

SCALAR CURVATURE ESTIMATES BY PARALLEL ALTERNATING TORSION

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ABSTRACT. We generalize Llarull’s scalar curvature comparison to Riemannian manifolds admitting metric connections with parallel and alternating torsion and having a nonnegative curvature operator on $\Lambda^2 TM$. As a by-product, we show that the Euler number and signature of such manifolds are determined by their global holonomy representation. Our result holds in particular for all quotients of compact Lie groups of equal rank, equipped with a normal homogeneous metric.

We also correct a mistake in the treatment of odd-dimensional spaces in our earlier papers.

There is a well-known relation between the existence of metrics of positive scalar curvature on a closed manifold M and the topology of M . If there exist metrics with positive scalar curvature κ , one would like to measure how large κ can become. This could be done using the Yamabe number or σ -invariant $\sigma(M)$, which is defined by taking the infimum of the integral of κ for all metrics of volume one in a conformal class, and then the supremum over all conformal classes. As announced by Ammann, Dahl and Humbert in [Am] and explained in [ADH], there exists a constant $c_m > 0$ for $m = \dim M$ such that $\min(\sigma(M), c_m)$ is a spin bordism invariant over $B\pi_1(M)$. In this note, however, we will consider the pointwise scalar curvature instead.

Let g and \bar{g} be two Riemannian metrics on a closed manifold M , and let κ and $\bar{\kappa}$ be their scalar curvature. Using the K -area inequalities, Gromov showed in [Gr] that there is a finite upper bound for $\min \bar{\kappa}$ if $\bar{g} \geq g$ on $\Lambda^2 TM$. A first example of a sharp upper bound was given by Llarull in [Ll2]. If g denotes the round metric on S^n and $\bar{g} \geq g$ on $\Lambda^2 TM$, then $\bar{\kappa}(p) \leq \kappa(p)$ for some $p \in M$. In fact, $\bar{\kappa} \geq \kappa$ on M implies $\bar{\kappa} = \kappa$ and $\bar{g} = g$. In other words, the round metric on the sphere is strongly area-extremal in the following sense.

Definition. A metric g on M is called *area-extremal* if for all metrics \bar{g} on M with $\bar{g} \geq g$ on $\Lambda^2 TM$, the inequality $\bar{\kappa} \geq \kappa$ everywhere on M implies that $\bar{\kappa} = \kappa$. We call g *strongly area-extremal* if $\bar{g} \geq g$ on $\Lambda^2 TM$ and $\bar{\kappa} \geq \kappa$ on M also imply that $\bar{g} = g$.

In [Gr], Gromov asked which manifolds possess area-extremal metrics and what such metrics may look like. He conjectured that Riemannian symmetric spaces

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should have area-extremal metrics. He also proposed to investigate not only variations of the metric on M itself, but to also consider area-nonincreasing spin maps of nonvanishing \hat{A} -degree from other Riemannian manifolds to M ; see Section 8 for an explanation of these terms.

Definition. A metric g on M is called *area-extremal in the sense of Gromov* if for all smooth spin maps $f: (N, \bar{g}) \rightarrow (M, g)$ of nonzero \hat{A} -degree with $\bar{g} \geq f^*g$ on $\Lambda^2 TN$, the inequality $\bar{\kappa} \geq \kappa \circ f$ everywhere on N implies that $\bar{\kappa} = \kappa \circ f$. We call g *strongly area-extremal in the sense of Gromov* if $\bar{g} \geq f^*g$ on $\Lambda^2 TN$ and $\bar{\kappa} \geq \kappa \circ f$ on N also imply that f is a Riemannian submersion.

One might think of stronger versions of area-extremality. However, even if M is a point, one has to assume that f is spin and has nonzero degree in one sense or another. Llarull proved in [L11] that spheres are also area-extremal in the sense of Gromov. Since then, several other manifolds have been shown to be area-extremal, some of them even in the sense of Gromov; see [GS2] for an overview of related results.

So far, the only method to prove area-extremality combines the nonvanishing of an analytic index with a careful investigation of the curvature term in the Bochner-Lichnerowicz-Weitzenböck formula for a twisted Dirac operator \bar{D} with respect to the metric \bar{g} . The relevant indices will be discussed in Section 2. As observed in [GS1], [GS2], the curvature conditions needed for the scalar curvature estimates are satisfied in the following cases.

- (1) (M, g) is a Fano manifold; i.e., it is Kähler and has nonnegative Ricci curvature [GS1].
- (2) The curvature operator on $\Lambda^2 TM$ is nonnegative; see [GS2]. Then M is homeomorphic to a Riemannian symmetric space G/H .

In the following situation, one expects the estimate to hold as well.

- (3) M is quaternionic Kähler with positive scalar curvature.¹

However, it is conjectured that quaternionic Kähler manifolds with positive scalar curvature are symmetric with $\text{rk } G = \text{rk } H$. If this is true, then (3) is just a special case of (2).

The estimate of [GS2] has recently been strengthened as follows.

Theorem 1 (Listing, [Li]). *Let (M, g) be a compact, connected, oriented Riemannian manifold with nonnegative curvature operator on $\Lambda^2(TM)$. Assume that there exists a parallel Dirac subbundle W of $\Lambda^\bullet T^*M$ such that the Hodge-Dirac operator has nonvanishing index on W . Let \bar{g} be another Riemannian metric on M with scalar curvature $\bar{\kappa}$, and let f be a positive function on M such that $f\bar{g} \geq g$ on $\Lambda^2 TM$. Then $\bar{\kappa} \geq f\kappa$ implies $\bar{\kappa} = f\kappa$. If, moreover, the Ricci curvature of g satisfies $\rho > 0$ and $2\rho - \kappa < 0$, then $\bar{\kappa} \geq f\kappa$ implies $f\bar{g} = g$ and f is constant.*

In the present paper, we replace the Levi-Civita connection of the metric \bar{g} in the definition of the twisted Dirac operator \bar{D} on M (or N if we are considering maps $f: N \rightarrow M$) by a different connection $\tilde{\nabla}$. This is still a generalised Dirac operator in the sense that its square is the sum of a generalised Laplacian and a zero-order operator. The generalised Laplacian here is induced by a connection ∇' on TM . We assume that ∇' has parallel and alternating torsion tensor T , as

¹Note added in proof. Case (3) has recently been established by Weinert [W] in his diploma thesis.

explained in Section 1. Unless $T = 0$, this implies that the holonomy group of ∇' is a proper subgroup of $SO(TM)$.

By Lemma 1.2 (4), the curvature R' of ∇' acts as a symmetric endomorphism on $\Lambda^2 TM$ because T is parallel. Hence, there exists a “curvature operator” as in the Riemannian case. For our estimates, this operator must be nonnegative.

Let us also define the kernel of the torsion tensor T as

$$\ker T_p := \{ V \in T_p M \mid T(V, W) = 0 \text{ for all } W \in T_p M \}$$

for all $p \in M$. Note that $\ker T_p$ defines a ∇' -parallel subbundle of TM . In particular, the universal covering of the triple (M, g, ∇') always splits as a product $(M_0, g_0, \nabla'_0) \times (M_1, g_1, \nabla'_1)$ such that ∇'_0 is the Levi-Civita connection and ∇'_1 has parallel and alternating torsion T_1 with $\ker T_1 = 0$. This splitting will not be used in this paper.

Theorem 2. *Let (M, g) be a closed connected Riemannian spin manifold with scalar curvature κ . Assume that there exists a metric connection ∇' on TM with parallel and alternating torsion that induces a nonnegative curvature operator R' on $\Lambda^2 TM$. Assume also that there exists a ∇' -parallel Dirac subbundle $W \subset \Lambda^\bullet T^*M$ such that the restriction of the Hodge-Dirac operator to W has nonzero index. Then (M, g) is area-extremal.*

Moreover, if either

- (1) the Ricci tensor ρ of g is positive definite on $\ker T$ and $T \neq 0$, or
- (2) we have $\rho > 0$ and $2\rho - \kappa g < 0$,

then (M, g) is strongly area-extremal.

The index above will be explained in Section 2, and the theorem will be proved in Section 7.

As mentioned above, our proof relies on a generalisation of the Bochner-Lichnerowicz-Weitzenböck formula for a certain modified twisted Dirac operator \tilde{D} . For connections with parallel and alternating torsion, this formula is stated in Corollary 5.1. Let us state a simple consequence of this formula, which we prove in Section 5.

