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THE CAUCHY PROBLEM FOR p-EVOLUTION EQUATIONS

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Abstract. In this paper we deal with the Cauchy problem for evolution equations with real characteristics. We show that the problem is well-posed in Sobolev spaces assuming a suitable decay of the coefficients as the space variable $x \to \infty$. In some cases, such a decay may also compensate a lack of regularity with respect to the time variable t.

1. Introduction and main results

Let us consider the Cauchy problem in $[0,T] \times \mathbb{R}_x$,

$$\begin{cases} Lu = 0 \\ u(0,x) = u_0, \ \partial_t u(0,x) = u_1 \end{cases} \tag{1.1}$$

for the operator

$$L := D_t^2 + 2b_p(t)D_tD_x^p + a_{2p}(t)D_x^{2p} + \sum_{j=0}^{p-1} b_j(t,x)D_tD_x^j + \sum_{k=0}^{2p-1} a_k(t,x)D_x^k, \tag{1.2}$$

where $p \geq 2$ is a positive integer, $D = \frac{1}{i}\partial$. For the coefficients a_k, $k < 2p$, and b_j, $j < p$, in general complex valued functions, we assume

$$a_k, b_j \in C\left([0,T];\mathcal{B}^\infty\right), \ k = 0, \dots, 2p-1, \ j = 0, \dots, p-1, \tag{1.3}$$

where $\mathcal{B}^\infty = \mathcal{B}^\infty(\mathbb{R}_x)$ denotes the space of all functions $f(x)$ which are bounded in $\mathbb{R}_x$ together with all their derivatives. The leading coefficients a_{2p} and b_p are assumed to be real valued continuous functions such that

$$\Delta(t) := b_p^2(t) - a_{2p}(t) \geq \lambda_0 > 0; \tag{1.4}$$

hence L is a p-evolution operator with real distinct characteristic roots $\tau = \left(-b_p(t) \pm \sqrt{\Delta(t)}\right)\xi^p$. In view of the Lax-Mizohata theorem, for any $p \geq 1$, to have real roots is a necessary condition in order to solve uniquely (1.1) in Sobolev spaces in a neighborhood of $t = 0$, [9]. As far as applications are concerned, for $p = 2$ and $b_p = 0$ we have a vibrating beam equation. Still for $p = 2$, the related first order (in time) equation $L_1 u = 0$,

$$L_1 := D_t + e_p(t)D_x^p + \sum_{j=0}^{p-1} e_j(t,x)D_x^j, \ e_p(t) \in \mathbb{R}, \tag{1.5}$$

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is a Schrödinger equation, while for $p = 3$, the principal part of $L_1 u = 0$ is the same as in the Korteweg-De Vries equation.

The kovalevskian case $p = 1$ in (1.2) is that of a strictly hyperbolic operator. Just in this latest case, by well-known results starting from [3], we know that there is a strict and deep relation between the well-posedness of the Cauchy problem (1.1) and the modulus of Hölder continuity of the coefficients with respect to the time variable. For $p \geq 2$ these topics have been studied in [1]. In particular, from the results we have obtained there, we have the following theorem:

Theorem 1.1 ([1])**.** *Besides* (1.3) *and* (1.4)*, let us assume*

$$(1.6) \qquad a_{j+p} = a_{j+p}(t),\ b_j = b_j(t),\ a_{j+p}, b_j \in C^{0,j/p}([0,T];\mathbb{R}), \quad j = 1, \dots, p.$$

Then, for every choice of Cauchy data $u_0 \in H^s, u_1 \in H^{s-p}$ *the problem* (1.1) *has a unique solution* $u \in C([0,T];H^s) \cap C^1([0,T];H^{s-p})$*.*

When the conclusions of Theorem 1.1 hold, we say that the problem (1.1) is well-posed in the Sobolev space H^s. We notice that the leading coefficients a_{2p} and b_p are Lipschitz continuous in (1.6).

Remark 1.2. In view of the counterexamples we showed in [1] and [2], the regularity given by (1.6) is the optimal one for the well-posedness in H^s.

Remark 1.3 ([1]). If one weakens the hypothesis (1.6) by assuming

$$(1.7) \quad a_{j+p} = a_{j+p}(t) \in \mathbb{R},\ \sup_{t_1 \neq t_2} \frac{|a_{j+p}(t_1) - a_{j+p}(t_2)|}{|t_1 - t_2|^{j/p} |\log |t_1 - t_2||} < +\infty,$$
$$b_j = b_j(t) \in \mathbb{R},\ \sup_{t_1 \neq t_2} \frac{|b_j(t_1) - b_j(t_2)|}{|t_1 - t_2|^{j/p} |\log |t_1 - t_2||} < +\infty,$$
$$j = 1, \dots, p,$$

then there exists $\delta > 0$ such that for every choice of Cauchy data $u_0 \in H^s, u_1 \in H^{s-p}$, the problem (1.1) has a unique solution $u \in C([0,T];H^{s-\delta}) \cap C^1([0,T];H^{s-p-\delta})$. In this case we say that the problem is well-posed in H^∞ with a loss of derivatives. Suitable counterexamples in [1] and [2] show that the assumption (1.7) is the optimal one for the well-posedness of (1.1) in H^∞. We notice that the leading coefficients a_{2p}, b_p have the so-called log-Lipschitz regularity in (1.7).

Remark 1.4 ([1]). If one has

$$a_{j+p} = a_{j+p}(t),\ a_{j+p} \in C^{0,\alpha_{j+p}}([0,T];\mathbb{R}),$$
$$b_j = b_j(t),\ b_j \in C^{0,\beta_j}([0,T];\mathbb{R}), j = 1, \dots, p,$$

with $\alpha_{j+p} < j/p$ or $\beta_j < j/p$, even for a single coefficient a_{j+p} or b_j, then the problem (1.1) may in general be well-posed only in Gevrey spaces.

Remark 1.5. In the limit hyperbolic case $p = 1$, the results of [1] that we recall here in Remark 1.3 are in line with the well-known results of [3] for coefficients depending only on the variable t and of [4] for the equation

$$u_{tt} - \partial_x(a(t,x)u_x) = 0$$

with a log-Lipschitz coefficient $a(t,x)$ with respect to both variables (t,x). See also [5] for the study of a general second order hyperbolic operator with log-Lipschitz coefficients.

After the sharp regularity in t for the well-posedness in Sobolev spaces has been established with real coefficients $a_{p+j}(t), b_j(t)$, $1 \le j \le p-1$, some natural questions arise looking at the results above. The first question is which coefficients a_k with $k > p$ and b_j with $j > 0$ may also depend on the space variable x, and, together with this question, we have the following two:

- *Question A:*

Can the coefficients a_k with $p < k < 2p$ and b_j with $0 < j < p$ be complex valued as those coefficients with $k \le p$ and $j = 0$ are?

- *Question B:*

How could we compensate for a lack of Hölder continuity $\alpha_k < (k-p)/p$ or $\beta_j < j/p$ of a coefficient a_k or b_j?

For instance, the necessity of a positive answer to the first problem arises in a natural way in the Euler-Bernoulli model of the vibrating beam, which corresponds to $p = 2$ and $b_p = 0$ in (1.1) and (1.2). In this case, $u = u(t, x)$ represents the displacement, and the shear force is proportional to $\partial_x^3 u = -iD_x^3 u$, so that one has to deal with a complex coefficient of D_x^3 in (1.2).

