TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 363, Number 1, January 2011, Pages 417–437
S 0002-9947(2010)05099-8
Article electronically published on August 27, 2010

EXTENSIONS OF THE FROBENIUS TO THE RING OF DIFFERENTIAL OPERATORS ON A POLYNOMIAL ALGEBRA IN PRIME CHARACTERISTIC

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Abstract. Let K be a field of characteristic $p > 0$. It is proved that each automorphism $\sigma \in \mathrm{Aut}_K(\mathcal{D}(P_n))$ of the ring $\mathcal{D}(P_n)$ of differential operators on a polynomial algebra $P_n = K[x_1, \ldots, x_n]$ is *uniquely* determined by the elements $\sigma(x_1), \ldots, \sigma(x_n)$, and that the set $\mathrm{Frob}(\mathcal{D}(P_n))$ of all the extensions of the Frobenius (homomorphism) from certain maximal commutative polynomial subalgebras of $\mathcal{D}(P_n)$, such as P_n, to the ring $\mathcal{D}(P_n)$ is equal to $\mathrm{Aut}_K(\mathcal{D}(P_n)) \cdot \mathcal{F}$ where $\mathcal{F}$ is the set of all the extensions of the Frobenius from P_n to $\mathcal{D}(P_n)$ that leave invariant the subalgebra of scalar differential operators. The set $\mathcal{F}$ is found explicitly; it is large (a typical extension depends on *countably* many independent parameters).

Contents

Received by the editors August 21, 2008 and, in revised form, May 3, 2009.
2000 *Mathematics Subject Classification.* Primary 13A35, 13N10, 16S32, 16W20, 16W22.
Key words and phrases. Extensions of the Frobenius, ring of differential operators, Frobenius polynomial subalgebra, group of automorphisms.

1. Introduction

Throughout, ring means an associative ring with 1, $\mathbb{N} := \{0, 1, \dots\}$ is the set of natural numbers, p is a prime number, $\mathbb{F}_p := \mathbb{Z}/\mathbb{Z}p$ is the field that contains p elements, K is an arbitrary field of characteristic $p > 0$ (if it is not stated otherwise), $P_n := K[x_1, \dots, x_n]$ is a polynomial algebra, $\mathcal{D}(P_n) = \bigoplus_{\alpha\in\mathbb{N}^n} P_n\partial^{[\alpha]}$ is the ring of differential operators on P_n, where $\partial^{[\alpha]} := \prod_{i=1}^n \frac{\partial_i^{\alpha_i}}{\alpha_i!}$, and $\Delta_n = \bigoplus_{\alpha\in\mathbb{N}^n} K\partial^{[\alpha]}$ is the algebra of *scalar* differential operators on P_n.

Rigidity of the group of automorphisms $\mathrm{Aut}_K(\mathcal{D}(P_n))$. In characteristic zero, there is a strong connection between the groups $\mathrm{Aut}_K(P_n)$ and $\mathrm{Aut}_K(\mathcal{D}(P_n))$ as, for example, the (essential) equivalence of the Jacobian Conjecture for P_n and the Dixmier Problem/Conjecture for $\mathcal{D}(P_n)$ shows (see [1], [8], [6]; see also [5]). Moreover, in the class of all the associative algebras, conjecture such as the two mentioned conjectures makes sense only for the algebras $P_m \otimes \mathcal{D}(P_n)$ as was proved in [3] (the two conjectures can be reformulated in terms of locally nilpotent derivations that satisfy certain conditions, and the algebras $P_m \otimes \mathcal{D}(P_n)$ are the only associative algebras that have such derivations). This general conjecture is true iff either the JC or the DC is true; see [3].

In prime characteristic, relations between the two groups, $\mathrm{Aut}_K(P_n)$ and $\mathrm{Aut}_K(\mathcal{D}(P_n))$, are even tighter, as the following result shows.

Theorem 1.1 (Rigidity of the group $\mathrm{Aut}_K(\mathcal{D}(P_n))$)**.** *Let K be a field of characteristic $p > 0$, and $\sigma, \tau \in \mathrm{Aut}_K(\mathcal{D}(P_n))$. Then $\sigma = \tau$ iff $\sigma(x_1) = \tau(x_1), \dots, \sigma(x_n) = \tau(x_n)$.*

Remark. Theorem 1.1 does not hold in characteristic zero (e.g., the automorphisms $\sigma : x \mapsto x + \partial$, $\partial \mapsto \partial$, and $\tau = \mathrm{id}_{\mathcal{D}(P_1)}$ of the first *Weyl* algebra $\mathcal{D}(P_1) = K\langle x, \partial \,|\, \partial x - x\partial = 1\rangle$ are distinct but $\sigma(x) = \tau(x)$). In general, in prime characteristic, Theorem 1.1 does not hold for localizations of the polynomial algebra P_n, e.g., $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, [4]. Note that the K-algebra $\mathcal{D}(P_n)$ is not finitely generated and is neither left nor right Noetherian.

Extensions of the Frobenius to the ring of differential operators $\mathcal{D}(P_n)$. In the paper, we are interested in the question of extending the *Frobenius homomorphism* (the Frobenius, for short),

$$F : P_n \to P_n, \ \ a \mapsto a^p,$$

to the ring of differential operators $\mathcal{D}(P_n)$. The canonical one is

$$F_x : \mathcal{D}(P_n) \to \mathcal{D}(P_n), \ \ \partial^{[\alpha]} \mapsto \partial^{[p\alpha]}, \ \ \alpha \in \mathbb{N}^n,$$

which is called the *canonical Frobenius* on $\mathcal{D}(P_n)$ that corresponds to the set of generators $(x) = (x_1, \dots, x_n)$ for the polynomial algebra P_n. As a curious fact, note that, by a trivial reason, the Frobenius F *cannot* be extended from the polynomial algebra P_n to the Weyl algebra A_n. To the contrary, as it is proved in the paper, the set $\mathrm{Frob}(\mathcal{D}(P_n), P_n)$ of all extensions of the Frobenius F on P_n to the algebra $\mathcal{D}(P_n)$ is massive (a typical extension depends on countably many parameters).

A ring homomorphism $F' : \mathcal{D}(P_n) \to \mathcal{D}(P_n)$ is called a *Frobenius* if it is an extension of the Frobenius from a certain polynomial subalgebra $P_n = K[x_1', \dots, x_n']$ of the ring $\mathcal{D}(P_n)$ (which is called a *Frobenius polynomial subalgebra*) that satisfies natural $\mathrm{Aut}_K(\mathcal{D}(P_n))$-invariant conditions like the canonical Frobenius F_x does (see

Section 4 for the details). Let $\mathrm{Frob}(\mathcal{D}(P_n))$ be the set of all the Frobenius homomorphisms on $\mathcal{D}(P_n)$ and $\mathrm{FPol}(\mathcal{D}(P_n))$ be the set of all the Frobenius polynomial subalgebras of $\mathcal{D}(P_n)$. It is shown that each Frobenius polynomial subalgebra is a maximal commutative subalgebra of $\mathcal{D}(P_n)$ (Corollary 4.1.(1)), and so one Frobenius polynomial subalgebra cannot properly contain another Frobenius polynomial subalgebra.

Definition. Let $\mathrm{Frob}(\mathcal{D}(P_n), P_n, \Delta_n)$ be the set of all the extensions F' of the Frobenius F on the polynomial algebra P_n to $\mathcal{D}(P_n)$ such that $F'(\Delta_n) \subseteq \Delta_n$.

The next theorem describes the sets $\mathrm{Frob}(\mathcal{D}(P_n))$, $\mathrm{Frob}(\mathcal{D}(P_n), P_n)$ and $\mathrm{FPol}(\mathcal{D}(P_n))$ up to the action of the groups $\mathrm{Aut}_K(\mathcal{D}(P_n))$ and $\mathrm{Aut}_K(P_n)$.

Theorem 1.2. (1) $\mathrm{Frob}(\mathcal{D}(P_n)) = \mathrm{Aut}_K(\mathcal{D}(P_n))\mathrm{Frob}(\mathcal{D}(P_n), P_n, \Delta_n)$, *i.e. for each Frobenius F' on $\mathcal{D}(P_n)$ there exists an automorphism $\sigma \in \mathrm{Aut}_K(\mathcal{D}(P_n))$ and a Frobenius $F'' \in \mathrm{Frob}(\mathcal{D}(P_n), P_n, \Delta_n)$ such that $F' = \sigma F'' \sigma^{-1}$.*

(2) $\mathrm{Frob}(\mathcal{D}(P_n), P_n) = \mathrm{Aut}_K(P_n)\mathrm{Frob}(\mathcal{D}(P_n), P_n, \Delta_n)$; *i.e. for each Frobenius $F' \in \mathrm{Frob}(\mathcal{D}(P_n), P_n)$ there exists an automorphism $\sigma \in \mathrm{Aut}_K(P_n)$ and a Frobenius $F'' \in \mathrm{Frob}(\mathcal{D}(P_n), P_n, \Delta_n)$ such that $F' = \sigma F'' \sigma^{-1}$.*

(3) $\mathrm{FPol}(\mathcal{D}(P_n)) = \mathrm{Aut}_K(\mathcal{D}(P_n)) \cdot P_n \simeq \mathrm{Aut}_K(\mathcal{D}(P_n))/\mathrm{Aut}_K(P_n) := \{\sigma \mathrm{Aut}_K(P_n) \,|\, \sigma \in \mathrm{Aut}_K(\mathcal{D}(P_n))\}$; *i.e. for each Frobenius polynomial subalgebra P_n' of $\mathcal{D}(P_n)$ there exists an automorphism $\sigma \in \mathrm{Aut}_K(\mathcal{D}(P_n))$ such that $\sigma(P_n) = P_n'$ and the automorphism σ is unique up to $\mathrm{Aut}_K(P_n)$.*

The set $\mathrm{Frob}(\mathcal{D}(P_n), P_n, \Delta_n)$ is found explicitly (Theorems 5.1 and 5.2). Each Frobenius $F' \in \mathrm{Frob}(\mathcal{D}(P_n))$ is not an $\mathbb{F}_p$-algebra isomorphism (though it is a monomorphism since the ring $\mathcal{D}(P_n)$ is simple). Moreover, the ring $\mathcal{D}(P_n)$ is a left and right free finitely generated $KF'(\mathcal{D}(P_n))$-module of rank p^{2n} (Corollary 5.3). Theorem 1.2 shows that the problem of finding the groups $\mathrm{Aut}_K(P_n)$ and $\mathrm{Aut}_K(\mathcal{D}(P_n))$ is closely related to the problem of finding all the extensions of the Frobenius from certain polynomial subalgebras to the ring $\mathcal{D}(P_n)$.

Iterative δ-descents. The question of finding all the extensions of the Frobenius (as well as many other difficult questions such as the Jacobian Conjecture or the Dixmier Conjecture) can be reformulated as a question about iterative δ-descents.

Definition. Let A be a ring and $\delta = (\delta_1, \dots, \delta_n)$ be an n-tuple of commuting derivations of the ring A. A multi-sequence $\{y^{[\alpha]}, \alpha \in \mathbb{N}^n\}$ in A is called an **iterative δ-descent** if

$$y^{[\alpha]} y^{[\beta]} = \binom{\alpha+\beta}{\alpha} y^{[\alpha+\beta]} \text{ and } \delta^{\alpha}(y^{[\beta]}) = y^{[\beta-\alpha]} \text{ for all } \alpha, \beta \in \mathbb{N}^n.$$

Example. $\{\partial^{[\alpha]}, \alpha \in \mathbb{N}^n\}$ is the iterative δ-descent in the ring $\mathcal{D}(P_n)$, where $\delta_1 = -\mathrm{ad}(x_1), \dots, \delta_n = -\mathrm{ad}(x_n)$.

This concept first appeared in [2] where using it the simple derivations of differentially simple Noetherian commutative rings were classified. One of the key results of the paper [2] is the existence and uniqueness of an iterative δ-descent (Theorem 3.8, [2]). In the paper, this result is extended to the case of several commuting derivations and infinite iterative descents (Theorem 2.10). This result is used in the proofs of almost all main results of the paper.

In Section 2, a theory of iterative δ-descents is developed. These results are used freely in the paper.

2. Existence and uniqueness of iterative δ-descent

The main result of this section is Theorem 2.10 on *existence* and *uniqueness* of an *iterative δ-descent*. Necessary and sufficient conditions are given (Corollary 2.8) for a multi-sequence to be an iterative δ-descent.

