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RATIONALITY AND RECIPROCITY FOR THE GREEDY NORMAL FORM OF A COXETER GROUP

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Abstract. We show that the characteristic series for the greedy normal form of a Coxeter group is always a rational series and prove a reciprocity formula for this series when the group is right-angled and the nerve is Eulerian. As corollaries we obtain many of the known rationality and reciprocity results for the growth series of Coxeter groups as well as some new ones.

1. Introduction

Background. Let G be a finitely generated group with a finite generating set S. The *growth series* of G relative to S is the power series

$$\gamma(t) = \sum_{g \in G} t^{|g|},$$

where $|g|$ denotes the word length of g with respect to S. Growth series are an important measure of complexity and size for infinite groups. If G satisfies certain algorithmic properties (for example, has a geodesic normal form recognized by a finite state automaton), then γ is known to be a rational function. Moreover, if G is a discrete group acting geometrically on a manifold or cell complex M and S is suitably related to the geometry of M, then one often finds interesting connections with the topology of M. Two notable examples are cases where (1) special values of $\gamma(t)$ are related to the Euler-Poincaré characteristic of M or G and (2) reciprocity formulas for γ (see, e.g., [6, 10, 22] for Coxeter groups, [11] for Fuchsian groups). For example, in the case where W is a Coxeter group and S is the standard generating set, Serre [22] showed that $\boldsymbol{\chi}(W) = 1/\gamma(1)$ and that for affine Coxeter groups, $\gamma(1/t) = \pm\gamma(t)$.

To describe more general formulas involving Coxeter groups, we recall some standard terminology and constructions. Let W be a Coxeter group with standard generating set S. Given a subset $\sigma \subset S$, let W_σ denote the *parabolic subgroup* of W generated by σ (by convention $W_\emptyset = \{1\}$). Let $N = N(W, S)$ be the set of all *spherical subsets* $\sigma \subset S$; i.e., N consists of all σ such that W_σ is finite. Each such finite W_σ is itself a Coxeter group and has a unique element of longest length, which we denote by w_σ. The collection N, partially ordered by inclusion, is an abstract simplicial complex, called the *nerve* of W. Let $W\mathcal{S}$ denote the collection of cosets $\{wW_\sigma \mid w \in W,\ \sigma \in N\}$ with partial order defined by inclusion. The *Davis complex*

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[7] associated to (W, S) is the geometric realization of $W\mathcal{S}$ and is usually denoted by Σ. It is a contractible complex on which W acts properly with finite stabilizers.

In [6], Charney and Davis generalized the reciprocity formula for affine Coxeter groups, showing that $\gamma(1/t) = \pm\gamma(t)$ whenever the nerve N is an *Eulerian sphere*, i.e., has the property that the link of every simplex σ has the Euler characteristic of a sphere (of the appropriate dimension). More recently, Dymara [10] has shown that for any positive real number q, one has $\boldsymbol{\chi}_2^q(\Sigma) = 1/\gamma(q)$, where $\boldsymbol{\chi}_2^q(\Sigma)$ denotes the "q-weighted" ℓ_2-Euler characteristic of Σ. When Σ is a (generalized homology) manifold, he also obtains the reciprocity formula $\gamma(1/q) = \pm\gamma(q)$ as a consequence of Poincaré duality for weighted ℓ_2-cohomology.

It was observed by Serre that, for Coxeter groups, there is a natural extension of the growth series to a multivariable series $\gamma(\mathbf{t})$ where $\mathbf{t}$ is a tuple of indeterminants, one for each conjugacy class of generators in S. The ℓ_2-Euler characteristic and the reciprocity formulas above also have natural interpretations in terms of this multivariable series $\gamma(\mathbf{t})$ (see [8, 9]).

Rationality results. Although explicit rational formulas for the growth series of a Coxeter group were known to Steinberg [24], rationality itself follows on general principles from the fact that Coxeter groups are automatic (the relevant property of an automatic group is that it has a normal form that is recognized by a finite state automaton). The proof of automaticity for Coxeter groups was completed by Brink and Howlett in [3]. However, in the special case of right-angled Coxeter groups, automaticity (in fact, bi-automaticity) also follows from the more recent results of Niblo and Reeves [18] and the fact that the Davis complex is a CAT(0) cube complex. The normal form for groups acting on cube complexes turns out to be different from the normal form recognized by the automaton in [3] and is in some sense more canonical (see Section 8, below). This canonical normal form for right-angled Coxeter groups is a special case of a normal form that makes sense for any Coxeter group. For each element $w \in W$, this normal form specifies a unique product representation of the form $w = w_{\sigma_n} \cdots w_{\sigma_1}$ where each letter w_{σ_i} is an element of longest length in the (finite) parabolic subgroup W_{σ_i}. The left-most letter w_σ appearing in the representation for w has the property that σ is the largest subset such that

$$|w| = |w_\sigma| + |w_\sigma^{-1} w|.$$

For this reason, we call this normal form the *(left) greedy normal form* and denote it by $\mathcal{L}(W, S)$.

The main purpose of this article is to describe rationality and reciprocity theorems for $\mathcal{L}(W, S)$. To avoid confusion between words in $\mathcal{L}(W, S)$ and group elements in W, we let $\mathcal{A}^*$ denote the free monoid on the set $\mathcal{A} = \{\sigma \in N \mid \sigma \neq \emptyset\}$ and regard $\mathcal{L}(W, S)$ as a subset of $\mathcal{A}^*$ (in other words, $\mathcal{L}(W, S)$ is a *language* with respect to the *alphabet* $\mathcal{A}$). Given any language $\mathcal{L} \subset \mathcal{A}^*$, we let $\chi_\mathcal{L}$ denote the *characteristic series* defined by

$$\chi_\mathcal{L} = \sum_{\alpha \in \mathcal{L}} \alpha.$$

We regard it as a formal power series over $\mathbb{Q}$ (with the elements of $\mathcal{A}$ representing noncommuting formal variables). Such a series is *rational* if it can be obtained from a finite set of polynomials using a finite sequence of additions, multiplications and quasi-inversions (see Section 3). It is a classical result in automata theory that $\mathcal{L}$ is a regular language (i.e., recognized by a finite state automaton) if and only if $\chi_{\mathcal{L}}$ is rational (see, e.g., [21]). Thus, it follows from Niblo and Reeves' proof of automaticity for groups acting on CAT(0) cube complexes, that in the case of a right-angled Coxeter group the characteristic series for the greedy normal form $\mathcal{L}(W,S)$ is rational. More generally, we prove the following.

Theorem 1. *Let W be any Coxeter group and let $\mathcal{L} = \mathcal{L}(W,S)$ be its greedy normal form. Then the characteristic series $\chi_{\mathcal{L}}$ is rational.*

Since the greedy normal form consists of minimal length words, substituting the appropriate commuting parameters for each letter $\sigma \in \mathcal{A}$ converts $\chi_{\mathcal{L}}$ into the usual growth series (single or multi-parameter) of the Coxeter group W relative to the generating set S. Thus, rationality of the noncommutative series $\chi_{\mathcal{L}}$ implies rationality for the usual growth series $\gamma(t)$ and its multivariate versions.

The proof of Theorem 1 (in Section 8) implicitly constructs an automaton that recognizes the greedy normal form. The states of the automaton are in bijection with certain regions (of the Tits' cone) cut out by minimal hyperplanes, and in this respect the automaton is similar to the canonical automaton described in Björner and Brenti [1, page 120]. The latter, however, has the set of all reduced expressions (as words in the standard generators) as its accepted language and thus does not have the uniqueness property – a given group element has many reduced expressions. On the other hand, the greedy normal form (defined over the larger alphabet $\mathcal{A}$) does have the uniqueness property, and our automaton appears to be new.

Reciprocity results. To make sense of reciprocity in the context of multivariate noncommutative power series, we work over the ring of formal Laurent series. In particular, in this ring each formal parameter $\sigma \in \mathcal{A}$ has an inverse σ^{-1}. We use the fact that for any regular language, the characteristic series $\chi_{\mathcal{L}}$ has a representation of the form

$$\chi_{\mathcal{L}} = A(I - Q)^{-1}B,$$

where A is a row vector with entries in $\mathbb{Q}$, B is a column vector with entries in $\mathbb{Q}$, and Q is a square matrix whose entries are linear combinations of the variables $\sigma \in \mathcal{A}$. (Roughly speaking, with respect to the automaton that recognizes $\mathcal{L}$, A records the accept states, B records the start state, and Q encodes the transition function.) By replacing each σ appearing in Q with σ^{-1}, we obtain a new matrix $\overline{Q}$, and we define the reciprocal $\chi^*_{\mathcal{L}}$ of $\chi_{\mathcal{L}}$ to be

$$\chi^*_{\mathcal{L}} = A(I - \overline{Q})^{-1}B$$

provided that the matrix $I - \overline{Q}$ is invertible *over the ring of Laurent series.* By using a result of Schützenberger [21], we show that such a series, if it exists, does not depend on the choice of A,B, and Q (see Proposition 4.1, below). The main theorem of this paper is the following.

Theorem 2. *Let W be a right-angled Coxeter group and let $\mathcal{L} = \mathcal{L}(W,S)$ be its greedy normal form. If the nerve $N(W,S)$ is an Eulerian sphere of dimension k, then $\chi^*_{\mathcal{L}}$ exists and*

$$\chi^*_{\mathcal{L}} = (-1)^{k+1}\chi_{\mathcal{L}}.$$

Again, appropriate substitutions for the letters $\sigma \in \mathcal{A}$ yield all of the known reciprocity formulas *in the right-angled case.* In fact, in the generality in which we work, we also obtain a new reciprocity formula for the *complete growth series* of a right-angled Coxeter group. By definition, the complete growth series of a group G with respect to a generating set S is the series

$$\widetilde{\gamma}(t) = \sum_{g \in G} g t^{|g|},$$

a power series in t with coefficients in the group ring $\mathbb{Q}[G]$. The complete growth series has been studied by several authors [13, 14, 16]. Rationality for hyperbolic groups and formulas for surface groups were obtained by Grigorchuk and Nagnibeda in [14]. For Coxeter groups, rationality of the complete growth series is known [5, 17], but also follows from our Theorem 1 and the substitutions $\sigma \to w_\sigma t^{|\sigma|}$ (for all $\sigma \in \mathcal{A}$) into $\chi_{\mathcal{L}}$. More notable, however, is the fact that this same substitution yields the reciprocity formula

$$\widetilde{\gamma}(1/t) = (-1)^{k+1} \widetilde{\gamma}(t)$$

for the complete growth series of a right-angled Coxeter group whose nerve is an Eulerian k-sphere.

Theorem 2 fails to hold in general for non-right-angled Coxeter groups with Eulerian nerve (even though the usual growth series does satisfy reciprocity [6]). In these cases, it would be interesting to find an identity involving $\chi_{\mathcal{L}}$ that specializes to the usual reciprocity formulas under an appropriate substitution.

The proof of Theorem 2 reduces to showing that for a simplicial complex K a certain matrix J is an involution when K is Eulerian. If $\mathcal{P}(K)$ denotes the poset of faces of K, then J is a $\mathcal{P}(K) \times \mathcal{P}(K)$ matrix that takes nonzero values ± 1 if and only if the corresponding pair of simplices are sufficiently far apart in K. For this reason we call J the *anti-incidence matrix* for K. The proof that J is an involution is fairly technical, but may be of independent interest to combinatorialists. For this reason, we include it as a separate section.

Organization. Section 2 introduces notation and standard facts for noncommutative series. Our reciprocity formulas require working with formal inverses of generators, so we work over the ring of formal noncommutative Laurent series. Sections 3 and 4 discuss rationality and reciprocity for formal series. In Section 5 we describe the greedy normal form for a Coxeter group. In Section 6 we show that the characteristic series for the greedy normal form is rational, essentially by describing a finite state automaton that recognizes the language $\mathcal{L}(W, S)$. Section 7 discusses some properties of Eulerian simplicial complexes, concluding with the proof that the anti-incidence matrix is an involution. In Section 8, we combine the rationality results and Eulerian complex results to prove our reciprocity theorem. Finally, in Section 9, we apply our main theorems to obtain rationality and reciprocity results for growth series of Coxeter groups relative to both the standard generating set S and the larger generating set $A = \{w_\sigma \mid \sigma \in \mathcal{A}\}$.

2. Noncommutative formal series

In this section, we recall some basic facts from the theory of noncommutative series.

Let $\mathcal{A}$ be a finite set, called the *alphabet*. We let $\mathcal{A}^+$ and $\mathcal{A}^-$ denote two copies of $\mathcal{A}$, and let $\mathcal{A}^\pm$ denote the disjoint union $\mathcal{A}^+ \cup \mathcal{A}^-$. For each element $a \in \mathcal{A}$, we denote the corresponding elements in $\mathcal{A}^+$ and $\mathcal{A}^-$ by a and a^{-1}, respectively. We let $\mathcal{A}^*$ (respectively, $(\mathcal{A}^\pm)^*$) denote the free monoid generated by $\mathcal{A}$ (resp., $\mathcal{A}^\pm$)

consisting of *words* over $\mathcal{A}$ (resp., over $\mathcal{A}^{\pm}$). A word over $\mathcal{A}^{\pm}$ is *reduced* if it is of the form

$$\alpha = a_1^{m_1} \cdots a_n^{m_n},$$

where $a_i \in \mathcal{A}$, $m_i \in \{\pm 1\}$, and there are no consecutive pairs of the form aa^{-1} or $a^{-1}a$ appearing in the expression. We let $F(\mathcal{A})$ denote the set of reduced words over $\mathcal{A}^{\pm}$, and endow it with the usual multiplication to form the free group over $\mathcal{A}$. We then define the *order* of an element $\alpha = a_1^{m_1} \cdots a_n^{m_n} \in F(\mathcal{A})$, which we denote by $\mathrm{Ord}(\alpha)$, to be the sum of all -1's appearing as exponents; that is, if $\alpha = a_1^{m_1} \cdots a_n^{m_n}$, then $\mathrm{Ord}(\alpha) = -\operatorname{Card}\{i \mid m_i = -1\}$. In particular, $\mathcal{A}^*$ can be identified with the submonoid of $F(\mathcal{A})$ consisting of all elements α such that $\mathrm{Ord}(\alpha) = 0$.