Actually, we cannot give positive answers to A or B if the coefficients do not depend on x, since a suitable decay is needed as $|x| \to +\infty$. In fact, already for a first order operator L_1 as in (1.5) one needs decay assumptions at least on the imaginary parts e_j'' of the coefficients $e_j = e_j' + ie_j''$, $e_j', e_j'' \in \mathbb{R}$. Precisely, denoting hereafter $\langle x \rangle$ for $\sqrt{1 + x^2}$, one has to assume

$$|e_j''(t, x)| \le C \langle x \rangle^{-j/(p-1)}, \ j = 1, \dots, p-1; \tag{1.8}$$

see, for example, [7] for $p = 2$ and Section 2 of this paper for $p \ge 2$. Indeed, in the Schrödinger case $p = 2$, the necessity of the condition (1.8) for the well-posedness in H^∞ has been fully proved; see, e.g., [6]. Still for $p = 2$, one needs the stronger condition

$$|e_{p-1}''(t, x)| \le C \langle x \rangle^{-\sigma}, \ \sigma > 1,$$

for the H^s well-posedness.

We state our results concerning these questions in the main two cases $p = 2$ and $p = 3$. We could give similar results for general p but the number of cases that one has to consider grows very fast with p, particularly for Question B. Besides (1.4), we also need to assume that the characteristic roots do not vanish for $\xi \neq 0$, that is,

$$|a_{2p}(t)| \ge \lambda_1 > 0. \tag{1.9}$$

First we state our results for Question A, assuming the sharp Hölder continuity of the coefficients:

Theorem 1.6. *Let us consider the problem* (1.1) *in the case $1 < p \le 3$ under the assumptions* (1.4), (1.9), (1.3). *Let the leading coefficients $a_{2p}(t)$, $b_p(t)$ be Lipschitz continuous functions and*

$$a_{p+j}, b_j \in C^{0,j/p}([0, T]; \mathcal{B}^\infty), \ 1 \le j \le p-1. \tag{1.10}$$

Let us write $a_k = a_k' + ia_k''$, $b_j = b_j' + ib_j''$, with real a_k', a_k'', b_j', b_j'', and let us assume that

$$|a_{2p-1}''| + |b_{p-1}''| \le C_0 \langle x \rangle^{-\sigma}, \ \sigma \ge 1, \tag{1.11}$$

and, for $p > 2$, that

$$|\partial_x a'_{2p-1}| + |\partial_x b'_{p-1}| + |a''_{2p-2}| + |b''_{p-2}| \leq C\langle x\rangle^{-(p-2)/(p-1)}. \tag{1.12}$$

Then the Cauchy problem (1.1) *is well-posed in:*

H^s *for $\sigma > 1$ in* (1.11);

H^∞ *with a loss of derivatives for $\sigma = 1$ in* (1.11), *provided that* (1.9) *is specified by*

$$a_{2p}(t) \geq \lambda_1 > 0. \tag{$(1.9)_+$}$$

Remark 1.7. The condition $(1.9)_+$ means that the two characteristic roots have the same sign at any $t \in [0, T]$ and $\xi \neq 0$. In Theorem 1.10, we deal with the remaining case of characteristic roots of different signs and $\sigma = 1$ in (1.11).

It is also interesting to point out that under the assumption $(1.9)_+$, Hölder continuity is required only for the real parts of the coefficients. On the other hand, for $p > 2$, we have to assume some extra decay conditions for the derivatives in x of the imaginary parts in order to allow them to be merely continuous functions of t. In fact, we have:

Proposition 1.8. *Let* (1.9) *be fulfilled in the case* $(1.9)_+$. *Maintaining all the other assumptions, the conclusions of Theorem* 1.6 *hold true, weakening* (1.10) *into*

$$a'_{p+j}, b'_j \in C^{0,j/p}([0,T]; \mathcal{B}^\infty), \ 1 \leq j \leq p-1, \tag{$(1.10)_\Re$}$$

provided that, for $p > 2$,

$$|\partial_x a''_{2p-1}| + |\partial_x b''_{p-1}| + |\partial_{xx} a''_{2p-1}| + |\partial_{xx} b''_{p-1}| \leq C\langle x\rangle^{-(p-2)/(p-1)} \tag{$(1.12)'$}$$

is added to (1.12) *and that the further condition*

$$|\partial_{xxx} a''_{2p-1}| + |\partial_{xxx} b''_{p-1}| \leq C\langle x\rangle^{-(p-2-1/2)/(p-1)} \tag{1.13}$$

is assumed.

Remark 1.9. If in Theorem 1.6 or Proposition 1.8, the coefficients satisfy (1.7), uniformly with respect to the variable x, instead of (1.10) or $(1.10)_\Re$ for $j < p$, then the problem (1.1) is well-posed in H^∞ with a loss of derivatives for any $\sigma \geq 1$ in (1.11). This time, a loss comes from the logarithm in the modulus of continuity with respect to t, independently of the behaviour for $|x| \to \infty$; cf. Remark 1.3.

The set of conditions (1.4), $(1.9)_+$ does not allow us to consider the case $b_p(t) = 0$. If (1.9) is satisfied in the opposite case,

$$-\, a_{2p}(t) \geq \lambda_1 > 0, \tag{$(1.9)_-$}$$

then (1.4) is fulfilled for any coefficient $b_p(t)$, but now we need more than Hölder regularity with respect to the time variable of the coefficients in dealing with the case $\sigma = 1$ in (1.11). It seems that for characteristic roots of opposite signs, the loss of derivatives coming from $\sigma = 1$ has to be compensated for by a higher regularity in t of the coefficients. In fact, we have:

Theorem 1.10. *Let us consider the problem* (1.1) *in the case $1 < p \leq 3$ under the assumptions* (1.3), $(1.9)_-$. *Assume that* (1.11) *is satisfied with $\sigma = 1$ and, for $p > 2$, assume also* (1.12). *Let us denote*

$$N_0 := \frac{C_0}{p}\left(\frac{1}{\sqrt{\lambda_0}} + \frac{2}{\lambda_1}\right) \tag{1.14}$$

with C_0, λ_0 and λ_1 the constants in (1.11), (1.4) *and* (1.9), *respectively.*

If the coefficients are such that

$$(1.15)\quad a_{2p}, b_p \in C^{N+p}([0,T];\mathbb{R}_+),$$
$$a_k \in C^{N+k-p}\left([0,T];\mathcal{B}^\infty\right),\ b_j \in C^{N+j}\left([0,T];\mathcal{B}^\infty\right),$$
$$0 \le k \le 2p-1,\ 0 \le j \le p-1,$$

for $N \ge N_0$*, then the Cauchy problem* (1.1) *is well-posed in* H^∞ *with a loss of* N_0 *derivatives.*

Passing to Question B, Proposition 1.8 says that the regularity in t and the behaviour for $|x| \to +\infty$ are essentially independent for the imaginary parts of the coefficients, at least in the case of characteristic roots of the same sign. On the other hand, from Theorem 1.6 we do not need any condition as $|x| \to +\infty$ for real parts which have the sharp Hölder regularity. Now we state for the real parts a'_{p+j}, b'_j, $j = 1, \dots, p-1$, a compensation between the regularity in t and the decay as $|x| \to +\infty$ which cannot take place for the imaginary parts a''_k, b''_j. In order to better underline this interesting effect, we state our answers to Question B for real coefficients. These results hold true for complex coefficients with imaginary parts which satisfy the same hypotheses as in Theorem 1.6 or Proposition 1.8.

