

## MODULAR FORMS ON NON-LINEAR DOUBLE COVERS OF ALGEBRAIC GROUPS

HUNG YEAN LOKE AND GORDAN SAVIN

ABSTRACT. We construct automorphic representations of non-linear two-fold covers of simply connected Chevalley groups via residues of Eisenstein series. In the process, we establish some basic results in representation theory of local groups.

### 1. INTRODUCTION

Let  $\underline{G}$  be a split, simply connected algebraic group corresponding to an irreducible root system  $\Phi$ . The group  $\underline{G}$  can be constructed as a Chevalley group, which is defined over  $\mathbb{Z}$ . Over a local field  $\mathbb{R}$ ,  $\mathbb{Q}_p$  or the ring of adèles  $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ , the group  $\underline{G}$  has a unique non-trivial 2-fold central extension denoted by  $G$ :

$$1 \rightarrow \mu_2 \rightarrow G \rightarrow \underline{G} \rightarrow 1.$$

An irreducible representation of  $G$  (local or global) is called genuine if the central subgroup  $\mu_2$  acts via the unique non-trivial character on the representation. The central extension  $G(\mathbb{A})$  splits over the group of rational points  $\underline{G}(\mathbb{Q})$ . Thus it is natural to study the space  $L_{\text{gen}}^2(\underline{G}(\mathbb{Q}) \backslash G(\mathbb{A}))$  where the subscript gen indicates that we consider only the functions  $f$  such that  $f(\epsilon g) = \epsilon f(g)$  for every  $\epsilon$  in  $\mu_2$ . This problem is a natural continuation of the study of classical modular forms of half integral weight. One purpose of this paper is to define Eisenstein series on  $G(\mathbb{A})$  and to construct residual representation(s)  $\Theta$  which, if  $\underline{G} = \text{SL}_2$ , correspond to the classical theta series  $1 + 2 \sum_{n>0} q^{n^2}$  or its anti-holomorphic analogue. Along the way, we study principal series representations of groups  $G(\mathbb{Q}_p)$  where  $p$  is any prime.

In order to explain our results here, let  $\underline{T}$  be a maximal split torus in  $\underline{G}$ . Then its inverse image  $T$  in  $G$  is not necessarily commutative. Since the Weyl group acts by conjugation on irreducible genuine representations of  $T(\mathbb{Q}_p)$ , a natural question is whether there are Weyl group-invariant representations. A need for such representations is obvious: If  $V$  is a genuine representation of  $T(\mathbb{Q}_p)$ , then we can define a family of representations  $i(\chi) = V \otimes \chi$  by twisting with unramified characters of the torus  $\underline{T}(\mathbb{Q}_p)$ . If  $V$  is Weyl group-invariant, then the conjugation action of the Weyl group on  $i(\chi)$  reduces to the conjugation action on the character  $\chi$ . In this way, at least, one can express some basic results on principal series in a neat way. For example, if  $\underline{G} = \text{Sp}_{2n}$ , then Weyl group invariant  $V$  can be constructed using the Weil index [W] [Rao]. On the other hand, in [Sa2] an explicit construction of

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such representations is given for simply laced groups. However, the corresponding Weyl group invariance was obtained by a somewhat tedious check using relations in the Steinberg group. In this paper we present a more natural construction of those representations of  $T(\mathbb{Q}_p)$ . Their Weyl group invariance will follow from a simple global argument. More precisely, our result is based on an observation that the analogous problem for real groups already has a solution for real groups, as given by Adams, Barbasch, Paul, Vogan and Trapa in [A-V]. Let  $K_\infty$  be a maximal compact subgroup of  $G(\mathbb{R})$ . Recall that  $T(\mathbb{R})$  has a decomposition  $MA$ , where  $M$  is the centralizer of  $A$  in  $K_\infty$ . The group  $K_\infty$  has certain small genuine representations, called pseudo-spherical representations, whose property is that they reduce irreducibly to  $M$ . In particular, Weyl group invariance of such representations of  $M$  is now obvious. Next, we consider the space

$$L_{\text{gen}}^2(AT(\mathbb{Q})\backslash T(\mathbb{A}))$$

of automorphic representations of  $T(\mathbb{A})$ . Given a pseudo-spherical type  $\delta$ , one easily sees that there is only one automorphic representation  $\pi = \bigotimes \pi_v$  of  $T(\mathbb{A})$  such that  $\pi_\infty \cong \delta$  and  $\pi_p$  is unramified for all primes  $p$ . The uniqueness of  $\pi$  and the Weyl group invariance of  $\delta$  immediately imply the Weyl group invariance of all  $\pi_p$ . If  $\underline{G} = \text{Sp}_{2n}$ , then one easily sees that our construction gives a Weil index.

We use  $\pi$  to define local principal series representations, the corresponding Eisenstein series and a global residual representation  $\Theta$  of Eisenstein series. Moreover, if  $p \neq 2$  we use the central character  $\gamma_p$  of  $\pi_p$  to normalize Hecke operators in the Iwahori Hecke algebra  $\mathcal{H}_-$  of  $G(\mathbb{Q})$ . Following [Sa2] this Hecke algebra is isomorphic to the Iwahori Hecke algebra of an algebraic group  $\underline{G}^l$ . This isomorphism allows us to (Shimura) lift genuine representations of  $G(\mathbb{Q}_p)$  with Iwahori fixed vectors to the linear group  $\underline{G}^l(\mathbb{Q}_p)$ . We show that the Shimura lift sends unitary representations to unitary representations. For example, the local component  $\Theta_p$  of  $\Theta$  lifts to the trivial representation of  $\underline{G}^l(\mathbb{Q}_p)$ . In particular, if  $\underline{G} \neq \text{SL}_2$ , it follows that  $\Theta_p$  is isolated in the unitary dual of  $G(\mathbb{Q}_p)$ . We emphasize once again that the representation  $\Theta$  and the isomorphism of Hecke algebra depend on the choice of the pseudo-spherical type  $\delta$ .

## 2. AN ADÈLIC GROUP

Let  $\Phi$  be a root system with simple roots  $\Delta = \{\alpha_1, \dots, \alpha_l\}$ . Let  $(\alpha|\beta)$  denote the inner product on  $\Phi$  normalized such that  $(\alpha|\alpha) = 2$  for a long root. We set  $\alpha^\vee := \frac{2\alpha}{(\alpha|\alpha)}$  and  $\langle \alpha, \beta \rangle := (\alpha|\beta^\vee)$ . We extend  $\langle \cdot, \cdot \rangle$  to a pairing between the root lattice and the co-root lattice  $\Lambda$ .

Let  $\mathfrak{g}$  be the corresponding simple Lie algebra over  $\mathbb{Q}$ . Fix a Chevalley basis in  $\mathfrak{g}$ . It defines a simply connected group Chevalley group  $\underline{G}$ . It is an algebraic group defined over  $\mathbb{Z}$ . For any field  $F$  the group  $\underline{G}(F)$  is generated by one-parameter subgroups  $\underline{U}_\alpha \simeq F$  for every root  $\alpha$  in  $\Phi$ . More precisely, the choice of Chevalley basis fixes an isomorphism of  $F$  and  $\underline{U}_\alpha$ ,  $t \mapsto \underline{e}_\alpha(t)$  for every  $t \in F$ . For example, if  $G = \text{SL}_2$ , then  $\underline{e}_\alpha(t)$  and  $\underline{e}_{-\alpha}(t)$  are

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

respectively. Define elements

$$\begin{cases} \underline{w}_\alpha(t) = \underline{e}_\alpha(t) \underline{e}_{-\alpha}(-t^{-1}) \underline{e}_\alpha(t), \\ \underline{h}_\alpha(t) = \underline{w}_\alpha(t) \underline{w}_\alpha(-1). \end{cases}$$

If  $G = \text{SL}_2$ , then these elements are

$$\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

By a result of Steinberg (Theorem 8(b), page 66 in [St]), the group  $\underline{G}(F)$  is abstractly generated by the one-parameter groups  $\underline{U}_\alpha$  modulo relations

$$(1) \quad [\underline{e}_\alpha(t), \underline{e}_\beta(u)] = \begin{cases} \prod_{i,j \geq 1} \underline{e}_{i\alpha+j\beta}(c_{ij}t^i u^j) & \text{if } \alpha + \beta \text{ is a root,} \\ 1 & \text{if not, and } -\alpha \neq \beta \end{cases}$$

and

$$(2) \quad \underline{h}_\alpha(s)\underline{h}_\alpha(t) = \underline{h}_\alpha(st),$$

where  $c_{ij}$  are integers depending on  $\alpha, \beta$ .

Now assume that  $F = \mathbb{R}$  or  $\mathbb{Q}_p$ . Let  $(\cdot, \cdot)$  be the Hilbert symbol<sup>1</sup> over  $F$ . It defines a two-fold central extension  $G(F)$ ,

$$1 \rightarrow \mu_2 \rightarrow G(F) \xrightarrow{\text{pr}} \underline{G}(F) \rightarrow 1,$$

by replacing the relation (2) by

$$(3) \quad h_\alpha(s)h_\alpha(t) = h_\alpha(st) \cdot (s, t)^{\frac{1}{2}(\alpha^\vee | \alpha^\vee)}.$$

Indeed, Steinberg (Theorem 12, page 86 in [St]) shows that a two-fold central extension of  $\underline{G}(F)$  is necessarily defined by these generators and relations, while Matsumoto [Ma] proves existence of the central extension.

Let  $U_\alpha$  be the subgroup of  $G(F)$  generated by  $e_\alpha(t)$ . Then  $U_\alpha \simeq \underline{U}_\alpha$  and the splitting is unique since  $F$  is two-divisible. Important to us will be the subgroups  $G_\alpha$  generated by  $U_\alpha$  and  $U_{-\alpha}$ . Let  $\underline{G}_\alpha \cong \text{SL}_2$  be the projection of  $G_\alpha$  in  $\underline{G}$ . Since  $[h_\alpha(t), e_\alpha(u)] = e_\alpha((t^2 - 1)u)$ , the group  $G_\alpha$  is perfect. Thus  $G_\alpha$  is a central extension of  $\underline{G}_\alpha$  of degree  $m_\alpha$ . It follows from (3) that  $m_\alpha = 2$  except when  $\alpha$  is a short root in  $B_n, C_n$  or  $F_4$ ; then  $m_\alpha = 1$ . Indeed, if  $\alpha$  is a short root in  $B_n, C_n$  or  $F_4$ , then  $(\alpha^\vee | \alpha^\vee) = 4$  and there is no Hilbert symbol in (3).