Lemma 3. *Let (M, g) be a connected closed Riemannian manifold admitting a connection ∇' with parallel and alternating torsion and nonnegative curvature operator on $\Lambda^2 TM$. Then \tilde{D} preserves the subspace of $\Omega^\bullet(M)$ consisting of ∇' -parallel forms, and all \tilde{D} -harmonic forms are ∇' -parallel. In particular, the Euler characteristic, the signature and the Kervaire semicharacteristic of M —whenever they are defined—can be read off from the global holonomy representation of ∇' .*

Under slightly stronger assumptions than in Theorem 2, we can even prove that (M, g) is area-extremal in the sense of Gromov.

Theorem 4. *Let (M, g) be as in Theorem 2. If M is oriented and has positive Euler number, then (M, g) is area-extremal in the sense of Gromov.*

Moreover, if either

- (1) the Ricci tensor ρ of g is positive definite on $\ker T$ and $T \neq 0$, or
- (2) we have $\rho > 0$ and $2\rho - \kappa g < 0$,

then (M, g) is strongly area-extremal in the sense of Gromov.

This will be proved in Section 8. In Section 6, we apply Theorems 2 and 4 to quotients of compact Lie groups.

Corollary 5. *Let $M = G/H$ be a quotient of compact connected Lie groups, and let g be a normal homogeneous metric. If $\text{rk } G = \text{rk } H$, then (M, g) is area-extremal in the sense of Gromov, and strongly area-extremal in the sense of Gromov if moreover $\dim M > 2$.*

*If $\text{rk } G - \text{rk } H = 1$, $\dim M = 4k + 1$ and $W \subset \Lambda^\bullet T^*M$ is a homogeneous Dirac subbundle such that $D|_W$ has nonvanishing index, then (M, g) is area-extremal. If moreover (M, g) does not contain a Euclidean local de Rham factor and $\dim M > 2$, then (M, g) is strongly area-extremal.*

Note that the proofs of the main results in [GS2] and [G3] for symmetric spaces with $\text{rk } G - \text{rk } H = 1$ only work with the same index-theoretic condition as in the corollary above; see Section 2 for an explanation.

Let us now relate the various assumptions on ∇' in Theorem 2 to more geometric properties of (M, g) .

Remark 6. If $T = 0$, then ∇' is the Levi-Civita connection, and we are in the situation of [GS2]. Hence we assume that $T \neq 0$ is ∇' -parallel and alternating. In particular, it is fixed by the holonomy group H of ∇' . Let π denote the holonomy representation of H .

If we assume that π is irreducible, then by a result of Cleyton and Swann [CS], there are three possibilities.

- (1) (M, g) is locally isometric to a nonsymmetric, isotropy irreducible homogeneous space G/H ;
- (2) (M, g) is locally isometric to one of the irreducible symmetric spaces $H \times H/H$ or $H^{\mathbb{C}}/H$;
- (3) (M, g) has weak holonomy $SU(3)$ or G_2 .

We are not aware of a general classification result in the case where H does not act irreducibly. If we assume that $\dim M = 2n$ and $H \subset U(n)$, then $T \neq 0$ is alternating and ∇' -parallel iff M is nearly Kähler by a result of Kirichenko [K]; these manifolds have been further classified by Nagy [N] and others.

Remark 7. If ∇' has parallel and alternating torsion, we have a well-defined symmetric “curvature operator” $R' \in \text{End}(\Lambda^2 TM)$. We assume that R' is nonnegative. In the case where $T = 0$ and M is locally irreducible, one knows that M is either homeomorphic to a sphere, biholomorphic to a complex projective space, or isometric to a Riemannian symmetric space of compact type by results of Gallot and Meyer [GaM], Cao and Chow [CC] and Tachibana [T]. If $T = 0$ and the curvature operator is strictly positive, then M is actually diffeomorphic to a sphere by a recent result of Böhm and Wilking [BW].

We will see in Section 6 that on all normal homogeneous spaces, the reductive connection has parallel and alternating torsion and its curvature operator is nonnegative. Most normal homogeneous metrics do not have nonnegative Riemannian curvature operator, so we really obtain some new examples of area-extremal metrics. But by Lemma 1.2 (5) below, nonnegativity of R' still implies that the Riemannian sectional curvatures of (M, g) are nonnegative. In particular, we are still far away from the goal stated in [GS2] to generalise Llarull’s theorem to Ricci positive manifolds.

Remark 8. Let us now regard our sufficient conditions for strong area-extremality in Theorems 2 and 4. Obviously, condition (1) follows from (2) if $T \neq 0$. We will

see that condition (1) also implies condition (2), which was used in [GS2] in the case $T = 0$.

Assume that $U \notin \ker T$. Using Lemma 1.2 (5), we see that

$$\rho(U, U) = \sum_{i=1}^m \langle R_{U, e_i} e_i, U \rangle = \sum_{i=1}^m (\langle R'_{U, e_i} e_i, U \rangle + \|T(U, e_i)\|^2) > 0$$

because $\langle R'_{U, e_i} e_i, U \rangle \geq 0$ and $\|T(U, e_i)\|^2 > 0$ for at least one i . So (M, g) is indeed Ricci positive.

Let U now be an arbitrary unit vector and extend $U = e_1$ to a g -orthonormal frame. Then

$$(\kappa g - 2\rho)(U, U) = \sum_{i,j=2}^m R_{ijji} = \sum_{i,j=2}^m \left(R'_{ijji} + \frac{1}{4} \sum_{k=1}^m \tau_{ijk}^2 \right) > 0$$

because $R'_{ijji} \geq 0$ by assumption, and at least one τ_{ijk} does not vanish. Thus $2\rho - \kappa$ is indeed negative definite.

1. METRIC CONNECTIONS WITH PARALLEL AND ALTERNATING TORSION

We collect some elementary properties of connections with parallel alternating torsion. Let $\nabla = \nabla^{TM}$ denote the Levi-Civita connection on M .

Definition 1.1. Let (M, g) be a Riemannian manifold, and let ∇' be a metric connection on TM with torsion tensor T . If $\langle T(\cdot, \cdot), \cdot \rangle$ is alternating, we say that ∇' has *alternating torsion*. If $\nabla' T = 0$, we say that ∇' has *parallel torsion*.

We will omit the superscript TM whenever no ambiguity can arise. Let X, Y, Z, W be smooth vector fields on M . Let us define $\tau \in \Omega^3(M)$ by

$$\tau(X, Y, Z) = \langle T(X, Y), Z \rangle .$$

The curvature R' of ∇' acts on $\Lambda^2 TM$ such that

$$\langle R'(X \wedge Y), Z \wedge W \rangle = -\langle R'_{X, Y} Z, W \rangle .$$

For the Levi-Civita connection, this gives the Riemannian curvature operator.

Lemma 1.2. *Assume that ∇' has parallel and alternating torsion. Then:*

- (1) *The form $\nabla\tau = \frac{1}{4} d\tau$ is fully alternating.*
- (2) *The form $d\tau$ is given by*

$$(d\tau)(X, Y, Z, W) = 2(\langle T(X, Y), T(Z, W) \rangle + \langle T(Y, Z), T(X, W) \rangle + \langle T(Z, X), T(Y, W) \rangle) .$$

- (3) *The curvature of R' is given by*

$$R'_{X, Y} Z = R_{X, Y} Z + (\nabla_X T)(Y, Z) + \frac{1}{4} T(X, T(Y, Z)) - \frac{1}{4} T(Y, T(X, Z)) .$$

- (4) *The action of R' on $\Lambda^2 TM$ is symmetric.*
- (5) *We have the relation*

$$\langle R'_{X, Y} Y, X \rangle = \langle R_{X, Y} Y, X \rangle - \frac{1}{4} \|T(X, Y)\|^2 .$$

- (6) *The tensor $S = R' - T(T(\cdot, \cdot), \cdot)$ has the same symmetries as the Riemannian curvature tensor.*

Assertion (6) will not be needed later on.

