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EXTENDING LOCAL ANALYTIC CONJUGACIES

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Abstract. We prove that if two globally-defined one-dimensional complex dynamics are locally analytically conjugate, then we extend the conjugacy to obtain global conjugacy by a correspondence. The most important case occurs when two rational maps have analytically conjugate polynomial-like restrictions. In this case, we prove that there exists another rational map which is semiconjugate to them both by some rational maps.

1. Introduction

Polynomial-like mappings were introduced by Douady and Hubbard [DH]. The definition of a polynomial-like mapping is very simple and it contains large classes of maps: A map $f : U' \to U$ is *polynomial-like* if it is proper and holomorphic, U' and U are topological disks in $\mathbb{C}$, and U' is a relatively compact subset of U. However, they proved that polynomial-like mappings actually behave like polynomials. Namely, any polynomial-like map is hybrid equivalent to some polynomial of the same degree (the straightening theorem). In other words, the set of hybrid equivalence classes of polynomial-like mappings of a given degree d is very small; it is just the hybrid equivalence classes of polynomials of degree d.

However, in this paper, we prove there are plenty of classes in the sense of analytic equivalence, by showing that we can distinguish polynomial-like renormalizations (or restrictions) of rational maps by their analytic equivalence classes except when they have some global analytic correspondence. We use this theorem to show that straightening maps between renormalizable polynomials of a given combinatorics and the corresponding family of polynomials is always discontinuous [I].

Theorem 1. *For $i = 1, 2$, let f_i be a rational map or an entire map. Assume that there exist polynomial-like restrictions $f_1 : U_1' \to U_1$ and $f_2 : U_2' \to U_2$ of degree not less than two which are analytically conjugate. Then there exist rational or entire maps g, ϕ_1 and ϕ_2 such that*

$$f_1 \circ \phi_1 = \phi_1 \circ g, \qquad f_2 \circ \phi_2 = \phi_2 \circ g \tag{1}$$

and g has a polynomial-like restriction $g : V' \to V$ analytically conjugate to $f_1 : U_1' \to U_1$ by ϕ_1.

Furthermore,

- *if both of the degrees $d_1 = \deg f_1$ and $d_2 = \deg f_2$ are finite, then g, ϕ_1, ϕ_2 are also of finite degrees. In particular, we have $d_1 = d_2$.*

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- *If f_1 is a polynomial and f_2 is a rational map, then f_2 is a polynomial by taking a Möbius conjugate and we can take g, ϕ_1 and ϕ_2 to be polynomials.*

Here we give several examples such that the conclusion holds. In the following, we say a map g is *semiconjugate* to another map f if there exists a map h such that $h \circ g = f \circ h$. We call such an h a *semiconjugacy* from g to f. In this paper, we always assume that f, g and h are holomorphic and mainly we are interested in *global* semiconjugacies, i.e., f, g and h are polynomials, rational maps, or entire maps.

Example. (1) Let $h(z)$ be a polynomial and let $f(z) = z(h(z))^d$, $g(z) = zh(z^d)$ and $\phi(z) = z^d$. Then $\phi(g(z)) = f(\phi(z))$.
(2) For rational maps R_1 and R_2, let $f = R_1 \circ R_2$ and $g = R_2 \circ R_1$. Then f and g are semiconjugate to each other by R_1 and R_2.
(3) Similarly, let R_1, R_2 and R_3 be rational maps and let $f_1 = R_1 \circ R_2 \circ R_3$, $f_2 = R_2 \circ R_3 \circ R_1$ and $g = R_3 \circ R_1 \circ R_2$. Then g is semiconjugate to f_i by ϕ_i, where $\phi_1 = R_3$ and $\phi_2 = R_3 \circ R_1$.
(4) Let $f_1(z) = z(c + z^2)^3$, $f_2(z) = z(c + z^3)^2$ and $g(z) = z(c + z^6)$. Then $f_1(z^3) = (g(z))^3$ and $f_2(z^2) = (g(z)))^2$. Furthermore, f_1 and f_2 are not semiconjugate to each other except when $c = 0$ (see Appendix A).

Ritt [R] gave a description of decompositions of polynomials by composition. It gives a classification of semiconjugate polynomials. See Appendix A for more details. A classification of semiconjugate rational maps is an open problem because we do not have such a nice theory of decompositions of rational maps. Transcendental entire maps seem to be more difficult. We do not even know whether there exist a transcendental entire map f_1 and a rational map f_2 which satisfy the conclusion of the theorem. A transcendental entire map cannot be semiconjugate to a polynomial by a (transcendental) entire map (Proposition 4), but that is almost all that we know and the following are open:

Conjecture. (1) *A polynomial of degree greater than one cannot be semiconjugate to a transcendental entire map by an entire map.*
(2) *A transcendental entire map cannot be semiconjugate to a rational map by a transcendental meromorphic map defined on $\mathbb{C}$.*

Observe that the inverse of a linearizing coordinate of a repelling fixed point gives a semiconjugacy from a linear map (polynomial of degree one) to a transcendental entire map.

The conclusion (1) in the theorem can be considered as a relation between f_1 and f_2. It becomes clear by describing it in terms of correspondences [BS]. A *holomorphic correspondence* $h : X \to Y$ is a multi-valued map between Riemann surfaces which has the form $h = \tilde{Q}_+ \circ \tilde{Q}_-^{-1}$, where $\tilde{Q}_- : Z \to X$ and $\tilde{Q}_+ : Z \to Y$ are holomorphic maps between complex manifolds. That is, $z \mapsto w$ if there exists some $x \in Z$ such that $z = \tilde{Q}_-(x)$ and $w = \tilde{Q}_+(x)$. If X and Y are compact, then the graph $\Gamma = \{(\tilde{Q}_-(x), \tilde{Q}_+(x));\ x \in Z\}$ is a (singular) one-dimensional complex manifold. Conversely, any (singular) one-dimensional complex manifold defines a holomorphic correspondence.