The group  $\underline{G}(\mathbb{Z}_p)$  is a (preferred) hyperspecial maximal compact subgroup of  $\underline{G}(\mathbb{Q}_p)$ . It stabilizes the Chevalley lattice and is generated by  $e_\alpha(t)$  with  $t$  in  $\mathbb{Z}_p$ . By reducing modulo  $p$  we have an exact sequence

$$1 \rightarrow K_p^1 \rightarrow \underline{G}(\mathbb{Z}_p) \rightarrow \underline{G}(\mathbb{F}_p) \rightarrow 1.$$

**Proposition 2.1.** *The central extension splits over  $\underline{G}(\mathbb{Z}_p)$  for  $p \neq 2$ . The splitting homomorphism  $s : \underline{G}(\mathbb{Z}_p) \rightarrow G(\mathbb{Q}_p)$  is unique and its image is henceforth denoted by  $K_p$ .*

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<sup>1</sup>For reference: The Hilbert symbol over  $\mathbb{Q}_2$  is given by  $(2^\alpha u, 2^\beta v)_2 = (-1)^r$ , where  $u, v \in 1 + 2\mathbb{Z}_2$  and  $r = \frac{u-1}{2} \frac{v-1}{2} + \alpha \frac{v^2-1}{8} + \beta \frac{u^2-1}{8}$ . The symbol over  $\mathbb{Q}_p$  is  $(p^\alpha u, p^\beta v)_p = (-1)^r \left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha$ , where  $u, v \in \mathbb{Z}_p^\times$  and  $r = \alpha\beta \left(\frac{p-1}{2}\right)$ .

*Proof.* As the proof of Lemma 11.3 in [Mo] shows, the central extension splits over the pro- $p$  subgroup  $K_p^1$ . Hence the central extension of  $\underline{G}(\mathbb{Q}_p)$  gives rise to a central extension of the finite group  $\underline{G}(\mathbb{F}_p)$ . However, the group  $\underline{G}(\mathbb{F}_p)$  has no Schur multipliers of order 2 if  $p$  is odd and the group is not of type  $B_3$  [Gr]. This proves that the central extension splits over the hyperspecial maximal compact subgroup except perhaps for the type  $B_3$ . However, a splitting for the type  $B_4$  implies a splitting for the type  $B_3$ , by inclusion of the corresponding groups.

It remains to show that the splitting is unique. Any two splittings differ by a homomorphism from  $\underline{G}(\mathbb{Z}_p)$  to  $\mu_2$ . Such a homomorphism is clearly trivial on the pro- $p$  group  $K_p^1$ , and then it must be trivial on  $\underline{G}(\mathbb{F}_p)$  since it has no normal subgroups of index  $Z$ . (Both arguments rely on the fact that  $p \neq 2$ .)  $\square$

**Proposition 2.2.** *If  $p$  is odd, then  $K_p$  contains  $e_\alpha(t)$  for all  $t \in \mathbb{Z}_p$  and, therefore,  $h_\alpha(t)$  for all  $t \in \mathbb{Z}_p^\times$ .*

*Proof.* Note that  $U_\alpha$  and  $K_p$  give two splittings of  $\underline{U}_\alpha(\mathbb{Z}_p)$ . They differ by a quadratic character of  $\mathbb{Z}_p$ . Since  $\mathbb{Z}_p$  is 2-divisible if  $p \neq 2$ , the character is trivial.  $\square$

Let  $S$  be any finite set of places containing  $\{\infty, 2\}$ . Let

$$\mu_S = \left\{ (\epsilon_1, \dots, \epsilon_{|S|}) \in \mu_2^{|S|} : \epsilon_1 \cdots \epsilon_{|S|} = 1 \right\}.$$

Define

$$G_S = \left( \prod_{v \in S} G(\mathbb{Q}_v) \right) / \mu_S \times \prod_{v \notin S} K_v.$$

If  $S \subseteq S'$ , then  $G_S \subseteq G_{S'}$ . We define  $G(\mathbb{A})$  as a direct limit of all  $G_S$ . We have a central extension

$$1 \rightarrow \mu_2 \rightarrow G(\mathbb{A}) \rightarrow \underline{G}(\mathbb{A}) \rightarrow 1.$$

For every  $\alpha \in \Phi$  and  $t \in \mathbb{Q}$ ,  $e_\alpha(t)$  can be viewed as an element in  $G(\mathbb{A})$  by diagonal embedding. This is well-defined by Proposition 2.2. These elements clearly satisfy the relations in (1). Moreover, corresponding  $h_\alpha(t)$ 's satisfy the relations in (2) by quadratic reciprocity for the Hilbert symbol. In particular, we have an explicit splitting of the extension over  $\underline{G}(\mathbb{Q})$ .

**Maximal compact  $K_\infty$ .** There is an automorphism  $\sigma$  of  $G(\mathbb{R})$  such that  $\sigma : e_\alpha(t) \mapsto e_{-\alpha}(-t)$  for every root  $\alpha$  and  $t \in \mathbb{R}$  (see Thm. 16 in [St]). The fixed points of  $\sigma$  on  $\underline{G}(\mathbb{R})$  is a maximal compact subgroup  $K_\infty$ . Similarly there is an automorphism  $\underline{\sigma}$  of  $\underline{G}(\mathbb{R})$ , and its fixed points  $\underline{K}_\infty$  are a maximal compact subgroup of  $\underline{G}(\mathbb{R})$ .

### 3. THE TORUS

Let  $\underline{T} \subseteq \underline{G}$  be the maximal split torus. If  $R$  is a ring, then  $\underline{T}(R)$  is generated by  $\underline{h}_\alpha(t)$  with  $t \in R^\times$ . If  $\Lambda$  is the co-root lattice, then  $\underline{T}(R) \simeq \Lambda \otimes_{\mathbb{Z}} R^\times$  with the isomorphism given by

$$\underline{h}_\alpha(t) \mapsto \alpha^\vee \otimes t.$$

Let  $T(F) \subset G(F)$  be the inverse image of  $\underline{T}(F)$ . Then  $T(F)$  is generated by  $h_\alpha(t)$ . The following commutator formula ([Ma], Lemme 5.4) is crucial to us throughout the paper:

$$[h_\alpha(s), h_\beta(t)] = (s, t)^{(\alpha^\vee | \beta^\vee)}.$$

The goal of this section is to describe the structure of  $T(F)$  for  $F = \mathbb{R}$  and  $F = \mathbb{Q}_p$  and to define pseudo-spherical representations of  $T(\mathbb{R})$  and  $T(\mathbb{Q}_2)$ , and unramified representations of  $T(\mathbb{Q}_p)$  for  $p$  odd.

*Case  $F = \mathbb{Q}_p$ , with  $p$  odd.* Define  $T_p = T(\mathbb{Q}_p) \cap K_p$ . Then by Proposition 2.2,  $T_p$  is generated by  $h_\alpha(t)$  for all  $t \in \mathbb{Z}_p^\times$  and is isomorphic to  $\underline{T}(\mathbb{Z}_p)$  by  $h_\alpha(t) \mapsto \underline{h}_\alpha(t)$ . Note that the symbol  $(\cdot, \cdot)$  is tame here, i.e.  $h_\alpha(s)h_\alpha(t) = h_\alpha(st)$  for all  $s, t \in \mathbb{Z}_p^\times$ . Let  $T_p^2$  be the set of squares in  $T_p$ . Critical to us are the genuine representations of  $T(\mathbb{Q}_p)$  which are trivial on  $T_p^2$ . A genuine representation of  $T(\mathbb{Q}_p)$  is unramified if it has a non-zero vector fixed by  $T_p$ .

*Case  $F = \mathbb{R}$ .* We note that  $(-1, -1) = -1$ . In this case  $\underline{T}(\mathbb{R}) = \underline{M}\underline{A}$ , where  $\underline{M} \simeq \Lambda \otimes \{\pm 1\}$  and  $\underline{A} \simeq \Lambda \otimes \mathbb{R}^+$ . Then  $T(\mathbb{R}) = MA$ , where  $M$  is generated by  $h_\alpha(-1)$  and contains the kernel  $\mu_2$  of the central extension. On the other hand  $A$  is generated by  $h_\alpha(t)$  for  $t \in \mathbb{R}^+$  and  $A \simeq \underline{A}$ . Note also that  $A$  is in the center of  $T(\mathbb{R})$ . Thus it is natural to concentrate on genuine representations of  $M$ . Let  $M_s$  be the subgroup of  $M$  generated by  $h_\alpha(-1)$  for all roots  $\alpha$  such that  $m_\alpha = 1$ . Since  $h_\alpha(-1)h_\alpha(-1) = 1$  for such roots,  $M_s$  does not contain the central subgroup  $\mu_2 \subset M$ . An irreducible genuine representation of  $M$  trivial on the normal subgroup  $M_s$  is called a pseudo-spherical representation. An important feature of pseudo-spherical representations of  $M$  is that they are invariant under the conjugation action of the Weyl group. See Lemma 4.11(3) in [A-V].

*Case  $F = \mathbb{Q}_2$ .* This is the most interesting case. The Hilbert symbol is ramified. The group  $\mathbb{Z}_2^\times$  has a filtration

$$\mathbb{Z}_2^\times = 1 + 2\mathbb{Z}_2 \supseteq 1 + 4\mathbb{Z}_2 \supseteq 1 + 8\mathbb{Z}_2.$$

Note that  $1 + 8\mathbb{Z}_2 = (\mathbb{Z}_2^\times)^2$ . In particular,  $1 + 8\mathbb{Z}_2$  is in the kernel of the Hilbert symbol. Since  $\mathbb{Z}_2^\times / (1 + 8\mathbb{Z}_2) \simeq (\mathbb{Z}/8\mathbb{Z})^\times = \{\pm 1, \pm 5\}$ , all values of the symbol are easily obtained from the following table.

	2	-1	5
2	1	1	-1
-1	1	-1	1
5	-1	1	1

Observe that the kernel of the symbol  $(\cdot, \cdot)$  when restricted to  $\mathbb{Z}_2^\times$  is  $1 + 4\mathbb{Z}_2$ . For every integer  $i \geq 1$ , let  $\underline{T}_2^i$  be the subgroup of  $\underline{T}(\mathbb{Z}_2)$  isomorphic to

$$\underline{T}_2^i \simeq \Lambda \otimes (1 + 2^{1+i}\mathbb{Z}_2).$$

Let  $T(\mathbb{Z}_2) \subset G(\mathbb{Q}_2)$  be the inverse image of  $\underline{T}(\mathbb{Z}_2)$ . Since the Hilbert symbol is trivial on  $1 + 4\mathbb{Z}_2$ , for every  $i \geq 1$ , elements  $h_\alpha(t)$  for  $t \in 1 + 2^{1+i}\mathbb{Z}_2$  generate a subgroup  $T_2^i \subset T(\mathbb{Z}_2)$  isomorphic to  $\underline{T}_2^i$ . Note that  $T_2^1$  is contained in the center of  $T(\mathbb{Z}_2)$ , while  $T_2^2$  is contained in the center of  $T(\mathbb{Q}_2)$ .

Since  $(-1, -1)_2 = (-1, -1)_\infty = -1$ , the subgroup of  $T(\mathbb{Z}_2)$  generated by  $h_\alpha(-1)$  is isomorphic to  $M$  of the real case! Moreover, since the non-trivial coset of  $1 + 4\mathbb{Z}_2$  in  $1 + 2\mathbb{Z}_2 = \mathbb{Z}_2^\times$  is represented by  $-1$ , we have an isomorphism

$$T(\mathbb{Z}_2) \simeq M \times T_2^1.$$

As in the real case, let  $M_s$  be the subgroup of  $M$  generated by  $h_\alpha(-1)$  for all roots  $\alpha$  such that  $m_\alpha = 1$ . Then  $M_s T_2^1$  is a commutative subgroup of  $T(\mathbb{Q}_2)$ . Note that

this group is generated by  $h_\alpha(t)$ , where  $t \in 1 + 4\mathbb{Z}_2$  if  $\alpha$  is long and  $t \in \mathbb{Z}_2^\times$  if  $\alpha$  is short. We say that a genuine representation of  $T(\mathbb{Q}_2)$  is pseudo-spherical if it has a vector invariant under  $M_s T_2^1$ .

