TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 363, Number 2, February 2011, Pages 887–900
S 0002-9947(2010)05180-3
Article electronically published on August 24, 2010

NEW HYPERGEOMETRIC-LIKE SERIES FOR $1/\pi^2$ ARISING FROM RAMANUJAN'S THEORY OF ELLIPTIC FUNCTIONS TO ALTERNATIVE BASE 3

NAYANDEEP DEKA BARUAH AND NARAYAN NAYAK

Dedicated to Professor Bruce C. Berndt on the occasion of his 70th birthday

Abstract. By using certain representations for Eisenstein series, we find new hypergeometric-like series for $1/\pi^2$ arising from Ramanujan's theory of elliptic functions to alternative base 3.

1. Introduction

Let $(a)_0 = 1$ and, for a positive integer n,

$$(a)_n := a(a+1)(a+2)...(a+n-1)$$

and

$${}_pF_{p-1}(a_1,...,a_p;b_1,...,b_{p-1};x) := \sum_{n=0}^{\infty} \frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_{p-1})_n}\frac{x^n}{n!}, \ |x|<1.$$

In his monumental paper "Modular equations and approximations to π" [14], [15, pp. 23–39], S. Ramanujan offered 17 beautiful series representations for $1/\pi$. Three of his series belong to the classical theory of elliptic functions, while the remaining fourteen series depend on Ramanujan's alternative theories of elliptic functions in which the classical nome "q" is replaced by

$$q_r := q_r(x) := \exp\left(-\pi\csc(\pi/r)\frac{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;1-x)}{{}_2F_1(\frac{1}{r},\frac{1-r}{r};1;x)}\right), \tag{1.1}$$

where $r = 3$, 4, or 6. In particular, two of these series

$$\frac{27}{4\pi} = \sum_{k=0}^{\infty}(15k+2)\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k\left(\frac{2}{3}\right)_k}{(k!)^3}\left(\frac{2}{27}\right)^k \tag{1.2}$$

and

$$\frac{15\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty}(33k+4)\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k\left(\frac{2}{3}\right)_k}{(k!)^3}\left(\frac{4}{125}\right)^k, \tag{1.3}$$

Received by the editors April 13, 2009.
2010 *Mathematics Subject Classification.* Primary 33C05; Secondary 33E05, 11F11, 11R29.
Key words and phrases. Hypergeometric series, cubic theta functions, cubic modular equations, Eisenstein series.

belong to his theory of q_3, where

$$q_3 := q_3(x) := \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-x)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;x)}\right). \tag{1.4}$$

It was only in 1987 that proofs of all 17 series representations for $1/\pi$ were found by J.M. and P.B. Borwein [8]. These authors and several other authors in the past two decades have found many new series for $1/\pi$ as well as for $1/\pi^2$. We refer to [4] for a recent survey on Ramanujan's series for $1/\pi$.

Although Ramanujan never published another paper on the theory of elliptic functions to alternative bases, six pages in his second notebook [2] are devoted to developing the subject matter. All of the results on these six pages have been proved in a paper by B.C. Berndt, S. Bhargava and F.G. Garvan [7]. See also [5, Chapter 33]. These theories are now known as "*Ramanujan's theories of elliptic functions to alternative bases* 3*,* 4*, or* 6" or "*Ramanujan's elliptic functions in the theories of signatures* 3*,* 4*, or* 6".

In [3], N.D. Baruah and Berndt used hypergeometric identities and certain representations for Eisenstein series to derive several new series representations for $1/\pi^2$. In particular, they found series for $1/\pi^2$ that are analogues of Ramanujan's series for $1/\pi$ in the classical theory and theories of q_4 and q_6. In this paper we apply Ramanujan's theory of q_3 and certain representations for Eisenstein series to find several new series for $1/\pi^2$. In the process we find two series representations for $1/\pi^2$ which are perfect analogues of (1.2) and (1.3).

2. Preliminary definitions and results

In his second notebook, Ramanujan [16, p. 258] recorded the fundamental inversion formula

$$z := z(q_3) := {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right) = a(q_3), \tag{2.1}$$

where q_3 is given by (1.4) and

$$a(q_3) := \sum_{m,n=-\infty}^{\infty} q_3^{m^2+mn+n^2}. \tag{2.2}$$

This result was first proved in print by the Borweins [9, p. 695, Theorem 2.3] and then later by Berndt, Bhargava, and Garvan [7], [5, p. 99]. In the sequel, we will write $q = q_3$ and often emphasize that x is also a function of q when writing $x = x(q)$.

Now we define a modular equation of degree n in the theory of signature 3 or a cubic modular equation of degree n. Suppose that the equality

$$n\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-k^2\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;k^2\right)} = \frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-l^2\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;l^2\right)} \tag{2.3}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l that is implied by (2.3). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α. The corresponding multiplier m is defined by

$$m := m(\alpha,\beta) := \frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right)} = \frac{z(q)}{z(q^n)}. \tag{2.4}$$

We conclude this section by defining Ramanujan's Eisenstein series

$$P(q) := 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}, \quad |q|<1, \tag{2.5}$$

which satisfies the identity [7, Lemma 4.1]

$$P(q) = (1-4x)z^2 + 3H(q)z\frac{dz}{dx}, \tag{2.6}$$

where $H(q) = 4x(q)(1-x(q))$.