We work over the rational numbers $\mathbb{Q}$ and consider all formal series λ of the form

$$\lambda = \sum_{\alpha \in F(\mathcal{A})} r_\alpha \alpha,$$

where $r_\alpha \in \mathbb{Q}$. The element r_α is called the *coefficient* of α in λ and will also be denoted by $\langle \lambda, \alpha \rangle$. We define the *support* of a series λ to be the set

$$\mathrm{Supp}(\lambda) = \{\alpha \in F(\mathcal{A}) \mid \langle \lambda, \alpha \rangle \neq 0\},$$

and we define the *order* of λ to be

$$\mathrm{Ord}(\lambda) = \inf\{\mathrm{Ord}(\alpha) \mid \alpha \in \mathrm{Supp}(\lambda)\}.$$

The sum of two series is defined in the usual way,

$$\lambda_1 + \lambda_2 = \sum_{\alpha \in F(\mathcal{A})} (\langle \lambda_1, \alpha \rangle + \langle \lambda_2, \alpha \rangle)\alpha,$$

but the formal product

$$\lambda_1 \lambda_2 = \sum_{\alpha \in F(\mathcal{A})} \left(\sum_{\alpha_1 \alpha_2 = \alpha} \langle \lambda_1, \alpha_1 \rangle \langle \lambda_2, \alpha_2 \rangle \right) \alpha \tag{2.1}$$

is not always well-defined (the inner sum might be infinite). To remedy this, we define a *Laurent series over* $\mathcal{A}$ to be any formal series λ such that $\mathrm{Ord}(\lambda) > -\infty$. If λ_1 and λ_2 are both Laurent series, then the inner sum in (2.1) will always be a finite sum and the resulting product $\lambda_1 \lambda_2$ will also be a Laurent series. Hence the set of formal Laurent series forms an associative ring with unit. We denote this ring by $\mathbb{Q}((\mathcal{A}))$.

A Laurent series λ is called a *power series* if $\mathrm{Supp}(\lambda) \subset \mathcal{A}^*$, a *Laurent polynomial* if $\mathrm{Supp}(\lambda)$ is finite, and a *polynomial* if it is both a Laurent polynomial *and* a power series. The *degree* of a polynomial λ is the length of the longest word in $\mathrm{Supp}(\lambda)$. A polynomial of degree 1 is called *linear*, and a polynomial with all terms of the same degree is called *homogeneous*. We let $\mathbb{Q}\langle\langle \mathcal{A} \rangle\rangle$ (respectively, $\mathbb{Q}(\mathcal{A})$, $\mathbb{Q}\langle \mathcal{A} \rangle$) denote the set of all power series (resp., Laurent polynomials, polynomials). All three of these sets are subrings of $\mathbb{Q}((\mathcal{A}))$.

3. Rational power series

A power series $\lambda \in \mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle$ is *quasi-regular* if the constant term is zero, i.e., if $1 \notin \mathrm{Supp}(\lambda)$. For a quasi-regular element λ, we define its *quasi-inverse* λ^+ by

$$\lambda^+ = \sum_{n=1}^{\infty} \lambda^n = \lim_{N\to\infty} \sum_{n=1}^{N} \lambda^N.$$

Note that if λ is quasi-regular, then $1-\lambda$ is invertible and $(1-\lambda)^{-1} = 1+\lambda^+$. A power series $\lambda \in \mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle$ is *rational* if it can be obtained from a finite set of polynomials by a finite sequence of additions, multiplications, and quasi-inversions.

The notation and terminology above also extend to vectors and matrices. Given a ring R, we let $R^{m\times n}$ denote the set of $m\times n$ matrices with entries in R. As in the case of quasi-regular power series, if a matrix $Q \in \mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle^{n\times n}$ has all quasi-regular entries, then $I-Q$ is invertible (over $\mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle$) with inverse given by

$$(I-Q)^{-1} = I + Q^+ = I + Q + Q^2 + Q^3 + \cdots.$$

Important examples of power series come from the theory of formal languages. By definition a *language* over $\mathcal{A}$ is any subset $\mathcal{L} \subset \mathcal{A}^*$. Given a language over $\mathcal{A}$, we define its *characteristic series* $\chi_{\mathcal{L}} \in \mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle$ by

$$\chi_{\mathcal{L}} = \sum_{\alpha\in\mathcal{L}} \alpha.$$

A language $\mathcal{L}$ is *regular* if it is the language accepted by a finite state automaton. This turns out to be equivalent to its characteristic series $\chi_{\mathcal{L}}$ being rational. In fact, more generally, we have the following standard characterization of rational power series, due to Schützenberger [20, 21].

Theorem 3.1. *A power series $\lambda \in \mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle$ is rational if and only if there exist a matrix $Q \in \mathbb{Q}\langle\mathcal{A}\rangle^{n\times n}$ with linear homogeneous entries and vectors $A \in \mathbb{Q}^{1\times n}$ and $B \in \mathbb{Q}^{n\times 1}$ such that*

$$\lambda = A(I-Q)^{-1}B.$$

Proof. Any series of such a form $A(I+Q^+)B$ is rational by Theorem 1.2 in [20]. (The proof there shows this if $A = [1\ 0 \cdots 0]$. The general case follows by a change of basis.) Conversely, any rational series is of this form by Theorem 1.4 in [20]. $\square$

Remark 3.2. Series of the form $A(I-Q)^{-1}B$ are usually called *recognizable*; thus Schützenberger's Theorem says that a series is rational if and only if it is recognizable. The connection to regular languages is fairly straightforward. Given a finite state automaton, let $\mathcal{S}$ denote the set of states and let $\mu : \mathcal{A}\times\mathcal{S} \to \mathcal{S}$ denote the transition function. Let A be the $1\times\mathcal{S}$ row vector whose entry corresponding to $s \in S$ is 1 if s is an accept state and is 0 otherwise. Let B be the $\mathcal{S}\times 1$ column vector whose entry corresponding to $s \in \mathcal{S}$ is 1 if s is the start state and is 0 otherwise, and let Q be the $\mathcal{S}\times\mathcal{S}$ matrix whose (s,t)-entry is $\sum_{\mu(a,t)=s} a$. Then the characteristic series for the language $\mathcal{L}$ recognized by the automaton is

$$\chi_{\mathcal{L}} = A(I-Q)^{-1}B = A(I+Q+Q^2+Q^3+\cdots)B.$$

See, e.g., [19] for more details.

Given a rational power series λ, a triple (A,Q,B) as in Theorem 3.1 is called a *representation* for λ. The *dimension* of a representation is the number n. Two representations (A,Q,B) and (A',Q',B') are *equivalent* if there exists an invertible matrix $P \in \mathbb{Q}^{n\times n}$ such that $A' = AP^{-1}$, $Q' = PQP^{-1}$, and $B' = PB$. A representation for λ is *minimal* if it has the minimum dimension among all representations for λ.

Example 3.3. Let $\mathcal{A} = \{x, y\}$ and let $\mathcal{L} \subset \mathcal{A}^*$ be the language consisting of all square-free words in x and y. Then the characteristic series for $\mathcal{L}$ is

$$\chi_{\mathcal{L}} = 1 + x + y + xy + yx + xyx + yxy + \cdots .$$

This series is rational since it can be written as

$$\chi_{\mathcal{L}} = A(I - Q)^{-1}B = A(I + Q + Q^2 + \cdots)B,$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & x \\ y & y & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

In other words, (A, Q, B) is a representation for $\chi_{\mathcal{L}}$. Using row reduction to invert $I - Q$, one obtains

$$(I - Q)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ x + x(1 - yx)^{-1}y(1 + x) & 1 + x(1 - yx)^{-1}y & x \\ (1 - yx)^{-1}y(1 + x) & (1 - yx)^{-1}y & (1 - yx)^{-1} \end{bmatrix}.$$

This gives a rational expression for the power series $\chi_{\mathcal{L}}$:

$$\chi_{\mathcal{L}} = A(I - Q)^{-1}B = 1 + x + x(1 - yx)^{-1}y(1 + x) + (1 - yx)^{-1}y(1 + x).$$

4. The reciprocal of a rational power series

The goal of this section is to make sense (when possible) of the series obtained from a rational power series by replacing all letters with their inverses.

Let $\lambda \mapsto \overline{\lambda}$ denote the involution on the set of Laurent polynomials defined by replacing each letter $x \in \mathcal{A}$ with x^{-1} and each letter x^{-1} with x. Similarly, for any matrix M with Laurent polynomial entries, we define $\overline{M}$ to be the same matrix with each entry λ replaced by $\overline{\lambda}$.

Now suppose λ is a rational power series and that (A, Q, B) is a representation for λ. Suppose further that the matrix $I - \overline{Q}$ is invertible over the ring $\mathbb{Q}((\mathcal{A}))$. Then we obtain a new Laurent series

$$\lambda^*(A, Q, B) = A(I - \overline{Q})^{-1}B.$$

Proposition 4.1. *The series $\lambda^*(A, Q, B)$, if it exists, does not depend on the choice of representation (A, Q, B).*

For the proof of Proposition 4.1, we use the following result due to Schützenberger [21, Theorem III.B.1].

Lemma 4.2. *Let λ be a rational series in $\mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle$, and let (A_0, Q_0, B_0) be a minimal representation for λ. Then any other representation for λ is equivalent to one of the form $(\widehat{A}, \widehat{Q}, \widehat{B})$, where these matrices have the block form*

$$\widehat{A} = \begin{bmatrix} * & A_0 & 0 \end{bmatrix}, \qquad \widehat{Q} = \begin{bmatrix} Q_1 & 0 & 0, \\ * & Q_0 & 0 \\ * & * & Q_2 \end{bmatrix}, \qquad \widehat{B} = \begin{bmatrix} 0 \\ B_0 \\ * \end{bmatrix}.$$

In particular, any two minimal representations are equivalent.

Proof of Proposition 4.1. Let (A, Q, B) be a representation for the series λ with respect to which the reciprocal $\lambda^*(A, Q, B)$ exists. Let (A_0, Q_0, B_0) be a minimal representation, and let $(\widehat{A}, \widehat{Q}, \widehat{B})$ be a representation equivalent to (A, Q, B) in the form specified by Lemma 4.2. It follows easily from the definition of equivalent representations that $\lambda^*(A, Q, B) = \lambda^*(\widehat{A}, \widehat{Q}, \widehat{B})$. Calculating the latter, we have

$$\begin{aligned}
\lambda^*(\widehat{A}, \widehat{Q}, \widehat{B}) &= \begin{bmatrix} * & A_0 & 0 \end{bmatrix} \left(I - \overline{\begin{bmatrix} Q_1 & 0 & 0 \\ * & Q_0 & 0 \\ * & * & Q_2 \end{bmatrix}} \right)^{-1} \begin{bmatrix} 0 \\ B_0 \\ * \end{bmatrix} \\
&= \begin{bmatrix} * & A_0 & 0 \end{bmatrix} \begin{bmatrix} I - \overline{Q}_1 & 0 & 0 \\ \overline{*} & I - \overline{Q}_0 & 0 \\ \overline{*} & \overline{*} & I - \overline{Q}_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_0 \\ * \end{bmatrix} \\
&= \begin{bmatrix} * & A_0 & 0 \end{bmatrix} \begin{bmatrix} (I - \overline{Q}_1)^{-1} & 0 & 0 \\ ** & (I - \overline{Q}_0)^{-1} & 0 \\ ** & ** & (I - \overline{Q}_2)^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ B_0 \\ * \end{bmatrix} \\
&= A_0 (I - \overline{Q}_0)^{-1} B_0 \\
&= \lambda^*(A_0, Q_0, B_0).
\end{aligned}$$

Since any two minimal representations are equivalent, the result follows. □

Since the series $\lambda^*(A, Q, B)$ does not depend on the representation, we denote it simply by λ^* and call it the *reciprocal* of λ.

Example 4.3. Let χ be the characteristic series for the square-free words in x and y as in Example 3.3. Then Q is the matrix

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & x \\ y & y & 0 \end{bmatrix},$$

so we have

$$\begin{aligned}
I - \overline{Q} &= \begin{bmatrix} 1 & 0 & 0 \\ -x^{-1} & 1 & -x^{-1} \\ -y^{-1} & -y^{-1} & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -x^{-1} & 0 \\ 0 & 0 & -y^{-1} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & -x & 1 \\ 1 & 1 & -y \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -x^{-1} & 0 \\ 0 & 0 & -y^{-1} \end{bmatrix} (I - Q) \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.
\end{aligned}$$

Inverting this matrix, we obtain

$$(I - \overline{Q})^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} (I - Q)^{-1} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & -y \end{bmatrix}.$$

We then have

$$\begin{aligned}\chi^* &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} (I-\overline{Q})^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} (I-Q)^{-1} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \\ &= -1 - x - y - xy - yx - xyx - yxy - \cdots .\end{aligned}$$

5. Coxeter groups and the greedy normal form

Let (W, S) be a Coxeter system as defined in [2]. In particular, W is a group, S is a generating set, and W admits a presentation of the form

$$W = \langle S \mid (ss')^{m(s,s')} = 1 \text{ for all } s, s' \in S\rangle,$$

where $m : S \times S \to \{1, 2, \ldots, \infty\}$ is a symmetric function such that $m(s, s') = 1$ if and only if $s = s'$. Any element $w \in W$ has a *length*, denoted $|w|$, defined to be the minimal number n such that $w = s_1 \cdots s_n$ with all $s_i \in S$.