**Weyl groups.** Assume that  $F = \mathbb{R}$  or  $\mathbb{Q}_p$ . Let  $W_F$  denote the subgroup of  $G(F)$  generated by  $w_\alpha(1)$  for all simple roots  $\alpha$ . Let  $T_F(\mathbb{Z})$  denote the subgroup generated by  $h_\alpha(-1)$  for all simple roots  $\alpha$ . Let  $W$  denote the Weyl group of  $\underline{G}(\mathbb{Q})$ . Then we have an exact sequence

$$(4) \quad 1 \rightarrow T_F(\mathbb{Z}) \rightarrow W_F \rightarrow W \rightarrow 1.$$

Conjugation action of  $W_F$  on  $T(F)$  does not descend to that of  $W$  because  $T_F(\mathbb{Z})$  does not lie in the center of  $T(F)$ . Suppose  $(\pi, V)$  is a representation of  $T(F)$  and  $w \in W_F$ . Let  $V^w$  denote the representation defined by  $t \mapsto \pi(w^{-1}tw)$ . Note that the isomorphism class of  $V^w$  depends only on the projection of  $w$  into the Weyl group  $W$ . In other words, we have a conjugation action of the Weyl group on the set of isomorphism classes of irreducible representations of  $T(F)$ . The following lemma implies that the classes of pseudo-spherical and unramified representations are preserved under the conjugation action of the Weyl group.

**Proposition 3.1.** *The following subgroups of  $T(F)$  are normalized by  $W_F$ :*

- (i)  $T_p$  if  $F = \mathbb{Q}_p$  and  $p$  is an odd prime.
- (ii)  $T_2^1$  and  $M_s$  if  $F = \mathbb{Q}_2$ .
- (iii)  $A$  and  $M_s$  if  $F = \mathbb{R}$ .

*Proof.* Combining (3) and Lemma 37(c) in [St] gives

$$w_\alpha(1)h_\beta(t)w_\alpha(-1) = h_\kappa(t) \cdot (c, t)^{\frac{1}{2}(\beta^\vee|\beta^\vee)}$$

where  $\kappa = w_\alpha(\beta)$  and  $c = \pm 1$ , which depends on structure coefficients for the Chevalley basis. In order to prove the proposition we need to show that the sign after  $h_\kappa(t)$  is trivial for  $h_\alpha(t)$  generating the relevant groups. If  $h_\alpha(t)$  is in  $T_p, T_2^1$  or  $A$ , then  $(c, t) = 1$  by elementary properties of the Hilbert symbol. Finally, recall that  $M_s$  is generated by  $h_\beta(-1)$ , where  $\beta$  is a root such that  $(\beta^\vee|\beta^\vee) = 4$ . Thus the sign is trivial in here, too. □

#### 4. REPRESENTATIONS OF $T(F)$

Assume that  $H$  is a subgroup of  $G$  which is the inverse image of an abelian subgroup  $\underline{H}$  in  $\underline{G}$ . Assume furthermore that the center  $Z(H)$  of  $H$  has finite index in  $H$ . Let  $\mathbf{H} = H/Z(H)$  and  $q : H \rightarrow \mathbf{H}$  denote the quotient map. Since  $\underline{H}$  is abelian, the square of any element of  $H$  is contained in  $\mu_2 \subseteq Z(H)$ . It follows that  $\mathbf{H} \simeq (\mathbb{Z}/2\mathbb{Z})^r$ , and we may consider  $\mathbf{H}$  as a vector space over  $\mathbb{Z}/2\mathbb{Z}$ . Given  $\mathbf{x} = q(x), \mathbf{y} = q(y) \in \mathbf{H}$  for some  $x, y \in H$ , we define  $B(\mathbf{x}, \mathbf{y}) = xyx^{-1}y^{-1} \in \mu_2$ . The definition of  $B$  is independent of the choice of  $x, y$ , and  $B$  could be interpreted as a symplectic non-degenerate form on  $\mathbf{H}$ . In particular, we may write  $\mathbf{H} = \mathbf{H}_1 \oplus \mathbf{H}_2$  as a direct sum of isotropic subspaces with respect to  $B$ , and  $\dim \mathbf{H} = r$  is even. We define  $H_1 = q^{-1}\mathbf{H}_1$ , which is an abelian subgroup of  $H$  containing  $Z(H)$ .

Recall that an irreducible representation of  $H$  (resp.  $Z(H)$ ) is called genuine if it is non-trivial on the kernel  $\mu_2$  of the covering map. Let  $\text{Irr}_{\text{gen}}(H)$  be the set of equivalence classes of irreducible genuine finite dimensional representations of  $H$ , and let  $\text{Irr}_{\text{gen}}(Z(H))$  be the set of genuine characters of  $Z(H)$ .

**Proposition 4.1.** *Given  $H$  and  $Z(H)$  as above. Then there is a one-to-one correspondence between  $\text{Irr}_{\text{gen}}(H)$  and  $\text{Irr}_{\text{gen}}(Z(H))$  given by sending an irreducible genuine representation of  $H$  to its central character. Moreover, the dimension of every genuine irreducible representation is equal to the square root of the index of  $Z(H)$  in  $H$ .  $\square$*

*Proof.* This is essentially Proposition 2.2 in [A-V]. Let  $V \in \text{Irr}_{\text{gen}}(H)$ . Let  $\chi_V$  denote its character, which is well defined since  $V$  is finite dimensional. The exact same argument as in [A-V] shows that  $\chi_V$  is supported on  $Z(H)$ . By Proposition 3 in Chapter 8, Section 12 in [Bou], the isomorphism class of  $V$  is uniquely determined by  $\chi_V$ . Hence the isomorphism class of  $V$  is uniquely determined by its central character in  $\text{Irr}_{\text{gen}}(Z(H))$ .

Conversely, given  $\chi \in \text{Irr}_{\text{gen}}(Z(H))$ , we can extend  $\chi$  to a one dimensional character  $\tilde{\chi}$  of  $H_1$ . Indeed we may choose  $\tilde{\chi}$  to be an irreducible  $H_1$ -submodule of  $\text{Ind}_{Z(H)}^{H_1} \chi$ . By Mackey theory,  $\text{Ind}_{H_1}^H \tilde{\chi}$  is an irreducible representation of  $H$  of dimension  $[H : Z(H)]^{1/2}$  with central character  $\chi$ .  $\square$

We apply this proposition to the group  $M$ , which is the inverse image of  $\underline{M}$ . In order to describe the center  $Z(M)$  of  $M$  we need to consider the commutator map on  $M$ , which induces a (symmetric)  $\mu_2$ -valued pairing on  $\underline{M} \cong \Lambda \otimes \{\pm 1\} \cong \Lambda/2\Lambda$ . Since the commutator is given by

$$[h_\alpha(-1), h_\beta(-1)] = (-1, -1)_2^{(\alpha^\vee | \beta^\vee)},$$

the pairing is (the same as) the bilinear form  $(\cdot | \cdot)$  reduced modulo 2. The kernel is given by the lattice  $\Lambda \cap 2\Lambda^*$ , where  $\Lambda^* \supseteq \Lambda$  is the dual lattice with respect to the form  $(\cdot | \cdot)$ . In particular, the index of  $\mu_2$  in  $Z(M)$  is equal to the index  $[\Lambda \cap 2\Lambda^* : 2\Lambda]$ , and the index of  $Z(M)$  in  $M$  is equal to the index  $[\Lambda : \Lambda \cap 2\Lambda^*]$ . By Proposition 4.1 we have proved the following:

**Proposition 4.2.** *The number of irreducible genuine representations of  $M$  is equal to the index  $[\Lambda \cap 2\Lambda^* : 2\Lambda]$ . The dimension of each such representation is a square root of the index of  $[\Lambda : \Lambda \cap 2\Lambda^*]$ .*

In the following table we give the index of  $\Lambda \cap 2\Lambda^*$  in  $\Lambda$  in the simply laced case and  $G_2$ :

$\Phi$	$A_{2n-1}$	$A_{2n}$	$D_{2n-1}$	$D_{2n}$	$E_6$	$E_7$	$E_8$	$G_2$
$[\Lambda : \Lambda \cap 2\Lambda^*]$	$4^{n-1}$	$4^n$	$4^{n-1}$	$4^{n-1}$	$4^3$	$4^3$	$4^4$	4

The index for types  $B_l$ ,  $C_l$  and  $F_4$  is the same as the index for  $A_{l-1}$ ,  $A_1$  and  $A_2$ , respectively. In other words, it is the same as the index for the subsystem generated by simple long roots.

In order to discuss genuine irreducible representations of  $T(\mathbb{Q}_p)$ , we need to describe the center of  $T(\mathbb{Q}_p)$ . We fix a choice of simple roots  $\Delta = \{\alpha_1, \dots, \alpha_l\}$ . If  $\lambda = n_1\alpha_1^\vee + \dots + n_l\alpha_l^\vee$  is an element in the co-root lattice  $\Lambda$ , then we define

$$\eta(\lambda) := h_{\alpha_1}(p^{n_1}) \cdots h_{\alpha_l}(p^{n_l}) \in T(\mathbb{Q}_p).$$

We shall use  $\eta_p$  instead of  $\eta$  if there is a need to distinguish between primes. Note that the order of multiplication is important, as the  $h_{\alpha_i}(p^{n_i})$ 's may not commute with one another. Indeed, the commutator is given by

$$[\eta(\lambda), \eta(\lambda')] = (p, p)^{(\lambda | \lambda')},$$

which may be non-trivial since  $(p, p) = -1$  if  $p \equiv 3 \pmod{4}$ . If  $\Lambda'$  is a subset of  $\Lambda$ , then we set  $\eta(\Lambda') := \{\eta(\lambda) : \lambda \in \Lambda'\}$ .

*Case  $p$  is odd.* Note that we have a decomposition  $T(\mathbb{Q}_p) = T_p \cdot \eta(\Lambda) \cdot \mu_2$ . The commutator of  $h_\alpha(p)$  in  $\eta(\Lambda)$  and  $h_\beta(t)$  in  $T_p$  is

$$[h_\alpha(p), h_\beta(t)] = (p, t)_p^{(\alpha^\vee | \beta^\vee)}.$$

Since  $(p, t)_p = 1$  if and only if  $t$  is a square in  $\mathbb{Z}_p^\times$ , it follows that the commutator defines a pairing of  $\Lambda \times T_p/T_p^2 \cong \Lambda \times \Lambda/2\Lambda$  which is simply the restriction of the bilinear form  $(\cdot | \cdot)$  modulo 2. This shows that the centralizer of  $T_p$  in  $\eta(\Lambda)$  is  $\eta(\Lambda \cap 2\Lambda^*)$  and the centralizer of  $\eta(\Lambda)$  in  $T_p$  is the group  $C_p$  containing  $T_p^2$ , and such that  $C_p/T_p^2 \cong (\Lambda \cap 2\Lambda^*)/2\Lambda$ . It follows that the center of  $T(\mathbb{Q}_p)$  is  $Z_p = C_p \cdot \eta(\Lambda \cap 2\Lambda^*) \cdot \mu_2$ . Note that the index of  $Z_p$  in  $T(\mathbb{Q}_p)$  is  $[\Lambda : \Lambda \cap 2\Lambda^*]^2$ . The next proposition follows from Proposition 4.1.

