THE CAUCHY PROBLEM FOR p-EVOLUTION EQUATIONS

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Abstract. In this paper we deal with the Cauchy problem for evolution equations with real characteristics. We show that the problem is well-posed in Sobolev spaces assuming a suitable decay of the coefficients as the space variable \( x \to \infty \). In some cases, such a decay may also compensate a lack of regularity with respect to the time variable \( t \).

1. Introduction and main results

Let us consider the Cauchy problem in \([0,T] \times \mathbb{R}_x\),

\[
\begin{cases}
Lu = 0 \\
u(0,x) = u_0, \quad \partial_t u(0,x) = u_1
\end{cases}
\]

for the operator

\[
L := D_t^2 + 2b_p(t)D_tD_x^p + a_{2p}(t)D_x^{2p} + \sum_{j=0}^{p-1} b_j(t,x)D_tD_x^j + \sum_{k=0}^{2p-1} a_k(t,x)D_x^k,
\]

where \( p \geq 2 \) is a positive integer, \( D = \frac{1}{i} \partial_t \). For the coefficients \( a_k, k < 2p \), and \( b_j, j < p \), in general complex valued functions, we assume

\[
a_k, b_j \in C([0,T];B^{\infty}), \quad k = 0, \ldots, 2p-1, \quad j = 0, \ldots, p-1,
\]

where \( B^{\infty} = B^{\infty}(\mathbb{R}_x) \) denotes the space of all functions \( f(x) \) which are bounded in \( \mathbb{R}_x \) together with all their derivatives. The leading coefficients \( a_{2p} \) and \( b_p \) are assumed to be real valued continuous functions such that

\[
\Delta(t) := b_p^2(t) - a_{2p}(t) \geq \lambda_0 > 0;
\]

hence \( L \) is a \( p \)-evolution operator with real distinct characteristic roots \( \tau = \left( -b_p(t) \pm \sqrt{\Delta(t)} \right) \xi^p \). In view of the Lax-Mizohata theorem, for any \( p \geq 1 \), to have real roots is a necessary condition in order to solve uniquely (1.1) in Sobolev spaces in a neighborhood of \( t = 0 \). As far as applications are concerned, for \( p = 2 \) and \( b_p = 0 \) we have a vibrating beam equation. Still for \( p = 2 \), the related first order (in time) equation \( L_1 u = 0 \),

\[
L_1 := D_t + e_p(t)D_x^p + \sum_{j=0}^{p-1} e_j(t,x)D_x^j, \quad e_p(t) \in \mathbb{R},
\]
is a Schrödinger equation, while for \( p = 3 \), the principal part of \( L_1 u = 0 \) is the same as in the Korteweg-De Vries equation.

The kovalevskian case \( p = 1 \) in (1.2) is that of a strictly hyperbolic operator. Just in this latest case, by well-known results starting from [3], we know that there is a strict and deep relation between the well-posedness of the Cauchy problem (1.1) and the modulus of Hölder continuity of the coefficients with respect to the time variable. For \( p \geq 2 \) these topics have been studied in [1]. In particular, from the results we have obtained there, we have the following theorem:

**Theorem 1.1** ([1]). Besides (1.3) and (1.4), let us assume

\[
(1.6) \quad a_{j+p} = a_{j+p}(t), \quad b_j = b_j(t), \quad a_{j+p}, b_j \in C^{0,j/p}([0,T];\mathbb{R}), \quad j = 1, \ldots, p.
\]

Then, for every choice of Cauchy data \( u_0 \in H^s, u_1 \in H^{s-p} \) the problem (1.1) has a unique solution \( u \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-p}) \).

When the conclusions of Theorem 1.1 hold, we say that the problem (1.1) is well-posed in the Sobolev space \( H^s \). We notice that the leading coefficients \( a_{2p} \) and \( b_p \) are Lipschitz continuous in (1.6).

**Remark 1.2.** In view of the counterexamples we showed in [1] and [2], the regularity given by (1.6) is the optimal one for the well-posedness in \( H^s \).

**Remark 1.3** ([1]). If one weakens the hypothesis (1.6) by assuming

\[
(1.7) \quad a_{j+p} = a_{j+p}(t) \in \mathbb{R}, \quad \sup_{t_1 \neq t_2} \frac{|a_{j+p}(t_1) - a_{j+p}(t_2)|}{|t_1 - t_2|^{j/p} \log |t_1 - t_2|} < +\infty, \\
\quad b_j = b_j(t) \in \mathbb{R}, \quad \sup_{t_1 \neq t_2} \frac{|b_j(t_1) - b_j(t_2)|}{|t_1 - t_2|^{j/p} \log |t_1 - t_2|} < +\infty,
\]

\( j = 1, \ldots, p, \)

then there exists \( \delta > 0 \) such that for every choice of Cauchy data \( u_0 \in H^s, u_1 \in H^{s-p} \), the problem (1.1) has a unique solution \( u \in C([0,T]; H^{s-\delta}) \cap C^1([0,T]; H^{s-p-\delta}) \). In this case we say that the problem is well-posed in \( H^\infty \) with a loss of derivatives. Suitable counterexamples in [1] and [2] show that the assumption (1.7) is the optimal one for the well-posedness of (1.1) in \( H^\infty \). We notice that the leading coefficients \( a_{2p}, b_p \) have the so-called log-Lipschitz regularity in (1.7).

**Remark 1.4** ([1]). If one has

\[
a_{j+p} = a_{j+p}(t), \quad a_{j+p} \in C^{0,\alpha_{j+p}}([0,T];\mathbb{R}), \\
\quad b_j = b_j(t), \quad b_j \in C^{0,\beta_j}([0,T];\mathbb{R}), j = 1, \ldots, p,
\]

with \( \alpha_{j+p} < j/p \) or \( \beta_j < j/p \), even for a single coefficient \( a_{j+p} \) or \( b_j \), then the problem (1.1) may in general be well-posed only in Gevrey spaces.

**Remark 1.5.** In the limit hyperbolic case \( p = 1 \), the results of [1] that we recall here in Remark 1.3 are in line with the well-known results of [3] for coefficients depending only on the variable \( t \) and of [4] for the equation

\[
u_{tt} - \partial_x(a(t,x)u_x) = 0
\]

with a log-Lipschitz coefficient \( a(t,x) \) with respect to both variables \( (t,x) \). See also [5] for the study of a general second order hyperbolic operator with log-Lipschitz coefficients.
After the sharp regularity in $t$ for the well-posedness in Sobolev spaces has been established with real coefficients $a_{p+j}(t)$, $b_j(t)$, $1 \leq j \leq p-1$, some natural questions arise looking at the results above. The first question is which coefficients $a_k$ with $k > p$ and $b_j$ with $j > 0$ may also depend on the space variable $x$, and, together with this question, we have the following two:

- **Question A:**
  Can the coefficients $a_k$ with $p < k < 2p$ and $b_j$ with $0 < j < p$ be complex valued as those coefficients with $k \leq p$ and $j = 0$ are?

- **Question B:**
  How could we compensate for a lack of Hölder continuity $\alpha_k < (k-p)/p$ or $\beta_j < j/p$ of a coefficient $a_k$ or $b_j$?