A subset $\sigma \subset S$ is called *spherical* if it is either empty or generates a finite subgroup of W. The *nerve* of a Coxeter system (W, S) is the set $N(W, S)$ consisting of all spherical subsets of S. We can regard $N = N(W, S)$ as an abstract simplicial complex on the vertex set S (so $\emptyset$ corresponds to the empty simplex). For any nontrivial $\sigma \in N(W, S)$, we let W_σ denote the subgroup generated by σ, and we let w_σ denote the element of longest length in W_σ. Note that w_σ is always an involution ([2, Ex. 22, p. 43]). For any $w \in W$, we define the *(left) descent set* of w to be the set $\mathrm{Desc}(w) = \{s \in S \mid |sw| < |w|\}$. A proof of the following can be found, for example, in [8] (Lemma 4.7.2).

Proposition 5.1. *Let w be any element in W, and let $\sigma = \mathrm{Desc}(w)$ be the left descent set. Then*

1. *σ is a spherical subset of S, and*
2. *w factors as $w = w_\sigma v$, where $|w| = |w_\sigma| + |v|$.*

Now let $\mathcal{A}$ be the set of of all nontrivial spherical subsets of S, which we call the *proper nerve* of W. Given a word $\alpha = \sigma_n \cdots \sigma_1 \in \mathcal{A}^*$, let $\pi(\alpha) \in W$ be the product $w_{\sigma_n} \cdots w_{\sigma_1}$ (and let $\pi(1) = 1$). Then $\pi : \mathcal{A}^* \to W$ is a monoid homomorphism, and since the standard generators for W are of the form w_σ with $\sigma = \{s\}$, π is surjective.

Definition 5.2. Let (W, S) be a Coxeter system and let $\mathcal{A}$ be the proper nerve. Then the *left greedy normal form* for W is the language $\mathcal{L} \subset \mathcal{A}^*$ consisting of all words $\alpha = \sigma_n\sigma_{n-1} \cdots \sigma_1 \in \mathcal{A}^*$ such that σ_i is the left descent set of $w_{\sigma_i} w_{\sigma_{i-1}} \cdots w_{\sigma_1}$ for all $i \in \{1, \ldots, n\}$.

We then have the following.

Proposition 5.3. *Let $\mathcal{L}$ be the left greedy normal form for the Coxeter group (W, S). Then π restricts to a bijection $\mathcal{L} \to W$, and for any $\alpha = \sigma_n \cdots \sigma_1 \in \mathcal{L}$, we have $|\pi(\alpha)| = |w_{\sigma_n}| + \cdots + |w_{\sigma_1}|$.*

Proof. The equality $|\pi(\alpha)| = |w_{\sigma_n}| + \cdots + |w_{\sigma_1}|$ follows by induction on n and Proposition 5.1.

To prove that π is injective, note first that only the trivial word maps to the identity. Assume by induction that any two words (in $\mathcal{L}$) that map to a group element of length $< k$ are equal, and suppose that $\alpha = \sigma_n \cdots \sigma_1$ and $\alpha' = \sigma'_m \cdots \sigma'_1$ in $\mathcal{L}$ both map to the same group element w with $|w| = k$. Then σ_n and σ'_m are both the left descent set of w, hence are equal, so $\sigma_{n-1} \cdots \sigma_1$ and $\sigma'_{m-1} \cdots \sigma'_1$ both map to the same group element (of length $< k$). By induction, these must be the same word.

For surjectivity (and the length property), assume by induction that every $w \in W$ of length $< k$ has a greedy representation $\alpha = \sigma_n \cdots \sigma_1 \in \mathcal{L}$ and that $|\pi(\alpha)| = |w_{\sigma_n}| + \cdots + |w_{\sigma_1}|$. Suppose $w \in W$ is an element of length k and let σ be the left descent set. Then by Proposition 5.1, $w = w_\sigma v$ and $|w| = |w_\sigma| + |v|$. Since $|v| < k$, the result follows by induction. □

6. Rationality of the greedy normal form

Let (W, S) be a Coxeter system, let $\mathcal{L} \subset \mathcal{A}^*$ be its left greedy normal form, and let $\chi = \chi_{\mathcal{L}}$ be the characteristic series for $\mathcal{L}$. In this section we shall prove that χ is a rational power series (equivalently, that $\mathcal{L}$ is a regular language). Our argument essentially constructs a finite state automaton whose accepted language is $\mathcal{L}$. The automaton is a modification of the "canonical automaton" described in [1], which is an automaton whose accepted language is the set of all reduced expressions for elements of W. The language of reduced expressions is canonical in that it does not depend on an ordering of the generators S, but it does not have the uniqueness property; i.e., a given element $w \in W$ can have many reduced expressions. The greedy normal form $\mathcal{L}$ is also canonical, but *does* have the uniqueness property. The trade-off that makes this possible seems to be that the set of reduced expressions is defined over the alphabet S, whereas $\mathcal{L}$ is defined over the larger alphabet $\mathcal{A}$.

To describe our representation for χ, we use the *geometric representation* for (W, S). Details can be found in any standard reference on Coxeter groups ([1, 2, 8, 15]). Let V be a real vector space of dimension $|S|$, and let $(h_s)_{s \in S}$ be a basis for the dual space V^*. For each $s \in S$, let $v_s \in V$ be the unique vector such that $h_{s'}(v_s) = -\cos(\pi/m(s, s'))$ for all $s' \in S$. Then $\rho_s(v) = v - 2h(v)v_s$ defines a linear reflection $\rho_s : V \to V$ across the hyperplane $\{v \mid h_s(v) = 0\}$. Moreover, $s \mapsto \rho_s$ defines a faithful representation $\rho : W \to GL(V)$, called the *geometric representation* for W. To simplify notation, we identify W with its image in $GL(V)$.

Let $C \subset V$ be the cone

$$C = \{v \mid h_s(v) \geq 0 \text{ for all } s \in S\}$$

and let $U \subset V$ be the union of W-translates of C:

$$U = \bigcup_{w \in W} wC.$$

By a theorem of Tits, U is a convex cone, on which W acts discretely with fundamental domain C. Moreover, a subgroup of W is finite if and only if it is contained in the stabilizer of a point in the interior of U. In particular, the finite parabolic subgroups W_σ are precisely the stabilizers of points in the fundamental chamber C that lie in the interior of U. The cone U is usually refered to as the *Tits cone*, the translates wC as *chambers*, and W as the *fundamental chamber*. Since C is a fundamental domain, $w \leftrightarrow wC$ defines a bijection between W and the set of chambers.

For each $s \in S$, we define the *fundamental hyperplane* H_s by $H_s = \{v \in U \mid h_s(v) = 0\}$, and we let $\mathcal{H}$ denote the collection of all W-translates of the fundamental hyperplanes:

$$\mathcal{H} = \{wH_s \mid w \in W, s \in S\}.$$

Note that if $H = wH_s$ is any hyperplane in $\mathcal{H}$, it is the fixed point set of the reflection $r = wsw^{-1} \in W$, and separates U into two connected components. We shall refer to any $H \in \mathcal{H}$ as a *hyperplane* in U and to (the closure of) a connected component of $U - H$ as a *halfspace*. Since hyperplanes do not intersect the interiors of chambers, any intersection of halfspaces is a union of chambers. For each $H \in \mathcal{H}$, let H^+ (respectively, H^-) denote the halfspace bounded by H that contains (resp., does not contain) the fundamental chamber C. We recall the following simple criterion for determining which fundamental halfspaces contain a given chamber (see [2]).

Proposition 6.1. *The chamber wC is contained in the fundamental halfspace H_s^- if and only if $|sw| < |w|$ (i.e., if and only if $s \in \mathrm{Desc}(w)$).*

Given a spherical subset $\sigma \subset S$, we let $\mathcal{H}_\sigma$ denote the collection of hyperplanes $\{wH_s \mid w \in W_\sigma,\ s \in \sigma\}$, and we let A_σ^+ and A_σ^- denote the subsets of U defined by

$$A_\sigma^+ = \bigcap_{s\in\sigma} H_s^+ \qquad \text{and} \qquad A_\sigma^- = \bigcap_{s\in\sigma} H_s^-$$

(see, e.g., Figure 1). In particular, the hyperplanes $\mathcal{H}_\sigma$ are permuted by W_σ, and

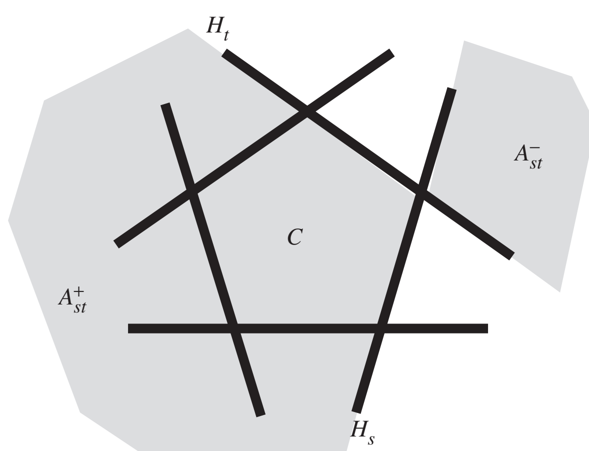

Figure 1

A_σ^+ is a fundamental domain for the action of W_σ on this arrangement. By definition of w_σ, $|sw_\sigma| < |w_\sigma|$ for every $s \in \sigma$; hence by Proposition 6.1 the chamber $w_\sigma C$ (and hence $w_\sigma A_\sigma^+$) lies in A_σ^-.

It also follows from Proposition 6.1 that $A_\sigma^- \cap A_\tau^- = A_{\sigma\cup\tau}^-$ if $\sigma \cup \tau$ is spherical (and is empty otherwise). This leads to a decomposition of the Tits cone cut out by the fundamental hyperplanes; namely, we define $C_\emptyset = C$ and for $\sigma \in \mathcal{A}$ we define $C_\sigma \subset U$ to be the subset

$$C_\sigma = \overline{A_\sigma^- - \bigcup_{\sigma \subset \tau \in \mathcal{A}} A_\tau^-}$$

(see Figure 2). Note that C_σ can also be obtained by intersecting A_σ^- with the

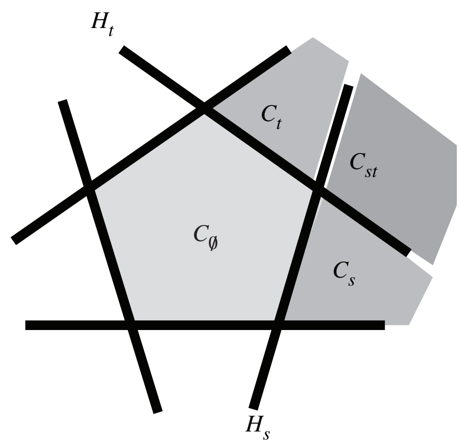

FIGURE 2

halfspaces H_s^+, where s ranges over all elements of S such that $s \notin \sigma$ and $\sigma \cup \{s\}$ is spherical. Thus, the C_σ's are also intersections of halfspaces, hence are unions of chambers. More precisely, we have the following decomposition.

Proposition 6.2. *The set C_σ is the union of chambers wC such that* $\mathrm{Desc}(w) = \sigma$.

Proof. By Proposition 6.1, $wC \subset H_s^-$ if and only if $l(sw) < l(w)$, i.e., if and only if $s \in \mathrm{Desc}(w)$. Thus, A_σ^- is the union of those chambers wC such that $\mathrm{Desc}(w) \supset \sigma$. Since C_σ is obtained from A_σ^- by removing the interior of every A_τ^- for $\tau \supset \sigma$, the result follows. □

Next we describe a finer decomposition (than the C_σ's) of U. We define a partial order $\leq$ on $\mathcal{H}$ by: $H_1 \leq H_2$ if and only if $H_1^+ \subset H_2^+$. In other words, $H_1 < H_2$ if and only if H_1 separates the hyperplane H_2 from the fundamental chamber C. Let $\mathcal{H}_{min}$ denote the set of minimal elements in the poset $\mathcal{H}$. In particular, each fundamental hyperplane H_s is in $\mathcal{H}_{min}$, and more generally, for any spherical σ, the hyperplanes in $\mathcal{H}_\sigma$ are all in $\mathcal{H}_{\min}$ (since they intersect the fundamental chamber in the interior of U). We recall some fundamental properties of the set of minimal hyperplanes. The first statement below is the main result in [3]; the second is Proposition 2.2 in [4].

Proposition 6.3.

(1) *The set $\mathcal{H}_{min}$ is finite.*
(2) *If $s \in S$ and $H \in \mathcal{H}_{\min}$, then either $sH \in \mathcal{H}_{\min}$ or $H_s \leq sH$.*

Let $\mathcal{R}$ be the collection of regions of U cut out by $\mathcal{H}_{\min}$. Since each fundamental hyperplane is in $\mathcal{H}_{\min}$, the fundamental chamber C is one of the regions in $\mathcal{R}$. More generally, each region R in $\mathcal{R}$ is a union of chambers. We are interested in how these regions behave under translations by the elements w_σ. Recall that for each spherical subset σ, the subgroup W_σ acts on U preserving the collection of hyperplanes $\mathcal{H}_\sigma$ and with fundamental domain A_σ^+. In particular, the translates wA_σ^+ for $w \in W_\sigma$ are the pieces cut out by $\mathcal{H}_\sigma$ and since these hyperplanes are all minimal, these pieces are further divided into regions in $\mathcal{R}$. We can now state the key technical observation for the construction of our representation.

