

DIRICHLET REGULARITY AND DEGENERATE DIFFUSION

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ABSTRACT. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set and let $m: \Omega \rightarrow (0, \infty)$ be measurable and locally bounded. We study a natural realization of the operator $m\Delta$ in $C_0(\Omega) := \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. If Ω is Dirichlet regular, then the operator generates a positive contraction semigroup on $C_0(\Omega)$ whenever $\frac{1}{m} \in L^p_{\text{loc}}(\Omega)$ for some $p > \frac{N}{2}$. If $m(x)$ does not go fast enough to 0 as $x \rightarrow \partial\Omega$, then Dirichlet regularity is necessary. However, if $|m(x)| \leq c \cdot \text{dist}(x, \partial\Omega)^2$, then we show that $m\Delta_0$ generates a semigroup on $C_0(\Omega)$ without any regularity assumptions on Ω . We show that the condition for degeneration of m near the boundary is optimal.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $m \in L^\infty_{\text{loc}}(\Omega)$ be strictly positive. The aim of this paper is to investigate when a natural realization of the operator $m\Delta$ in $C_0(\Omega) := \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ generates a C_0 -semigroup. If Ω is Dirichlet regular, then it suffices that $\frac{1}{m} \in L^p_{\text{loc}}(\Omega)$ for some $\frac{N}{2} < p \leq \infty$. If $\frac{1}{m} \in L^p(\Omega)$, then Dirichlet regularity is a necessary condition. However, if the *diffusion is weak at a point* $z \in \partial\Omega$ in the sense that $m(x) \leq c \cdot \text{dist}(x, \partial\Omega)^2$ in a neighbourhood of z , then Dirichlet regularity is not needed.

In fact, these phenomena are of local nature. Our main result (Theorem 7.1) says the following. Let $m \in L^\infty(\Omega)$ be strictly positive such that $\frac{1}{m} \in L^p_{\text{loc}}(\Omega)$ for some $\frac{N}{2} < p \leq \infty$. Assume that for each $z \in \partial\Omega$ one of the following conditions is satisfied:

- (a) z is a regular point (in the sense of Wiener) or
- (b) the diffusion is weak at z .

Then $m\Delta_0$ generates a positive C_0 -semigroup on $C_0(\Omega)$. Here $m\Delta_0$ is the natural realization of $m\Delta$ in $C_0(\Omega)$ (see Section 4).

Our notion of weak diffusion is optimal. We show that it does not suffice that $m(x) \leq c \cdot \text{dist}(x, \partial\Omega)^\beta$ for some $\beta < 2$ to ensure that $m\Delta_0$ generates a semigroup.

It is much easier to study the operator in the setting of L^p spaces, by which we also start. However, there are good reasons to consider the operator on the space $C_0(\Omega)$. One reason is that we obtain a Feller semigroup in this way with the corresponding relations to stochastic processes (see [14], [16], [17] and [33] for the role of $C_0(\Omega)$ in the theory of Markov processes). Another reason concerns possible applications to non-linear problems and dynamical systems. For semilinear problems

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the space $C_0(\Omega)$ is much better suited than $L^p(\Omega)$ -spaces since composition with a locally Lipschitz continuous function is locally Lipschitz continuous on $C_0(\Omega)$ but never on $L^p(\Omega)$; see the treatise of Cazenave-Haraux [10], for example. Studying arbitrary measurable functions m seems to be useful for possible applications to quasilinear equations.

In the present paper nowhere do we suppose that the function m satisfies any regularity assumptions other than measurability. Generation results on $C_0(\Omega)$ for bounded *continuous* functions m have been given previously by Lumer [23] (see also [22]). He uses barriers with respect to the new operator $m\Delta$ (instead of the Laplacian). The methods we use here are very different from those employed in [23].

In the case where $\frac{1}{m} \in L^p(\Omega)$ for some $p > \frac{N}{2}$ we use techniques from [2]. The special case where m is a smooth version of the distance to the boundary had been considered by Davies [12] and Pang [30]. These results were inspiring for us, and we use a smooth version of the distance as comparison when the diffusion is weak at a boundary point.

Our results show in particular that for m larger than a positive constant (even $\frac{1}{m} \in L^p(\Omega)$, $p > \frac{N}{2}$ suffices) the regular points of $m\Delta$ are the same as for Δ . The operator $m\Delta$ is a very special kind of elliptic operator in non-divergence form. For general elliptic operators in non-divergence form this is no longer true in both directions. In fact Miller [26] showed that there may be regular points for the Laplacian which are non-regular for a particular elliptic operator in non-divergence form and vice versa. This is in sharp contrast with the situation for uniformly elliptic operators in *divergence* form; see the results of Littman, Stampacchia, Weinberger [21].

The operator $m\Delta$ obtained further attention in the literature. McIntosh and Nahmod [25] proved H^∞ -calculus. Duong and Ouhabaz [15] investigated Gaussian estimates for the semigroup generated by this operator. In both results m is assumed to be larger than a positive constant. We should also point out that non-divergence operators in *one* dimension (also degenerate ones) and their probabilistic interpretation are studied by Mandl [24]. An application to mathematical finance is contained in Cannarsa et al. [9].

2. PRELIMINARIES

Here we fix some notation and explain arguments which are frequently used. Let $\Omega \subset \mathbb{R}^N$ be open and bounded. We write $\omega \Subset \Omega$ if ω is an open subset of \mathbb{R}^N such that $\bar{\omega} \subset \Omega$. The space $C_c(\Omega)$ denotes continuous functions on Ω with values in \mathbb{R} having compact support. $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ is the space of all test functions and $\mathcal{D}(\Omega)'$ the space of all distributions.

We denote by

$$H^1(\Omega) := \{u \in L^2(\Omega) : D_j u \in L^2(\Omega), j = 1, \dots, d\}$$

the first Sobolev space and by $H_0^1(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. We let

$$L_{\text{loc}}^p(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable s.t. } \int_\omega |u(x)|^p dx < \infty \text{ whenever } \omega \Subset \Omega \right\},$$

where $1 \leq p < \infty$. Similarly,

$$H_{\text{loc}}^1(\Omega) := \{u \in L_{\text{loc}}^2(\Omega) : D_j u \in L_{\text{loc}}^2(\Omega) \text{ for } j = 1, \dots, d\}.$$

We let

$$C_0(\Omega) := \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\},$$

where $\partial\Omega$ is the boundary of Ω .

Then $H^1(\Omega) \cap C_0(\Omega) \subset H_0^1(\Omega)$, but

$$H_0^1(\Omega) \cap C(\overline{\Omega}) \subset C_0(\Omega) \quad \text{if and only if } \Omega \text{ is regular in capacity}$$

(see [8]). The spaces $H_0^1(\Omega)$ and $H^1(\Omega)$ are sublattices of $L^2(\Omega)$. More precisely,

$$u \in H^1(\Omega) \quad \text{implies } D_j u^+ \in H^1(\Omega) \text{ and } D_j u^+ = \chi_{\{u>0\}} D_j u, \quad j = 1, \dots, d,$$

where by χ_A we denote the characteristic function of a set A . If $u \in H_0^1(\Omega)$, then also $u^+ \in H_0^1(\Omega)$.

If $u \in L_{loc}^1(\Omega)$, then the Laplacian Δu is a distribution. By

$$-\Delta u \leq 0 \quad \text{in } \mathcal{D}(\Omega)'$$

we mean that

$$-\langle \Delta u, v \rangle \leq 0 \quad \text{whenever } 0 \leq v \in \mathcal{D}(\Omega).$$

If $u \in H_{loc}^1(\Omega)$, this is equivalent to

$$(2.1) \quad \int_{\Omega} \nabla u(x) \nabla v(x) \, dx \leq 0 \quad \text{for } 0 \leq v \in \mathcal{D}(\Omega)$$

and if $u \in H^1(\Omega)$, both inequalities remain true for all $0 \leq v \in H_0^1(\Omega)$. In fact, the cone $\mathcal{D}(\Omega)_+$ of all positive test functions is dense in $H_0^1(\Omega)_+ := \{u \in H_0^1(\Omega) : u \geq 0\}$.

We frequently use the following **maximum principle**: Let $u \in H^1(\Omega)$ such that

$$-\Delta u \leq 0.$$

If $u^+ \in H_0^1(\Omega)$, then $u \leq 0$.