Theorem 1.11. *Let us consider the problem* (1.1) *in the case* $1 < p \le 3$ *under the assumptions* (1.4), $(1.9)_+$, (1.3)*. Let the leading coefficients* $a_{2p}(t)$, $b_p(t)$ *be Lipschitz continuous functions and let us take real valued* $a_{p+j}(t,x), b_j(t,x)$, $1 \le j \le p-1$*, such that*

$$(1.16)\quad a_{p+j} \in C^{0,\alpha_{p+j}}([0,T];\mathcal{B}^\infty),\ b_j \in C^{0,\beta_j}([0,T];\mathcal{B}^\infty),$$
$$|a_{p+j}| \le C\langle x\rangle^{-(j-p\alpha_{p+j})/(p-1)},\ |b_j| \le C\langle x\rangle^{-(j-p\beta_j)/(p-1)},$$
$$0 < \alpha_{p+j}, \beta_j < j/p,\ j = 1, \dots, p-1.$$

For $p > 2$*, let* $\partial_x a_{2p-1}, \partial_x b_{p-1}$ *be such that*

$$(1.17)\qquad |\partial_x a_{2p-1}| + |\partial_x b_{p-1}| \le \langle x\rangle^{-(p-2)/(p-1)}.$$

Furthermore, still for $p > 2$*, if* $\alpha_{2p-1} < (p-2)/p$*, respectively* $\beta_{p-1} < (p-2)/p$*, then let us assume*

$$(1.18)\quad |\partial_{xx} a_{2p-1}| \le C\langle x\rangle^{-(p-2-p\alpha_{2p-1})/(p-1)},$$
$$\textit{respectively}\quad |\partial_{xx} b_{p-1}| \le C\langle x\rangle^{-(p-2-p\beta_{p-1})/(p-1)},$$

and, if $\alpha_{2p-1} < (p-2-1/2)/p$*, respectively* $\beta_{p-1} < (p-2-1/2)/p$*, also*

$$(1.19)\quad |\partial_{xxx} a_{2p-1}| \le C\langle x\rangle^{-(p-2-1/2-p\alpha_{2p-1})/(p-1)},$$
$$\textit{respectively}\quad |\partial_{xxx} b_{p-1}| \le C\langle x\rangle^{-(p-2-1/2-p\beta_{p-1})/(p-1)}.$$

Then, the problem (1.1) *is well-posed in* H^s*.*

Remark 1.12. For $\alpha_{p+j} = \beta_j = j/p$ in (1.16) (sufficient Hölder regularity, no necessity of decay for a_{p+j} and b_j), the result of Theorem 1.11 reduces to the particular case $a''_{p+j} = b''_j = 0$ of real coefficients in Theorem 1.6. For $\alpha_{p+j} = \beta_j = 0$ (no Hölder continuity, fastest decay) the well-posedness holds true in H^∞ with complex a_k, b_j. For merely continuous coefficients, there is not any gain from assuming them to be real valued; cf. the assumptions on the imaginary parts a''_{p+j}, b''_j in Proposition 1.8.

2. Preliminary results and first order equations

In this section we state some preliminary results and deal with first order (in time) p-evolution operators as in (1.5).

We need to introduce pseudo-differential operators $p(x, D_x)$ of order m on $\mathbb{R}$ with symbols $p(x, \xi)$ in the standard class S^m defined by

$$|p_{(\beta)}^{(\alpha)}(x,\xi)| \leq C_{\alpha,\beta,h}\langle\xi\rangle_h^{m-\alpha},\ p_{(\beta)}^{(\alpha)} := \partial_\xi^\alpha D_x^\beta p,\ \langle\xi\rangle_h := \sqrt{h^2+\xi^2}, h \geq 1. \tag{2.1}$$

These are bounded operators from H^{s+m} to H^s for any s. In particular, we also use families of symbols $\Lambda(x, \xi)$ such that

$$|\Lambda(x,\xi)| \leq C + \delta \log\langle\xi\rangle_h,\quad |\Lambda_{(\beta)}^{(\alpha)}(x,\xi)| \leq \delta_{\alpha,\beta}\langle\xi\rangle_h^{-\alpha},\ \alpha+\beta \geq 1, \tag{2.2}$$

with constants C, δ and $\delta_{\alpha,\beta}$ independent of the family parameter h.

Proposition 2.1. *Let $\Lambda(x, \xi)$ satisfy (2.2). Then, the operator e^Λ with symbol $e^{\Lambda(x,\xi)} \in S^\delta$ is invertible for a large enough h.*

Proof. Let us take $e^{-\Lambda}$ with symbol $e^{-\Lambda(x,\xi)}$. We have

$$e^\Lambda e^{-\Lambda} = I - r(x, D_x)$$

with principal symbol of r given by

$$r_{-1}(x,\xi) = D_x\Lambda(x,\xi)\partial_\xi\Lambda(x,\xi). \tag{2.3}$$

We have $|r_{(\beta)}^{(\alpha)}(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle_h^{-1-\alpha}$ with $C_{\alpha,\beta}$ independent on h, so we also have

$$|r_{(\beta)}^{(\alpha)}(x,\xi)| \leq C_{\alpha,\beta}h^{-1}\langle\xi\rangle_h^{-\alpha}$$

and we can fix a large h in order to have a bounded operator $r(x, D_x)$ on H^s with norm $\|r(x, D_x)\| < 1$. From this, $I - r(x, D_x)$ is invertible by Neumann series and its inverse operator

$$q(x, D_x) = \sum_{j=0}^{\infty} r^j(x, D_x)$$

has symbol in S^0. The operator $e^{-\Lambda}q$ is the right inverse of e^Λ. By similar arguments one proves the existence of a left inverse, so $e^{-\Lambda}q$ is the (two-sided) inverse operator. □

We use also the following result for $k \times k$ matrix operators which shows that an operator of order m with positive Hermitian symbol is a positive operator modulo an error of order $m-1$. The asymptotic expansion of the error term is very important for our applications in the case $p > 2$.

Theorem 2.2 ([8], page 134)**.** *Let $Q(x, \xi)$ be a $k \times k$ matrix of symbols in S^m, $k \geq 1$, and assume that its Hermitian part satisfies*

$$(Q(x,\xi) + Q^*(x,\xi))/2 \geq 0. \tag{2.4}$$

Then there is a positive $k \times k$ matrix operator $P(x, D_x)$ of order m,

$$2\Re(Pu, u) \geq 0,\ u \in H^m,$$

such that

$$
\begin{aligned}
(2.5) \quad & Q(x,D_x) = P(x,D_x) + R(x,D_x), \quad R(x,\xi) \in S^{m-1}, \\
& R(x,\xi) \sim \psi_1(\xi) D_x Q(x,\xi) + \sum_{\alpha+\beta\geq 2} \psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta Q(x,\xi), \\
& \psi_1 \in S^{-1},\ \psi_{\alpha,\beta} \in S^{(\alpha-\beta)/2}.
\end{aligned}
$$

This implies the well-known sharp Gårding inequality

$$2\Re(Qu,u) \geq -C\|u\|^2_{(m-1)/2}$$

for a matrix operator satisfying (2.4). We deal with matrices in the next section. For scalar operators with positive symbol, the stronger Fefferman-Phong inequality

$$2\Re(Qu,u) \geq -C\|u\|^2_{(m-2)/2}$$

holds true.

Let us now consider the Cauchy problem for an operator L_1 as in (1.5), that is,

$$
\begin{cases} D_t u + e_p(t) D_x^p u + \sum_{j=0}^{p-1} e_j(t,x) D_x^j u = 0 \\ u(0,x) = u_0. \end{cases} \tag{2.6}
$$

We say that the problem is well-posed in H^s if for any $u_0 \in H^s$ there is a unique solution $u \in C([0,T];H^s)$. If there is a unique solution $u \in C([0,T];H^{s-\delta})$, $\delta > 0$, then we say that (2.6) is well-posed in H^∞ (with a loss of derivatives).