In our case, $h = \phi_2 \circ \phi_1^{-1}$ is a correspondence conjugating f_1 and f_2. Furthermore, if f_1 and f_2 are rational, then the graph of h is an irreducible algebraic set in $\hat{\mathbb{C}}^2$ (see Proposition 3). Hence we can say that f_1 *and* f_2 *are conjugate by* *an irreducible holomorphic correspondence*. In the case of (transcendental) entire maps, the graph can have a very wild shape, but we still require Z to be a Riemann surface (i.e. connected one-dimensional complex manifold). It is an open problem whether this relation is an equivalence relation or not. We can compose conjugacies as correspondences to get a conjugacy, but a composition of two irreducible holomorphic correspondences is not irreducible in general.

The proof of Theorem 1 depends on the "graph pushing-forward argument", which is a generalization of the classical extension technique by a functional equation. If we have a local conjugacy and one of the dynamics is univalent (like a linearization of a fixed point), we can extend the conjugacy from univalent dynamics to the other globally by using a functional equation.

Even when both dynamics have critical points, we can still extend a local conjugacy by pushing forward the conjugacy as a correspondence. The precise statement is given in Theorem 2.

Roughly speaking, consider the graph $\Gamma_0 \subset U_1 \times U_2$ of the analytic conjugacy ϕ and the product dynamics $F = (f_1, f_2)$. Then $\Gamma_n := F^n(\Gamma_0)$ forms an increasing sequence of local analytic sets. The union $\Gamma = \bigcup_n \Gamma_n$ is an invariant local analytic set, so we have a dynamics $F : \Gamma \to \Gamma$. By normalizing (desingularizing) Γ, we obtain a Riemann surface X and a dynamics $g : X \to X$ on it. We also naturally have semiconjugacies $\phi_i : X \to \hat{\mathbb{C}}$ from g to f_i. Since g has a chaotic dynamics coming from the original polynomial-like maps, X cannot be a hyperbolic Riemann surface. Hence we can classify X and study the dynamics g to obtain the theorem. Note that this proves that Γ is not only analytic but also algebraic in fact (Proposition 3). Furthermore, X is minimal in the sense that if there exist rational maps $\hat{g}$ and $\hat{\phi}_i$ such that $\hat{\phi}_i \circ \hat{g} = f_i \circ \hat{\phi}_i$, then $\hat{g}$ is semiconjugate to g (Proposition 6).

If we allow polynomial-like restrictions $f_i : U_i' \to U_i$ to have degree one, then they are simply neighborhoods of repelling fixed points and they are analytically conjugate if and only if those fixed points have the same multiplier. Our construction still works well in that case, but usually we have that $g : \mathbb{C} \to \mathbb{C}$ is a linear map and $\phi_1 : \mathbb{C} \to \hat{\mathbb{C}}$ and $\phi_2 : \mathbb{C} \to \hat{\mathbb{C}}$ are the inverses of linearizing coordinates for those fixed points. Thus ϕ_1 and ϕ_2 are transcendental. Further when f_1 and f_2 are rational maps, we have the following classification for a triple (f_i, g, ϕ_i) for each $i = 1, 2$:

(1) The conclusion of Theorem 1 as in the above examples.
(2) Power maps: $X = \mathbb{C}^*$ and $g(z) = z^{\pm d}$. If further we assume that ϕ_i is not a rational map, then f_i is an integral Lattès example.
(3) Chebyshev maps: $X = \mathbb{C}$ and $g(z + z^{-1}) = z^d + z^{-d}$. Similar to the case of power maps, if ϕ_i is not rational, f_i is an integral Lattès example.
(4) Linear maps on tori: $X = \mathbb{C}/\Lambda$ is a torus and $g([z]) = [\lambda z]$ for some λ. Then f_i is a Lattès example.

In particular, g is not transcendental even in this case. Note that the definition of the Chebyshev polynomial is slightly different from the usual definition (use $(z+z^{-1})/2$ instead of $z+z^{-1}$), but they are linearly conjugate and we use the above definition to simplify the notation. A *Lattès example* is a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that there exists a doubly periodic meromorphic function $\theta : \mathbb{C} \to \hat{\mathbb{C}}$ and an affine map $\ell(t) = at + b$ such that $\ell \circ \theta = f \circ \ell$. We say f is *integral* if $a \in \mathbb{Z}$. For more details, see [M] and [BE].

2. Constructing a complex manifold

Let R_1 and R_2 be Riemann surfaces and let $M = R_1 \times R_2$. In the following, we denote by $p_i : M \to R_i$ the projection to the i-th coordinate for $i = 1, 2$.

In this section, we prove the following desingularization theorem:

Theorem 2. *Let R_1 and R_2 be Riemann surfaces and let $M = R_1 \times R_2$. Let $f_i : R_i \to R_i$ be a holomorphic self-map on R_i for $i = 1, 2$, $\phi : U_1 \to U_2$ be a nonconstant holomorphic map between open sets $U_i \subset R_i$ and $\Gamma_0 = \{(z, \phi_z);\ z \in U_1\} \subset M$ be the graph of ϕ. Consider a forward invariant set*

$$\Gamma = \bigcup_{n \geq 0} F^n(\Gamma_0)$$

by the product map $F(z_1, z_2) = (f_1(z_1), f_2(z_2))$ on M. Then there exist a one-dimensional complex manifold X, possibly having infinitely many connected components, and holomorphic maps $g : X \to X$ and $\pi : X \to M$ such that $\pi(X) = \Gamma$ and $\pi \circ g = F \circ \pi$. Furthermore,

(1) *let us denote $\phi_i = p_i \circ \pi$. There exists an open set $U \subset X$ such that $\pi(U) = \Gamma_0$ and $\phi_i : U \to U_1$ is an isomorphism.*
(2) *If the degrees of f_1 and f_2 are finite, then $\deg g$ is not greater than $\deg f_1 \cdot \deg f_2$.*
(3) *If U_1 is connected and $\phi \circ f_1 = f_2 \circ \phi$ on some open set $U_1' \subset U_1$ with $f_1(U_1') \subset U_1$, then X is a Riemann surface (i.e., it is connected) and there exists an open set $U' \subset U$ such that $\phi_1(U') = U$, $g(U') \subset U$ and $\phi_1 \circ g = f_1 \circ \phi_1$ on U'.*

Note that Γ can be weird, especially when f_i is transcendental. It might have a lot of self-intersections and might accumulate to itself like (un)stable manifolds of two-dimensional dynamics, and so on. This theorem implies that we can still "desingularize" Γ.

