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REDUCTION (MOD q) OF FUSION SYSTEM AMALGAMS

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Abstract. We use representation theory to construct finite homomorphic images of infinite groups realising fusion systems on finite p-groups.

Introduction

This paper originated in the (unexpected) coalescence of parts of our recent work [6] on realising fusion systems on finite p-groups via amalgams with generalisations of some of the representation-theoretic methods of [5]. In [5], we studied a particular subgroup $G_{4,8}$ of $\mathrm{SO}_3(\mathbb{R})$ generated by a pair of rotations

$$x = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

of respective orders 4 and 8. Our original motivation for that work came from consideration of complex linear groups. A well-known theorem of Blichfeldt asserts that if G is a finite primitive complex linear group, and $g \in G$ has an eigenvalue λ such that all eigenvalues of g lie within $\frac{\pi}{3}$ of λ on the unit circle S^1, then $g \in Z(G)$.

The group $G_{4,8}$ is easily seen to be (irreducible and) primitive, but the rotation y is not central, yet has eigenvalues $1, e^{\frac{\pi i}{4}}, e^{\frac{-\pi i}{4}}$. Hence $G_{4,8}$ is infinite. On the other hand, $G_{4,8}$ is quite close to the borderline of Blichfeldt's result.

In fact, the group $G_{4,8}$ had already been considered as one of an infinite family of subgroups of $\mathrm{SO}_3(\mathbb{R})$, each generated by a pair of rotations of finite order, by Radin and Sadun in [4], groups which they explicitly identified as amalgams. In particular, as proved in [4], the group $G_{4,8}$ is isomorphic to the amalgam $D_{16} *_{D_8} S_4$, where D_{16} is a dihedral group of order 16. By the results of [6], such an amalgam realises (via its conjugation action on its 2-subgroups), the fusion system of any finite group X which has a dihedral Sylow 2-subgroup of order 16 and which has a normal subgroup of index 2, but no normal 2-complement (for example, the group $X = \mathrm{PGL}(2,7)$ has these properties).

Much of the paper [5] exploited the fact that since the entries of the given generators of $G_{4,8}$ have entries which are (at worst) half an algebraic integer, the group $G_{4,8}$ admits "reduction (mod p)" for all odd primes p (strictly speaking, we

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need to adjoin $\sqrt{2}$ to the relevant ground ring if 2 is not a quadratic residue (mod p)). This leads to a finite almost simple epimorphic image of $G_{4,8}$ for each odd prime p (one of $\mathrm{PSL}(2,p), \mathrm{PGL}(2,p)$ or $\mathrm{PGL}(2,p^2)$, depending on the congruence (mod 16) of p). These epimorphic images always have dihedral Sylow 2-subgroups, but the order of their Sylow 2-subgroups can be arbitrarily large with a suitable choice of p.

In fact, there is a "global"explanation for these epimorphic images. In private communications to the author (2006 and 2007), J-P. Serre proved (by considering action on an appropriate tree) that (in the given 3-dimensional representation), the group $G_{4,8} = \mathrm{SO}_3(\mathbb{Z}[\frac{1}{\sqrt{2}}])$. The given representation of $G_{4,8}$ clearly gives $G_{4,8} \subseteq \mathrm{SO}_3(\mathbb{Z}[\frac{1}{\sqrt{2}}])$. The rightmost group also admits reductions (mod p) for odd primes p (after adjoining $\sqrt{2}$ to the field of p elements for $p \equiv \pm 3 \pmod 8$), yielding a homomorphism into $\mathrm{SO}_3(p)$ or $\mathrm{SO}_3(p^2)$. One interpretation of the results of [5] is that the image of both groups under reduction (mod p) is the same for each odd prime p. Serre's result may be interpreted as saying that there is a single global isomorphism yielding all the local isomorphisms.

Serre also proved that this global phenomenon persists in the sense that $G_{4,16} = \mathrm{SO}_3(\mathbb{Z}[\cos(\frac{\pi}{8})])$. However, for $n > 4$, Serre proved (by an argument involving Tamagawa numbers) that $G_{4,2^n}$ has infinite index in $\mathrm{SO}_3(\mathbb{Z}[\cos(\frac{\pi}{2^{n-1}})])$, so that although, for each odd prime p, the respective images of $G_{4,2^n}$ and $\mathrm{SO}_3(\mathbb{Z}[\cos(\frac{\pi}{2^{n-1}})])$ after reduction (mod p) are the same, the groups themselves are not isomorphic.

There are certain general principles which are illustrated by these examples. We will see in this paper that the amalgams associated to fusion systems on a finite p-group P, constructed in [6], admit certain finite-dimensional complex representations with free kernels, and that these representations may be chosen to be realised over cyclotomic number fields. The representations need not be realisable over the ring of integers of the given cyclotomic fields. However, we will see that whenever we take a rational prime $q \neq p$, we may localise at a prime ideal (of the appropriate ring of integers) containing q and assume that the representation is realised over that local ring. Then we may reduce modulo the unique maximal ideal of the local ring to obtain a finite-dimensional representation of our group (associated to a fusion system) over a finite field. Under suitable circumstances, the epimorphic image obtained is close to simple, and the kernel of the representation is a free normal subgroup.

An alternative strategy for reduction is to note that representations realised over cyclotomic fields are "almost" realised over the ring of integers of that field, in the sense that it is only necessary to invert a finite number of primes, as we are dealing with finitely generated groups. Hence we obtain a single representation which admits reduction (mod π) for all but a finite number of prime ideals π (the case of $G_{4,8}$ being a case when only a prime ideal containing 2 needs to be inverted). One advantage of this approach over the approach of working with one prime at a time is that in the latter approach, the representations which admit respective reductions (mod q) and (mod r) for different primes q and r need not be equivalent as complex representations.

In passing, we will make some remarks about permutation representations and representations with other integrality properties.

We will freely draw (sometimes without explicit reference) on results from Serre's book [7] for basic properties of amalgams established there.

1. Review of Alperin fusion systems

We retain the notation and terminology of [6] (and its references) for general statements and results about fusion systems.

As in [6], we refer to a fusion system $\mathcal{F}$ on a non-trivial finite p-group P as an *Alperin* fusion system if there is a finite set of finite groups $L_1, L_2, \ldots, L_n$ such that:
i) Each L_i has a Sylow p-subgroup P_i isomorphic to (and identified with) a subgroup of P and, furthermore, $P_1 = P$.
ii) $U_i = F^*(L_i) = O_p(L_i)$ and $L_i/U_i \cong \mathrm{Out}_{\mathcal{F}}(U_i) = \mathrm{Mor}_{\mathcal{F}}(U_i, U_i)/\mathrm{Inn}(U_i)$ for each i.
iii) $\mathcal{F}$ is generated by $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$, where $\mathcal{F}_i$ is the fusion system on P_i induced by the conjugation action of L_i on its p-subgroups.

To avoid obvious redundancy, we assume from now on that there are no inclusions between any of the groups L_i.

Any saturated fusion system on a finite p-group is an Alperin fusion system, using the results of Broto, Castellana, Grodal, Levi and Oliver [1]. However, an Alperin fusion system on a finite p-group need not be saturated, as noted explicitly in (Leary and Stancu, [3]).

Given an Alperin fusion system $\mathcal{F}$, we showed in [6] that the conjugation action on its p-subgroups of the iterated amalgam

$$X = X_{\mathcal{F}} = (\ldots((L_1 *_{P_2} L_2) *_{P_3}) \ldots) *_{P_n} L_n$$

realises the fusion system $\mathcal{F}$ on P. We also recall from that paper that the amalgam $X_{\mathcal{F}}$ always has free normal subgroups of finite index. If K is any such free normal subgroup of X, then the fusion system associated to the conjugation action of X/K on its p-subgroups contains the fusion system $\mathcal{F}$, since each L_i embeds isomorphically in X/K. We also note that since X is generated by the L_i, $O_p(X)$ (the largest normal p-subgroup of X) is the largest subgroup of P which is normal in each L_i, since the hypotheses readily imply that $O_p(X) \subseteq U_i$ for $1 \leq i \leq n$. A subgroup Q of P is strongly closed in P with respect to $\mathcal{F}$ if and only if $Q^x \cap P \leq Q$ for each $x \in X$.