We assume that the leading coefficient is real and such that

$$e_p \in C([0,T];\mathbb{R}),\ e_p(t) \geq 0. \tag{2.7}$$

The lower order coefficients are complex valued and such that

$$e_j \in C([0,T];\mathcal{B}^\infty), \quad j = 0,\dots,p-1. \tag{2.8}$$

Theorem 2.3. *Let us consider the problem* (2.6) *with* $1 < p \leq 3$ *under the assumptions* (2.7) *and* (2.8)*. Let us write* $e_j = e_j' + ie_j''$ *with real* e_j', e_j''*. For the imaginary part* e_j''*, let us assume*

$$|e''_{p-1}| \leq Ce_p(t)\langle x\rangle^{-\sigma},\ \sigma \geq 1, \tag{2.9}$$

and, for $p > 2$*,*

$$|\partial_x e'_{p-1}| + |e''_{p-2}| \leq Ce_p(t)\langle x\rangle^{-(p-2)/(p-1)}. \tag{2.10}$$

Then the Cauchy problem (2.6) *is well-posed in Sobolev spaces. In particular, if* (2.9) *is fulfilled with* $\sigma > 1$*, then the problem* (2.6) *is well-posed in* H^s*. If* (2.9) *is fulfilled with* $\sigma = 1$*, then the problem* (2.6) *is well-posed in* H^∞ *with a loss of derivatives.*

Remark 2.4. If the leading coefficient $e_p(t)$ never vanishes in $[0,T]$, then (2.9) and (2.10) reduce to

$$|e''_{p-1}| \leq C'\langle x\rangle^{-\sigma},\ |\partial_x e'_{p-1}| + |e''_{p-2}| \leq C'\langle x\rangle^{-(p-2)/(p-1)}.$$

Proof of Theorem 2.3. Our aim is to obtain the well-posedness in H^s or H^∞ of problem (2.6) for the operator L_1 as in (1.5), by proving the well-posedness in H^s of the Cauchy problem for a transformed operator

$$L_1^\Lambda := (e^\Lambda)^{-1} L_1 e^\Lambda, \tag{2.11}$$

where Λ is real valued and satisfies (2.2). The symbols of $e^\Lambda, (e^\Lambda)^{-1}$ are in S^δ and this brings a loss of derivatives for $\delta > 0$. In the case $\sigma > 1$ in (2.9), we can take $\delta = 0$ in the change of variable, so that the well-posedness in H^s holds for (2.6).

Let us write $iL_1 = \partial_t + iK(t, x, D_x)$, that is,

$$K(t, x, D_x) = e_p(t) D_x^p + \sum_{j=0}^{p-1} e_j(t, x) D_x^j. \tag{2.12}$$

Since

$$iL_1^\Lambda = \partial_t + iK^\Lambda, \quad K^\Lambda = (e^\Lambda)^{-1} K e^\Lambda,$$

we seek $\Lambda(x, \xi)$ such that iK^Λ is bounded from below in L^2 uniformly for $t \in [0, T]$:

$$2\Re\left(iK^\Lambda u, u\right) \geq -C\|u\|^2, \ u \in H^p. \tag{2.13}$$

From this, the energy method gives the well-posedness in L^2 of the Cauchy problem for L_1^Λ. Provided that $\langle D_x\rangle^s iK^\Lambda \langle D_x\rangle^{-s}$ also satisfies (2.13), the well-posedness in H^s follows.

Let us write iK as the sum

$$iK = H_K + A_K, \quad H_K = (iK + (iK)^*)/2, \ A_K = (iK - (iK)^*)/2$$

of its Hermitian and anti-Hermitian parts. In order to obtain (2.13), we consider two zones in the phase space: the support of $\varrho\left(\langle x\rangle/\langle\xi\rangle_h^{p-1}\right)$ and the support of $\left(1 - \varrho\left(\langle x\rangle/\langle\xi\rangle_h^{p-1}\right)\right)$ with $\varrho \in C_0^\infty$ a cutoff function, $0 \leq \varrho(y) \leq 1$, $\varrho(y) = 1$ in a neighborhood of $y = 0$. Conditions (2.9) and (2.10) imply

$$\left(1 - \varrho\left(\langle x\rangle/\langle\xi\rangle_h^{p-1}\right)\right) H_K(t, x, \xi) \in C([0, T]; S^0);$$

hence, we have to transform only ϱH_K into a bounded from below operator. We define the symbol Λ as a sum of $p-1$ terms Λ_{p-j}, $j = 1, \ldots, p-1$, each one related to the part of positive order $j - h$ in the expansion of the Hermitian part H_K. The first term is

$$\Lambda_{p-1} = M_{p-1}\omega(\xi/h) \int_0^x \langle y\rangle^{-\sigma} \varrho\left(\langle y\rangle/\langle\xi\rangle_h^{p-1}\right) dy \tag{2.14}$$

with M_{p-1} a large constant and $\omega(y)$ a smooth function with $\omega(y) = 0$ for $|y| \leq 1$, $\omega(y) = |y|^{p-1}/y^{p-1}$ for $|y| \geq 2$. Such a symbol Λ_{p-1} satisfies (2.2) for $\sigma = 1$ whereas for $\sigma > 1$ it belongs to S^0, still with $\sup_{x,\xi} |\Lambda_{(\beta)}^{(\alpha)}(x, \xi)|\langle\xi\rangle_h^\alpha$ independent of h. Also taking

$$\langle x\rangle^{-1}\partial_\xi \Lambda_{p-1} \in S^{-p}, \langle x\rangle^\sigma \partial_x \Lambda_{p-1} \in S^0$$

and (2.3) into account, the Hermitian part $H_{K^{\Lambda_{p-1}}}$ of $iK^{\Lambda_{p-1}}$ is given by

$$\begin{aligned} &H_{K^{\Lambda_{p-1}}}(t, x, \xi) \\ &\quad = \varrho\left(\langle x\rangle/\langle\xi\rangle_h^{p-1}\right)\left[pM_{p-1}e_p(t)|\xi|^{p-1}\langle x\rangle^{-\sigma} - e''_{p-1}(t, x)\xi^{p-1}\right] + Q_{p-2}(t, x, \xi), \end{aligned}$$

with $Q_{p-2}(t,x,\xi) \in C([0,T];S^{p-2})$. Taking a sufficiently large M_{p-1}, in view of (2.9) and thanks to Theorem 2.2, there are a positive operator $P_{p-1}(t,x,D_x)$ of order $p-1$ and a remainder $R_{p-2}(t,x,D_x)$ of order $p-2$ such that

$$H_{K^{\Lambda_{p-1}}} = P_{p-1} + R_{p-2};$$

hence, considering also the anti-Hermitian part $ie_p(t)D_x^p + A_{p-1}$, we get

$$\begin{aligned}(2.15)\quad iK^{\Lambda_{p-1}} &= ie_p(t)D_x^p + A_{p-1} + P_{p-1} + R_{p-2},\\ &2\Re\,(A_{p-1}u,u) = 0,\ 2\Re\,(P_{p-1}u,u) \geq 0,\ R_{p-2} \in C([0,T];S^{p-2}).\end{aligned}$$

In the case $p = 2$, this is already sufficient to get (2.13) with $\Lambda = \Lambda_{p-1}$.