3. Series for $1/\pi^2$ arising from Ramanujan's cubic theory

In [3], Baruah and Berndt derived many series for $1/\pi^2$ by combining two different representations for $P(e^{-2\pi/\sqrt{n}})P(e^{-2\pi\sqrt{n}})$. Their method utilized the classical theory of elliptic functions. Unfortunately, their ideas do not apply in the theory of signature 3, since the necessary transformation formula for $P(q)$ takes a different form (see equation (3.30)). In this paper, we use certain representations for $P^2(e^{-2\pi/\sqrt{3n}})$ and $P^2(e^{-2\pi\sqrt{n/3}})$ along with some hypergeometric series identities and obtain our series for $1/\pi^2$ by appealing to various cubic singular moduli defined in (3.16). We explain our method in the remainder of this section.

From Lemma 3.1 of [3], for $|x|<1$, we note that

$${}_3F_2^2(a_1,a_2,a_3;1,1;x) = \sum_{k=0}^{\infty}U_k x^k, \tag{3.1}$$

where

$$U_k = \sum_{n=0}^{k}\frac{(a_1)_n(a_2)_n(a_3)_n(a_1)_{k-n}(a_2)_{k-n}(a_3)_{k-n}}{(n!)^3((k-n)!)^3}. \tag{3.2}$$

Setting $a_1 = 1/3, \;\; a_2 = 2/3$ and $a_3 = 1/2$ in (3.1), we obtain

$${}_3F_2^2\left(\frac{1}{3},\frac{2}{3},\frac{1}{2};1,1;x\right) = \sum_{k=0}^{\infty}U_k x^k, \tag{3.3}$$

where

$$U_k = \sum_{n=0}^{k}\frac{(\frac{1}{3})_n(\frac{2}{3})_n(\frac{1}{2})_n(\frac{1}{3})_{k-n}(\frac{2}{3})_{k-n}(\frac{1}{2})_{k-n}}{(n!)^3((k-n)!)^3}. \tag{3.4}$$

Now, if z is defined by (2.1), then by a special case of Clausen's formula [8, p. 178, Proposition 5.6(b)]

$$z^2 = {}_3F_2\left(\frac{1}{3},\frac{2}{3},\frac{1}{2};1,1;H\right), \quad 0 \le x \le \frac{1}{2}, \tag{3.5}$$

where $H = 4x(1-x)$.

From (3.3) and (3.5), we find that

$$z^4 = {}_3F_2^2\left(\frac{1}{3},\frac{2}{3},\frac{1}{2};1,1;H\right) = \sum_{k=0}^{\infty}U_k H^k, \tag{3.6}$$

where U_k is defined by (3.4).

Differentiating (3.6) with respect to x, we find that

$$z^3\frac{dz}{dx}=\frac{1-2x}{H}\sum_{k=0}^{\infty}U_k kH^k. \tag{3.7}$$

But, z satisfies the hypergeometric differential equation [1, p. 75, Eq. (2.3.5)]

$$x(1-x)\frac{d^2z}{dx^2}+(1-2x)\frac{dz}{dx}-\frac{2}{9}z=0, \tag{3.8}$$

which can be rewritten as

$$z^3\frac{d^2z}{dx^2}=\frac{8z^4}{9H}-\frac{4(1-2x)}{H}\ z^3\frac{dz}{dx}. \tag{3.9}$$

Now,

$$\frac{d}{dx}\left(z^3\frac{dz}{dx}\right)=3z^2\left(\frac{dz}{dx}\right)^2+z^3\frac{d^2z}{dx^2}. \tag{3.10}$$

Employing (3.9) and (3.7) in (3.10), we obtain

$$\frac{d}{dx}\left(z^3\frac{dz}{dx}\right)=3z^2\left(\frac{dz}{dx}\right)^2+\frac{8z^4}{9H}-\frac{4(1-H)}{H^2}\sum_{k=0}^{\infty}U_k kH^k. \tag{3.11}$$

Again, differentiating (3.7) with respect to x, we find that

$$\frac{d}{dx}\left(z^3\frac{dz}{dx}\right)=\frac{4(1-2x)^2}{H^2}\sum_{k=0}^{\infty}U_k k^2H^k+\frac{2(H-2)}{H^2}\sum_{k=0}^{\infty}U_k kH^k. \tag{3.12}$$

From (3.11) and (3.12), we deduce that

$$3z^2\left(\frac{dz}{dx}\right)^2=\sum_{k=0}^{\infty}\left\{\frac{4(1-H)}{H^2}k^2-\frac{2}{H}k-\frac{8}{9H}\right\}U_kH^k. \tag{3.13}$$

Now, squaring both sides of (2.6), we obtain

$$P^2(q)=(1-4x)^2z^4+9H(q)^2z^2\left(\frac{dz}{dx}\right)^2+6H(q)(1-4x)z^3\frac{dz}{dx}. \tag{3.14}$$

Replacing q by q^n in (3.14), and then employing (3.6), (3.7), and (3.13), we deduce that

(3.15)

$$P^2(q^n)=\sum_{k=0}^{\infty}\left[12(1-H)k^2+\{6(1-2\beta)(1-4\beta)-6H\}k+(1-4\beta)^2-\frac{8H}{3}\right]U_kH^k.$$

Now, we set

$$x_n:=x(e^{-2\pi\sqrt{n/3}})\quad\text{and}\quad z_n:=z(e^{-2\pi\sqrt{n/3}}). \tag{3.16}$$

The numbers x_n are *cubic singular moduli*. It can also be shown that [12, Eqs. (3.11) and (3.7)]

$$1-x_n=x_{1/n},\quad z_{1/n}=\sqrt{n}z_n,\quad\text{and}\quad m(x_{1/n})=\sqrt{n}, \tag{3.17}$$

where $m(x(q))=m(x(q),x(q^n))$ is the multiplier defined by (2.4).