In fact, taking $v = u$ in (2.1) we obtain $\int_{\Omega} |\nabla u(x)^+|^2 \, dx \leq 0$. By Poincaré's inequality, this implies that $u^+ = 0$.

3. THE SEMIGROUP ON $L^2(\Omega, \frac{dx}{m(x)})$

Let $m : \Omega \rightarrow (0, \infty)$ be measurable such that $\frac{1}{m} \in L_{loc}^1(\Omega)$. We consider the Hilbert space $L^2(\Omega, \frac{dx}{m(x)})$ with the scalar product

$$\langle u|v \rangle = \int_{\Omega} u(x)v(x) \frac{dx}{m(x)}.$$

On $L^2(\Omega, \frac{dx}{m(x)})$ we define the operator $m\Delta_2$ by

$$\mathcal{D}(m\Delta_2) := \left\{ u \in H_0^1(\Omega) \cap L^2\left(\Omega, \frac{dx}{m(x)}\right) : \exists f \in L^2\left(\Omega, \frac{dx}{m(x)}\right) \text{ such that } \Delta u = \frac{f}{m} \right\},$$

$$(m\Delta_2)u := f.$$

Note that $\frac{f}{m} \in L_{loc}^1(\Omega)$ since for $\omega \Subset \Omega$

$$\int_{\omega} \frac{|f(x)|}{m(x)} \, dx \leq \left(\int_{\omega} |f(x)|^2 \frac{dx}{m(x)} \right)^{\frac{1}{2}} \left(\int_{\omega} \frac{dx}{m(x)} \right)^{\frac{1}{2}}.$$

Thus the identity $\Delta u = \frac{f}{m}$ is well defined in $\mathcal{D}(\Omega)'$. The expression $m\Delta_2$ is purely symbolic and has to be understood in the sense of the above definition. In fact, in general Δu is merely in $\mathcal{D}(\Omega)'$ and $m\Delta u$ cannot be defined as a distribution.

We will prove the following theorem.

Theorem 3.1. *The operator $m\Delta_2$ is self-adjoint and generates a positive, contractive C_0 -semigroup T_2 on $L^2(\Omega, \frac{dx}{m(x)})$. Moreover, the semigroup is submarkovian.*

Here, an operator S on $L^2(\Omega, \frac{dx}{m(x)})$ is called *submarkovian* if $f(x) \leq 1$ a.e. implies $Sf(x) \leq 1$ a.e. This is equivalent to saying that S is positive and

$$\|Sf\|_\infty \leq \|f\|_\infty \text{ for all } f \in L^2(\Omega, \frac{dx}{m(x)}) \cap L^\infty(\Omega).$$

To say that the semigroup T_2 is *submarkovian* means that each $T_2(t), t \geq 0$, is submarkovian.

We set $V := H_0^1(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$. Then V is a Hilbert space for the norm

$$\|u\|_V^2 = \|u\|_{H^1(\Omega)}^2 + \|u\|_{L^2(\Omega, \frac{dx}{m(x)})}^2.$$

We let $\mathcal{D}(\Omega)_+ := \{v \in \mathcal{D}(\Omega) : v \geq 0\}$ and $V_+ := \{u \in V : u \geq 0 \text{ a.e.}\}$.

Proposition 3.2. *$\mathcal{D}(\Omega)$ is dense in V and $\mathcal{D}(\Omega)_+$ is dense in V_+ .*

Proof. We prove the second assertion. The first assertion then follows since $V = V_+ - V_+$.

a) Let $u \in V_+$. There exists a sequence $\varphi_n \in \mathcal{D}(\Omega)$ s.t. $\varphi_n \rightarrow u$ in $H^1(\Omega)$. Let $u_n := (\varphi_n \wedge u) \vee 0$. Then $0 \leq u_n \leq u$ and $u_n \rightarrow u$ in $H^1(\Omega)$. Moreover $u_n \rightarrow u$ a.e. (for a subsequence which we denote also by u_n). Hence $u_n \rightarrow u$ in $L^2(\Omega, \frac{dx}{m(x)})$ by the dominated convergence theorem. We have shown that $V_+ \cap L_c^\infty(\Omega)$ is dense in V_+ , where

$$L_c^\infty(\Omega) := \{u \in L^\infty(\Omega) : \text{supp } u \subset \Omega \text{ is compact}\}.$$

b) Let $u \in V_+ \cap L_c^\infty(\Omega), u_n := \rho_n * u$, where ρ_n is a mollifier. Then $u_n \in \mathcal{D}(\Omega)$, $\text{supp } u_n \subset K \Subset \Omega$ (for $n \geq n_0$) and $\|u_n\|_\infty \leq c$ (for $n \geq n_0$), $u_n \rightarrow u$ in $H^1(\Omega)$ and $u_n \rightarrow u$ a.e. after choosing a subsequence. Hence $u_n \rightarrow u$ in $L^2(\Omega, \frac{dx}{m(x)})$. \square

Proof of Theorem 3.1. Let $a: V \times V \rightarrow \mathbb{R}$ be given by

$$a(u, v) = \int_\Omega \nabla u(x) \nabla v(x) \, dx.$$

Then a is continuous, symmetric and bilinear. Moreover, a is *accretive*, i.e., $a(u, u) \geq 0$ for all $u \in V$ and *elliptic* with respect to $L^2(\Omega, \frac{dx}{m(x)})$, i.e.,

$$a(u, u) + \omega \|u\|_{L^2(\Omega, \frac{dx}{m(x)})}^2 \geq \alpha \|u\|_V^2$$

for some $\omega \in \mathbb{R}$ and $\alpha > 0$.

This follows from Poincaré’s inequality, which asserts that $\sqrt{\int_\Omega |\nabla u(x)|^2 \, dx}$ defines an equivalent norm on $H_0^1(\Omega)$.

Let A be the operator associated with a . Then A is self-adjoint and $-A$ generates a contractive semigroup T_2 on $L^2(\Omega, \frac{dx}{m(x)})$. We show that $-m\Delta_2 = A$. In fact, for $u, f \in L^2(\Omega, \frac{dx}{m(x)})$ we have by definition,

$$\begin{aligned} u \in \mathcal{D}(A) \text{ and } -Au = f & \quad \text{if and only if} \\ a(u, v) = - \int_\Omega f(x)v(x) \frac{dx}{m(x)} & \quad \text{for all } v \in V. \end{aligned}$$

Taking $v \in \mathcal{D}(\Omega)$, this implies that $\Delta u = \frac{f}{m}$. Hence $u \in \mathcal{D}(m\Delta_2)$ and $(m\Delta_2)u = f$. Conversely, if $u \in \mathcal{D}(m\Delta_2)$ and $(m\Delta_2)u = f$, then $\Delta u = \frac{f}{m}$ in $\mathcal{D}(\Omega)'$. Since $u \in H_0^1(\Omega)$, this implies that

$$\int_{\Omega} \nabla u(x) \nabla v(x) \, dx = -\langle \Delta u, v \rangle = - \int_{\Omega} f(x)v(x) \frac{dx}{m(x)}$$

for all $v \in \mathcal{D}(\Omega)$. Since $\mathcal{D}(\Omega)$ is dense in V it follows that $u \in \mathcal{D}(A)$ and $Au = f$.

It follows from the Beurling-Deny criterion ([11], Theorem 1.3.3) or ([28], Corollary 2.17) that the semigroup is submarkovian. \square

As a consequence we find a consistent family T_p , $1 \leq p \leq \infty$, of semigroups on $L^p(\Omega, \frac{dx}{m(x)})$, such that T_2 is the given semigroup generated by $m\Delta_2$. Here T_p is a positive, contractive C_0 -semigroup for $1 \leq p < \infty$ and $T_{\infty}(t) = T_1'(t)$ for all $t \geq 0$. We denote the generator of T_p by $m\Delta_p$. Thus $m\Delta_{\infty} = (m\Delta_1)'$.

We note that consistency of the semigroups implies consistency of the resolvents. In particular,

$$(3.1) \quad R(\lambda, m\Delta_{\infty})f = R(\lambda, m\Delta_2)f$$

for all $\lambda > 0$, $f \in L^{\infty}(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$. We also note that

$$R(\lambda, m\Delta_{\infty}) \geq 0 \quad \text{for all } \lambda > 0.$$

Finally, we will frequently use the following local regularity of the Laplacian.