Proof. If ∇' has alternating torsion, then

$$(1.1) \quad \nabla'_X Y = \nabla_X Y + \frac{1}{2} T(X, Y) .$$

By a routine computation, the curvature tensor R' of ∇' is given as

$$(1.2) \quad R'_{X,Y} Z = R_{X,Y} Z + \frac{1}{2} (\nabla_X T)(Y, Z) - \frac{1}{2} (\nabla_Y T)(X, Z) \\ + \frac{1}{4} T(X, T(Y, Z)) - \frac{1}{4} T(Y, T(X, Z)) .$$

If ∇' has parallel torsion, then

$$(1.3) \quad 0 = (\nabla_X T)(Y, Z) + \frac{1}{2} \left(T(X, T(Y, Z)) + T(Y, T(Z, X)) + T(Z, T(X, Y)) \right) ,$$

which shows that ∇T is fully alternating. Because ∇ is metric and T is alternating, the form $\nabla\tau = \langle (\nabla \cdot T)(\cdot, \cdot), \cdot \rangle$ is also fully alternating. Thus

$$(d\tau)(X, Y, Z, W) = (\nabla_X \tau)(Y, Z, W) - (\nabla_Y \tau)(X, Z, W) \\ + (\nabla_Z \tau)(X, Y, W) - (\nabla_W \tau)(X, Y, Z) = 4(\nabla_X \tau)(Y, Z, W) ,$$

which gives (1). Inserting this into (1.3) gives

$$(d\tau)(X, Y, Z, W) = 4 \langle (\nabla_X T)(Y, Z), W \rangle \\ = 2(\langle T(X, Y), T(Z, W) \rangle + \langle T(Y, Z), T(X, W) \rangle \\ + \langle T(Z, X), T(Y, W) \rangle) ,$$

which is (2).

From (1.2), we get (3) by

$$\langle R'_{X,Y} Z, W \rangle = \langle R_{X,Y} Z, W \rangle + \langle (\nabla_X T)(Y, Z), W \rangle \\ + \frac{1}{4} \langle T(X, Z), T(Y, W) \rangle - \frac{1}{4} \langle T(Y, Z), T(X, W) \rangle = \langle R'_{Z,W} X, Y \rangle ,$$

which implies (4) because $\nabla\tau$ is fully alternating. Inserting (1.3) gives

$$(1.4) \quad R'_{X,Y} Z = R_{X,Y} Z - \frac{1}{2} T(Z, T(X, Y)) - \frac{1}{4} T(X, T(Y, Z)) - \frac{1}{4} T(Y, T(Z, X)) ,$$

from which (5) follows. Summing over all cyclic permutations proves the Bianchi identity in (6). The other symmetries of S are obvious. \square

Remark 1.3. All assertions we need to prove Theorem 2 follow if ∇' has alternating torsion and the conclusions (1) and (4) of Lemma 1.2 hold. However, from the computations in the proof above it is easy to see that these assumptions already imply that ∇' has parallel torsion.

2. INDEX-THEORETICAL PRELIMINARIES

We explain the index-theoretical condition used in [GS1] and in Theorem 2. We also correct a mistake in the treatment of odd-dimensional spaces in [GS2] and [G3].

Recall that $\Lambda^\bullet T^*M$ is a Dirac bundle with Clifford multiplication $c(X) = \langle X, \cdot \rangle \wedge -\iota_X$ and with a connection ∇ induced by the Levi-Civita connection ∇ . A Dirac subbundle would be a subbundle of $\Lambda^\bullet T^*M$ that is closed under Clifford multiplication and parallel with respect to the Levi-Civita connection. We allow one to choose connections $\tilde{\nabla}$ on TM and on $\Lambda^\bullet T^*M$ such that

$$\tilde{\nabla}_X(c(Y)\alpha) = c(\tilde{\nabla}_X Y)\alpha + c(Y)\tilde{\nabla}_X \alpha$$

for all vector fields X, Y and all forms α . Note that this does not imply that the connection on $\Lambda^\bullet T^*M$ is induced by the connection on TM . We call a subbundle $W \subset \Lambda^\bullet T^*M$ a *generalised Dirac subbundle* if W is closed under Clifford multiplication and if there exist connections $\tilde{\nabla}$ as above such that W becomes $\tilde{\nabla}$ -parallel. In the following, we will call the operator $\tilde{D} = c \circ \tilde{\nabla}$ a *generalised Hodge-Dirac operator* on W .

Definition 2.1. We say that a generalised Hodge-Dirac operator \tilde{D} on a generalised Dirac subbundle $W \subset \Lambda^\bullet T^*M$ has *nonvanishing index* if either

- (1) M is even-dimensional, W splits as $W^+ \oplus W^-$ such that Clifford multiplication with odd elements exchanges W^+ and W^- , the operator \tilde{D} splits as $\tilde{D}^\pm: \Gamma(W^\pm) \rightarrow \Gamma(W^\mp)$, and $0 \neq \text{ind}(\tilde{D}^+) \in \mathbb{Z}$, or
- (2) M is $(4k + 1)$ -dimensional, oriented, and $\dim_{\mathbb{R}}(\ker \tilde{D}|_{W \cap \Lambda^{\text{ev}} T^*M})$ is odd.

In situation (1), the analytic index is invariant under perturbations and thus independent of our choice of $\tilde{\nabla}$. In situation (2), let e_1, \dots, e_m denote a local oriented orthonormal frame, and let $\omega_{\mathbb{R}} = c(e_1) \cdots c(e_m)$ denote the real Clifford volume element. Then the operator $\omega_{\mathbb{R}} \tilde{D}$ acts on $W \cap \Lambda^{\text{ev}} T^*M$ as a real, skew-adjoint Fredholm operator. In particular, the parity of

$$\dim \ker(\tilde{D}|_{W \cap \Lambda^{\text{ev}} T^*M}) = \dim \ker(\omega_{\mathbb{R}} \tilde{D}|_{W \cap \Lambda^{\text{ev}} T^*M})$$

is again preserved under deformations.

Example 2.2. There are three typical situations where the full Hodge-Dirac operator on $W = \Lambda^\bullet T^*M$ has nonvanishing index.

- (1) If M is even-dimensional and the Euler number $\chi(M)$ is nonzero, we may take $W^\pm = \Lambda^{\text{ev/odd}} T^*M$. This happens for instance if G/H is a quotient of compact Lie groups of equal rank.
- (2) If M is oriented, $4k$ -dimensional, and its signature $\sigma(M)$ is nonzero, we let W^\pm be the selfadjoint and anti-selfadjoint forms. In both cases, we are in situation (1).
- (3) On the other hand, if M is $(4k + 1)$ -dimensional and oriented and the Kervaire semicharacteristic $k(M)$ does not vanish, then we are in situation (2). The sphere S^{4k+1} is an example.

At this point, we want to point out a mistake in [GS2] and [G3]. In those papers, we suggested to “stabilise” manifolds M of odd dimension by S^{2n} to obtain

a manifold of dimension $8k + 1$. Unfortunately, this does not help if $\dim M \equiv -1 \pmod{4}$.

Example 2.3. There are situations where a suitable Dirac subbundle with a Hodge-Dirac operator of nonvanishing index does not exist. Assume that M and N are oriented Riemannian manifolds of dimensions $m \equiv -1$ and $n \equiv 2 \pmod{4}$, respectively. Then $\dim(M \times N) = 4k + 1$. Let $\tilde{\nabla}$ be the tensor product of Clifford connections on $\Lambda^\bullet T^*M$ and $\Lambda^\bullet T^*N$, and let $W \subset \Lambda^\bullet T^*(M \times N)$ be a $\tilde{\nabla}$ -parallel Dirac subbundle. Then the deformed Dirac operators \tilde{D}_M and \tilde{D}_N both act on W and anticommute. Thus, we have

$$\tilde{D}_{M \times N}^2 = \tilde{D}_M^2 + \tilde{D}_N^2 \quad \text{and} \quad \ker(\tilde{D}_{M \times N}^2) = \ker(\tilde{D}_M^2) \cap \ker(\tilde{D}_N^2).$$

The real Clifford volume element $\omega_{N, \mathbb{R}}$ of N acts on $\Lambda^\bullet T^*(M \times N)$. It is even, $\tilde{\nabla}$ -parallel and has square $\omega_{N, \mathbb{R}}^2 = -1$. It commutes with \tilde{D}_M and anticommutes with \tilde{D}_N , so it acts both on $\ker(\tilde{D}_M^2)$ and on $\ker(\tilde{D}_N^2)$. We may regard $\omega_{N, \mathbb{R}}$ as a complex structure on the real vector space $\ker(\tilde{D}_{M \times N}^2|_{W \cap \Lambda^{\text{ev}} T^* M})$, which is therefore even-dimensional. In particular, there are symmetric spaces G/H with $\text{rk } G - \text{rk } H = 1$ to which the arguments of [GS2] and [G3] do not apply.

3. MODIFIED DIRAC OPERATORS

We construct a modified Dirac operator on Dirac bundles (W, ∇^W) over Riemannian manifolds, in particular on the bundle of exterior forms, starting from a connection ∇' with parallel and alternating torsion. In the special case of homogeneous spaces, we recover the reductive Dirac operators of [G1] and [G2], also known as Kostant's cubic Dirac operator. A similar modification of the untwisted Dirac operator has been considered by Agricola and Friedrich; see [AF].