Setting $q = e^{-2\pi/\sqrt{3n}}$ in (3.15), so that $\beta = x_n$, $H_n = 4x_n(1-x_n)$, we find that

$$P^2\left(e^{-2\pi\sqrt{n/3}}\right) = \sum_{k=0}^{\infty}\Big[12(1-H_n)k^2 + \{6(1-2x_n)(1-4x_n) - 6H_n\}k + (1-4x_n)^2 - \frac{8H_n}{3}\Big]U_k H_n^k. \tag{3.18}$$

Next, we derive a similar expression for $P^2\left(e^{-2\pi/\sqrt{3n}}\right)$. To this end, we note from (2.4) that if m is the multiplier connecting x and $\beta = x(q^n)$, then

$$z = mz(q^n). \tag{3.19}$$

Thus,

$$\frac{dz}{dx} = \frac{dz}{d\beta}\cdot\frac{d\beta}{dx} = \frac{d\beta}{dx}\left\{\frac{dm}{d\beta}z(q^n) + m\frac{dz(q^n)}{d\beta}\right\}. \tag{3.20}$$

Now, from Theorem 2.3 of [11], we recall that

$$\frac{d\beta}{dx} = \frac{n}{m^2}\cdot\frac{\beta(1-\beta)}{x(1-x)}. \tag{3.21}$$

Employing (3.21) in (3.20), we find that

$$\frac{dz}{dx} = \frac{n\beta(1-\beta)}{m^2x(1-x)}z(q^n)\frac{dm}{d\beta} + \frac{n\beta(1-\beta)}{mx(1-x)}\frac{dz(q^n)}{d\beta}. \tag{3.22}$$

Invoking (3.22) and (3.19) in (2.6), we deduce that

$$P(q) = Dz^2(q^n) + 3nHz(q^n)\frac{dz(q^n)}{d\beta}, \tag{3.23}$$

where

$$D = (1-4x)m^2 + \frac{12n\beta(1-\beta)}{m}\cdot\frac{dm}{d\beta}. \tag{3.24}$$

Again, employing (3.24), (3.22), and (3.19) in (3.14), we deduce that

$$P^2(q) = \sum_{k=0}^{\infty}\left[12n^2(1-H)k^2 + \{6nD(1-2\beta) - 6n^2H\}k + D^2 - \frac{8H^2}{3}\right]U_kH^k. \tag{3.25}$$

Setting $q = e^{-2\pi/\sqrt{3n}}$ in (3.25), we arrive at

$$P^2\left(e^{-2\pi/\sqrt{3n}}\right) = \sum_{k=0}^{\infty}\Big[12n^2(1-H_n)k^2 + \{6nD_n\sqrt{1-H_n} - 6n^2H_n\}k + D_n^2 - \frac{8H_n^2}{3}\Big]U_kH_n^k, \tag{3.26}$$

where

$$D_n = n(4x_n - 3) + 3\sqrt{n}H_n\left[\frac{dm}{d\beta}\right]_{q=e^{-2\pi/\sqrt{3n}}}. \tag{3.27}$$

Next, we recall two further identities from H.H. Chan and W.-C. Liaw's paper [12, Eqs. (3.12) and (3.17)], namely,

$$\begin{aligned} nP\big(e^{-2\pi\sqrt{n/3}}\big) - P\big(e^{-2\pi/\sqrt{3n}}\big) &= 4z_n^2\big\{n\sqrt{1-H_n} - \frac{3\sqrt{n}}{4}H_n\frac{dm}{dx}(1-x_n, x_n)\big\} \\ &= L_n z_n^2, \end{aligned} \tag{3.28}$$

where

$$L_n = 4\big\{n\sqrt{1-H_n} - \frac{3\sqrt{n}}{4}H_n\frac{dm}{dx}(1-x_n, x_n)\big\} \tag{3.29}$$

and

$$nP\big(e^{-2\pi\sqrt{n/3}}\big) + P\big(e^{-2\pi/\sqrt{3n}}\big) = \frac{6\sqrt{3n}}{\pi} - 2nz_n^2. \tag{3.30}$$

Squaring both (3.28) and (3.30), and then adding the resulting identities, we find that

$$2n^2P^2\big(e^{-2\pi\sqrt{n/3}}\big) + 2P^2\big(e^{-2\pi/\sqrt{3n}}\big) = \frac{108n}{\pi^2} + (4n^2 + L_n^2)z_n^4 - \frac{24n\sqrt{3n}}{\pi}z_n^2. \tag{3.31}$$

Again, multiplying (3.28) and (3.30), we obtain

$$n^2P^2\big(e^{-2\pi\sqrt{n/3}}\big) - P^2\big(e^{-2\pi/\sqrt{3n}}\big) = \frac{6\sqrt{3n}}{\pi}L_n z_n^2 - 2nL_n z_n^4. \tag{3.32}$$

Multiplying both sides of (3.32) by $4n/L_n$, and adding the resulting identity to (3.31), we deduce that

$$\left(2n^2 + \frac{4n^3}{L_n}\right)P^2\big(e^{-2\pi\sqrt{n/3}}\big) + \left(2 - \frac{4n}{L_n}\right)P^2\big(e^{-2\pi/\sqrt{3n}}\big) + (4n^2 - L_n^2)z_n^4 = \frac{108n}{\pi^2}. \tag{3.33}$$

Employing (3.18), (3.26), and (3.6) in (3.33), we arrive at the following theorem.