Proof. First, observe that Γ_n is a one-dimensional local analytic set in M for each n. In fact, it is trivial for the graph Γ_0 of ϕ, and Γ_n is so because it is the image of a local analytic set by a proper map.

Let $\tilde{\Gamma}_k = \bigcup_{k=0}^n \Gamma_k$ and let $\pi_n : X_n \to \tilde{\Gamma}_k$ be the normalization (desingularization) of $\tilde{\Gamma}_n$; i.e., X_n is a one-dimensional complex manifold and $\pi_n : X_n \to M$ is a finite proper holomorphic map which is biholomorphic to its image except on a discrete set (see, e.g., [C, §6]). We have a natural inclusion $\iota_n : X_n \to X_{n+1}$ induced from the inclusion $\tilde{\Gamma}_k \subset \tilde{\Gamma}_k$, which is holomorphic by the removable singularity theorem. Similarly, since we have $F(\tilde{\Gamma}_k) \subset \tilde{\Gamma}_{k+1}$, there exists a holomorphic map $g_n : X_n \to X_{n+1}$ such that $\pi_{n+1} \circ g_n = F \circ \pi_n$. We can take the direct limit of $\{X_n\}$:

$$X = \varinjlim X_n.$$

Then X is a complex manifold and π_n and g_n induce holomorphic maps $\pi : X \to \Gamma \subset M$ (the normalization of Γ) and $g : X \to X$ such that $\pi \circ g = F \circ \pi$.

Let $U = \{[x] \in X;\ x \in X_0\}$. It is easy to see that U, X_0, Γ_0 and U_i are naturally isomorphic. This proves (1). The property (2) follows from the facts that π is one-to-one except on a discrete set and $\deg F = \deg f_1 \cdot \deg f_2$.

Under the assumption in (3), we have $\Gamma_{n+1} \cap \Gamma_n \supset F^{n+1}(\Phi(U_1'))$, where $\Phi(z) = (z, \phi(z))$ for $z \in U_1$. Therefore, $\tilde{\Gamma}_n$ is an irreducible local analytic set and it follows that X_n is connected. Since $\phi_1 : U \to U_1$ is an isomorphism by (1), it is easy to check that $U' = (\phi_1|_U)^{-1}(U_1')$ satisfies the desired property. □

Although we use the removable singularity theorem to prove the analyticity of g, we can check directly the analyticity of the map g (or g_n) at a singular point. For a point $x_0 \in X_n$, there is a neighborhood $V \subset X_n$ of x_0 such that π_n is biholomorphic on $V \setminus x_0$. Take a point $z_0 \in U_1$ and $0 \leq k \leq n$ such that there exists a neighborhood V_1 of z_0 such that $F^k(\Phi(z_0)) = \pi_n(x_0)$ and

$$F^k \circ \Phi : (V_1 \setminus z_0) \to F^k \circ \Phi(V_1) \subset \pi_n(V \setminus x_0)$$

is a proper map. Let a_1 be the local degree of f_1^k at z_0. Then there exist conformal maps ψ_0 and ψ_1 defined near z_0 and $f_1^k(z_0)$, respectively, such that $\psi_0(z_0) = \psi_1(f_1^k(z_0)) = 0$ and

$$\psi_1 \circ f_1^k \circ \psi_0^{-1}(w) = w^{a_1}.$$

Therefore, by the equality $F^k \circ \Phi(z) = (f_1^k(z), f_2^k(\phi(z)))$, there exists some $b_1 > 0$ dividing a_1 such that $F^k \circ \Phi(\psi_0^{-1}(w)) = F^k \circ \Phi(\psi_0^{-1}(\tilde{w}))$ if and only if $w^{b_1} = \tilde{w}^{b_1}$. Thus it follows that w^{b_1} gives a local coordinate (complex structure) of X in a neighborhood of x_0.

Similarly, there exists a conformal map ψ_2 at $f_1^{k+1}(z_0)$ such that $\psi_2(f_1^{k+1}(z_0)) = 0$ and $\psi_2 \circ f_1^{k+1} \circ \psi_0^{-1}(w) = w^{a_2}$ where $a_2 = \deg_{z_0}(f_1^{k+1})$ by replacing ψ_0 if necessary and w^{b_2} gives a local coordinate in a neighborhood of $g(x_0)$ for some b_2 which divides a_2. Thus $w^{b_1} \mapsto w^{b_2}$ is holomorphic because b_1 divides b_2. This proves that g is holomorphic at x_0.

3. Proof of the theorem

3.1. **Basic classification.** Now assume that f_1 and f_2 are rational maps or entire maps, and $f_1 : U_1' \to U_1$ and $f_2 : U_2' \to U_2$ are polynomial-like mappings of degree $d' \geq 2$ with analytic conjugacy $\phi : U_1 \to U_2$. By Theorem 2, there exists a Riemann surface X and a holomorphic map $g : X \to X$ such that $\phi_i \circ g = f_i \circ \tilde{\pi}_i$.