We assume that M is spin with a fixed spin structure, and let $SM \rightarrow M$ denote its complex spinor bundle. Then (W, ∇^W) is the tensor product of the spinor bundle (SM, ∇^{SM}) and another Hermitian vector bundle (V, ∇^V) . If W is a Dirac subbundle of $\Lambda^\bullet T^*M$, then we may regard V as a subbundle of SM because

$$(3.1) \quad \Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C} = \begin{cases} SM \otimes_{\mathbb{C}} SM & \text{if } \dim M \text{ is even, and} \\ 2(SM \otimes_{\mathbb{C}} SM) & \text{if } \dim M \text{ is odd.} \end{cases}$$

If M is not spin, then all Dirac bundles locally still look like twisted spinor bundles. Since all our constructions in the next chapters are local, there is no loss of generality in assuming M to be spin.

Let ∇^{TM} denote the Levi-Civita connection on M . Let us assume that we have a fixed $q \in P_{\text{Spin}}$ over $p \in M$, and let us assume that we have a local section s of P_{Spin} such that $\text{im } d_p s$ is horizontal. Let us choose a coordinate system near p such that the coordinate vectors at p are precisely e_1, \dots, e_m and $\nabla_{e_i}^{TM} e_j = 0$ at p for all i, j . Let us write c_i for Clifford multiplication with the vector e_i .

The Levi-Civita connection induces a connection ∇^{SM} on SM that is compatible with Clifford multiplication. Let (V, ∇^V) be as above, and let R^V denote the curvature of ∇^V . Note that even if M is not spin, the curvature $R^V \in \Omega^2(M; \text{End } W)$ and the twisted Riemannian Dirac operator D^W on $\Gamma(W)$ are still well-defined.

Let $\tau \in \Omega^3(M)$ be an alternating form with coefficients

$$(3.2) \quad \tau(e_i, e_j, e_k) = \tau_{ijk}.$$

Then the connection

$$(3.3) \quad \nabla'_{e_i}{}^{TM} e_j = \nabla_{e_i}{}^{TM} e_j + \frac{1}{2} \sum_{k=1}^m \tau_{ijk} e_k$$

has alternating torsion T with $\tau = \langle T(\cdot, \cdot), \cdot \rangle$.

The connection $\nabla'{}^{TM}$ induces a connection $\nabla'{}^{SM}$ on SM such that

$$(3.4) \quad \nabla'_{e_i}{}^{SM} = \nabla_{e_i}{}^{SM} + \frac{1}{8} \sum_{j,k=1}^m \tau_{ijk} c_j c_k .$$

We also consider another connection $\tilde{\nabla}{}^{SM}$ on SM that is given by

$$\tilde{\nabla}_{e_i}{}^{SM} = \nabla_{e_i}{}^{SM} + \frac{1}{24} \sum_{j,k=1}^m \tau_{ijk} c_j c_k .$$

We then consider the modified Dirac twisted operator

$$(3.5) \quad \tilde{D}^W = \sum_{i=1}^m c_i (\tilde{\nabla}{}^{SM} \otimes \nabla^V)_{e_i} = \sum_{i=1}^m c_i (\nabla{}^{SM} \otimes \nabla^V)_{e_i} + \frac{1}{24} \sum_{i,j,k=1}^m \tau_{ijk} c_i c_j c_k .$$

The square of this operator has been computed by Agricola and Friedrich.

Lemma 3.1 ([AF], Theorem 6.2). *The square of \tilde{D}^W is given by*

$$\begin{aligned} (\tilde{D}^W)_p^2 &= (\nabla'{}^{SM} \otimes \nabla^V)_p^* (\nabla'{}^{SM} \otimes \nabla^V) + \frac{\kappa}{4} + \frac{1}{2} \sum_{i,j=1}^m c_i c_j R_{e_i, e_j}^V \\ &\quad + \frac{1}{96} \sum_{i,j,k,l=1}^m (d\tau)_{ijkl} c_i c_j c_k c_l - \frac{1}{48} \sum_{i,j,k=1}^m \tau_{ijk}^2 . \quad \square \end{aligned}$$

4. SOME CURVATURE FORMULAS

Let $p \in M$ and let $\bar{e}_1(p), \dots, \bar{e}_m(p)$ be an orthonormal basis of $(T_p M, \bar{g}_p)$. We may assume that there exist $\lambda_1(p), \dots, \lambda_m(p) > 0$ such that $\bar{e}_1(p) = \lambda_1(p) e_1(p), \dots, \bar{e}_m(p) = \lambda_m(p) e_m(p)$, where $e_1(p), \dots, e_m(p)$ is an orthonormal basis with respect to g as before. Because $\bar{g} \geq g$ on $\Lambda^2 T M$, we have $\lambda_i(p) \lambda_j(p) \leq 1$ for all $p \in M$ and all $i \neq j$. From now on, we fix p and write simply $\bar{e}_i = \lambda_i e_i$. We may identify the spinor spaces for both metrics in such a way that the Clifford actions at p are related by

$$c(e_i) = \bar{c}(\bar{e}_i) =: c_i .$$

If ∇' is a metric connection with parallel and alternating torsion T on (M, g) , we consider the alternating form

$$\tau = g(T(\cdot, \cdot), \cdot) .$$

Let us put $\bar{\tau} = \tau$ for the moment; we will consider a different $\bar{\tau}$ in Section 8. With respect to the new metric \bar{g} , the coefficients of this form at p are given by

$$\bar{\tau}_{ijk} = \tau(\bar{e}_i, \bar{e}_j, \bar{e}_k) = \lambda_i \lambda_j \lambda_k \tau_{ijk} .$$

Let $\widehat{SM} \rightarrow M$ denote a second copy of the spinor bundle with respect to the old metric g , equipped with the connection $\nabla'{}^{SM}$ as in (3.4). Let $V \rightarrow M$ be a $\nabla'{}^{SM}$ -parallel subbundle of \widehat{SM} , and let $W = SM \otimes V$. Then $W \subset \Lambda^\bullet T^* M \otimes_{\mathbb{R}} \mathbb{C}$ by (3.1). The bundle W is a $\nabla'{}^{SM}$ -parallel Dirac subbundle of $\Lambda^\bullet T^* M_{\mathbb{R}} \mathbb{C}$ as in

Theorem 2. We will write \hat{c} for the Clifford multiplication of TM on \widehat{SM} . Then at the point p , the curvature of $\hat{\nabla}'|_V$ is given by

$$\hat{R}'_{\bar{e}_i, \bar{e}_j} = \frac{1}{4} \sum_{k,l=1}^m \langle R'_{\bar{e}_i, \bar{e}_j} e_k, e_l \rangle \hat{c}_k \hat{c}_l = \frac{1}{4} \lambda_i \lambda_j \sum_{k,l=1}^m R'_{ijkl} \hat{c}_k \hat{c}_l .$$

Let $\bar{\nabla}$ denote the Levi-Civita connection of the metric \bar{g} . We consider the connection

$$(4.1) \quad \bar{\nabla}'_U V = \bar{\nabla}_U V + \frac{1}{2} \sum_{i=1}^m \bar{\tau}(U, V, \bar{e}_i) \bar{e}_i$$

on TM . Note that $\bar{\nabla}'$ has alternating torsion \bar{T} with $\bar{g}(\bar{T}(\cdot, \cdot), \cdot) = \bar{\tau}$, but in general $\bar{\nabla}'\bar{T} \neq 0$. Let \bar{D}^W denote the Riemannian Dirac operator on (M, \bar{g}) twisted by (V, ∇') . We modify \bar{D}^W as in (3.5), obtaining the operator

$$\tilde{D}^W = \bar{D}^W + \frac{1}{24} \sum_{i,j,k=1}^m \bar{\tau}_{ijk} c_i c_j c_k = \bar{D}^W + \frac{1}{24} \sum_{i,j,k=1}^m \lambda_i \lambda_j \lambda_k \tau_{ijk} c_i c_j c_k .$$

The Bochner-Lichnerowicz-Weitzenböck formula in Lemma 3.1 gives

$$\begin{aligned} (\tilde{D}^W)_p^2 &= (\bar{\nabla}' \otimes \hat{\nabla}')^* (\bar{\nabla}' \otimes \hat{\nabla}') + \frac{\bar{\kappa}}{4} + \frac{1}{8} \sum_{i,j,k,l=1}^m \lambda_i \lambda_j R'_{ijkl} c_i c_j \hat{c}_k \hat{c}_l \\ &\quad + \frac{1}{96} \sum_{i,j,k,l=1}^m (d\bar{\tau})(\bar{e}_i, \bar{e}_j, \bar{e}_k, \bar{e}_l) c_i c_j c_k c_l - \frac{1}{48} \sum_{i,j,k=1}^m \bar{\tau}_{ijk}^2 . \end{aligned}$$