Iterative δ-descents. Consider the free abelian group of rank n, $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i$, where the set of elements $e_1 = (1,0,\dots,0),\dots,e_n = (0,\dots,0,1)$ is the standard free basis for $\mathbb{Z}^n$. For each element $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n$, we have $\alpha = \sum_{i=1}^n \alpha_i e_i$. For elements $\alpha, \beta \in \mathbb{Z}^n$, we write $\alpha \geq \beta$ if $\alpha_1 \geq \beta_1,\dots,\alpha_n \geq \beta_n$; we write $\alpha > \beta$ if $\alpha \geq \beta$ and $\alpha \neq \beta$. For a pair of integers i and j such that $i \leq j$ (resp. $i < j$), let $[i,j]_{dis} := \mathbb{Z} \cap [i,j] = \{i, i+1,\dots,j\}$ (resp. $[i,j)_{dis} := \mathbb{Z} \cap [i,j) = \{i,i+1,\dots,j-1\}$).

Definition. Let A be a ring and let $\delta = (\delta_1,\dots,\delta_n)$ be an n-tuple of commuting derivations of the ring A. For each element $\alpha \in \mathbb{N}^n$, let $\delta^\alpha := \delta_1^{\alpha_1}\cdots\delta_n^{\alpha_n}$. A multi-sequence of elements of the ring A, $\mathbf{y} := \{y^{[\alpha]}, \alpha \in I := \prod_{i=1}^n [0, l_i)_{dis}\}$, $y^{[0]} := 1$, where $l_i \in \mathbb{N} \cup \{\infty\}$, is called a δ-*descent* if

$$\delta^\alpha(y^{[\beta]}) = y^{[\beta-\alpha]} \text{ for all } \alpha \in \mathbb{N}^n, \ \beta \in I,$$

where $y^{[\gamma]} := 0$ for all $\gamma \in \mathbb{Z}^n \backslash \mathbb{N}^n$.

Definition. Suppose that A is an $\mathbb{F}_p$-algebra. A multi-sequence $\{y^{[\alpha]}, \alpha \in I := \prod_{i=1}^n [0, p^{d_i})\}$ where $d_i \in \mathbb{N} \cup \{\infty\}$ is called an *iterative* multi-sequence if

$$y^{[\alpha]}y^{[\beta]} = \binom{\alpha+\beta}{\beta} y^{[\alpha+\beta]} \text{ for all } \alpha, \beta \in I,$$

where $\binom{\alpha+\beta}{\beta} := \prod_{i=1}^n \binom{\alpha_i+\beta_i}{\beta_i}$ are the multi-binomial coefficients. In prime characteristic, we view the multi-binomial coefficients as elements of the field $\mathbb{F}_p$.

Definition ([2])**.** Let A be an $\mathbb{F}_p$-algebra. An iterative multi-sequence $\{y^{[\alpha]}, \alpha \in I = \prod_{i=1}^n [0, p^{d_i})\}$, $d_i \in \mathbb{N} \cup \{\infty\}$ in A which is a δ-descent is called an **iterative δ-descent** of *rank* n and *exponent* $(d_1,\dots,d_n)$.

Example. Let K be a field of characteristic $p > 0$, $\mathcal{D}(P_n) = \bigoplus_{\alpha,\beta \in \mathbb{N}^n} Kx^\alpha \partial^{[\beta]}$ be the ring of differential operators on the polynomial algebra $P_n = K[x_1,\dots,x_n]$. Consider the inner derivations $\delta_1 := -\mathrm{ad}(x_1),\dots,\delta_n := -\mathrm{ad}(x_n)$ of the ring $\mathcal{D}(P_n)$ ($\mathrm{ad}(x)(y) := xy - yx$) and let $\delta := (\delta_1,\dots,\delta_n)$. Then the multi-sequence $\{\partial^{[\alpha]} := \frac{\partial_1^{\alpha_1}}{\alpha_1!}\cdots\frac{\partial_n^{\alpha_n}}{\alpha_n!}, \alpha \in \mathbb{N}^n\}$ is an iterative δ-descent (of rank n and exponent $(\infty,\dots,\infty)$).

Note that any truncation $\{y^{[\alpha]}, \alpha \in I' = \prod_{i=1}^n [0, p^{d_i'})\}$, $d_i' \leq d_i$, of the iterative δ-descent $\{y^{[\alpha]}, \alpha \in I\}$ is an iterative δ-descent of rank n and of exponent $(d_1',\dots,d_n')$.

The nil ring of commuting derivations. A ring S is a *positively filtered* ring if S is a union of its abelian subgroups, $S = \bigcup_{i\geq 0} S_i$, such that $S_0 \subseteq S_1 \subseteq \cdots$ and $S_iS_j \subseteq S_{i+j}$ for all $i,j \geq 0$. Let A be a ring and let δ be a derivation of the ring A. Recall that, for any elements $a,b \in A$ and a natural number n, we have the equality

$$\delta^n(ab) = \sum_{i=0}^n \binom{n}{i} \delta^i(a)\delta^{n-i}(b).$$

It follows directly from this equality that the union of the abelian groups $N := N(\delta, A) = \bigcup_{i\geq 0} N_i$, $N_i := \ker \delta^{i+1}$, is a positively filtered ring ($N_iN_j \subseteq N_{i+j}$ for all $i, j \geq 0$), so-called the *nil ring* of δ. Clearly, $N_0 = A^\delta := \ker \delta$ is the subring (of *constants* for δ) of A, and $N = \{a \in A \mid \delta^n(a) = 0$ for some natural $n = n(a)\}$. For all natural numbers i and j, $\delta^i(N_j) \subseteq N_{j-i}$, where $N_k := 0$ for all $\mathbb{Z}\backslash\mathbb{N}$. In general, little is known about the rings $N(\delta, A)$; these rings have a complicated structure and they are not easy objects to deal with. Even for the first Weyl algebra there are old open problems about these rings; see the paper of J. Dixmier, [7]. A derivation δ of a ring A is called a *locally nilpotent* derivation if $A = N(\delta, A)$ or, equivalently, for each element a of A, $\delta^i(a) = 0$ for all $i \gg 0$. In this case, the ring $A = \bigcup_{i\geq 0} N(\delta, A)_i$ is a positively filtered ring.

Let $\delta = (\delta_1, \dots, \delta_n)$ be an n-tuple of commuting derivations of a ring A. The intersection $N(\delta, A) := \bigcap_{i=1}^n N(\delta_i, A)$ is called the *nil ring* of δ. It is an $\mathbb{N}^n$-filtered ring, $N(\delta, A) = \bigcup_{\alpha\in\mathbb{N}^n} N(\delta, A, \alpha)$, where

$$N_\alpha := N(\delta, A, \alpha) := \bigcap_{i=1}^n N(\delta_i, A, \alpha_i) \quad (N_\alpha N_\beta \subseteq N_{\alpha+\beta} \text{ for all } \alpha, \beta \in \mathbb{N}^n).$$

Note that $N_0 := A^\delta := \bigcap_{i=1}^n A^{\delta_i}$ is the subring of A, so-called the ring of δ-constants for δ. For $\alpha, \beta \in \mathbb{N}^n$, $\delta^\alpha(N_\beta) \subseteq N_{\beta-\alpha}$, where $N_\gamma := 0$ for all $\gamma \in \mathbb{Z}^n\backslash\mathbb{N}^n$.

Let $\delta = (\delta_1, \dots, \delta_n)$ be an n-tuple of commuting, locally nilpotent derivations of a ring A. Then $A = N(\delta, A)$, and the ring $A = \bigcup_{\alpha\in\mathbb{N}^n} N(\delta, A, \alpha)$ is an $\mathbb{N}^n$-filtered ring.

Example. Let K be a field of characteristic $p > 0$, and $P_n = K[x_1, \dots, x_n]$ be a polynomial algebra over K. Then $\delta := (-\mathrm{ad}(x_1), \dots, -\mathrm{ad}(x_n))$ is the n-tuple of commuting, locally nilpotent derivations of the ring $\mathcal{D}(P_n) = \bigoplus_{\alpha\in\mathbb{N}^n} P_n\partial^{[\alpha]}$, and so the ring $\mathcal{D}(P_n) = \bigcup_{\alpha\in\mathbb{N}^n} N_\alpha$ is $\mathbb{N}^n$-filtered, where $N_\alpha := \bigoplus_{0\leq\beta\leq\alpha} P_n\partial^{[\alpha]}$.

The following lemma establishes a relation between the filtration $\{N_\alpha\}_{\alpha\in\mathbb{N}^n}$ of the nil ring $N(\delta, A)$ of $\delta = (\delta_1, \dots, \delta_n)$ and δ-descents.

Lemma 2.1. *Let $\delta = (\delta_1, \dots, \delta_n)$ be an n-tuple of commuting derivations of a ring A, and let $\{x^{[\alpha]}, \alpha \in I := \prod_{i=1}^n [0, d_i]_{dis}\}$ be a δ-descent, where $d_i \in \mathbb{N}\cup\{\infty\}$. Then, for each $\alpha \in I$, $N_\alpha = \bigoplus_{0\leq\beta\leq\alpha} A^\delta x^{[\beta]} = \bigoplus_{0\leq\beta\leq\alpha} x^{[\beta]} A^\delta$.*

Proof. Let us prove the first equality. Let $N'_\alpha := \bigoplus_{0\leq\beta\leq\alpha} A^\delta x^{[\beta]}$ for each element $\alpha \in I$. The inclusion $N'_\alpha \subseteq N_\alpha$ is obvious. To prove the reverse inclusion we use induction on $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $|\alpha| = 0$, i.e. $\alpha = 0$, then there is nothing to prove since $N_0 = A^\delta$. Suppose that $s := |\alpha| > 0$ and the result is true for all α' with $|\alpha'| < s$. Then, up to order, $\alpha_n > 0$. Let $u \in N_\alpha$. We have to show that $u \in N'_\alpha$. If $\delta_n(u) = 0$, then $u \in N_{\alpha-e_n}$, and, by induction, $u \in N_{\alpha-e_n} \subseteq N'_{\alpha-e_n} \subseteq N'_\alpha$ since $|\alpha - e_n| = |\alpha| - 1 < |\alpha|$.

If $\delta_n(u) \neq 0$, then $\delta_n(u) \in N_{\alpha-e_n} = N'_{\alpha-e_n}$ (by induction), and so $\delta_n(u) = \sum_{0\leq\beta\leq\alpha-e_n} \lambda_\beta x^{[\beta]}$ for some elements $\lambda_\beta \in A^\delta$, and not all of them are equal to zero. Note that the element $v := \sum_{0\leq\beta\leq\alpha-e_n} \lambda_\beta x^{[\beta+e_n]}$ belongs to the set N'_α, and $\delta_n(u - v) = 0$. Then $u - v \in A^{\delta_n} \cap N_\alpha = N_{\alpha'}$ where $\alpha' := (\alpha_1, \dots, \alpha_{n-1}, 0)$. Since $|\alpha'| < |\alpha|$, $N_{\alpha'} \subseteq N'_{\alpha'}$ and $v \in N'_\alpha$, we see that $u = (u - v) + v \in N'_\alpha$, as required.

The second equality can be proved by similar arguments. □

The following lemma gives all the δ-descents provided a single δ-descent is known.

Lemma 2.2. *Let $\delta = (\delta_1, \dots, \delta_n)$ be an n-tuple of commuting derivations of a ring A, and let $\{x^{[\alpha]}, \alpha \in I := \prod_{i=1}^n [0, d_i]_{dis}\}$ be a δ-descent, where $d_i \in \mathbb{N} \cup \{\infty\}$. Let $\{x'^{[\alpha]}, \alpha \in I\} \subseteq A$. Then the following statements are equivalent.*

(1) *$\{x'^{[\alpha]}, \alpha \in I\}$ is a δ-descent.*
(2) *$x'^{[0]} := 1$ and, for each element $0 \neq \alpha \in I$, $x'^{[\alpha]} = x^{[\alpha]} + \sum_{0 \neq \beta \leq \alpha} \lambda_\beta x^{[\alpha-\beta]}$, where $\lambda_\gamma \in A^\delta$, $0 \neq \gamma \in I$.*
(3) *$x'^{[0]} := 1$ and, for each element $0 \neq \alpha \in I$, $x'^{[\alpha]} = x^{[\alpha]} + \sum_{0 \neq \beta \leq \alpha} x^{[\alpha-\beta]} \mu_\beta$, where $\mu_\gamma \in A^\delta$, $0 \neq \gamma \in I$.*

Proof. $(1 \Leftarrow 2)$ This implication is obvious.