If $p-2 > 0$, then we have to add a term Λ_{p-2} in the expansion of Λ and to specify the principal part of the remainder R_{p-2}. Since here we consider only $p \leq 3$ and we are dealing with scalar operators, it is important to observe that, taking directly

$$(2.16)\quad P'_{p-1}(t,x,\xi) = \varrho\left(\langle x\rangle/\langle\xi\rangle_h^{p-1}\right)\left[pM_{p-1}e_p(t)|\xi|^{p-1}\langle x\rangle^{-\sigma} - e''_{p-1}(t,x)\xi^{p-1}\right],$$

the Fefferman-Phong inequality also gives

$$\begin{aligned}(2.17)\quad iK^{\Lambda_{p-1}} &= ie_p(t)D_x^p + A_{p-1} + P'_{p-1} + R'_{p-2},\\ &2\Re\left(P'_{p-1}u,u\right) \geq -C\|u\|^2_{(p-3)/2},\ R'_{p-2} \in C([0,T];S^{p-2}),\end{aligned}$$

where the principal part of R'_{p-2} now does not contain the real term $\psi_{0,2}(\xi)\partial_{xx}e''_{p-1}\xi^{p-1}$ which comes in R_{p-2} of (2.15) from (2.5). This allows us to not include $\partial_{xx}e''_{p-1}$ in (2.10) for the present case $p \leq 3$. We define Λ_{p-2} by

$$(2.18)\qquad \Lambda_{p-2}(x,\xi) = M_{p-2}\langle\xi\rangle_h^{-1}\omega(\xi/h)\int_0^x \langle y\rangle^{-(p-2)/(p-1)}\varrho\left(\langle y\rangle/\langle\xi\rangle_h^{p-1}\right)dy,$$

which belongs to S^0 with semi-norms independent of h and compute the expansion of the Hermitian part H_{K^Λ}, $\Lambda = \Lambda_{p-1} + \Lambda_{p-2}$. Also taking

$$\langle x\rangle^{(p-2)/(p-1)}\partial_x\Lambda_{p-2},\ \langle x\rangle^{-1/(p-1)}\Lambda_{p-2} \in S^{-1}$$

and (2.3) into account, we have

$$\begin{aligned}H_{K^\Lambda}(t,x,\xi) = P'_{p-1}(t,x,\xi) &+ \varrho\left(\langle x\rangle/\langle\xi\rangle_h^{p-1}\right)Q'_{p-2}(t,x,\xi)\\ &+ i(p-1)\partial_x e''_{p-1}(t)\xi^{p-2}/2 + R_{p-3}(t,x,\xi)\end{aligned}$$

with P'_{p-1} as in (2.16),

$$\begin{aligned}Q'_{p-2}(t,x,\xi) &= pM_{p-2}e_p(t)|\xi|^{p-1}\langle\xi\rangle_h^{-1}\langle x\rangle^{-(p-2)/(p-1)}\\ &+ \xi^{p-2}\left[-e''_{p-2}(t) - (p-1)\partial_x e'_{p-1}(t)/2 + (p-1)M_{p-1}e'_{p-1}(t)\langle x\rangle^{-\sigma}\right],\end{aligned}$$

and

$$R_{p-3} \in C([0,T];S^{p-3}).$$

From (2.10), we can choose the constant M_{p-2} in order to have

$$Q'_{p-2}(t,x,\xi) \geq 0.$$

Hence, denoting by $ie_p(t)D_x^p + A'_{p-1}$ the anti-Hermitian part of iK^Λ and by A_{p-2} the anti-Hermitian part of $i(p-1)\partial_x e''_{p-1}(t)D_x^{p-2}/2$, there is a positive operator P_{p-2} of order $p-2$ such that

$$
\begin{aligned}
(2.19)\quad iK^\Lambda &= ie_p(t)D_x^p + A'_{p-1} + P'_{p-1} + A_{p-2} + P_{p-2} + R'_{p-3},\\
&2\Re\left(ie_p(t)D_x^p u + A'_{p-1}u + A_{p-2}u, u\right) = 0,\\
&2\Re\left(P'_{p-1}u, u\right) \geq -C\|u\|^2_{(p-3)/2},\ 2\Re\left(P_{p-2}u, u\right) \geq 0,
\end{aligned}
$$

and $R'_{p-3} \in C([0,T]; S^{p-3})$. The expansion (2.19) gives inequality (2.13) also for $p = 3$.

In order to complete the proof for any H^s, we observe that it is sufficient to choose constants $M_{p-1} = M_{p-1}(s)$, $M_{p-2} = M_{p-2}(s)$ in order to have the same bound (2.13) for $\langle D_x\rangle^s iK^\Lambda \langle D_x\rangle^{-s}$. □

3. Proofs of the main results

In this section we consider an operator L as in (1.2) and we prove Theorem 1.6, Proposition 1.8, Theorem 1.10 and Theorem 1.11.

We reduce the equation $Lu = 0$ to a first order (in ∂_t) 2×2 system. In doing so, we need a partial factorization of L. We describe in detail the procedure under the assumptions of Theorem 1.6; then, in the proofs of the other results, we sketch the necessary changes.

Lemma 3.1. *Let us consider the operator L given by* (1.2) *under the assumptions of Theorem* 1.6 *and let*

$$(3.1)\qquad e_p^\pm(t) = b_p(t) \pm \sqrt{b_p^2(t) - a_{2p}(t)}$$

such that $-e_p^+(t)\xi^p$ and $-e_p^-(t)\xi^p$ are the real distinct roots of the principal symbol of L. Then

$$(3.2)\qquad L = L_1^-(t,x,D_t,D_x)L_1^+(t,x,D_t,D_x) + R_p(t,x,D_x) + R_0(t,x,D_x)D_t,$$

where

$$(3.3)\qquad L_1^\pm = D_t + e_p^\pm(t)D_x^p + \sum_{j=1}^{p-1} e_j^\pm(t,x,D_x)D_x^j$$

with $e_j^\pm \in C([0,T]; S^0)$ such that

$$(3.4)\qquad \left|\Im e_{p-1}^\pm\right| \leq C\langle x\rangle^{-\sigma},$$

and, for $p > 2$, such that

$$(3.5)\qquad \left|\partial_x \Re e_{p-1}^\pm\right| + \left|\Im e_{p-2}^\pm\right| \leq C\langle x\rangle^{-(p-2)/(p-1)},$$

and where

$$(3.6)\qquad R_p \in L^\infty([0,T]; S^p), R_0 \in L^\infty([0,T]; S^0).$$

Proof. The first step in the proof is to factorize the principal part of L obtaining

$$L = (D_t + e_p^-(t)D_x^p)(D_t + e_p^+(t)D_x^p) + R_{2p-1}(t,x,D_x) + R_{p-1}(t,x,D_x)D_t,$$

where the principal part of the remainder is

$$a_{2p-1}(t,x)D_x^{2p-1} + b_{p-1}(t,x)D_x^{p-1}D_t.$$

If also a_{2p-1} and b_{p-1} were Lipschitz continuous in t, then one could perform directly a second step of factorization. Since these coefficients are only $C^{0,(p-1)/p}$, we introduce the regularization

$$\tilde{a}_{2p-1}(t,x,\xi) = \int_{-\infty}^{+\infty} a_{2p-1}(\tau,x) g((t-\tau)\langle\xi\rangle_h^p)\langle\xi\rangle_h^p d\tau, \tag{3.7}$$

where $g(y)$ is a cutoff function, $0 \le g(y) \le 1$, $\int_{-\infty}^{+\infty} g(y)dy = 1$, and $a_{2p-1}(\tau,x) = a_{2p-1}(0,x)$ for $\tau \le 0$, $a_{2p-1}(\tau,x) = a_{2p-1}(T,x)$ for $\tau \ge T$. We also define $\tilde{b}_{p-1}$ in the same way. From the Hölder continuity in t of $a_{2p-1}(t,x)$ and $b_{p-1}(t,x)$, we have

$$\begin{aligned} &\tilde{a}_{2p-1}, \tilde{b}_{p-1} \in C([0,T];S^0) \cap C^1([0,T];S^1), \\ &\qquad\qquad \tilde{a}_{2p-1} - a_{2p-1}, \tilde{b}_{p-1} - b_{p-1} \in C([0,T];S^{-p+1}). \end{aligned} \tag{3.8}$$