Theorem 3.1. *If $H_n = 4x_n(1-x_n)$ and U_k is defined by* (3.4), *then*

$$\frac{108n}{\pi^2} = \sum_{k=0}^{\infty}\big\{A(x_n)k^2 + B(x_n)k + C(x_n)\big\}U_k H_n^k, \tag{3.34}$$

where

$$A(x_n) = 48n^2(1-H_n), \tag{3.35}$$

$$B(x_n) = 4n^2(1-2x_n)\{3(1-4x_n)+6\} - 24n^2H_n + 12nD_n(1-2x_n) \tag{3.36}$$

and

$$C(x_n) = 2n^2\big(1+\frac{2n}{L_n}\big)(1-4x_n)^2 - \frac{32n^2H_n}{3} + 4n^2 - L_n^2 + \big(2-\frac{4n}{L_n}\big)D_n^2, \tag{3.37}$$

where D_n is defined by (3.27).

In the next few sections, we present our new series for $1/\pi^2$, obtained from (3.34).

4. Example: $n = 2$

Theorem 4.1. *If* U_k *is defined by* (3.4)*, then*

$$\frac{81}{\pi^2} = \sum_{k=0}^{\infty} \big(36k^2 - 5\big) U_k \left(\frac{1}{2}\right)^k. \tag{4.1}$$

Proof. Setting $n = 2$ in (3.34), we obtain

$$\frac{216}{\pi^2} = \sum_{k=0}^{\infty} \big\{A(x_2)k^2 + B(x_2)k + C(x_2)\big\} U_k H_2^k. \tag{4.2}$$

By using two cubic modular equations of degree 2 [5, p. 120, Theorem 7.1(i) and Theroem 7.1(iii)] and (6.11) in Section 6, Baruah and Berndt [2, Eqs. (5.15) and (5.23)] found that

$$x_2 = \frac{\sqrt{2}-1}{2\sqrt{2}}, \quad H_2 = \frac{1}{2}, \quad \text{and} \quad \left[\frac{dm}{d\beta}\right]_{q=e^{-2\pi/\sqrt{6}}} = \frac{4}{3}. \tag{4.3}$$

Setting $n = 2$ in (3.27) and (3.29), and then employing (4.3), we obtain

$$D_2 = -2 \quad \text{and} \quad L_2 = 2\sqrt{2}. \tag{4.4}$$

Next, setting $n = 2$ in (3.35), (3.36), (3.37), and then using (4.3) and (4.4), we find that

$$A(x_2) = 96, \quad B(x_2) = 0, \quad \text{and} \quad C(x_2) = -\frac{40}{3}. \tag{4.5}$$

Employing (4.5) and (4.3) in (4.2), we readily arrive at (4.1). □

5. Example: $n = 3$

Theorem 5.1. *If* U_k *is defined by* (3.4)*, then*

(5.1)

$$\frac{3\sqrt{3}}{\pi^2} = \sum_{k=0}^{\infty} \big\{2(38\sqrt{3}-63)k^2 + 15(7\sqrt{3}-12)k + 37\sqrt{3} - 64\big\} U_k \left(\frac{3\sqrt{3}(2-\sqrt{3})^2}{2}\right)^k.$$

Proof. Setting $n = 3$ in (3.34), we obtain

$$\frac{324}{\pi^2} = \sum_{k=0}^{\infty} \big\{A(x_3)k^2 + B(x_3)k + C(x_3)\big\} U_k H_3^k. \tag{5.2}$$

Now, from [2, Eq. (5.32)], we note that

$$x_3 = \frac{3\sqrt{3}-5}{4} \quad \text{and} \quad H_3 = \frac{3\sqrt{3}(2-\sqrt{3})^2}{2}. \tag{5.3}$$

Again, from [5, p. 123, Lemma 7.4], we recall that

$$m = 1 + 2\beta^{1/3}, \tag{5.4}$$

where m is the multiplier connecting x and $\beta = x(q^3)$. Differentiating (5.4) with respect to β, we obtain

$$\frac{dm}{d\beta} = \frac{2}{3\beta^{2/3}}. \tag{5.5}$$

Setting $q=e^{-2\pi/3}$ in (5.5), so that $\beta=x_3=\frac{3\sqrt{3}-5}{4}$, we find that

$$\left[\frac{dm}{d\beta}\right]_{q=e^{-2\pi/3}}=\frac{4(2+\sqrt{3})}{3}. \tag{5.6}$$

Next, setting $n=3$ in (3.27) and (3.29), and then employing (5.3) and (5.6), we obtain

$$D_3=12-9\sqrt{3} \quad \text{and} \quad L_3=6. \tag{5.7}$$

Finally, setting $n=3$ in (3.35), (3.36), (3.37), and then employing (5.3) and (5.7), we find that

(5.8)

$$A(x_3)=216(38-21\sqrt{3}), \quad B(x_3)=1620(2-\sqrt{3})^2 \quad \text{and} \quad C(x_3)=36(111-64\sqrt{3}).$$

Employing (5.3) and (5.8) in (5.2), we readily arrive at (5.1). □

6. Example: $n=4$

Theorem 6.1. *If U_k is defined by* (3.4)*, then*

$$\frac{3}{4\pi^2}=\sum_{k=0}^{\infty}\left\{225k^2+81k+8\right\}U_k\left(\frac{2}{27}\right)^{k+2}. \tag{6.1}$$

The above series for $1/\pi^2$ is a perfect analogue of Ramanujan's series (1.2).