$(1 \Rightarrow 2)$ Suppose that $\{x'^{[\alpha]}, \alpha \in I\}$ is a δ-descent. Then $x'^{[0]} := 1$. Let $0 \neq \alpha \in I$. By Lemma 2.1 and the fact that $\delta^\alpha(x^{[\alpha]}) = 1$, we have $x'^{[\alpha]} = x^{[\alpha]} + \sum_{0 \neq \beta \leq \alpha} \lambda_\beta x^{[\alpha-\beta]}$ for some elements $\lambda_\beta \in A^\delta$ that depend on α. For each $\gamma \in I$ such that $\gamma \leq \alpha$,

$$x'^{[\alpha-\gamma]} = \delta^\gamma(x'^{[\alpha]}) = x^{[\alpha-\gamma]} + \sum_{0 \neq \beta \leq \alpha-\gamma} \lambda_\beta x^{[\alpha-\beta-\gamma]},$$

and the implication $(1 \Rightarrow 2)$ is obvious since the equality above is true for all elements $\gamma, \alpha \in I$ such that $\gamma \leq \alpha$.

$(1 \Leftrightarrow 3)$ Repeat the arguments above making obvious modifications. $\square$

Lemma 2.3. *Let $\delta = (\delta_1, \dots, \delta_n)$ be an n-tuple of commuting derivations of a ring A such that $\delta^\alpha(y^{[\alpha]}) = 1$ for some elements $y^{[\alpha]} \in N(\delta, A, \alpha)$, $\alpha \in I := \prod_{i=1}^n [0, d_i]_{dis}$, where $d_i \in \mathbb{N} \cup \{\infty\}$. Then*

(1) *there exists a unique δ-descent $\{x^{[\alpha]}, \alpha \in I\}$ such that, for each nonzero element $\alpha \in I$, $x^{[\alpha]} = y^{[\alpha]} + \sum_{0 \leq \beta < \alpha} c_{\alpha\beta} y^{[\beta]}$ for some elements $c_{\alpha\beta} \in A^\delta$ such that $c_{\alpha,0} = 0$.*
(2) *there exists a unique δ-descent $\{z^{[\alpha]}, \alpha \in I\}$ such that, for each nonzero element $\alpha \in I$, $z^{[\alpha]} = y^{[\alpha]} + \sum_{0 \leq \beta < \alpha} y^{[\beta]} b_{\alpha\beta}$ for some elements $b_{\alpha\beta} \in A^\delta$ such that $b_{\alpha,0} = 0$.*

Proof. 1. One can easily prove that $N_\alpha := N(\delta, A, \alpha) = \bigoplus_{0 \leq \beta \leq \alpha} A^\delta y^{[\beta]}$, $\alpha \in I$ (by modifying slightly the arguments of the proof of Lemma 2.1). Let us prove that there exists at least one δ-descent, say $\{x'^{[\alpha]}, \alpha \in I\}$. If the set I is finite, then it contains the largest element with respect to $\geq$, say $\beta \in I$. Then the multi-sequence $\{x'^{[\alpha]} := \delta^{\beta-\alpha}(y^{[\beta]}), \beta \in I\}$ is a δ-descent. If the set I is infinite, then we fix a strictly ascending infinite sequence of elements of I, $\beta_1 < \beta_2 < \cdots$ such that $I = \bigcup_{i \geq 1} I_i$, where $I_i := \{\alpha \in I \mid \alpha \leq \beta_i\}$. For each $\alpha, \beta \in I$ such that $\alpha \leq \beta$, we have

$$\delta^\alpha(y^{[\beta]}) - y^{[\beta-\alpha]} \in \sum_{\gamma < \beta-\alpha} A^\delta y^{[\gamma]}.$$

Using this fact we can find elements $x'^{[\beta_i]}, i \geq 1$, such that $x'^{[\beta_1]} := y^{[\beta_1]}$ and $\delta^{\beta_i - \beta_{i-1}}(x'^{[\beta_i]}) = x'^{[\beta_{i-1}]}$ for all $i \geq 2$. Let $I_0 := \emptyset$. For each $\alpha \in I$, there exists a unique index, say i, such that $\alpha \in I_i \backslash I_{i-1}$. Let $x'^{[\alpha]} := \delta^{\beta_i - \alpha}(x'^{[\beta_i]})$. Then it is obvious that $\{x'^{[\alpha]}, \alpha \in I\}$ is the δ-descent.

Let, for a moment, a multi-sequence $\{x'^{[\alpha]}, \alpha \in I\}$ be an arbitrary δ-descent in A. By Lemma 2.1, for each $\alpha \in I$, $x'^{[\alpha]} \in N_\alpha$; i.e., $x'^{[\alpha]} = y^{[\alpha]} + \sum_{0 \leq \beta < \alpha} c'_{\alpha\beta} y^{[\beta]}$, for some elements $c'_{\alpha\beta} \in A^\delta$. Let $\{x^{[\alpha]}, \alpha \in I\}$ be another δ-descent, and so $x^{[\alpha]} = y^{[\alpha]} + \sum_{0\leq\beta<\alpha} c_{\alpha\beta} y^{[\beta]}$, $\alpha \in I$, for some elements $c_{\alpha\beta} \in A^\delta$. By Lemma 2.2,

$$x^{[\alpha]} = x'^{[\alpha]} + \sum_{0\neq\gamma\leq\alpha} \lambda_\gamma x'^{[\alpha-\gamma]} \text{ for all } \alpha \in I,$$

for some elements $\lambda_\gamma \in A^\delta$. We have to prove that the defining conditions

$$c_{\alpha,0} = 0 \text{ for all } 0 \neq \alpha \in I$$

of the δ-descent from Lemma 2.3(1) *uniquely* determine the elements λ_γ. For each $i = 1, \ldots, n$, the equality $c_{e_i,0} = 0$ yields the equalities

$$y^{[e_i]} = x^{[e_i]} = x'^{[e_i]} + \lambda_{e_i} = y^{[e_i]} + c'_{e_i,0} + \lambda_{e_i};$$

hence $\lambda_{e_i} = -c'_{e_i,0}$. Let s be a natural number such that $s \geq 2$. Suppose that, using the equalities $c_{\alpha,0} = 0$, $0 \neq \alpha \in I$, $|\alpha| < s$, we have already found unique elements λ_γ with $|\gamma| < s$. Take any element $\alpha \in I$ with $|\alpha| = s$. Then the element λ_α can be found uniquely from the equality

$$x^{[\alpha]} = x'^{[\alpha]} + \sum_{0\neq\gamma<\alpha} \lambda_\gamma x'^{[\alpha-\gamma]} + \lambda_\alpha.$$

For, we have to equate to zero the coefficient $c_{\alpha,0}$ of the elements $y^{[0]} := 1$ after we substitute the sum for each $x'^{[\alpha-\gamma]}$ above (via $y^{[\beta]}$) into the above equality:

$$x^{[\alpha]} = y^{[\alpha]} + \sum_{0\neq\beta<\alpha} c_{\alpha\beta} y^{[\beta]} + (c'_{\alpha,0} + \sum_{0\neq\gamma<\alpha} \lambda_\gamma c'_{\alpha-\gamma,0} + \lambda_\alpha) y^{[0]},$$

that is, $\lambda_\alpha := -c'_{\alpha,0} - \sum_{0\neq\gamma<\alpha} \lambda_\gamma c'_{\alpha-\gamma,0}$. Therefore, for this unique choice of the elements $\{\lambda_\alpha\}$, we have $c_{\alpha,0} = 0$, $0 \neq \alpha \in I$, for the δ-descent $\{x^{[\alpha]}, \alpha \in I\}$. This proves the first statement of the lemma.

2. Repeat the arguments of the first statement making obvious modifications. □

Structure of the iterative sequence. The structure of the iterative sequence of rank 1 is given by the following proposition.

Proposition 2.4 (Structure of iterative sequence of rank 1, Proposition 3.5, [2])**.** *Let A be an $\mathbb{F}_p$-algebra and $\{x^{[i]}, 0 \leq i < p^d\}$ be an iterative sequence. Then*

1. *for each $i = 1, \ldots, p^d - 1$, written p-adically as $i = \sum_k i_k p^k$, $x^{[i]} = \prod_k \frac{x^{[p^k]i_k}}{i_k!}$. This means that the iterative sequence is determined by the elements $\{x^{[0]}, x^{[p^j]} \mid j = 0, 1, \ldots, d-1\}$.*
2. *For each $j = 0, 1, \ldots, d-1$, $x^{[p^j]p} = 0$ (hence $x^{[i]p} = 0$ for all $i = 1, \ldots, p^d - 1$, by statement 1).*
3. *$x^{[0]} x^{[p^j]} = x^{[p^j]}$, $j = 0, 1, \ldots, n-1$, and $x^{[0]} x^{[0]} = x^{[0]}$.*

Conversely, given commuting elements $\{x^{[0]}, x^{[p^j]} \mid j = 0, 1, \ldots, d-1\}$ in A that satisfy the conditions of statements 2 and 3 above, then the elements $\{x^{[i]}, 0 \leq i < p^d\}$ defined as in statement 1 form an iterative sequence.

Remark. To make formulae more readable we often use the notation $x^{[p^k]j}$ for $(x^{[p^k]})^j$.

Let A be an $\mathbb{F}_p$-algebra. For each $i = 1, \ldots, n$, let $\{x_i^{[j]}, j \in [0, p^{d_i})_{dis}\}$ be an iterative sequence of rank 1 in A where $d_i \in \mathbb{N} \cup \{\infty\}$. Suppose that these sequences *commute* ($x_i^{[j]}x_k^{[l]} = x_k^{[l]}x_i^{[j]}$) and have a common initial element, that is, $x_1^{[0]} = x_2^{[0]} = \cdots = x_n^{[0]}$. It is easy to verify that their product $\{x^{[\alpha]} := x_1^{[\alpha_1]}\cdots x_n^{[\alpha_n]}, \alpha \in \prod_{i=1}^n [0, p^{d_i})_{dis}\}$ is an iterative multi-sequence of rank n and of exponent $(d_1, \dots, d_n)$.

Definition. The iterative multi-sequence $\{x^{[\alpha]} := x_1^{[\alpha_1]}\cdots x_n^{[\alpha_n]}, \alpha \in \prod_{i=1}^n [0, p^{d_i})_{dis}\}$ of rank n and exponent $(d_1, \dots, d_n)$ is called the *product* of n iterative sequences $\{x_i^{[j]}, j \in [0, p^{d_i})_{dis}\}$ of rank 1 and exponent d_i.

Corollary 2.5 (Structure of iterative multi-sequence of rank n)**.** *Let A be an $\mathbb{F}_p$-algebra and $\{x^{[\alpha]}, \alpha \in I\}$ be an iterative multi-sequence of rank n in A, where $I := \prod_{i=1}^n [0, p^{d_i})_{dis}$, $d_i \in \mathbb{N} \cup \{\infty\}$. Then the multi-sequence $\{x^{[\alpha]}, \alpha \in I\}$ is the product of n iterative sequences $\{x_i^{[j]} := x^{[je_i]}, j \in [0, p^{d_i})_{dis}\}$ of rank 1, and vice versa.*

Proof. It is obvious that, for each $i = 1, \dots, n$, the sequence $\{x_i^{[j]}, j \in [0, p^{d_i})_{dis}\}$ is iterative of rank 1. It is obvious that $x_1^{[0]} = x_2^{[0]} = \cdots = x_n^{[0]}$ and that $x^{[\alpha]} := x_1^{[\alpha_1]}\cdots x_n^{[\alpha_n]}$. Now, the result follows. □

Necessary and sufficient conditions for an iterative multi-sequence to be a δ-descent. In the case of a sequence of rank 1, such conditions are given in the next proposition.

Proposition 2.6 (Corollary 3.6, [2])**.** *Let A be an $\mathbb{F}_p$-algebra, δ be a derivation of A, and $\{x^{[i]}, 0 \leq i < p^d\}$ be an iterative sequence in A with $x^{[0]} = 1$, where $d \in \mathbb{N} \cup \{\infty\}$. Then the iterative sequence $\{x^{[i]}, 0 \leq i < p^d\}$ is a δ-descent iff $\delta(x^{[p^j]}) = x^{[p^j-1]}$, $0 \leq j \leq d-1$.*

Corollary 2.7. *Let $\delta = (\delta_1, \dots, \delta_n)$ be an n-tuple of commuting derivations of an $\mathbb{F}_p$-algebra A, and let $\{x^{[\alpha]}, \alpha \in I\}$ be an iterative multi-sequence of rank n in A with $x^{[0]} = 1$, where $I := \prod_{i=1}^n [0, p^{d_i})_{dis}$, $d_i \in \mathbb{N} \cup \{\infty\}$. Then the iterative multi-sequence $\{x^{[\alpha]}, \alpha \in I\}$ is a δ-descent if and only if, for each $i = 1, \dots, n$, the iterative sequence $\{x_i^{[j]} := x^{[je_i]}, j \in [0, p^{d_i})_{dis}\}$ is a δ_i-descent and $\{x_i^{[j]}, j \in [0, p^{d_i})_{dis}\} \subseteq \bigcap_{k \neq i} A^{\delta_k}$ if and only if $\delta_i(x_j^{[p^{k_j}]}) = \delta_{i,j} x_j^{[p^{k_j}-1]}$ for all $i, j = 1, \dots, n$ and $k_j \in [0, d_j)_{dis}$, where $\delta_{i,j}$ is the Kronecker delta.*

Proof. The first 'if and only if' follows from Corollary 2.5. The second 'if and only if' follows from Proposition 2.6. □

Combining Proposition 2.4, Corollary 2.5 and Corollary 2.7, we have necessary and sufficient conditions for a multi-sequence to be an iterative δ-descent.