Furthermore, U in the conclusion of Theorem 2 is naturally isomorphic to U_i under ϕ_i. Namely, g has a polynomial-like restriction analytically conjugate to $f_i : U_i' \to U_i$. This implies that g has a chaotic dynamics (for example, g has a repelling periodic point). Hence X cannot be hyperbolic. Therefore, X is conformally isomorphic to either the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$, the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, or a torus $\mathbb{C}/\Lambda$. Therefore, g is

(1) a rational map if $X \cong \hat{\mathbb{C}}$,
(2) an entire map if $X \cong \mathbb{C}$,
(3) $g(z) \cong z^{\pm \deg g} e^{h(z)}$ for some holomorphic map $h : \mathbb{C}^* \to \mathbb{C}$ if $X \cong \mathbb{C}^*$, and
(4) a linear map if X is a torus.

However, if X is a torus, g cannot have a polynomial-like restriction of degree $d' \geq 2$, so it is not the case.

Since U_i is an open set intersecting the Julia set of f_i, we have

$$\phi_i(X) = \bigcup_n f_i^n(U_i) \supset \hat{\mathbb{C}} \setminus \mathcal{E}_i,$$

where $\mathcal{E}_i$ is the exceptional set for f_i. Hence we may assume either

(1) $\phi_i(X) = \hat{\mathbb{C}}$,
(2) $\phi_i(X) = \mathbb{C}$ and f_i is an entire map, or
(3) $\phi_i(X) = \mathbb{C}^*$.

Hence for each i, we can divide into cases according to X and $\phi_i(X)$.

3.2. **The case of rational maps.** First, we consider the case of rational maps (including polynomials), i.e., $d_i = \deg f_i < \infty$. Then $d = \deg g$ is also finite by Theorem 2. Hence if $X \cong \mathbb{C}^*$, then $g(z) = z^{\pm d}$ does not have a polynomial-like restriction in $\mathbb{C}^*$. Hence we may assume $X = \hat{\mathbb{C}}$ or $\mathbb{C}$.

Similarly, nor is $\phi_i(X) = \mathbb{C}^*$ the case. Thus we need only consider the cases $\phi_i(X) = \hat{\mathbb{C}}$ and $\phi_i(X) = \mathbb{C}$.

Case I. $\phi_i(X) = \hat{\mathbb{C}}$ and $X = \hat{\mathbb{C}}$. Then clearly $\phi_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map and we are done.

Case II. $\phi_i(X) = \hat{\mathbb{C}}$ and $X = \mathbb{C}$. A priori, ϕ_i can be transcendental. However, Eremenko [Er] proved that ϕ_i is transcendental if g is either a power map or a Chebyshev map, and f_i is an integral Lattès example. Since any integral Lattès example does not have a polynomial-like restriction, this cannot happen. Thus ϕ_i is a polynomial. In particular, we have $\phi_i(X) = \mathbb{C}$, so Case II does not occur indeed.

Case III. $\phi_i(X) = \mathbb{C}$ and $X = \hat{\mathbb{C}}$. Then ϕ_i must be a constant, which is a contradiction. Therefore, we do not have this case either.

Case IV. $\phi_i(X) = \mathbb{C}$ and $X = \mathbb{C}$. Again by [Er], ϕ_i cannot be a transcendental entire function, so ϕ_i is a polynomial.

In any case, ϕ_i is a rational map. Moreover, if f_1 is a polynomial, then we have $\phi_1(X) = \mathbb{C}$ by construction. This implies that $X = \mathbb{C}$ and $\phi_2(X) = \mathbb{C}$. This proves Theorem 1 for rational maps.

Proposition 3. *Under the assumption of Theorem* 1*, if* f_1 *and* f_2 *are rational maps, then* $\overline{\Gamma} \subset \hat{\mathbb{C}}^2$ *is an irreducible algebraic set.*

Proof. First, observe that $\Gamma = \{(\phi_1(z), \phi_2(z));\ z \in X\}$. Since ϕ_1 and ϕ_2 are rational maps (including polynomials), we have

$$\overline{\Gamma} = \{(\phi_1(z), \phi_2(z);\ z \in \hat{\mathbb{C}}\},$$

and it is an irreducible algebraic set. □

3.3. **The case of (transcendental) entire maps.** Now consider the case $f_1 : \mathbb{C} \to \mathbb{C}$ is a transcendental entire map. By the same argument as in Case III above, X cannot be isomorphic to the Riemann sphere. However, we cannot exclude the case $X \cong \mathbb{C}^*$:

Example. Consider the (complexified) Arnold family $f_{a,b}(x) = x + a + b\sin(2\pi x)$ defined on $\mathbb{C}/\mathbb{Z}$. It is conjugate to $g_{a,b} : \mathbb{C}^* \to \mathbb{C}^*$ defined by

$$g_{a,b}(z) = \lambda z \exp\left(-\frac{bi}{2}\left(z - \frac{1}{z}\right)\right),$$

where $\lambda = e^a$. When $a = 0$, $f_{0,b}$ is odd; $f_{0,b}(-x) = -f_{0,b}(x)$. Therefore, $g_{0,b}(1/z) = 1/g_{0,b}(z)$ and $g_{0,b}$ is semiconjugate to an entire map

$$h_b(w) = w\cos\left(b\sqrt{w^2-1}\right) - i\sqrt{w^2-1}\sin\left(b\sqrt{w^2-1}\right).$$

Observe that $\cos z$ and $z\sin z$ are entire functions of z^2.

Even when $X \cong \mathbb{C}^*$, we can get a semiconjugacy from an entire map to g by taking a proper lift to the universal covering so that it still has a polynomial-like restriction. Hence we need only consider the case $X = \mathbb{C}$ and g is an entire map.

However, as we stated in the Introduction, we do not know whether g is transcendental or not, nor whether f_2 can be a polynomial or even a rational map.

We conclude this section by the following proposition and its application to Theorem 1.