**Proposition 4.3.** *There is a bijection between genuine irreducible representations  $V$  of  $T(\mathbb{Q}_p)$  and genuine characters  $\gamma$  of  $Z_p$ , the center of  $T(\mathbb{Q}_p)$ . Moreover, any such representation  $V$  has the dimension equal to the index  $[\Lambda : \Lambda \cap 2\Lambda^*]$ .  $\square$*

If  $\gamma$  is a genuine character of  $Z_p$ , the corresponding representation of  $T(\mathbb{Q}_p)$  will be henceforth denoted by  $V(\gamma)$ . Let  $\text{Irr}_{\text{gen}}^2(T(\mathbb{Q}_p))$  be the set of isomorphism classes of genuine representations of  $T(\mathbb{Q}_p)$  with non-zero  $T_p^2$ -fixed vectors. Define an equivalence relation on  $\text{Irr}_{\text{gen}}^2(T(\mathbb{Q}_p))$ , where two representations  $V$  and  $V'$  are equivalent if  $V'$  is isomorphic to a twist of  $V$  by an unramified character of the algebraic torus  $\underline{T}(\mathbb{Q}_p)$ .

**Proposition 4.4.** *Two genuine representations  $V(\gamma)$  and  $V(\gamma')$  in  $\text{Irr}_{\text{gen}}^2(T(\mathbb{Q}_p))$  are equivalent if and only if  $\gamma|_{C_p} = \gamma'|_{C_p}$ . The number of equivalence classes is equal to the index  $[\Lambda \cap 2\Lambda^* : 2\Lambda]$ . Only one of these classes, the class where  $\gamma|_{C_p} = 1$ , consists of unramified representations of  $T(\mathbb{Q}_p)$ .*

*Proof.* Since  $Z_p = C_p \cdot \eta(\Lambda \cap 2\Lambda^*) \cdot \mu_2$ , it easily follows that any two genuine characters of  $Z_p$  which coincide on  $C_p$  are unramified twists of one another. It follows that the equivalence classes are parameterized by characters of the finite group  $C_p/T_p^2$ . Since the order of this group is  $[\Lambda \cap 2\Lambda^* : 2\Lambda]$ , we have proved the first two statements. If  $V(\gamma)$  is unramified, that is, it contains a vector fixed by  $T_p$ , then the central character must be trivial on  $C_p$ . The proposition is proved.  $\square$

*Case  $p = 2$ .* The set  $T^1(\mathbb{Q}_2) := T_2^1 \cdot \eta_2(\Lambda) \cdot \mu_2$  is a normal subgroup of  $T(\mathbb{Q}_2)$  and commutes with  $M$ , as can be seen from the values of the Hilbert symbol  $(\cdot, \cdot)_2$ . Thus

$$T(\mathbb{Q}_2) = (M \times T^1(\mathbb{Q}_2))/\mu_2.$$

It follows that any genuine representation of  $T(\mathbb{Q}_2)$  is a tensor product of genuine representations of  $M$  and  $T^1(\mathbb{Q}_2)$ . Moreover, we have the following key proposition which reduces the study of representations of  $T(\mathbb{Q}_2)$  to that of  $M$  and  $T(\mathbb{Q}_p)$  for  $p \equiv 1 \pmod{4}$ .

**Proposition 4.5.** *Assume that  $p \equiv 1 \pmod{4}$ . Pick a non-square  $\zeta$  in  $\mathbb{F}_p^\times$ . The map given by  $h_\alpha(2) \mapsto h_\alpha(p)$  and  $h_\alpha(5) \mapsto h_\alpha(\zeta)$  induces an isomorphism*

$$T^1(\mathbb{Q}_2)/T_2^2 \cong T(\mathbb{Q}_p)/T_p^2.$$

*Proof.* This is obvious since the tame symbol  $(\cdot, \cdot)_p$  takes the following values:

	$p$	$\zeta$
$p$	1	-1
$\zeta$	-1	1

□

5. MODULAR FORMS ON  $T(\mathbb{A})$

We are interested in studying Eisenstein series on  $G(\mathbb{A})$ . To that end we need to understand the space  $\mathcal{A} = L^2_{gen}(A\mathbb{T}(\mathbb{Q})\backslash T(\mathbb{A}))$ . It is natural to look for maximally unramified representations in  $\mathcal{A}$  first. Recall that  $T_p = K_p \cap T(\mathbb{Q}_p)$  if  $p$  is odd and  $T_2^1$  is generated by  $h_\alpha(t)$  for all simple roots  $\alpha$  and  $t \in 1 + 4\mathbb{Z}_2$ .

**Proposition 5.1.** *Let  $\mathcal{A}_0$  be the space of all right  $T_2^1 \prod_{p \neq 2} T_p$ -invariant functions in  $\mathcal{A}$ . Note that this is naturally an  $M \times M$  module where the two factors sit in  $T(\mathbb{R})$  and  $T(\mathbb{Z}_2)$ . As such it is isomorphic to the genuine part of the regular representation of the finite group  $M$ :*

$$\mathcal{A}_0 \cong L^2_{gen}(M).$$

*Proof.* In the proof,  $h_{\alpha, \mathbb{Q}}(t)$ ,  $h_{\alpha, \infty}(t)$  and  $h_{\alpha, p}(t)$  denote elements of the global group  $\mathbb{T}(\mathbb{Q})$  and the local groups  $T(\mathbb{R})$  and  $T(\mathbb{Q}_p)$ , respectively. Let  $I$  be the group of invertible adeles. In view of the decomposition

$$I = \mathbb{Q}^\times \cdot \mathbb{R}^+ \times \prod_p \mathbb{Z}_p^\times$$

the space  $\mathcal{A}_0$  is indeed isomorphic to  $L^2_{gen}(M)$ , where here  $M$  is considered as a subgroup of  $T(\mathbb{Z}_2)$ . In order to finish the proof we need to determine the action of  $h_{\alpha, \infty}(-1)$  for this identification. Let  $f$  be in  $\mathcal{A}_0$ . Since  $f$  is left  $\mathbb{T}(\mathbb{Q})$  and right  $T_p$ -invariant, for every  $p \neq 2$ , for every  $m$  in  $T(\mathbb{Z}_2)$  we have

$$f(mh_{\alpha, \infty}(-1)) = f(h_{\alpha, \mathbb{Q}}(-1)^{-1}mh_{\alpha, \infty}(-1)) = f(h_{\alpha, 2}(-1)^{-1}m).$$

□

Recall that  $M_s \subseteq M$  is generated by  $h_{\alpha, 2}(-1)$  for all roots  $\alpha$  such that  $m_\alpha = 1$ . In particular it is a central subgroup. Now let  $\mathcal{A}_{00}$  be the subspace of  $\mathcal{A}_0$  consisting of  $M_s$ -invariant functions. Let  $\bar{M} = M/M_s$  be the quotient group. By the Peter-Weyl theorem, we have

$$\mathcal{A}_{00} \cong L^2_{gen}(\bar{M}) = \bigoplus_{\delta} \delta \otimes \delta^\vee,$$

where the sum is taken over irreducible genuine representations  $\delta$  of  $\bar{M}$  or, equivalently, over the pseudo-spherical representations of  $M$ . Thus we have the following corollary:

**Corollary 5.2.** *Let  $\delta$  be a pseudo-spherical representation of  $M$ . Then there exists a unique representation  $\pi \subseteq L^2_{gen}(A\mathbb{T}(\mathbb{Q})\backslash T(\mathbb{A}))$  such that  $\pi_\infty \cong \delta$  and  $\pi_p$  is unramified at all primes. The isomorphism class of  $\pi_p$  is invariant under the conjugation of the Weyl group.*

*Proof.* The uniqueness is obvious. Now consider a Weyl group conjugate  $\pi^w$ . Note that  $\pi^w$  is again unramified at all primes. Since  $\delta^w \cong \delta$  it follows that  $\pi^w \cong \pi$  by the uniqueness of  $\pi$ .  $\square$

Let  $\pi$  be the global representation as in the previous corollary. We would like to determine the local components  $\pi_p$ . To that end we need to determine the corresponding central characters. A large part of the center acts trivially on  $\pi$ , independent of the choice of  $\delta$ :

**Proposition 5.3.** *Let  $p$  be any prime. For any  $t$  in  $\mathbb{Q}_p^\times$  the central element  $h_{\alpha,p}(t^{m_\alpha})$  acts trivially on  $\mathcal{A}_{00}$ .*

*Proof.* Since  $\mathcal{A}_{00}$  is  $(M_s T_2^1) \prod_{p \neq 2} T_p$ -right invariant, it suffices to check this for  $t = p$ . Assume first that  $p$  is odd. Let  $f$  be in  $\mathcal{A}_{00}$ . Note that  $f$  is right  $h_{\alpha,q}(p^{m_\alpha})$ -invariant for every  $q \neq p$ . Indeed,  $h_{\alpha,q}(p^{m_\alpha})$  is contained in  $T_q$  if  $q \neq 2$  and in  $M_s T_2^1$  if  $q = 2$ . (This is clear if  $m_\alpha = 1$ ; otherwise it follows from  $p^2 \equiv 1 \pmod{4}$  for every odd  $p$ .) Using left  $h_{\alpha,\mathbb{Q}}(p^{m_\alpha})$ -invariance of  $f$  we have

$$f(mh_{\alpha,p}(p^{m_\alpha})) = f(h_{\alpha,\mathbb{Q}}(p^{m_\alpha})^{-1}mh_{\alpha,p}(p^{m_\alpha})) = f(m).$$

Now assume that  $p = 2$ . Then, analogously,

$$f(mh_{\alpha,2}(2^{m_\alpha})) = f(h_{\alpha,\mathbb{Q}}(2^{m_\alpha})^{-1}mh_{\alpha,2}(2^{m_\alpha})) = f(m).$$

$\square$

In order to determine the central character of  $\pi_p$ , we need to determine the action of the full center of  $T(\mathbb{Q}_p)$  on  $\delta \otimes \delta^\vee \subseteq \mathcal{A}_{00}$ . Observe that  $(p, p)_p = (p, p)_2 = (-1)^{(p-1)/2}$  for any odd prime. This allows us to define a homomorphism

$$\varphi : \eta_p(\Lambda) \cdot \mu_2 \rightarrow T(\mathbb{Z}_2)$$

by sending  $h_{\alpha,p}(p)$  to  $h_{\alpha,2}(p)$ . The restriction of  $\varphi$  to  $\eta_p(\Lambda \cap 2\Lambda^*)$  has the image in the center of  $T(\mathbb{Z}_2)$ . Thus, if  $\gamma_\infty$  is the central character of  $\delta$ , then the composite

$$(5) \quad \gamma_p = \gamma_\infty \circ \varphi$$

defines an unramified central character for  $T(\mathbb{Q}_p)$ . We also define  $\gamma_2$ , an unramified central character of  $T^1(\mathbb{Q}_2)$ , by  $\gamma_2(\eta_2(\lambda)) = 1$  for any  $\lambda$  in  $\Lambda \cap 2\Lambda^*$ .