Lemma 6.4. *Suppose $\sigma \subset S$ is spherical, $w \in W_\sigma$, and $s \in S$ satisfies $|sw| > |w|$. Then for every region $R \in \mathcal{R}$ such that $R \subseteq wA_\sigma^+$ there exists a region $R' \in \mathcal{R}$ such that $sR \subseteq R' \subseteq swA_\sigma^+$. In particular, if $R \subset A_\sigma^+$, then there exists an $R' \in \mathcal{R}$ such that $w_\sigma R \subseteq R' \subseteq w_\sigma A_\sigma^+ = A_\sigma^-$.*

Proof. Since $l(sw) > l(w)$, we have $wC \subseteq H_s^+$ (Figure 3). Since wC and wA_σ^+

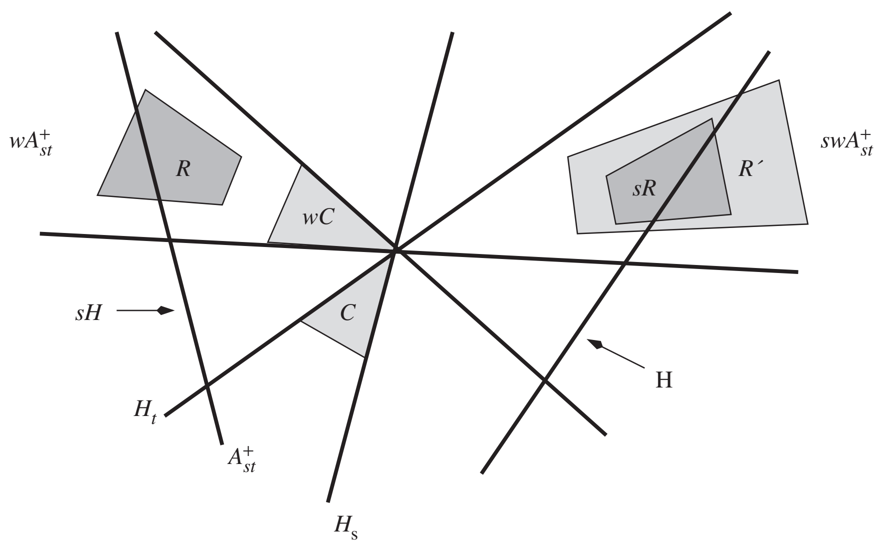

FIGURE 3

lie on the same side of every hyperplane in $\mathcal{H}_\sigma$, we also know $R \subseteq wA_\sigma^+ \subseteq H_s^+$. Now suppose on the contrary that sR were *not* contained in some region R'. This means that (the interior of) sR would have to be cut by some minimal hyperplane $H \in \mathcal{H}_{min}$. Multiplying by s, we would then have sH intersecting the interior of the region R. Since the interior of R is disjoint from every minimal hyperplane, this means sH cannot be minimal; hence by (2) of Proposition 6.3, we must have $sH \geq H_s$ or, equivalently, $H_s^+ \subseteq (sH)^+$. But this implies $R \subset (sH)^+$, which is impossible since the hyperplane sH intersects the interior of the region R. It follows that there must exist a region R' such that $sR \subseteq R'$. Moreover, since $sR \subseteq swA_\sigma^+$ and the latter is bounded by minimal hyperplanes, we must have $R' \subseteq swA_\sigma^+$ as well.

The final claim of the lemma follows by applying the first part of the lemma iteratively to any reduced expression $s_k \dots s_1$ for w_σ. $\square$

We can now describe the inductive lemma we need to construct our rational representation for $\chi_{\mathcal{L}}$. We set $W^R = \{w \in W \mid wC \subset R\}$ and $\mathcal{L}_R = \{\alpha \in$

$\mathcal{L} \mid \pi(\alpha) \in W^R\}$. Thus, W^R is the set of group elements that move the fundamental chamber into the region R, and $\mathcal{L}_R$ is the set of greedy representatives for those elements. As a point of clarification, we note that the *length* of a word α is its length as a string of letters. (This is not the same as $|\pi(\alpha)|$, the length of $\pi(\alpha)$ in W.)

Lemma 6.5. *Suppose α is a word of length k in $\mathcal{L}_R$. Then $\sigma\alpha$ is a word of length $k+1$ in $\mathcal{L}$ if and only if $w_\sigma R \subset C_\sigma$. Moreover, in this case, there exists a unique region $R' \in \mathcal{R}$ such that $w_\sigma R \subseteq R'$ (that is, such that $\sigma\alpha \in \mathcal{L}_{R'}$).*

Proof. Let $w = \pi(\alpha)$. Then $\sigma\alpha$ is greedy if and only if σ is the left descent set of $w_\sigma w$. By Proposition 6.2, this is equivalent to $w_\sigma wC \subseteq C_\sigma$. Since $wC \subseteq R$ (because $w \in \mathcal{L}_R$) and C_σ is a union of regions in $\mathcal{R}$, this is also equivalent to $w_\sigma R \subseteq C_\sigma$. It remains to show that $w_\sigma R$ is contained in a unique region $R' \in \mathcal{R}$. For this we note that $w_\sigma R \subseteq C_\sigma$ implies $R \subseteq w_\sigma C_\sigma \subseteq w_\sigma A_\sigma^- = A_\sigma^+$. Thus, by Lemma 6.4, there exists an $R' \in \mathcal{R}$ such that $w_\sigma R \subseteq R'$. $\square$

Let $\chi \in \mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle$ be the characteristic series for $\mathcal{L}$ and let χ_R denote the characteristic series for $\mathcal{L}_R$. Since the regions are disjoint, we have

$$\chi = \sum_{R\in\mathcal{R}} \chi_R.$$

Let $n = |\mathcal{R}|$ and fix an ordering of $\mathcal{R}$ (with the fundamental chamber $R = C$ first in the order). We can then index the entries of vectors in $\mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle^{n\times 1}$ by the set $\mathcal{R}$ and the entries of matrices in $\mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle^{n\times n}$ by the set $\mathcal{R}\times\mathcal{R}$.

Theorem 6.6 (Rationality)**.** *Let $Q = (Q_{R',R}) \in \mathbb{Q}\langle\mathcal{A}\rangle^{n\times n}$ be the matrix given by*

$$Q_{R',R} = \begin{cases} \sigma & \textit{if } w_\sigma R \subset R',\ R' \subset C_\sigma, \textit{ and } R' \neq C_\emptyset, \\ 0 & \textit{otherwise,} \end{cases}$$

and let $A \in \mathbb{Q}^{1\times n}$, $B \in \mathbb{Q}^{n\times 1}$ be the vectors

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \qquad \textit{and} \qquad B = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T.$$

Then $\chi = A(I-Q)^{-1}B$. In particular, χ is rational (by Theorem 3.1*).*

Proof. For each $R \in \mathcal{R}$, let $\mathcal{L}_R^k$ denote the set of words in $\mathcal{L}_R$ of length k, and let χ_R^k be the corresponding characteristic series. Let $X_k \in \mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle^{n\times 1}$ be the vector whose Rth entry is χ_R^k, and let X be the sum

$$X = X_1 + X_2 + \cdots.$$

In other words, X is the vector whose Rth entry is χ_R, the characteristic series for $\mathcal{L}_R$. Since there is only one word of length 1 (corresponding to the region C), we know the initial vector $X_1 = B$. In general, by Lemma 6.5, X_k satisfies the recurrence

$$X_{k+1} = QX_k.$$

It follows that $X = (I + Q + Q^2 + \cdots)B = (I-Q)^{-1}B$. Since $\chi = \sum_R \chi_R = AX$, the result follows. $\square$

Remark 6.7. In proving rationality, we have essentially constructed a finite state automaton that recognizes the language $\mathcal{L}$. The state set is the set $\mathcal{R} \cup \{F\}$, where F is a single fail state. The start state is C, and the transition function $\mu : \mathcal{A} \times \mathcal{R} \cup \{F\} \to \mathcal{R} \cup \{F\}$ is defined by

$$\mu(\sigma, R) = \begin{cases} R' & \text{if } R \neq F \text{ and } w_\sigma R \subset R' \subset C_\sigma, \\ F & \text{otherwise.} \end{cases}$$

We illustrate Theorem 6.6 with the following example.

Example 6.8. Let W be the affine Coxeter group $\widetilde{B}_2$. That is, $S = \{s_1, s_2, s_3\}$, and the Coxeter relations are

$$(s_1 s_2)^4 = 1, \qquad (s_1 s_3)^4 = 1, \qquad (s_2 s_3)^2 = 1.$$

There are 8 minimal hyperplanes $H_1, H_2, \ldots, H_8$ cutting U into the 25 regions $R_\emptyset = C$, R_1, R_2, R_3, R_{23}, R_{27}, R_{14}, R_{15}, R_{36}, R_{368}, R_{238}, R_{278}, R_{247}, R_{147}, R_{145}, R_{156}, R_{356}, R_{3568}, R_{2368}, R_{2378}, R_{2478}, R_{1247}, R_{1457}, R_{1456}, R_{1356} (see Figure 4). The regions are indexed by the set of minimal hyperplanes that separate them from

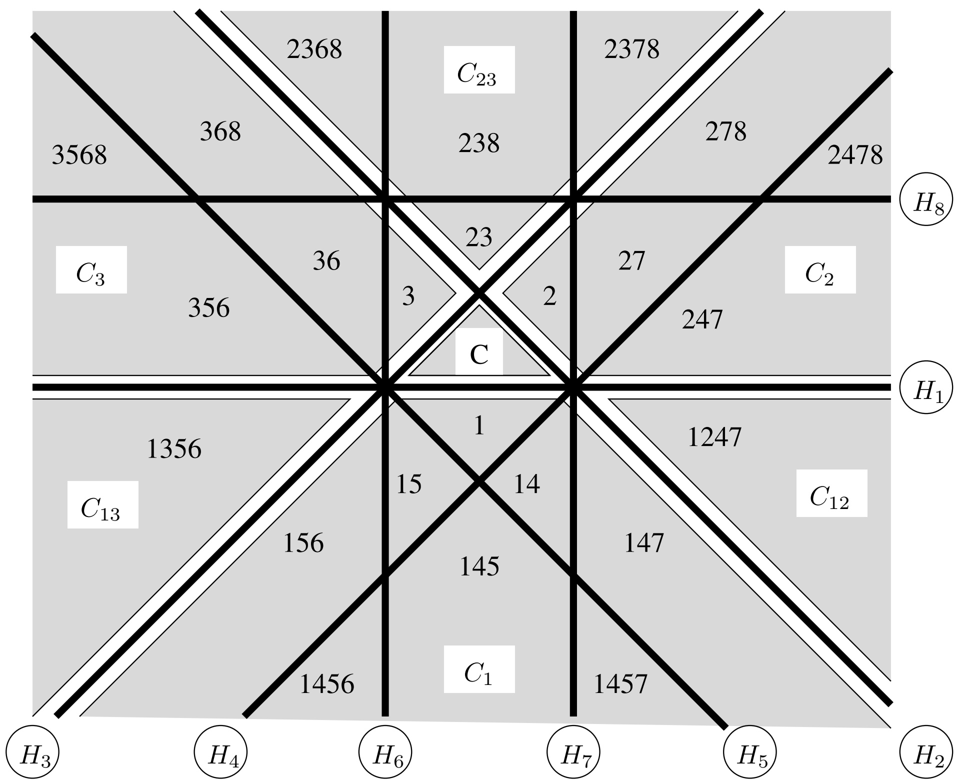

FIGURE 4

the fundamental chamber. The spherical subsets are $a = \{s_1\}$, $b = \{s_2\}$, $c = \{s_3\}$, $x = \{s_2, s_3\}$, $y = \{s_1, s_3\}$, and $z = \{s_1, s_2\}$. The longest elements w_a, w_b, and w_c act on the figure by reflecting across the hyperplanes H_1, H_2, and H_3, respectively. The longest elements w_x, w_y, and w_z act on the figure by rotating 180° about the points $H_2 \cap H_3$, $H_1 \cap H_3$, and $H_1 \cap H_2$, respectively.

With respect to this ordering of $\mathcal{R}$, the matrix Q of Theorem 6.6 is shown in Figure 5. (To see where the entries of this matrix came from, consider the region R_{1456}. The $w_x = s_2s_3$ translate of R_{1456} lies in the region R_{2378}. Since this region lies in C_{23}, the (R_{2378}, R_{1456})-entry of Q is $x = \{s_2, s_3\}$. On the other hand, although the $w_c = s_3$ translate of R_{1456} lands in R_{1356}, the latter region is not contained in C_3, so the corresponding entry of Q is zero. Both of these entries are indicated in bold in the matrix.)

$$Q = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
a&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
b&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
c&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
x&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&b&0&0&0&0&0&b&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&a&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&a&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&c&0&0&0&0&c&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&c&0&0&0&0&0&0&0&0&0&0&0\\
0&x&0&0&0&0&x&x&0&0&0&0&0&0&x&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&b&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&b&0&0&0&0&0&0&0&b&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&a&0&0&0&0&0&a&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&a&0&0&0&0&0&a&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&a&a&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&c&0&0&0&0&0&0&c&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&c&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&x&0&0&0&0&0&0&0&0&x&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&x&0&0&0&0&0&0&0&\underline{\mathbf{x}}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&b&0\\
z&0&0&z&0&0&0&0&z&z&0&0&0&0&0&0&z&z&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&a&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&a&0&0&0&0&0&0\\
y&0&y&0&0&y&0&0&0&0&0&y&y&0&0&0&0&0&0&0&y&0&0&\underline{\mathbf{0}}&0
\end{bmatrix}$$

FIGURE 5

This gives

$$\begin{aligned}
\chi &= A(I-Q)^{-1}B \\
&= A(I+Q+Q^2+\cdots)B \\
&= 1 + (a+b+c+x+y+z) + (ab+ba+ac+ca+ax+xa+zc+yb) + \cdots .
\end{aligned}$$

By formally inverting $I - Q$ using row reduction and inverses of polynomials, one can obtain a rational expression for χ, but it will be a complicated nested expression in noncommuting polynomials and their inverses. Instead, by replacing each letter with the single variable t, we obtain the (rational) series:

$$\overline{\chi} = 1 + 6t + 8t^2 + \cdots = \frac{1 + 4t - 3t^2 + 4t^3 - 2t^4}{1 - 2t + t^2}.$$

The series $\overline{\chi}$ is the "growth series" for the language $\mathcal{L}(W, S)$; that is, it is the generating function for the sequence $a(n)$ where $a(n)$ is the number of group elements whose left greedy representations have length n (in the monoid $\mathcal{A}^*$).