For instance, the necessity of a positive answer to the first problem arises in a natural way in the Euler-Bernoulli model of the vibrating beam, which corresponds to $p = 2$ and Section 2 of this paper for $p \geq 2$. Indeed, in the Schrödinger case $p = 2$, the necessity of the condition (1.8) for the well-posedness in $H^\infty$ has been fully proved; see, e.g., [6]. Still for $p = 2$, one needs the stronger condition

$$|a''_{p-1}(t, x)| \leq C(x)^{-\sigma}, \quad \sigma > 1,$$

for the $H^s$ well-posedness.

We state our results concerning these questions in the main two cases $p = 2$ and $p = 3$. We could give similar results for general $p$ but the number of cases that one has to consider grows very fast with $p$, particularly for Question B. Besides (1.4), we also need to assume that the characteristic roots do not vanish for $\xi \neq 0$, that is,

$$|a_{2p}(t)| \geq \lambda_1 > 0.$$

First we state our results for Question A, assuming the sharp Hölder continuity of the coefficients:

**Theorem 1.6.** Let us consider the problem (1.1) in the case $1 < p \leq 3$ under the assumptions (1.4), (1.9), (1.10). Let the leading coefficients $a_{2p}(t)$, $b_p(t)$ be Lipschitz continuous functions and

$$a_{p+j}, b_j \in C^{0,j/p}([0, T], B^\infty), \quad 1 \leq j \leq p-1.$$

Let us write $a_k = a'_k + ia''_k$, $b_j = b'_j + ib''_j$, with real $a'_k, a''_k, b'_j, b''_j$, and let us assume that

$$|a''_{2p-1}| + |b''_{p-1}| \leq C_0(x)^{-\sigma}, \quad \sigma \geq 1,$$

see, for example, [7] for $p = 2$ and Section 2 of this paper for $p \geq 2$. Precisely, denoting hereafter $\langle x \rangle$ for $\sqrt{1 + x^2}$, one has to assume

$$|e''_j(t, x)| \leq C(x)^{-j/(p-1)}, \quad j = 1, \ldots, p - 1;$$

see, for example, [7] for $p = 2$ and Section 2 of this paper for $p \geq 2$. Indeed, in the Schrödinger case $p = 2$, the necessity of the condition (1.8) for the well-posedness in $H^\infty$ has been fully proved; see, e.g., [6]. Still for $p = 2$, one needs the stronger condition

$$|e''_p(t, x)| \leq C(x)^{-\sigma}, \quad \sigma > 1,$$

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We state our results concerning these questions in the main two cases $p = 2$ and $p = 3$. We could give similar results for general $p$ but the number of cases that one has to consider grows very fast with $p$, particularly for Question B. Besides (1.4), we also need to assume that the characteristic roots do not vanish for $\xi \neq 0$, that is,

$$|a_{2p}(t)| \geq \lambda_1 > 0.$$
and, for $p > 2$, that
\begin{equation}
|\partial_x a''_{p-1}| + |\partial_x b''_{p-1}| + |a''_{p-2}| + |b''_{p-2}| \leq C(x)^{-(p-2)/(p-1)}.
\end{equation}

Then the Cauchy problem (1.1) is well-posed in:
- $H^s$ for $\sigma > 1$ in (1.11);
- $H^\infty$ with a loss of derivatives for $\sigma = 1$ in (1.11), provided that (1.9) is specified by
\begin{equation}
a_{2p}(t) \geq \lambda_1 > 0.
\end{equation}

Remark 1.7. The condition (1.9) means that the two characteristic roots have the same sign at any $t \in [0, T]$ and $\xi \neq 0$. In Theorem 1.10 we deal with the remaining case of characteristic roots of different signs and $\sigma = 1$ in (1.11).

It is also interesting to point out that under the assumption (1.9), Hölder continuity is required only for the real parts of the coefficients. On the other hand, for $p > 2$, we have to assume some extra decay conditions for the derivatives in $x$ of the imaginary parts in order to allow them to be merely continuous functions of $t$. In fact, we have:

**Proposition 1.8.** Let (1.9) be fulfilled in the case (1.9)$.+$ Maintaining all the other assumptions, the conclusions of Theorem 1.6 hold true, weakening (1.10) into
\begin{equation}
|a''_{p+j}, b''_j| \in C^{0,j/p}(0, T; B^\infty), \quad 1 \leq j \leq p-1,
\end{equation}
provided that, for $p > 2$,
\begin{equation}
|\partial_x a''_{p-1}) + |\partial_x b''_{p-1}) + |\partial_x a''_{2p-1}) + |\partial_x b''_{2p-1}) \leq C(x)^{-(p-2)/(p-1)}
\end{equation}
is added to (1.12) and that the further condition
\begin{equation}
|\partial_x a''_{2p-1}) + |\partial_x b''_{2p-1}) \leq C(x)^{-(p-2-2)/(p-1)}
\end{equation}
is assumed.

Remark 1.9. If in Theorem 1.6 or Proposition 1.8 the coefficients satisfy (1.7), uniformly with respect to the variable $x$, instead of (1.10) or (1.10)$_R$ for $j < p$, then the problem (1.1) is well-posed in $H^\infty$ with a loss of derivatives for any $\sigma \geq 1$ in (1.11). This time, a loss comes from the logarithm in the modulus of continuity with respect to $t$, independently of the behaviour for $|x| \to \infty$; cf. Remark 1.3.

The set of conditions (1.4), (1.9)$_+ \notin$ does not allow us to consider the case $b_p(t) = 0$. If (1.9) is satisfied in the opposite case,
\begin{equation}
a_{2p}(t) \geq \lambda_1 > 0,
\end{equation}
then (1.4) is fulfilled for any coefficient $b_p(t)$, but now we need more than Hölder regularity with respect to the time variable of the coefficients in dealing with the case $\sigma = 1$ in (1.11). It seems that for characteristic roots of opposite signs, the loss of derivatives coming from $\sigma = 1$ has to be compensated for by a higher regularity in $t$ of the coefficients. In fact, we have:

**Theorem 1.10.** Let us consider the problem (1.1) in the case $1 < p \leq 3$ under the assumptions (1.3), (1.9)$_-$. Assume that (1.11) is satisfied with $\sigma = 1$ and, for $p > 2$, assume also (1.12). Let us denote
\begin{equation}
N_0 := \frac{C_0}{p} \left( \frac{1}{\sqrt{\lambda_0}} + \frac{2}{\lambda_1} \right)
\end{equation}
with $C_0$, $\lambda_0$ and $\lambda_1$ the constants in (1.11), (1.4) and (1.9), respectively.
If the coefficients are such that
\[(1.15) \quad a_{2p}, b_p \in C^{N+p}([0, T]; \mathbb{R}_+),
\]
\[
a_k \in C^{N+k-p}([0, T]; \mathcal{B}^\infty), \quad b_j \in C^{N+j}([0, T]; \mathcal{B}^\infty),
\]
\[0 \leq k \leq 2p - 1, \quad 0 \leq j \leq p - 1,
\]
for \(N \geq N_0\), then the Cauchy problem \((1.1)\) is well-posed in \(H^\infty\) with a loss of \(N_0\) derivatives.