**Proposition 5.4.** *Fix a pseudo-spherical representation  $\delta$  of  $M$ . Let  $\pi \subseteq \mathcal{A}$  be the unique representation such that  $\pi_\infty \cong \delta$  and that  $\pi_p$  is unramified for all primes  $p$ , as in Corollary 5.2. Let  $\gamma_p$  be the central character defined by (5). Then  $\pi_2 \cong \delta^\vee \otimes V(\gamma_2)$  and  $\pi_p \cong V(\gamma_p)$  for  $p$  odd.*

*Proof.* The proof is completely analogous to the proof of Proposition 5.3. We leave the details to the reader.  $\square$

For uniformity, we set  $\gamma_\infty$  to be the central character of  $\pi_\infty = \delta$  extended trivially to  $A$ . We set  $V(\gamma_\infty)$  to be the representation  $\delta$  extended trivially to  $A$ .

6. PRINCIPAL SERIES REPRESENTATIONS OF  $G(\mathbb{Q}_v)$

In this section we define principal series representations of  $G(\mathbb{Q}_v)$ , where  $v = \infty$  or  $p$ . Let  $B = TU$  denote the Borel subgroup of  $G$ , where  $U$  is generated by  $e_\alpha(t)$  for all positive roots  $\alpha$ . Let  $\bar{U}$  be the group generated by  $e_\alpha(t)$  for all negative roots  $\alpha$ .

Fix a pseudo-spherical representation  $\delta$  of  $M$ . It gives rise to a global representation  $\pi$  of  $T(\mathbb{A})$ , such that  $\pi_\infty \cong \delta$  as in Corollary 5.2. Let  $\chi$  be an unramified character of  $\underline{T}(\mathbb{Q}_v)$ . If  $v = \infty$  an unramified character is a character trivial on  $\underline{M}$ . Let  $i(\chi)$  be the twist of  $\pi_v$  by  $\chi$ . Since  $\pi_v$  is Weyl group invariant, we have  $i(\chi)^w \cong i(\chi^w)$  for every  $w$  in  $W$ . In this section we study induced representations (normalized induction)

$$I(\chi) = \text{Ind}_B^G(i(\chi)).$$

Let  $\alpha$  be a simple root. A character  $\chi$  is called  $\alpha$ -dominant if  $\chi(\underline{h}_\alpha(t)) = |t|^s$  with  $\Re(s) > 0$ . A character  $\chi$  is called dominant if it is  $\alpha$ -dominant for all simple roots. For every  $w$  in  $W_{\mathbb{Q}_v}$  we have an intertwining map  $A_w : I(\chi) \rightarrow I(\chi^w)$  defined by

$$A_w(f)(g) = \int_{U \cap w\bar{U}w^{-1}} f(w^{-1}ug)du.$$

**Proposition 6.1.** *The operator  $A_w$  is absolutely convergent if  $\chi$  is dominant.*

*Proof.* Our proof is, of course, based on the corresponding result for algebraic groups. (See, for example, Section 2.1 of [Sh].) Let  $\ell(w)$  denote the length of the projection of  $w$  into the Weyl group. The proof of the proposition is on induction on the length  $\ell(w)$ . We consider the case of  $\ell(w) = 1$ . Then  $w$  corresponds to a simple root, so we shall denote it by  $w_\alpha$ .

**Lemma 6.2.** *Let  $\alpha$  be a simple root and  $\chi$  an unramified  $\alpha$ -dominant character of  $\underline{T}$ . Then  $A_{w_\alpha}$  is absolutely convergent.*

*Proof.* The proof of this lemma is a reduction to  $\text{SL}_2$ . Let  $s \in \mathbb{C}$  such that  $\chi(\underline{h}_\alpha(t)) = |t|^s$ . Then  $\Re(s) > 0$  since  $\chi$  is  $\alpha$  dominant. In the formula for  $A_{w_\alpha}(f)$  we can assume that  $g = 1$ , replacing  $f$  if necessary. Note that  $U \cap w_\alpha\bar{U}w_\alpha^{-1} = U_\alpha$ ; thus the question of convergence is answered by working in  $G_\alpha$ . Let  $B_\alpha = B \cap G_\alpha = T_\alpha U_\alpha$ . The restriction of  $f$  to  $G_\alpha$  belongs to the induced representation  $\text{Ind}_{B_\alpha}^{G_\alpha}(i(\chi))$ . Note that  $T_\alpha$ , the group generated by elements  $h_\alpha(t)$ , is commutative. Decompose  $i(\chi) = \bigoplus \mu_i$  as a sum of characters of  $T_\alpha$ . It follows that  $\text{Ind}_{B_\alpha}^{G_\alpha}(i(\chi)) = \bigoplus I_i$ , where  $I_i$  are principal series representations induced from the characters  $\mu_i$ . Recall that  $i(\chi) = \pi_p \otimes \chi$ . Since Proposition 5.3 describes the action of  $h_\alpha(t)$  on  $\pi_p$ , it follows that

$$|\mu_i(h_\alpha(t))| = |t|^{\Re(s)}$$

for every  $i$  and  $\alpha$ . Thus, if we write  $f = \bigoplus f_i$  with  $f_i$  in  $I_i(s)$ , then  $|f_i|$  belongs to a principal series representation  $I(\Re(s))$  of  $\underline{G}_\alpha \cong \text{SL}_2$ , induced from the character  $\underline{h}_\alpha(t) \mapsto |t|^{\Re(s)}$ . The convergence of the integral for  $|f_i|$  can be easily calculated. If  $\mathbb{Q}_v = \mathbb{R}$  the integral is bounded by a multiple of

$$\int_{\mathbb{R}} \left( \frac{1}{1+x^2} \right)^{\frac{\Re(s)+1}{2}} dx,$$

while if  $v = p$ , then the integral is bounded by a multiple of

$$\sum_{i=n}^{\infty} \frac{1}{p^{i\Re(s)}}.$$

Both of these converge if  $\Re(s) > 0$ . □

Now we can easily finish the proof of the proposition. Assume that  $\chi$  is dominant and  $A_w$  is absolutely convergent for some  $w$  in  $W$ . If  $\ell(w_\alpha w) = \ell(w) + 1$ , then  $\chi^w$  is  $\alpha$ -dominant. In particular, the composite  $A_{w_\alpha} \circ A_w$  is absolutely convergent. It is equal to  $A_{w_\alpha w}$  by Fubini's theorem. The proposition is proved. □

Recall that  $m_\alpha$  is the degree of the central extension  $G_\alpha$  of  $\underline{G}_\alpha \cong \mathrm{SL}_2$ . This number is equal to 2, except when  $\alpha$  is the short root in the root systems  $C_n, B_n$  and  $F_4$ . A character  $\chi_0 : \underline{T}(\mathbb{Q}_v) \rightarrow \mathbb{R}^+$  such that  $\chi_0(\underline{h}_\alpha(t)) = |t|^{\frac{1}{m_\alpha}}$  for every simple root  $\alpha$  is called exceptional. Note that  $\chi_0$  is unique and dominant.

**Proposition 6.3.** *The induced representation  $I(\chi_0)$  has a unique quotient. We denote the quotient by  $\Theta(\gamma_v)$ .*

*Proof.* When  $v$  is the archimedean place,  $\Theta(\gamma_\infty)$  is the Langlands quotient of  $I(\chi_0)$ .

Suppose  $v = p$ . In this case this is a standard result for induced representations with a regular inducing character. More precisely, we say that  $i(\chi)$  is regular if  $i(\chi)$  is not isomorphic to  $i(\chi^w)$  for any non-trivial element  $w$  in the Weyl group. If that is the case, then  $I(\chi)$  has a unique irreducible submodule and, dually, a unique irreducible quotient. This can be seen as follows. By the geometric lemma in [BZ], the semi-simplification of the (unnormalized) Jacquet module  $I(\chi)_U$  is

$$I(\chi)_U \cong \bigoplus_{w \in W} [\rho_U \cdot i(\chi^w)],$$

where  $\rho_U$  is the square root of the modular character with respect to  $U$ . Suppose  $V$  is an irreducible submodule of  $I(\chi)$ . Then, by Frobenius reciprocity,  $\mathrm{Hom}_G(V, I(\chi)) = \mathrm{Hom}_T(V_U, \rho_U \cdot i(\chi))$ , so  $\rho_U \cdot i(\chi)$  must be a summand of  $V_U$ . By exactness of the Jacquet functor and regularity of  $i(\chi)$ ,  $V$  must be unique. This proves the proposition. □

*Remark.* For  $G(\mathbb{Q}_v)$  of type  $C_n$ , the exceptional representation  $\Theta(\gamma_v)$  is an even component of the oscillator representation [W]. The representation  $\pi_v = V(\gamma_v) = \gamma_v$  is one dimensional and is the Weil index [Rao].

If  $v = p$ , then the Jacquet functor  $\Theta(\gamma_p)_U$  can be exactly described.

**Proposition 6.4.** *Let  $\chi_0$  be the exceptional character and  $w_0$  the longest element in the Weyl group. Then  $\Theta(\gamma_p)_U \cong \rho_U \cdot i(\chi_0^{w_0})$ .*

*Proof.* Let  $\alpha$  be a simple root. Let  $P_\alpha = G_\alpha \cdot B$  be a parabolic subgroup, where  $G_\alpha$  is the group generated by one-parameter subgroups  $U_\alpha$  and  $U_{-\alpha}$ . We need the following lemma:

**Lemma 6.5.** *For every simple root  $\alpha$ , the induced representation  $\mathrm{Ind}_B^{P_\alpha}(i(\chi_0))$  is reducible.*

*Proof.* Let us restrict this representation to  $G_\alpha$ . Decompose  $i(\chi) = \bigoplus \mu_i$  as a sum of characters of  $T_\alpha = G_\alpha \cap T$ . It follows that  $\mathrm{Ind}_B^{P_\alpha}(i(\chi_0)) = \bigoplus I_i$ , where  $I_i$  are principal series representations of  $G_\alpha$ , parabolically induced from the characters

$\mu_i$ . The characters  $\mu_i$  are determined as follows. Recall that  $i(\chi_0)$  is a twist, by  $\chi_0$ , of a Weyl-group invariant representation of  $T(\mathbb{Q}_p)$  appearing as a local component of a representation in  $\mathcal{A}$ . Hence, if  $m_\alpha = 1$ , then Proposition 5.3 implies that  $\mu_i(h_\alpha(t)) = \chi_0(h_\alpha(t)) = |t|$ . It follows that each  $I_i$  has the Steinberg representation as a submodule and the trivial representation as a quotient. Since  $T$  normalizes  $G_\alpha$ , the sum of all Steinberg submodules is a proper submodule for  $P_\alpha$ . A similar argument works if  $m_\alpha = 2$ . Then Proposition 5.3 implies that  $\mu_i(h_\alpha(t^2)) = \chi_0(h_\alpha(t^2)) = |t|$ . It follows that each  $I_i$  reduces with a discrete series representation as a submodule and a quotient isomorphic to an even component of an oscillator representation [GS]. Again, the sum of discrete series representations is a  $P_\alpha$ -submodule. The lemma is proved.  $\square$