Let $\frac{N}{2} < p \leq \infty$. Then

$$(3.2) \quad u \in L_{\text{loc}}^1(\Omega), \Delta u \in L_{\text{loc}}^p(\Omega) \quad \text{implies } u \in C(\Omega).$$

See ([13], II.3 Proposition 6). To avoid confusion in the case $N = 1$ we shall tacitly assume $p \geq 1$ throughout the paper.

If $m \equiv 1$, then the operator $\Delta_p := m\Delta_p$ is just the Dirichlet Laplacian on $L^p(\Omega)$. We need the following properties of this operator.

Proposition 3.3. *The operator Δ_p is invertible. Moreover, for $\frac{N}{2} < p \leq \infty$ the following holds:*

- (a) $\mathcal{D}(\Delta_p) = \{u \in H_0^1(\Omega) : \Delta u \in L^p(\Omega)\}$ and $\Delta_p u = \Delta u$ in $\mathcal{D}(\Omega)'$ for all $u \in \mathcal{D}(\Delta_p)$.
- (b) $\mathcal{D}(\Delta_p) \subset C^b(\Omega) := \{u : \Omega \rightarrow \mathbb{R} : u \text{ is bounded and continuous}\}$.

Proof. The invertibility follows from ([11], Theorem 1.6.3), for example. Note that for $\frac{N}{2} < p \leq \infty$

$$\|T_p(t)\|_{\mathcal{L}(L^p(\Omega), L^{\infty}(\Omega))} \leq ct^{-\frac{N}{2p}} e^{-\omega t} \quad (t \geq 0)$$

for some $c > 0$, $\omega > 0$ (see e.g. [28] Lemma 6.5). Thus

$$R(0, \Delta_p) = \int_0^{\infty} T_p(t) \, dt \in \mathcal{L}(L^p(\Omega), L^{\infty}(\Omega)).$$

Let $f \in L^p(\Omega)$, $u = R(0, \Delta_p)f$. Then $u \in L^{\infty}(\Omega)$. Moreover, $-\Delta u = f$ in $\mathcal{D}(\Omega)'$. In fact, let $f_k \rightarrow f$ in $L^p(\Omega)$ where $f_k \in L^2(\Omega) \cap L^p(\Omega)$. Then $u_k := R(0, \Delta_p)f_k \rightarrow u$ in $L^{\infty}(\Omega)$. Moreover, since $R(0, \Delta_p)f_k = R(0, \Delta_2)f_k$, one has $u_k \in H_0^1(\Omega)$ and $-\Delta u_k = f_k$ in $\mathcal{D}(\Omega)'$. Since $u_k \rightarrow u$ in $L^{\infty}(\Omega) \hookrightarrow \mathcal{D}(\Omega)'$, it follows that $\Delta u_k \rightarrow \Delta u$

in $\mathcal{D}(\Omega)'$. Thus $-\Delta u = f$. It follows from (3.2) that $u \in C(\Omega)$. Finally, by the definition of Δ_2 , one has

$$\int_{\Omega} |\nabla u_k(x)|^2 dx = \int_{\Omega} f_k(x)u_k(x) dx \leq \|f_k\|_{L^p(\Omega)} \|u_k\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{p'}},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Thus $(u_k)_{k \in \mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Taking a subsequence, we may assume that $u_k \rightharpoonup w \in H_0^1(\Omega)$. Since $u_k \rightarrow u \in L^\infty(\Omega)$, it follows that $u = w \in H_0^1(\Omega)$. Thus (b) and one inclusion in (a) are proved.

Let $u \in H_0^1(\Omega)$ such that $f := \Delta u \in L^p(\Omega)$. It remains to show that $u \in \mathcal{D}(\Delta_p)$ and $\Delta_p u = \Delta u$. Let $w = R(0, \Delta_p)f$. Then $w \in H_0^1(\Omega)$ and $-\Delta w = f$ by what has been proved above. Thus $u + w \in H_0^1(\Omega)$ and $\Delta(u + w) = 0$. By the maximum principle (see the Introduction) this implies $u + w = 0$. \square

Now we can add the following local regularity of the Laplacian.

Let $\frac{N}{2} < p \leq \infty$. Then

$$(3.3) \quad u \in L_{loc}^1(\Omega), \Delta u \in L_{loc}^p(\Omega) \text{ implies } u \in H_{loc}^1(\Omega).$$

In fact, let $u \in L_{loc}^1(\Omega)$ such that $\Delta u \in L_{loc}^p(\Omega)$. Let $\omega \Subset \Omega$ be arbitrary and $f = \Delta u|_{\omega} \in L^p(\omega)$. Consider the operator Δ_p on $L^p(\omega)$. Then $w := \Delta_p^{-1}f \in H_0^1(\omega)$ by Proposition 3.3. Since $\Delta w = f = \Delta u$ in $\mathcal{D}(\omega)'$, the function $u - w$ is harmonic and hence in $C^\infty(\omega)$. Thus $u \in H^1(\omega)$.

In the following we again consider a function $m: \Omega \rightarrow (0, \infty)$ satisfying $\frac{1}{m} \in L_{loc}^1(\Omega)$. We first show how $m\Delta_\infty$ operates on functions.

Proposition 3.4. (a) *Let $u \in \mathcal{D}(m\Delta_\infty)$, $f = (m\Delta_\infty)u$. Then*

$$\Delta u = \frac{f}{m} \text{ in } \mathcal{D}(\Omega)'.$$

(b) *If $\frac{1}{m} \in L_{loc}^p(\Omega)$ for some $p > \frac{N}{2}$, then*

$$\mathcal{D}(m\Delta_\infty) \subset C^b(\Omega) \cap H_{loc}^1(\Omega).$$

(c) *If $m \in L_{loc}^\infty(\Omega)$, then $\mathcal{D}(\Omega) \subset \mathcal{D}(m\Delta_\infty)$ and $(m\Delta_\infty)u = m \cdot \Delta u$ for $u \in \mathcal{D}(\Omega)$.*

Proof. (a) Let $\lambda > 0$. Define $g := \lambda u - f \in L^\infty(\Omega)$. Then $u = R(\lambda, m\Delta_\infty)g$. If $g \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$, then the claim follows from the fact that $R(\lambda, m\Delta_\infty)g = R(\lambda, m\Delta_2)g$. In the general case there exist $g_k \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$ such that $g_k \rightarrow g$ for $\sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}))$. Let $u_k = R(\lambda, m\Delta_\infty)g_k$. Then

$$-\Delta u_k = \frac{g_k - \lambda u_k}{m}.$$

Now we use the fact that $R(\lambda, m\Delta_\infty) = R(\lambda, m\Delta_1)'$ is continuous for the *weak**-topology $\sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}))$. Hence $u_k \rightarrow u$ for $\sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}))$. Since $\mathcal{D}(\Omega) \subset L^1(\Omega, \frac{dx}{m(x)})$ we conclude that $u_k \rightarrow u$ in $\mathcal{D}(\Omega)'$. Hence $\Delta u_k \rightarrow \Delta u$ in $\mathcal{D}(\Omega)'$. Since $g_k - \lambda u_k \rightarrow g - \lambda u$ for $\sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}))$, it follows that $\frac{g_k - \lambda u_k}{m} \rightarrow \frac{g - \lambda u}{m}$ in $\mathcal{D}(\Omega)'$. Thus

$$-\Delta u = \frac{g - \lambda u}{m} = -\frac{f}{m}.$$

The proof of (a) is complete.

(b) This follows now from (3.2) and (3.3).

(c) Assume that $m \in L^\infty_{\text{loc}}(\Omega)$. Let $u \in \mathcal{D}(\Omega)$, $f = m \cdot \Delta u$. Then $u \in H^1_0(\Omega)$, $f \in L^2(\Omega, \frac{dx}{m(x)})$ and $\Delta u = \frac{f}{m}$. Thus $u \in \mathcal{D}(m\Delta_2)$ and $(m\Delta_2)u = f$. Let $\lambda > 0$ and set $g := \lambda u - f$. Then $g \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$ and $R(\lambda, m\Delta_\infty)g = R(\lambda, m\Delta_2)g = u$. Thus $u \in \mathcal{D}(m\Delta_\infty)$ and $\lambda u - (m\Delta_\infty)u = g = \lambda u - f$, i.e., $(m\Delta_\infty)u = f$. \square

In Proposition 3.4, the boundary condition is not incorporated. But if $\frac{1}{m} \in L^1(\Omega)$, then $L^\infty(\Omega) \subset L^2(\Omega, \frac{dx}{m(x)})$, and the operator $m\Delta_\infty$ is just the part of $m\Delta_2$ in $L^\infty(\Omega)$. Thus, if $\frac{1}{m} \in L^1(\Omega)$, then

$$(3.4) \quad \begin{aligned} \mathcal{D}(m\Delta_\infty) &= \left\{ u \in H^1_0(\Omega) \cap L^\infty(\Omega) : \exists f \in L^\infty(\Omega) \text{ s.t. } \Delta u = \frac{f}{m} \right\} \\ (m\Delta_\infty)u &= f. \end{aligned}$$

If $\frac{1}{m} \in L^p(\Omega)$ for some $\infty \geq p > \frac{N}{2}$, we can even assert more.