Passing to Question B, Proposition \([1.8]\) says that the regularity in \(t\) and the behaviour for \(|x| \to +\infty\) are essentially independent for the imaginary parts of the coefficients, at least in the case of characteristic roots of the same sign. On the other hand, from Theorem \([1.6]\) we do not need any condition as \(|x| \to +\infty\) for real parts which have the sharp Hölder regularity. Now we state for the real parts \((1.14)\)

| \begin{align*}
|a_{p+j}| &\leq C \langle x \rangle^{-(j-p\alpha_{p+j})/(p-1)}, \quad |b_j| \leq C \langle x \rangle^{-(j-p\beta_j)/(p-1)}, \\
0 &< \alpha_{p+j} < j/p, \quad j = 1, \ldots, p - 1.
\end{align*} |

For \(p > 2\), let \(\partial_x a_{2p-1}, \partial_x b_{p-1}\) be such that
\[(1.17) \quad |\partial_x a_{2p-1}| + |\partial_x b_{p-1}| \leq \langle x \rangle^{-(p-2)/(p-1)}.
\]
Furthermore, still for \(p > 2\), if \(\alpha_{p-1} < (p-2)/p\), respectively \(\beta_{p-1} < (p-2)/p\), then let us assume
\[(1.18) \quad |\partial_{xx} a_{2p-1}| \leq C \langle x \rangle^{-(p-2-p\alpha_{2p-1})/(p-1)},
\]
respectively
\[
|\partial_{xx} b_{p-1}| \leq C \langle x \rangle^{-(p-2-p\beta_{p-1})/(p-1)},
\]
and, if \(\alpha_{p-1} < (p-2 - 1/2)/p\), respectively \(\beta_{p-1} < (p-2 - 1/2)/p\), also
\[
|\partial_{xxx} a_{2p-1}| \leq C \langle x \rangle^{-(p-2-1/2-p\alpha_{2p-1})/(p-1)},
\]
respectively
\[
|\partial_{xxx} b_{p-1}| \leq C \langle x \rangle^{-(p-2-1/2-p\beta_{p-1})/(p-1)}.
\]

Then, the problem \((1.1)\) is well-posed in \(H^*\).

**Remark 1.12.** For \(\alpha_{p+j} = \beta_j = j/p\) in \((1.10)\) (sufficient Hölder regularity, no necessity of decay for \(a_{p+j}\) and \(b_j\)), the result of Theorem \([1.11]\) reduces to the particular case \(a''_{p+j} = b''_j = 0\) of real coefficients in Theorem \([1.6]\). For \(\alpha_{p+j} = \beta_j = 0\) (no Hölder continuity, fastest decay) the well-posedness holds true in \(H^\infty\) with complex \(a_k, b_j\). For merely continuous coefficients, there is not any gain from assuming them to be real valued; cf. the assumptions on the imaginary parts \(a''_{p+j}, b''_j\) in Proposition \([1.8]\).
2. Preliminary results and first order equations

In this section we state some preliminary results and deal with first order (in time) $p$-evolution operators as in [13].

We need to introduce pseudo-differential operators $p(x,D_x)$ of order $m$ on $\mathbb{R}$ with symbols $p(x,\xi)$ in the standard class $S^m$ defined by

\begin{equation}
|p^{(\alpha)}(x,\xi)| \leq C_{\alpha,\beta,h} \langle \xi \rangle_h^{-m-\alpha}, \quad p^{(\alpha)} := \partial_\xi^{\alpha} D_x^{\beta} p, \quad \langle \xi \rangle_h := \sqrt{h^2 + \xi^2}, h \geq 1.
\end{equation}

These are bounded operators from $H^{s+m}$ to $H^s$ for any $s$. In particular, we also use families of symbols $\Lambda(x,\xi)$ such that

\begin{equation}
|\Lambda(x,\xi)| \leq C + \delta \log \langle \xi \rangle_h, \quad |\Lambda^{(\alpha)}(x,\xi)| \leq \delta_{\alpha,\beta} \langle \xi \rangle_h^{-\alpha}, \quad \alpha, \beta \geq 1,
\end{equation}

with constants $C, \delta$ and $\delta_{\alpha,\beta}$ independent of the family parameter $h$.

**Proposition 2.1.** Let $\Lambda(x,\xi)$ satisfy (2.2). Then, the operator $e^{\Lambda}$ with symbol $e^{\Lambda(x,\xi)} \in S^0$ is invertible for a large enough $h$.

**Proof.** Let us take $e^{-\Lambda}$ with symbol $e^{-\Lambda(x,\xi)}$. We have

\[ e^{\Lambda} e^{-\Lambda} = I - r(x,D_x) \]

with principal symbol of $r$ given by

\begin{equation}
|\Lambda(x,\xi)| \leq C + \delta \log \langle \xi \rangle_h, \quad |\Lambda^{(\alpha)}(x,\xi)| \leq \delta_{\alpha,\beta} \langle \xi \rangle_h^{-\alpha}, \quad \alpha, \beta \geq 1,
\end{equation}

with constants $C, \delta$ and $\delta_{\alpha,\beta}$ independent of the family parameter $h$.

We use also the following result for $k \times k$ matrix operators which shows that an operator of order $m$ with positive Hermitian symbol is a positive operator modulo an error of order $m-1$. The asymptotic expansion of the error term is very important for our applications in the case $p > 2$.

**Theorem 2.2** ([8], page 134). Let $Q(x,\xi)$ be a $k \times k$ matrix of symbols in $S^m$, $k \geq 1$, and assume that its Hermitian part satisfies

\begin{equation}
|Q(x,\xi)| + Q^*(x,\xi)/2 \geq 0.
\end{equation}

Then there is a positive $k \times k$ matrix operator $P(x,D_x)$ of order $m$,

\[ 2\Re(Pu,u) \geq 0, \quad u \in H^m, \]
such that
\begin{equation}
Q(x, D_x) = P(x, D_x) + R(x, D_x), \quad R(x, \xi) \in S^{m-1},
\end{equation}
\begin{equation}
R(x, \xi) \sim \psi_1(\xi) D_x Q(x, \xi) + \sum_{\alpha+\beta \geq 2} \psi_{\alpha,\beta}(\xi) \partial_\xi^\alpha D_x^\beta Q(x, \xi),
\end{equation}
\begin{equation}
\psi_1 \in S^{-1}, \quad \psi_{\alpha,\beta} \in S^{(\alpha-\beta)/2}.
\end{equation}

This implies the well-known sharp Gårding inequality
\begin{equation}
2\Re(Q u, u) \geq -C \|u\|^2_{(m-1)/2}
\end{equation}
for a matrix operator satisfying (2.4). We deal with matrices in the next section.

For scalar operators with positive symbol, the stronger Fefferman-Phong inequality
\begin{equation}
2\Re(Q u, u) \geq -C \|u\|^2_{(m-2)/2}
\end{equation}
holds true.

Let us now consider the Cauchy problem for an operator \(L_1\) as in (1.5), that is,
\begin{equation}
\begin{cases}
D_t u + e_p(t) D_x^p u + \sum_{j=0}^{p-1} e_j(t, x) D_x^j u = 0 \\
u(0, x) = u_0.
\end{cases}
\end{equation}

We say that the problem is well-posed in \(H^s\) if for any \(u_0 \in H^s\) there is a unique solution \(u \in C([0, T]; H^s)\). If there is a unique solution \(u \in C([0, T]; H^{s-\delta})\), \(\delta > 0\), then we say that (2.6) is well-posed in \(H^\infty\) (with a loss of derivatives).