Proposition 4 (Bergweiler)**.** *A transcendental entire map f cannot be semiconjugate to a polynomial P by an entire map g.*

Corollary 5. *Let f_1 be a polynomial and f_2 be a (transcendental) entire map. Assume f_1 and f_2 satisfy the assumption of Theorem* 1*. Then the entire map g in the conclusion is a polynomial.*

This proposition will be proved by comparing the speed of divergence at infinity. Let us recall some notation and facts before the proof. For an entire map φ, let us denote for $r > 0$,

$$M(r,\varphi) = \max_{|z|=r} |\varphi(z)|.$$

Then we have

(1) $\log M(r,\varphi)$ is an increasing convex function of $\log r$ (Hadamard three-circle theorem).

(2) $\dfrac{\log M(r,\varphi)}{\log r}$ tends to infinity if and only if φ is transcendental.

(3) Let f, g be entire maps and let $a = f(0)$. There exists a constant $K > 0$ such that

$$M(r, g\circ f) \geq M\left(KM\left(\frac{1}{2}r, f\right), g\right)$$

for sufficiently large $r > 0$. (See, e.g., [Ha].)

Proof. We prove by contradiction. Assume $g \circ f = P \circ g$ for a polynomial P, a transcendental entire map f and an entire map g. Then g must be transcendental also. Let $d = \deg P$. Then there exists some $c > 0$ such that

$$\log M(r, P\circ g) \leq d\log M(r,g) + c$$

for sufficiently large $r > 0$. By assumption and fact (3) above, we have

$$\log M(r_1, g) \leq d\log M(r,g) + c,$$

where $r_1 = KM\left(\frac{1}{2}r, f\right)$. In particular, $\log M(r_1,g)/\log M(r,g)$ is bounded above.

On the contrary, we also have

$$\frac{\log M(r_1,g)}{\log M(r,g)} = \left(\frac{\log M(r_1,g)}{\log r_1}\Big/\frac{\log M(r,g)}{\log r}\right)\cdot\frac{\log r_1}{\log r}.$$

Let $r_0 > 0$. By (1), there exists some $C > 0$ such that for any $r > r_0$,

$$\begin{aligned}\frac{\log M(r,g)}{\log r} &\le \frac{\log r_1 - \log r}{\log r_1 - \log r_0}\frac{\log M(r_0,g)}{\log r} + \frac{1-\frac{\log r_0}{\log r}}{1-\frac{\log r_0}{\log r_1}}\frac{\log M(r_1,g)}{\log r_1} \\ &\le \frac{\log M(r_0,g)}{\log r} + \frac{\log M(r_1,g)}{\log r_1};\end{aligned}$$

hence $\left(\frac{\log M(r_1,g)}{\log r_1}\Big/\frac{\log M(r,g)}{\log r}\right)$ is bounded from below for sufficiently large r by (2). Since $\frac{\log r_1}{\log r}$ tends to infinity by (2), it follows that $\frac{\log M(r_1,g)}{\log M(r,g)}$ tends to infinity, so this is a contradiction. □

3.4. **Minimality of g.** We conclude this section by showing the minimality of g:

Proposition 6. *The g constructed above is minimal. More precisely, if rational or entire maps $\hat{g}$ and $\hat{\phi}_i$ satisfy $\hat{\phi}_i \circ \hat{g} = f_i \circ \hat{\phi}_i$ for $i = 1, 2$ and there is a well-defined branch of $\hat{\phi}_1^{-1}$ on U_1 such that $\hat{\phi}_2 \circ \hat{\phi}_1^{-1}$ is equal to the original analytic conjugacy ϕ, then there exists a rational or entire map ψ such that $\psi \circ \hat{g} = g \circ \psi$ and $\hat{\phi}_i = \phi_i \circ \psi$.*

Proof. First assume that X in the proof of Theorem 1 is not isomorphic to $\mathbb{C}^*$. Let $\hat{X} = \hat{\mathbb{C}}$ or $\mathbb{C}$ according to whether $\hat{g}$ is rational or entire (a polynomial or a transcendental entire map). Roughly speaking (or when $\hat{g}$ is not transcendental), a map

$$\begin{array}{cccc}\tilde{\psi}: & \hat{X} & \to & \hat{\mathbb{C}} \times \hat{\mathbb{C}} \\ & z & \mapsto & (\hat{\phi}_1(z), \hat{\phi}_2(z))\end{array}$$

gives a map from $\hat{X} \to \Gamma$ (or $\overline{\Gamma}$), so we can get a semiconjugacy $\hat{\phi} = \pi^{-1} \circ \tilde{\psi}$ via the normalization $\pi : X \to \Gamma$.

More precisely, by assumption, there exists a topological disk $\hat{U} \subset \hat{X}$ such that $\hat{\phi}_1 : \hat{U} \to U_1$ is an isomorphism and $\hat{\phi}_i \circ \hat{g} = f_i \circ \hat{\phi}_i$ on $\hat{U}' = \hat{g}^{-1}(\hat{U}) \cap \hat{U}$ for $i = 1, 2$. Then $\tilde{\psi}(\hat{U})$ is equal to the graph of ϕ. Therefore $\psi_0 = \pi^{-1} \circ \tilde{\psi} : \hat{U} \to U = X_0$ is a conjugacy between $\hat{g}$ and g. Similarly, we can define a holomorphic semiconjugacy $\psi_n = \pi_n^{-1} \circ \tilde{\psi} : \hat{g}^n(\hat{U}) \to g^n(U) = X_n$ (where π_n and X_n are in the proof of Theorem 2) and by taking the direct limit, we obtain a holomorphic semiconjugacy $\psi = \lim_{\longrightarrow} \psi_n : \hat{X} \to X$ with $\pi \circ \psi = \tilde{\psi}$. It is easy to check that ψ has the desired property.