Modulo a term of order zero, the principal symbol of $R_{2p-1} + R_{p-1}D_t$ can be represented by

$$\tilde{a}_{2p-1}(t,x,\xi)\xi^{2p-1} + \tilde{b}_{p-1}(t,x,\xi)\xi^{p-1}\tau,$$

which is differentiable in t. Taking the solution $e_{p-1}^{\pm}(t,x,\xi)$ of the linear system

$$\begin{cases} e_p^- e_{p-1}^+ + e_p^+ e_{p-1}^- = \tilde{a}_{2p-1} \\ e_{p-1}^+ + e_{p-1}^- = \tilde{b}_{p-1} \end{cases} \tag{3.9}$$

we have

$$L = (D_t + e_p^- D_x^p + e_{p-1}^- D_x^{p-1})(D_t + e_p^+ D_x^p + e_{p-1}^+ D_x^{p-1}) + R_{2p-2} + R_{p-2}D_t,$$

which gives (3.2) together with (3.4) for $p = 2$, taking (1.11) into account.

For $p > 2$, we take the symbols $\tilde{a}_{2p-2}$ and $\tilde{b}_{p-2}$, which are such that

$$\begin{aligned} &\tilde{a}_{2p-2}, \tilde{b}_{p-2} \in C^1([0,T];S^0) \cap C^1([0,T];S^2), \\ &\qquad\qquad \tilde{a}_{2p-2} - a_{2p-2}, \tilde{b}_{p-2} - b_{p-2} \in C([0,T];S^{-p+2}), \end{aligned} \tag{3.10}$$

and represent the principal symbols of $R_{2p-2} + R_{p-2}D_t$ by

$$r_{2p-2}^0 \xi^{2p-2} + \tilde{b}_{p-2}\xi^{p-2}\tau, \quad r_{2p-2}^0 = \tilde{a}_{2p-2} + e_{p-1}^- e_{p-1}^+ + p e_p^- D_x e_{p-1}^+.$$

Since we have $\left|\partial_x \Re e_{p-1}^{\pm}\right| \le C\langle x\rangle^{-(p-2)/(p-1)}$ from (1.12), it follows that the imaginary part of r_{2p-2}^0 also satisfies

$$\left|\Im r_{2p-2}^0\right| \le C\langle x\rangle^{-(p-2)/(p-1)}$$

as $\Im\tilde{b}_{p-2}$ does, still from (1.12). Now taking the solution $e_{p-2}^{\pm}(t,x,\xi)$ of the linear system

$$\begin{cases} e_p^- e_{p-2}^+ + e_p^+ e_{p-2}^- = r_{2p-2}^0 \\ e_{p-2}^+ + e_{p-2}^- = \tilde{b}_{p-2} \end{cases} \tag{3.11}$$

we have

$$\begin{aligned} L = (D_t + e_p^- D_x^p + e_{p-1}^- D_x^{p-1} + e_{p-2}^- D_x^{p-2})(D_t + e_p^+ D_x^p + e_{p-1}^+ D_x^{p-1+} e_{p-2}^+ D_x^{p-2}), \\ + R_{2p-3} + R_{p-3}D_t, \end{aligned}$$

which gives (3.2) together with (3.4) and (3.5) for $p = 3$. Since we are considering $1 < p \le 3$, we can stop the factorization procedure. □

Lemma 3.2. *Let us consider the operator L given by* (1.2) *under the assumptions of Theorem* 1.6 *and let us denote by*

$$L_1^{\pm} = D_t + K_1^{\pm}(t, x, D_x)$$

the operators in (3.3), *that is,*

$$K_1^{\pm}(t, x, D_x) = e_p^{\pm}(t) D_x^p + \sum_{j=1}^{p-1} e_j^{\pm}(t, x, D_x) D_x^j. \tag{3.12}$$

Then, the scalar equation $Lu = 0$ is equivalent to the 2×2 system $SU = 0$,

$$S = D_t + K(t, x, D_x) + E_0(t, x, D_x), \tag{3.13}$$

with

$$K = \begin{pmatrix} K^+ & 0 \\ 0 & K^- \end{pmatrix}, \quad K^+ = K_1^+ + \left[\langle D_x \rangle^p, K_1^+\right] \langle D_x \rangle^{-p}, \; K^- = K_1^-, \tag{3.14}$$

$\left[\langle D_x \rangle^p, K_1^+\right]$ *the commutator of $\langle D_x \rangle^p$ and K_1^+, and with*

$$E_0(t, x, \xi) = \begin{pmatrix} e_{11}^0(t, x, \xi) & e_{12}^0(t, x, \xi) \\ e_{21}^0(t, x, \xi) & e_{22}^0(t, x, \xi) \end{pmatrix}, \; e_{jk}^0(t, x, \xi) \in L^\infty([0, T]; S^0). \tag{3.15}$$

Remark 3.3. The term $\left[\langle D_x \rangle^p, K_1^+\right] \langle D_x \rangle^{-p}$ in K^+ is of order $p-2$ since e_p^+ does not depend on x. For $p = 2$, one can move it from the diagonal part $D_t + K$ of the system S into the matrix E_0. For $p > 2$, we observe that, from (1.12) and (3.9), its principal imaginary part $p(1+\xi^2)^{-1}\xi^p \partial_x \Re e_{p-1}^+$ is also bounded by $C\langle x \rangle^{-(p-2)/(p-1)} \langle \xi \rangle^{p-2}$ as $\Im e_{p-2}^{\pm} \xi^{p-2}$.

Proof of Lemma 3.2. For a scalar unknown u we define the vector $U = {}^t(u_0, u_1)$ by

$$u_0 = \langle D_x \rangle^p u, \; u_1 = L_1^+ u$$

so that, from (3.2), the scalar equation $Lu = 0$ is equivalent to the system $S_1 U = 0$ with

$$S_1 = D_t + \begin{pmatrix} K^+ & -\langle D_x \rangle^p \\ 0 & K^- \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \left(R_p - R_0 K_1^+\right) \langle D_x \rangle^{-p} & R_0 \end{pmatrix}. \tag{3.16}$$

Now we perform $p - 1$ steps of diagonalization. In view of (1.4), the matrix

$$\begin{pmatrix} K^+ & -\langle \xi \rangle^p \\ 0 & K^- \end{pmatrix}$$

can be smoothly diagonalized for large $|\xi|$, say $|\xi| \geq R$, by

$$N_0(t, x, \xi) = \begin{pmatrix} 1 & n_0(t, x, \xi) \\ 0 & 1 \end{pmatrix}, \quad n_0(t, x, \xi) = \langle \xi \rangle^p / (K^+ - K^-), \tag{3.17}$$

which is in $C^1([0, T]; S^0)$ by (3.8) and (3.10). For the system S_1 in (3.16) we have

$$N_0^{-1} S_1 N_0 = S$$

with S as in (3.13) modulo a term

$$\begin{pmatrix} 0 & z_{p-2} \\ 0 & 0 \end{pmatrix},$$

where the principal part

$$z_{p-2}^0 = \partial_\xi K^+ D_x n_0 - \partial_\xi n_0 D_x K^-$$

of z_{p-2} is of order $p-2$, since $e_p^\pm$ does not depend on x. For $p=2$, this first step of diagonalization is sufficient. For $p>2$, we perform a second step by means of the operator with symbol

$$N_1 = \begin{pmatrix} 1 & -z_{p-2}^0/(K^+ - K^-) \\ 0 & 1 \end{pmatrix}, \quad |\xi| \geq R, \tag{3.18}$$

obtaining an error of order $p-3$. Since we are considering $1 < p \leq 3$, we can stop the diagonalization procedure. □

Proof of Theorem 1.6. We prove the well-posedness of the Cauchy problem (1.1) for the scalar operator L by proving the well-posedness of the equivalent first order (in time) problem

$$\begin{cases} SU(t,x) = 0 \\ U(0,x) = U_0(x) \end{cases} \tag{3.19}$$

for the system S in (3.13). In doing so, we can apply Theorem 2.3 (in its pseudo-differential form) to the diagonal part $D_t + K$ of $S = D_t + K + E_0$ thanks to (3.4), (3.5) and, for $p>2$, also taking Remark 3.3 into account.