Since the notion of a simple saturated fusion system is taken, we find it convenient here to coin the term *episimple* Alperin fusion system. The Alperin fusion system $\mathcal{F}$ is said to be *episimple* if there is some i such that P is not normal in L_i, no proper non-trivial subgroup of P is strongly closed in P with respect to $\mathcal{F}$ and $\mathcal{F}$ is generated by the fusion systems induced by the conjugation actions of $O^{p'}(L_i)$ on P_i. Clearly, if there is an episimple Alperin fusion system on a finite p-group P, then P is non-Abelian, since otherwise $P = F^*(L_i)$ for each i, contrary to the first condition. We note that if we had allowed the case that $P \lhd L_i$ for each i, then the last condition would guarantee that the fusion system on P is the trivial one.

Then by (slight variants of) the results of [6], an episimple Alperin fusion system $\mathcal{F}$ on P is realised (by conjugation action on p-subgroups) within the iterated amalgam

$$Y = ((\ldots((O^{p'}(L_1) *_{P_2} O^{p'}(L_2)) *_{P_3}) \ldots) *_{P_n} O^{p'}(L_n)).$$

Unless otherwise stated, we suppose from now on that $n > 1$, and that there are no inclusions among the $O^{p'}(L_i)$. We also remark that each proper normal subgroup of Y is free under these hypotheses. For suppose that $N \lhd Y$. If $P \cap N = 1$, then the hypotheses on $\mathcal{F}$ imply that $L_i \cap N = 1$ for each i, so that (as already noted in [6]), N is free. On the other hand, if $P \cap N \neq 1$, then $P \cap N$ is strongly closed in P with respect to $\mathcal{F}$, so that $P \cap N = P$ by hypothesis. Then $O^{p'}(L_i) = \langle P_i^{L_i} \rangle \leq N$ for each i, so $Y \leq N$. Hence a maximal free normal subgroup of Y is in fact a maximal normal subgroup. We note also that Y is perfect, for otherwise $[Y, Y]$ is free, so that $P \cap [Y, Y] = 1$, and P is Abelian, a contradiction.

Lemma 1.1. *If $\mathcal{F}$ is an episimple Alperin fusion system, and Y is defined as above, then every proper normal subgroup of Y is free, $Y = [Y, Y]$ and Y/K is a finite simple group whenever K is a maximal free normal subgroup of Y of finite index.*

2. Representing iterated amalgams

In this section, we work in somewhat more generality than is necessary for our immediate purpose, though our primary interest is still with (iterated) amalgams associated to fusion systems. Suppose that we have n (not necessarily finite) groups $A_1, A_2, \ldots, A_n$ and groups $C_2, C_3, \ldots, C_n$ such that, for $2 \leq i \leq n$, C_i is identified with a subgroup of both A_1 and A_i. We consider the iterated amalgam

$$X = ((A_1 *_{C_2} A_2) \ldots A_{n-1}) *_{C_n} A_n.$$

Henceforth, we simply write

$$X = A_1 *_{C_2} A_2 *_{C_3} \ldots *_{C_{n-1}} A_{n-1} *_{C_n} A_n.$$

We note that, by an inductive argument (the case $n = 2$ being true by virtue of the defining property of an amalgam of two groups), if T is any group, and

$$\{\sigma_i : A_i \to T : 1 \leq i \leq n\}$$

is a set of n homomorphisms such that

$$\mathrm{Res}^{A_1}_{C_i}(\sigma_1) = \mathrm{Res}^{A_i}_{C_i}(\sigma_i)$$

for each $i > 1$, then there is a unique group homomorphism $\sigma : X \to T$ which, for each j, restricts to σ_j on (the natural copy of) A_j in X.

This allows us to extend various kinds of representations of the groups A_i to representations of X. We will sometimes wish to have the freedom to define the representations σ_i up to equivalence only. We will still obtain existence results, but we will lose uniqueness of the extension (even up to equivalence) in general. We will see later that this lack of uniqueness can sometimes be used to our advantage.

In the case of linear representations, if we choose a ring R and a positive integer m, and only specify above that we have n homomorphisms

$$\{\tau_i : A_i \to \mathrm{GL}(m, R) : 1 \leq i \leq n\}$$

such that $\mathrm{Res}^{A_1}_{C_i}(\tau_1)$ is equivalent to $\mathrm{Res}^{A_i}_{C_i}(\tau_i)$ for each $i > 1$, then for each i we may choose a representation σ_i from the equivalence type of τ_i so that $\mathrm{Res}^{A_1}_{C_i}(\sigma_1) = \mathrm{Res}^{A_i}_{C_i}(\sigma_i)$ for each $i > 1$ and obtain a group homomorphism $\sigma : X \to \mathrm{GL}(m, R)$ which restricts to σ_i on (the natural copy of) A_i in X.

It will occasionally be necessary for us to work with projective representations (in Schur's sense). Given a ring R and an R-free R-module V of rank h, there is a homomorphism $\sigma : X \to \mathrm{Aut}(\mathrm{End}_R(V))$ if and only if there are n homomorphisms

$$\{\sigma_i : A_i \to \mathrm{Aut}(\mathrm{End}_R(V)) : 1 \leq i \leq n\}$$

such that

$$\mathrm{Res}^{A_1}_{C_i}(\sigma_1) = \mathrm{Res}^{A_i}_{C_i}(\sigma_i)$$

for each $i > 1$.

Returning to genuine representations, in the particular case where the A_i are finite groups, and R is a field of characteristic zero, the equivalence class of the representation σ_i is determined by the character it affords. Hence any n characters $\{\alpha_i : 1 \leq i \leq n\}$ afforded by finite-dimensional representations over R of the respective A_i such that $\mathrm{Res}^{A_1}_{C_i}(\alpha_1) = \mathrm{Res}^{A_i}_{C_i}(\alpha_i)$ for each $i > 1$ yield an R-representation of X (and conversely, any finite-dimensional representation of X over R yields n such characters). We refer to an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of characters with this property as a *compatible* n-tuple of characters. The degree of α is $h = \alpha_1(1)$.

A compatible n-tuple α of complex characters is said to be *irreducible* if it can't be properly decomposed as a sum of two compatible n-tuples. The n-tuple α is said to be *faithful* if each α_i is faithful (this does not, in general, guarantee that a complex representation of X which affords α is faithful in the usual sense, only that it has a free kernel. For example, the free product $Q_8 * S_3$ has a 2-dimensional complex representation (with image isomorphic to $\mathrm{GL}(2,3)$) which is faithful in the above sense, but its kernel is a non-trivial free group). We say that the compatible n-tuple α has *finite type* if it may be afforded by a complex representation of X with finite image; otherwise, we say that α is of *infinite type*.

In the case of permutation representations, if we choose a finite set Ω and a positive integer m, and only specify that we have n homomorphisms

$$\{\tau_i : A_i \to \mathrm{Sym}(\Omega) : 1 \leq i \leq n\}$$

such that $\mathrm{Res}^{A_1}_{C_i}(\tau_1)$ is permutation equivalent to $\mathrm{Res}^{A_i}_{C_i}(\tau_i)$ for each $i > 1$, then for each i we may choose a representation σ_i from the equivalence type of τ_i so that $\mathrm{Res}^{A_1}_{C_i}(\sigma_1) = \mathrm{Res}^{A_i}_{C_i}(\sigma_i)$ for each $i > 1$ and obtain a group homomorphism $\sigma : X \to \mathrm{Sym}(\Omega)$ which restricts to σ_i on (the natural copy of) A_i in X.