Proof. Setting $n=4$ in (3.34), we obtain

$$\frac{432}{\pi^2}=\sum_{k=0}^{\infty}\left\{A(x_4)k^2+B(x_4)k+C(x_4)\right\}U_kH_4^k. \tag{6.2}$$

Next, setting $n=4$ in (3.27), we find that

$$D_4=4(4x_4-3)+3\sqrt{4}H_4\left[\frac{dm}{d\beta}\right]_{q=e^{-2\pi/\sqrt{12}}}. \tag{6.3}$$

First, to calculate x_4 and hence H_4, we recall the following cubic modular equation of degree 4 [5, p. 121].

Let γ be of degree 4 over α, and let m be the associated multiplier in the theory of signature 3. Then

$$m=\left(\frac{\gamma}{\alpha}\right)^{1/3}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/3}-\frac{4}{m}\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/3}. \tag{6.4}$$

Setting $q=e^{-2\pi/\sqrt{12}}=e^{-\pi/\sqrt{3}}$ in (6.4), so that $\alpha=x_{1/4}=1-x_4$, $\gamma=x_4$, and $m=2$, we find that

$$\left(\frac{x_4}{1-x_4}\right)^{1/3}=2-\sqrt{3}, \tag{6.5}$$

from which we readily deduce that

$$x_4=\frac{9-5\sqrt{3}}{18} \quad \text{and} \quad H_4=\frac{2}{27}. \tag{6.6}$$

Secondly, to evaluate $\frac{dm}{dx}(1-x_4,x_4)$, we recall two more modular equations from [5, p. 120].

If α, β, and γ have degrees 1, 2, and 4, respectively, in the theory of signature 3 and m_1 and m_2 are the multipliers associated with pairs α, β and β, γ, respectively, then

$$\left(\frac{(1-\beta)^2}{1-\alpha}\right)^{1/3}+\left(\frac{\beta^2}{\alpha}\right)^{1/3}=m_1^2 \tag{6.7}$$

and

$$\frac{\sqrt{3}\{\beta(1-\beta)\}^{1/6}}{\{\alpha(1-\gamma)\}^{1/3}-\{\gamma(1-\alpha)\}^{1/3}}=\frac{m_1}{m_2}. \tag{6.8}$$

Using (6.7) and (6.8), we find that

$$\begin{aligned}\sqrt{3}m\{x(q^2)(1-x(q^2))\}^{1/6}=&\left\{\frac{(1-x(q^2))^2}{1-x(q)}\right\}^{1/3}\{x(q)(1-x(q^4))\}^{1/3}\\&-\{(1-x(q^2))^2x(q^4)\}^{1/3}+\{x^2(q^2)(1-x(q^4))\}^{1/3}\\&-\left\{\frac{x^2(q^2)}{x(q)}x(q^4)(1-x(q))\right\}^{1/3},\end{aligned} \tag{6.9}$$

where m is the multiplier connecting $x(q)$ and $x(q^4)$. Differentiating (6.9) with respect to $x:=x(q)$, we find that

(6.10)

$$\begin{aligned}&\sqrt{3}\left\{x(q^2)(1-x(q^2))\right\}\frac{dm}{dx}+\frac{\sqrt{3}}{6}m\left\{x(q^2)(1-x(q^2))\right\}\left\{\frac{1}{x(q^2)}-\frac{1}{(1-x(q^2))}\right\}\frac{dx(q^2)}{dx(q)}\\&=\frac{1}{3}\left\{\frac{(1-x(q^2))^2}{1-x(q)}\;x(q)(1-x(q^4))\right\}^{1/3}\left\{\frac{-2}{1-x(q^2)}\frac{dx(q^2)}{dx(q)}+\frac{1}{x(q)}+\frac{1}{(1-x(q))}\right.\\&\left.-\frac{1}{1-x(q^4)}\frac{dx(q^4)}{dx(q)}\right\}-\frac{1}{3}\left\{(1-x(q^2))^2x(q^4)\right\}^{1/3}\left\{\frac{-2}{1-x(q^2)}\frac{dx(q^2)}{dx(q)}\right.\\&\left.+\frac{1}{x(q^4)}\frac{dx(q^4)}{dx(q)}\right\}+\frac{1}{3}\left\{x^2(q^2)(1-x(q^4))\right\}^{1/3}\left\{\frac{2}{x(q^2)}\frac{dx(q^2)}{dx(q)}-\frac{1}{1-x(q^4)}\frac{dx(q^4)}{dx(q)}\right\}\\&-\frac{1}{3}\left\{\frac{x^2(q^2)}{x(q)}x(q^4)(1-x(q))\right\}^{1/3}\left\{\frac{2}{x(q^2)}\frac{dx(q^2)}{dx(q)}-\frac{1}{x(q)}+\frac{1}{x(q^4)}\frac{dx(q^4)}{dx(q)}\right.\\&\left.-\frac{1}{1-x(q)}\right\}.\end{aligned}$$

But, by Theorem 2.3 of [11],

$$\frac{dx(q^n)}{dx(q)}=\frac{n}{m^2}\cdot\frac{x(q^n)(1-x(q^n))}{x(q)(1-x(q))}, \tag{6.11}$$

where m is the multiplier connecting $x(q)$ and $x(q^n)$. In particular, if $n=4$, then

$$\frac{dx(q^4)}{dx(q)}=\frac{4}{m^2}\cdot\frac{x(q^4)(1-x(q^4))}{x(q)(1-x(q))}, \tag{6.12}$$

where m is the multiplier connecting $x(q)$ and $x(q^4)$. Setting $q = e^{-\pi/\sqrt{3}}$ in (6.12), so that, by (4.6), $x = x_{1/4} = 1 - x_4$, $x(q^4) = x_4$, and $m = \sqrt{4} = 2$, we deduce that

$$\left[\frac{dx(q^4)}{dx(q)}\right]_{q=e^{-2\pi/\sqrt{12}}} = 1. \tag{6.13}$$

Again, if $n = 2$ in (6.11), then

$$\frac{dx(q^2)}{dx(q)} = \frac{2}{m_2^2} \cdot \frac{x(q^2)(1 - x(q^2))}{x(q)(1 - x(q))}, \tag{6.14}$$

where m_2 is the multiplier connecting $x(q)$ and $x(q^2)$.