Even when $X \cong \mathbb{C}^*$, the same construction works onto $\mathbb{C}^*$ and by taking the lift to the universal cover which analytically conjugates the corresponding polynomial-like restrictions, we can get the same conclusion. □

4. The case of degree one polynomial-like mappings

Here we consider the case when $f_i : U_i' \to U_i$ $(i = 1, 2)$ are polynomial-like mappings of degree one. By the Schwarz lemma, there exists a repelling fixed point $x_i \in U_i'$ for f_i and the forward orbit of any $z \in U_i'$ except x_i escapes from U_i'. Now if $f_1 : U_1' \to U_1$ and $f_2 : U_2' \to U_2$ are analytically conjugate, then $f_1'(x_1) = f_2'(x_2)$. On the contrary, if $\lambda = f_1'(x_1) = f_2'(x_2)$, then let ψ_i be the inverse of a linearizing coordinate, i.e., a holomorphic map defined near the origin such that $\psi_i(0) = x_i$, $\psi_i'(0) \neq 0$ and $\psi_i(\lambda x) = f_i(\psi_i(x))$. Then $\psi_2 \circ \psi_1^{-1}$ gives an analytic conjugacy between the polynomial-like restrictions. Furthermore, it is well known that we can extend the domain of definition of ψ_i to the whole complex plane. This implies that we can always take $X = \mathbb{C}$, $g(z) = \lambda z$ and $\phi_i = \psi_i$ in the conclusion of Theorem 1. Namely, there exists a linear map $z \mapsto \lambda z$ which is semiconjugate to both f_1 and f_2.

However, we may still apply our construction of $g : X \to X$ which is semiconjugate to both f_1 and f_2 in this case to get the minimal g. Since we still have expanding dynamics, X is still parabolic or elliptic. We can have all the cases above because if two polynomial-like maps of degree greater than one are analytically conjugate and have repelling fixed points, then we get analytically conjugate polynomial-like maps of degree one by restricting them to some neighborhoods of the fixed points.

However, we have some other cases because we cannot exclude the case when $\deg g = 1$, nor the case when there do not exist any polynomial-like restrictions (linear maps on tori, Lattès maps, g or f_i are equal to $z^{\pm d} : \mathbb{C}^* \to \mathbb{C}^*$) in the proof. When $\deg g = 1$, then X must be isomorphic to $\mathbb{C}$, as we have already seen (i.e., $\psi_i = \phi_i$ is the inverse of the linearizing coordinate).

When f_i is a rational map, we can classify the rest of the cases for a triple (f_i, g, ψ_i) as follows:

- If $g : X \to X$ is a linear map on a torus, then f_i is a Lattès map.
- If $g(z) = z^{\pm d}$ and $X = \mathbb{C}^*$, then f_i can be either
  - a power map ($\phi_i(z) = z$),
  - a Chebyshev map $\phi_i(z) = z + 1/z$, or
  - an integral Lattès example (ϕ_i is transcendental).
- If $\phi_i(X) = \mathbb{C}^*$, then $f_i(z) = z^{\pm d_i}$ and g is either a power map ($g = f_i$, $X = \mathbb{C}^*$) or a linear map $g(z) = \pm d_i z$, $X = \mathbb{C}$).
- If $X = \mathbb{C}$ and $\phi_i(X) = \hat{\mathbb{C}}$, then ϕ_i is transcendental when g is Chebyshev and f_i is an integral Lattès example. Note that if g is a power map, then it is not semiconjugate to a Chebyshev map or an integral Lattès example by our construction because $0 \in X = \bigcup_{n \geq 0} V'$, which implies that the superattractive fixed point 0 lies in the domain of definition of the polynomial-like restriction $g : V' \to V$.

Appendix A. Semiconjugacies of polynomials

In this Appendix, we give a classification of semiconjugate polynomials. It is based on the results by Ritt [R] and Engstrom [En] on decompositions of polynomials in terms of composition. Let $\mathcal{S}$ be the set of all affine conjugacy classes of triples (f, g, h) of polynomials of degree at least two such that $f \circ h = h \circ g$, where we say that two triples (f_1, g_1, h_1) and (f_2, g_2, h_2) are *affinely conjugate* if there exist affine maps σ_1, σ_2 such that

$$f_2 = \sigma_1 \circ f_1 \circ \sigma_1^{-1},$$
$$g_2 = \sigma_2 \circ g_1 \circ \sigma_2^{-1},$$
$$h_2 = \sigma_1 \circ h_1 \circ \sigma_2^{-1},$$

and we denote $(f_1, g_1, h_1) \sim (f_2, g_2, h_2)$. The aim of this Appendix is to classify $[(f, g, h)] \in \mathcal{S}$.

Theorem 7. *Let $[(f,g,h)] \in \mathcal{S}$. If $\gcd(\deg f, \deg h) = d > 1$, then there exist polynomials $g_1, h_1, f_1, \hat{h}_1, \alpha_1$ and β_1 such that*

$$f \circ h_1 = h_1 \circ g_1, \qquad f_1 \circ \hat{h}_1 = \hat{h}_1 \circ g, \qquad h = h_1 \circ \beta_1 = \alpha_1 \circ \hat{h_1},$$
$$\deg f_1 = \deg g_1 = \deg f, \quad \deg \alpha_1 = \deg \beta_1 = d, \quad \deg h_1 = \deg \hat{h_1} = \deg h/d.$$

In particular, if $d < \deg h$, then $[(f, g_1, h_1)],\ [(f_1, g, \hat{h}_1)] \in \mathcal{S}$.

This theorem implies that we need only consider the case when $\deg f = \deg g$ and $\deg h$ are coprime. More precisely, for a given $[(f_0, g_0, h_0)] \in \mathcal{S}$, we can construct a sequence $\{(f_0, g_n, h_n)\}_{n=1,\dots,N}$ and $\{[(f_n, g_0, \hat{h}_n)]\}_{n=1,\dots,N}$ by applying this theorem repeatedly until we have $\gcd(\deg f_0, \deg h_N) = 1$ and $\gcd(\deg g_0, \deg \hat{h}_N) = 1$. Furthermore, if $\deg h_N = \deg \hat{h}_N > 1$, then $[(f_0, g_N, h_N)]$ and $[(f_N, g_0, \hat{h}_N)]$ lie in $\mathcal{S}$. Otherwise, f and g_N, and f_N and g are affinely conjugate.