We recall that when Ω is a finite set and C is any group, then two permutation representations $\sigma_1, \sigma_2 : C \to \mathrm{Sym}(\Omega)$ are permutation equivalent if and only if $\ker\sigma_1 = \ker\sigma_2$ and $|\Omega^{D\sigma_1}| = |\Omega^{D\sigma_2}|$ for every subgroup D of C containing $\ker\sigma_1$.

A particular case is worth special mention. If each A_i is finite, and we have a finite set Ω such that each A_i acts semi-regularly on Ω, then there is an action of X on Ω in which each A_i acts semi-regularly. This is because, for each $i > 1$, the action of each C_i on Ω is semi-regular, so no non-identity subgroup of C_i fixes any point of Ω in either of the two given actions of C_i restricted respectively from A_1 and A_i (while the identity subgroup fixes all points in either action). The kernel of this permutation representation of X is torsion free, since every element of finite order in X is conjugate to an element of some A_i, while each A_i acts semi-regularly. Since X is an iterated amalgam, the kernel of the permutation action is thus free.

Notice in this case that any set Ω on which each A_i acts semi-regularly has cardinality divisible by $\mathrm{lcm}\{|A_i| : 1 \leq i \leq n\}$. On the other hand, any set Ω of cardinality $\mathrm{lcm}\{|A_i| : 1 \leq i \leq n\}$ may be endowed with a semi-regular permutation action of each A_i, and thus endowed with a permutation action of X with free kernel such that each A_i acts semi-regularly.

3. Euler characteristics and iterated amalgams

The Euler(-Poincaré) characteristic $\chi(A)$, of a finitely generated group A is a useful invariant that (as noted by C.T.C. Wall) can be defined whenever A has a (finitely generated) free subgroup of finite index. We only need to note some formal properties (which may be found, for example, in (Serre [7])), and which we list below.

i) If A is free of rank r (the possibility $r = 0$ is allowed), then $\chi(A) = 1 - r$.

ii) If A has a free subgroup B of index s, then $\chi(A) = \frac{\chi(B)}{s}$.

iii) If $A = B *_C D$ for groups B, D with a common subgroup C, then $\chi(A) = \chi(B) + \chi(D) - \chi(C)$ whenever the three rightmost quantities are defined.

Notice that $\chi(A) > 0$ if and only if A is finite, and $\chi(A) = 0$ if and only if A has an infinite cyclic subgroup of finite index.

These few properties enable us to calculate the rank of certain free subgroups of $X = X_{\mathcal{F}}$ for an Alperin fusion system $\mathcal{F}$ on P, and to show that if $O_p(X) = 1$, then X/K embeds in $\mathrm{Out}(K)$ whenever K is a free normal subgroup of finite index of X.

We note that for our iterated amalgam X, we have

$$\chi(X) = \left(\sum_{i=1}^{n} \frac{1}{|L_i|}\right) - \left(\sum_{i=2}^{n} \frac{1}{|P_i|}\right) \leq 0$$

(in fact, the inequality is strict under our current assumptions). Since L_2 is not a p-group, if $P_1 \neq P_2$, we have $|L_i| \geq 2|P_2|$ for $i = 1, 2$. In that case, if, furthermore, both inequalities are equalities, then $p = 2$, and L_2 is a p-group, a contradiction. Hence we only need to exclude the case that $P_1 = P_2$. If $|L_i| = 2|P|$ for each i, then $L_1 = L_2$, contrary to assumption. If $L_1 = P_1$, then $L_1 \leq L_2$, contrary to assumption. Hence we have

$$\chi(X) = \frac{d_{\mathcal{F}}}{|P|\mathrm{lcm}\{[L_i : P_i] : 1 \leq i \leq n\}}$$

for some negative integer $d_{\mathcal{F}}$.

We remark that whenever K is a free normal subgroup of finite index of X, the index $[G : K]$ is divisible by $|P|\mathrm{lcm}\{[L_i : P_i] : 1 \leq i \leq n\}$, since each L_i is isomorphic to a subgroup of G/K and $P = P_1$. We set $i_K = \frac{[X:K]}{|P|\mathrm{lcm}\{[L_i:P_i]:1 \leq i \leq n\}}$. Then if K has rank r_K, we have $(1 - r_K) = [X : K]\chi(X) = i_K d_{\mathcal{F}}$. Hence $r_K = 1 - i_K d_{\mathcal{F}}$.

Theorem 3.1. *a) Any free (not necessarily normal) subgroup of finite index of X has index divisible by $|P|\mathrm{lcm}\{[L_i : P_i] : 1 \leq i \leq n\}$. Furthermore, X has a free subgroup of this minimal possible index, and any such free subgroup has rank $1 - d_{\mathcal{F}}$.*
b) Suppose that $O_p(X) = 1$ (in particular, this occurs if $\mathcal{F}$ is episimple). Let K be a free normal subgroup of finite index of X. Then $C_X(K) = 1$ and X/K is isomorphic to a subgroup of $\mathrm{Out}(K)$.

Proof. a) If F is a free subgroup of finite index of X, then each L_i acts semi-regularly on right cosets of F in X, for if $u \in L$ and $Ftu = Ft$ for some $t \in X$, then tut^{-1} is an element of finite order in the free group F, so $u = 1_X$.

On the other hand, we saw in Section 2 that there is a set Ω of cardinality $h = |P|\mathrm{lcm}\{[L_j : P_j] : 1 \leq j \leq n\}$ which admits a permutation action of X with each L_i acting semi-regularly.

All point stabilizers in the permutation action must be torsion free (and hence free) and clearly of finite index, since each element of finite order in X is conjugate to an element of some L_i. Thus all point stabilizers have index divisible by h. Hence the permutation action is transitive, and a point stabilizer, F say, is a free subgroup of index h. Another Euler characteristic calculation shows that F has rank $1 - d_{\mathcal{F}}$.
b) Since $\chi(X) < 0$, we know that K is non-Abelian, so that $Z(K) = 1$, as K is free. Hence $C_X(K)$ is isomorphic to a subgroup of X/K and is thus finite. Since $C_X(K) \lhd X$ and $n > 1$, Theorem 2 of [6] allows us to conclude that $C_X(K)$ is a finite p-group, so is contained in $O_p(X) = 1$. □

4. A focal subgroup theorem

In this section, we characterize (for a general Alperin fusion system $\mathcal{F}$ which is not assumed to be episimple) the focal subgroup of $X = X_{\mathcal{F}}$ in a manner inspired by a well-known proof of D.G. Higman's focal subgroup theorem for finite groups, a proof which makes use of Brauer's characterization of characters.

The focal subgroup is a Sylow p-subgroup of the smallest normal subgroup of X such that the factor group is an Abelian p-group, as in the finite group case (however, the fact that such a subgroup exists needs some argument in this situation). We note that if $H = X/F$, where F is a free normal subgroup of finite index of X, then P is isomorphic to a p-subgroup of H, but (as we have already seen in examples) P need not be isomorphic to a Sylow p-subgroup of H in general. However, the focal subgroup theorem below shows that any Abelian p-factor group of H is an epimorphic image of P.

Let $X = X_{\mathcal{F}}$ be as above.

Remark 4.1. Let N be any normal subgroup of G such that X/N is a p-group (in the usual sense that each of its elements has order a (finite) power of p). Then $[X : N]$ is finite (in fact, of order at most $|P|$), and $X = PN$. Hence there is unique normal subgroup M of X which is minimal (that is, of maximal index) subject to X/M being a p-group.

To see these facts, notice that (since P_i is a Sylow p-subgroup of L_i), $L_i = P_i(L_i \cap N)$ for each i, so that $L_i \leq PN$ for each i. Thus $X = PN$, as X is generated by the L_i. In particular, $[X : N] \leq |P|$. If A and B are two minimal choices of normal subgroup with $X = PA = PB$, then $X/(A \cap B)$ is also a finite p-group, so that $X = P(A \cap B)$ as above. Hence $(A \cap B) = A = B$ by the maximality of $[X : A]$ and $[X : B]$.