7. The anti-incidence matrix

We shall now assume that W is a *right-angled Coxeter group*; that is, the function m in the defining presentation satisfies $m(s,s') \in \{2,\infty\}$ for all $s \neq s'$. As before, we let N denote the nerve of W, $\mathcal{L}$ denote the greedy normal form, and χ denote its characteristic series. Our goal for the remainder of the paper is to show that for certain right-angled Coxeter groups, the reciprocal χ^* exists and (up to sign) coincides with χ. The proof of this formula reduces to showing that a certain integer matrix is an involution. This matrix can be defined for any poset $\mathcal{P}$ and is reminiscent of the zeta function (cf. [23]) except that instead of taking values on pairs of related elements, it takes values on pairs that are sufficiently far apart in the geometric realization of $\mathcal{P}$. For this reason, we call it the *anti-incidence matrix* for $\mathcal{P}$.

Let K be a finite simplicial complex with vertex set V. For convenience, we shall identify K with its standard realization in $\mathbb{R}^V$: that is, we identify V with the standard basis of $\mathbb{R}^V$ and each simplex of K with the convex hull of its vertices. In keeping with the rest of the paper, we shall use boldface Greek letters $\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\rho}$ for geometric simplices and nonbold letters σ, τ, ρ for their respective vertex sets. The *dimension* of a simplex $\boldsymbol{\sigma}$, denoted by $\dim \boldsymbol{\sigma}$, is $m-1$, where m is the number of vertices of $\boldsymbol{\sigma}$.

We let $\mathcal{P}(K)$ denote the set of simplices in K (including the empty simplex $\emptyset$, which has dimension -1), partially ordered by inclusion. We let $\boldsymbol{\chi}(K)$ denote the usual Euler characteristic:

$$\boldsymbol{\chi}(K) = \sum_{\boldsymbol{\sigma} \in \mathcal{P}, \boldsymbol{\sigma} \neq \emptyset} (-1)^{\dim \boldsymbol{\sigma}},$$

and we let $\overline{\boldsymbol{\chi}}(K)$ denote the reduced Euler characteristic:

$$\overline{\boldsymbol{\chi}}(K) = \sum_{\boldsymbol{\sigma} \in \mathcal{P}} (-1)^{\dim \boldsymbol{\sigma}} = \chi(K) - 1.$$

(We use the boldface symbol $\boldsymbol{\chi}$ to distinguish the Euler characteristic from the characteristic series χ of a formal language.)

A *subcomplex* of K is any subset $L \subseteq K$ that can be expressed as a union of simplices in $\mathcal{P}(K)$. Given any subcomplex L of K, we let $\mathrm{Vert}(L)$ denote the set of vertices in L. A subcomplex L is called *full* or *induced* if it is maximal among all subcomplexes with vertex set $\mathrm{Vert}(L)$. In other words, L is a full subcomplex if it contains every simplex $\boldsymbol{\sigma}$ of K whose vertices are all in L.

For any simplex $\boldsymbol{\sigma}$ in K, we recall the following standard subcomplexes.

1. $\mathrm{St}(\boldsymbol{\sigma}, K)$, the *star* of $\boldsymbol{\sigma}$ in K, is the union of all simplices $\boldsymbol{\tau}$ such that $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ span a simplex in K.
2. $\mathrm{Lk}(\boldsymbol{\sigma}, K)$, the *link* of $\boldsymbol{\sigma}$ in K, is the union of all simplices $\boldsymbol{\tau}$ in $\mathrm{St}(\boldsymbol{\sigma}, K)$ such that $\boldsymbol{\tau} \cap \boldsymbol{\sigma} = \emptyset$.

Note that $\mathrm{St}(\boldsymbol{\sigma}, K)$ can be identified with the join $\boldsymbol{\sigma} * \mathrm{Lk}(\boldsymbol{\sigma}, K)$. Our primary object of study in this section is the following matrix, which is determined entirely by the combinatorial structure of K.

Definition 7.1. Let K be a simplicial complex and let $\mathcal{P} = \mathcal{P}(K)$. Then the *anti-incidence* matrix of K is the $\mathcal{P} \times \mathcal{P}$ matrix J defined by

$$J_{\boldsymbol{\sigma}, \boldsymbol{\tau}} = \begin{cases} (-1)^{\dim \boldsymbol{\sigma}} & \text{if } \boldsymbol{\tau} \cap \mathrm{St}(\boldsymbol{\sigma}) = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

In light of the definition of this matrix, we single out some additional subcomplexes. First we give some more notation: given a subcomplex L, we let $K - L$ denote the subcomplex of K obtained by deleting all open stars of vertices in L (the *open star* of a simplex $\boldsymbol{\sigma}$ is the union of all relative interiors of simplices that have nontrivial intersection with the relative interior of $\boldsymbol{\sigma}$), and we let $\mathrm{Nb}(L, K)$ denote the union of all (closed) stars of vertices in L. Given any simplex $\boldsymbol{\sigma}$ in K, we define the following two subcomplexes:

3. $B(\boldsymbol{\sigma}, K) = \mathrm{Nb}(\boldsymbol{\sigma}, K)$. Equivalently, $B(\boldsymbol{\sigma}, K)$ is the union of all $\mathrm{St}(v, K)$ such that $v \in \mathrm{Vert}(\boldsymbol{\sigma})$.
4. $E(\boldsymbol{\sigma}, K) = K - \mathrm{St}(\boldsymbol{\sigma}, K)$. Equivalently, $E(\boldsymbol{\sigma}, K)$ is the subcomplex spanned by all simplices $\boldsymbol{\tau}$ such that $\boldsymbol{\tau} \cap \mathrm{St}(\boldsymbol{\sigma}, K) = \emptyset$. Note that $B(\emptyset, K) = E(\emptyset, K) = \emptyset$. When the simplicial complex K is understood, we will drop it from the notation and refer to these subcomplexes simply as $\mathrm{St}(\boldsymbol{\sigma})$, $\mathrm{Lk}(\boldsymbol{\sigma})$, $B(\boldsymbol{\sigma})$, and $E(\boldsymbol{\sigma})$ (Figure 6).

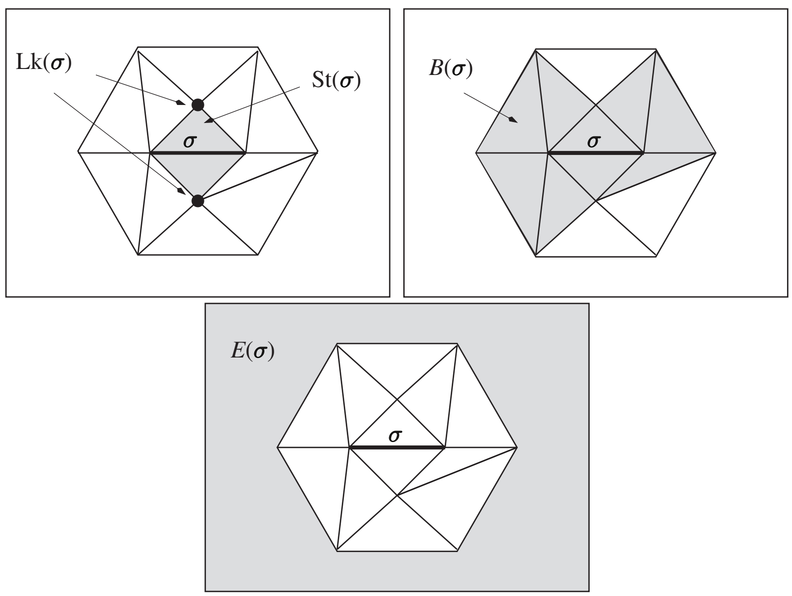

FIGURE 6

Of particular interest to us are simplicial complexes that are determined by their vertices and edges in the following sense.

Definition 7.2. Let K be a simplicial complex with vertex set V. We say that two vertices $v, w \in V$ are *adjacent* if they span an edge in K. A simplicial complex K is a *flag complex* if any set of pairwise adjacent vertices spans a simplex in K.

Remark 7.3. Recall that for a Coxeter group (W, S), the nerve $N = N(W, S)$ can be regarded as an abstract simplicial complex with vertex set S. This means that we can form a geometric simplicial complex (called the *realization of* N, and denoted by $|N|$) as follows. We identify S with the standard basis for $\mathbb{R}^S$ and we let $|N| \subset \mathbb{R}^S$ be the union of all convex hulls of subsets $\sigma \in N$. It is a standard result in topology that $\mathcal{P}(|N|) = N$. In the right-angled case, a subset $\sigma \subset S$ generates a finite group if and only if any pair of elements in σ commute; thus the (geometric realization of the) nerve of a right-angled Coxeter group is always a flag complex. Note that we have been using bold letters $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$, etc., to denote geometric simplices and nonbold letters for their respective vertex sets. This is consistent with the rest of the paper since any spherical subset $\sigma \in N$ spans a unique geometric simplex $\boldsymbol{\sigma} \subseteq |N|$.

Any full subcomplex of a flag complex is a flag complex, and it is easy to check from the definitions that if K is a flag complex, then $\mathrm{Lk}(\boldsymbol{\sigma})$, $\mathrm{St}(\boldsymbol{\sigma})$, and $E(\boldsymbol{\sigma})$ are all full subcomplexes (in fact $E(\boldsymbol{\sigma})$ is a full subcomplex regardless of whether K is a flag complex). Hence these subcomplexes are also flag. The subcomplex $B(\boldsymbol{\sigma})$ need be neither full nor flag (take K to be the boundary complex of the octahedron and $\boldsymbol{\sigma}$ to be one of the faces). The next two propositions collect the key properties of flag complexes that we shall need.

Proposition 7.4. *Let K be a flag simplicial complex, and let $\boldsymbol{\sigma}$ be a nonempty simplex in K. Then*

$$\mathrm{St}(\boldsymbol{\sigma}) = \bigcap_{v \in \mathrm{Vert}(\boldsymbol{\sigma})} \mathrm{St}(v).$$

Proof. The inclusion $\mathrm{St}(\boldsymbol{\sigma}) \subseteq \bigcap_{v \in \mathrm{Vert}(\boldsymbol{\sigma})} \mathrm{St}(v)$ is clear. We prove the opposite inclusion by induction on $\dim \boldsymbol{\sigma}$. The case $\dim \boldsymbol{\sigma} = 0$ is clear. If $\dim \boldsymbol{\sigma} > 0$, then $\boldsymbol{\sigma}$ can be written as the span of some vertex v_0 and some simplex $\boldsymbol{\tau}$ such that $\dim \boldsymbol{\tau} < \dim \boldsymbol{\sigma}$. By induction, we have

$$\mathrm{St}(\boldsymbol{\tau}) = \bigcap_{v \in \mathrm{Vert}(\boldsymbol{\tau})} \mathrm{St}(v),$$

so it suffices to prove $\mathrm{St}(v_0) \cap \mathrm{St}(\boldsymbol{\tau}) \subseteq \mathrm{St}(\boldsymbol{\sigma})$. Suppose that $\boldsymbol{\rho}$ is a simplex in $\mathrm{St}(v_0) \cap \mathrm{St}(\boldsymbol{\tau})$. Then $\boldsymbol{\rho}$ and v_0 span a simplex in K, and $\boldsymbol{\rho}$ and $\boldsymbol{\tau}$ span a simplex in K. Since v_0 and $\boldsymbol{\tau}$ also span a simplex, we know that $\{v_0\} \cup \mathrm{Vert}(\boldsymbol{\tau}) \cup \mathrm{Vert}(\boldsymbol{\rho})$ is a set of pairwise adjacent vertices. Since K is a flag complex, this set of vertices must span a simplex $\widetilde{\boldsymbol{\rho}}$. But since $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ also span the simplex $\widetilde{\boldsymbol{\rho}}$, we have $\boldsymbol{\rho} \subseteq \mathrm{St}(\boldsymbol{\sigma})$, which completes the proof. $\square$

Proposition 7.5. *Let K be a flag complex and let $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ be any two simplices of K. Set $\boldsymbol{\tau}' = \boldsymbol{\tau} \cap E(\boldsymbol{\sigma})$ and $\boldsymbol{\sigma}' = \boldsymbol{\sigma} - (\boldsymbol{\sigma} \cap \boldsymbol{\tau})$. Then*

$$E(\boldsymbol{\sigma}) \cap \mathrm{St}(\boldsymbol{\tau}) = \begin{cases} \boldsymbol{\tau}' * (\mathrm{Lk}(\boldsymbol{\tau}) \cap E(\boldsymbol{\sigma})) & \text{if } \boldsymbol{\tau}' \neq \emptyset, \\ E(\boldsymbol{\sigma}', \mathrm{Lk}(\boldsymbol{\tau})) & \text{if } \boldsymbol{\tau}' = \emptyset. \end{cases}$$

Proof. Since $\mathrm{St}(\boldsymbol{\tau}) = \boldsymbol{\tau} * \mathrm{Lk}(\boldsymbol{\tau})$ and $E(\boldsymbol{\sigma})$ is a full subcomplex, we have

$$E(\boldsymbol{\sigma}) \cap \mathrm{St}(\boldsymbol{\tau}) = (\boldsymbol{\tau} \cap E(\boldsymbol{\sigma})) * (\mathrm{Lk}(\boldsymbol{\tau}) \cap E(\boldsymbol{\sigma})) = \boldsymbol{\tau}' * (\mathrm{Lk}(\boldsymbol{\tau}) \cap E(\boldsymbol{\sigma}))$$

(Figure 7). It suffices, then, to show that if $\boldsymbol{\tau}' = \emptyset$, then

$$\mathrm{Lk}(\boldsymbol{\tau}) \cap E(\boldsymbol{\sigma}) = E(\boldsymbol{\sigma}', \mathrm{Lk}(\boldsymbol{\tau})). \tag{7.1}$$

Suppose $\boldsymbol{\tau}' = \emptyset$. Then $\boldsymbol{\tau} \subseteq \mathrm{St}(\boldsymbol{\sigma})$, so $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ span a simplex in K. By definition of $\boldsymbol{\sigma}'$, this simplex can be written as the join $\boldsymbol{\tau} * \boldsymbol{\sigma}'$. (Figure 8 gives an illustration for the case where $\boldsymbol{\tau}$ is a face of $\boldsymbol{\sigma}$, in which case $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}'$ span the simplex $\boldsymbol{\sigma}$.) To prove (7.1), we suppose $\boldsymbol{\rho}$ is a simplex in $\mathrm{Lk}(\boldsymbol{\tau})$. We need to show that $\boldsymbol{\rho} \cap \mathrm{St}(\boldsymbol{\sigma}) = \emptyset$ if and only if $\boldsymbol{\rho} \cap \mathrm{St}(\boldsymbol{\sigma}') = \emptyset$. Since $\mathrm{St}(\boldsymbol{\sigma}) \subseteq \mathrm{St}(\boldsymbol{\sigma}')$, one implication ($\Leftarrow$) is obvious. For the other direction, suppose $\boldsymbol{\rho} \cap \mathrm{St}(\boldsymbol{\sigma}') \neq \emptyset$, and pick a vertex v in $\boldsymbol{\rho} \cap \mathrm{St}(\boldsymbol{\sigma}')$.