Corollary 2.8. *Let $\delta = (\delta_1, \dots, \delta_n)$ be an n-tuple of commuting derivations of an $\mathbb{F}_p$-algebra A, $\{x_\nu^{[p^{k_\nu}]}, \nu = 1, \dots, n; k_\nu = [0, d_\nu)_{dis}\}$ be commuting elements of the algebra A, where $d_\nu \in \mathbb{N} \cup \{\infty\}$, and $x_\nu^{[0]} := 1$ for all ν. Let $x_\nu^{[i]} := \prod_k \frac{x_\nu^{[p^k]i_k}}{i_k!}$ for each $i = \sum_k i_k p^k$, $0 \leq i_k < p$, such that $0 \leq i < p^{d_\nu}$. Let $x^{[\alpha]} := \prod_{\nu=1}^n x_\nu^{[\alpha_\nu]}$.*

Then the multi-sequence $\{x^{[\alpha]}, \alpha \in I := \prod_{i=1}^{n}[0, p^{d_\nu})_{dis}\}$ *is an iterative* δ*-descent iff* $\delta_\nu(x_\mu^{[p^{k_\mu}]}) = \delta_{\nu,\mu}x_\mu^{[p^{k_\mu}-1]}$ *and* $x_\mu^{[p^{k_\mu}]p} = 0$ *for* $\nu, \mu = 1, \dots, n$ *and* $k_\mu \in [0, d_\mu)_{dis}$ *(where* $\delta_{\nu,\mu}$ *is the Kronecker delta).*

Let A be a commutative $\mathbb{F}_p$-algebra and δ be a derivation of A. Let $\mathrm{ID}(\delta, d)$ be the set of all iterative δ-descents $\{x^{[i]}, 0 \le i < p^d\}$ of exponent d in A, where $d \in \mathbb{N} \cup \{\infty\}$. Let $C(\delta, d)$ be the set of all d-tuples $(\lambda_0, \lambda_1, \dots, \lambda_{d-1})$ such that $\lambda_j \in A^\delta$ and $\lambda_j^p = 0$ for $0 \le j \le d-1$. Note that if A^δ is a *reduced* ring, then $C(\delta, d) = \{(0, \dots, 0)\}$; i.e. $C(\delta, d)$ contains a single element. In the case when $n = 1$, by Lemma 2.1 and Proposition 2.4, for each iterative δ-descent, say $\{x^{[i]}, 0 \le i < p^d\}$,

(1)
$$N(\delta, A)_{p^d-1} = \bigoplus_{i=0}^{p^d-1} A^\delta x^{[i]} \simeq A^\delta[x^{[1]}, x^{[p]}, \dots, x^{[p^{d-1}]}]/(x^{[1]p}, x^{[p]p}, \dots, x^{[p^{d-1}]p}).$$

So, $N(\delta, A)_{p^d-1}$ is the *subring* of A that contains A^δ, and the decomposition (1) holds for *all* iterative δ-descents in A of exponent d. In particular, all iterative δ-descents in A of exponent d belong to the ring $N(\delta, A)_{p^d-1}$. Then

$$N(\delta, A)_{p^d-1} = A^\delta \oplus \mathfrak{m}, \quad \mathfrak{m} := (x^{[1]}, x^{[p]}, \dots, x^{[p^{d-1}]}). \tag{2}$$

If $\{y^{[i]}, 0 \le i < p^d\}$ is an iterative δ-descent in A, then $\{y^{[i]}, 0 \le i < p^d\} \subseteq N(\delta, A)_{p^d-1}$. Therefore, the following map is well-defined:

$$r = r_d : \mathrm{ID}(\delta, d) \to C(\delta, d), \quad \{y^{[i]}, 0 \le i < p^d\} \mapsto (\lambda_0, \lambda_1, \dots, \lambda_{d-1}), \tag{3}$$

where $\lambda_j \equiv y^{[p^j]} \mod \mathfrak{m}$, $j = 0, 1, \dots, d-1$. Note that the map r depends on the choice of the iterative δ-descent $\{x^{[i]}, 0 \le i < p^d\}$ since the decomposition (2) does.

If $d = \infty$, then $N(\delta, A)_{p^\infty - 1} = N(\delta, A)$, and so the equalities (1) and (2) are as follows:

$$\begin{aligned} N(\delta, A) &= \bigoplus_{i \ge 0} A^\delta x^{[i]} \simeq A^\delta[x^{[1]}, x^{[p]}, \dots, x^{[p^i]}, \dots]/(x^{[1]p}, x^{[p]p}, \dots, x^{[p^i]p}, \dots) \\ N(\delta, A) &= A^\delta \oplus \mathfrak{m}, \quad \mathfrak{m} := (x^{[1]}, x^{[p]}, \dots, x^{[p^i]}, \dots). \end{aligned}$$

Let $\mathrm{ID}(\delta) := \mathrm{ID}(\delta, \infty)$. Then the map (3) takes the form

$$r = r_\infty : \mathrm{ID}(\delta) \to C(\delta, \infty), \quad \{y^{[i]}, i \in \mathbb{N}\} \mapsto (\lambda_0, \lambda_1, \dots, \lambda_j, \dots), \tag{4}$$

where $\lambda_j \equiv y^{[p^j]} \mod \mathfrak{m}$, $j \in \mathbb{N}$.

The following theorem is a key (and difficult) result of the paper [2].

Theorem 2.9 (Existence and uniqueness of an iterative δ-descent when $n = 1$, Theorem 3.8, [2])**.** *Let A be a commutative algebra over a field K of characteristic $p > 0$ and δ be a K-derivation of the algebra A such that there exists a finite sequence of elements $y_0, y_1, \dots, y_{d-1}$ of A such that $y_k^p = 0$ and $\delta^{p^k}(y_k) = 1$ for all $0 \le k \le d-1$. Then*

1. *(Existence) The following sequence $\{x^{[i]}, 0 \le i < p^d\}$ is an iterative δ-descent, where $x^{[0]} := 1$, $x^{[1]} := y_0$, and, for $i \ge 2$ written p-adically as $i = \sum_{k=0}^{t} i_k p^k$ $(0 \le i_k \le p-1)$ the element $x^{[i]}$ is defined as $x^{[i]} :=$*

$\prod_{k=0}^{t} \frac{(x^{[p^k]})^{i_k}}{i_k!}$, *where*

$$x^{[p]} := (-1)^{p-1}\phi_0(y_1), \quad \phi_0(z) := \sum_{j=0}^{p-1}(-1)^j \frac{(x^{[1]})^j}{j!}\delta^j(z),$$

and then recursively, for each k such that $1 \leq k \leq n-2$, the element $x^{[p^{k+1}]}$ is defined by the rule

$$x^{[p^{k+1}]} := (-1)^{p-1}\delta^{p^k-1}\Big(\prod_{l=0}^{k-1}\frac{(x^{[p^l]})^{p-1}}{(p-1)!}\cdot\phi_k(y_{k+1})\Big), \quad \phi_k(z) := \sum_{j=0}^{p-1}(-1)^j\frac{(x^{[p^k]})^j}{j!}\delta^{p^k j}(z).$$

(2) *(Almost uniqueness) Let $\{x^{[i]}, 0 \leq i < p^d\}$ be an arbitrary iterative δ-descent (not necessarily as in statement 1, and d here is not necessarily as in statement 1 either). Then the map* (3) *is a bijection.*

(3) *(Uniqueness) If, in addition, the ring A^δ is reduced, then $\{x^{[i]}, 0 \leq i < p^d\}$ from statement 1 is the only iterative δ-descent.*

Remark. Note that Theorem 2.9 also holds for $d = \infty$ since each infinite iterative δ-descent $\{x^{[i]}, i \in \mathbb{N}\}$ is a union of finite iterative δ-descents $\{x^{[i]}, i \in [0, p^d)_{dis}\}$, $d \in \mathbb{N}$.

Let A be an $\mathbb{F}_p$-algebra and $\delta = (\delta_1, \ldots, \delta_n)$ be an n-tuple of commuting derivations of the algebra A. Let $\mathrm{ID}(\delta, d)$ be the set of all the iterative δ-descents $\mathbf{x} := \{x^{[\alpha]}\}$ of exponent $d := (d_1, \ldots, d_n)$, where $d_i \in \mathbb{N} \cup \{\infty\}$. Let $C(\delta, d) := C'(d_1) \times \cdots \times C'(d_n)$, where $C'(d_i)$ is the set of all d_i-tuples $(\lambda_0, \lambda_1, \ldots, \lambda_{d_i-1})$ such that $\lambda_j \in A^\delta$ and $\lambda_j^p = 0$ for all $j \in [0, d_i)_{dis}$. By Corollary 2.5, each $\mathbf{x}$ is the product $\prod_{i=1}^n \mathbf{x}_i$ of the iterative δ_i-descents $\mathbf{x}_i := \{x_i^{[j]} := x^{[je_i]}, j \in [0, p^{d_i})_{dis}\}$. For each $i = 1, \ldots, n$, let $r_{d_i} : \mathrm{ID}(\delta_i, d_i) \to C(\delta, d_i)$ be the map (3) for the derivation δ_i. Consider the map

$$(5) \qquad r_d : \mathrm{ID}(\delta, d) \to C(\delta, d), \quad \mathbf{x} \mapsto (r_{d_1}(\mathbf{x}_1), \ldots, r_{d_n}(\mathbf{x}_n)).$$

Theorem 2.10 (Existence and uniqueness of an iterative δ-descent)**.** *Let A be a commutative $\mathbb{F}_p$-algebra and $\delta = (\delta_1, \ldots, \delta_n)$ be an n-tuple of commuting derivations of the algebra A. Suppose that there exists a set of elements of the algebra A, $\{y_{ik}, i = 1, \ldots, n; k \in [0, d_i)_{dis}\}$, where $d_i \in \mathbb{N} \cup \{\infty\}$, such that $y_{ik}^p = 0$ and $\delta_i^{p^k}(y_{ik}) = 1$ for all $i = 1, \ldots, n$ and $k \in [0, d_i)_{dis}$, and $\delta_j(y_{ik}) = 0$ for all $i \neq j$. Note that, for each $i = 1, \ldots, n$, the elements $\{y_{ik}, k \in [0, d_i)_{dis}\}$ satisfy the assumptions of Theorem* 2.9(1), *and let $\mathbf{x}_i := \{x_i^{[j]}, j \in [0, p^{d_i})_{dis}\}$ be the corresponding δ_i-descent of rank* 1 *and exponent d_i as in Theorem* 2.9(1). *Then*

(1) *(Existence) The product $\mathbf{x} := \prod_{i=1}^n \mathbf{x}_i = \{x^{[\alpha]} := x_1^{[\alpha_1]} \cdots x_n^{[\alpha_n]}, \alpha \in \mathbb{N}^n\}$ of the iterative δ_i-descents $\mathbf{x}_i$ is an iterative δ-descent.*

(2) *(Almost uniqueness) The map* (5) *is a bijection.*

(3) *(Uniqueness) If, in addition, the ring A^δ is reduced, then the iterative δ-descent $\mathbf{x}$ from statement 1 is the only iterative δ-descent in A.*

Proof. 1. Statement 1 follows from Corollary 2.6.

2. Statement 2 follows from statement 1 and Theorem 2.9(2).

3. Statement 3 follows from statement 2 since the set $C(\delta, d)$ contains a single element (since the ring A^δ is reduced). $\square$

3. Rigidity of the group $\mathrm{Aut}_K(\mathcal{D}(P_n))$

In this section, the rigidity of the group $\mathrm{Aut}_K(\mathcal{D}(P_n))$ is proved (Theorem 3.1).