We also note that when $X = PN$ and $N \lhd X$, then $P \cap N$ is a Sylow p-subgroup of N. To see this let Q be any maximal p-subgroup of N. Then $Q^g \leq P$ for some $g \in X$, as P is (up to conjugacy) the unique maximal p-subgroup of X. Hence we may suppose that $Q \leq P$, in which case $Q = P \cap N$ by maximality of Q. Then $Q \lhd P$. If S is any other maximal p-subgroup of N, then $S^x \leq P \cap N (= Q)$ for some $x \in X$, so $S^x = Q$ by maximality of S and Q. Now $x^{-1} = un$ for some $u \in P$, some $n \in N$, so that $S = Q^n$ is an N-conjugate of Q.

Theorem 4.2. *Let* $P_0 = \langle x^{-1}x^g : x, x^g \in P, g \in X \rangle$. *Then:*
a) X *has a normal subgroup* H *with* $X = PH$ *and* $P \cap H = P_0$ *(so that* $X/H \cong P/P_0$*). Furthermore,* $P \cap H$ *is a Sylow* p*-subgroup of* H *and* $P \cap H = P \cap [X, X]$.
b) $P \cap [X, X] = \langle P_i \cap [L_i, L_i] : 1 \leq i \leq n \rangle$.

Proof. Let $P_\infty = \langle P_i \cap [L_i, L_i] : 1 \le i \le n \rangle$. By the focal subgroup theorem for finite groups, P_0 contains $P_i \cap [L_i, L_i]$ for each i, so that $P_\infty \le P_0 \le P \cap [X, X]$. Let λ be any (complex) linear character of P/P_∞, inflated to a linear character of P. Then, for each i, $\mathrm{Res}^P_{P_i}(\lambda)$ extends to a linear character, say μ_i, of L_i. As X is an iterated amalgam of the L_i, there is a common extension of the μ_i to a linear character, μ say, of X. Furthermore, $X/\mathrm{ker}\mu$ is a finite Abelian p-group.

Performing this construction for each linear character λ of P/P_∞, and letting H be the intersection of the kernels of all the corresponding linear characters μ of X, we find that $X = HP$, and that $H \cap P = P_\infty$ (the former equality following from Remark 4.1, and the latter because the intersection of the kernels of all such linear characters λ of P is P_∞). Now X/H is Abelian, so that $P \cap [X, X] \le P \cap H = P_\infty$. We now have

$$P_\infty = P \cap H \le P_0 \le P \cap [X, X] \le P_\infty,$$

so that $P_0 = P \cap [X, X] = P_\infty$. By the discussion preceding the theorem, $P \cap H$ is a Sylow p-subgroup of H. □

Example 4.3. As remarked in the Introduction, the amalgam $X = D_{16} *_{D_8} S_4$ is isomorphic to $\mathrm{SO}_3(\mathbb{Z}[\frac{1}{\sqrt{2}}])$. The above focal subgroup theorem shows directly that X has a normal subgroup of index 2, since $X = X_{\mathcal{F}}$, where $\mathcal{F}$ is the Alperin fusion system on the dihedral group P of order 16 corresponding to $n = 2, L_1 = P, L_2 = S_4$. Then

$$P \cap [X, X] = \langle [P, P], P_2 \cap [L_2, L_2] \rangle \ne P,$$

as $[P, P] \le \Phi(P)$ and $P_2 \ne P$.

5. Reduction (mod q) for primes $q \ne p$; lifting representations

We discuss further some themes begun in Section 2, and consider questions of integrality of representations. As in that section, we work in somewhat more generality than necessary for the fusion system amalgams. We consider n finite groups $A_1, A_2, \ldots, A_n$ and $n-1$ groups $C_2, C_3, \ldots, C_n$ such that C_i is identified with a subgroup of both A_1 and A_i for each i. We again set $X = A_1 *_{C_2} A_2 *_{C_3} A_3 \ldots *_{C_n} A_n$. We set $d = \mathrm{lcm}\{|A_i| : 1 \le i \le n\}$, and we let ω be a complex primitive d-th root of unity. Let q be a prime which does not divide $\mathrm{lcm}\{|C_i| : 2 \le i \le n\}$. Let π be a prime ideal of $\mathbb{Z}[\omega]$ which contains q, and let $R = \mathbb{Z}[\omega]_{(\pi)}$ (the incomplete localization); we retain the notation π for the unique maximal ideal of R. We let $\alpha = \{(\alpha_1, \alpha_2, \ldots, \alpha_n)\}$ be a compatible n-tuple of characters of the respective A_i, and we let h be its degree.

Theorem 5.1. *The compatible n-tuple α may be realised by a $\mathbb{Q}[\omega]X$-module. If the n-tuple is faithful, then the subgroup of elements of X acting trivially on this module is free.*

Proof. The first claim follows from the discussion of Section 2, using the fact (first proved by R. Brauer) that each representation of each A_i is realizable over $\mathbb{Q}[\omega]$. Furthermore, the kernel of the action of X on this module is torsion free, since every element of finite order of X is conjugate to an element of some L_i, but the action of each L_i is faithful. □

We now begin to address issues of integrality.

Theorem 5.2. *i) The compatible n-tuple α may be realized by an RX-module M. Furthermore, if α_i is faithful and $O_q(A_i) = 1$ for each i, then the kernel of the action of X on $M/\pi M$ is a free normal subgroup of finite index. If the n-tuple is faithful and irreducible, and q does not divide d, then the action of X on $M/\pi M$ is absolutely irreducible.*

ii) Suppose that q does not divide d, and let M be an absolutely irreducible kX-module, for a finite field k of characteristic q. Then M lifts to an R-free RX-module.

Proof. i) Since R is a principal ideal domain whose field of fractions is a splitting field for each A_i, each of the irreducible characters α_i may be afforded by an RA_i-module. Since q does not divide $|C_i|$ for any $i > 1$, two R-free RC_i-modules (for any i) are isomorphic if and only if they afford the same character. Hence, by the discussions of Section 2, we obtain an R-free RX-module M which affords α.

We claim that if α_i is faithful and $O_q(A_i) = 1$ for each i, then K, the kernel of the action of X on $M/\pi M$, is torsion free, hence free.

Any element of finite order in K is conjugate to an element of some A_i, and $K \cap A_i \leq O_q(A_i) = 1$, so the second claim follows (that the index is finite is clear because $\mathrm{GL}(h, R/\pi R)$ is finite).

Now suppose that the n-tuple is faithful and irreducible. If $q \not| d$, then each A_i acts completely reducibly on $M/\pi M$. If the representation of X on $M/\pi M$ is not absolutely irreducible, then, since each A_i has order prime to q, we see that the n-tuple α is not irreducible, on lifting the (restriction to each A_i of the) Brauer character from characteristic q, for this properly decomposes α as a sum of compatible n-tuples, contrary to irreducibility. Part ii) follows by a similar argument. $\square$

The following three lemmas may be well known, but we include a proof for convenience. They are motivated by a desire to understand the interplay between integrality of representations and representability by unitary matrices.

Lemma 5.3. *Let σ be a representation of the finite group G over the ring of algebraic integers R of a number field K. Suppose that σ affords an irreducible complex character χ (of degree h, say) of G. Let $M = [m_{ij}]$ be a matrix in $\mathrm{GL}(h, K)$ which normalizes $G\sigma$. Then:*
a) For each (i, j) the matrix

$$m_{ij} \frac{[G : Z(G)]}{\chi(1)} M^{-1}$$

has all its entries in R.
b) If M has finite order, then

$$\frac{[G : Z(G)]}{\chi(1)} M$$

has all its entries in R.