Proposition 3.5. *Assume that $\frac{1}{m} \in L^p(\Omega)$, where $\frac{N}{2} < p \leq \infty$. Then $m\Delta_\infty$ is invertible.*

Proof. Let $f \in L^\infty(\Omega)$. Then $\frac{f}{m} \in L^p(\Omega)$. Thus by Proposition 3.3 there exists $u \in H^1_0(\Omega)$ such that $\Delta u = \frac{f}{m}$. This shows that $m\Delta_\infty$ is surjective. If $u \in \mathcal{D}(m\Delta_\infty)$, $(m\Delta_\infty)u = 0$, then by (3.4) we have $u \in H^1_0(\Omega)$ and $\Delta u = 0$. This implies that $u = 0$. Thus $(m\Delta_\infty)$ is injective. Since the operator is closed, the proof is finished. \square

The positive semigroups T_p generated by $m\Delta_p$ on $L^p(\Omega, \frac{dx}{m(x)})$ have many interesting properties. We just mention that they are always irreducible if Ω is connected (where we assume only $0 < m, \frac{1}{m} \in L^1_{\text{loc}}(\Omega)$ as before). This means that

$$(e^{t(m\Delta_p)} f)(x) > 0 \text{ a.e. for all } 0 \leq f \in L^p\left(\Omega, \frac{dx}{m(x)}\right), f \neq 0, \text{ and for all } t > 0.$$

For $p = 2$ this follows from Ouhabaz' simple criterion that

$$\chi_C \cdot H^1_0(\Omega) \subset H^1_0(\Omega) \text{ implies } |C| = 0 \text{ or } |\Omega \setminus C| = 0$$

for each Borel set $C \subset \Omega$ (see [28], Section 4.2 or [3]). For another proof of irreducibility we refer to [18], and for consequences we refer to [4].

4. THE OPERATOR $m\Delta_0$ ON $C_0(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Let $m: \Omega \rightarrow (0, \infty)$ be a measurable function such that $m \in L^\infty_{\text{loc}}(\Omega)$ and $\frac{1}{m} \in L^p_{\text{loc}}$, where $p > \frac{N}{2}$. We want to define a maximal realization of $m\Delta$ in $C_0(\Omega)$. If $u \in C_0(\Omega)$, then $\Delta u \in \mathcal{D}(\Omega)'$, but $m\Delta u$ may not be defined as a distribution. Thus the following definition is natural.

Definition 4.1. We define the operator $m\Delta_0$ on $C_0(\Omega)$ by

$$\begin{aligned} \mathcal{D}(m\Delta_0) &:= \left\{ u \in C_0(\Omega) : \exists f \in C_0(\Omega) \text{ s.t. } \Delta u = \frac{f}{m} \right\}, \\ (m\Delta_0)u &:= f. \end{aligned}$$

Since $\frac{f}{m} \in L^1_{\text{loc}} \subset \mathcal{D}(\Omega)'$, this definition makes sense. The notation $(m\Delta_0)$ is purely symbolic. But if $u \in C_0(\Omega) \cap C^2(\Omega)$ such that $m \cdot \Delta u \in C_0(\Omega)$, then $u \in \mathcal{D}(m\Delta_0)$ and $(m\Delta_0)u = m \cdot \Delta u$.

Proposition 4.2. *The operator $m\Delta_0$ is closed and dissipative. Moreover, if*

$$R(\lambda_0, m\Delta_\infty)C_0(\Omega) \subset C_0(\Omega)$$

for some $\lambda_0 > 0$, then $m\Delta_0$ generates a C_0 -semigroup of positive contractions on $C_0(\Omega)$. In that case

$$\begin{aligned} (0, \infty) &\subset \rho(m\Delta_0), \\ R(\lambda, m\Delta_\infty)C_0(\Omega) &\subset C_0(\Omega) \quad \text{for all } \lambda > 0 \quad \text{and} \\ R(\lambda, m\Delta_0) &= R(\lambda, m\Delta_\infty)|_{C_0(\Omega)}. \end{aligned}$$

Note that in general, $\mathcal{D}(\Omega) \not\subseteq \mathcal{D}(m\Delta_0)$, since we do not assume that m is continuous. Thus in Proposition 4.2 density of the domain (which is necessary for the generation property) needs a separate argument.

Since $m\Delta_0$ is dissipative, it follows in particular that no proper restriction of $m\Delta_0$ may generate a C_0 -semigroup on $C_0(\Omega)$.

We first prove dissipativity.

Lemma 4.3. *Let $\lambda > 0$, $u \in \mathcal{D}(m\Delta_0)$, and $f = \lambda u - (m\Delta_0)u$. Let $c > 0$ be such that*

$$f(x) \leq c \quad \text{for all } x \in \Omega.$$

Then $\lambda u(x) \leq c$ for all $x \in \Omega$.

Proof. By the definition of the operator we have

$$\lambda \frac{u}{m} - \Delta u = \frac{f}{m} \leq \frac{c}{m}.$$

Since by (3.3) $u \in H_{\text{loc}}^1(\Omega)$, this implies that for $0 \leq v \in \mathcal{D}(\Omega)$

$$(4.1) \quad \int_{\Omega} \frac{(\lambda u(x) - c)v(x)}{m(x)} dx + \int_{\Omega} \nabla u(x) \nabla v(x) dx \leq 0.$$

Since $u \in C_0(\Omega)$, $(\lambda u - c)^+$ has compact support. Let $\omega \Subset \Omega$ such that $\text{supp}(\lambda u - c)^+ \subset \omega$. Then $(\lambda u - c)^+ \in H_0^1(\omega)$ and $(\lambda u - c) \in H^1(\omega)$. Now (4.1) implies that

$$\int_{\omega} \frac{(\lambda u(x) - c)v(x)}{m(x)} dx + \frac{1}{\lambda} \int_{\omega} \nabla(\lambda u(x) - c) \nabla v(x) dx \leq 0$$

for all $0 \leq v \in H_0^1(\omega)$. Taking, in particular, $v := (\lambda u - c)^+$, we see that

$$\int_{\omega} \frac{(\lambda u(x) - c)^2}{m(x)} dx + \frac{1}{\lambda} \int_{\omega} |\nabla(\lambda u(x) - c)^+|^2 dx \leq 0.$$

This implies that $(\lambda u - c)^+ = 0$, i.e., $\lambda u \leq c$. □

Applying Lemma 4.3 to $\pm u$, we see that

$$\|\lambda u\|_{L^\infty(\Omega)} \leq \|\lambda u - (m\Delta_0)u\|_\infty$$

for all $u \in \mathcal{D}(m\Delta_0)$, i.e., $m\Delta_0$ is dissipative. But in fact, Lemma 4.3 shows that the operator $m\Delta_0$ is *dispersive*. We refer to ([5], [27], Chapter II) for this notion.

Proof of Proposition 4.2. The dissipativity has been proved above, and the closedness is easy to see. Now let $R(\lambda, m\Delta_\infty)C_0(\Omega) \subset C_0(\Omega)$ for some $\lambda > 0$. We show

that $\lambda \in \rho(m\Delta_0)$ and $R(\lambda, m\Delta_0) = R(\lambda, m\Delta_\infty)|_{C_0(\Omega)}$. Let $f \in C_0(\Omega)$ and consider $u = R(\lambda, m\Delta_\infty)f \in C_0(\Omega)$. Then (by Proposition 3.4)

$$\lambda \frac{u}{m} - \Delta u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

It follows that $u \in \mathcal{D}(m\Delta_0)$ and $(\lambda u - (m\Delta_0)u) = f$. We have shown that $\lambda - (m\Delta_0)$ is surjective. Since the injectivity of $(\lambda - m\Delta_0)$ follows from the dissipativity of $m\Delta_0$, the closed graph theorem now implies that $\lambda \in \rho(m\Delta_0)$. The calculation above also shows that $R(\lambda, m\Delta_0)f = u = R(\lambda, m\Delta_\infty)f$.