The ring $\mathcal{D}(P_n)$ of differential operators. The ring $\mathcal{D}(P_n)$ of differential operators on a polynomial algebra $P_n := K[x_1, \dots, x_n]$ is a K-algebra generated by the elements $x_1, \dots, x_n$ and *commuting* higher derivations $\partial_i^{[k]} := \frac{\partial_i^k}{k!}$, $i = 1, \dots, n$; $k \geq 1$, that satisfy the following defining relations:

$$(6)\quad [x_i, x_j] = 0,\ [\partial_i^{[k]}, \partial_j^{[l]}] = 0,\quad \partial_i^{[k]}\partial_i^{[l]} = \binom{k+l}{k}\partial_i^{[k+l]},\quad [\partial_i^{[k]}, x_j] = \delta_{ij}\partial_i^{[k-1]},$$

for all $i, j = 1, \dots, n$; $k, l \geq 1$, where δ_{ij} is the Kronecker delta, $\partial_i^{[0]} := 1$, $\partial_i^{[-1]} := 0$, and $\partial_i^{[1]} = \partial_i = \frac{\partial}{\partial x_i} \in \mathrm{Der}_K(P_n)$, $i = 1, \dots, n$. The action of the higher derivation $\partial_i^{[k]}$ on the polynomial algebra $P_n = K \otimes_{\mathbb{Z}} \mathbb{Z}[x_1, \dots, x_n]$ should be understood as the action of the element $1 \otimes_{\mathbb{Z}} \frac{\partial_i^k}{k!}$.

The algebra $\mathcal{D}(P_n)$ is a *simple* algebra. Note that the algebra $\mathcal{D}(P_n)$ is not finitely generated and neither left nor right Noetherian; it does not satisfy finitely many defining relations.

$$\mathcal{D}(P_n) = \bigoplus_{\alpha,\beta\in\mathbb{N}^n} Kx^\alpha\partial^{[\beta]} = \bigoplus_{\alpha,\beta\in\mathbb{N}^n} K\partial^{[\beta]}x^\alpha,$$

where $x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\partial^{[\beta]} := \partial_1^{[\beta_1]}\cdots\partial_n^{[\beta_n]}$. For each $i = 1, \dots, n$, let $\mathcal{D}(P_1)(i) := \mathcal{D}(K[x_i])$. Then

$$\mathcal{D}(P_n) = \bigotimes_{i=1}^n \mathcal{D}(P_1)(i) \simeq \mathcal{D}(P_1)^{\otimes n}.$$

For each $i = 1, \dots, n$ and $j \in \mathbb{N}$ written p-adically as $j = \sum_k j_k p^k$, $0 \leq j_k < p$,

$$(7)\qquad \partial_i^{[j]} = \prod_k \partial_i^{[j_k p^k]} = \prod_k \frac{\partial_i^{[p^k]j_k}}{j_k!},\quad \partial_i^{[j_k p^k]} = \frac{\partial_i^{[p^k]j_k}}{j_k!},$$

where $\partial_i^{[p^k]j_k} := (\partial_i^{[p^k]})^{j_k}$. For $\alpha, \beta \in \mathbb{N}^n$,

$$(8)\qquad \partial^{[\alpha]}(x^\beta) = \binom{\beta}{\alpha}x^{\beta-\alpha},\quad \binom{\beta}{\alpha} := \prod_i \binom{\beta_i}{\alpha_i},$$

where, in the formula above, $x_i^t := 0$ for all negative integers t and all i.

For $\alpha, \beta \in \mathbb{N}^n$,

$$(9)\qquad \partial^{[\alpha]}\partial^{[\beta]} = \binom{\alpha+\beta}{\beta}\partial^{[\alpha+\beta]}.$$

The algebra $\mathcal{D}(P_n)$ has two natural filtrations: the *canonical filtration* $F = \{F_i\}_{i\geq 0}$, where $F_i := \bigoplus_{|\alpha|+|\beta|\leq i} Kx^\alpha\partial^{[\beta]}$ and $|\alpha| := |\alpha_1| + \cdots + |\alpha_n|$, and the *order filtration* $\{\mathcal{D}(P_n)_i\}_{i\geq 0}$, where $\mathcal{D}(P_n)_i := \bigoplus_{|\beta|\leq i} P_n\partial^{[\beta]}$. The first one is a finite dimensional filtration but the second one is not. For both filtrations the associated graded algebras are isomorphic to the tensor product $P_n \otimes \Delta_n$ of the polynomial algebra P_n and the algebra $\Delta_n = \bigoplus_{\alpha\in\mathbb{N}^n} K\partial^{[\alpha]}$ of scalar differential operators. In particular, these algebras are commutative but not finitely generated, not Noetherian, and not domains.

Defining relations for $\mathcal{D}(P_n)$. As an abstract K-algebra, the ring $\mathcal{D}(P_n)$ of differential operators on P_n is generated by the elements x_i, $\partial_i^{[p^k]}$, $i=1,\dots,n$, $k\in\mathbb{N}$, that satisfy the defining relations:

$$(10)\qquad [x_i,x_j]=0,\ [\partial_i^{[p^k]},\partial_j^{[p^l]}]=0,\ (\partial_i^{[p^k]})^p=0,\ [\partial_i^{[p^k]},x_j]=\delta_{ij}\prod_{l=0}^{k-1}\frac{\partial_i^{[p^l](p-1)}}{(p-1)!}$$

for all $1\leq i,j\leq n$ and $k,l\in\mathbb{N}$.

In characteristic $p>0$, the next theorem proves the existence and uniqueness of iterative $(-\mathrm{ad}(x_1),\dots,-\mathrm{ad}(x_n))$-descent in $\mathcal{D}(P_n)$. In characteristic zero, this result is not true; there are many iterative $(-\mathrm{ad}(x_1),\dots,-\mathrm{ad}(x_n))$-descents in $\mathcal{D}(P_n)$: $\{y^{[\alpha]}:=\prod_{i=1}^n\frac{(\partial_i+a_i)^{\alpha_i}}{\alpha_i!},\alpha\in\mathbb{N}^n\}$ is the iterative $(-\mathrm{ad}(x_1),\dots,-\mathrm{ad}(x_n))$-descent in $\mathcal{D}(P_n)$, where $a_i\in K[x_i]$, $i=1,\dots,n$.

Since the ring of invariants $\bigcap_{i=1}^n\Delta_n^{\mathrm{ad}(x_i)}=K$ is reduced, the multi-sequence $\{\partial^{[\alpha]},\alpha\in\mathbb{N}^n\}$ is the only iterative $(-\mathrm{ad}(x_1),\dots,-\mathrm{ad}(x_n))$-descent in the algebra Δ_n, by Theorem 2.10. The next theorem proves the same but for the ring $\mathcal{D}(P_n)$. The ring $\mathcal{D}(P_n)$ is noncommutative, so we cannot apply Theorem 2.10 directly.

Theorem 3.1. *Let K be a field of characteristic $p>0$. Then $\{\partial^{[\alpha]},\alpha\in\mathbb{N}^n\}$ is the only iterative $(-\mathrm{ad}(x_1),\dots,-\mathrm{ad}(x_n))$-descent in $\mathcal{D}(P_n)$.*

Proof. Let $\delta_1:=-\mathrm{ad}(x_1),\dots,\delta_n:=-\mathrm{ad}(x_n)$ and $\delta:=(\delta_1,\dots,\delta_n)$. Let $\{y^{[\alpha]},\alpha\in\mathbb{N}^n\}$ be an iterative δ-descent in $\mathcal{D}(P_n)$. We have to show that $y^{[\alpha]}=\partial^{[\alpha]}$ for all α or, equivalently, that $y_i^{[p^k]}=\partial_i^{[p^k]}$ for all $i=1,\dots,n$ and $k\in\mathbb{N}$. To prove this fact we use induction on k.

Let $k=0$ and $y_i:=y_i^{[p^0]}$. Let us prove that $y_i=\partial_i$ for all i. Note that

$$y_i\in\bigcap_{j\neq i}\ker(\delta_j)\cap\ker(\delta_i^2)=P_n\oplus P_n\partial_i.$$

Since $\delta_i(y_i)=1$, we see that $y_i=\partial_i+\lambda_i$ for some polynomial $\lambda_i\in P_n$. If $\lambda_i=0$, we are done. Supposing that $\lambda_i\neq0$, we seek a contradiction. Then, using the defining relations (10) for $\mathcal{D}(P_n)$, we obtain that

$$0=y_i^p=(\partial_i+\lambda_i)^p=\partial_i^p+\sum_{j=1}^{p-1}a_j\partial_i^{[j]}+a_0+\lambda_i^p$$

for some polynomials $a_k\in P_n$ such that $\deg_{x_i}(a_0)<\deg_{x_i}(\lambda_i^p)$, where $\deg_{x_i}(q)$ is the x_i-degree of a polynomial $q\in P_n$. Since $\partial_i^p=0$, the above equality yields $a_1=\cdots=a_{p-1}=0$ and $a_0+\lambda_i^p=0$. The last equality is impossible since $\deg_{x_i}(a_0)<\deg_{x_i}(\lambda_i^p)$.

Suppose that $k\geq1$ and that $y_i^{[p^s]}=\partial_i^{[p^s]}$ for all $s=0,\dots,k-1$ and $i=1,\dots,n$. To finish the proof by induction it remains to show that $y_i^{[p^k]}=\partial_i^{[p^k]}$ for all i. Note that

$$y_i^{[p^k]}\in\bigcap_{j\neq i}\ker(\delta_j)\cap\ker(\delta_i^{[p^{k+1}]})=P_n\oplus K[\partial_i,\partial_i^{[p]},\dots,\partial_i^{[p^k]}]\qquad\text{(Lemma 6.1.(6))},$$

$$y_i^{[p^k]}\in\bigcap_{j=1}^n\ker\mathrm{ad}(\partial_j^{[p^{k-1}]}))=K[x_1^{p^k},\dots,x_n^{p^k}]\otimes\Delta_n\qquad\text{(Lemma 6.1.(2))},$$

and so $y_i^{[p^k]} \in K[x_1^{p^k}, \dots, x_n^{p^k}] \otimes K[\partial_i, \partial_i^{[p]}, \dots, \partial_i^{[p^k]}]$. Since $\delta_i^{[p^k]}(y_i^{[p^k]}) = 1$,

$$y_i^{[p^k]} = \partial_i^{[p^k]} + \sum_{j=0}^{p^k-1} \mu_j \partial_i^{[j]}$$

for some polynomials $\mu_j \in K[x_1^{p^k}, \dots, x_n^{p^k}]$. On the one hand, $[y_i^{[p^k]}, x_i] = y_i^{[p^k-1]} = \partial_i^{[p^k-1]}$. On the other hand,

$$[y_i^{[p^k]}, x_i] = \partial_i^{[p^k-1]} + \sum_{j=1}^{p^k-1} \mu_j \partial_i^{[j-1]}.$$

Hence, $\mu_1 = \dots = \mu_{p^k-1} = 0$, i.e. $y_i^{[p^k]} = \partial_i^{[p^k]} + \mu_0$. If $\mu_0 = 0$, we are done. Supposing that $\mu_0 \neq 0$, we seek a contradiction. Then

$$0 = (y_i^{[p^k]})^p = (\partial_i^{[p^k]} + \mu_0)^p = (\partial_i^{[p^k]})^p + \sum_{j=1}^{p^k-1} b_j \partial_i^{[j]} + b_0 + \mu_0^p$$

for some polynomials $b_s \in K[x_1^{p^k}, \dots x_n^{p^k}]$ such that $\deg_{x_i^{p^k}}(b_0) < \deg_{x_i^{p^k}}(\mu_0^p)$. Since $(\partial_i^{[p^k]})^p = 0$, the above equality yields $b_1 = \dots = b_{p^k-1} = 0$ and $b_0 + \mu_0^p = 0$. The last equality is impossible since $\deg_{x_i^{p^k}}(b_0) < \deg_{x_i^{p^k}}(\mu_0^p)$. This proves that $y_i^{[p^k]} = \partial_i^{[p^k]}$ for all i and $k \in \mathbb{N}^n$. The proof of the theorem is complete. □

Each automorphism $\sigma \in \mathrm{Aut}_K(P_n)$ can be naturally extended (by change of variables) to a K-automorphism, say σ, of the ring $\mathcal{D}(P_n)$ of differential operators on the polynomial algebra P_n by the rule:

(11) $$\sigma(a) := \sigma a \sigma^{-1}, \;\; a \in \mathcal{D}(P_n).$$

Then the group $\mathrm{Aut}_K(P_n)$ can be seen as a subgroup of $\mathrm{Aut}_K(\mathcal{D}(P_n))$ via (11). In characteristic zero, there are many other extensions of the automorphism σ. For example, the automorphisms of $\mathcal{D}(K[x])$:

$$\sigma_f : x \mapsto x, \;\; \partial \mapsto \partial + f, \;\; f \in K[x],$$

are extensions of the identity automorphism of $K[x]$. This is not the case in prime characteristic as Theorem 1.1 shows.

The set $\{x_i, \partial_i^{[p^k]} \,|\, 1 \le i \le n, k \in \mathbb{N}\}$ is a set of K-algebra generators for $\mathcal{D}(P_n)$. Each automorphism σ of $\mathcal{D}(P_n)$ is uniquely determined by its action on this *infinite* set. Theorem 1.1 (which is not true in characteristic zero) says that σ is uniquely determined by the elements $\sigma(x_1), \dots, \sigma(x_n)$.