We rewrite this as

$$\begin{aligned} (\tilde{D}^W)_p^2 &= (\bar{\nabla}' \otimes \hat{\nabla}')^* (\bar{\nabla}' \otimes \hat{\nabla}') + \frac{\bar{\kappa}}{4} \\ &\quad + \frac{1}{16} \sum_{i,j,k,l=1}^m R'^{TM}_{ijkl} (\lambda_i \lambda_j c_i c_j + \hat{c}_i \hat{c}_j) (\lambda_k \lambda_l c_k c_l + \hat{c}_k \hat{c}_l) \\ (4.2) \quad &\quad - \frac{1}{16} \sum_{i,j,k,l=1}^m \lambda_i \lambda_j \lambda_k \lambda_l R'^{TM}_{ijkl} c_i c_j c_k c_l - \frac{1}{16} \sum_{i,j,k,l=1}^m R'^{TM}_{ijkl} \hat{c}_i \hat{c}_j \hat{c}_k \hat{c}_l \\ &\quad + \frac{1}{96} \sum_{i,j,k,l=1}^m \lambda_i \lambda_j \lambda_k \lambda_l (d\tau)_{ijkl} c_i c_j c_k c_l - \frac{1}{48} \sum_{i,j,k=1}^m \lambda_i^2 \lambda_j^2 \lambda_k^2 \tau_{ijk}^2 . \end{aligned}$$

We will now use the various formulas in Lemma 1.2 to simplify the right hand side of (4.2).

Proposition 4.1. *We have*

$$\begin{aligned} \frac{1}{16} \sum_{i,j,k,l=1}^m \lambda_i \lambda_j \lambda_k \lambda_l R'^{TM}_{ijkl} c_i c_j c_k c_l &= \frac{\kappa}{8} - \frac{1}{32} \sum_{i,j,k=1}^m \tau_{ijk}^2 - \frac{1}{8} \sum_{i,j=1}^m (1 - \lambda_i^2 \lambda_j^2) R'_{ijji} \\ (1) \quad &\quad + \frac{1}{96} \sum_{i,j,k,l=1}^m \lambda_i \lambda_j \lambda_k \lambda_l (d\tau)_{ijkl} c_i c_j c_k c_l , \end{aligned}$$

$$(2) \quad \frac{1}{16} \sum_{i,j,k,l=1}^m R'^{TM}_{ijkl} \hat{c}_i \hat{c}_j \hat{c}_k \hat{c}_l = \frac{\kappa}{8} + \frac{1}{96} \sum_{i,j,k=1}^m \tau_{ijk}^2 - \left(\frac{1}{12} \sum_{i,j,k=1}^m \tau_{ijk} \hat{c}_i \hat{c}_j \hat{c}_k \right)^2 .$$

Proof. Using Lemma 1.2 and the well-known computation of the scalar curvature in the classical Schrödinger-Lichnerowicz formula, we compute

$$\begin{aligned} \sum_{i,j,k,l=1}^m \lambda_i \lambda_j \lambda_k \lambda_l R'_{ijkl} c_i c_j c_k c_l &= 2 \sum_{i,j=1}^m \lambda_i^2 \lambda_j^2 R_{ijji} \\ &+ \frac{1}{4} \sum_{i,j,k,l=1}^m \lambda_i \lambda_j \lambda_k \lambda_l (d\tau)_{ijkl} c_i c_j c_k c_l \\ &+ \frac{1}{4} \sum_{i,j,k,l,p=1}^m \lambda_i \lambda_j \lambda_k \lambda_l (\tau_{jkp} \tau_{ipl} - \tau_{ikp} \tau_{jpl}) c_i c_j c_k c_l . \end{aligned}$$

By the Clifford relations and Lemma 1.2 (2), this becomes

$$\begin{aligned} &\sum_{i,j,k,l=1}^m \lambda_i \lambda_j \lambda_k \lambda_l R'_{ijkl} c_i c_j c_k c_l \\ &= 2 \sum_{i,j=1}^m \lambda_i^2 \lambda_j^2 R_{ijji} + \frac{1}{4} \sum_{i,j,k,l=1}^m \lambda_i \lambda_j \lambda_k \lambda_l (d\tau)_{ijkl} c_i c_j c_k c_l - \frac{1}{2} \sum_{i,j,k=1}^m \lambda_i^2 \lambda_j^2 \tau_{ijk}^2 \\ &\quad - \frac{1}{6} \sum_{i,j,k,l,p=1}^m \lambda_i \lambda_j \lambda_k \lambda_l (\tau_{ijp} \tau_{pkl} + \tau_{ikp} \tau_{plj} + \tau_{ilp} \tau_{pjk}) c_i c_j c_k c_l \\ &= 2 \sum_{i,j=1}^m \lambda_i^2 \lambda_j^2 R_{ijji} - \frac{1}{2} \sum_{i,j,k=1}^m \lambda_i^2 \lambda_j^2 \tau_{ijk}^2 + \frac{1}{6} \sum_{i,j,k,l=1}^m \lambda_i \lambda_j \lambda_k \lambda_l (d\tau)_{ijkl} c_i c_j c_k c_l . \end{aligned}$$

Using Lemma 1.2 (5), we obtain (1).

To prove (2), we may take formulas established above and replace all Clifford variables c_i by \hat{c}_i and vice versa. Either by direct computation or by collecting those terms in Lemma 3.1 where τ occurs quadratically, we obtain

$$\left(\frac{1}{24} \sum_{i,j,k=1}^m \tau_{ijk} \hat{c}_i \hat{c}_j \hat{c}_k \right)^2 = - \sum_{i=1}^m \left(\frac{1}{8} \sum_{j,k=1}^m \tau_{ijk} \hat{c}_j \hat{c}_k \right)^2 - \frac{1}{48} \sum_{i,j,k=1}^m \tau_{ijk}^2 .$$

Using equation (1.4) for R' and proceeding as above, we find

$$\begin{aligned} \sum_{i,j,k,l=1}^m R'_{ijkl} \hat{c}_i \hat{c}_j \hat{c}_k \hat{c}_l &= 2\kappa - \frac{1}{4} \sum_{i,j,k,l,p=1}^m (2\tau_{ijp} \tau_{kpl} + \tau_{jkp} \tau_{ipl} + \tau_{kip} \tau_{jpl}) \hat{c}_i \hat{c}_j \hat{c}_k \hat{c}_l \\ &= 2\kappa + \sum_{i=1}^m \left(\sum_{j,k=1}^m \tau_{ijk} \hat{c}_i \hat{c}_j \right)^2 + \frac{3}{2} \sum_{i,j,k=1}^m \tau_{ijk}^2 \\ &= 2\kappa - \left(\frac{1}{3} \sum_{i,j,k=1}^m \tau_{ijk} \hat{c}_i \hat{c}_j \hat{c}_k \right)^2 + \frac{1}{6} \sum_{i,j,k=1}^m \tau_{ijk}^2 , \end{aligned}$$

which proves (2). \square

5. A BOCHNER-LICHNEROWICZ-WEITZENBÖCK FORMULA

We now establish Lemma 3. This chapter is not needed in the proof of Theorem 2. We consider the special case $\bar{g} = g$. Then the operator \tilde{D}^W becomes a modified

Euler or signature operator \tilde{D}^W . In this case, we can use (4.2) and Proposition 4.1 to simplify the Bochner-Lichnerowicz-Weitzenböck formula in Lemma 3.1 as follows.

Corollary 5.1. *Assume that ∇' has parallel and alternating torsion. Then the square of the modified signature operator \tilde{D}^W is given by*

$$\begin{aligned} (\tilde{D}^W)_p^2 &= (\nabla' \otimes \hat{\nabla}')^* (\nabla' \otimes \hat{\nabla}') + \left(\frac{1}{12} \sum_{i,j,k=1}^m \tau_{ijk} \hat{c}_i \hat{c}_j \hat{c}_k \right)^2 \\ &\quad + \frac{1}{16} \sum_{i,j,k,l=1}^m R'_{ijkl} (c_i c_j + \hat{c}_i \hat{c}_j) (c_k c_l + \hat{c}_k \hat{c}_l). \quad \square \end{aligned}$$

If R' acts nonnegatively on $\Lambda^2 TM$, then this formula implies that all \tilde{D}^W -harmonic forms are ∇' -parallel. We will now prove Lemma 3, which allows us to compute some global topological invariants of M at a single point.