We now follow an argument of Rodier [Ro]. Let  $V_\alpha$  be the unique quotient of  $\text{Ind}_B^{P_\alpha}(i(\chi_0))$ . Then, by induction in stages,  $\text{Ind}_{P_\alpha}^G(V_\alpha)$  is a quotient of  $I(\chi_0)$ . Since  $\Theta(\gamma_p)$  is the unique irreducible quotient of  $I(\chi_0)$ , it must also be a quotient of  $\text{Ind}_{P_\alpha}^G(V_\alpha)$ . Since

$$\text{Ind}_{P_\alpha}^G(V_\alpha)_U = \bigoplus_{w \in W, w(\alpha) < 0} [\rho_U \cdot i(\chi_0^w)],$$

it follows that  $\Theta(\gamma_p)_U$  is a sum of  $\rho_U \cdot i(\chi_0^w)$  for  $w$  in the Weyl group such that  $w(\alpha)$  is negative for all simple roots  $\alpha$ . But this holds only for  $w = w_0$ , the longest element in the Weyl group. The proposition is proved.  $\square$

Assume that  $p$  is odd. Let  $v^\circ$  be a non-zero element in  $i(\chi)$  fixed by  $T_p$ . Note that  $v^\circ$  is unique up to a non-zero scalar. Then the representation  $I(\chi)$  contains a unique  $K_p$ -fixed vector  $f_\chi^\circ$  normalized by  $f_\chi^\circ(1) = v^\circ$ . The action of the intertwining operators on the spherical vector has been computed in [Sa2].

**Proposition 6.6.** *Assume that  $p \neq 2$ . Let  $\alpha$  be a simple root. Then*

$$A_{w_\alpha}(f_\chi^\circ) = \frac{1 - p^{-1}(\chi(h_\alpha(p^{m_\alpha})))}{1 - \chi(h_\alpha(p^{m_\alpha}))} f_{\chi^{w_\alpha}}^\circ.$$

Note that the formula for  $A_{w_\alpha(1)}(f_\chi^\circ)$  depends on the projection of  $w_\alpha$  into the Weyl group  $W$ . Thus, for a general element in  $W_{\mathbb{Q}_p}$  we have the following corollary.

**Corollary 6.7.** *Let  $\underline{w}$  be in  $W$  and  $w$  be a preimage of  $\underline{w}$  in  $W_{\mathbb{Q}_p}$ . Then*

$$A_w(f_\chi^\circ) = \prod_{\alpha > 0, \underline{w}(\alpha) < 0} \frac{1 - p^{-1}(\chi(h_\alpha(p^{m_\alpha})))}{1 - \chi(h_\alpha(p^{m_\alpha}))} f_{\chi^w}^\circ. \quad \square$$

7. EISENSTEIN SERIES

Recall that  $B = TU$  denotes the Borel subgroup of  $G$ , where  $U$  is generated by  $e_\alpha(t)$  for all positive root  $\alpha$ . In the same fashion, we define the Borel subgroup  $\underline{B} = \underline{T}\underline{U}$  of  $\underline{G}$ .

We identify  $\mathbb{A}^l \simeq \underline{T}(\mathbb{A})$  by  $(x_1, \dots, x_l) \mapsto \prod_{i=1}^l h_{\alpha_i}(x_i)$ . For  $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{C}^l$ , we define the Hecke character  $\chi_{\mathbf{s}}$  of  $\underline{T}(\mathbb{Q}) \backslash \underline{T}(\mathbb{A})$  by  $\chi_{\mathbf{s}}(h_{\alpha_i}(x_i)) = |x_i|^{s_i}$  for every simple root  $\alpha_i$ . Here  $|x_i| = \prod_v |x_i|_v$ . We extend this to a function on  $\underline{G}(\mathbb{A})$  by  $\chi_{\mathbf{s}}(utk) = \chi_{\mathbf{s}}(t)$ , where  $u \in \underline{U}(\mathbb{A})$ ,  $t \in \underline{T}(\mathbb{A})$  and  $k \in \prod_p K_\infty \underline{G}(\mathbb{Z}_p)$ . The square root of the modular function is given by  $\rho = \chi_{(1, \dots, 1)} = \chi_{\mathbf{1}}$ , where  $\mathbf{1} = (1, \dots, 1)$ .

Similarly, for a place  $v$  of  $\mathbb{Q}$ , we define a character  $\chi_{\mathbf{s},v}$  of  $\underline{T}(\mathbb{Q}_v)$  by  $\chi_{\mathbf{s},v}(h_{\alpha_i}(t)) = |t|_v^{s_i}$  for every simple root  $\alpha_i$ . We extend this to a function on  $\underline{G}(\mathbb{Q}_v)$  by  $\chi_{\mathbf{s},v}(utk) = \chi_{\mathbf{s},v}(t)$ , where  $u \in \underline{U}(\mathbb{Q}_v)$ ,  $t \in \underline{T}(\mathbb{Q}_v)$  and  $k \in \underline{K}_v$ .

Let  $\pi$  be as in Corollary 5.2. Let  $K = K_\infty \prod_p K_p$ . Let  $\mathcal{J}$  denote the space of functions on  $G(\mathbb{A})$  satisfying the following conditions:

- (1)  $f(ubag) = f(g)$  for  $u \in U(\mathbb{A})$ ,  $b \in B(\mathbb{Q})$ ,  $a \in A$ ,  $g \in G(\mathbb{A})$ ,
- (2)  $f$  is  $K$ -finite, and for each  $k \in K$ , the function  $t \mapsto f(tk)$  is a function in  $\pi$ .

Let  $I(\chi_{\mathbf{s}})$  denote the representation of  $G(\mathbb{A})$  on functions of the form

$$g \mapsto f(g)\chi_{\mathbf{s}+1}(g),$$

where  $f \in \mathcal{J}$ . We have

$$I(\chi_{\mathbf{s}}) = \text{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} \pi \chi_{\mathbf{s}} = \left( \text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} \pi_\infty \chi_{\mathbf{s},\infty} \right) \bigotimes_p \text{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \pi_p \chi_{\mathbf{s},p},$$

where all the induced representations are normalized inductions. We form an Eisenstein series:

$$E(g, \mathbf{s}, f) = \sum_{x \in B(\mathbb{Q}) \backslash \underline{G}(\mathbb{Q})} f(xg)\chi_{\mathbf{s}+1}(g),$$

where  $g \in G(\mathbb{A})$ ,  $\mathbf{s} \in \mathbb{C}^l$ ,  $f \in \mathcal{J}$ . The above sum converges absolutely and uniformly on compact sets contained in the region  $\text{Re}(s_i) > 1$  for all  $i$ . The Eisenstein series has a meromorphic continuation to  $\mathbb{C}^l$ ; see [MW]. We define the constant term of the above Eisenstein series by

$$E(g, \mathbf{s}, f)_U = \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} E(ug, \mathbf{s}, f) du.$$

A standard computation in the domain of convergence of  $E(g, \mathbf{s}, f)$  gives

$$E(g, \mathbf{s}, f)_U = \sum_{\underline{w} \in W} (A_w(\mathbf{s})f)(g),$$

where

$$(A_w(\mathbf{s})f)(g) = \int_{(U(\mathbb{Q}) \cap w\bar{U}(\mathbb{Q})w^{-1}) \backslash (U(\mathbb{A}) \cap w\bar{U}(\mathbb{A})w^{-1})} f(w^{-1}ug)\chi_{\mathbf{s}+1}(w^{-1}ug) du$$

and  $w \in W_{\mathbb{Q}}$  is an (arbitrary) element such that  $\text{pr}(w) = \underline{w}$ . Suppose  $S$  is a finite set of primes including 2 and  $\infty$  and  $f = (\bigotimes_{v \in S} f_v) \otimes (\bigotimes_{p \notin S} f_p^\circ)$ . Then by Corollary 6.7

$$(A_w(\mathbf{s})f)(g) = \left( \bigotimes_{v \in S} A_{w,v}(\mathbf{s})f_v \right) \otimes \left( c_S(\underline{w}, \mathbf{s}) \bigotimes_{p \notin S} f_{w(\mathbf{s}),p}^\circ \right),$$

where

$$c_S(\underline{w}, \mathbf{s}) = \prod_{p \notin S} \prod_{\alpha > 0, \underline{w}(\alpha) < 0} \frac{1 - p^{-1}(\chi_{\mathbf{s},p}(h_\alpha(p^{m_\alpha})))}{1 - \chi_{\mathbf{s},p}(h_\alpha(p^{m_\alpha}))} = \prod_{\alpha > 0, \underline{w}(\alpha) < 0} \frac{\zeta_S(m_\alpha \alpha(\mathbf{s}))}{\zeta_S(1 + m_\alpha \alpha(\mathbf{s}))}.$$

Here  $\zeta_S(z) = \prod_{p \notin S} (1 - p^{-z})^{-1}$  is the partial Riemann zeta function, and  $\alpha(\mathbf{s}) = \sum_{i=1}^l n_i s_i$  if  $\alpha = \sum_{i=1}^l n_i \alpha_i$  as a sum of simple roots. Therefore as  $\mathbf{s}$  tends to  $\mathbf{s}_0 = (m_{\alpha_1}^{-1}, \dots, m_{\alpha_l}^{-1})$ , each term  $(\prod_{i=1}^l (s_i - m_{\alpha_i}^{-1}))A_w(\mathbf{s})f$  vanishes except for the term where  $\underline{w} = \underline{w}_0$  is the longest element of  $W$ . Furthermore, if we set  $S = \{2, \infty\}$ , then  $A_{w,v}(\mathbf{s}_0)$  for  $v \in S$  are non-zero intertwining operators, so we may arrange  $f$  such that  $(\prod_{i=1}^l (s_i - m_{\alpha_i}^{-1}))A_w(\mathbf{s})f$  is non-zero.

For  $f \in \mathcal{J}$ , we define

$$\theta_f(g) = \lim_{\mathbf{s} \rightarrow \mathbf{s}_0} \left( \prod_{i=1}^l (s_i - m_{\alpha_i}^{-1}) \right) E(g, \mathbf{s}, f).$$

Then

$$\int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} \theta_f(ug) du = A_{w_0}(\mathbf{s}_0)(f),$$

and, by the criterion of Jacquet (see [J] and [MW]),  $\theta_f(g)$  is a square integrable function in  $L^2(\underline{G}(\mathbb{Q}) \backslash G(\mathbb{A}))$ . Let  $\Theta$  denote the span of  $\{\theta_f : f \in \mathcal{J}\}$ . We now recall the exceptional representation  $\Theta(\gamma_v)$  defined in Section 6.