Proof of Theorem 1.1. (⇒) This implication is trivial.

(⇐) By considering the automorphism $\sigma\tau^{-1}$ of the algebra $\mathcal{D}(P_n)$, it suffices to show that if an automorphism, say σ, of the algebra $\mathcal{D}(P_n)$ satisfies the conditions that $\sigma(x_1) = x_1, \dots, \sigma(x_n) = x_n$, then σ is the identity map. Using the defining relations (6), we see that $\{\sigma(\partial^{[\alpha]}), \alpha \in \mathbb{N}^n\}$ is the iterative $(-\mathrm{ad}(x_1), \dots, -\mathrm{ad}(x_n))$-descent. By Theorem 3.1, $\sigma(\partial^{[\alpha]}) = \partial^{[\alpha]}$ for all elements $\alpha \in \mathbb{N}^n$, i.e. the automorphism σ is the identity map, as required. □

Corollary 3.2. $\mathrm{Aut}_K(P_n) = \{\tau \in \mathrm{Aut}_K(\mathcal{D}(P_n)) \mid \tau(P_n) = P_n\}$ *and, for each automorphism* $\sigma \in \mathrm{Aut}_K(P_n)$*, the equation* (11) *gives the only extension of the automorphisms* σ *to an automorphism of the* K*-algebra* $\mathcal{D}(P_n)$*.*

4. Extensions of the Frobenius to the ring $\mathcal{D}(P_n)$

In this section, Theorem 1.2 is proved and the concept of a Frobenius on $\mathcal{D}(P_n)$ is introduced.

The canonical extension of the Frobenius F to $\mathcal{D}(P_n)$. Let A be an algebra over a field K of characteristic $p > 0$. For each $x \in A$,

$$(\mathrm{ad}\, x)^p = \mathrm{ad}(x^p) \tag{12}$$

since, for any $a \in A$, $\mathrm{ad}(x^p)(a) = [x^p, a] = \sum_{i=1}^p \binom{p}{i} (\mathrm{ad}\, x)^i(a) x^{p-i} = (\mathrm{ad}\, x)^p(a)$.

Using the defining relations (10) for the algebra $\mathcal{D}(P_n)$ and the equality (12), we see that the *Frobenius* $\mathbb{F}_p$-algebra monomorphism

$$F : P_n \to P_n, \;\; a \mapsto a^p,$$

can be lifted to the $\mathbb{F}_p$-algebra monomorphism of the ring $\mathcal{D}(P_n)$ by the rule

$$F = F_x : \mathcal{D}(P_n) \to \mathcal{D}(P_n), \;\; \partial^{[\alpha]} \mapsto \partial^{[p\alpha]}, \;\; \alpha \in \mathbb{N}^n. \tag{13}$$

In more detail, it suffices to check that

$$[\partial_i^{[p^{k+1}]}, x_i^p] = \prod_{l=1}^k \frac{\partial_i^{[p^l](p-1)}}{(p-1)!}, \;\; 1 \le i \le n, \;\; k \ge 0.$$

By (12) and (8),

$$\begin{aligned} [\partial_i^{[p^{k+1}]}, x_i^p] &= (-\mathrm{ad}\, x_i)^p (\partial_i^{[p^{k+1}]}) = (-\mathrm{ad}\, x_i)^{p-1} (\prod_{l=0}^k \frac{\partial_i^{[p^l](p-1)}}{(p-1)!}) \\ &= (-\mathrm{ad}\, x_i)^{p-1} (\frac{\partial_i^{p-1}}{(p-1)!}) \cdot (\prod_{l=1}^k \frac{\partial_i^{[p^l](p-1)}}{(p-1)!}) = \prod_{l=1}^k \frac{\partial_i^{[p^l](p-1)}}{(p-1)!}. \end{aligned}$$

Definition. The $\mathbb{F}_p$-algebra monomorphism $F = F_x$ in (13) is called the **canonical Frobenius** with respect to the choice $x = (x_1, \dots, x_n)$ of free generators for the polynomial algebra P_n.

Note that the Frobenius F in (13) is *not* the 'obvious' extension of the Frobenius (which makes no sense as can be easily seen).

Definition. A ring homomorphism $F' : \mathcal{D}(P_n) \to \mathcal{D}(P_n)$ is called a **Frobenius** (homomorphism) if the following conditions hold:

1. there exists an F'-invariant polynomial subalgebra $P_n' = K[x_1', \dots, x_n']$ of the ring $\mathcal{D}(P_n)$ such that the restriction $F'|_{P_n'}$ is the Frobenius on P_n': $a \mapsto a^p$;
2. the inner derivations $\mathrm{ad}(x_1'), \dots, \mathrm{ad}(x_n')$ of the ring $\mathcal{D}(P_n)$ are locally nilpotent derivations with $\bigcap_{i=1}^n \mathcal{D}(P_n)^{\mathrm{ad}(x_i')} = P_n'$;
3. there exist a commutative F'-invariant K-subalgebra Δ' of the ring $\mathcal{D}(P_n)$ such that $[x_i', \Delta'] \subseteq \Delta'$ for all i and elements $y_1, \dots, y_n \in \Delta'$ such that $y_1^p = \cdots = y_n^p = 0$, $[y_i, x_j'] = \delta_{ij}$ (the Kronecker delta), and $[F'^k(y_i), x_j'] = 0$ for all $i \ne j$ and $k \ge 1$.

Example 1. The canonical Frobenius F_x is a Frobenius, where $x_i' = x_i$, $\partial_i' = \partial_i$ and $\Delta' = \Delta_n$, the algebra of scalar differential operators.

Example 2. For each K-algebra automorphism $\sigma \in \mathrm{Aut}_K(\mathcal{D}(P_n))$, $F' := \sigma F_x \sigma^{-1}$ is a Frobenius on $\mathcal{D}(P_n)$, where $x_i' = \sigma(x_i)$, $\partial_i = \sigma(\partial_i)$ and $\Delta' = \sigma(\Delta_n)$.

Let $\mathrm{Frob}(\mathcal{D}(P_n))$ be the set of all the Frobenius homomorphisms on $\mathcal{D}(P_n)$. The polynomial algebra P_n' above is called a *Frobenius polynomial subalgebra* of the ring $\mathcal{D}(P_n)$. Let $\mathrm{FPol}(\mathcal{D}(P_n))$ be the set of all the Frobenius polynomial subalgebras of the ring $\mathcal{D}(P_n)$. We will see later that each Frobenius subalgebra is a maximal commutative subalgebra of the algebra $\mathcal{D}(P_n)$ (Corollary 4.1(1)), and so one Frobenius polynomial subalgebra cannot be a proper subalgebra of another. We will see later that the algebra Δ' is unique for each Frobenius F' on $\mathcal{D}(P_n)$ and it is also a maximal commutative subalgebra of $\mathcal{D}(P_n)$ (Corollary 4.1(1,3)). Let $\mathrm{Frob}(\mathcal{D}(P_n), P_n')$ be the set of all the Frobenius homomorphisms F' with the Frobenius polynomial subalgebra P_n'. Then

$$\mathrm{Frob}(\mathcal{D}(P_n)) = \bigcup_{P_n' \in \mathrm{FPol}(\mathcal{D}(P_n))} \mathrm{Frob}(\mathcal{D}(P_n), P_n'). \tag{14}$$

This is the disjoint union since each Frobenius polynomial subalgebra is a maximal commutative subalgebra. The group $\mathrm{Aut}_K(\mathcal{D}(P_n))$ acts naturally on the sets $\mathrm{Frob}(\mathcal{D}(P_n))$ and $\mathrm{FPol}(\mathcal{D}(P_n))$: $(\sigma, F') \mapsto \sigma F' \sigma^{-1}$ and $(\sigma, P_n') \mapsto \sigma(P_n')$. Clearly, $\sigma \mathrm{Frob}(\mathcal{D}(P_n), P_n')\sigma^{-1} = \mathrm{Frob}(\mathcal{D}(P_n), \sigma(P_n'))$.

Definition. $\mathrm{Frob}(\mathcal{D}(P_n), P_n, \Delta_n) := \{F' \in \mathrm{Frob}(\mathcal{D}(P_n), P_n) \mid F'(\Delta_n) \subseteq \Delta_n\}$.

Proof of Theorem 1.2. Let $F' \in \mathrm{Frob}(\mathcal{D}(P_n))$ and let x_i', y_i and Δ' be as in the definition of a Frobenius homomorphism on $\mathcal{D}(P_n)$ above. Let $y_{ik} := F'^k(y_i)$, $k \geq 0$, and $\delta_i := -\mathrm{ad}(x_i')$. Then the conditions of Theorem 2.10(3) hold for the algebra $A := \Delta'$, $\delta = (\delta_1, \ldots, \delta_n)$ and the elements $\{y_{ik}\}$:

$$\begin{aligned}
y_{ik}^p &= F'^k(y_i)^p = F'^k(y_i^p) = 0; \\
\delta_i^{p^k}(y_{ik}) &= -\mathrm{ad}(x_i')^{p^k}(y_{ik}) = -\mathrm{ad}(x_i'^{p^k})(y_{ik}) = [y_{ik}, x_i'^{p^k}] = [F'^k(y_i), F'^k(x_i')] \\
&= F'^k([y_i, x_i']) = F'^k(1) = 1; \\
\delta_j(y_{ik}) &= [F'^k(y_i), x_j'] = 0, \quad i \neq j, \quad k \geq 0;
\end{aligned}$$

the algebra $\Delta'^{\delta} = \Delta' \cap \mathcal{D}(P_n)^{\delta} = \Delta' \cap P_n'$ is reduced. By Theorem 2.10(3), in Δ' there exists a unique iterative δ-descent, say $\{\partial'^{[\alpha]}\}$. Since the derivations δ_i of the algebra $\mathcal{D}(P_n)$ are locally nilpotent and $\mathcal{D}(P_n)^{\delta} = P_n'$, by using Lemma 2.1 we have

$$\mathcal{D}(P_n) = N(\delta, \mathcal{D}(P_n)) = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{D}(P_n)^{\delta} \partial'^{[\alpha]} = \bigoplus_{\alpha \in \mathbb{N}^n} P_n' \partial'^{[\alpha]} \simeq \mathcal{D}(P_n') \simeq \mathcal{D}(P_n).$$

Consider the K-algebra automorphism σ of $\mathcal{D}(P_n)$ given by the rule: $x_i \mapsto x_i'$, $\partial^{[\alpha]} \mapsto \partial'^{[\alpha]}$. Then $\sigma(P_n) = P_n'$, and so $\mathrm{FPol}(\mathcal{D}(P_n)) = \mathrm{Aut}_K(\mathcal{D}(P_n)) \cdot P_n$. The algebra $\Delta_n = \bigoplus_{\alpha \in \mathbb{N}^n} K\partial^{[\alpha]}$ of scalar differential operators is a *maximal* commutative subalgebra of $\mathcal{D}(P_n)$; hence $\bigoplus_{\alpha \in \mathbb{N}^n} K\partial'^{[\alpha]}$ is the maximal commutative subalgebra of $\mathcal{D}(P_n)$ which is contained in the commutative subalgebra Δ'. Therefore, $\Delta' = \bigoplus_{\alpha \in \mathbb{N}^n} K\partial'^{[\alpha]}$. This means that $\sigma(\Delta_n) = \Delta'$, and so $\sigma^{-1}F'\sigma \in \mathrm{Frob}(\mathcal{D}(P_n), P_n, \Delta_n)$ and the algebra Δ' in the definition of a Frobenius F' of $\mathcal{D}(P_n)$ is unique, and it is a maximal commutative subalgebra of $\mathcal{D}(P_n)$. This proves statement 1 and Corollary 4.1(1,2)).

The stabilizer of the polynomial algebra P_n under the action of the group $\mathrm{Aut}_K(\mathcal{D}(P_n))$ on the set $\mathrm{FPol}(\mathcal{D}(P_n))$ is equal to the set $\{\sigma \in \mathrm{Aut}_K(\mathcal{D}(P_n)) \,|\, \sigma(P_n) = P_n\} = \mathrm{Aut}_K(P_n)$, Corollary 3.2. Now, $\mathrm{Aut}_K(\mathcal{D}(P_n)) \cdot P_n \simeq \mathrm{Aut}_K(\mathcal{D}(P_n))/\mathrm{Aut}_K(P_n)$. This completes the proof of statement 3.