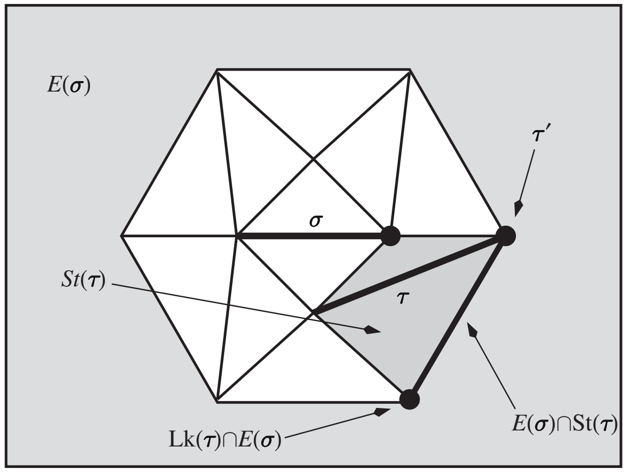

FIGURE 7

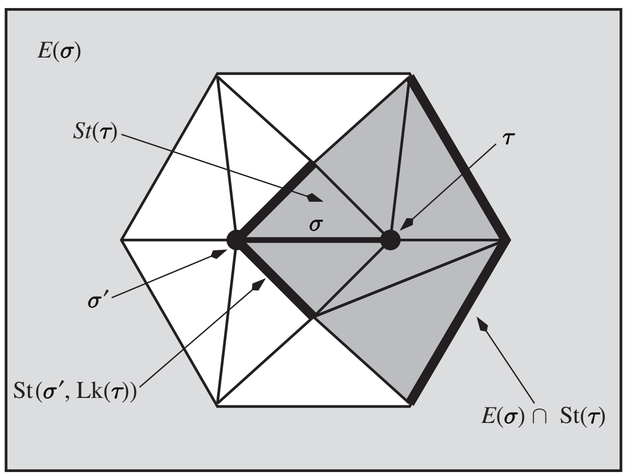

FIGURE 8

Then v and $\boldsymbol{\tau}$ span a simplex in K as do v and $\boldsymbol{\sigma}'$. But since $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}'$ span a simplex and K is a flag complex, we know that v, $\boldsymbol{\tau}$, and $\boldsymbol{\sigma}'$ span a simplex. But $\boldsymbol{\sigma}$ must be a face of this simplex (since it's a face of $\boldsymbol{\tau} * \boldsymbol{\sigma}'$). It follows that v and $\boldsymbol{\sigma}$ span a simplex, so $v \in \mathrm{St}(\boldsymbol{\sigma})$. Hence $\boldsymbol{\rho} \cap \mathrm{St}(\boldsymbol{\sigma}) \neq \emptyset$, and this completes the proof. $\square$

Surprisingly, the anti-incidence matrix turns out to be an involution when K is a sphere, and this fact lies at the heart of our reciprocity formulas. In fact, K need only resemble a sphere up to Euler characteristics in the following sense.

Definition 7.6. A simplicial complex K is an *Eulerian sphere* of dimension k if $\overline{\chi}(K) = (-1)^k$ and for every $\boldsymbol{\sigma} \in K$, $\overline{\chi}(\mathrm{Lk}(\boldsymbol{\sigma})) = (-1)^{k-1-\dim \boldsymbol{\sigma}}$. If, in addition, K is a flag complex, we shall call it an *Eulerian flag complex.*

Remark 7.7. If K is an Eulerian sphere, then so is the link of any simplex in K.

Remark 7.8. A simplicial complex K is an Eulerian sphere if and only if the poset $\mathcal{P}(K)$ (with a single maximal element $\widehat{1}$ added) is an *Eulerian poset* in the sense of [23].

Example 7.9. Obviously if K is a triangulation of a sphere, then it is Eulerian. For a different example, one can take K to be two disjoint circles (or their suspension).

The connection between Coxeter groups whose nerve is Eulerian and reciprocity formulas for growth series of those Coxeter groups is well known (see, e.g., [6]) when the series is defined over a *commutative* ring. But the matrix J seems to be new, and not only yields new proofs of some of these known formulas, but also provides the key to reciprocity formulas for noncommutative power series. Not surprisingly, the relevant facts about J are related to various Euler characteristics of K and its subcomplexes.

Lemma 7.10. *Let K be a flag complex and let $\boldsymbol{\sigma}$ be a nonempty simplex in K. Then $\overline{\chi}(B(\boldsymbol{\sigma})) = 0$. If in addition K is Eulerian, then $\overline{\chi}(E(\boldsymbol{\sigma})) = 0$.*

Proof. Applying inclusion-exclusion and Proposition 7.4 to

$$B(\boldsymbol{\sigma}) = \bigcup_{v \in \mathrm{Vert}(\boldsymbol{\sigma})} \mathrm{St}(v),$$

we obtain

$$\overline{\chi}(B(\boldsymbol{\sigma})) = \sum_{\boldsymbol{\tau} \subseteq \boldsymbol{\sigma}, \boldsymbol{\tau} \neq \emptyset} (-1)^{\dim \boldsymbol{\tau}} \overline{\chi}(\mathrm{St}(\boldsymbol{\tau})).$$

Since $\overline{\chi}(\mathrm{St}(\boldsymbol{\tau})) = 0$ ($\mathrm{St}(\boldsymbol{\tau})$ is contractible), this simplifies to 0.

The proof that $\overline{\chi}(E(\boldsymbol{\sigma})) = 0$ follows from Proposition 3.14.5 of [23], which says that up to sign, the reduced Euler characteristic of a subcomplex Q of an Eulerian sphere K coincides with the reduced Euler characteristic of the (topological) complement of Q in K (we denote this open subset of K by $K \setminus Q$ to distinguish it from the closed subcomplex $K - Q$). In the present case, we take Q to be $\mathrm{St}(\boldsymbol{\sigma})$, which has reduced Euler characteristic 0, hence so does $K \setminus \mathrm{St}(\boldsymbol{\sigma})$. We claim that $K \setminus \mathrm{St}(\boldsymbol{\sigma})$ has a piecewise linear deformation retract onto $E(\boldsymbol{\sigma}) = K - \mathrm{St}(\boldsymbol{\sigma})$. We define this retract simplex by simplex. If $\boldsymbol{\tau} \subset E(\boldsymbol{\sigma})$, then the retract restricted to $\boldsymbol{\tau}$ is the identity map. If $\boldsymbol{\tau} \not\subset E(\boldsymbol{\sigma})$, then $\boldsymbol{\tau}$ must have a nontrivial intersection with $\mathrm{St}(\boldsymbol{\sigma})$. In fact, since $\mathrm{St}(\boldsymbol{\sigma})$ is a full subcomplex of K, this intersection must be a face of $\boldsymbol{\tau}$. It follows that $\boldsymbol{\tau}$ can be written as the join $\boldsymbol{\tau}_1 * \boldsymbol{\tau}_2$, where $\boldsymbol{\tau}_1 = \boldsymbol{\tau} \cap \mathrm{St}(\boldsymbol{\sigma})$ and $\boldsymbol{\tau}_2 = \boldsymbol{\tau} \cap E(\boldsymbol{\sigma})$. The intersection of $\boldsymbol{\tau}$ with $K \setminus \mathrm{St}(\boldsymbol{\sigma})$ is $\boldsymbol{\tau} \setminus \boldsymbol{\tau}_1$, and the retract restricted to $\boldsymbol{\tau} \setminus \boldsymbol{\tau}_1$ is the standard retract onto the face $\boldsymbol{\tau}_2$ along the join lines. These retracts on simplices glue together to give a well-defined deformation retract from $K \setminus \mathrm{St}(\boldsymbol{\sigma})$ onto $E(\boldsymbol{\sigma})$. It follows that $E(\boldsymbol{\sigma})$ also has reduced Euler characteristic 0, which completes the proof. □

In fact, when K is Eulerian we can also compute Euler characteristics for the subcomplexes $E(\boldsymbol{\sigma}) \cap B(\boldsymbol{\tau})$. It turns out that these are precisely the numbers that appear when we square the anti-incidence matrix. First we consider the Euler characteristics of the intersections $E(\boldsymbol{\sigma}) \cap \mathrm{St}(\boldsymbol{\tau})$ when K is Eulerian.

Lemma 7.11. *Let K be an Eulerian flag complex, and let $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ be any two nonempty simplices in K. Then*

$$\overline{\chi}(E(\boldsymbol{\sigma}) \cap \mathrm{St}(\boldsymbol{\tau})) = \begin{cases} -1 & \text{if } \boldsymbol{\sigma} \subseteq \boldsymbol{\tau}, \\ 0 & \text{if } \boldsymbol{\sigma} \not\subseteq \boldsymbol{\tau}. \end{cases}$$

Proof. If $\boldsymbol{\tau} \subseteq \mathrm{St}(\boldsymbol{\sigma})$, then by Proposition 7.5, $E(\boldsymbol{\sigma}) \cap \mathrm{St}(\boldsymbol{\tau})$ is $\emptyset$ when $\boldsymbol{\sigma} \subseteq \boldsymbol{\tau}$ (hence has reduced Euler characteristic -1) and is of the form $E(\boldsymbol{\sigma}', \mathrm{Lk}(\boldsymbol{\tau}))$ with $\boldsymbol{\sigma}' \neq \emptyset$ otherwise. Since $\mathrm{Lk}(\boldsymbol{\tau})$ is also an Eulerian flag complex, it follows from Lemma 7.10 that in the latter case, the reduced Euler characteristic is 0.

If $\boldsymbol{\tau} \not\subseteq \mathrm{St}(\boldsymbol{\sigma})$, then $\boldsymbol{\tau}' = \boldsymbol{\tau} \cap E(\boldsymbol{\sigma}) \neq \emptyset$, so by Proposition 7.5, $E(\boldsymbol{\sigma}) \cap \mathrm{St}(\boldsymbol{\tau})$ is the join of $\boldsymbol{\tau}'$ with some other subcomplex. Such a join is always contractible, hence has reduced Euler characteristic 0. □

We can now use inclusion-exclusion and the previous lemma to compute the Euler characteristics of $E(\boldsymbol{\sigma}) \cap B(\boldsymbol{\tau})$.

Lemma 7.12. *Let K be an Eulerian flag complex, and let $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ be any two nonempty simplices in K. Then $\overline{\chi}(E(\boldsymbol{\sigma}) \cap B(\boldsymbol{\tau}))$ is equal to 0 if $\boldsymbol{\sigma} \neq \boldsymbol{\tau}$ and is equal to $(-1)^{1+\dim \boldsymbol{\sigma}}$ if $\boldsymbol{\sigma} = \boldsymbol{\tau}$.*

Proof. We first calculate the Euler characteristic of the union

$$E(\boldsymbol{\sigma}) \cup B(\boldsymbol{\tau}) = \bigcup_{v \in \mathrm{Vert}(\boldsymbol{\sigma})} (E(\boldsymbol{\sigma}) \cup \mathrm{St}(v)).$$

Applying inclusion-exclusion and Proposition 7.4, we have

$$\overline{\chi}(E(\boldsymbol{\sigma}) \cup B(\boldsymbol{\tau})) = \sum_{\boldsymbol{\rho} \subseteq \boldsymbol{\tau}, \boldsymbol{\rho} \neq \emptyset} (-1)^{\dim \boldsymbol{\rho}} \overline{\chi}(E(\boldsymbol{\sigma}) \cup \mathrm{St}(\boldsymbol{\rho})). \tag{7.2}$$

Using the previous lemma and the fact that

$$\begin{aligned} \overline{\chi}(E(\boldsymbol{\sigma}) \cup \mathrm{St}(\boldsymbol{\rho})) &= \overline{\chi}(E(\boldsymbol{\sigma})) + \overline{\chi}(\mathrm{St}(\boldsymbol{\rho})) - \overline{\chi}(E(\boldsymbol{\sigma}) \cap \mathrm{St}(\boldsymbol{\rho})) \\ &= -\chi(E(\boldsymbol{\sigma}) \cap \mathrm{St}(\boldsymbol{\rho})), \end{aligned}$$

we have

$$\overline{\chi}(E(\boldsymbol{\sigma}) \cup \mathrm{St}(\boldsymbol{\rho})) = \begin{cases} 0 & \text{if } \boldsymbol{\sigma} \not\subseteq \boldsymbol{\rho}, \\ 1 & \text{if } \boldsymbol{\sigma} \subseteq \boldsymbol{\rho}. \end{cases}$$

Substituting into (7.2) gives

$$\overline{\chi}(E(\boldsymbol{\sigma}) \cup B(\boldsymbol{\tau})) = \sum_{\boldsymbol{\sigma} \subseteq \boldsymbol{\rho} \subseteq \boldsymbol{\tau}} (-1)^{\dim \boldsymbol{\rho}}.$$

This evaluates to 0 if $\boldsymbol{\sigma} \neq \boldsymbol{\tau}$ and evaluates to $(-1)^{\dim \boldsymbol{\sigma}}$ if $\boldsymbol{\sigma} = \boldsymbol{\tau}$. To complete the proof, we use the identity

$$\begin{aligned} \overline{\chi}(E(\boldsymbol{\sigma}) \cap B(\boldsymbol{\tau})) &= \overline{\chi}(E(\boldsymbol{\sigma})) + \overline{\chi}(B(\boldsymbol{\tau})) - \overline{\chi}(E(\boldsymbol{\sigma}) \cup B(\boldsymbol{\tau})) \\ &= -\overline{\chi}(E(\boldsymbol{\sigma}) \cup B(\boldsymbol{\tau})). \end{aligned}$$

□

We now come to the main result of the section.