Proof. We claim that for each (i, j) the matrix $m_{ij} \frac{[G:Z(G)]}{\chi(1)} M^{-1}$ has all entries algebraic integers, from which a) of the lemma follows. The integrality (or otherwise) of these quantities is unaffected by replacing M by a non-zero scalar multiple (even if the scalar lies outside K).

Conjugation by M induces a permutation of the elements of $G\sigma$. Hence some finite power of M centralizes $G\sigma$. Thus some finite power of M is a scalar matrix. Hence we may suppose (possibly after extending K and R) that M has finite order.

Now choose a prime ideal π of R. Since M has finite order, there is some (i,j) such that the valuation $\nu_\pi(m_{ij}) \leq 0$. Set $g\tau = Mg\sigma M^{-1}$ for each $g \in G$ (these matrices are all in $\mathrm{GL}(n,R)$). Let E_{ji} be the $n \times n$ matrix whose (j,i)-entry is 1 and whose other entries are all 0.

Now we have

$$\sum_{g \in G/Z(G)} (g\sigma)^{-1} E_{ji} g\tau = \sum_{g \in G/Z(G)} (g\sigma)^{-1} E_{ji} M g\sigma M^{-1}$$

(by consideration of trace)

$$= \frac{[G:Z(G)]\mathrm{trace}(E_{ji}M)}{\chi(1)} M^{-1}.$$

Since this matrix has entries in R, since π was an arbitrary prime ideal of R, and since $\mathrm{trace}(E_{ji}M) = m_{ij}$, our claim of integrality (and, in fact, the integrality of

$$\frac{[G:Z(G)]}{\chi(1)} M^{-1}$$

in the case that M has finite order) is established. Part b) follows on replacing M by M^{-1}. □

Lemma 5.4. *Let H be an irreducible monomial finite subgroup of* $\mathrm{GL}(s,\mathbb{Z}[\omega])$ *for some root of unity ω and suppose that $H \lhd T$ for some larger finite subgroup, T, of* $\mathrm{GL}(s,\mathbb{C})$. *Then T consists of unitary matrices, and*

$$\frac{[H:Z(H)]}{s} T \subseteq \mathrm{M_s}(\mathbb{Z}[\eta]),$$

for some complex root of unity η.

Proof. The given representation of T is equivalent to some unitary representation, and the equivalence may be achieved via a unitary conjugating matrix, since H is a unitary irreducible subgroup of T. Thus T itself consists of unitary matrices. It follows from Lemma 5.3 that T consists of matrices which have integral entries after multiplication by $\frac{[H:Z(H)]}{s}$. Since we know as in earlier discussions that

$$t \frac{[H:C_H(t)]\mathrm{trace}(t^{-1})}{s} = \sum_{y \in H/C_H(t)} [t,y],$$

for each $t \in T$, we see that the entries of t lie in a cyclotomic field for each $t \in T$ (notice that this issue of realizability only depends on the coset tH, and that since $\sum_{h \in H} |\mathrm{trace}(th)|^2 = |H|$, we may choose a coset representative for tH of non-zero trace). □

Remark 5.5. Let α be a faithful compatible n-tuple of infinite type of X. Let σ be an associated matrix representation of X affording α via a $\mathbb{Q}[\omega]X$-module, as in Theorem 4. Then, as X is finitely generated, this may be viewed as a homomorphism $\sigma : X \to \mathrm{GL}(m, \mathbb{Z}[\omega, \frac{1}{t}])$ for some positive integer t. Notice that this single representation allows us to perform reduction (mod π) for all but a finite number of prime ideals π of $\mathbb{Z}[\omega]$.

The next lemma is obvious, but we consider it worth recording explicitly.

Lemma 5.6. *For each prime ideal π of $\mathbb{Z}[\omega]$ not containing the integer td, let $\sigma_\pi : X \to \mathrm{GL}(h, \mathbb{Z}[\omega]/\pi)$ denote the composition of σ with reduction (mod π), and let K_π denote its kernel. Then each K_π is a free normal subgroup of finite index. Furthermore, there are infinitely many distinct normal subgroups K_π, and infinitely many non-isomorphic possibilities for X/K_π.*

Proof. Choose a prime ideal π. Since σ_π has finite image, the assertion that K_π has finite index is clear. The fact that K_π is free follows from Lemma 5.2. Let q be the unique rational prime contained in π.

If q does not divide $[X : K_\pi]$, then the given characteristic q representation lifts to a complex representation of the finite group X/K_π, contrary to the fact that α has infinite type. Thus $[X : K_\pi]$ is divisible by q. $\square$

As π was arbitrary, the claims of the lemma follow.

Remark 5.7. In the case that $X = X_{\mathcal{F}}$ is the amalgam of Section 1, reduction (mod q) may be performed for any prime $q \neq p$. We remark that in the cases where K_π is a free normal subgroup, it need not be maximal.

The next result illustrates that if we can realise a faithful representation of an infinite group via unitary matrices with entries in a cyclotomic number field, then the representation can't be realised over the ring of algebraic integers of that field.

Lemma 5.8. *Let $\mathbb{K}$ be a subfield of a finite cyclotomic extension of $\mathbb{Q}$, and let R be the ring of algebraic integers in $\mathbb{K}$. Let G be an infinite subgroup of $\mathrm{U}(h, \mathbb{K})$ for some positive integer h. Then G is not conjugate within $\mathrm{GL}(h, \mathbb{C})$ to any subgroup of $\mathrm{GL}(h, R)$.*

Proof. Let $s = [\mathbb{K} : \mathbb{Q}]$ and set $\mathcal{G} = \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$. As usual, let $\|A\| = \max\{\|vA\| : \|v\| = 1\}$ for each $A \in M_h(\mathbb{C})$, where $\|v\|$ denotes the usual Euclidean norm of the vector $v \in \mathbb{C}^h$, and $\mathbb{C}^h$ is viewed as the space of h-long complex row vectors. Then all unitary $h \times h$ matrices are in the unit ball of $M_h(\mathbb{C})$. Let B denote the direct product of s copies of $M_h(\mathbb{C})$, endowed with the (componentwise) max norm $\|(A_1, A_2, \ldots A_s)\|' = \max\{\|A_i\|\}$. Then B is a finite-dimensional complex Banach algebra, and its closed unit ball is compact. $\square$

We note that $\mathcal{G}$ acts as a group of automorphisms of $\mathrm{U}(h, \mathbb{K})$. Let us label the elements of $\mathcal{G}$ as $\{\tau_1, \ldots, \tau_s\}$, according to some chosen fixed ordering. Then $\phi : M_h(\mathbb{K}) \to B$ defined by $A\phi = (A^{\tau_1}, \ldots, A^{\tau_s})$ is an injection which maps $\mathrm{U}(h, \mathbb{K})$ into the unit ball of B. We also note that $\|A\phi\|' \geq 1$ whenever $A = [a_{ij}]$ is a non-zero matrix in $M_h(R)$, since for all $\tau \in \mathcal{G}$, we have $\|A^\tau\| \geq \max\{|a_{ij}^\tau| : 1 \leq i, j \leq h\}$, and $\prod_{\tau \in \mathcal{G}} |a_{k\ell}^\tau| \geq 1$ whenever $a_{k\ell} \neq 0$.

Now suppose that $T^{-1}GT \subseteq \mathrm{GL}(h, R)$ for some $T \in \mathrm{GL}(h, \mathbb{C})$. Then, since G is unitary, hence completely reducible, T may be assumed to lie in $\mathrm{GL}(h, \mathbb{K})$, by the usual argument using elementary linear algebra.

For $g \neq h \in G$, we have

$$1 \leq \|(T^{-1}(g-h)T)\phi\|' \leq \|(g-h)\phi\|' \|T\phi\|' \|(T^{-1})\phi\|'.$$

Since $G\phi$ is infinite, and the unit ball of B is compact, this is a contradiction.