Example. Let us consider the example (4) in the Introduction. Namely, let $f_1(z) = z(c+z^2)^3$ and $f_2(z) = z(c+z^3)^2$. Since $\deg f_1 = \deg f_2 = 7$ is a prime number, they are prime polynomials (i.e., f_i cannot be decomposed into a composition of polynomials of degree at least two).

Assume there exists a polynomial h such that $f_1 \circ h = h \circ f_2$. If $\deg h$ is divisible by 7, then by Theorem 7, h can be written as

$$h = f_1 \circ h_1 = h_1 \circ g_1$$

for some h_1 and $\hat{h}_1$. Repeating this argument, we can write $h = f_1^n \circ h_n = \hat{h}_n \circ f_2^n$ for some $n \geq 0$, h_n and $\hat{h}_n$ such that $\deg h_n$ and $\deg f_1 = \deg f_2$ are coprime. However, Theorem 8 below implies that such an h_n exists if and only if $c = 0$.

Here we give a complete classification for the case $\gcd(\deg f, \deg h) = 1$.

Theorem 8. *Assume $[(f,g,h)] \in \mathcal{S}$ satisfies $\gcd(\deg f, \deg h) = 1$. Then there exists a representative (f_0, g_0, h_0) of $[(f,g,h)]$ such that either*

- *$f(z) = z^c P^b(z)$, $g(z) = z^c P(z^b)$ and $h(z) = z^b$, where $c \equiv a \mod b$, or*
- *$f = g = T_a$, $h = T_b$ are Chebyshev polynomials,*

where $a = \deg f (= \deg g)$ and $b = \deg h$.

To prove these theorems, we need the following two facts on decompositions of polynomials with respect to composition. The first one is proved by Ritt [R]:

Theorem 9 (The second Ritt theorem)**.** *Let $\varphi_1, \varphi_2, \psi_1$ and ψ_2 be polynomials of degree at least two such that*

$$F = \varphi_1 \circ \varphi_2 = \psi_1 \circ \psi_2.$$

Assume $a = \deg \varphi_1 = \deg \psi_2$ and $b = \deg \varphi_2 = \deg \psi_1$ are coprime and $a > b$. Then there exist affine maps $\mu, \nu, \sigma_1, \sigma_2$ such that

$$\hat{\varphi}_1 = \mu \circ \varphi_1 \circ \sigma_1^{-1}, \qquad \hat{\varphi}_2 = \sigma_1 \circ \varphi_2 \circ \nu,$$
$$\hat{\psi}_1 = \mu \circ \psi_1 \circ \sigma_2^{-1}, \qquad \hat{\psi}_2 = \sigma_2 \circ \psi_2 \circ \nu$$

satisfy either

(1) $\hat{\varphi}_1 = \hat{\psi}_2 = T_a$ *and* $\hat{\varphi}_2 = \hat{\psi}_1 = T_b$*, where* T_n *is the Chebyshev polynomial of degree* n*, or*

(2) *let* $0 < c < b$ *satisfy* $a \equiv c \mod b$*. There exists some polynomial* P *such that* $\hat{\varphi}_2(z) = \hat{\psi}_1(z) = z^b$ *and*

$$\hat{\varphi}_1(z) = z^c(P(z))^b, \qquad \hat{\psi}_2(z) = z^c P(z^b).$$

Remark 10. In the second Ritt theorem, both cases of the conclusion hold only if $b = 2$. In particular, if we replace the conclusion (1) by

(1′) $b \geq 3$, $\hat{\varphi}_1 = \hat{\psi}_2 = T_a$ and $\hat{\varphi}_2 = \hat{\psi}_1 = T_b$,

then exactly one of the conclusions (1′) or (2) holds.

The second one is proved by Engstrom [En], which is a stronger version of "the first Ritt theorem" [R]:

Theorem 11. *Assume* $\varphi_1, \varphi_2, \psi_1, \psi_2$ *are nonconstant polynomials such that*

$$F = \varphi_1 \circ \varphi_2 = \psi_1 \circ \psi_2.$$

Then there exist polynomials $\alpha, \beta, \hat{\varphi}_1, \hat{\varphi}_2, \hat{\psi}_1, \hat{\psi}_2$*, such that*

$$\varphi_1 = \alpha \circ \hat{\varphi}_1, \qquad \psi_1 = \alpha \circ \hat{\psi}_1, \qquad \deg \alpha = \gcd(\deg \varphi_1, \deg \psi_1),$$
$$\varphi_2 = \hat{\varphi}_2 \circ \beta, \qquad \psi_2 = \hat{\psi}_2 \circ \beta, \qquad \deg \beta = \gcd(\deg \varphi_2, \deg \psi_2).$$

Indeed, we can say more:

Corollary 12. *In the conclusion of Theorem* 11*, we can also add the condition* $\hat{\phi}_1 \circ \hat{\phi}_2 = \hat{\psi}_1 \circ \hat{\psi}_2$.