We assume that the leading coefficient is real and such that
\begin{equation}
e_p \in C([0, T]; \mathbb{R}), \quad e_p(t) \geq 0.
\end{equation}
The lower order coefficients are complex valued and such that
\begin{equation}
e_j \in C([0, T]; \mathcal{B}^\infty), \quad j = 0, \ldots, p-1.
\end{equation}

\textbf{Theorem 2.3.} Let us consider the problem (2.6) with \(1 < p \leq 3\) under the assumptions (2.7) and (2.8). Let us write \(e_j = e'_j + i e''_j\) with real \(e'_j, e''_j\). For the imaginary part \(e''_j\), let us assume
\begin{equation}
|e''_{p-1}| \leq C e_p(t) \langle x \rangle^{-\sigma}, \quad \sigma \geq 1,
\end{equation}
and, for \(p > 2\),
\begin{equation}
|\partial_x e'_{p-1}| + |e''_{p-2}| \leq C e_p(t) \langle x \rangle^{-(p-2)/(p-1)}.
\end{equation}
Then the Cauchy problem (2.6) is well-posed in Sobolev spaces. In particular, if (2.4) is fulfilled with \(\sigma > 1\), then the problem (2.6) is well-posed in \(H^s\). If (2.4) is fulfilled with \(\sigma = 1\), then the problem (2.6) is well-posed in \(H^\infty\) with a loss of derivatives.

\textbf{Remark 2.4.} If the leading coefficient \(e_p(t)\) never vanishes in \([0, T]\), then (2.4) and (2.10) reduce to
\begin{equation}
|e''_{p-1}| \leq C' \langle x \rangle^{-\sigma}, \quad |\partial_x e'_{p-1}| + |e''_{p-2}| \leq C' \langle x \rangle^{-(p-2)/(p-1)}.
\end{equation}
Proof of Theorem 2.3. Our aim is to obtain the well-posedness in $H^s$ or $H^\infty$ of problem (2.6) for the operator $L_1$ as in (1.9), by proving the well-posedness in $H^s$ of the Cauchy problem for a transformed operator

\[(2.11) \quad L_1^\Lambda := (e^\Lambda)^{-1}L_1e^\Lambda,\]

where $\Lambda$ is real valued and satisfies (2.2). The symbols of $e^\Lambda, (e^\Lambda)^{-1}$ are in $S^\delta$ and this brings a loss of derivatives for $\delta > 0$. In the case $\sigma > 1$ in (2.9), we can take $\delta = 0$ in the change of variable, so that the well-posedness in $H^s$ holds for (2.6).

Let us write $iL_1 = \partial_t + iK(t, x, D_x)$, that is,

\[(2.12) \quad K(t, x, D_x) = e_p(t)D_x^p + \sum_{j=0}^{p-1} e_j(t, x)D_x^j.\]

Since

\[iL_1^\Lambda = \partial_t + iK^\Lambda, \quad K^\Lambda = (e^\Lambda)^{-1}Ke^\Lambda,\]

we seek $\Lambda(x, \xi)$ such that $iK^\Lambda$ is bounded from below in $L^2$ uniformly for $t \in [0, T]$:

\[(2.13) \quad 2\Re (iK^\Lambda u, u) \geq -C\|u\|^2, \quad u \in H^p.\]

From this, the energy method gives the well-posedness in $L^2$ of the Cauchy problem for $L_1^\Lambda$. Provided that $\langle D_x \rangle^s iK^\Lambda \langle D_x \rangle^{-s}$ also satisfies (2.13), the well-posedness in $H^s$ follows.

Let us write $iK$ as the sum

\[iK = H_K + A_K, \quad H_K = (iK + (iK)^*)/2, \quad A_K = (iK - (iK)^*)/2\]

of its Hermitian and anti-Hermitian parts. In order to obtain (2.13), we consider two zones in the phase space: the support of $\varrho \left( \langle x \rangle / \langle \xi \rangle_h^{p-1} \right)$ and the support of $\left( 1 - \varrho \left( \langle x \rangle / \langle \xi \rangle_h^{p-1} \right) \right)$ with $\varrho \in C_0^\infty$ a cutoff function, $0 \leq \varrho(y) \leq 1$, $\varrho(y) = 1$ in a neighborhood of $y = 0$. Conditions (2.9) and (2.10) imply

\[\left( 1 - \varrho \left( \langle x \rangle / \langle \xi \rangle_h^{p-1} \right) \right) H_K(t, x, \xi) \in C([0, T]; S^0);\]

hence, we have to transform only $\varrho H_K$ into a bounded from below operator. We define the symbol $\Lambda$ as a sum of $p - 1$ terms $\Lambda_{p-j}$, $j = 1, \ldots, p-1$, each one related to the part of positive order $j - h$ in the expansion of the Hermitian part $H_K$. The first term is

\[(2.14) \quad \Lambda_{p-1} = M_{p-1}\omega(\xi/h) \int_0^\infty \langle y \rangle^{-\sigma} \varrho \left( \langle y \rangle / \langle \xi \rangle_h^{p-1} \right) dy\]

with $M_{p-1}$ a large constant and $\omega(y)$ a smooth function with $\omega(y) = 0$ for $|y| \leq 1$, $\omega(y) = |y|^{p-1}/y^{p-1}$ for $|y| \geq 2$. Such a symbol $\Lambda_{p-1}$ satisfies (2.2) for $\sigma = 1$ whereas for $\sigma > 1$ it belongs to $S^0$, still with $\sup_{x, \xi} |\Lambda_{p-1}^{(j)}(x, \xi)| \langle \xi \rangle_h^\sigma$ independent of $h$. Also taking

\[\langle x \rangle^{-1} \partial_x \Lambda_{p-1} \in S^{-r}, \langle x \rangle^\sigma \partial_x \Lambda_{p-1} \in S^0\]

and (2.3) into account, the Hermitian part $H_{K_{\Lambda_{p-1}}}$ of $iK_{\Lambda_{p-1}}$ is given by

\[H_{K_{\Lambda_{p-1}}}(t, x, \xi) \]

\[= \varrho \left( \langle x \rangle / \langle \xi \rangle_h^{p-1} \right) \left[ pM_{p-1}e_p(t)\xi^{p-1}\langle x \rangle^{-\sigma} - e_{p-1}''(t, x)\xi^{p-1} \right] + Q_{p-2}(t, x, \xi),\]
with \( Q_{p-2}(t, x, \xi) \in C([0, T]; S^{p-2}) \). Taking a sufficiently large \( M_{p-1} \), in view of (2.9) and thanks to Theorem 2.2 there are a positive operator \( P_{p-1}(t, x, D_x) \) of order \( p-1 \) and a remainder \( R_{p-2}(t, x, D_x) \) of order \( p-2 \) such that

\[
H_{K^{\Lambda_{p-1}}} = P_{p-1} + R_{p-2};
\]

hence, considering also the anti-Hermitian part \( ie_p(t)D_x^p + A_{p-1} \), we get

\[
iK^{\Lambda_{p-1}} = ie_p(t)D_x^p + A_{p-1} + P'_{p-1} + R'_{p-2},
\]

where the principal part of \( R'_{p-2} \) now does not contain the real term \( \psi_{0,2}(\xi)\partial_x e''_{p-1} x^{p-1} \) which comes in \( R_{p-2} \) of (2.15) from (2.5). This allows us to not include \( \partial_x e''_{p-1} \) in (2.10) for the present case \( p \leq 3 \). We define \( \Lambda_{p-2} \) by