6. Minimal projective representations

The next lemma, which will be useful in the remainder of this section, is related to some results of (Feit-Tits [2]), but the presence of a large prime divisor allows a slightly cleaner conclusion.

Lemma 6.1. *Let G be a finite subgroup of $\mathrm{GL}(h, K)$ for an algebraically closed field K, and let S be a simple non-Abelian section of G. Then S is isomorphic to a subgroup of $\mathrm{PGL}(h, F)$ for some field F. If, furthermore, $|S|$ is divisible by a prime $q > h+1$, then we may take $F = K$.*

Proof. By induction, we may suppose G is irreducible, and that $S \cong G/M$ for some maximal normal subgroup M of G which has no section isomorphic to S. Also, by repeated Frattini arguments and use of the Schur-Zassenhaus theorem, we may suppose that $M = \Phi(G)$ (so is nilpotent), and that all prime divisors of $|M|$ are divisors of $|S|$. We may further suppose that $G = G'$, and that $M \not\leq Z(G)$. We are done in case some covering group of S is a component (that is, quasi-simple subnormal subgroup) of G. From now on, then, we also suppose that $F^*(G) = \Phi(G)$. Hence $O_r(G) \not\leq Z(G)$ for some prime r, and S is isomorphic to a subgroup of $\mathrm{Out}(O_r(G))$ for any such prime r. If the given representation is (up to equivalence) induced from a representation of a proper subgroup H containing M, then S is isomorphic to a subgroup of the symmetric group $S_{[G:H]}$ and $[G : H] \leq h$, so the result is clear. Thus we may suppose that the representation is primitive, and hence that the underlying KG-module is homogeneous as a KM-module.

Now h is divisible by r^w for some integer w such that S is isomorphic to a subgroup of $\mathrm{Sp}(2w, r)$, and the first claim is established, since $2w \leq r^w$. For the second claim, notice that when $|S|$ is divisible by a prime $q > h+1$, we have $q > h \geq r^w \geq r$. Thus q divides $r^j \pm 1$ for some $j \leq w$, since $|\mathrm{Sp}(2w, r)| = r^{w^2} \prod_{i=1}^{w} (r^{2i} - 1)$. In particular, $q \leq h+1$, a contradiction. □

Remark 6.2. For the rest of this section, we confine our discussion to the case of an episimple Alperin fusion system $\mathcal{F}$ on a p-group P, and we define the amalgam Y as before. Given a finite-dimensional $\mathbb{C}$-vector space V, and n injective homomorphisms $\sigma_i : L_i \to \mathrm{Aut}(\mathrm{End}_{\mathbb{C}}(V))$ such that σ_1 and σ_i agree on P_i for $2 \leq i \leq n$, there is a unique homomorphism $\sigma : Y \to \mathrm{Aut}(\mathrm{End}_{\mathbb{C}}(V))$ which extends each σ_i, and the kernel of σ, say K, is a free normal subgroup.

In that case, V has the structure of a projective (in Schur's sense) $\mathbb{C}Y$-module. If, in addition, V is chosen to have minimal dimension, h say, subject to this, we say that σ affords a *minimal faithful projective complex representation* of Y. We note that there is always such a projective representation of Y, since Y has a free subgroup of finite index, which gives rise to a complex permutation representation (of finite degree) of Y with free kernel.

If σ may be chosen so that $[Y : K]$ is finite, we say that $\mathcal{F}$ has a *finite complex linear minimal projective realisation*, and Lemma 6.1 (and/or [2]) may be applied in that situation. We suppose in what follows that Y has no finite complex linear minimal projective realisation.

Slight modifications of earlier arguments show that we may choose a perfect central extension $\tilde{Y}$ of Y and a complex root of unity ω such that V is equivalent to an absolutely irreducible $\mathbb{Q}[\omega]\tilde{Y}$-module (and such that, for each i, $\tilde{L}_i$ has finite centre). We let $\tilde{K} \lhd \tilde{Y}$ consist of all those elements of $\tilde{Y}$ which act as scalars on V.

The minimality of the dimension of V shows that V is primitive when viewed as a $\mathbb{C}\tilde{Y}$-module (for if $\tilde{Y}$ has a proper subgroup H of finite index (less than or equal to h), then $\tilde{Y} \neq HZ(\tilde{Y})$, so we may, and do, suppose that H contains $Z(\tilde{Y})$. The image of the associated permutation representation of Y acts faithfully on some complement to a (one-dimensional) trivial submodule. By minimality of the choice of dimension, the kernel of the permutation representation is not free, so contains P by episimplicity, and then is all of Y, a contradiction).

As in earlier discussions, there is an integer t such that V is afforded by a $\mathbb{Z}[\omega, \frac{1}{t}]\tilde{Y}$-module, and we may reduce the representation (mod q) for each prime q which does not divide pt. We now consider such primes q which are also greater than $h+1$, for convenience. For each such choice of q, this yields a normal subgroup of finite index, say $\tilde{K}(q)$, of $\tilde{Y}$. Namely, $\tilde{K}(q)$ consists of all the elements which act trivially in the representation of $\tilde{Y}$ afforded by the direct sum of all composition factors of (a) reduction (mod q) of V. By unimodularity (recall that $\tilde{Y}$ is perfect), the image of $\tilde{K}(q)$ in its action on V contains only finitely many scalar matrices, each of which has order dividing h. But since q does not divide h, we see that this image of $\tilde{K}(q)$ contains no non-identity scalar matrix, as elements of finite order in the kernel of a reduction (mod q) have order a power of q, and any element of order prime to q which acts trivially on all the composition factors of the reduced module act trivially on the reduced module itself. Now $\tilde{K}(q)$ is isomorphic to a (normal) subgroup, $K(q)$ say, of Y. This (proper) subgroup contains no element of order p, since $\mathcal{F}$ is episimple. Hence $\tilde{K}(q)$ is free (of finite index). We note also that $O_q(\tilde{Y}/\tilde{K}(q)) = 1$, since the leftmost group acts trivially on the direct sum of all composition factors of the (chosen) reduction (mod q) of V.

The minimality of the dimension of V, and standard Clifford theory, show that if $\tilde{K}(q)$ does not act trivially on V, then (as it is not represented by scalars) it must act irreducibly. For otherwise, we would obtain a non-trivial projective representation of $\tilde{Y}/\tilde{K}(q)$ of dimension smaller than $h = \dim(V)$). This would lead to a projective representation of Y of dimension less than h, with free kernel. On the other hand, if $\tilde{K}(q)$ does act trivially on V, then $\tilde{Y}$ is finite, contrary to current assumptions.

Since $\tilde{Y}$ is infinite, we now know that $\tilde{K}(q)$ acts irreducibly on V for each choice of prime $q > h+1$ not dividing pt. Let $M(q)$ be a maximal free normal subgroup of Y containing $K(q)$, and let $\tilde{M}(q)$ be the full pre-image of $M(q)$ in $\tilde{Y}$. We know that $Y/M(q)$ is a non-Abelian finite simple group. Suppose that this group has order prime to q. Then (as in Lemma 5.5), we may choose a supplement $W(q)/K(q)$ to $M(q)/K(q)$ in $Y/K(q)$ such that $(W(q) \cap M(q))/K(q)$ is a nilpotent q'-group (in fact, contained in the Frattini subgroup of $Y/K(q)$). Now $Y = W(q)M(q)$ and $Y/M(q) \cong W(q)/(W(q) \cap M(q))$. Hence $W(q)/K(q)$ is a q'-group. Let $\tilde{W}(q)$ be the full pre-image of $W(q)$ in $\tilde{Y}$.