Setting $q = e^{-\pi/\sqrt{3}}$ in (6.7), so that $\alpha = 1 - x_4$ and $\beta = x_1 = 1/2$, we find that

$$m_2^2 = 3. \tag{6.15}$$

Thus, with $q = e^{-2\pi/\sqrt{12}}$ in (6.14) and with the aid of $x_1 = 1/2$, (6.6), and (6.15), we find that

$$\left[\frac{dx(q^2)}{dx(q)}\right]_{q=e^{-2\pi/\sqrt{12}}} = 9. \tag{6.16}$$

Finally, setting $q = e^{-2\pi/\sqrt{12}}$ in (6.10), and then using (6.6), (6.13), and (6.16), we find that

$$\left[\frac{dm}{dx}\right]_{q=e^{-2\pi/\sqrt{12}}} = \frac{24}{\sqrt{3}}. \tag{6.17}$$

The remaining part of the proof of (6.1) follows along the same lines as those in previous sections, and so we omit the detailed proof. □

7. Example: $n = 5$

Theorem 7.1. *If U_k is defined by* (3.4), *then*

$$\frac{2025}{4\pi^2} = \sum_{k=0}^{\infty} \left\{1089k^2 + 378k + 40\right\} U_k \left(\frac{4}{125}\right)^k. \tag{7.1}$$

The above series is an analogue of Ramanujan's series (1.3).

Proof. From [2, Eq. (5.38)], we have

$$x_5 = \frac{5\sqrt{5} - 11}{10\sqrt{5}}, \quad H_5 = \frac{4}{125}. \tag{7.2}$$

Next, to evaluate $\frac{dm}{dx}(1 - x_5, x_5)$, we rewrite (6.11) in the form

$$\frac{d\beta}{d\alpha} = \frac{n}{m^2} \cdot \frac{\beta(1-\beta)}{\alpha(1-\alpha)}, \tag{7.3}$$

where β has degree n over α and m is the multiplier connecting α and β. Differentiating (7.3) with respect to α, we obtain

$$m^2 \frac{d^2\beta}{d\alpha^2} + 2m \frac{dm}{d\alpha} \cdot \frac{d\beta}{d\alpha} = n \cdot \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \left\{ \left(\frac{1}{\beta} - \frac{1}{1-\beta}\right) \frac{d\beta}{d\alpha} - \frac{1}{\alpha} + \frac{1}{1-\alpha} \right\}. \tag{7.4}$$

Setting $q = e^{-2\pi/\sqrt{3n}}$ in (7.4), so that $\alpha = x_{1/n} = 1 - x_n$, $\beta = x_n$, $\frac{d\beta}{d\alpha} = 1$, and $m = \sqrt{n}$, we find that

$$n\frac{d^2\beta}{d\alpha^2} + 2\sqrt{n}\frac{dm}{d\alpha} = 2n\left(\frac{1-2\beta}{\alpha\beta}\right). \tag{7.5}$$

In particular, if $n = 5$, then

$$5\frac{d^2\beta}{d\alpha^2} + 2\sqrt{5}\frac{dm}{d\alpha} = 550\sqrt{5}. \tag{7.6}$$

To calculate $\frac{d^2\beta}{d\alpha^2}$, we recall from [5, p. 124] the following modular equation of degree 5 in the theory of signature 3. If β has degree 5 over α, then

$$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} + 3\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \tag{7.7}$$

Differentiating (7.7) twice with respect to α and then setting $q = e^{-2\pi/\sqrt{15}}$, so that $\alpha = 1 - \beta$ and $\left[\frac{d\beta}{d\alpha}\right]_{q=e^{-2\pi/\sqrt{15}}} = 1$, we deduce that

$$\left[\frac{d^2\beta}{d\alpha^2}\right]_{q=e^{-2\pi/\sqrt{15}}} = \frac{290\sqrt{5}}{3}. \tag{7.8}$$

Employing (7.8) in (7.6), we obtain

$$\left[\frac{dm}{d\alpha}\right]_{q=e^{-2\pi/\sqrt{15}}} = \frac{100}{3}. \tag{7.9}$$

Thus,

$$\left[\frac{dm}{d\beta}\right]_{q=e^{-2\pi/\sqrt{15}}} = \left[\frac{dm}{d\alpha}\cdot\frac{d\alpha}{d\beta}\right]_{q=e^{-2\pi/\sqrt{15}}} = \frac{100}{3}. \tag{7.10}$$

The remaining part of the proof of (7.1) follows along the same lines as those in previous sections, and so we skip the detailed proof. □

8. Example: $n = 6$

Theorem 8.1. *If U_k is defined by* (3.4)*, then*

$$\frac{15625}{\pi^2} = \sum_{k=0}^{\infty}\left\{\frac{4}{3}(18749 + 4914\sqrt{6})k^2 + 6(148 + 853\sqrt{6})k \right.$$
$$\left. + 25(-101 + 64\sqrt{6})\right\}U_k\left(\frac{-27(-463 + 182\sqrt{6})}{31250}\right)^k. \tag{8.1}$$

Proof. Setting $n = 6$ in (3.34), we obtain

$$\frac{648}{\pi^2} = \sum_{k=0}^{\infty}\{A(x_6)k^2 + B(x_6)k + C(x_6)\}U_k H_6^k, \tag{8.2}$$

where $H_6 = 4x_6(1 - x_6)$. To calculate x_6 and hence H_6, we recall the following identity from [10, Eq. (2.7)].