\[
\Lambda_{p-2}(x, \xi) = M_{p-2}(\xi)\frac{1}{\hbar} \omega(\xi/\hbar) \int_{0}^{x} (y)^{-(p-2)/(p-1)} \varrho \left( \langle y \rangle / \langle \xi \rangle^{p-1}_{\hbar} \right) dy,
\]

which belongs to \( S^0 \) with semi-norms independent of \( \hbar \) and compute the expansion of the Hermitian part \( H_{K^{\Lambda}} = \Lambda_{p-1} + \Lambda_{p-2} \). Also taking

\[
\langle x \rangle^{(p-2)/(p-1)} \partial_x \Lambda_{p-2}, \langle x \rangle^{-1/(p-1)} \Lambda_{p-2} \in S^{-1}
\]

and (2.3) into account, we have

\[
H_{K^{\Lambda}}(t, x, \xi) = P'_{p-1}(t, x, \xi) + \varrho \left( \langle x \rangle / \langle \xi \rangle^{p-1}_{\hbar} \right) Q'_{p-2}(t, x, \xi) + i(p-1)\partial_x e''_{p-1}(t)\xi^{p-2}/2 + R_{p-3}(t, x, \xi)
\]

with \( P'_{p-1} \) as in (2.10),

\[
Q'_{p-2}(t, x, \xi) = pM_{p-2}e_p(t)|\xi|^{p-1}(\xi)_{\hbar}^{-1}(x)^{-(p-2)/(p-1)} + \xi^{p-2} [-e''_{p-2}(t) - (p-1)\partial_x e''_{p-1}(t)/2 + (p-1)M_{p-1}e''_{p-1}(t)\langle x \rangle^{-1}}
\]

and

\[
R_{p-3} \in C([0, T]; S^{p-3}).
\]

From (2.10), we can choose the constant \( M_{p-2} \) in order to have

\[
Q'_{p-2}(t, x, \xi) \geq 0.
\]
3. Proofs of the main results

In this section we consider an operator $L$ as in (1.2) and we prove Theorem 1.6 Proposition 1.8, Theorem 1.10 and Theorem 1.11.

We reduce the equation $Lu = 0$ to a first order (in $\partial_t$) $2 \times 2$ system. In doing so, we need a partial factorization of $L$. We describe in detail the procedure under the assumptions of Theorem 1.6 then, in the proofs of the other results, we sketch the necessary changes.

**Lemma 3.1.** Let us consider the operator $L$ given by (1.2) under the assumptions of Theorem 1.6 and let

$$
(3.1) \quad e_p^\pm(t) = b_p(t) \pm \sqrt{b_p^2(t) - a_{2p}(t)}
$$

such that $-e_p^+(t)\xi^p$ and $-e_p^-(t)\xi^p$ are the real distinct roots of the principal symbol of $L$. Then

$$
(3.2) \quad L = L_1^-(t, x, D_t, D_x) + R_p(t, x, D_x) + R_0(t, x, D_x)D_t,
$$

where

$$
(3.3) \quad L_1^\pm = D_t + e_p^\pm(t)D_x + \sum_{j=1}^{p-1} e_j^\pm(t, x, D_x)D_x^j
$$

with $e_j^\pm \in C([0, T]; S^0)$ such that

$$
(3.4) \quad |\Im e_{p-1}^\pm| \leq C\langle x \rangle^{-\sigma},
$$

and, for $p > 2$, such that

$$
(3.5) \quad |\partial_x \Re e_{p-1}^\pm| + |\Im e_{p-2}^\pm| \leq C\langle x \rangle^{-(p-2)/(p-1)},
$$

and where

$$
(3.6) \quad R_p \in L^\infty([0, T]; S^p), R_0 \in L^\infty([0, T]; S^0).
$$

**Proof.** The first step in the proof is to factorize the principal part of $L$ obtaining

$$
L = (D_t + e_p^-(t)D_x)(D_t + e_p^+(t)D_x) + R_{2p-1}(t, x, D_x) + R_{p-1}(t, x, D_x)D_t,
$$

where the principal part of the remainder is

$$
a_{2p-1}(t, x)D_x^{2p-1} + b_{p-1}(t, x)D_x^{p-1}D_t.
$$
If also \(a_{2p-1}\) and \(b_{p-1}\) were Lipschitz continuous in \(t\), then one could perform directly a second step of factorization. Since these coefficients are only \(C^0,(p-1)/p\), we introduce the regularization

\[
(3.7) \quad \tilde{a}_{2p-1}(t, x, \xi) = \int_{-\infty}^{\infty} a_{2p-1}(\tau, x) g((t - \tau)(\xi)^p)(\xi)^p d\tau,
\]

where \(g(y)\) is a cutoff function, \(0 \leq g(y) \leq 1\), \(\int_{-\infty}^{\infty} g(y) dy = 1\), and \(a_{2p-1}(\tau, x) = a_{2p-1}(0, x)\) for \(\tau \leq 0\), \(a_{2p-1}(\tau, x) = a_{2p-1}(T, x)\) for \(\tau \geq T\). We also define \(\tilde{b}_{p-1}\) in the same way. From the Hölder continuity in \(t\) of \(a_{2p-1}(t, x)\) and \(b_{p-1}(t, x)\), we have

\[
(3.8) \quad \tilde{a}_{2p-1}, \tilde{\tilde{b}}_{p-1} \in C([0, T]; S^0) \cap C^1([0, T]; S^1),
\]

\[
\tilde{a}_{2p-1} - a_{2p-1}, \tilde{\tilde{b}}_{p-1} - b_{p-1} \in C([0, T]; S^{-p+1}).
\]

Modulo a term of order zero, the principal symbol of \(R_{2p-1} + R_{p-1}D_t\) can be represented by

\[
\tilde{a}_{2p-1}(t, x, \xi)\xi^{2p-1} + \tilde{\tilde{b}}_{p-1}(t, x, \xi)\xi^{p-1} t,
\]

which is differentiable in \(t\). Taking the solution \(e_{p-1}^{\pm}(t, x, \xi)\) of the linear system

\[
(3.9)
\begin{align*}
\frac{\partial}{\partial t} e_{p-1}^+ &+ p_{p-1} e_{p-1}^+ = \tilde{a}_{2p-1} \\
\frac{\partial}{\partial t} e_{p-1}^- &+ p_{p-1} e_{p-1}^- = \tilde{\tilde{b}}_{p-1}
\end{align*}
\]

we have

\[
L = \left(D_t + e_p^{-} D_x + e_p^{-} D_x^{p-1} - e_p^{+} D_x^{p-1}\right)\left(D_t + e_p^{+} D_x + e_p^{+} D_x^{p-1}\right) + R_{2p-2} + R_{p-2} D_t,
\]

which gives (3.2) together with (3.4) for \(p = 2\), taking (1.11) into account.