By the resolvent identity (see [1], Proposition 3.II.2) for $0 \leq f \in C_0(\Omega)$ and $\lambda > \lambda_0$ we have

$$0 \leq R(\lambda, m\Delta_\infty)f \leq R(\lambda_0, m\Delta_\infty)f \in C_0(\Omega).$$

Since by Proposition 3.3 the function $R(\lambda, m\Delta_\infty)f$ is continuous, it follows from the domination property above that $R(\lambda, m\Delta_\infty)f \in C_0(\Omega)$. Thus $C_0(\Omega)$ is invariant for all $\lambda \geq \lambda_0$. Hence $[\lambda_0, \infty) \subset \rho(m\Delta_0)$.

Next we show that $\mathcal{D}(m\Delta_0)$ is dense in $C_0(\Omega)$. Since $m \in L^\infty_{\text{loc}}(\Omega)$, we have $\mathcal{D}(\Omega) \subset \mathcal{D}(m\Delta_\infty)$ by Proposition 3.4. Hence $C_0(\Omega) \subset \overline{\mathcal{D}(m\Delta_\infty)}$. Thus, for $f \in C_0(\Omega)$ one has

$$\lim_{\lambda \rightarrow \infty} \lambda R(\lambda, m\Delta_0)f = \lim_{\lambda \rightarrow \infty} \lambda R(\lambda, m\Delta_\infty)f = f.$$

Since $\lambda R(\lambda, m\Delta_0)f \in \mathcal{D}(m\Delta_0)$, density of the domain is proved. Now the Lumer-Phillips theorem implies that $m\Delta_0$ generates a contractive C_0 -semigroup. Since the resolvent of $m\Delta_0$ is positive, this semigroup is positive. It also follows that $(0, \infty) \subset \rho(m\Delta_0)$. □

We will now consider two cases which imply the invariance given in Proposition 4.2, namely that Ω is Dirichlet regular or that the diffusion coefficient $m(x)$ tends to 0 fast enough as x approaches the boundary. We start by discussing Dirichlet regularity.

5. REGULAR POINTS

Let $\Omega \subset \mathbb{R}^N$ be open, bounded and let $\frac{N}{2} < p \leq \infty$. Let $m: \Omega \rightarrow (0, \infty)$ be measurable such that $m \in L^\infty_{\text{loc}}(\Omega)$ and $\frac{1}{m} \in L^p_{\text{loc}}(\Omega)$.

Theorem 5.1. *If Ω is Dirichlet regular, then $m\Delta_0$ generates a positive contractive C_0 -semigroup on $C_0(\Omega)$.*

Thus in the case of a Dirichlet regular set, no condition on $m(x)$ as x approaches the boundary is needed. We merely impose a (very weak) regularity condition on m in the interior of Ω .

It will be useful to prove an individual version of Theorem 5.1 first. For this we have to recall the notion of regular points.

Consider the Dirichlet problem

$$(5.1) \quad \begin{cases} h \in C(\overline{\Omega}) \cap C^2(\Omega), \\ \Delta h = 0 \text{ in } \Omega, \\ h|_{\partial\Omega} = \varphi, \end{cases}$$

where $\varphi \in C(\partial\Omega)$ is given. Recall that Ω is called *Dirichlet regular* if for each $\varphi \in C(\partial\Omega)$ a (necessarily unique) solution of (5.1) exists. If Ω has Lipschitz boundary,

then Ω is Dirichlet regular. Much weaker geometric properties of the boundary suffice, though. In dimension $N = 1$ each bounded open subset Ω of \mathbb{R} is Dirichlet regular. If $N = 2$, then each simply connected bounded open set is Dirichlet regular. This is no longer true in \mathbb{R}^3 . The Lebesgue cusp gives an example of a simply connected domain with continuous boundary, which is not Dirichlet regular (see [6] for more information).

A function $u \in C(\overline{\Omega})$ is called a *subsolution* if

$$-\Delta u \leq 0 \text{ in } \mathcal{D}(\Omega)' \quad \text{and} \quad \limsup_{x \rightarrow z, x \in \Omega} u(x) \leq \varphi(z) \quad \text{for all } z \in \partial\Omega.$$

A function $u \in C(\overline{\Omega})$ is called a *supersolution* if

$$-\Delta u \geq 0 \text{ in } \mathcal{D}(\Omega)' \quad \text{and} \quad \liminf_{x \rightarrow z, x \in \Omega} u(x) \geq \varphi(z) \quad \text{for all } z \in \partial\Omega.$$

Theorem 5.2 (Perron). *Let $\varphi \in C(\partial\Omega)$. Then for all $x \in \Omega$*

$$h_\varphi(x) := \sup \{u(x) : u \text{ is a subsolution}\}$$

exists. Moreover,

$$h_\varphi(x) = \inf \{v(x) : v \text{ is a supersolution}\}.$$

The function h_φ is harmonic and

$$\inf_{\partial\Omega} \varphi \leq h_\varphi(x) \leq \sup_{\partial\Omega} \varphi$$

for all $x \in \Omega$. If (5.1) has a solution h , then $h_\varphi = h$.

The function h_φ is called the *Perron solution* of (5.1).

A point $z \in \partial\Omega$ is called *regular* if

$$\lim_{x \rightarrow z, x \in \Omega} h_\varphi(x) = \varphi(z)$$

for all $\varphi \in C(\partial\Omega)$. Thus Ω is Dirichlet regular if and only if each point $z \in \partial\Omega$ is regular. It is possible to characterize regular points by the existence of a barrier or by a capacity condition (Wiener's theorem). We refer to [20].

Now we can formulate the local version of Theorem 5.1, which we want to prove.

Theorem 5.3. *Let Ω be bounded and open. Let $z \in \partial\Omega$ be a regular point. Let $\lambda > 0$, $f \in C_0(\Omega)$, and $u = R(\lambda, m\Delta_\infty)f$. Then*

$$\lim_{x \rightarrow z, x \in \Omega} u(x) = 0.$$

Thus, if Ω is Dirichlet regular, then $C_0(\Omega)$ is invariant under $R(\lambda, m\Delta_\infty)$ and Theorem 5.1 follows from Proposition 4.2.

For the proof of Theorem 5.3 we use the following variational characterization of the Perron solution (see [7]).

Theorem 5.4. *Let $\Phi \in C(\overline{\Omega})$ be such that $\Delta\Phi \in H^{-1}(\Omega)$. Let $\varphi = \Phi|_{\partial\Omega}$. Let u be the unique solution of*

$$\begin{aligned} u &\in H_0^1(\Omega), \\ -\Delta u &= \Delta\Phi. \end{aligned}$$

Then $h_\varphi = \Phi + u$.

For our purposes the following consequence is important. Recall that by Proposition 3.3, for all $f \in L^p(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in } \mathcal{D}(\Omega)'.$$

In fact, $u = R(0, \Delta_p)f$, where Δ_p denotes the Dirichlet Laplacian on $L^p(\Omega)$. Moreover, one has $u \in C^b(\Omega)$.

Corollary 5.5. *Let $f \in L^p(\Omega)$, $u = R(0, \Delta_p)f$. Then*

$$\lim_{x \rightarrow z, x \in \Omega} u(x) = 0$$

for each regular point $z \in \partial\Omega$. Thus, if Ω is Dirichlet regular, then $u \in C_0(\Omega)$.

Proof. It follows from the Sobolev embedding theorem that $L^p(\Omega) \subset H^{-1}(\Omega)$. Let $f \in L^p(\Omega)$. Let $\Phi = E * f$, where E is the Newtonian potential. Then (by [13], II.3, Proposition 6) $\Phi \in C(\mathbb{R}^N)$, and in $\mathcal{D}(\Omega)'$ we have

$$\Delta \Phi = f \in L^p(\Omega) \subset H^{-1}(\Omega).$$

Let $u = R(0, \Delta_p)f$. Then it follows from Theorem 5.4 that $h_\varphi = \Phi + u$. Thus

$$\lim_{x \rightarrow z, x \in \Omega} h_\varphi(x) = \varphi(z) \quad \text{if } z \in \partial\Omega \text{ is regular.}$$

Consequently,¹ $\lim_{x \rightarrow z} u(x) = 0$. □

Remark. a) In [2] a more special case of Corollary 5.5 is proved with the help of H^1 -barriers (the proof of Theorem 3.8 in [2]).

b) Special cases of Theorem 5.4 were obtained previously by Hildebrandt [19] and Simader [31].