Proof of Lemma 3. We choose $(V, \nabla^V) = (SM, \nabla'^{SM})$. The operator

$$\sum_{i,j,k=1}^m \tau_{ijk} \hat{c}_i \hat{c}_j \hat{c}_k$$

is selfadjoint, and hence has a positive square.

If R' acts nonnegatively on $\Lambda^2 TM$, it possesses a symmetric square root that we will denote by B , with coefficients B_{ijkl} . More precisely,

$$-R_{ijkl} = \sum_{p,q=1}^m B_{ijpq} B_{pqkl}.$$

Thus,

$$(5.1) \quad \sum_{i,j,k,l=1}^m R'_{ijkl} (c_i c_j + \hat{c}_i \hat{c}_j) (c_k c_l + \hat{c}_k \hat{c}_l) = - \sum_{i,j=1}^m \left(\sum_{k,l=1}^m B_{ijkl} (c_k c_l + \hat{c}_k \hat{c}_l) \right)^2 \geq 0$$

because the square of a skewadjoint operator is negative.

As all operators on the right hand side in the corollary are nonnegative, each \tilde{D}^W -harmonic form lies in the kernel of the connection Laplacian and is thus ∇' -parallel. Note that

$$\tilde{D} = \sum_{i=1}^m c_i (\nabla' \otimes \hat{\nabla}')_{e_i} - \frac{1}{12} \sum_{i,j,k=1}^m \tau_{ijk} c_i c_j c_k.$$

Clearly, the first term on the right hand side vanishes on ∇' -parallel forms. Because the torsion is ∇' -parallel and ∇' is compatible with Clifford multiplication, we see that the second term on the right hand side is ∇' -parallel. In particular, the operator \tilde{D} preserves the space $\ker \nabla'$ of ∇' -parallel forms.

Assuming that m is even, we conclude

$$\begin{aligned} \chi(M) &= \text{ind}(\tilde{D}) = \text{ind}(\tilde{D}|_{\ker \nabla'}) = \dim(\ker \nabla' \cap \Omega^{\text{ev}}(M)) \\ &\quad - \dim(\ker \nabla' \cap \Omega^{\text{odd}}(M)) = \dim(\Lambda^{\text{ev}} \pi)^H - \dim(\Lambda^{\text{odd}} \pi)^H, \end{aligned}$$

because ∇' -parallel forms are determined by their value at a single point. The signature and the Kervaire semicharacteristic can be computed similarly if they are defined. \square

6. NORMAL HOMOGENEOUS SPACES

We regard homogeneous spaces as a special case of Riemannian manifolds that admit a connection with nonvanishing parallel and alternating torsion.

Let G be a Lie group, let $H \subset G$ be a closed subgroup, and let $\mathfrak{h} \subset \mathfrak{g}$ denote their Lie algebras. In the following, we consider the homogeneous space $M = G/H$, with tangent bundle $TM \cong G \times_H (\mathfrak{g}/\mathfrak{h})$. Let g be a metric on M . Then g is invariant under the natural action of G iff g is induced by an Ad_H -invariant metric $g_{\mathfrak{g}/\mathfrak{h}}$ on $\mathfrak{g}/\mathfrak{h}$. In this case, we call (M, g) a Riemannian homogeneous space.

Definition 6.1. A Riemannian homogeneous space $(G/H, g)$ is called *naturally reductive* if there exists an Ad_H -invariant linear complement \mathfrak{p} of \mathfrak{h} in \mathfrak{g} with an Ad_H -invariant metric $g_{\mathfrak{p}}$ inducing the Riemannian metric g on $T(G/H) \cong G \times_H \mathfrak{p}$, such that the three-form $g_{\mathfrak{p}}([\cdot, \cdot]_{\mathfrak{p}}, \cdot)$ on \mathfrak{p} is fully alternating.

A naturally reductive Riemannian homogeneous space $(G/H, g)$ is called *normal* if there exists an Ad_G -invariant metric $g_{\mathfrak{g}}$ on \mathfrak{g} such that $\mathfrak{p} = \mathfrak{h}^{\perp}$ and $g_{\mathfrak{p}} = g_{\mathfrak{g}}|_{\mathfrak{p}}$.

Let $(M, g) = (G/H, g)$ be naturally reductive. Then we have the isotropy representation

$$\pi = \text{Ad}|_{H \times \mathfrak{p}} : H \rightarrow \text{SO}(\mathfrak{p}) .$$

We identify \mathfrak{g} with the left-invariant vector fields on G . Then the action of H on G is generated by the left-invariant vector fields corresponding to elements of $\mathfrak{h} \subset \mathfrak{g}$. We represent a vector field X on M by its horizontal lift \hat{X} to G . Then \hat{X} is a map $\hat{X} : G \rightarrow \mathfrak{p} \subset \mathfrak{g}$ that is H -equivariant with respect to the isotropy representation. Then the *reductive connection* ∇' and the Levi-Civita connection ∇^{TM} are given by

$$\widehat{\nabla_X^{TM} Y} = \hat{X}(\hat{Y}) \quad \text{and} \quad \widehat{\nabla_X^{TM} Y} = \hat{X}(\hat{Y}) + \frac{1}{2} [\hat{X}, \hat{Y}]_{\mathfrak{p}} ,$$

where $[\hat{X}, \hat{Y}]_{\mathfrak{p}}$ is the pointwise algebraic Lie bracket projected to \mathfrak{p} ; see Cheeger and Ebin [CE]. These connections are different unless (M, g) is locally symmetric. Let us recall some well-known facts about normal homogeneous metrics.

Lemma 6.2. *Let $M = G/H$ be a homogeneous space with a normal metric g . Then the reductive connection ∇' has parallel and alternating torsion, and its curvature operator R' on $\Lambda^2 TM$ is nonnegative.*

Proof. One easily checks that the torsion tensor T of ∇'^{TM} satisfies

$$T(\widehat{X}, \widehat{Y}) = -[\hat{X}, \hat{Y}]_{\mathfrak{p}} .$$

In particular, ∇'^{TM} has alternating torsion by natural reductivity. The torsion is parallel because

$$\hat{X}([\hat{Y}, \hat{Z}]_{\mathfrak{p}}) = [\hat{X}(\hat{Y}), \hat{Z}]_{\mathfrak{p}} + [\hat{Y}, \hat{X}(\hat{Z})]_{\mathfrak{p}} .$$

The reductive connection has curvature

$$\widehat{R'_{X,Y} Z} = -[[\hat{X}, \hat{Y}]_{\mathfrak{h}}, \hat{Z}]_{\mathfrak{p}} .$$

On a normal homogeneous space, we may write

$$(6.1) \quad \langle R'_{X,Y} Z, W \rangle = -\langle [\hat{X}, \hat{Y}]_{\mathfrak{h}}, [\hat{Z}, \hat{W}]_{\mathfrak{h}} \rangle ;$$

in particular, the action of R' on $\Lambda^2 TM$ is nonnegative. This is not true in general for naturally reductive spaces. \square

For the rigidity statements in Theorems 2 and 4, we need to control the Ricci curvature.

Lemma 6.3. *Let $M = G/H$ and g be as before. Then the Ricci curvature ρ is positive definite unless G/H contains a Euclidean local de Rham factor, which implies in particular that $\text{rk } G > \text{rk } H$. Moreover, the form $2\rho - \kappa g$ is negative definite unless M is covered by \mathbb{R}^m or $S^2 \times \mathbb{R}^{m-2}$.*

Proof. The Riemannian curvature tensor of (M, g) satisfies

$$\langle R_{X,Y}X, X \rangle = \|[\hat{X}, \hat{Y}]_{\mathfrak{h}}\|^2 + \frac{1}{4} \|[\hat{X}, \hat{Y}]_{\mathfrak{p}}\|^2 \geq 0.$$

This implies that the Ricci tensor is nonnegative.

Assume that $X(p) \in \ker \rho$ is a unit vector and extend $\hat{X}(p)$ to a basis $e_1 = \hat{X}(p)$, e_2, \dots, e_m of \mathfrak{p} . Because

$$0 = \rho(X, X)(p) = \sum_{i=2}^m \left(\| [e_1, e_i]_{\mathfrak{h}} \|^2 + \frac{1}{4} \| [e_1, e_i]_{\mathfrak{p}} \|^2 \right)$$

and $\langle R_{X,Y}X, X \rangle = 0$ implies $[\hat{X}, \hat{Y}]_{\mathfrak{g}} = 0$, we conclude that $[e_1, w]_{\mathfrak{g}} = 0$ for all $w \in \mathfrak{p}$. For $v \in \mathfrak{h}$, we have $[e_1, v] \in \mathfrak{p}$ and

$$\langle [e_1, v], w \rangle = -\langle [e_1, w], v \rangle = 0$$

for all $w \in \mathfrak{p}$. Hence e_1 lies in the center \mathfrak{z} of \mathfrak{g} . But then the vector field X_0 with $\hat{X}_0 \equiv e_1$ is a parallel vector field that spans a one-dimensional Euclidean local de Rham factor of M . If G and H are of equal rank, then $\mathfrak{z} \subset \mathfrak{h}$, so $\mathfrak{z} \cap \mathfrak{p} \neq \emptyset$ implies $\text{rk } G > \text{rk } H$.