For $q = e^{-2\pi\sqrt{n/3}}$, if

$$\mu_n = \frac{f^6(-q)}{3\sqrt{3q}f^6(-q^3)},$$

where

$$f(-q)=\prod_{k=1}^{\infty}(1-q^k),$$

then

$$\frac{1}{x_n}=\mu_n^2+1, \tag{8.3}$$

where x_n's are the cubic singular moduli defined in (3.16). The parameter μ_n was introduced by K.G. Ramanathan [13, eq. (51)].

Now, from [6, Theorem 4.6], we note that

$$\mu_6=9+3\sqrt{6}. \tag{8.4}$$

Employing (8.4) in (8.3), we readily find that

$$x_6=\frac{1}{136+54\sqrt{6}}, \quad \text{and hence,} \quad H_6=\frac{-27(-463+182\sqrt{6})}{31250}. \tag{8.5}$$

To calculate $\frac{dm}{dx}(1-x_6,x_6)$, we recall from (5.4) that

$$m_1=1+2x^{1/3}(q^3), \tag{8.6}$$

where m_1 is the multiplier connecting $x(q)$ and $x(q^3)$.

Replacing q by q^2 in (8.6) and using Theorem 7.1 (iii) in [5, p. 120], we obtain that

$$m=\{1+2x^{1/3}(q^6)\}\left[\left\{\frac{(1-x(q^2))^2}{1-x(q)}\right\}^{1/3}-\left\{\frac{x^2(q^2)}{x(q)}\right\}^{1/3}\right], \tag{8.7}$$

where m is the multiplier connecting $x(q)$ and $x(q^6)$. Differentiating (8.7) with respect to $x:=x(q)$, and employing Theorem 7.1 (v) of [5, p. 120] and (8.5), we deduce that

$$\left[\frac{dm}{dx}\right]_{q=e^{-2\pi/\sqrt{18}}}=\frac{4(81+34\sqrt{6})}{9}. \tag{8.8}$$

The remainder of the proof of (8.1) follows along the same lines as those in previous sections, and so we do not give the detailed proof. □

9. Example: $n=9$

Theorem 9.1. *If U_k is defined by* (3.4)*, then*

$$\frac{15625}{2\pi^2}=\sum_{k=0}^{\infty}\Big\{2(-8468-20556\cdot 2^{1/3}+31473\cdot 2^{2/3})k^2+3(-15603-19476\cdot 2^{1/3}$$
$$+27083\cdot 2^{2/3})k+20(-917-989\cdot 2^{1/3}+1387\cdot 2^{2/3})\Big\}U_kH_9^k, \tag{9.1}$$

where $H_9=\dfrac{-9(-2677-2284\cdot 2^{1/3}+3497\cdot 2^{2/3})}{15625}$.

Proof. Setting $n=9$ in (3.34), we obtain

$$\frac{972}{\pi^2}=\sum_{k=0}^{\infty}\big\{A(x_9)k^2+B(x_9)k+C(x_9)\big\}D_kH_9^k. \tag{9.2}$$

To calculate x_9 and hence H_9, we replace q by q^3 in (8.6) to arrive at

$$m = \left(1 + 2x^{1/3}(q^9)\right)\left(1 + 2x^{1/3}(q^3)\right), \tag{9.3}$$

where m is the multiplier connecting $x(q)$ and $x(q^9)$. Setting $q = e^{-2\pi/\sqrt{27}}$ in (9.3), so that $x(q^3) = x_1 = 1/2$, $x(q^9) = x_9$, and $m = \sqrt{9} = 3$, we deduce that

$$x_9 = \frac{1}{250}\left(187 - 171 \cdot 2^{1/3} + 18 \cdot 2^{2/3}\right) \tag{9.4}$$

and

$$L_9 = \frac{-9}{15626}\left(-2677 - 2284 \cdot 2^{1/3} + 3497 \cdot 2^{2/3}\right). \tag{9.5}$$

Next, to evaluate $\frac{dm}{dx}(1 - x_9, x_9)$, we differentiate (9.3) with respect to $x = x(q)$, to obtain

(9.6)
$$\frac{dm}{dx} = \left(1 + 2x^{1/3}(q^3)\right) \cdot \frac{2}{3}x^{-2/3}(q^9)\frac{dx(q^9)}{dx(q)} + \left(1+2x^{1/3}(q^9)\right) \cdot \frac{2}{3}x^{-2/3}(q^3)\frac{dx(q^3)}{dx(q)}.$$