**Theorem 7.1.** *The span  $\Theta$  lies in  $L^2(\underline{G}(\mathbb{Q}) \backslash G(\mathbb{A}))$ . It is an irreducible automorphic representation of  $G(\mathbb{A})$ , and it is isomorphic to  $\bigotimes_v \Theta(\gamma_v)$ .*

*Proof.* For every  $f \in \mathcal{J}$ , the map  $f\chi_{\mathbf{s}_0+1} \mapsto \theta_f$  defines a non-zero intertwining operator from the induced representation to  $L^2(\underline{G}(\mathbb{Q}) \backslash G(\mathbb{A}))$ . Thus the image  $\Theta$  must decompose as a direct sum of irreducible representations. On the other hand, at each local place  $v$  the exceptional representation  $\Theta(\gamma_v)$  is a unique quotient of the local induced representation. This implies that  $\Theta \cong \bigotimes_v \Theta(\gamma_v)$ , as desired.  $\square$

**Corollary 7.2.** *The exceptional representation  $\Theta(\gamma_v)$  is unitarizable.*  $\square$

In a terminology of [A-V],  $\Theta(\gamma_\infty)$  corresponds to the trivial representation of a split group  $\underline{G}^l(\mathbb{R})$ , which will be introduced in the next section. The unitarity of  $\Theta(\gamma_\infty)$  was proved and studied for classical groups of type  $B_l$  in [Kn], [LS] and [T]. The unitarity for other groups may be new.

### 8. IWAHORI-HECKE ALGEBRAS

We will fix an odd prime  $p$  in this section. We fix an Iwahori subgroup  $I$  of  $K_p$  such that  $I$  contains  $U_\alpha(\mathbb{Z}_p)$  for all positive  $\alpha$  and  $I \cap T(\mathbb{Q}_p) = T_p$ . We recall that  $\mu_2$  is the kernel of the covering map  $\text{pr} : G(\mathbb{Q}_p) \rightarrow \underline{G}(\mathbb{Q}_p)$ . Let  $\mathcal{H}_- = \mathcal{H}_-(G(\mathbb{Q}_p))$  denote the algebra of all compactly supported  $I$ -bi-invariant functions on  $G(\mathbb{Q}_p)$  such that  $f(\epsilon g) = \epsilon f(g)$  for all  $\epsilon \in \mu_2$ . The multiplicative structure of  $\mathcal{H}_-$  is defined by a convolution of functions,

$$(f' \cdot f'')(g) = \int_G f'(h) f''(h^{-1}g) dh,$$

where  $dh$  is a Haar measure on  $G$  so that the volume of  $\mu_2 \times I$  is one. We call  $\mathcal{H}_-$  the Iwahori-Hecke algebra of  $G$ . The following is Proposition 6.1 in [Sa2].

**Proposition 8.1.** *Let  $N'$  denote the normalizer in  $G$  of  $T_p$ . Then the support of the Hecke algebra is  $\text{supp}(\mathcal{H}_-) = IN'I$ .*  $\square$

One can easily describe  $N'$ . Recall that, if  $\underline{N}(\mathbb{Q}_p)$  is the normalizer of  $\underline{T}(\mathbb{Z}_p)$  in  $\underline{G}(\mathbb{Q}_p)$ , then the quotient of the two is isomorphic to the affine Weyl group  $\Lambda \rtimes W$ . The group  $N'$  is smaller than the inverse image of  $\underline{N}(\mathbb{Q}_p)$ . Recall that  $\eta_p(\lambda)$  centralizes (or normalizes)  $T_p$  if and only if  $\lambda$  is in

$$\Lambda' := \Lambda \cap 2\Lambda^*.$$

In particular, we have an exact sequence

$$1 \rightarrow \mu_2 \times T_p \rightarrow N' \xrightarrow{\phi} \Lambda' \rtimes W \rightarrow 1,$$

where  $\phi$  is defined by sending  $w_\alpha(1)$  to the reflection  $w_\alpha$  in  $W$  and  $\eta_p(\lambda)$  to  $\lambda$  in  $\Lambda'$ .

We now define a normalization of elements in the Hecke algebra. Let  $\pi_p$  be an unramified, Weyl group invariant, irreducible genuine representation of  $T(\mathbb{Q}_p)$  as in Corollary 5.2. Let  $\gamma_p$  be the central character of  $\pi_p$ . Recall that  $\eta_p(\lambda)$  is in the center of  $T(\mathbb{Q}_p)$  for every  $\lambda$  in  $\Lambda'$ . In particular,  $\gamma_p(\eta_p(\lambda))$  is well defined for every  $\lambda$  in  $\Lambda'$ . The Weyl group invariance of the central character of  $\pi_p$  implies that we can extend  $\gamma_p$  to  $N'$  by setting

$$\gamma_p(w_\alpha(1)) = 1.$$

Thus,  $\gamma_p$  is a character of  $N'$ , which is trivial on  $T_p$ . For  $w$  in  $\Lambda' \rtimes W$ , we define  $e_w \in \mathcal{H}_-$  by its values for every  $x$  in  $N'$  as follows:

$$e_w(IxI) = \begin{cases} \overline{\gamma_p(x)} & \text{if } \phi(x) = w, \\ 0 & \text{otherwise.} \end{cases}$$

We note some elementary properties of elements  $e_w$ . Let  $\ell(w)$  denote the usual length function on the affine Weyl group  $\Lambda \rtimes W$ . If  $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$  for two elements in  $\Lambda' \rtimes W$ , then  $e_{w_1w_2} = e_{w_1} \cdot e_{w_2}$ . (See [Sa2]. A key here is the multiplicative property of  $\gamma_p$ .)

Let  $\mathcal{L}$  denote the  $\mathbb{C}$ -span of  $e_\lambda$  where  $\lambda$  is dominant in  $\Lambda'$ . Note that  $\ell(\lambda) = \langle \rho, \lambda \rangle$  for dominant  $\lambda$ . It follows that  $\ell(\lambda + \lambda') = \ell(\lambda) + \ell(\lambda')$  for dominant  $\lambda, \lambda'$  in  $\Lambda'$ . Hence  $e_\lambda \cdot e_{\lambda'} = e_{\lambda + \lambda'}$ . In particular,  $\mathcal{L}$  is a commutative subalgebra in  $\mathcal{H}_-$ .

Let  $H$  denote the subalgebra consisting of functions supported on  $\mu_2 \times K_p$ . It has basis  $\{e_w : w \in W\}$ . If  $\alpha$  is a simple root and  $w_\alpha$  is the corresponding simple reflection, then we denote  $e_{w_\alpha}$  by  $e_\alpha$ . These elements satisfy the following relations:

- (1)  $(e_\alpha - p)(e_\alpha + 1) = 0$  and
- (2)  $e_\alpha \cdot e_\beta \cdot e_\alpha \cdots = e_\beta \cdot e_\alpha \cdot e_\beta \cdots$ , where the number of factors on each side is equal to the order  $m_{\alpha\beta}$  of the element  $w_\alpha w_\beta$  in  $W$ .

Conversely,  $H$  is the  $\mathbb{C}$ -algebra generated by the set of  $e_\alpha$  for all simple roots  $\alpha$  satisfying the above two relations. One easily sees that

$$\mathcal{H}_- = H \cdot \mathcal{L} \cdot H.$$

An important result is that for a positive  $\lambda \in \Lambda'$ ,  $e_\lambda$  is an invertible element in  $\mathcal{H}_-$ . This implies that if  $V$  is an admissible genuine  $G$ -module generated by the subspace  $V^I$ , then every submodule  $V_1$  of  $V$  is also generated by its subspace  $V_1^I$ .

Given  $\lambda \in \Lambda'$ , we write  $\lambda = \lambda_1 - \lambda_2$ , where  $\lambda_1, \lambda_2$  are positive in  $\Lambda'$ . We define

$$t_\lambda = p^{-\frac{1}{2}\langle \rho, \lambda \rangle} e_{\lambda_1} \cdot e_{\lambda_2}^{-1}.$$

This definition does not depend on the choice of  $\lambda_1$  and  $\lambda_2$ . We state the main results of [Sa1] and [Sa2]. (Note that we have already explained the first three relations.)

**Theorem 8.2.** *Let  $\alpha, \beta$  be two simple roots, and let  $\lambda, \lambda' \in \Lambda'$ . Then  $e_\alpha, e_\beta, t_\lambda$  and  $t_{\lambda'}$  satisfy the following relations:*

- (1)  $(e_\alpha - p)(e_\alpha + 1) = 0$ .
- (2)  $e_\alpha \cdot e_\beta \cdot e_\alpha \cdots = e_\beta \cdot e_\alpha \cdot e_\beta \cdots$ , where there are  $m_{\alpha\beta}$  factors on each side.
- (3)  $t_\lambda \cdot t_{\lambda'} = t_{\lambda + \lambda'}$ .
- (4)  $e_\alpha \cdot t_\lambda - t_{w_\alpha(\lambda)} \cdot e_\alpha = (q - 1) \frac{t_\lambda - t_{w_\alpha(\lambda)}}{1 - t_{-m_\alpha \alpha^\vee}}$ .

Conversely, let  $\mathcal{H}'_-$  be the  $\mathbb{C}$ -algebra abstractly generated by  $e_\alpha$  for all simple roots  $\alpha$ , and  $t_\lambda$  for all  $\lambda \in \Lambda'$ . These generators satisfy the relations (1) to (4) above; then  $\mathcal{H}'_- = \mathcal{H}_-$ .  $\square$

*Remark.* The above theorem was stated in [Sa2] for simply laced  $\underline{G}$ , and for any degree central extension. However, the proof of relation (4) takes place in the Levi factor of the parabolic subgroup  $P_\alpha$ . Thus the calculation given there (relying on  $\gamma_p(h_\alpha(p^{m_\alpha})) = 1$ ; Proposition 5.3) is applicable to our situation.

**Definition of  $\underline{G}^l$ .** We will define an algebraic split group  $\underline{G}^l(\mathbb{Q}_p)$ . In order to do this, it suffices to define its co-roots  $\Psi^\vee$  and its co-character lattice  $\Lambda_c$ . We recall that  $\Lambda$  is the co-root lattice of  $G$ , and we define

$$\Psi^\vee := \left\{ \frac{m_\alpha}{2} \alpha^\vee \in \Lambda \otimes \mathbb{R} \mid \alpha^\vee \in \Phi^\vee \right\}$$

and  $\Lambda_c := \frac{1}{2} \Lambda'$ . Note that the root system  $\Psi$  is dual to the root system  $\Phi$ . The isogeny class of  $\underline{G}^l$  is determined by the lattice  $\Lambda_c$ . Let  $\Lambda_{cr}$  be the  $\mathbb{Z}$ -span of co-roots in  $\Psi^\vee$ . The group  $\underline{G}^l$  is a split, algebraic group obtained by taking a quotient of the split, simply connected algebraic group corresponding to  $\Psi$  by the central subgroup isomorphic to  $\Lambda_c / \Lambda_{cr}$ . It is an elementary 2-group. Its order is equal to the number of pseudo-spherical representations of  $M$ . The following table lists all cases when this 2-group is non-trivial:

$\Phi$	$A_{2n-1}$	$D_{2n-1}$	$D_{2n}$	$C_n$	$B_{2n}$	$E_7$
$\Psi$	$A_{2n-1}$	$D_{2n-1}$	$D_{2n}$	$B_n$	$C_{2n}$	$E_7$
$[\Lambda_c : \Lambda_{cr}]$	2	2	4	2	2	2

The Iwahori-Hecke algebra  $\mathcal{H}(\underline{G}^l)$  of  $\underline{G}^l$  is similarly generated by  $\underline{t}_\lambda$  and  $\underline{e}_w$ , where  $\lambda \in \Lambda_c$  and  $w \in W$ .