Theorem 7.13. *Let K be a flag complex and let $J = J(K)$ be the anti-incidence matrix for K. Then the columns of J all have sum $\overline{\chi}(K)$. If, in addition, K is Eulerian, then $J^2 = \mathrm{Id}$.*

Proof. The column of J corresponding to $\boldsymbol{\tau}$ has sum

$$\sum_{\boldsymbol{\tau}\cap\mathrm{St}(\boldsymbol{\sigma})=\emptyset}(-1)^{\dim\boldsymbol{\sigma}}=\sum_{\boldsymbol{\sigma}\subset K}(-1)^{\dim\boldsymbol{\sigma}}-\sum_{\boldsymbol{\tau}\cap\mathrm{St}(\boldsymbol{\sigma})\neq\emptyset}(-1)^{\dim\boldsymbol{\sigma}}.$$

This simplifies to $\overline{\chi}(K)-\overline{\chi}(B(\boldsymbol{\tau}))$ if $\boldsymbol{\tau}\neq\emptyset$ and to $\overline{\chi}(K)-0$ if $\boldsymbol{\tau}=\emptyset$. Thus, by Lemma 7.10, we obtain $\overline{\chi}(K)$ in either case.

Now assume that K is Eulerian. The $(\boldsymbol{\sigma},\boldsymbol{\tau})$-entry of J^2 is

$$\sum_{\boldsymbol{\rho}\cap\mathrm{St}(\boldsymbol{\sigma})=\emptyset,\boldsymbol{\tau}\cap\mathrm{St}(\boldsymbol{\rho})=\emptyset}(-1)^{\dim\boldsymbol{\sigma}}(-1)^{\dim\boldsymbol{\rho}}.$$

If $\boldsymbol{\sigma}=\boldsymbol{\tau}=\emptyset$, this entry simplifies to 1. If $\boldsymbol{\sigma}=\emptyset\neq\boldsymbol{\tau}$, this entry simplifies to 0. If $\boldsymbol{\sigma}\neq\emptyset=\boldsymbol{\tau}$, this entry simplifies to

$$(-1)^{\dim\boldsymbol{\sigma}}\sum_{\boldsymbol{\rho}\cap\mathrm{St}(\boldsymbol{\sigma})=\emptyset}(-1)^{\dim\boldsymbol{\rho}}=(-1)^{\dim\boldsymbol{\sigma}}\overline{\chi}(E(\boldsymbol{\sigma})),$$

which is equal to 0 by Lemma 7.10. If $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are both nontrivial, this entry simplifies to

$$\begin{aligned}&(-1)^{\dim\boldsymbol{\sigma}}\left(\sum_{\boldsymbol{\rho}\cap\mathrm{St}(\boldsymbol{\sigma})=\emptyset}(-1)^{\dim\boldsymbol{\rho}}-\sum_{\boldsymbol{\rho}\cap\mathrm{St}(\boldsymbol{\sigma})=\emptyset,\boldsymbol{\tau}\cap\mathrm{St}(\boldsymbol{\rho})\neq\emptyset}(-1)^{\dim\boldsymbol{\rho}}\right)\\&=(-1)^{\dim\boldsymbol{\sigma}}(\overline{\chi}(E(\boldsymbol{\sigma}))-\overline{\chi}(E(\boldsymbol{\sigma})\cap B(\boldsymbol{\tau})))\\&=(-1)^{1+\dim\boldsymbol{\sigma}}\overline{\chi}(E(\boldsymbol{\sigma})\cap B(\boldsymbol{\tau})),\end{aligned}$$

which by Lemma 7.12 is equal to 0 when $\boldsymbol{\sigma}\neq\boldsymbol{\tau}$ and is equal to 1 when $\boldsymbol{\sigma}=\boldsymbol{\tau}$. It follows that J^2 is the identity matrix. □

8. Reciprocity for right-angled Coxeter groups

We now return to the proof of our reciprocity formula and combine the results of the previous two sections. An important observation that makes the series $\chi_{\mathcal{L}}$ for right-angled Coxeter groups much more manageable than for arbitrary Coxeter groups is that the regions of the Tits cone cut out by minimal hyperplanes are actually in bijection with the nerve. More precisely, we have the following (with notation as in Section 6).

Proposition 8.1. *Assume that W is a right-angled Coxeter group. Then the set $\mathcal{H}_{min}$ of minimal hyperplanes is precisely the set of fundamental hyperplanes. Moreover, the correspondence $\sigma\mapsto C_\sigma$ defines a bijection from the nerve N to the set of regions $\mathcal{R}$ cut out by the minimal hyperplanes.*

Proof. For a right-angled Coxeter group, the regions C_σ are precisely those cut out by the fundamental hyperplanes; hence the second statement follows immediately from the first. To show that the fundamental hyperplanes are the only minimal ones, suppose H is a minimal hyperplane, and let wC be a chamber having H as a wall (i.e., $H\cap wC$ is a codimension-one face of wC). Assume that wC is chosen so that w has minimal length. Since $w^{-1}H$ is a wall of the fundamental chamber, it is a fundamental hyperplane, say H_s. If $|w|=0$, we're done since $wC=C$, so H is a fundamental hyperplane. Otherwise, there exists an $s'\in S$ such that ws' is shorter than w. Let H' be the hyperplane fixed by the reflection $r=ws'w^{-1}$. Then H and

H' are w translates of the fundamental hyperplanes H_s and $H_{s'}$, hence are either perpendicular or parallel. If they were parallel, then H' would separate H from the fundamental chamber, contradicting the minimality of H. On the other hand, if H and H' were perpendicular, then the chamber $ws'C$ would still intersect H in a codimension-one face, contradicting the minimality of the length of w. Hence wC must, in fact, be the fundamental chamber and H a fundamental hyperplane. $\square$

Since $\mathcal{R} = \{C_\sigma \mid \sigma \in N\}$, the recurrence described in Theorem 6.6 has a simpler description, depending only on the combinatorics of the nerve. Recall that the nerve N can be regarded as an abstract simplicial complex with geometric realization $|N|$. As in Remark 7.3, for any abstract simplex $\sigma \in N$, we let $\boldsymbol{\sigma}$ denote the corresponding geometric simplex in $|N|$.

Replacing R with C_τ and using the fact that $R' \subset C_\sigma$ implies $R' = C_\sigma$, we obtain the following simplification of Lemma 6.5.

Proposition 8.2. *For $\sigma, \tau \in N$, $w_\sigma C_\tau \subseteq C_\sigma$ if and only if* $\mathrm{St}(\boldsymbol{\sigma}) \cap \boldsymbol{\tau} = \emptyset$.

Proof. Note that for a right-angled Coxeter group, $\sigma \in N$ implies that $w_\sigma = \prod_{s\in\sigma} s$ (the order of multiplication doesn't matter). First we show that the condition $\mathrm{St}(\boldsymbol{\sigma}) \cap \boldsymbol{\tau} = \emptyset$ is equivalent to the statement $\mathrm{Desc}(w_\sigma w_\tau) = \sigma$. Indeed, $s \in \mathrm{St}(\boldsymbol{\sigma}) \cap \boldsymbol{\tau}$ implies that either (1) $s \notin \sigma$ and $\{s\} \cup \sigma \in N$ in which case $w_\sigma w_\tau$ has a reduced expression starting with s (hence $s \in \mathrm{Desc}(w_\sigma w_\tau)$) or (2) $s \in \sigma \cap \tau$ in which case the factors of s in w_σ and w_τ cancel in the product $w_\sigma w_\tau$ (hence $s \notin \mathrm{Desc}(w_\sigma)$). In either case, $\sigma \neq \mathrm{Desc}(w_\sigma w_\tau)$. Conversely, if $\mathrm{St}(\boldsymbol{\sigma}) \cap \boldsymbol{\tau} = \emptyset$, then (1) the product $\prod_{s\in\sigma} s \prod_{s\in\tau} s$ is a reduced expression for $w_\sigma w_\tau$ (there is no cancellation) and (2) no element of τ can be moved to the front of the product. Thus the set of $s \in S$ that shorten $w_\sigma w_\tau$ is precisely σ.

To finish the proof note that $w_\sigma C_\tau \subset C_\sigma$ is equivalent to $w_\sigma w_\tau C \subseteq C_\sigma$, which by Proposition 6.2 is equivalent to $\mathrm{Desc}(w_\sigma w_\tau) = \sigma$. $\square$

As before, we let χ denote the characteristic series for the greedy normal form for (W, S). Again, we fix a total ordering on N (with $\emptyset$ as the first element in the order). By Proposition 8.2 and Theorem 6.6, we have

$$\chi = A(I - Q)^{-1}B,$$

where

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$$

and Q is the $N \times N$ matrix

$$Q_{\sigma,\tau} = \begin{cases} \sigma & \text{if } \sigma \neq \emptyset \text{ and } \mathrm{St}(\boldsymbol{\sigma}) \cap \boldsymbol{\tau} = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Example 8.3. Let W be the group generated by reflections across the sides of a right-angled pentagon. That is, W has 5 generators s_1,s_2,s_3,s_4,s_5, and the only relations specify that each s_i is an involution and that s_i and s_j commute whenever $j = i + 1 \bmod 5$. Figure 9 shows the Tits cone (on the left) and the nerve N (on the right). The nerve N consists of the subsets $\emptyset$, $\sigma_1 = \{s_1\}$, $\sigma_2 = \{s_2\}$, $\sigma_3 = \{s_3\}$, $\sigma_4 = \{s_4\}$, $\sigma_5 = \{s_5\}$, $\sigma_{12} = \{s_1, s_2\}$, $\sigma_{15} = \{s_1, s_5\}$, $\sigma_{23} = \{s_2, s_3\}$, $\sigma_{34} = \{s_3, s_4\}$,

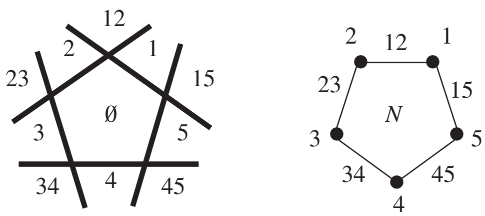

FIGURE 9

and $\sigma_{45} = \{s_4, s_5\}$. With respect to this order, the matrix Q in this case is

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma_1 & 0 & 0 & \sigma_1 & \sigma_1 & 0 & 0 & 0 & 0 & \sigma_1 & 0 \\ \sigma_2 & 0 & 0 & 0 & \sigma_2 & \sigma_2 & 0 & 0 & 0 & 0 & \sigma_2 \\ \sigma_3 & \sigma_3 & 0 & 0 & 0 & \sigma_3 & 0 & \sigma_3 & 0 & 0 & 0 \\ \sigma_4 & \sigma_4 & \sigma_4 & 0 & 0 & 0 & \sigma_4 & 0 & 0 & 0 & 0 \\ \sigma_5 & 0 & \sigma_5 & \sigma_5 & 0 & 0 & 0 & 0 & \sigma_5 & 0 & 0 \\ \sigma_{12} & 0 & 0 & \sigma_{12} & \sigma_{12} & \sigma_{12} & 0 & 0 & 0 & \sigma_{12} & \sigma_{12} \\ \sigma_{15} & 0 & \sigma_{15} & \sigma_{15} & \sigma_{15} & 0 & 0 & 0 & \sigma_{15} & \sigma_{15} & 0 \\ \sigma_{23} & \sigma_{23} & 0 & 0 & \sigma_{23} & \sigma_{23} & 0 & \sigma_{23} & 0 & 0 & \sigma_{23} \\ \sigma_{34} & \sigma_{34} & \sigma_{34} & 0 & 0 & \sigma_{34} & \sigma_{34} & \sigma_{34} & 0 & 0 & 0 \\ \sigma_{45} & \sigma_{45} & \sigma_{45} & \sigma_{45} & 0 & 0 & \sigma_{45} & 0 & \sigma_{45} & 0 & 0 \end{bmatrix}.$$

In principle, one could obtain a rational expression for χ by formally inverting the matrix $I - Q$ using row-reduction. The resulting expression would be rather unwieldy. On the other hand, if we let $\mathbb{Q}[[\mathbf{t}]]$ denote the multivariable power series ring with *commuting* parameters $\mathbf{t} = (t_\sigma)$ (in this case, 10 parameters), then we can consider the image χ^{abel} of χ under the substitution homomorphism $\mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle \to \mathbb{Q}[[\mathbf{t}]]$ that takes each σ to the corresponding t_σ. On general principles, the resulting power series will be a rational function of the form

$$\chi^{abel}(\mathbf{t}) = \frac{p(\mathbf{t})}{q(\mathbf{t})},$$

which can be computed easily using a computer algebra package. In this case, the polynomials p and q we obtained both have degree 10, p with 244 terms and q with 154. A more manageable substitution is to put every $\sigma_i = x$ and every $\sigma_{ij} = y$, and treat x and y as commuting variables. In this case, the resulting series $\overline{\chi}(x, y)$ has the rational expression

$$\overline{\chi}(x, y) = \frac{xy + 3x + 3y + 1}{xy - 2x - 2y + 1}.$$

We consider this example further in Example 8.8, below.