To prove statement 2, let $F' \in \mathrm{Frob}(\mathcal{D}(P_n), P_n)$. By statement 1, $F' = \sigma F'' \sigma^{-1}$ for some $F'' \in \mathrm{Frob}(\mathcal{D}(P_n), P_n, \Delta_n)$ and $\sigma \in \mathrm{Aut}_K(\mathcal{D}(P_n))$ such that $\sigma(P_n) = P_n$. Then $\sigma \in \mathrm{Aut}_K(P_n)$, by Corollary 3.2, as required. □

Corollary 4.1. (1) *For each Frobenius $F' \in \mathrm{Frob}(\mathcal{D}(P_n))$, the corresponding algebras P_n' and Δ' are unique; they are maximal commutative subalgebras of the algebra $\mathcal{D}(P_n)$.*

(2) *Let $F', F'' \in \mathrm{Frob}(\mathcal{D}(P_n))$ and $F'' = \sigma F' \sigma^{-1}$ for some automorphism $\sigma \in \mathrm{Aut}_K(\mathcal{D}(P_n))$. Then $P_n'' = \sigma(P_n')$ and $\Delta'' = \sigma(\Delta')$, where P_n'' and Δ'' are the corresponding algebras for the Frobenius F''.*

(3) *One Frobenius polynomial algebra cannot be a proper subalgebra of another Frobenius polynomial algebra.*

(4) *The union* (14) *is a disjoint union.*

5. The sets $\mathrm{Frob}_s(\mathcal{D}(P_n), P_n, \Delta_n), s \geq 1$

Definition. For each natural number $s \geq 1$, let $\mathrm{Frob}_s(\mathcal{D}(P_n), P_n))$ be the set of all ring endomorphisms F' of $\mathcal{D}(P_n)$ such that $F'(P_n) \subseteq P_n$ and $F'|_{P_n} = F^s$, i.e. $F'(a) = a^{p^s}$ for all $a \in P_n$.

Definition. For each natural number $s \geq 1$, let

$$\mathrm{Frob}_s(\mathcal{D}(P_n), P_n, \Delta_n) := \{G \in \mathrm{Frob}_s(\mathcal{D}(P_n), P_n) |\, G(\Delta_n) \subseteq \Delta_n\}.$$

This set contains precisely all the extensions G of F^s on the polynomial algebra P_n that respect the subalgebra Δ_n of scalar differential operators of $\mathcal{D}(P_n)$. In this section, the sets $\mathrm{Frob}_s(\mathcal{D}(P_n), P_n, \Delta_n)$ are found explicitly (Theorem 5.1 and Theorem 5.2).

Let $\delta := (\delta_1, \dots, \delta_n)$ where $\delta_i := -\mathrm{ad}(x_i)$ is the inner derivation of the ring $\mathcal{D}(P_n)$. For each natural number $s \geq 1$, $\delta_i^{p^s} = -\mathrm{ad}(x_i)^{p^s} = -\mathrm{ad}(x_i^{p^s})$ is the inner derivation of the ring $\mathcal{D}(P_n)$. For each natural number $s \geq 1$, $\delta^{p^s} := (\delta_1^{p^s}, \dots, \delta_n^{p^s})$ is the n-tuple of commuting inner derivations of $\mathcal{D}(P_n)$ such that $\delta_i^{p^s}(\Delta_n) \subseteq \Delta_n$ for all i. Let $\mathrm{ID}(\delta^{p^s}, \Delta_n)$ be the set of all iterative δ^{p^s}-descents in the algebra Δ_n of exponent $(\infty, \dots, \infty)$. It follows from the defining relations (6) for the K-algebra $\mathcal{D}(P_n)$ that the map

$$\mathrm{Frob}_s(\mathcal{D}(P_n), P_n, \Delta_n) \to \mathrm{ID}(\delta^{p^s}, \Delta_n), \;\; G \mapsto \{G(\partial^{[\alpha]})\}_{\alpha \in \mathbb{N}^n}, \tag{15}$$

is a *bijection*, where the inverse map is given by the rule

$$\{y^{[\alpha]}\} \mapsto G : \partial^{[\alpha]} \mapsto y^{[\alpha]}, \;\; \alpha \in \mathbb{N}^n.$$

For each $i = 1, \dots, n$, let us introduce the K-linear map

$$\int_i^* : \Delta_n \to \Delta_n, \;\; \partial^{[\alpha]} \mapsto \partial^{[\alpha + e_i]}, \;\; \alpha \in \mathbb{N}^n, \tag{16}$$

where $e_1, \dots, e_n$ is the standard $\mathbb{Z}$-basis of $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z} e_i$. The map $\int_i^*$ is a right inverse of the map δ_i, $\delta_i \int_i^* = \mathrm{id}_{\Delta_n}$. We call the map $\int_i^*$ the i'th *dual integration* on

Δ_n by analogy with the usual i'th integration $\int_i$ on P_n when $\mathrm{char}(K) = 0$, which is the K-*linear* map

$$\int_i : P_n \to P_n, \quad x^{[\alpha]} \mapsto x^{[\alpha+e_i]}, \quad x^{[\alpha]} := \prod_{i=1}^{n} \frac{x^{\alpha_i}}{\alpha_i!}, \quad \alpha \in \mathbb{N}^n.$$

The map $\int_i$ is a right inverse to $\partial_i := \frac{\partial}{\partial x_i}$ in P_n as $\partial_i \int_i = \mathrm{id}_{P_n}$.

For each $i \neq j$, the maps $\int_i^*$ and δ_j commute. The commutator $[\delta_i, \int_i^*] = 1 - \int_i^* \delta_i$ is the projection onto the subalgebra $\Delta_{n,\widehat{i}} := \bigotimes_{j \neq i} \Delta_1(j)$ in the decomposition $\Delta_n = \bigoplus_{j \geq 0} \Delta_{n,\widehat{i}} \partial_i^{[j]}$.

Definition. For each natural number s such that $s \geq 1$, let $\mathcal{M}_{n,s}$ be the set of all $n \times \mathbb{N}$ matrices $u = (u_{ij})$, $i = 1, \dots, n$, $j \in \mathbb{N}$, such that

$$u_{ij} \in \bigoplus_{0 \neq \alpha \in C_s^n} K\partial^{[\alpha]} \subset \bigoplus_{\alpha \in C_s^n} K\partial^{[\alpha]} = \Delta_n \cap \bigcap_{i=1}^{s} \ker(\mathrm{ad}(x_i^{p^s})) \qquad \text{(Lemma 6.1(6))}$$

where $C_s^n := \{\alpha \in \mathbb{N}^n \,|\, \text{all } \ \alpha_i < p^s\}$. The set C_s^n is the *n-dimensional discrete cube* of size p^s. It contains p^{ns} elements. An element $a \in \Delta_n$ satisfies the condition that $a^p = 0$ iff $a \in \Delta_{n,+} := \bigoplus_{0 \neq \alpha \in \mathbb{N}^n} K\partial^{[\alpha]}$. Since $u_{ij} \in \Delta_{n,+}$, we have $u_{ij}^p = 0$.

The next theorem gives explicitly all the elements of the set $\mathrm{ID}(\delta^{p^s}, \Delta_n)$.

Theorem 5.1. *Let K be a field of characteristic $p > 0$ and $s \geq 1$. Then the map*

$$\mathcal{M}_{n,s} \to \mathrm{ID}(\delta^{p^s}, \Delta_n), \quad u = (u_{ij}) \mapsto y_u := \{y_u^{[\alpha]}\}_{\alpha \in \mathbb{N}^n},$$

is a bijection, where $y_u^{[\alpha]} := \prod_{i=1}^n y_{u,i}^{[\alpha_i]}$, $y_{u,i}^{[j]} := \prod_{k \geq 0} \frac{y_{u,i}^{[p^k] j_k}}{j_k!}$ (for each $j \in \mathbb{N}$ written p-adically as $j = \sum j_k p^k$), $y_{u,i}^{[1]} := \partial_i^{[p^s]} + u_{i0}$, and then recursively, for each $k \geq 1$,

$$y_{u,i}^{[p^k]} := u_{ik} + \int_i^{*p^s} \prod_{l=0}^{k-1} \frac{y_{u,i}^{[p^l](p-1)}}{(p-1)!}.$$

The map $\{z^{[\alpha]}\}_{\alpha \in \mathbb{N}^n} \mapsto u = (u_{ik})$, given by the rule

$$u_{ik} := z_i^{[p^k]} - \int_i^{*p^s} z_i^{[p^k - 1]},$$

is the inverse map of the map $u \mapsto y_u$.

Remark. Note that $y_{u,i}^{[p^k]} := u_{ik} + \int_i^{*p^s} y_{u,i}^{[p^k-1]}$, $k \geq 1$.

Proof. First, we prove that the map $u \mapsto y_u$ is well-defined, i.e. $y_u \in \mathrm{ID}(\delta^{p^s}, \Delta_n)$. The ring Δ_n is commutative. By Corollary 2.8, $y_u \in \mathrm{ID}(\delta^{p^s}, \Delta_n)$ iff, for each $i, j = 1, \dots, n$ and $k \in \mathbb{N}$,

$$y_{u,j}^{[p^k]p} = 0 \ \text{ and } \ \delta_i^{p^s}(y_{u,j}^{[p^k]}) = \delta_{i,j} y_{u,j}^{[p^k - 1]},$$

where $\delta_{i,j}$ is the Kronecker delta. By the very definition, the elements $y_{u,i}^{[p^k]}$ belong to the set $\Delta_{n,+}$, and so the first type of equalities holds, i.e. $y_{u,j}^{[p^k]p} = 0$. To verify that the second type of equalities holds, we use induction on k. The case $k = 0$ is trivially true. Suppose that $k \geq 1$ and that the equalities hold for all k' such that $k' < k$. Then

$$\delta_i^{p^s}(y_{u,i}^{[p^k]}) = \delta_i^{p^s}(u_{ik} + \int_i^{*p^s} y_{u,i}^{[p^k-1]}) = 0 + \delta_i^{p^s} \int_i^{*p^s} y_{u,i}^{[p^k-1]} = y_{u,i}^{[p^k-1]},$$

and, for $i \neq j$,

$$\delta_j^{p^s}(y_{u,i}^{[p^k]}) = \delta_j^{p^s}(u_{ik} + \int_i^{*p^s} y_{u,i}^{[p^k-1]}) = 0 + \int_i^{*p^s} \delta_j^{p^s} y_{u,i}^{[p^k-1]} = \int_i^{*p^s} 0 = 0.$$

By induction, the equalities hold for all $k \in \mathbb{N}$. This shows that the map $u \mapsto y_u$ is well-defined.

By the very definition, the map $u \mapsto y_u$ is injective. To prove that it is surjective we have to show that each element $y = \{y^{[\alpha]}\} \in \mathrm{ID}(\delta^{p^s}, \Delta_n)$ is equal to y_u for some $u = (u_{ik}) \in \mathcal{M}_{n,s}$. Let us set $u_{ik} := y_i^{[p^k]} - \int_i^{*p^s} y_i^{[p^k-1]}$. Clearly, $u_{ik}^p = 0$, $\delta_i^{p^s}(u_{ik}) = y_i^{[p^k-1]} - y_i^{[p^k-1]} = 0$ and, for each $j \neq i$,

$$\delta_j^{p^s}(u_{ik}) = 0 - \int_i^{*p^s} \delta_j^{p^s} y_i^{[p^k-1]} = 0.$$

Therefore, $u_{ik} \in \sum_{0 \neq \alpha \in C_s^n} K\partial^{[\alpha]}$. By the very definition, $y = y_u$. This proves that the map $u \mapsto y_u$ is a bijection and $\{z^{[\alpha]}\} \mapsto u = (u_{ik})$ is its inverse map. □

As a consequence of Theorem 5.1, we can find explicitly the sets $\mathrm{Frob}_s(\mathcal{D}(P_n), P_n, \Delta_n)$, $s \geq 1$.

Theorem 5.2. *Let K be a field of characteristic $p > 0$ and $s \geq 1$. Then the map*

$$\mathcal{M}_{n,s} \to \mathrm{Frob}_s(\mathcal{D}(P_n), P_n, \Delta_n), \quad u \mapsto G_u : \partial^{[\alpha]} \mapsto y_u^{[\alpha]}, \quad \alpha \in \mathbb{N}^n,$$

is a bijection, where the elements $y_u^{[\alpha]}$ are defined in Theorem 5.1, and the map

$$G \mapsto (u_{ik}(G)), \quad u_{ik}(G) := G(\partial_i^{[p^k]}) - \int_i^{*p^s} G(\partial_i^{[p^k-1]}),$$

is its inverse map.