For \(p > 2\), we take the symbols \(\tilde{a}_{2p-2}\) and \(\tilde{\tilde{b}}_{p-2}\), which are such that

\[
(3.10) \quad \tilde{a}_{2p-2}, \tilde{\tilde{b}}_{p-2} \in C^1([0, T]; S^{0}) \cap C^{1}([0, T]; S^{2}),
\]

\[
\tilde{a}_{2p-2} - a_{2p-2}, \tilde{\tilde{b}}_{p-2} - b_{p-2} \in C([0, T]; S^{-p+2}),
\]

and represent the principal symbols of \(R_{2p-2} + R_{p-2} D_t\) by

\[
r_{2p-2}^0 e_{2p-2} + \tilde{b}_{2p-2} e_{2p-2}, \quad r_{2p-2}^0 = \tilde{a}_{2p-2} + e_p^- e_{p-1} + p e_p^- D_x e_{p-1}^+.
\]

Since we have \(|\partial_x Re_{p-1}^\pm| \leq C(\langle x \rangle)^{-(p-2)/(p-1)}\) from (1.12), it follows that the imaginary part of \(r_{2p-2}^0\) also satisfies

\[
|\Im r_{2p-2}^0| \leq C(\langle x \rangle)^{-(p-2)/(p-1)}
\]
as \(\Im \tilde{b}_{p-2}\) does, still from (1.12). Now taking the solution \(e_{p-2}^\pm(t, x, \xi)\) of the linear system

\[
(3.11)
\begin{align*}
\frac{\partial}{\partial t} e_{p-2}^+ &+ p_{p-2} e_{p-2}^+ = r_{2p-2}^0 \\
\frac{\partial}{\partial t} e_{p-2}^- &+ p_{p-2} e_{p-2}^- = \tilde{b}_{p-2}
\end{align*}
\]

we have

\[
L = \left(D_t + e_p^- D_x + e_p^- D_x^{p-1} + e_p^+ D_x^{p-1}\right)\left(D_t + e_p^+ D_x + e_p^+ D_x^{p-1} + e_p^- D_x^{p-2}\right) + R_{2p-3} + R_{p-3} D_t,
\]

which gives (3.2) together with (3.4) and (3.5) for \(p = 3\). Since we are considering \(1 < p \leq 3\), we can stop the factorization procedure.
Lemma 3.2. Let us consider the operator \( L \) given by (1.2) under the assumptions of Theorem 1.6 and let us denote by

\[ L^\pm_t = D_t + K^\pm_1(t, x, D_x) \]

the operators in (3.3), that is,

\[ K^\pm_1(t, x, D_x) = e^\pm_p(t) D_x^p + \sum_{j=1}^{p-1} e^\pm_j(t, x, D_x) D_x^j. \]

Then, the scalar equation \( Lu = 0 \) is equivalent to the 2 \times 2 system \( SU = 0 \),

\[ S = D_t + K(t, x, D_x) + E_0(t, x, D_x), \]

with

\[ K = \begin{pmatrix} K^+ & 0 \\ 0 & K^- \end{pmatrix}, \quad K^+ = K^+_1 + [(D_x)^p, K^+_1] (D_x)^{-p}, \quad K^- = K^-_1, \]

\[ [(D_x)^p, K^+_1] \text{ the commutator of } (D_x)^p \text{ and } K^+_1, \]

and

\[ E_0(t, x, \xi) = \begin{pmatrix} e^{01}_1(t, x, \xi) & e^{02}_1(t, x, \xi) \\ e^{01}_2(t, x, \xi) & e^{02}_2(t, x, \xi) \end{pmatrix}, \quad e^{0j}_k(t, x, \xi) \in L^\infty([0, T], S^0). \]

Remark 3.3. The term \([D_x]^p, K^+_1]\) \((D_x)^{-p}\) in \( K^+ \) is of order \( p - 1 \) since \( e^+_p \) does not depend on \( x \). For \( p = 2 \), one can move it from the diagonal part \( D_t + K \) of the system \( S \) into the matrix \( E_0 \). For \( p > 2 \), we observe that, from (1.12) and (3.2), its principal imaginary part \( p(1 + \xi^2)^{-1} \xi^p \partial_x Re e^+_{p-1} \) is also bounded by \( C\langle x\rangle^{-p-2}/(\langle p-2\rangle)\langle \xi\rangle^{-p-2} \) as \( |e^\pm_{p-2}| \langle \xi\rangle^{-p-2} \).

Proof of Lemma 3.2. For a scalar unknown \( u \) we define the vector \( U = (u_0, u_1) \) by

\[ u_0 = \langle D_x \rangle^p u, \quad u_1 = L^+_1 u \]

so that, from (3.2), the scalar equation \( Lu = 0 \) is equivalent to the system \( S_1 U = 0 \) with

\[ S_1 = D_t + \begin{pmatrix} K^+ & -\langle D_x \rangle^p \\ 0 & K^- \end{pmatrix} + \begin{pmatrix} 0 \\ (R_p - R_0 K^-_1) \langle D_x \rangle^{-p} \end{pmatrix} \begin{pmatrix} 0 \\ R_0 \end{pmatrix}. \]

Now we perform \( p - 1 \) steps of diagonalization. In view of (1.4), the matrix

\[ \begin{pmatrix} K^+ & -\langle \xi \rangle^p \\ 0 & K^- \end{pmatrix} \]

can be smoothly diagonalized for large \( |\xi| \), say \( |\xi| \geq R \), by

\[ N_0(t, x, \xi) = \begin{pmatrix} 1 & n_0(t, x, \xi) \\ 0 & 1 \end{pmatrix}, \quad n_0(t, x, \xi) = \langle \xi \rangle^p/(K^+ - K^-), \]

which is in \( C^1([0, T]; S^0) \) by (3.8) and (3.10). For the system \( S_1 \) in (3.10) we have

\[ N_0^{-1} S_1 N_0 = S \]

with \( S \) as in (3.10) modulo a term

\[ \begin{pmatrix} 0 & z_{p-2} \\ 0 & 0 \end{pmatrix}, \]

where the principal part

\[ z_{p-2}^0 = \partial_k K^+ D_x n_0 - \partial_k n_0 D_x K^- \]
of \( z_{p-2} \) is of order \( p - 2 \), since \( e^{\pm \frac{t}{p}} \) does not depend on \( x \). For \( p = 2 \), this first step of diagonalization is sufficient. For \( p > 2 \), we perform a second step by means of the operator with symbol

\[
N_1 = \begin{pmatrix} 1 & -z_0^{p-2}/(K^+ - K^-) \\ 0 & 1 \end{pmatrix}, \quad |\xi| \geq R,
\]

obtaining an error of order \( p - 3 \). Since we are considering \( 1 < p \leq 3 \), we can stop the diagonalization procedure. □

**Proof of Theorem 1.6.** We prove the well-posedness of the Cauchy problem (1.1) for the scalar operator \( L \) by proving the well-posedness of the equivalent first order (in time) problem

\[
\begin{cases}
SU(t, x) = 0 \\
U(0, x) = U_0(x)
\end{cases}
\]

for the system \( S \) in (3.13). In doing so, we can apply Theorem 2.3 (in its pseudo-differential form) to the diagonal part \( D_t + K \) of \( S = D_t + K + E_0 \) thanks to (3.4), (3.5) and, for \( p > 2 \), also taking Remark 3.3 into account.