Proof. Assume the conclusion of Theorem 11. Then we have $\alpha \circ \hat{\phi}_1 \circ \hat{\phi}_2 \circ \beta = \alpha \circ \hat{\psi}_1 \circ \hat{\psi}_2 \circ \beta$. Since β is surjective, we have $\alpha \circ \hat{\phi}_1 \circ \hat{\phi}_2 = \alpha \circ \hat{\psi}_1 \circ \hat{\psi}_2$. By applying Theorem 11 again to the decompositions $\alpha \circ (\hat{\phi}_1 \circ \hat{\phi}_2) = \alpha \circ (\hat{\psi}_1 \circ \hat{\psi}_2)$, we can see that there exist polynomials Φ, Ψ, β_1 such that

$$\hat{\phi}_1 \circ \hat{\phi}_2 = \Phi \circ \beta_1, \qquad \hat{\psi}_1 \circ \hat{\psi}_2 = \Psi \circ \beta_1$$

and

$$\deg \beta_1 = \gcd(\deg \hat{\phi}_1 \circ \hat{\phi}_2, \hat{\psi}_1 \circ \hat{\psi}_2) = \deg \hat{\phi}_1 \circ \hat{\phi}_2 = \deg \hat{\psi}_1 \circ \hat{\psi}_2.$$

In particular, Φ and Ψ are affine. Therefore, we have

$$\hat{\phi}_1 \circ \hat{\phi}_2 = \Phi \circ \Psi^{-1} \circ \hat{\psi}_1 \circ \hat{\psi}_2.$$

It is easy to check that the conclusion holds if we replace $\hat{\psi}_1$ by $\Phi \circ \Psi^{-1} \circ \hat{\psi}_1$. □

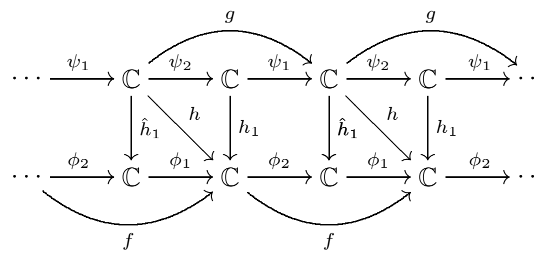
Now we prove Theorem 7. First of all, the following lemma easily follows from Corollary 12:

Lemma 13. *Let* $[(f, g, h)] \in \mathcal{S}$*. Then there exist polynomials* $\phi_1, \phi_2, \psi_1, \psi_2, h_1, \hat{h}_1$ *such that*

$$\begin{aligned} &f = \phi_1 \circ \phi_2, && g = \psi_1 \circ \psi_2, \\ &h = \phi_1 \circ \hat{h}_1 = h_1 \circ \psi_2, && \phi_2 \circ h_1 = \psi_1 \circ \hat{h}_1, \\ &\deg \phi_1 = \deg \psi_2 = \gcd(\deg f, \deg h), && \deg \psi_1 = \deg \phi_2, \\ &\deg h_1 = \deg \hat{h}_1. \end{aligned} \tag{2}$$

In particular, the following diagram commutes:

(3)



and if $\deg h_1 \geq 2$*, we have* $[(f, g_1, h_1)], [(f_1, g, \hat{h}_1)] \in \mathcal{S}$ *where* $f_1 = \phi_2 \circ \phi_1$ *and* $g_1 = \psi_2 \circ \psi_1$*.*

Theorem 7 is an immediate consequence of the diagram (3).

Proof of Theorem 8. Let $[(f, g, h)] \in \mathcal{S}$ with $\gcd(\deg f, \deg h) = 1$. By the second Ritt theorem, there exist affine maps $\mu, \nu, \sigma_1, \sigma_2$ such that

$$\hat{\varphi}_1 = \mu \circ f \circ \sigma_1^{-1}, \qquad \hat{\varphi}_2 = \sigma_1 \circ h \circ \nu,$$
$$\hat{\psi}_1 = \mu \circ h \circ \sigma_2^{-1}, \qquad \hat{\psi}_2 = \sigma_2 \circ g \circ \nu$$

satisfy the conclusion of Theorem 9 (note that we do not know whether $a = \deg f > b = \deg h$ or not). By taking an affine conjugacy, we may further assume that μ and σ_2 are the identity, so it follows that $h = \hat{\psi}_1$.

First assume $a > b$. Then we have

$$\hat{\varphi}_2 = \hat{\psi}_2 = h = \sigma_1^{-1} \circ \hat{\varphi}_2 \circ \nu$$

are either the power map or the Chebyshev polynomial of degree b. If they are Chebyshev, then ν and σ_1 are also equal to the identity and we are done. If they are power maps, then $\nu(z) = \nu \cdot z$ and $\sigma_1(z) = \sigma \cdot z$ are linear maps with $\sigma^b = \nu$. Let us denote $\hat{\phi}_1(z) = z^c(P(z))^b$ and $\hat{\psi}_2(z) = z^c P(z^b)$. Then $f(z) = z^c(P_1(z))^b$ and $g(z) = z^c(P_1(z^b))$, where $P_1(z) = \sigma^c P(\sigma^b z)$, so we are also done.

Now assume $a < b$. If $\hat{\phi}_i$ and $\hat{\psi}_i$ are Chebyshev, then we have the same conclusion as above. So the remaining case is the following:

(3) Let $0 < c < a$ satisfy $b \equiv c \mod a$. There exists some polynomial P such that $\hat{\varphi}_1(z) = \hat{\psi}_2(z) = z^a$ and

$$\hat{\varphi}_2(z) = z^c(P(z))^a, \qquad \hat{\psi}_1(z) = z^c P(z^a).$$

Even in this case, by considering $(f^{\circ n}, g^{\circ n}, h)$ for sufficiently large n, we have $\deg f^{\circ n} = a^n > b$, and $\gcd(a^n, b) = 1$. Thus again we can apply the second Ritt theorem and h has the form

$$h(z) = A(z + B)^b + C.$$

(Observe that h is not Chebyshev because then f and g must be Chebyshev.) Furthermore, $h(z)$ has a symmetry: For an a-th root of unity ω, we have $h(\omega z) = \omega^c h(z)$ $(= \omega^b h(z))$. This implies $B = C = 0$, i.e., $h(z) = Az^b$. Hence $P(z) = Az^{(b-c)/a}$ and $\hat{\phi}_2(z) = Az^b = h(z)$. Therefore, the triple (f, g, h) is affinely conjugate to (z^a, z^b, z^a), so it is the first case of the conclusion of Theorem 8. $\square$

A generalization of Ritt's theorem for rational maps is an open problem. Hence we cannot classify semiconjugacies of rational maps. A generalization to Laurent polynomials was recently done by Zieve [Z].

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