From (1.9), the functions $e_p^\pm(t)$ do not vanish. Let

$$\delta_\pm = e_p^\pm(t)/\left|e_p^\pm(t)\right| \tag{3.20}$$

be the respective signs and, for Λ as in (2.11) constructed in the proof of Theorem 2.3, let us consider the transformed system

$$S^\Lambda := \begin{pmatrix} e^{\delta_+\Lambda} & 0 \\ 0 & e^{\delta_-\Lambda} \end{pmatrix}^{-1} S \begin{pmatrix} e^{\delta_+\Lambda} & 0 \\ 0 & e^{\delta_-\Lambda} \end{pmatrix}. \tag{3.21}$$

We have

$$S^\Lambda = D_t + K^\Lambda + E_0^\Lambda,$$

where

$$2\Re\left(iK^\Lambda U, U\right) \geq -C\|U\|^2, \ U \in H^p.$$

The energy estimate gives the well-posedness in H^s of the Cauchy problem for S^Λ, which corresponds to the well-posedness of (3.19) in H^s for $\sigma > 1$ in (1.11) and in H^∞ for $\sigma = 1$, provided that E_0^Λ is of order zero. This is true, independently of the signs $\delta_\pm$, for $\sigma > 1$ because Λ is of order 0 in this case. For $\sigma = 1$, here we need to specify (1.9) into $(1.9)_+$ in order to avoid terms of positive order in the anti-diagonal part of E_0^Λ. In fact, for $\delta_+ = \delta_-$, the change of variable in (3.21) is of scalar type and we have $E_0^\Lambda = E_0 + E_{-1}$ with E_{-1} a matrix operator of order -1. □

Proof of Proposition 1.8. Here the assumption (1.10) is weakened into $(1.10)_\Re$. In Lemma 3.1, this allows us to include only the real parts of $a_{p+j}D_x^{p+j}$, $b_jD_x^jD_t$, $1 \leq j \leq p-1$, in the factorization of L and to leave the imaginary parts in the remainder. Consequently, in Lemma 3.2, the system S is now given by

$$S = D_t + K(t,x,D_x) + \sum_{j=0}^{p-1} F_j(t,x,D_x), \tag{3.22}$$

where:

- the operators $K_1^{\pm}$ in the diagonal matrix K defined by (3.14) have real full symbols; in particular, the Hermitian part $(iK+(iK)^*)/2$ is of order $p-2$ and, from (1.12), the principal part H_K^0 is such that

$$|H_K^0| \le C\langle x\rangle^{-(p-2)/(p-1)}\langle\xi\rangle^{p-2} \tag{3.23}$$

for $p>2$;

- the full matrices F_j, $0\le j\le p-1$, are of order j with

$$|F_{p-1}| \le C\langle x\rangle^{-\sigma}\langle\xi\rangle^{p-1} \tag{3.24}$$

from (1.11) and with

$$\langle\xi\rangle^{-1}|\partial_x F_{p-1}| + \langle\xi\rangle^{-1}|\partial_{xx}F_{p-1}| + |F_{p-2}| \le C\langle x\rangle^{-(p-2)/(p-1)}\langle\xi\rangle^{p-2}, \tag{3.25}$$

$$|\partial_{xxx}F_{p-1}| \le C\langle x\rangle^{-(p-2-1/2)/(p-1)}\langle\xi\rangle^{p-1} \tag{3.26}$$

from (1.12), (1.12)$'$ and (1.13) for $p>2$.

In view of $(1.9)_+$, we have $\delta_+=\delta_-$ in (3.20), say $\delta_+=\delta_-=+1$: if $\delta_+=\delta_-=-1$, then one just changes Λ into $-\Lambda$ in what follows. This means that the change of variable in (3.21) is given by the scalar operator e^{Λ} and for a full matrix operator $F=F(x,D_x)$ we have the same expansion

$$F^{\Lambda} = F + \text{ lower order terms}$$

of $F^{\Lambda}=(e^{\Lambda})^{-1}Fe^{\Lambda}$ as we have for a scalar F. From this, we can follow the same strategy as in the proof of Theorem 2.3 since the sharp Gårding inequality holds true for the system. We cannot use the Fefferman-Phong inequality here, so the expansion of the error term in (2.5) leads us to assume (1.12)$'$ and (1.13) besides (1.12) in order to provide (3.25) and (3.26) for $p>2$.

Let us take Λ_{p-1} as in (2.14) with a sufficiently large constant M_{p-1}. From (3.24), Theorem 2.2 and taking the order $p-2$ of $(iK+(iK)^*)/2$ into account, there are an anti-Hermitian matrix operator A_p of order p, a positive matrix operator P_{p-1} of order $p-1$ and a matrix operator Q_{p-2} of order $p-2$ such that

$$(e^{\Lambda_{p-1}})^{-1}iSe^{\Lambda_{p-1}} = \partial_t + A_p + P_{p-1} + Q_{p-2}$$

for the system S in (3.22). For $p=2$, this is already sufficient to obtain the desired well-posedness of (3.19), hence of (1.1), according to the value of σ in (1.11). For $p>2$, (3.23) and (2.5), together with (3.25) and (3.26), give

$$Q_{p-2} = G_{p-2} + G_{p-2-1/2} + G_{p-3},$$

where each G_q is of order q and with symbol such that

$$|G_q| \le C\langle x\rangle^{-q/(p-1)}\langle\xi\rangle^{q},\ q>p-3.$$

Following this expansion of Q_{p-2}, as in the proof of Theorem 2.3, we add terms to Λ_{p-1} by defining

$$\Lambda = \Lambda_{p-1} + \Lambda_{p-2} + \Lambda_{p-2-1/2}$$

with Λ_q for $q>p-1$ defined as in (2.18) by

$$\Lambda_q(x,\xi) = M_q\langle\xi\rangle_h^{q-p+1}\omega(\xi/h)\int_0^x \langle y\rangle^{-q/(p-1)}\varrho\left(\langle y\rangle/\langle\xi\rangle_h^{p-1}\right)dy. \tag{3.27}$$

Since the sum of anti-Hermitian operators is anti-Hermitian and the sum of positive operators is positive, we obtain

$$(e^\Lambda)^{-1} iSe^\Lambda = \partial_t + A + P + Q_{p-3}$$

with A anti-Hermitian of order p, P positive of order $p-1$ and Q_{p-3} of order $p-3$. This concludes the proof for $p \le 3$. $\square$