Now $\overline{W}(q) = \tilde{W}(q)/\tilde{K}(q)$ is also a q'-group, and "lifts" to a finite complex linear group which has $Y/M(q)$ as an epimorphic image. In fact, each composition factor of the reduction (mod q) of V lifts to a $\mathbb{C}\tilde{Y}$-module which yields a homomorphism from $\tilde{Y}$ to a finite complex linear group. At least one of these finite images has a composition factor isomorphic to $Y/M(q)$. By the minimality of h, the reduction (mod q) of V is thus absolutely irreducible. But now $\mathcal{F}$ has a finite complex linear minimal projective realisation, contrary to assumption. Thus the simple group $Y/M(q)$ does have order divisible by q. Since $q > h+1$, we may conclude from Lemma 6.1 that some finite central extension of $Y/M(q)$ has an irreducible representation of degree at most h over an algebraically closed field of characteristic q.

Theorem 6.3. *Let $\mathcal{F}$ be an episimple Alperin fusion system on a p-group P. Let Y be the iterated amalgam of the $O^{p'}(L_i)$ as before, and let h be the dimension of a minimal faithful complex projective representation of Y . Then one of the following occurs:*
a) $\mathcal{F}$ has no finite complex linear minimal projective realisation, and for all but a finite number of primes q, Y has a finite simple epimorphic image $Y(q)$ of order divisible by q which has a finite central extension having an absolutely irreducible representation σ of degree at most h in characteristic q.
b) Y has a finite complex linear minimal projective realisation.

Proof. We choose any prime $q > h+1$ which does not divide $p.t$. If $Y/M(q)$ has order divisible by q, then it has a faithful projective representation of degree at most h over an algebraically closed field of characteristic q, by Lemma 5.5 and the discussion preceding the theorem.

If $Y/M(q)$ has order prime to q, then $\mathcal{F}$ has a finite complex linear minimal projective realisation by the discussion preceding the theorem. □

Remark 6.4. This leads us to seek cases where the minimal degree faithful complex projective representation of the fusion system $\mathcal{F}$ can't be realised by a finite group. Since this representation is primitive, one method for ensuring that this is the case is to find a primitive complex representation such that some non-central elements have eigenvalues sufficiently close together on the unit circle S^1. In general, other *ad hoc* methods may be available in a particular case.

We illustrate some of the earlier discussions with an example.

Example 7.1: Twisted and untwisted images of a single amalgam. Let $p = 2$ and P_1 be a semi-dihedral 2-group of order 16. Let $\mathcal{F}$ denote the Alperin fusion system on P_1 with $L_1 = \mathrm{GL}(2,3)$ and $L_2 = S_4$, where P_2 is the dihedral subgroup of P_1 of order 8, and is identified with a Sylow 2-subgroup of L_2, while P_1 is identified with a Sylow 2-subgroup of L_1. Since neither S_4 nor $\mathrm{GL}(2,3)$ has a non-trivial epimorphic image of odd order, $\mathcal{F}$ is episimple with U_1 quaternion of order 8 and U_2 a Klein 4-group. We may realise $\mathcal{F}$ via the amalgam $Y = \mathrm{GL}(2,3) *_{D_8} S_4$. By Lemma 1.1, we know that $Y = [Y,Y]$ and that any maximal free normal subgroup of finite index is a maximal normal subgroup. We will exhibit a 3-dimensional complex representation of Y which is a minimal faithful projective complex representation of Y. Such a representation has kernel of infinite index, for otherwise the finite image of Y would be a perfect finite primitive complex linear group containing a subgroup H isomorphic to $\mathrm{GL}(2,3)$. An element of order 6 in H would be non-central, yet would have eigenvalues $1, e^{\frac{\pi i}{3}}$ and $e^{\frac{-\pi i}{3}}$, contrary to Blichfeldt's result.

We begin by writing $\mathrm{GL}(2,3)$ as a group of unitary 2×2 matrices with entries in $\mathbb{Z}[\frac{1}{\sqrt{-2}}]$, starting from the dihedral group D of order eight:

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle.$$

We wish to adjoin a unitary matrix of order 8 which normalises this group. Using the procedure of Lemma 5.4, for example, one such matrix is

$$\begin{pmatrix} \frac{-1}{\sqrt{-2}} & \frac{-1}{\sqrt{-2}} \\ \frac{1}{\sqrt{-2}} & \frac{-1}{\sqrt{-2}} \end{pmatrix}.$$

Then the group

$$P = \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{-2}} & \frac{-1}{\sqrt{-2}} \\ \frac{1}{\sqrt{-2}} & \frac{-1}{\sqrt{-2}} \end{pmatrix} \rangle$$

is semi-dihedral of order 16 (we note that the first matrix conjugates the second to its cube). This group has a unique quaternion subgroup of order 8, which is

$$Q = \langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{-2}} & \frac{-1}{\sqrt{-2}} \\ \frac{-1}{\sqrt{-2}} & \frac{1}{\sqrt{-2}} \end{pmatrix} \rangle.$$

We now seek a unitary matrix of order 3 which normalises Q. Such a matrix is

$$\begin{pmatrix} \frac{-1+\sqrt{-2}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-1-\sqrt{-2}}{2} \end{pmatrix}.$$

Then Q is normalised by

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{-2}} & \frac{-1}{\sqrt{-2}} \\ \frac{1}{\sqrt{-2}} & \frac{-1}{\sqrt{-2}} \end{pmatrix}, \begin{pmatrix} \frac{-1+\sqrt{-2}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-1-\sqrt{-2}}{2} \end{pmatrix} \rangle,$$

which is a group of order at least 48. This gives the desired unitary representation of $\mathrm{GL}(2,3)$. Another way to verify this is to notice that the product

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{-1+\sqrt{-2}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-1-\sqrt{-2}}{2} \end{pmatrix}$$

is a unitary matrix of determinant -1 and trace $\sqrt{-2}$, so is a matrix of order eight. Taking images modulo $\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$, we have an element of order 2 and an element of order 3 whose product has order 4, so we obtain a group isomorphic to S_4. Thus

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{-1+\sqrt{-2}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-1-\sqrt{-2}}{2} \end{pmatrix} \rangle$$

is isomorphic to $\mathrm{GL}(2,3)$ (it can't be the binary octahedral group, as it contains a non-central element of order 2).

Hence there is a (faithful in the earlier sense that its kernel is free) representation of Y as a (primitive, absolutely irreducible) subgroup of $\mathrm{SU}(3, \mathbb{Z}[\frac{1}{\sqrt{-2}}])$ via

$$G = \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{-1+\sqrt{-2}}{2} & \frac{1}{2} & 0 \\ \frac{-1}{2} & \frac{-1-\sqrt{-2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rangle.$$

The distinguished copy of $\mathrm{GL}(2,3)$ is that generated by the first two matrices. The distinguished copy of S_4 is

$$\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rangle$$

and our original dihedral group D is the intersection of these distinguished subgroups.

Now G is an infinite group by Blichfeldt's result, since it is primitive, and the distinguished $\mathrm{GL}(2,3)$ subgroup contains a non-central element of order 6 with eigenvalues $1, e^{\frac{\pi i}{3}}, e^{\frac{-\pi i}{3}}$. We notice that entrywise complex conjugation of its matrices induces an automorphism of order 2 of G.

This unitary complex representation of Y may be reduced (mod q) (strictly, modulo any prime ideal of $\mathbb{Z}[\sqrt{-2}]$ containing q) for each odd prime q. The kernel (in Y) of any one of these reductions (mod q) is a free normal subgroup of Y. If $q \equiv 1 \pmod 8$, or $q \equiv 3 \pmod 8$, then -2 is a quadratic residue (mod q), and the reduction is realised over $\mathrm{GF}(q)$. Otherwise -2 is a quadratic non-residue (mod q), the reduction as given is realised over $\mathrm{GF}(q^2)$, but (on consideration of the trace) is not realisable (even up to equivalence) over $\mathrm{GF}(q)$. Let $Y(q)$ denote (the image of) this reduction (mod q). Then $Y(q)$ is always a primitive absolutely irreducible group. We note that when -2 is not a quadratic residue (mod q), then the automorphism of $Y(q)$ induced by complex conjugation on G is that induced by the Frobenius automorphism of $\mathrm{GF}(q^2)$. Hence, in that case, $Y(q)$ is isomorphic to a subgroup of $\mathrm{SU}(3,q)$.