But, by (6.11) with $n = 9$, we have

$$\frac{dx(q^9)}{dx(q)} = \frac{9}{m^2} \cdot \frac{x(q^9)(1 - x(q^9))}{x(q)(1 - x(q))}, \tag{9.7}$$

where m is the multiplier connecting $x(q)$ and $x(q^9)$. Setting $q = e^{-2\pi/\sqrt{27}}$ in (9.7), so that, by (3.9), $x(q) = x_{1/9} = 1 - x_9$, $x(q^9) = x_9$, and $m = \sqrt{9} = 3$, we deduce that

$$\left[\frac{dx(q^9)}{dx(q)}\right]_{q=e^{-2\pi/\sqrt{27}}} = 1. \tag{9.8}$$

Again, if $n = 3$ in (6.11), then

$$\frac{dx(q^3)}{dx(q)} = \frac{3}{m_1^2}.\frac{x(q^3)(1 - x(q^3))}{x(q)(1 - x(q))}, \tag{9.9}$$

where m_1 is the multiplier connecting $x(q)$ and $x(q^3)$. Setting $q = e^{-2\pi/\sqrt{27}}$ in (8.6), so that $x(q^3) = x_1 = 1/2$, we find that

$$m_1 = 1 + 2^{2/3}. \tag{9.10}$$

Thus, setting $q = e^{-2\pi/\sqrt{27}}$ in (9.6) and using (9.4), (9.5), (9.8)–(9.10), we deduce that

$$x_9(1 - x_9)\left[\frac{dm}{dx}\right]_{q=e^{-2\pi/\sqrt{27}}} = \frac{6 \cdot 2^{-1/3}}{(1 + 2^{2/3})^3}. \tag{9.11}$$

The remaining part of the proof of (9.1) is similar to those in previous sections, and so we omit the detailed proof. □

10. Concluding remarks

From the above examples, we notice that in order to derive a series for $1/\pi^2$ from (3.34), it is essentially required to evaluate x_n and $\frac{dm}{dx}(1-x_n, x_n)$. In [11], Chan and Liaw tabulated 22 values of x_n and $a_n = \frac{2x_n(1-x_n)}{\sqrt{3}}\frac{dm}{dx}(1-x_n, x_n)$, that includes the cases for $n=2$ and 5 evaluated in Sections 4 and 7 above. By employing each of the remaining pair of values for x_n and a_n from [11] in (3.34), we can readily arrive at twenty further new series for $1/\pi^2$.

Acknowledgment

The authors would like to thank the referee for his/her helpful comments.

References

[1] G. E. Andrews, R. Askey, R. Roy, *Special functions: Encyclopedia of Mathematics and its Applications*, 71 Cambridge Univ. Press, Cambridge, 1999. MR1688958 (2000g:33001)
[2] N. D. Baruah, B. C. Berndt, *Ramanujan's series for $1/\pi$ arising from his cubic and quartic theories of elliptic functions*, J. Math. Anal. Appl. **341** (2008), 357–371. MR2394090 (2009d:33053)
[3] N. D. Baruah, B. C. Berndt, *Ramanujan's Eisenstein Series and new hypergeometric-like series for $1/\pi^2$*, J. Approx. Theory **160** (2009), 135–153. MR2558018
[4] N. D. Baruah, B. C. Berndt, H. H. Chan, *Ramanujan's series for $1/\pi$: a survey*, Amer. Math. Monthly **116** (2009), 567–587.
[5] B. C. Berndt, *Ramanujan's Notebooks*, Part V, Springer-Verlag, New York, 1998. MR1486573 (99f:11024)
[6] N. D. Baruah, N. Saikia, *Some new explicit values of Ramanujan's continued fractions*, Indian J. Math., **46** (2004), 357–371. MR2133498 (2006i:33019)
[7] B. C. Berndt, S. Bhargava, F. G. Garvan, *Ramanujan's theories of elliptic functions to alternative bases*, Trans. Amer. Math. Soc., **347** (1995), 4136–4244. MR1311903 (97h:33034)
[8] J. M. Borwein, P. B. Borwein, *Pi and the AGM; A Study in Analytic Number Theory and Computational Complexity*, Wiley, New York, 1987. MR877728 (89a:11134)
[9] J. M. Borwein, P. B. Borwein, *A cubic counterpart of Jacobi's identity and the AGM*, Trans. Amer. Math. Soc., **323** (1991), 691–701. MR1010408 (91e:33012)
[10] H. H. Chan, *On Ramanujan's cubic transformation formula for ${}_2F_1$ $(1/3, 2/3; 1; z)$*, Math. Proc. Cambridge Philos. Soc., **124** (1998), 193–204. MR1631107 (99f:11054)
[11] H. H. Chan, W. -C. Liaw, *Cubic modular equations and new Ramanujan-type series for $1/\pi$*, Pacific J. Math **343** (2000), 219–238 . MR1744566 (2001e:11041)
[12] H. H. Chan, W. -C. Liaw, V. Tan, *Ramanujan's class invariant λ_n and a new class of series for $1/\pi$*, London Math. Soc.(2), **64** (2001), 93–106. MR1840773 (2002d:33028)
[13] K. G. Ramanathan, *Some applications of Kronecker limit formula*, J. Indian Math. Soc., **52** (1987), 71–89. MR989231 (90j:11112)
[14] S. Ramanujan, *Modular equations and approximations to π*, Quart. J. Math. (Oxford) **45** (1914), 350–372.
[15] S. Ramanujan, *Collected Papers*, Cambridge Univ. Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by Amer. Math. Soc., Providence, RI, 2000.
[16] S. Ramanujan, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957. MR0099904 (20:6340)

Department of Mathematical Sciences, Tezpur University, Sonitpur, Pin-784028, India
E-mail address: nayan@tezu.ernet.in

Department of Mathematical Sciences, Tezpur University, Sonitpur, Pin-784028, India
E-mail address: narayan.nayak05@gmail.com