By Remark 8, we conclude that $2\rho - \kappa g$ is negative definite unless $T = 0$. Let us assume $T = 0$. Then $M = G/H$ is symmetric, and the universal covering \tilde{M} of M splits as a product of irreducible symmetric spaces of compact type and \mathbb{R}^k for some $k \geq 0$. Let ρ_1, \dots, ρ_m denote the eigenvalues of the Ricci tensor. As explained in [GS2], the form $(2\rho - \kappa)$ is negative definite unless

$$\rho_i \geq \sum_{j \neq i} \rho_j$$

for some index $i \in \{1, \dots, m\}$. But since every irreducible symmetric space is Einstein, this is only possible if $\tilde{M} = \mathbb{R}^m$ or $\tilde{M} = S^2 \times \mathbb{R}^{m-2}$. \square

Proof of Corollary 5. It remains to control the index of the Hodge-Dirac operator on suitable Dirac subbundles of $\Lambda^\bullet T^*M$ if $\text{rk } G = \text{rk } H$. One has the well-known formula

$$(6.2) \quad \chi(G/H) = \#(W_G/W_H) > 0,$$

where W_G and W_H are the Weyl groups of G and H with respect to a common maximal torus. In particular, the Euler operator itself has nonvanishing index; see Example 2.2 (1). \square

If $\text{rk } G - \text{rk } H = 1$ and $\dim M \equiv 1 \pmod{4}$, the situation is more complicated. There are examples such as S^{4k+1} where the Kervaire semicharacteristic does not vanish. In Example 2.3, we have seen that for product spaces such as $M = G/H \times S^{4k+2}$, there is no suitable ∇' -parallel Dirac subbundle $W \subset \Lambda^\bullet T^*M$ giving rise to a nontrivial index.

Lemma 3 suggests treating the case $\text{rk } G - \text{rk } H = 1$ and $\dim M = 4k + 1$ by representation-theoretic methods. First note that if H is connected and \mathfrak{h} contains no central elements of \mathfrak{g} , then H is the global holonomy group of the reductive connection on TM , in accordance with our notation in Lemma 3.

We regard the infinitesimal isotropy representation of M as a Lie algebra homomorphism $\pi_*: \mathfrak{h} \rightarrow \mathfrak{spin}(m)$. Pulling back the complex spinor representation of $\mathfrak{spin}(m)$ to \mathfrak{h} gives rise to the spinor representation $\tilde{\pi}_*$ of M . Let $\hat{\pi}_*$ be another copy of $\tilde{\pi}_*$. By (3.1), the complex exterior representation $\Lambda^\bullet \pi_* \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the sum of one or two copies of $\tilde{\pi}_* \otimes \hat{\pi}_*$. If M is G -equivariantly spin, this integrates to an isomorphism of H -representations.

Let $S \subset T$ be maximal tori of H and G , respectively. Let $\mathfrak{s} \subset \mathfrak{t}$ denote their Lie algebras. Because g is a normal metric, we may identify the dual \mathfrak{s}^* with a subspace of \mathfrak{t}^* . Let $\rho_H \in \mathfrak{s}^*$ and $\rho_G \in \mathfrak{t}^*$ denote the half sums of the positive (real) roots of H and G with respect to some Weyl chamber. By abuse of notation, we identify each irreducible representation of H or G with its highest (real) weight in \mathfrak{s}^* or \mathfrak{t}^* , respectively.

Remark 6.4. Let us regard the Bochner-Lichnerowicz-Weitzenböck formula in Corollary 5.1. The group G acts on $\Lambda^\bullet T^*M$, and the restriction of the infinitesimal action to \mathfrak{h} is isomorphic to $\tilde{\pi}_* \otimes \hat{\pi}_*$. Using (6.1), we can identify the term

$$(\nabla' \otimes \hat{\nabla}')^*(\nabla' \otimes \hat{\nabla}') + \frac{1}{16} \sum_{i,j,k,l=1}^m R'_{ijkl} (c_i c_j + \hat{c}_i \hat{c}_j) (c_k c_l + \hat{c}_k \hat{c}_l)$$

with the action of the Casimir operator c_G . By a computation in [G1], [G2], we have

$$\left(\frac{1}{12} \sum_{i,j,k=1}^m \tau_{ijk} \hat{c}_i \hat{c}_j \hat{c}_k \right)^2 = \|\rho_G\|^2 - \|\rho_H\|^2 - c_H^{\hat{\pi}_*},$$

where $c_H^{\hat{\pi}_*}$ denotes the Casimir operator of H . If δ is an irreducible component of $\hat{\pi}_*$ acting on V^δ , let $V = G \times_H V^\delta \subset SM$, and consider the Dirac subbundle $W \subset \Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C}$. If γ is an irreducible representation of G , then $(\tilde{D}^W)^2$ acts on the γ -isotypical component of $\Gamma(W)$ by

$$c_G + \|\rho_G\|^2 - \|\rho_H\|^2 - c_H^\delta = \|\gamma + \rho_G\|^2 - \|\delta + \rho_H\|^2 .$$

This is the well-known generalisation of Parthasarathy’s formula [P] to homogeneous spaces.

Lemma 6.5. *Let $M = G/H$ be a quotient of compact Lie groups. Then $\Lambda^\bullet TM$ contains a Dirac subbundle W with $\ker \tilde{D}^W \neq 0$ if and only if there exists $w \in W_G$ such that $w(\rho_G) \in \mathfrak{s}^*$. The index of \tilde{D}^W vanishes if $\text{rk } G - \text{rk } H > 1$.*

Proof. By Lemma 4.4 in [G1], for each irreducible subrepresentation δ of $\tilde{\pi}$ on S , we have $\|\delta + \rho_H\| \leq \|\rho_G\|$. Let W be the associated Dirac subbundle of $\Lambda^\bullet T^*M$ as in Remark 6.4. Then on the γ -isotypical component, the operator $(\tilde{D}^W)^2$ acts by the scalar

$$(6.3) \quad \|\gamma + \rho_G\|^2 - \|\delta + \rho_H\|^2 \geq \|\gamma + \rho_G\| - \|\rho_G\|^2 = c_G^\gamma \geq 0 .$$

Equality can only occur if γ is the trivial representation and if $\|\delta + \rho_H\| = \|\rho_G\|$. By [G1], the equation $\|\delta + \rho_H\| = \|\rho_G\|$ holds for a subrepresentation δ of $\tilde{\pi}$

iff $w\rho_G \in \mathfrak{s}^*$ and $\delta = w\rho_G - \rho_H$ for some element w in the Weyl group of G . This is always the case if $\mathfrak{s} = \mathfrak{t}$, i.e., if $\text{rk } G = \text{rk } H$.

Now assume that there exists $\delta \in \tilde{\pi}$ such that $\ker(\tilde{D}^\delta) \neq 0$. By (6.3), one easily concludes that $\dim_{\mathbb{C}} \ker(\delta)$ equals the multiplicity of δ in $\tilde{\pi}$. By Lemma 4.4 in [G1], this multiplicity is 1 if $\text{rk } G \leq \text{rk } H + 1$ and no nonzero weight of $\text{ad}_{\mathfrak{g}}$ vanishes on \mathfrak{s} .

On the other hand, if $\text{rk } G \geq \text{rk } H + 2$, then the multiplicity is even, and if $\dim M$ is even, the multiplicity of δ in $\tilde{\pi}^+$ and $\tilde{\pi}^-$ is the same. In this case, the index of \tilde{D}^δ is zero. \square

Unfortunately, Lemma 6.5 only states a necessary condition. There are homogeneous spaces G/H where $w\rho_G \notin \mathfrak{s}$ for all $w \in W_G$. The Berger space $\text{SO}(5)/\text{SO}(3)$ considered in [G1], [G2] is an example.

Assume that $\text{rk } G - \text{rk } H = 1$, $\dim M = 4k + 1$, $w\rho_G \in \mathfrak{s}^*$ and the multiplicity of $\delta = w\rho_G - \rho_H$ in $\tilde{\pi}_*$ is one. The example $M = S^2 \times S^3$ shows that \tilde{D}^W can still have vanishing index. A careful investigation of the bundles of real differential forms and real spinors using the classification in [LM] shows that δ must be of quaternionic type if k is odd, and of real type if k is even. In the cases considered in Example 2.3, the corresponding representations are of complex type.