Proof of Theorem 1.10. Under the assumptions of Theorem 1.10, the differentiability of the coefficients with respect to the time variable in (1.15) allows us to perform $N+p$ steps of factorization in Lemma 3.1, instead of p steps, and without any need for the regularizations $\tilde{a}_k, \tilde{b}_j$ defined in (3.7). This leads to

$$L = L_1^-(t,x,D_t,D_x)L_1^+(t,x,D_t,D_x) + R_{p-N}(t,x,D_x) + R_{-N}(t,x,D_x)D_t,$$

where now, still denoting by D_x^j an operator with symbol ξ^j outside a neighborhood of $\xi = 0$ also for $j < 0$,

$$L_1^\pm = D_t + e_p^\pm(t)D_x^p + \sum_{j=-N+1}^{p-1} e_j^\pm(t,x,D_x)D_x^j$$

with $e_j^\pm \in C([0,T]; S^0)$ and where the remainders are now such that

$$R_{p-N} \in C([0,T]; S^{p-N}), R_{-N} \in C([0,T]; S^{-N}).$$

In particular, here $e_{p-1}^\pm(t,x,\xi)$ is the solution of the linear system

$$\begin{cases} e_p^- e_{p-1}^+ + e_p^+ e_{p-1}^- = a_{2p-1} \\ e_{p-1}^+ + e_{p-1}^- = b_{p-1} \end{cases}$$

with roots $e_p^\pm$ in (3.1) of different sign, $e_p^+ > 0$, $e_p^- < 0$, from $(1.9)_-$.

In Lemma 3.2, we still define K as in (3.14), now with

$$K_1^\pm(t,x,D_x) = e_p^\pm(t)D_x^p + \sum_{j=-N+1}^{p-1} e_j^\pm(t,x,D_x)D_x^j,$$

but, after $N+p-1$ steps of diagonalization, we now obtain a system

$$S = D_t + K(t,x,D_x) + E_{-N}(t,x,D_x)$$

with a non-diagonal part E_{-N} of negative order $-N$. We still apply Theorem 2.3 to the diagonal part $D_t + K$ of $S = D_t + K + E_{-N}$ as in the proof of Theorem 1.6, now with $\delta_+ = +1$, $\delta_- = -1$ in (3.21).

Provided that E_{-N}^Λ in (3.21) is of order less than or equal to zero, the energy method gives the well-posedness in H^s of the Cauchy problem for S^Λ, which here corresponds to the well-posedness of (3.19) in H^∞ with a loss of derivatives, since now $\sigma = 1$ in (1.11). From the different signs of the roots $e_p^\pm$ and the assumption $\sigma = 1$, the order of E_{-N}^Λ and the loss are given by $-N + 2M_{p-1}$ and $2M_{p-1}$, respectively, with M_{p-1} the constant in (2.14). From the proof of Theorem 2.3 and the expression of $e_{p-1}^\pm$, M_{p-1} has to be such that

$$p|e_p^\pm|M_{p-1} \ge C_0(1 + |e_p^\pm|)/(e_p^+ - e_p^-),$$

C_0 the constant in (1.11), and this is satisfied by taking

$$2M_{p-1} = N_0$$

with N_0 defined by (1.14).

We have E^{Λ}_{-N} in (3.21) of order less than or equal to zero thanks to $N \geq N_0$ in (1.15). The problem (3.19) is well-posed in H^∞ with a loss of N_0 derivatives. □

Proof of Theorem 1.11. Under the Hölder continuity given by (1.16), the properties (3.8) and (3.10) become

$$\tilde{a}_{p+j} \in C([0,T]; S^0) \cap C^1([0,T]; S^{p-p\alpha_{p+j}}),$$
$$\tilde{b}_j \in C([0,T]; S^0) \cap C^1([0,T]; S^{p-p\beta_j}),$$
$$\tilde{a}_{p+j} - a_{p+j} \in C([0,T]; S^{-p\alpha_{p+j}}),\ \tilde{b}_j - b_j \in C([0,T]; S^{-p\beta_j}).$$

Consequently the factorization (3.2) becomes

$$L = L_1^-(t,x,D_t,D_x)L_1^+(t,x,D_t,D_x) + \sum_{j=1}^{p-1} R_{p+h-p\alpha_{p+j}}(t,x,D_x) + R_{h-p\beta_j}(t,x,D_x)D_t$$

with R_q of order q. Also here, as in the proof of Proposition 1.8, the operators $L_1^\pm$ have real full symbols.

Now, the system S in Lemma 3.2 is given by

$$S = D_t + K + \sum_{j=1}^{p-1} \left(F_{j-p\alpha_{p+j}} + F_{h-p\beta_j}\right) + F_0,$$

where:

- the Hermitian part $(iK + (iK)^*)/2$ of the diagonal operator iK is of order $p-2$ and, from (1.17), with principal part H_K^0 which satisfies (3.23);

- the full matrices F_q are of order q for $q \geq 0$ with

$$|F_q| \leq C\langle x\rangle^{-q/(p-1)}\langle\xi\rangle^q \tag{3.28}$$

for $q > 0$ from the decay assumptions in (1.16). For $p > 2$, if there are orders $q > 1$, then we also have

$$|\partial_x F_q| + |\partial_{xx} F_q| \leq C\langle x\rangle^{-(q-1)/(p-1)}\langle\xi\rangle^q,\ q > 1, \tag{3.29}$$

from (1.18) and, eventually,

$$|\partial_{xxx} F_q| \leq C\langle x\rangle^{-(q-1-1/2)/(p-1)}\langle\xi\rangle^q,\ q > 1 + 1/2, \tag{3.30}$$

from (1.19).

Also here from $(1.9)_+$ we have $\delta_+ = \delta_-$ in (3.20) and we may take $\delta_+ = \delta_- = +1$ without loss of generality.

Let us take the symbol of order zero,

$$\Lambda = \Lambda_{p-1-p\alpha_{2p-1}} + \Lambda_{p-1-p\beta_{p-1}},$$

where each Λ_q is defined by (3.27) with a sufficiently large constant M_q. From (3.28), Theorem 2.2 and taking the order $p-2$ of $(iK+(iK)^*)/2$ into account, there are an anti-Hermitian matrix operator A_p of order p, positive matrix operators P_q of order q and a matrix operator Q_{p-2} of order $p-2$ such that

$$(e^\Lambda)^{-1} iSe^\Lambda = \partial_t + A_p + P_{p-1-p\alpha_{2p-1}} + P_{p-1-p\beta_{p-1}} + Q_{p-2}.$$

For $p = 2$, this is already sufficient to obtain the well-posedness in H^s of (3.19), hence of (1.1). For $p > 2$, (3.23) and (2.5), together with (3.29) and (3.30), give

$$(3.31)\quad Q_{p-2} = \sum_{q\in\mathcal{I},q\geq 0} G_q,$$
$$\mathcal{I} = \{p-3, p-5/2-p\alpha_{2p-1}, p-5/2-p\beta_{p-1}, p-2-p\alpha_{2p-1}, p-2-p\beta_{p-1}, p-2-p\alpha_{2p-2}, p-2-p\beta_{p-2}, p-2\},$$

where each G_q is of order q and with symbol such that

$$|G_q| \leq C\langle x\rangle^{-q/(p-1)}\langle\xi\rangle^q$$

for $q > 0$. Following (3.31), we add terms in the expansion of Λ, now taking

$$\Lambda = \Lambda_{p-1-p\alpha_{2p-1}} + \Lambda_{p-1-p\beta_{p-1}} + \sum_{q\in\mathcal{I},q>0} \Lambda_q,$$

all the terms still defined by (3.27). We obtain

$$(e^\Lambda)^{-1} iSe^\Lambda = \partial_t + A + P + Q_{p-3}$$

with A anti-Hermitian, P positive and Q_{p-3} of order $p-3$, concluding the proof for $p \leq 3$. □

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