From (1.9), the functions \( e^{\pm \frac{t}{p}} \) do not vanish. Let

\[
\delta_{\pm} = e^{\pm \frac{t}{p}}(t)/|e^{\pm \frac{t}{p}}(t)|
\]

be the respective signs and, for \( \Lambda \) as in (2.11) constructed in the proof of Theorem 2.3 let us consider the transformed system

\[
S^\Lambda := \begin{pmatrix} e^{\delta_+ \Lambda} & 0 \\ 0 & e^{\delta_- \Lambda} \end{pmatrix}^{-1} S \begin{pmatrix} e^{\delta_+ \Lambda} & 0 \\ 0 & e^{\delta_- \Lambda} \end{pmatrix}.
\]

We have

\[
S^\Lambda = D_t + K^\Lambda + E_0^\Lambda,
\]

where

\[
2\Re(iK^\Lambda U, U) \geq -C\|U\|^2, \quad U \in H^p.
\]

The energy estimate gives the well-posedness in \( H^s \) of the Cauchy problem for \( S^\Lambda \), which corresponds to the well-posedness of (3.19) in \( H^s \) for \( \sigma > 1 \) in (1.11) and in \( H^\infty \) for \( \sigma = 1 \), provided that \( E_0^\Lambda \) is of order zero. This is true, independently of the signs \( \delta_{\pm} \), for \( \sigma > 1 \) because \( \Lambda \) is of order 0 in this case. For \( \sigma = 1 \), here we need to specify (1.9) into (1.9)\( _+ \) in order to avoid terms of positive order in the anti-diagonal part of \( E_0^\Lambda \). In fact, for \( \delta_+ = \delta_- \), the change of variable in (3.21) is of scalar type and we have \( E_0^\Lambda = E_0 + E_{-1} \) with \( E_{-1} \) a matrix operator of order \(-1\). □

**Proof of Proposition 1.8.** Here the assumption (1.10) is weakened into (1.10)\( _R \). In Lemma 3.1 this allows us to include only the real parts of \( a_{p+j}D_x^{p+j}, b_jD_x^jD_t \), \( 1 \leq j \leq p - 1 \), in the factorization of \( L \) and to leave the imaginary parts in the remainder. Consequently, in Lemma 3.2, the system \( S \) is now given by

\[
S = D_t + K(t, x, D_x) + \sum_{j=0}^{p-1} F_j(t, x, D_x),
\]

where
where:
- the operators $K_{+}^{\pm}$ in the diagonal matrix $K$ defined by (3.13) have real full symbols; in particular, the Hermitian part $(iK + (iK)^*)/2$ is of order $p - 2$ and, from (1.12), the principal part $H_{K}^{0}$ is such that
\begin{equation}
|H_{K}^{0}| \leq C\langle x\rangle^{-(p-2)/(p-1)}\langle \xi\rangle^{p-2}
\end{equation}
for $p > 2$;
- the full matrices $F_{j}$, $0 \leq j \leq p - 1$, are of order $j$ with
\begin{equation}
|F_{p-1}| \leq C\langle x\rangle^{-\sigma}\langle \xi\rangle^{p-1}
\end{equation}
from (1.11) and with
\begin{equation}
\langle \xi\rangle^{-1}|\partial_{x}F_{p-1}| + \langle \xi\rangle^{-1}|\partial_{xx}F_{p-1}| + |F_{p-2}| \leq C\langle x\rangle^{-(p-2)/(p-1)}\langle \xi\rangle^{p-2},
\end{equation}
\begin{equation}
|\partial_{xxx}F_{p-1}| \leq C\langle x\rangle^{-(p-2)/(p-1)}\langle \xi\rangle^{p-1}
\end{equation}
from (1.12) and (1.13) for $p > 2$.

In view of (1.14), we have $\delta_{+} = \delta_{-}$ in (3.20), say $\delta_{+} = \delta_{-} = +1$: if $\delta_{+} = \delta_{-} = -1$, then one just changes $\Lambda$ into $-\Lambda$ in what follows. This means that the change of variable in (3.21) is given by the scalar operator $e^{\Lambda}$ and for a full matrix operator $F = F(x, D_{x})$ we have the same expansion
\begin{equation}
F^{\Lambda} = F + \text{ lower order terms}
\end{equation}
of $F^{\Lambda} = (e^{\Lambda})^{-1}Fe^{\Lambda}$ as we have for a scalar $F$. From this, we can follow the same strategy as in the proof of Theorem 2.3 since the sharp Gårding inequality holds true for the system. We cannot use the Fefferman-Phong inequality here, so the expansion of the error term in (2.5) leads us to assume (1.12) and (1.13) besides (1.12) in order to provide (3.25) and (3.26) for $p > 2$.

Let us take $\Lambda_{p-1}$ as in (2.14) with a sufficiently large constant $M_{p-1}$. From (3.21), Theorem 2.2 and taking the order $p - 2$ of $(iK + (iK)^*)/2$ into account, there are an anti-Hermitian matrix operator $A_{p}$ of order $p$, a positive matrix operator $F_{p-1}$ of order $p - 1$ and a matrix operator $Q_{p-2}$ of order $p - 2$ such that
\begin{equation}
(e^{\Lambda_{p-1}})^{-1}iSe^{\Lambda_{p-1}} = \partial_{x} + A_{p} + F_{p-1} + Q_{p-2}
\end{equation}
for the system $S$ in (3.22). For $p = 2$, this is already sufficient to obtain the desired well-posedness of (3.19), hence of (1.1), according to the value of $\sigma$ in (1.11). For $p > 2$, (3.24) and (2.2), together with (3.26) and (3.26), give
\begin{equation}
Q_{p-2} = G_{p-2} + G_{p-2-1/2} + G_{p-3},
\end{equation}
where each $G_{q}$ is of order $q$ and with symbol such that
\begin{equation}
|G_{q}| \leq C\langle x\rangle^{-q/(p-1)}\langle \xi\rangle^{q}, \quad q > p - 3.
\end{equation}
Following this expansion of $Q_{p-2}$, as in the proof of Theorem 2.3, we add terms to $\Lambda_{p-1}$ by defining
\begin{equation}
\Lambda = \Lambda_{p-1} + \Lambda_{p-2} + \Lambda_{p-2-1/2}
\end{equation}
with $\Lambda_{q}$ for $q > p - 1$ defined as in (2.18) by
\begin{equation}
\Lambda_{q}(x, \xi) = M_{q}(\xi)^{g-p+1} \omega(\xi/h) \int_{0}^{x} \langle y\rangle^{-q/(p-1)} g \left( \langle y\rangle/\langle \xi\rangle_{h}^{p-1} \right) dy.
\end{equation}
Since the sum of anti-Hermitian operators is anti-Hermitian and the sum of positive operators is positive, we obtain
\[(e^A)^{-1}iSe^A = \partial_t + A + P + Q_{p-3}\]
with \(A\) anti-Hermitian of order \(p\), \(P\) positive of order \(p-1\) and \(Q_{p-3}\) of order \(p-3\). This concludes the proof for \(p \leq 3\).