7. PROOF OF THE SCALAR CURVATURE ESTIMATE

We prove Theorem 2. As in Corollary 5.1 above, we insert the formulas of Proposition 4.1 into (4.2). Using (5.1), we obtain

$$(7.1) \quad \begin{aligned} (\tilde{D}^W)_p^2 &= (\bar{\nabla}' \otimes \hat{\nabla}')^* (\bar{\nabla}' \otimes \hat{\nabla}') + \frac{\bar{\kappa} - \kappa}{4} + \left(\frac{1}{12} \sum_{i,j,k=1}^m \tau_{ijk} \hat{c}_i \hat{c}_j \hat{c}_k \right)^2 \\ &\quad - \frac{1}{16} \sum_{i,j=1}^m \left(\sum_{k,l=1}^m B_{ijkl} (\lambda_k \lambda_l c_k c_l + \hat{c}_k \hat{c}_l) \right)^2 \\ &\quad + \frac{1}{8} \sum_{i,j=1}^m (1 - \lambda_i^2 \lambda_j^2) R'_{ijji} + \frac{1}{48} \sum_{i,j,k=1}^m (1 - \lambda_i^2 \lambda_j^2 \lambda_k^2) \tau_{ijk}^2. \end{aligned}$$

Because $\bar{g} \geq g$ on $\Lambda^2 TM$ by assumption, we have

$$(7.2) \quad \lambda_i \lambda_j \leq 1 \quad \text{for all } i, j \in \{1, \dots, m\} \text{ with } i \neq j.$$

Because we also assumed $R'_{ijji} \geq 0$, the last two terms above are nonnegative. Thus finally,

$$(7.3) \quad (\tilde{D}^W)_p^2 \geq \frac{\bar{\kappa}}{4} - \frac{\kappa}{4}.$$

By our assumptions in Theorem 2 and the considerations in Section 2, the operator \tilde{D}^W has a nontrivial kernel. We may thus apply (7.1) to a \tilde{D}^W -harmonic spinor $\sigma \neq 0$, obtaining

$$0 \geq \int_M \left(\frac{\bar{\kappa}}{4} - \frac{\kappa}{4} \right) \|\sigma\|^2 d\text{vol}_{\bar{g}}.$$

Note that a nontrivial harmonic spinor on a connected manifold is nonzero almost everywhere by [Bä]. Hence if we assume that $\bar{\kappa} \geq \kappa$ everywhere, we immediately get $\bar{\kappa} = \kappa$ almost everywhere, and hence everywhere by continuity. This completes the proof of the first part of Theorem 2.

We will now prove the second part of Theorem 2. Note that since the torsion T is assumed to be parallel and M is connected, either $T = 0$ or T vanishes nowhere. In the case $T = 0$, the hypotheses in Theorem 2 are the same as in [GS2]. Hence in this case, nothing is left to prove.

Thus, we may assume that $T \neq 0$ everywhere on M . At a point $p \in M$, choose an adapted g -orthonormal frame e_1, \dots, e_m as in Section 4. Then there exist indices $1 \leq p, q, r \leq m$ such that $\tau_{pqr} \neq 0$. We may assume that $p = 1, q = 2, r = 3$.

Now we assume that $\bar{\kappa} \geq \kappa$. This implies that we have equality in (7.3). In particular, the nonnegative term

$$\frac{1}{48} \sum_{i,j,k=1}^m (1 - \lambda_i^2 \lambda_j^2 \lambda_k^2) \tau_{ijk}^2$$

in (7.1) vanishes. For the indices 1, 2, 3, this implies

$$1 = \lambda_1^2 \lambda_2^2 \lambda_3^2 = (\lambda_1 \lambda_2) (\lambda_1 \lambda_3) (\lambda_2 \lambda_3)$$

because $\tau_{1,2,3} \neq 0$. Hence $\lambda_1 = \lambda_2 = \lambda_3 = 1$ by (7.2). For the same reason, we have $\lambda_i = 1$ whenever there exist j, k such that $\tau_{ijk} \neq 0$.

The assumption $\lambda_1 \lambda_i \leq 1$ implies

$$(7.4) \quad \lambda_i \leq 1$$

for all indices $i > 1$, hence for all i .

Now, consider an index i such that $e_i \in \ker \tau$, so $\tau_{ijk} = 0$ for all j, k . Because $\rho|_{\ker \tau}$ is positive definite, Lemma 1.2 (5) implies

$$0 < \sum_{j=1}^m R_{ijji} = \sum_{j=1}^m \left(R_{ijji} + \sum_{k=1}^m \alpha_{ijk}^2 \right) = \sum_{j=1}^m R'_{ijji}.$$

By Lemma 1.2 (5), all $R_{ijji} \geq 0$, as noted in Remark 7. By the above, there exists j such that $R_{ijji} > 0$. But by (7.1), equality in (7.3) implies

$$0 = (1 - \lambda_i^2 \lambda_j^2) R_{ijji};$$

hence $\lambda_i \lambda_j = 1$. By (7.4), we have $\lambda_i = \lambda_j = 1$. This completes the proof of the rigidity part of Theorem 2.

8. AREA NONINCREASING SPIN MAPS OF NONZERO \hat{A} -DEGREE

We now explain and prove Theorem 4. Recall that a map $f: (N, \bar{g}) \rightarrow (M, g)$ between Riemannian manifolds is called *area-nonincreasing* iff $\bar{g} \geq f^*g$ on $\Lambda^2 TN$. Riemannian submersions are a special case.

A *spin map* is a map $f: N \rightarrow M$ between differentiable manifolds such that their second Stiefel-Whitney classes are related by $w_2(N) = f^*w_2(M)$. This is the case precisely if there exists a Dirac bundle $W \rightarrow N$ that is locally isometric to $SN \otimes f^*SM$.

Finally, if M and N are oriented with orientation classes $(M) \in H^m(M; \mathbb{Z})$ and $[N] \in H_n(M; \mathbb{Z})$, the \hat{A} -degree of a map $f: N \rightarrow M$ is defined as

$$\deg_{\hat{A}} f = (\hat{A}(N) \wedge (M))[N].$$

If f is a spin map and \bar{D} is the Dirac operator on the bundle $W \rightarrow N$ above, then

$$(8.1) \quad \text{ind}(\bar{D}) = \deg_{\hat{A}} f \cdot \chi(M).$$

In Theorem 4, we have assumed that $f: (N, \bar{g}) \rightarrow (M, g)$ is an area-nonincreasing spin map of nonzero \hat{A} -genus, and that $\chi(M) \neq 0$.

Proof of Theorem 4. We modify the Dirac operator \bar{D} on $W \rightarrow N$ as in Section 4, now using the three-form $\bar{\tau} = f^*\tau \in \Omega^3(N)$. At a point $q \in N$, we may fix orthonormal bases e_1, \dots, e_m of $T_{f(q)}M$ and $\bar{e}_1, \dots, \bar{e}_n$ of T_qN such that there exist numbers $\lambda_1, \dots, \lambda_m \geq 0$ with

$$(8.2) \quad df_q \bar{e}_i = \begin{cases} \lambda_i e_i & \text{if } 1 \leq i \leq m, \text{ and} \\ 0 & \text{if } i > m. \end{cases}$$

Then the coefficients of $\bar{\tau}$ are again given by

$$\bar{\tau}_{ijk} = \begin{cases} \lambda_i \lambda_j \lambda_k \tau_{ijk} & \text{if } 1 \leq i, j, k \leq m, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

By (8.1) and our assumptions $\deg_{\hat{A}} f \neq 0$ and $\chi(M) \neq 0$, the modified operator \tilde{D}^W has nonzero index

$$\begin{aligned} \text{ind}(\tilde{D}^W) &= (\hat{A}(TN) \wedge \text{ch}(W/S))[N] \\ &= (\hat{A}(TN) \wedge f^*e(TM))[N] = \deg_{\hat{A}}(f) \cdot \chi(M) \neq 0. \end{aligned}$$

From this point on, the proof of area-extremality in the sense of Gromov proceeds as in Section 7.

To prove strong area-extremality in the sense of Gromov, we again consider the modified operator \tilde{D} on $W \rightarrow N$. As in the proof of strong area-extremality above, we conclude that $\bar{\kappa} \geq \kappa \circ f$ implies that $\lambda_1 = \dots = \lambda_m = 1$; hence f is a Riemannian submersion. \square

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