Proof. The theorem follows directly from Theorem 5.1 and (15). □

Example. Let $n = s = 1$, $u = (\lambda\partial, 0, 0, \ldots)$, $\lambda \in K$, and $F' := G_u$. Then $F'(\partial) = \partial^{[p]} + \lambda\partial$ and

$$\begin{aligned} F'(\partial^{[p]}) &= \int^{*p} \frac{(\partial^{[p]} + \lambda\partial)^{p-1}}{(p-1)!} = \frac{1}{(p-1)!} \int^{*p} \sum_{j=0}^{p-1} \binom{p-1}{j} \partial^{[p]j} (\lambda\partial)^{p-1-j} \\ &= \int^{*p} \sum_{j=0}^{p-1} \lambda^{p-1-j} \partial^{[jp]} \partial^{[p-1-j]} = \int^{*p} \sum_{j=0}^{p-1} \lambda^{p-1-j} \partial^{[p-1-j+jp]} \\ &= \sum_{j=0}^{p-1} \lambda^{p-1-j} \partial^{[p-1-j+(j+1)p]}. \end{aligned}$$

The ring $\mathcal{D}(P_n)$ is simple, so each Frobenius $F' \in \mathrm{Frob}(\mathcal{D}(P_n))$ is automatically a monomorphism.

Corollary 5.3. *Each Frobenius $F' \in \mathrm{Frob}(\mathcal{D}(P_n))$ is not an automorphism of the ring $\mathcal{D}(P_n)$. Moreover, the ring $\mathcal{D}(P_n)$ is a left and right free finitely generated $KF'(\mathcal{D}(P_n))$-module of rank p^{2n}.*

Proof. By Theorem 1.2(1), we may assume that $F' \in \mathrm{Frob}(\mathcal{D}(P_n), P_n, \Delta_n)$. Note that $\mathcal{D}(P_n) = P_n \otimes \Delta_n$, $F'(P_n) \subseteq P_n$ and $F'(\Delta_n) \subseteq \Delta_n$. The commutative algebra P_n is a free $KF'(P_n)$-module of rank p^n. By Theorem 5.1, the commutative algebra Δ_n is a free $KF'(\Delta_n)$-module of rank p^n. Then it is easy to deduce (using Theorem 5.1) that

$$\mathcal{D}(P_n) = \bigoplus_{\alpha,\beta\in C_1^n} KF'(\mathcal{D}(P_n))x^\alpha\partial^{[\beta]} = \bigoplus_{\alpha,\beta\in C_1^n} x^\alpha\partial^{[\beta]}KF'(\mathcal{D}(P_n)),$$

and the results follow. □

6. Appendix: Some technical results

In this section, some obvious technical results on the ring $\mathcal{D}(P_n)$ are collected. Proofs are included for reader's convenience. Let $\Delta_1(i) := \bigoplus_{j\geq 0} K\partial_i^{[j]}$.

Lemma 6.1. *Let K be a field of characteristic $p > 0$, $k, k_i \in \mathbb{N}$, and $\ker(\cdot) := \ker_{\mathcal{D}(P_n)}(\cdot)$. Then*

1. $\ker \mathrm{ad}(\partial_i^{[p^k]}) = \mathcal{D}(K[x_1,\dots,\widehat{x}_i,\dots,x_n]) \otimes K[x_i^{p^{k+1}}] \otimes \Delta_1(i)$.
2. $\bigcap_{i=1}^n \ker \mathrm{ad}(\partial_i^{[p^{k_i}]}) = \Delta_n \otimes K[x_1^{p^{k_1+1}},\dots,x_n^{p^{k_n+1}}]$.
3. $\bigcap_{k\geq 0} \ker \mathrm{ad}(\partial_i^{[p^k]}) = \mathcal{D}(K[x_1,\dots,\widehat{x}_i,\dots,x_n]) \otimes \Delta_1(i)$.
4. $\bigcap_{i=1}^n \bigcap_{k\geq 0} \ker \mathrm{ad}(\partial_i^{[p^k]}) = \Delta_n$.
5. $\ker \mathrm{ad}(x_i^{p^k}) = \mathcal{D}(K[x_1,\dots,\widehat{x}_i,\dots,x_n]) \otimes K[x_i] \otimes K[\partial_i, \partial_i^{[p]},\dots,\partial_i^{[p^{k-1}]}]$.
6. $\bigcap_{i=1}^n \ker \mathrm{ad}(x_i^{p^{k_i}}) = P_n \otimes \bigotimes_{i=1}^n K[\partial_i, \partial_i^{[p]},\dots,\partial_i^{[p^{k_i-1}]}]$.
7. $\bigcap_{i=1}^n \ker \mathrm{ad}(x_i) = P_n$.
8. *For each natural number $k \geq 1$, the centralizer $C(F_x^k(\mathcal{D}(P_n)), \mathcal{D}(P_n))$ of the ring $F_x^k(\mathcal{D}(P_n))$ in $\mathcal{D}(P_n)$ is equal to the direct sum $\bigoplus_\alpha K\partial^{[\alpha]}$, where $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, $0 \leq \alpha_1 < p^k, \dots, 0 \leq \alpha_n < p^k$.*

Proof. 1. The RHS of the equality in question is a subset of the LHS. The opposite inclusion follows at once from the following equality: for each natural number j written p-adically as $j = \sum_{s\geq l} j_s p^s$ with $j_l \neq 0$ and such that $l \leq k$, there is the equality

$$[\partial_i^{[p^k]}, x_i^j] = (j_l\partial_i^{[p^k-p^l]}x_i^{(j_l-1)p^l} + \cdots)x_i^{\sum_{s\geq k+1} j_s p^s}, \tag{17}$$

where the three dots means an element of the set $\sum_{0\leq t<p^k-p^l} \partial_i^{[t]}K[x_i]$. In more detail,

$$\begin{aligned}
[\partial_i^{[p^k]}, x_i^j] &= [\partial_i^{[p^k]}, \prod_{s\geq l} x_i^{j_s p^s}] = [\partial_i^{[p^k]}, \prod_{s=l}^{k} x_i^{j_s p^s}] \cdot \prod_{t\geq k+1} x_i^{j_t p^t} \\
&= ([\partial_i^{[p^k]}, x_i^{j_l p^l}] \prod_{\nu=l+1}^{k} x_i^{j_\nu p^\nu} + x_i^{j_l p^l}[\partial_i^{[p^k]}, x_i^{j_{l+1}p^{l+1}}] \prod_{\nu=l+2}^{k} x_i^{j_\nu p^\nu} + \cdots \\
&\quad + x_i^{j_l p^l} \cdots x_i^{j_{\mu-1}p^{\mu-1}}[\partial_i^{[p^k]}, x_i^{j_\mu p^\mu}] \prod_{\nu=\mu+1}^{k} x_i^{j_\nu p^\nu} + \cdots) \cdot \prod_{t\geq k+1} x_i^{j_t p^t}.
\end{aligned}$$

Since

$$\begin{aligned}[\partial_i^{[p^k]}, x_i^{j_\mu p^\mu}] &= F_x^\mu([\partial_i^{[p^{k-\mu}]}, x_i^{j_\mu}]) = F_x^\mu(j_\mu \partial_i^{[p^{k-\mu}-1]} x_i^{j_\mu -1} + a_\mu) \\ &= j_\mu \partial_i^{[p^k - p^\mu]} x_i^{(j_\mu -1)p^\mu} + F_x^\mu(a_\mu)\end{aligned}$$

for some element $a_\mu \in \sum_{r<p^{k-\mu}-1} \partial_i^{[r]} K[x_i]$, the equality (17) is obvious.

2 and 3. Statements 2 and 3 follow from statement 1.

4. Statement 4 follows from statement 3.

5. The RHS of the equality in statement 5 is a subset of the LHS. The opposite inclusion follows directly from the following equality: for each natural number j written p-adically as $j = \sum_{0\leq s\leq l} j_s p^s$ with $j_l \neq 0$ and such that $l \geq k$ (i.e. $j \geq p^k$),

$$[\partial_i^{[j]}, x_i^{p^k}] = \partial_i^{[j-p^k]}. \tag{18}$$

In more detail, let $a := \partial_i^{[\sum_{s=0}^{k-1} j_s p^s]}$, $b := \partial_i^{[\sum_{s=k}^{l} j_s p^s]} = F_x^k(c)$ and $c := \partial_i^{[\sum_{s=k}^{l} j_s p^{s-k}]}$. Then

$$\begin{aligned}[\partial_i^{[j]}, x_i^{p^k}] &= [ab, x_i^{p^k}] = a[F_x^k(c), F_x^k(x_i)] = aF_x^k([c, x_i]) = aF_x^k(\partial_i^{[\sum_{s=k}^{l} j_s p^{s-k}-1]}) \\ &= a\partial_i^{[\sum_{s=k}^{l} j_s p^s - p^k]} = \partial_i^{[j-p^k]}.\end{aligned}$$

6. Statement 6 follows from statement 5.

7. Statement 7 is a particular case of statement 6.

8. Note that the centralizer $C := C(F_x^k(\mathcal{D}(P_n)), \mathcal{D}(P_n))$ is the centralizer of the set that is the image under F_x^k of the set of canonical generators for $\mathcal{D}(P_n)$. Let C' be the sum $\sum_\alpha K\partial^{[\alpha]}$ in statement 8 (we have to show that $C = C'$). By statements 1 and 5,

$$C_1 := \bigcap_{i=1}^n \ker(x_i^{p^k}) \cap \bigcap_{i=1}^n \ker(\mathrm{ad}\, \partial_i^{[p^k]}) = KF_x^{k+1}(P_n) \otimes C'.$$

Now, $C := C_1 \cap \bigcap_{i=1}^n \bigcap_{l\geq k+1} \ker(\mathrm{ad}\, \partial_i^{[p^l]}) = C'$. □

Lemma 6.2. *Let K be a field of characteristic $p > 0$; $k, k_i \in \mathbb{N}$, and $\ker(\cdot) := \ker_{P_n}(\cdot)$. Then*

(1) $\ker(\partial_i^{[p^k]}) = K[x_1, \ldots, \widehat{x}_i, \ldots, x_n] \otimes K[x_i^{p^{k+1}}]$.

(2) $\bigcap_{i=1}^n \ker(\partial_i^{[p^{k_i}]}) = K[x_1^{p^{k_1+1}}, \ldots, x_n^{p^{k_n+1}}]$,

(3) $\bigcap_{k\geq 0} \ker(\partial_i^{[p^k]}) = K[x_1, \ldots, \widehat{x}_i, \ldots, x_n]$.

(4) $\bigcap_{i=1}^n \bigcap_{k\geq 0} \ker(\partial_i^{[p^k]}) = K$.

Proof. 1. The RHS of the equality in question is a subset of the LHS. The opposite inclusion follows the equality

$$\partial_i^{[p^k]} * x_i^j = \binom{j}{p^k} x_i^{j-p^k}.$$

Note that $\binom{j}{p^k} = 0$ iff $j_k = 0$ in the p-adic sum $j = \sum j_s p^s$.

2 and 3. Statements 2 and 3 follow from statement 1.

4. Statement 4 follows from statement 3. □

Acknowledgements

A part of the paper was written during the author's two month visit to the IHES in 2007; the hospitality and support of the IHES is greatly acknowledged. The author would like to thank P. Cartier, O. Gabber and M. Kontsevich for several discussions on various aspects of rings of differential operators in prime characteristic and their modules and for their comments.

References

[1] H. Bass, E. H. Connell and D. Wright, The Jacobian Conjecture: reduction of degree and formal expansion of the inverse, *Bull. Amer. Math. Soc. (New Series)*, **7** (1982), 287–330. MR663785 (83k:14028)

[2] V. V. Bavula, Simple derivations of differentiably simple Noetherian commutative rings in prime characteristic, *Trans. Amer. Math. Soc.*, **360** (2008), no. 8, 4007-4027. MR2395162 (2009c:13060)

[3] V. V. Bavula, The inversion formulae for automorphisms of polynomial algebras and differential operators in prime characteristic, *J. Pure Appl. Algebra*, **212** (2008), 2320–2337. MR2426512

[4] V. V. Bavula, The group of order preserving automorphisms of the ring of differential operators on a Laurent polynomial algebra in prime characteristic, *Proc. Amer. Math. Soc.*, **137** (2009), 1891-1898. MR2480268

[5] V. V. Bavula, The Jacobian Conjecture$_{2n}$ implies the Dixmier Problem$_n$, ArXiv:math. RA/0512250.

[6] A. Belov-Kanel and M. Kontsevich, The Jacobian conjecture is stably equivalent to the Dixmier Conjecture, *Mosc. Math. J.*, **7** (2007), no. 2, 209–218. MR2337879 (2009f:16041)

[7] J. Dixmier, Sur les algèbres de Weyl. *Bull. Soc. Math. France*, **96** (1968), 209–242. MR0242897 (39:4224)

[8] Y. Tsuchimoto, Endomorphisms of Weyl algebra and p-curvatures. *Osaka J. Math.*, **42** (2005), no. 2, 435–452. MR2147727 (2006g:14101)

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