Since $Y(q)$ is not equivalent to a monomial group, we see that its Fitting subgroup is either central of order dividing 3, or else is an extra-special 3-group of order 27 and exponent 3. The latter case may be excluded, since otherwise, $Y(q)/F(Y(q))$ is isomorphic to a subgroup of $\mathrm{SL}(2,3)$, while $Y(q)/F(Y(q))$ has a subgroup isomorphic to $\mathrm{GL}(2,3)$. Hence $Y(q)$ has a central Fitting subgroup. Since the generalized Fitting subgroup of $Y(q)$ contains its centralizer, we see that $Y(q)$ has a component (that is to say, a quasi-simple subnormal subgroup) $L(q)$, which acts absolutely irreducibly by Clifford's theorem, so is unique, as distinct components centralize each other.

If $L(q)$ is a q'-group and $q > 3$, then the Hall-Higman theorem forces $Y(q)$ to be a q'-group. Otherwise, an element of order q centralizes $L(q)$ and hence is represented by a scalar matrix of determinant 1 (using the fact that $Y(q)$ is perfect). In that case, $Y(q)$ is liftable to a finite subgroup of $\mathrm{GL}(3,\mathbb{C})$, a possibility we excluded in the discussion at the beginning of this section.

For $q = 3$, a similar argument forces 3 to divide $|Y(3)|$ in the case that $L(3)$ is a $3'$-group. The Hall-Higman theorem then forces an element x of order 3 to act with cubic minimum polynomial, since $[L(3), x]$ is not a 2-group, so that x must act non-trivially on a Sylow r-subgroup of $L(3)$ for some odd prime r. In that case, $C_{L(3)}(x)$ consists of scalar matrices (which must then be trivial by unimodularity). But then $L(3)$ is nilpotent, a contradiction.

Hence in all cases, the component $L(q)$ has order divisible by q. Thus $O_{q'}(Y(q))$ is central (again of order dividing 3 on consideration of determinants). The component $L(q)$ is not isomorphic to $\mathrm{SU}(3,4)$, since $\mathrm{SU}(3,4)$ has no 3-dimensional representation over any field of odd characteristic.

We note also that (by unimodularity) $Y(q)$ contains no elementary Abelian subgroup of order 8. Also, since a Sylow 2-subgroup of $Y(q)$ has a 3-dimensional representation over a field of odd order (which becomes monomial (up to equivalence) over some extension field), a Sylow 2-subgroup, say $S(q)$, of $Y(q)$ has an Abelian (normal) subgroup of index 2. Since $Y(q)$ has no factor group of order 2 and $S(q)$ does contain a semi-dihedral subgroup of order 16, $S(q)$ is either semi-dihedral or wreathed. Let $S_0(q) = S(q) \cap L$. Then either $S_0(q)$ is generalised quaternion, or else $S_0(q)$ contains a Klein 4-group. In either case, consideration of the representation of $S_0(q)$, together with unimodularity, shows that $C_{Y(q)}(S_0(q))$ is Abelian. It follows by a Frattini argument that $Y(q)/L$ is solvable, as $\mathrm{Out}(S_0(q))$ is solvable in all cases. Since $Y(q)$ is perfect, we conclude that $Y(q) = L(q)$ is quasi-simple.

Since we have exhibited a 3-dimensional representation of $Y(q)$ over a field of q or q^2 elements, further analysis (such as the use of characterisation theorems, either of low-dimensional primitive linear groups, or of finite simple groups with semi-dihedral or wreathed Sylow 2-subgroups) leads to the conclusion that if $q \equiv 5 \pmod 8$ or $q \equiv 7 \pmod 8$, then $Y(q)$ is isomorphic to $\mathrm{SU}(3, q)$, while if $q \equiv 1 \pmod 8$, or $q \equiv 3 \pmod 8$, then $Y(q)$ is isomorphic to $\mathrm{SL}(3, q)$. It is interesting to note that $Y(q)$ has wreathed Sylow 2-subgroups if and only if $q \equiv \pm 1 \pmod 8$, whereas the Sylow 2-subgroup of Y is semi-dihedral of order 16.

We note that the case $Y(q) = M_{11}$ can't occur because M_{11} contains a Frobenius group of order 55 and a Frobenius group of order 20, so has no faithful 3-dimensional representation over any field.

Remark 7.2. The example above illustrates a general methodology, implicit in the results of this paper. Using the fact that $\mathrm{SL}(3, 3)$ is an epimorphic image of the amalgam Y, one way to view the example is that we have regarded the natural 3-dimensional representation of $\mathrm{SL}(3, 3)$ as a $\mathrm{GF}(3)Y$-module and lifted it to a $\mathbb{C}Y$-module. This yields a representation of Y which is realisable over $\mathbb{Z}[1/\sqrt{-2}]$, allowing reduction (mod q) for all other odd primes, yielding infinitely many non-isomorphic finite homomorphic images of Y.

The result is slightly atypical, in that the liftability of the representation of Y from characteristic 3 to characteristic 0 implicitly uses the liftabilty for S_4 and $\mathrm{GL}(2, 3)$, which, despite the presence of the prime 3, follows using the Fong-Swan theorem.

The general method is to take the iterated amalgam X associated to an Alperin fusion system $\mathcal{F}$, and a finite epimorphic image, say H, with a faithful (without loss of generality) representation over a field of characteristic q not dividing any $|L_i|$. The Brauer characters of the L_i afforded by the given representation then lift to complex characters. Furthermore, the complex characters so obtained are automatically compatible, since we started with a genuine representation of P in characteristic $q \neq p$. Hence we obtain a $\mathbb{C}X$-module, and the associated representation may be realised over a cyclotomic field. This representation may then be reduced (mod r) for all but finitely many primes r.

The condition that q does not divide $|L_i|$ for any i can be relaxed somewhat, as the example and discussion above illustrated. The method can be applied as long as $q \neq p$ and L_i is q-solvable whenever q divides $|L_i|$.

Remark 7.3. In a recent private communication to the author, J-P. Serre has now proved (using the above representation) that the group G above is isomorphic to both $\mathrm{SU}(3, \mathbb{Z}[\frac{1}{\sqrt{-2}}])$ and to $Y = \mathrm{GL}(2, 3) *_{D_8} S_4$.

References

[1] Broto, C.; Castellana, N.; Grodal, J.; Levi, R.; Oliver, R., *Subgroup families controlling p-local finite groups*, Proc. LMS,(3),91,(2005),325-354. MR2167090 (2007e:20111)

[2] Feit, W.; Tits, J.,*Projective representations of minimal degree of group extensions*,Can. Journal Math.,39,5,(1978),1092-1102. MR0498824 (58:16861)

[3] Leary, I.; Stancu, R., *Realising fusion systems*, Algebra Number Theory 1,1, (2007),17–34. MR2322922 (2008f:20030)

[4] Radin, C.; Sadun, L., *On* 2*-generator subgroups of* SO(3), Trans. Amer. Math. Soc., 351,11, (1999), 4469–4480. MR1624202 (2000a:51017)

[5] Robinson, G.R., *A subgroup of* $\mathrm{SO}(3,\mathbb{R})$ *generated by rotations of orders* 4 *and* 8, J. Algebra 306,1, (2006), 201–207. MR2271579 (2008g:20113)

[6] Robinson, G.R., *Amalgams, blocks, weights, fusion systems and finite simple groups*, J. Algebra,314,2,(2007),912-923. MR2344591 (2008f:20046)

[7] Serre, J-P., *Trees*, Springer-Verlag, Berlin, 1980. MR607504 (82c:20083)

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