**Proof of Theorem 1.10** Under the assumptions of Theorem 1.10 the differentiability of the coefficients with respect to the time variable in (1.15) allows us to perform \(N + p\) steps of factorization in Lemma 3.1 instead of \(p\) steps, and without any need for the regularizations \(\tilde{\partial}_k, \tilde{b}_j\) defined in (3.7). This leads to

\[L = L_0^- (t, x, D_t, D_x) L_0^+ (t, x, D_t, D_x) + R_{p-N} (t, x, D_x) + R_{-N} (t, x, D_x) D_t,\]

where now, still denoting by \(D_x^j\) an operator with symbol \(\xi^j\) outside a neighborhood of \(\xi = 0\) also for \(j < 0\),

\[L_0^\pm = D_t + e_p^\pm (t) D_x^p + \sum_{j=-N+1}^{p-1} e_j^\pm (t, x, D_x) D_x^j\]

with \(e_j^\pm \in C([0, T]; S^0)\) and where the remainders are now such that

\[R_{p-N} \in C([0, T]; S^{p-N}), R_{-N} \in C([0, T]; S^{-N}).\]

In particular, here \(e_{p-1}^\pm (t, x, \xi)\) is the solution of the linear system

\[\begin{cases} e_p^- e_p^- + e_p^+ e_{p-1} = a_{2p-1} \\ e_p^+ + e_{p-1}^- = b_{p-1} \end{cases}\]

with roots \(e_p^\pm\) in (5.1) of different sign, \(e_p^+ > 0, e_p^- < 0\), from (1.9). In Lemma 3.2 we still define \(K\) as in (3.14), now with

\[K_1^\pm (t, x, D_x) = e_p^\pm (t) D_x^p + \sum_{j=-N+1}^{p-1} e_j^\pm (t, x, D_x) D_x^j,\]

but, after \(N + p - 1\) steps of diagonalization, we now obtain a system

\[S = D_t + K(t, x, D_x) + E_{-N} (t, x, D_x)\]

with a non-diagonal part \(E_{-N}\) of negative order \(-N\). We still apply Theorem 2.3 to the diagonal part \(D_t + K\) of \(S = D_t + K + E_{-N}\) as in the proof of Theorem 1.6, now with \(\delta_+ = +1, \delta_- = -1\) in (3.21).

Provided that \(E_{-N}^A\) in (3.21) is of order less than or equal to zero, the energy method gives the well-posedness in \(H^\sigma\) of the Cauchy problem for \(S^A\), which here corresponds to the well-posedness of (3.19) in \(H^\infty\) with a loss of derivatives, since now \(\sigma = 1\) in (1.11). From the different signs of the roots \(e_p^\pm\) and the assumption \(\sigma = 1\), the order of \(E_{-N}^A\) and the loss are given by \(-N + 2M_{p-1}\) and \(2M_{p-1}\), respectively, with \(M_{p-1}\) the constant in (2.14). From the proof of Theorem 2.3 and the expression of \(e_{p-1}^\pm\), \(M_{p-1}\) has to be such that

\[p|e_p^\pm| M_{p-1} \geq C_0 (1 + |e_p^\pm|) / (e_p^+ - e_p^-),\]

\[C_0\] the constant in (1.11), and this is satisfied by taking

\[2M_{p-1} = N_0\]

with \(N_0\) defined by (1.14).
We have $E^\Lambda_N$ in (3.24) of order less than or equal to zero thanks to $N \geq N_0$ in (1.15). The problem (3.19) is well-posed in $H^\infty$ with a loss of $N_0$ derivatives.

Proof of Theorem 1.11 Under the H"older continuity given by (1.16), the properties (3.8) and (3.10) become

$$\tilde{a}_{p+j} \in C([0,T]; S^0) \cap C^1([0,T]; S^{p-p\alpha_{p+j}}),$$

$$\tilde{b}_j \in C([0,T]; S^0) \cap C^1([0,T]; S^{p-p\beta_j}),$$

$$\tilde{a}_{p+j} - a_{p+j} \in C([0,T]; S^{-p\alpha_{p+j}}), \quad \tilde{b}_j - b_j \in C([0,T]; S^{-p\beta_j}).$$

Consequently the factorization (3.2) becomes

$$L = L_{1+}^\pm (t, x, D_t, D_x) L_{1-}^\pm (t, x, D_t, D_x)$$

$$+ \sum_{j=1}^{p-1} R_{p+1-p\alpha_{p+j}}(t, x, D_x) + R_{h-p\beta_j}(t, x, D_x) D_t$$

with $R_q$ of order $q$. Also here, as in the proof of Proposition 1.8 the operators $L_1^\pm$ have real full symbols.

Now, the system $S$ in Lemma 3.2 is given by

$$S = D_t + K + \sum_{j=1}^{p-1} (F_{j-p\alpha_{p+j}} + F_{h-p\beta_j}) + F_0,$$

where:

- the Hermitian part $(iK + (iK)^*)/2$ of the diagonal operator $iK$ is of order $p - 2$ and, from (1.17), with principal part $R_0^0$ which satisfies (3.23);
- the full matrices $F_q$ are of order $q$ for $q \geq 0$ with

$$|F_q| \leq C(x)^{-q/(p-1)}(\xi)^q$$

for $q > 0$ from the decay assumptions in (1.16). For $p > 2$, if there are orders $q > 1$, then we also have

$$|\partial_x F_q| + |\partial_{xx} F_q| \leq C(x)^{-(q-1)/(p-1)}(\xi)^q, \quad q > 1,$$

from (1.18) and, eventually,

$$|\partial_{xxx} F_q| \leq C(x)^{-(q-1-1/2)/(p-1)}(\xi)^q, \quad q > 1 + 1/2,$$

from (1.19).

Also here from (1.9), we have $\delta_+ = \delta_-$ in (3.20) and we may take $\delta_+ = \delta_- = +1$ without loss of generality.

Let us take the symbol of order zero,

$$\Lambda = \Lambda_{p-1-p\alpha_{p-1}} + \Lambda_{p-1-p\beta_{p-1}},$$

where each $\Lambda_q$ is defined by (3.24) with a sufficiently large constant $M_q$. From (3.28), Theorem 2.2 and taking the order $p - 2$ of $(iK + (iK)^*)/2$ into account, there are an anti-Hermitian matrix operator $A_p$ of order $p$, positive matrix operators $P_q$ of order $q$ and a matrix operator $Q_{p-2}$ of order $p - 2$ such that

$$(e^\Lambda)^{-1} i S e^\Lambda = \partial_t + A_p + P_{p-1-p\alpha_{p-1}} + P_{p-1-p\beta_{p-1}} + Q_{p-2}. $$
For $p = 2$, this is already sufficient to obtain the well-posedness in $H^s$ of (3.19), hence of (11). For $p > 2$, (3.23) and (2.5), together with (3.29) and (3.30), give

\begin{equation}
Q_{p-2} = \sum_{q \in I, q \geq 0} G_q,
\end{equation}

\begin{equation}
I = \{p - 3, p - 5/2 - p\alpha_{2p-1}, p - 5/2 - p\beta_{p-1}, p - 2 - p\alpha_{2p-1},
\end{equation}

\begin{equation}
p - 2 - p\beta_{p-1}, p - 2 - p\alpha_{2p-2}, p - 2 - p\beta_{p-2}, p - 2\},
\end{equation}

where each $G_q$ is of order $q$ and with symbol such that

$$\|G_q\| \leq C(x)^{-q/(p-1)} \langle \xi \rangle^q$$

for $q > 0$. Following (3.31), we add terms in the expansion of $\Lambda$, now taking

$$\Lambda = \Lambda_{p-1 - p\alpha_{2p-1}} + \Lambda_{p-1 - p\beta_{p-1}} + \sum_{q \in I, q > 0} \Lambda_q,$$

all the terms still defined by (3.27). We obtain

$$(\varepsilon^\Lambda)^{-1} iS e^\Lambda = \partial_t + A + P + Q_{p-3}$$

with $A$ anti-Hermitian, $P$ positive and $Q_{p-3}$ of order $p - 3$, concluding the proof for $p \leq 3$. □

References


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