Our first result gives a condition under which the reciprocal χ^* exists and can be explicitly computed. As in the previous section, we let $J \in \mathbb{Q}^{n\times n}$ be the matrix given by

$$J_{\sigma,\tau} = \begin{cases} (-1)^{\dim \boldsymbol{\sigma}} & \text{if } \mathrm{St}(\boldsymbol{\sigma}) \cap \boldsymbol{\tau} = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Then the matrix Q has a factorization of the form

$$Q = D_0 J,$$

where D_0 is a quasi-regular diagonal matrix. More precisely, let $D \in \mathbb{Q}\langle K\rangle^{n\times n}$ be the diagonal matrix given by

$$D_{\sigma,\sigma} = \begin{cases} 1 & \text{if } \sigma = \emptyset, \\ (-1)^{\dim \boldsymbol{\sigma}}\sigma & \text{otherwise.} \end{cases}$$

Then D_0 is the matrix obtained from D by replacing the first entry with 0. We then have the following.

Lemma 8.4. *Let (W, S) be a right-angled Coxeter group, let N denote the nerve, and let χ denote the characteristic series of the greedy normal form. If the matrix $J = J(N)$ is invertible over $\mathbb{Q}$, then the reciprocal χ^* exists and is given by the formula*

$$\chi^* = -\frac{1}{\overline{\boldsymbol{\chi}}(N)} A(I - D_0 J^{-1})^{-1} B.$$

In particular, χ^* is a power series in $\mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle$.

Proof. First we note that $I - \overline{Q}$ can be written as

$$I - \overline{Q} = I - \overline{D_0}J = \overline{D}D - \overline{D_0}J.$$

Letting P denote the square matrix with all zeros except a 1 in the top left entry, we then have

$$\overline{D}D - \overline{D_0}J = (\overline{D}D - P) - (\overline{D_0}J - P) = \overline{D}D_0 - \overline{D}J.$$

Combining these equations, we obtain

$$I - \overline{Q} = \overline{D}D_0 - \overline{D}J = \overline{D}(-J + D_0).$$

If J is invertible, this gives

$$I - \overline{Q} = (-\overline{D})(I - D_0 J^{-1})J.$$

Since $D^{-1} = \overline{D}$ and $D_0 J^{-1}$ has quasi-regular entries, $I - \overline{Q}$ is invertible with inverse given by

$$(I - \overline{Q})^{-1} = J^{-1}(I - D_0 J^{-1})^{-1}(-D).$$

Substituting this into

$$\chi^* = A(I - \overline{Q})^{-1}B$$

gives

$$\chi^* = AJ^{-1}(I - D_0 J^{-1})^{-1}(-DB).$$

By Theorem 7.13, we have $AJ = \overline{\boldsymbol{\chi}}(N)A$. Multiplying by J^{-1} on the right and dividing by $\overline{\boldsymbol{\chi}}(N)$ (which is necessarily nonzero since J is invertible) gives $AJ^{-1} = \frac{1}{\overline{\boldsymbol{\chi}}(N)}A$. Substituting this and $-DB = -B$ into the previous expression for χ^* gives the desired formula. □

Example 8.5. Let (W, S) be the infinite dihedral group D_∞. That is, S has two noncommuting generators $\{x, y\}$, and the nerve is the simplicial complex consisting of two distinct points (i.e., the nerve is the 0-dimensional sphere). The matrix Q and the calculation of χ and χ^* were already illustrated in Examples 3.3 and 4.3. Note that in this case $\chi^* = -\chi$.

Example 8.6. Let (W, S) be the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. This time S consists of 3-elements $\{x, y, z\}$, no two of which commute. The nerve is 3 points and the matrices Q, J, and D are given by:

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ x & 0 & x & x \\ y & y & 0 & y \\ z & z & z & 0 \end{bmatrix}, \ J = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix}.$$

Since J is invertible with inverse

$$J^{-1} = \frac{1}{2} \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},$$

the reciprocal χ^* exists (by Lemma 8.4) and is given by

$$\begin{aligned} \chi^* &= -\frac{1}{2} A(I - D_0 J^{-1})^{-1} B \\ &= -\frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \left(I - \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ x & -x & x & x \\ y & y & -y & y \\ z & z & z & -z \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= -\frac{1}{2} - \frac{1}{4}(x + y + z) - \frac{1}{8}(xy + xz + yx + yz + zx + zy - x^2 - y^2 - z^2) - \cdots . \end{aligned}$$

We can now prove the main theorem.

Theorem 8.7 (Reciprocity)**.** *If (W, S) is a right-angled Coxeter group and the nerve N is an Eulerian k-sphere, then χ^* exists and*

$$\chi^* = (-1)^{k+1} \chi.$$

Proof. By Lemma 8.4, χ^* exists and is given by

$$\chi^* = -\frac{1}{\overline{\boldsymbol{\chi}}(N)} A(I - D_0 J^{-1})^{-1} B.$$

Since N is an Eulerian k-sphere, $\overline{\boldsymbol{\chi}}(N) = (-1)^k$ and J is an involution (by Theorem 7.13). Thus, we have

$$\chi^* = -(-1)^k A(I - D_0 J)^{-1} B = (-1)^{k+1} A(I - Q)^{-1} B = (-1)^{k+1} \chi.$$

$\square$

Example 8.8. The right-angled pentagonal Coxeter group considered in Example 8.3 has nerve equal to the boundary complex of a pentagon, hence is an Eulerian 1-sphere. By Theorem 8.7, the characteristic series χ satisfies $\chi^* = \chi$. It follows that the rational function $\chi^{abel}(\mathbf{t}) = \frac{p(\mathbf{t})}{q(\mathbf{t})}$ described in Example 8.3 satisfies the reciprocity formula

$$\chi^{abel}(\mathbf{t}^{-1}) = \chi^{abel}(\mathbf{t}),$$

where the expression on the left-hand side denotes the rational function obtained by replacing each commuting parameter t_σ in $\chi^{abel}(\mathbf{t})$ with its reciprocal $1/t_\sigma$. This formula can be verified easily with a computer algebra package. For an even simpler illustration of reciprocity, we can use the series $\overline{\chi}(x,y)$ obtained by the substitution $\sigma_i = x$, $\sigma_{ij} = y$. In this case, the resulting series has the rational expression

$$\overline{\chi}(x,y) = \frac{xy + 3x + 3y + 1}{xy - 2x - 2y + 1},$$

which clearly satisfies the reciprocity formula

$$\overline{\chi}(x^{-1}, y^{-1}) = \overline{\chi}(x,y).$$

The following example shows that the right-angled assumption in Theorem 8.7 cannot be removed.

Example 8.9. Consider the affine Coxeter group $\widetilde{B}_2$. The nerve N is the boundary complex of a triangle. Though not a flag complex, N *is* an Eulerian sphere, and the usual growth series with respect to the standard generators has the rational expression

$$\gamma(t) = \frac{t^4 + 2t^3 + 2t^2 + 2t + 1}{t^4 - t^3 - t + 1}.$$

Since $\widetilde{B_2}$ is an affine Coxeter group, this example falls into Serre's original class of examples for which the reciprocity formula $\gamma(1/t) = \gamma(t)$ holds. On the other hand, the characteristic series χ for the greedy normal form *does not* satisfy the reciprocity formula $\chi^* = \chi$. If it did, then substituting the single variable t for each $\sigma \in \mathcal{A}$ in χ would result in a series $\overline{\chi}(t)$ which satisfies the reciprocity formula. But as we computed in Example 6.8, this series has the rational representation

$$\overline{\chi} = \frac{1 + 4t - 3t^2 + 4t^3 - 2t^4}{1 - 2t + t^2},$$

and this does not satisfy $\overline{\chi}(1/t) = \overline{\chi}(t)$.

9. Applications to growth series

Standard generators. Let W be a Coxeter group with generating set S, and let $i : S \to I$ be any function that is constant on conjugacy classes. Let $\mathbf{t}$ denote the I-tuple $(t_i)_{i \in I}$, and let $\mathbb{Q}[[\mathbf{t}]]$ denote the ring of formal power series in the commuting variables t_i. For any $w \in W$, let $s_1 \cdots s_n$ be a reduced expression for w. Then the monomial $\prod_{j=1}^{n} t_{i(s_j)} \in \mathbb{Q}[[\mathbf{t}]]$ does not depend on the choice of reduced expression (this follows from Tits' solution to the word problem; see, e.g., [8]). We denote this monomial by t^w. We then define the *standard growth series* $\gamma(\mathbf{t}) \in \mathbb{Q}[[\mathbf{t}]]$ by

$$\gamma(\mathbf{t}) = \sum_{w \in W} t^w.$$

Similarly, we let $\mathbb{Q}[W][[\mathbf{t}]]$ denote the ring of formal power series with commuting parameters $\mathbf{t}$ and coefficients in the (noncommutative) group ring $\mathbb{Q}[W]$. The *standard complete growth series* $\widetilde{\gamma}(\mathbf{t}) \in \mathbb{Q}[W][[\mathbf{t}]]$ is then defined by

$$\widetilde{\gamma}(\mathbf{t}) = \sum_{w \in W} w t^w.$$

Note that when I is a singleton set, γ (respectively, $\widetilde{\gamma}$) is the single-variable standard (resp., complete) growth series. At the other extreme, if W is right-angled (or more generally if all $m(s,s')$ are even), then no two generators in S are conjugate in W, so we can take $I = S$ and $i : S \to S$ to be the identity map. In this case, we have a parameter t_s for each generator $s \in S$.

Now let $\mathcal{A}$ be the proper nerve of W, and let $\mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle$ denote the corresponding power series ring. We define homomorphisms $\phi : \mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle \to \mathbb{Q}[[\mathbf{t}]]$ and $\widetilde{\phi} : \mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle \to \mathbb{Q}[W][[\mathbf{t}]]$ by $\phi(\sigma) = t^{w_\sigma}$ and $\widetilde{\phi}(\sigma) = w_\sigma t^{w_\sigma}$. Let $\mathcal{L} \subseteq \mathcal{A}^*$ be the greedy normal form for W and let χ be its characteristic series. Then

$$\phi(\chi) = \gamma(\mathbf{t})$$

and

$$\widetilde{\phi}(\chi) = \widetilde{\gamma}(\mathbf{t}).$$

Since ϕ and $\widetilde{\phi}$ take rational series to rational series, Theorem 6.6 gives the following.

Corollary 9.1. *For any Coxeter group, both the standard growth series $\gamma(\mathbf{t})$ and the complete growth series $\widetilde{\gamma}(\mathbf{t})$ are rational.*

Since ϕ and $\widetilde{\phi}$ are homomorphisms, Theorem 8.7 gives the following.

Corollary 9.2. *Let W be a right-angled Coxeter group and assume the nerve is an Eulerian k-sphere. Then*

$$\gamma(\mathbf{t}^{-1}) = (-1)^{k+1}\gamma(\mathbf{t})$$

and

$$\widetilde{\gamma}(\mathbf{t}^{-1}) = (-1)^{k+1}\widetilde{\gamma}(\mathbf{t}),$$

where $\mathbf{t}^{-1}$ denotes the I-tuple $(t_i^{-1})_{i\in I}$.

Most of these rationality and reciprocity formulas are well known, with the possible exception of the reciprocity formula for the complete growth series.

Greedy generators. Let W be a Coxeter group and let $A \subset W$ be the generating set $A = \{w_\sigma \mid \sigma \in \mathcal{A}\}$. We define the (single variable) *(left) greedy growth series* $\gamma_A(t) \in \mathbb{Q}[[t]]$ by

$$\gamma_A(t) = \sum_{w\in W} t^{|w|},$$

where $|w|$ now denotes the word length of w *with respect to A*. This series $\gamma_A(t)$ was considered by the author and R. Glover in [12] under the name "automatic growth series". (In fact, it was the observation that rational expressions for some of these series had palindromic numerator and denominator which suggested a more general reciprocity formula.) Similarly, one defines the *greedy complete growth series* by

$$\widetilde{\gamma}_A(t) = \sum_{w\in W} wt^{|w|}.$$

For general Coxeter groups, the length (in $\mathcal{A}^*$) of a greedy word α need not coincide with the length of $w = \pi(\alpha)$.

Example 9.3. Let W be the dihedral group of order 8 with generators s, t. Then $\mathcal{A} = \{\{s\}, \{t\}, \{s,t\}\}$ and $A = \{s, t, stst\}$. The element $w = sts$ has $\alpha = \{s\}\{t\}\{s\}$ as its greedy representative, but can be written as a product of just two generators: $w = (stst)(t)$.

However, if W is right-angled and $\alpha \in \mathcal{L}$, then the length of α in $\mathcal{A}^*$ does coincide with $|\pi(\alpha)|$. This means that for right-angled Coxeter groups, summing $t^{|w|}$ over all $w \in W$ is the same as summing $t^{l(\alpha)}$ over all $\alpha \in \mathcal{L}$ (where $l(\alpha)$ denotes the length of α as a word in $\mathcal{A}^*$). Hence, if we define homomorphisms $\phi_A : \mathbb{Q}\langle\langle\mathcal{A}\rangle\rangle \to \mathbb{Q}[[t]]$

and $\widetilde{\phi}_A : \mathbb{Q}\langle\langle \mathcal{A} \rangle\rangle \to \mathbb{Q}[W][[t]]$ by $\phi_A(\sigma) = t$ and by $\widetilde{\phi}_A(\sigma) = w_\sigma t$, respectively, then we get

$$\phi_A(\chi) = \gamma_A(t)$$

and

$$\widetilde{\phi}_A(\chi) = \widetilde{\gamma}_A(t).$$

Now applying our main theorems gives the following.

Corollary 9.4. *Let W be a right-angled Coxeter group, and let $\gamma_A(t)$ and $\widetilde{\gamma}_A(t)$ denote, respectively, the greedy growth series and complete greedy growth series of W. Then $\gamma_A(t)$ and $\widetilde{\gamma}_A(t)$ are both rational series. Moreover, if the nerve of W is an Eulerian k-sphere, then*

$$\gamma_A(t^{-1}) = (-1)^{k+1}\gamma_A(t)$$

and

$$\widetilde{\gamma}_A(t^{-1}) = (-1)^{k+1}\widetilde{\gamma}_A(t).$$

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