Proof of Theorem 5.3. (a) Let $\lambda > 0$, $0 \leq f \in C_c(\Omega)$, and $u = R(\lambda, m\Delta_\infty)f$. Then $u \in H_0^1(\Omega)$ and

$$\lambda \frac{u}{m} - \Delta u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

Moreover, $0 \leq u \in C^b(\Omega)$. Observe that $0 \leq \frac{f}{m} \in L^p(\Omega)$. Let $w = R(0, \Delta_p)\frac{f}{m}$. Then we know that $0 \leq w \in H_0^1(\Omega) \cap C^b(\Omega)$ and, by Corollary 5.5, $\lim_{x \rightarrow z} w(x) = 0$ for all regular points $z \in \partial\Omega$. By definition,

$$-\Delta w = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

Thus $-\Delta(u - w) \leq 0$ in $\mathcal{D}(\Omega)'$. Since $u - w \in H^1(\Omega)$ and $(u - w)^+ \in H_0^1(\Omega)$, it follows from the maximum principle that $u \leq w$. Hence $\lim_{x \rightarrow z} u(x) = 0$ for each regular point $z \in \partial\Omega$.

(b) Let $z \in \partial\Omega$ be a regular point. Then by (a)

$$\lim_{x \rightarrow z, x \in \Omega} (R(\lambda, m\Delta_\infty)f)(x) = 0$$

for each $0 \leq f \in C_c(\Omega)$, hence also for each $f \in C_c(\Omega)$. Since $C_c(\Omega)$ is dense in $C_0(\Omega)$, this remains true for all $f \in C_0(\Omega)$. □

Next we show a converse of Theorem 5.1. If the diffusion coefficient m is not weak enough at the boundary, then Dirichlet regularity is necessary for $m\Delta_0$ to generate a C_0 -semigroup. More precisely, the following holds. Recall that $\frac{N}{2} < p \leq \infty$.

¹We will sometimes use the notation $\lim_{x \rightarrow z} f(x) := \lim_{x \rightarrow z, x \in \Omega} f(x)$ for $f: \Omega \rightarrow \mathbb{R}$.

Theorem 5.6. *Assume that $\frac{1}{m} \in L^p(\Omega)$. Then $m\Delta_0$ generates a C_0 -semigroup if and only if Ω is Dirichlet regular.*

For the proof we need the following.

Proposition 5.7. *Let $u \in C_0(\Omega)$ be such that $-\Delta u = f \in L^p(\Omega)$ for some $p > \frac{N}{2}$. Then $u \in H_0^1(\Omega)$, hence $u = R(0, \Delta_p)f$.*

This follows from [6], Corollary 1.4, since $L^p(\Omega) \subset H^{-1}(\Omega)$.

Proof of Theorem 5.6. Assume that $m\Delta_0$ generates a C_0 -semigroup. Since $\frac{1}{m} \in L^p(\Omega)$, we know from Proposition 3.5 that $[0, \infty) \subset \rho(m\Delta_\infty)$ and $R(\lambda, m\Delta_\infty) \geq 0$ for all $\lambda \geq 0$.

We now claim $R(\lambda, m\Delta_\infty)C_0(\Omega) \subset C_0(\Omega)$ and $R(\lambda, m\Delta_0) = R(\lambda, m\Delta_\infty)|_{C_0(\Omega)}$ for any $\lambda > 0$. Let $f \in C_0(\Omega)$ and $u = R(\lambda, m\Delta_0)f$. Then

$$-\Delta u = \frac{f}{m} - \lambda \frac{u}{m} \in L^p(\Omega).$$

Since $u \in C_0(\Omega)$, it follows from Proposition 5.7 that $u \in H_0^1(\Omega)$. Since $\frac{1}{m} \in L^p(\Omega)$ we have $L^\infty(\Omega) \subset L^2(\Omega, \frac{dx}{m(x)})$. Thus by (3.4) we have $u \in \mathcal{D}(m\Delta_\infty)$ and $\lambda u - (m\Delta_\infty)u = f$. Hence $u = R(\lambda, m\Delta_\infty)f$. This proves the claim.

Since $0 \in \rho(m\Delta_\infty)$, the claim implies that

$$\limsup_{\lambda \rightarrow 0} \|R(\lambda, m\Delta_0)\|_{\mathcal{L}(C_0(\Omega))} < \infty,$$

hence $0 \in \rho(m\Delta_0)$ and $R(0, m\Delta_0) \geq 0$.

Let $0 \leq f \in C_0(\Omega)$ and $f(x) > 0$ for all $x \in \Omega$ and $u = R(0, m\Delta_0)f$. Then $u \in C_0(\Omega)$ and $-\Delta u = \frac{f}{m}$ in $\mathcal{D}(\Omega)'$. Hence $R(0, \Delta_p)\frac{f}{m} = u \in C_0(\Omega)$ by Proposition 5.7. We deduce that $R(0, \Delta_p)g \in C_0(\Omega)$ for all $g \in L^p(\Omega)$ such that $|g| \leq \frac{f}{m}$ for some $0 \leq f \in C_0(\Omega)$. The space of all such functions g is dense in $L^p(\Omega)$. Thus $R(0, \Delta_p)L^p(\Omega) \subset C_0(\Omega)$. Now it follows from [2], Theorem 2.4, that Ω is Dirichlet regular. \square

6. POINTS OF WEAK DIFFUSION

Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $m: \Omega \rightarrow (0, \infty)$ be a bounded measurable function such that $\frac{1}{m} \in L_{loc}^p(\Omega)$ for some $\frac{N}{2} < p \leq \infty$. Instead of regularity we may assume that m is small in a neighbourhood of a boundary point. We say that $z \in \partial\Omega$ is a *point of weak diffusion* (for the operator $m\Delta$) if there exist $r > 0$ and $c > 0$ such that

$$(6.1) \quad m(x) \leq c \cdot \text{dist}(x, \partial\Omega)^2$$

for all $x \in \Omega \cap B(z, r)$. If $z \in \partial\Omega$ is a point of weak diffusion, then we show that

$$(6.2) \quad \lim_{x \rightarrow z, x \in \Omega} (R(\lambda, m\Delta_\infty)f)(x) = 0$$

for all $f \in C_0(\Omega)$. We will also show that condition (6.1) is optimal in the sense that

$$m(x) \leq c \cdot \text{dist}(x, \partial\Omega)^\alpha$$

for some $0 < \alpha < 2$ does not suffice to enforce (6.2).

We need the notion of a regularized distance function.

Lemma 6.1. *There exist a function $\sigma: \Omega \rightarrow (0, +\infty)$, which is of class $C^\infty(\Omega)$, and a constant $c_\sigma > 0$ such that*

$$\begin{aligned} c_\sigma^{-1}d(x) &\leq \sigma(x) \leq c_\sigma d(x), \\ |\nabla\sigma|^2 &\leq c_\sigma, \\ |\sigma\Delta\sigma| &\leq c_\sigma \end{aligned}$$

for all $x \in \Omega$, where $d(x) := \inf \{ \|x - y\|, y \in \mathbb{R}^d \setminus \Omega \}$.

See [32], Chapter 6, for a proof based on the Whitney decomposition of Ω .

Since $\sigma \in C_0(\Omega)$, it follows in particular that $\sigma \in H_0^1(\Omega)$. First we consider the case $m(x) := \sigma(x)^2$.