Let  $f(x) \in \mathcal{H}(G)$  (resp.  $\mathcal{H}(\underline{G}^l)$ ). We define  $f^*(x) = \overline{f(x^{-1})}$ . Hence  $*$ :  $\mathcal{H}_- \rightarrow \mathcal{H}_-$  (resp.  $*$ :  $\mathcal{H}(\underline{G}^l) \rightarrow \mathcal{H}(\underline{G}^l)$ ) satisfies  $(f^*)^* = f$  and  $f^* \cdot g^* = (g \cdot f)^*$ , i.e. it is an algebra anti-involution. We have  $e_\lambda^* = e_{-\lambda}$  and  $e_w^* = e_{w^{-1}}$  in  $\mathcal{H}_-$ . Similarly,  $\underline{e}_\lambda^* = \underline{e}_{-\lambda}$  and  $\underline{e}_w^* = \underline{e}_{w^{-1}}$  in  $\mathcal{H}(\underline{G}^l)$ .

**Theorem 8.3.** (i) There is an algebra homomorphism  $A : \mathcal{H}(\underline{G}^l) \rightarrow \mathcal{H}_-$  given by  $A(\underline{t}_\lambda) = t_{2\lambda}$  and  $A(\underline{e}_w) = e_w$  for  $\lambda \in \underline{\Lambda}_c$  and  $w \in W$ .

(ii) The algebra isomorphism  $A$  commutes with anti-involutions  $*$  on  $\mathcal{H}(\underline{G}^l)$  and  $\mathcal{H}_-$ .

*Proof.* Part (i) follows by comparing relations in  $\mathcal{H}(\underline{G}^l)$  in [Lu] and those for  $\mathcal{H}_-$  in Theorem 8.2. For (ii) we first have  $A(\underline{e}_w^*) = A(\underline{e}_{w^{-1}}) = e_{w^{-1}} = e_w^*$  for any  $w$  in  $W$ . By the decomposition  $\mathcal{H}_- = H \cdot \mathcal{L} \cdot H$ , it remains to show that  $A(\underline{e}_\lambda) = (A(\underline{e}_\lambda))^*$  for a dominant co-character  $\lambda$ . To that end, let  $w$  be the unique element in  $W$  such that  $w(\Delta) = -\Delta$ . Then  $\mu = -\lambda^w$  is again-dominant. Since

$$\begin{cases} \ell(\mu w) = \ell(\mu) + \ell(w), \\ \ell(-w\lambda) = \ell(w) + \ell(-\lambda) \end{cases}$$

we have  $\underline{e}_w \underline{e}_{-\lambda} = \underline{e}_{-w\lambda} = \underline{e}_\mu \underline{e}_w$ , and a similar statement exists for elements in  $\mathcal{H}_-$ . Now we have  $A(\underline{e}_\lambda^*) = A(\underline{e}_{-\lambda}) = A(\underline{e}_w^{-1} \underline{e}_\mu \underline{e}_w) = e_w^{-1} A(\underline{e}_\mu) e_w = p^{-\ell(\mu)/2} e_w^{-1} e_{2\mu} e_w = p^{-\ell(\mu)/2} e_{-2\lambda} = p^{-\ell(\lambda)/2} e_{2\lambda}^* = A(\underline{e}_\lambda)^*$ , as required.  $\square$

9. REPRESENTATIONS WITH IWAHORI FIXED VECTORS

Let  $I$  and  $I'$  denote the Iwahori subgroups of  $G$  and  $\underline{G}^l$ , respectively, which give rise to isomorphic Iwahori Hecke algebras  $\mathcal{H}_-$  and  $\mathcal{H} = \mathcal{H}(\underline{G}^l)$  in Theorem 8.3. Let  $\mathcal{R}(\mathcal{H}_-)$  and  $\mathcal{R}(\mathcal{H})$  denote the categories of finite dimensional representations of the Iwahori-Hecke algebras  $\mathcal{H}_-$  and  $\mathcal{H}$ , respectively.

Let  $\mathcal{R}_-^I(G)$  denote the category of admissible smooth *genuine* representations  $V$  of  $G$  such that  $V^I$  generates  $V$  as a  $G$ -module. Similarly we let  $\mathcal{R}^{I'}(\underline{G}^l)$  denote the category of admissible smooth representations  $V$  of  $\underline{G}^l$  such that  $V^{I'}$  generates  $V$  as a  $\underline{G}^l$ -module.

By [Bo] and [BZ], the functor  $V \mapsto V^{I'}$  is an equivalence of categories from  $\mathcal{R}^{I'}(\underline{G}^l)$  to  $\mathcal{R}(\mathcal{H})$ . Let  $C_c(\underline{G}^l/I')$  denote locally constant, compactly supported, complex valued functions on  $\underline{G}^l/I'$ . This is a right  $\mathcal{H}$ -module. Then the inverse functor is given by  $E \mapsto \mathbf{I}(E) := C_c(\underline{G}^l/I') \otimes_{\mathcal{H}} E$ .

Similarly the functor  $V \mapsto V^I$  is an equivalence of categories from  $\mathcal{R}_-^I(G)$  to  $\mathcal{R}(\mathcal{H}_-)$ . Let  $C_{c,-}(G/I)$  denote locally constant, compactly supported, complex valued functions on  $G/I$  such that  $f(\epsilon xI) = \epsilon f(xI)$  for  $\epsilon \in \mu_2, x \in G$ . This is a right  $\mathcal{H}_-$ -module. Then the inverse functor is given by  $E \mapsto \mathbf{I}(E) := C_{c,-}(G/I) \otimes_{\mathcal{H}_-} E$ .

We recall the isomorphism  $A : \mathcal{H} \rightarrow \mathcal{H}_-$  in Theorem 8.3. This establishes an equivalence of categories between  $\mathcal{R}(\mathcal{H})$  and  $\mathcal{R}(\mathcal{H}_-)$ . Hence the following four categories are equivalent:

$$(6) \quad \mathcal{R}^{I'}(\underline{G}^l) \simeq \mathcal{R}(\mathcal{H}) \simeq \mathcal{R}(\mathcal{H}_-) \simeq \mathcal{R}_-^I(G).$$

Suppose  $V$  is a representation in  $\mathcal{R}_-^I(G)$ ; then we call the corresponding representation in  $\mathcal{R}^{I'}(\underline{G}^l)$  the *local Shimura lift* of  $V$ . For example, the Shimura lift of  $\Theta(\gamma_p)$  is the trivial representation.

**Hermitian representations.** We gather some facts from [BM1] and [BM2]. Let  $(\pi, E)$  be a finite dimensional representation of  $\mathcal{H}$ . We say that  $E$  is a *Hermitian* representation of  $\mathcal{H}$  if there exists a Hermitian form  $\langle \cdot, \cdot \rangle$  on  $E$  such that

$$\langle \pi(f)v_1, v_2 \rangle = \langle v_1, \pi(f^*)v_2 \rangle$$

for all  $v_1, v_2 \in E$  and  $f \in \mathcal{H}$ . We say that  $E$  is a *unitary* representation of  $\mathcal{H}$  if the Hermitian form is positive definite. Similarly, we define Hermitian representations and unitary representations of  $\mathcal{H}_-$ .

Let  $V$  be a representation in  $\mathcal{R}^{I'}(\underline{G}^l)$  (resp.  $\mathcal{R}_-^I(G)$ ). Suppose  $\langle \cdot, \cdot \rangle$  is a non-degenerate  $\underline{G}^l$ -invariant (resp.  $G$ -invariant) Hermitian form on  $V$ . Then the restriction of the Hermitian form on  $V^I$  gives a Hermitian representation of  $\mathcal{H}$  (resp.  $\mathcal{H}_-$ ). Similarly, a unitary representation  $V$  gives rise to a unitary representation of the Iwahori-Hecke algebra  $\mathcal{H}$  (resp.  $\mathcal{H}_-$ ).

Conversely, if  $E$  is a Hermitian representation of  $\mathcal{H}$  (resp.  $\mathcal{H}_-$ ), then  $\mathbf{I}(E)$  exhibits an  $\underline{G}^l$ -invariant (resp.  $G$ -invariant) Hermitian form. Moreover, if  $E$  is a unitary representation of  $\mathcal{H}$ , then  $\mathbf{I}(E)$  is a unitary representation of  $\underline{G}^l$ . This non-trivial statement is due to Barbasch and Moy (see [BM1] and Thm 8.1 in [BM2]). This, combined with the equivalence of the four categories in (6) (with the middle isomorphism preserving the anti-involution  $*$ ), gives:

**Theorem 9.1.** *If  $V$  is an irreducible unitary representation in  $\mathcal{R}_-^I(G)$ , then its local Shimura lift to  $\underline{G}^l(\mathbb{Q}_p)$  is unitary.  $\square$*

Note that the Shimura lift of the exceptional representation  $\Theta(\gamma_p)$  is the trivial representation of  $\underline{G}^l(\mathbb{Q}_p)$ . We have proved unitarizability of  $\Theta(\gamma_p)$  by global methods.

**Corollary 9.2.** *Assume that  $\underline{G} \neq \mathrm{SL}_2$ . Then the unitary representation  $\Theta(\gamma_p)$  is isolated in the unitary dual  $G(\mathbb{Q}_p)$ .  $\square$*

*Proof.* Recall that the space of (equivalence classes of) smooth irreducible representations of  $G(\mathbb{Q}_p)$  is equipped with a Fell topology [Ta]. To every irreducible representation  $\Pi$  we can attach a point in the support  $\Omega$  of the Bernstein center of  $G(\mathbb{Q}_p)$ . (The support is a disjoint union of complex varieties of dimension less than or equal to the rank of  $G(\mathbb{Q}_p)$ .) Tadić in [Ta], Theorem 5.7, shows that this map is continuous and closed. Thus, the question as to whether  $\Theta_p$  is isolated with respect to Fell's topology is equivalent to the same question for the Bernstein center. Since our isomorphism of Hecke algebras gives an equivalence of categories,  $\Theta_p$  must be isolated in the unitary dual since the trivial representation in the unitary dual of  $\underline{G}^l(\mathbb{Q}_p)$  is isolated.  $\square$

*Remark.* Theorem 9.1 completes a part of [Hu]. Indeed, a key to Theorem 9.1 is that the isomorphism of Hecke algebras preserves  $*$ -structures. This was claimed but not verified in [Hu]. In retrospect, a verification of this statement at that time was impossible since normalizations of Hecke operators were not properly defined in [Sa1].

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DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 2 SCIENCE DRIVE 2, SINGAPORE 117543

*E-mail address:* [matlhy@nus.edu.sg](mailto:matlhy@nus.edu.sg)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH 84112

*E-mail address:* [savin@math.utah.edu](mailto:savin@math.utah.edu)