1.30).

Proof. We only need to prove that the norm on the left-hand side of (4.8) can be obtained by interpolation of norms on the left-hand sides of (4.1) and (4.3).

Let $d\mu$ be the measure $\tau^{-2}d\tau$ on $[0,T]$. Then the norm on the left-hand side of (4.1) is

$$\sup_{0\le\tau\le T}(\tau\|v_0(\tau)\|_{L^1(R)})=\|\tau v_0\|_{L^\infty(d\mu;L^1(R))}, \tag{4.9}$$

and the norm on the left-hand side of (4.3) is

$$\left\{\int_0^T \|v_0(\tau)\|^2_{L^2(R)} d\tau\right\}^{\frac{1}{2}} = \|\tau v_0\|_{L^2(d\mu; L^2(R))}, \tag{4.10}$$

while the norm on the left-hand side of (4.8) is

$$\left\{\int_0^T \tau^{\theta p'} \|v_0(\tau)\|^{p'}_{L^p(R)} d\tau\right\}^{\frac{1}{p'}} = \|\tau v_0\|_{L^{p'}(d\mu; L^p(R))}, \tag{4.11}$$

for $1 < p < 2$ (since $\theta p' = p' - 2$). □

We are now ready to prove Theorem 1.5.

Let us define the set

$$X = \{w | w(0) = u_0, \left\{\int_0^T \tau^{\theta p'} \|\partial_\tau w(\tau)\|^{p'}_{L^p(R)} d\tau\right\}^{\frac{1}{p'}} \le C_1 \|u_0\|^3_{L^p(R)}\}, \tag{4.12}$$

where θ, p' are defined by (1.30) and C_1 is a positive constant independent of the initial data and will be determined later. For any $w \in X$, define a map M by

$$(Mw)(t) = u_0 \pm \int_0^t S(-\tau)[S(-\tau)\bar{w}(\tau)(S(\tau)w(\tau))^2] d\tau. \tag{4.13}$$

We want to show that M maps X into itself and is a contraction.

For simplicity, we denote $v = Mw$. Obviously,

$$v(0) = u_0 \tag{4.14}$$

and

$$\partial_\tau v(\tau) = \pm S(-\tau)[S(-\tau)\bar{w}(\tau)(S(\tau)w(\tau))^2]. \tag{4.15}$$

Applying Lemma 4.3, we get

$$\left\{\int_0^T \tau^{\theta p'} \|\partial_\tau v(\tau)\|^{p'}_{L^p(R)} d\tau\right\}^{\frac{1}{p'}} \le C(\|u_0\|_{L^p} + \int_0^T \|\partial_\tau w(\tau)\|_{L^p(R)} d\tau)^3. \tag{4.16}$$

By Hölder's inequality, we obtain

$$\begin{aligned} \int_0^T \|\partial_\tau w(\tau)\|_{L^p} d\tau &\le \left\{\int_0^T \tau^{-\theta p}\right\}^{\frac{1}{p}} \left\{\int_0^T \tau^{\theta p'} \|\partial_\tau w(\tau)\|^{p'}_{L^p(R)} d\tau\right\}^{\frac{1}{p'}} \\ &= CT^{\frac{1}{p'}} \left\{\int_0^T \tau^{\theta p'} \|\partial_\tau w(\tau)\|^{p'}_{L^p(R)} d\tau\right\}^{\frac{1}{p'}} \\ &\le CC_1 T^{\frac{1}{p'}} \|u_0\|^3_{L^p(R)}. \end{aligned} \tag{4.17}$$

It then follows that

$$\begin{aligned} \left\{\int_0^T \tau^{\theta p'} \|\partial_\tau v(\tau)\|^{p'}_{L^p(R)} d\tau\right\}^{\frac{1}{p'}} &\le C(\|u_0\|_{L^p(R)} + C_1 T^{\frac{1}{p'}} \|u_0\|^3_{L^p(R)})^3 \\ &\le C_1 \|u_0\|^3_{L^p(R)} \end{aligned} \tag{4.18}$$

provided that C_1 is suitably large and T is sufficiently small. By a similar argument, we can show that M is a contraction. Moreover, it is not difficult to prove (1.28) and (1.31).

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