Proposition 6.2. *The operator $\sigma^2\Delta_0$ generates a strongly continuous semigroup of positive contractions on $C_0(\Omega)$.*

Proof. Let $\lambda \geq c_\sigma + 1$, where c_σ is a constant from Lemma 6.1. Set $u = R(\lambda, \sigma^2\Delta_\infty)\sigma$. Since $\sigma \in L^2(\Omega, \frac{dx}{\sigma(x)^2})$, it follows from (3.1) that $0 \leq u \in H_0^1(\Omega) \cap L^2(\Omega, \frac{dx}{\sigma(x)^2})$ and

$$\lambda \frac{u}{\sigma^2} - \Delta u = \frac{\sigma}{\sigma^2} \quad \text{in } \mathcal{D}(\Omega)'.$$

Since $\sigma\Delta\sigma \leq c_\sigma$, it follows that $\sigma \leq \lambda\sigma - c_\sigma\sigma \leq \lambda\sigma - \sigma^2\Delta\sigma$. Thus

$$\lambda \frac{u}{\sigma^2} - \Delta u = \frac{1}{\sigma^2}\sigma \leq \lambda \frac{\sigma}{\sigma^2} - \Delta\sigma \quad \text{in } \mathcal{D}(\Omega)'.$$

Hence

$$\lambda \frac{(u - \sigma)}{\sigma^2} - \Delta(u - \sigma) \leq 0 \quad \text{in } \mathcal{D}(\Omega)'.$$

Since $u - \sigma \in H^1(\Omega)$ and $(u - \sigma)^+ \leq u \in H_0^1(\Omega)$, it follows that $(u - \sigma)^+ \in H_0^1(\Omega)$. Now the maximum principle (see Section 2) implies that $(u - \sigma)^+ \leq 0$, i.e., $u \leq \sigma$.

We have shown that

$$(6.3) \quad R(\lambda, \sigma^2\Delta_\infty)\sigma \leq \sigma \quad (\lambda \geq \lambda_0 := 1 + c_\sigma).$$

Thus, for $f \in C_0(\Omega)$ such that $|f| \leq c\sigma$, one has

$$|R(\lambda, \sigma^2\Delta_\infty)f| \leq cR(\lambda, \sigma^2\Delta_\infty)\sigma \leq c\sigma.$$

Consequently, $R(\lambda, \sigma^2\Delta_\infty)f \in C_0(\Omega)$ for $\lambda \geq \lambda_0$. Since functions satisfying $|f| \leq c\sigma$ for some $c \geq 0$ are dense in $C_0(\Omega)$, we deduce that $R(\lambda, \sigma^2\Delta_\infty)C_0(\Omega) \subset C_0(\Omega)$ for $\lambda \geq \lambda_0$. Now the claim follows from Proposition 4.2. \square

We comment that the result of Proposition 6.2 may be alternatively deduced from [12], Theorem 5.4. However, our argument given here is quite different from [12].

We need a local extension of the resolvents of $\sigma^2\Delta$. Recall that $\frac{N}{2} < p \leq \infty$.

Lemma 6.3. *Let $\omega \Subset \Omega$, $\lambda > 0$. There exists an operator*

$$Q(\lambda, \omega) \in \mathcal{L}(L^p(\omega), C_0(\Omega))$$

such that

$$Q(\lambda, \omega)f = R(\lambda, \sigma^2\Delta_0)f \quad \text{for all } f \in L^p(\omega) \cap C_0(\Omega).$$

For $f \in L^p(\omega)$ the function $u = Q(\lambda, \omega)f$ is the unique solution of

$$(6.4) \quad \begin{aligned} u &\in C_0(\Omega), \\ \lambda \frac{u}{\sigma^2} - \Delta u &= \frac{f}{\sigma^2} \text{ in } \mathcal{D}(\Omega)'. \end{aligned}$$

Moreover, $u \in H_0^1(\Omega)$.

Here we consider $L^p(\omega)$ as a subspace of $L^p(\Omega)$ extending functions by 0 outside ω . Similarly, we consider $C_c(\omega) \subset C_0(\omega) \subset C_0(\Omega)$.

Proof. (a) Let $0 \leq f \in C_c(\omega)$. There exists $\delta > 0$ such that $\sigma^2 \geq \delta$ on ω . Let $u = R(\lambda, \sigma^2 \Delta_0)f = R(\lambda, \sigma^2 \Delta_2)f$. Then $0 \leq u \in H_0^1(\Omega)$ and

$$\lambda \frac{u}{\sigma^2} - \Delta u = \frac{f}{\sigma^2} \leq \frac{1}{\delta} f.$$

Let $w := \frac{1}{\delta} R(0, \Delta_p)f$, where Δ_p denotes the Dirichlet Laplacian on $L^p(\Omega)$. Then $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and

$$-\Delta w = \frac{1}{\delta} f \quad \text{in } \mathcal{D}(\Omega)'.$$

Moreover, $\|w\|_{L^\infty(\Omega)} \leq c_1 \|f\|_{L^p(\omega)}$, where $c_1 = \frac{1}{\delta} \|R(0, \Delta_p)\|_{\mathcal{L}(L^p(\Omega), L^\infty(\Omega))}$ (see Proposition 3.3 (b)). We show that $u \leq w$. In fact, we have

$$\begin{aligned} -\Delta u &\leq \lambda \frac{u}{\sigma^2} - \Delta u \leq \frac{1}{\delta} f \quad \text{and} \\ -\Delta w &= \frac{1}{\delta} f, \end{aligned}$$

hence $-\Delta(u - w) \leq 0$ in $\mathcal{D}(\Omega)'$. Consequently, by the maximum principle (see Section 2), $u \leq w$. Thus

$$\|u\|_{L^\infty(\Omega)} \leq \|w\|_{L^\infty(\Omega)} \leq c_1 \|f\|_{L^p(\omega)}.$$

We have shown that

$$(6.5) \quad \|R(\lambda, \sigma^2 \Delta_0)f\|_{L^\infty(\Omega)} \leq c_1 \|f\|_{L^p(\omega)}$$

for $0 \leq f \in C_c(\omega)$. Since for arbitrary $f \in C_c(\omega)$,

$$|R(\lambda, \sigma^2 \Delta_0)f| \leq R(\lambda, \sigma^2 \Delta_0)|f|,$$

the estimate (6.5) remains true for all $f \in C_c(\omega)$. By the density of $C_c(\omega)$ in $L^p(\omega)$, the first claim is proved.

(b) In order to prove the second claim, let $f \in L^p(\omega)$, $u = Q(\lambda, \omega)f$. Let $f_k \in C_c(\omega)$ be such that $f_k \rightarrow f$ in $L^p(\omega)$. Then $u_k := Q(\lambda, \omega)f_k \rightarrow u$ in $C_0(\Omega)$. We have $u_k \in H_0^1(\Omega) \cap C_0(\Omega)$ and

$$(6.6) \quad \lambda \frac{u_k}{\sigma^2} - \Delta u_k = \frac{f_k}{\sigma^2} \quad \text{in } \mathcal{D}(\Omega)'.$$

Passing to the limit as $k \rightarrow \infty$ shows that (6.4) holds.

It remains to show that $u \in H_0^1(\Omega)$. Multiplying (6.6) by u_k and integrating yields

$$\begin{aligned} \lambda \int_{\Omega} \frac{u_k(x)^2}{\sigma(x)^2} dx + \int_{\Omega} |\nabla u_k(x)|^2 dx &= \int_{\Omega} \frac{f_k(x)u_k(x)}{\sigma(x)^2} dx \\ &\leq \|u_k\|_{L^\infty(\Omega)} \frac{1}{\delta^2} \cdot |\Omega|^{\frac{1}{p'}} \|f_k\|_{L^p(\Omega)}. \end{aligned}$$

This shows that $(u_k)_{k \in \mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Thus, passing to a subsequence we may assume that $u_k \rightharpoonup w \in H_0^1(\Omega)$. Since $u_k \rightarrow u$ in $C_0(\Omega)$, it follows that $u = w \in H_0^1(\Omega)$. \square

Now we consider a more general function m satisfying the hypothesis formulated in the beginning of this section. We prove regularity of $m\Delta_\infty$ at points of weak diffusion.

Theorem 6.4. *Let $z \in \partial\Omega$ be a point of weak diffusion (in the sense of (6.1)). Let $f \in C_0(\Omega)$, $\lambda > 0$, and $u = R(\lambda, m\Delta_\infty)f$. Then*

$$\lim_{x \rightarrow z, x \in \Omega} u(x) = 0.$$

Proof. Let $r_1 > 0$ be a large radius such that $\bar{\Omega} + \bar{B}(0, r) \subset B(0, r_1)$. Consider the open set

$$\tilde{\Omega} := (\Omega \cap B(z, r)) \cup (B(0, r_1) \setminus \bar{B}(z, \frac{r}{2})).$$

Then $\Omega \subset \tilde{\Omega}$ and $\bar{B}(z, \frac{r}{2}) \cap \partial\Omega \subset \partial\tilde{\Omega}$. In particular, $z \in \partial\tilde{\Omega}$. Consider a regularized distance $\tilde{\sigma}$ with respect to $\tilde{\Omega}$. Then there exists a constant $c > 0$ such that

$$(6.7) \quad m(x) \leq c\tilde{\sigma}(x)^2 \quad \text{for all } x \in \Omega.$$

In fact, for $x \in B(z, r) \cap \Omega$ this follows from (6.1). But for $x \in \Omega \setminus B(z, \frac{3}{4}r)$, one has $\text{dist}(x, \partial\tilde{\Omega}) \geq \frac{r}{4}$. Since m is bounded, it follows that

$$m(x) \leq c_2(\frac{r}{4})^2 \leq c_2 \text{dist}(x, \partial\tilde{\Omega})^2$$

for all $x \in \Omega \setminus B(z, \frac{3}{4}r)$. This shows that (6.7) is valid for a suitable constant $c > 0$.

Now let $\lambda > 0$. Let $0 \leq f \in C_c(\Omega)$ and $u = R(\lambda, m\Delta_\infty)f$. Then $u \in C^b(\Omega) \cap H_0^1(\Omega)$ and

$$\lambda \frac{u}{m} - \Delta u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

Let $\rho := \frac{m}{\tilde{\sigma}^2}$. Then $0 < \rho \leq c$ on Ω and

$$\frac{1}{c} \leq \frac{1}{\rho} = \frac{\tilde{\sigma}^2}{m} \in L_{\text{loc}}^p(\Omega).$$

Hence

$$\frac{\lambda}{c} \frac{u}{\tilde{\sigma}^2} \leq \frac{\lambda}{\rho} \frac{u}{\tilde{\sigma}^2} = \frac{\lambda u}{m}.$$

Thus

$$\frac{\lambda}{c} \frac{u}{\tilde{\sigma}^2} - \Delta u \leq \frac{f}{m} = \frac{1}{\tilde{\sigma}^2} \frac{f}{\rho}.$$

Let $\omega \Subset \Omega$ be such that $\text{supp } f \subset \omega$. Consider $Q(\lambda, \omega) \in \mathcal{L}(L^p(\omega), C_0(\tilde{\Omega}))$ of Lemma 6.3 defined with respect to $\tilde{\sigma}$. Let $w = Q(\frac{\lambda}{c}, \omega) \frac{f}{\rho}$. Note that w is well defined, since $\frac{f}{\rho} \in L^p(\omega)$. Then $0 \leq w \in C_0(\tilde{\Omega}) \cap H_0^1(\tilde{\Omega})$ and, by (6.4),

$$\frac{\lambda}{c} \frac{w}{\tilde{\sigma}^2} - \Delta w = \frac{1}{\tilde{\sigma}^2} \frac{f}{\rho} \quad \text{in } \mathcal{D}(\tilde{\Omega})'$$

and hence also in $\mathcal{D}(\Omega)'$. Thus

$$\frac{\lambda}{c} \frac{(u-w)}{\tilde{\sigma}^2} - \Delta(u-w) \leq 0 \quad \text{in } \mathcal{D}(\Omega)'.$$

Recall that $u \in H_0^1(\Omega) \cap C^b(\Omega)$. Thus $(u - w) \in H^1(\Omega)$. Hence

$$(6.8) \quad \frac{\lambda}{c} \int_{\Omega} \frac{(u(x) - w(x))}{\tilde{\sigma}(x)^2} v(x) \, dx + \int_{\Omega} \nabla(u(x) - w(x)) \nabla v(x) \, dx \leq 0$$

for all $0 \leq v \in \mathcal{D}(\Omega)$. Since $(u - w)^+ \in H^1(\Omega)$ and $(u - w)^+ \leq u \in H_0^1(\Omega)$, it follows that $(u - w)^+ \in H_0^1(\Omega)$.

Since $u = R(\lambda, m\Delta_\infty)f = R(\lambda, m\Delta_2)f$, it follows that

$$u \in L^2\left(\Omega, \frac{dx}{m(x)}\right) \subset L^2\left(\Omega, \frac{dx}{\tilde{\sigma}(x)^2}\right)$$

because of (6.7). It follows (since also $w \in L^2(\Omega, \frac{dx}{\tilde{\sigma}(x)^2})$) that

$$v_1 := (u - w)^+ \in V := L^2\left(\Omega, \frac{dx}{\tilde{\sigma}(x)^2}\right) \cap H_0^1(\Omega).$$

Since $\mathcal{D}(\Omega)_+$ is dense in V_+ by Proposition 3.2, (6.8) remains true for $v := v_1$. This means that

$$\frac{\lambda}{c} \int_{\Omega} \frac{(u(x) - w(x))^+{}^2}{\tilde{\sigma}(x)^2} \, dx + \int_{\Omega} |\nabla(u(x) - w(x))^+|^2 \, dx \leq 0.$$

This implies that $(u - w)^+ = 0$. Hence $0 \leq u \leq w$.

Since

$$\lim_{x \rightarrow z, x \in \tilde{\Omega}} w(x) = 0,$$

it follows that

$$\lim_{x \rightarrow z, x \in \Omega} u(x) = 0.$$

We have proved the theorem for the case when $0 \leq f \in C_c(\Omega)$. Hence it is also true for arbitrary $f \in C_c(\Omega)$. Since $R(\lambda, m\Delta_\infty) \in \mathcal{L}(L^\infty(\Omega))$, and $C_c(\Omega)$ is dense in $C_0(\Omega)$, it follows that

$$\lim_{x \rightarrow z, x \in \Omega} (R(\lambda, m\Delta_\infty)f)(x) = 0$$

for all $f \in C_0(\Omega)$. □

Corollary 6.5. *Assume that each $z \in \partial\Omega$ is a point of weak diffusion (in the sense of (6.1)). Then $m\Delta_0$ generates a positive, contractive C_0 -semigroup on $C_0(\Omega)$.*

7. CONCLUSION

We may now formulate the following general generation theorem. Let $\Omega \subset \mathbb{R}^N$ be bounded, open and $\frac{N}{2} < p \leq \infty$. Let $m: \Omega \rightarrow (0, \infty)$ be bounded and such that $\frac{1}{m} \in L^p_{loc}(\Omega)$.

Theorem 7.1. *Assume that for each point $z \in \partial\Omega$ one of the following conditions is satisfied:*

- (a) z is a regular point or
- (b) z is a point of weak diffusion (in the sense of (6.1)).

Then $m\Delta_0$ generates a positive, contractive C_0 -semigroup on $C_0(\Omega)$.

Proof. Theorem 5.3 and Theorem 6.4 show that $C_0(\Omega)$ is invariant. Thus the claim follows from Proposition 4.2. □

Finally, we show that the condition (6.1) of being a point of weak diffusion is optimal.

Let $N = 2$ and $\Omega = \{x \in \mathbb{R}^2 : 0 < |x| < 2\}$. Then $\partial\Omega = \mathbb{T} \cup \{0\}$, where $\mathbb{T} = \{x \in \mathbb{R}^2 : |x| = 2\}$. The points in \mathbb{T} are regular, but 0 is not regular.

Consider the function d given by $d(x) = |x|$, $x \in \Omega$. Thus $d(x) = \text{dist}(x, \partial\Omega)$ for $0 < |x| < \frac{1}{2}$. Then $\frac{1}{d} \in L^q(\Omega)$ if and only if $q < 2$. Now let $0 < \beta < 2$. Then $\frac{1}{d^\beta} \in L^p(\Omega)$ for some $p > 1 = \frac{N}{2}$. Since Ω is not Dirichlet regular, it follows from Theorem 5.6 that $d^\beta \Delta_0$ is not a generator.

On the other hand, if $\beta \geq 2$, then for $m = d^\beta$, the point 0 is of weak diffusion. Since the other boundary points are regular, it follows from Theorem 7.1 that $d^\beta \Delta_0$ generates a C_0 -semigroup on $C_0(\Omega)$.

An interesting open set in \mathbb{R}^3 with continuous boundary and exactly one singular point is the Lebesgue cusp (see e.g. [7] for a detailed investigation).

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