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BOUNDARY PARTITIONS IN TREES AND DIMERS

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Abstract. Given a finite planar graph, a grove is a spanning forest in which every component tree contains one or more of a specified set of vertices (called nodes) on the outer face. For the uniform measure on groves, we compute the probabilities of the different possible node connections in a grove. These probabilities only depend on boundary measurements of the graph and not on the actual graph structure; i.e., the probabilities can be expressed as functions of the pairwise electrical resistances between the nodes, or equivalently, as functions of the Dirichlet-to-Neumann operator (or response matrix) on the nodes. These formulae can be likened to generalizations (for spanning forests) of Cardy's percolation crossing probabilities and generalize Kirchhoff's formula for the electrical resistance. Remarkably, when appropriately normalized, the connection probabilities are in fact integer-coefficient polynomials in the matrix entries, where the coefficients have a natural algebraic interpretation and can be computed combinatorially. A similar phenomenon holds in the so-called double-dimer model: connection probabilities of boundary nodes are polynomial functions of certain boundary measurements, and, as formal polynomials, they are specializations of the grove polynomials. Upon taking scaling limits, we show that the double-dimer connection probabilities coincide with those of the contour lines in the Gaussian free field with certain natural boundary conditions. These results have a direct application to connection probabilities for multiple-strand SLE_2, SLE_8, and SLE_4.

1. Introduction

1.1. **Grove partitions.** A **circular planar graph** $\mathcal{G}$ is a finite weighted planar graph with a set of vertices $\mathbf{N}$ on its outer face numbered $1, \ldots, n$ in counterclockwise order. The vertices in $\mathbf{N}$ are called **nodes**, and the remaining vertices are called **inner vertices**. Define a **grove** to be a spanning acyclic subgraph (a forest) of $\mathcal{G}$ such that each component tree contains at least one node. The weight of a grove is the product of the weights of the edges it contains. We study random groves where the probability of a grove is proportional to its weight.

The term **grove** comes from Carroll and Speyer [CS04] and Petersen and Speyer [PS05] who studied a special case of (our) groves, with a particular family of underlying graphs. Since we are dealing with a natural generalization we chose to keep their terminology, and refer to the special case they discuss as Carroll-Speyer groves, which we will discuss further in a subsequent paper [KW08].

The connected components of a grove partition the nodes into a planar (i.e. noncrossing) partition. For example, when $n = 4$, there are 14 planar partitions:

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1234, 1|234, 2|134, 3|124, 4|123, 12|34, 23|14, 1|2|34, 1|3|24, 1|4|23, 2|3|14, 2|4|13, 3|4|12, 1|2|3|4. There are no groves with the partition 13|24 because it is not planar (there is no way to connect nodes 1 and 3 and nodes 2 and 4 by disjoint paths within a circular planar graph). For general n, the number of noncrossing partitions is the nth Catalan number $C_n = (2n)!/(n!(n+1)!)$ (see [Sta99, ex. 6.19(pp), pg. 226]).

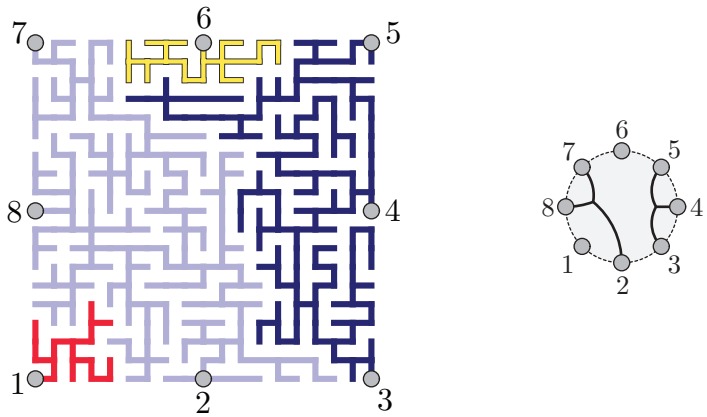


FIGURE 1. A random grove (left) of a rectangular grid with 8 nodes on the outer face. In this grove there are 4 trees (each colored differently), and the partition of the nodes is $\{\{1\},\{2,7,8\},\{3,4,5\},\{6\}\}$, which we write as 1|278|345|6 and illustrate schematically as shown on the right.

If σ is a planar partition of $1,\dots,n$, we let $\Pr(\sigma)$ denote the probability that a random grove of $\mathcal{G}$ partitions the nodes according to σ. Since groves with one component are trees, we refer to the partition $\sigma = 123\dots n$ as the **tree** partition. When there are n nodes we call the partition $\sigma = 1|2|3|\cdots|n$ the **uncrossing**, since groves with this partition type contain no crossings (i.e. paths) connecting the nodes. We show how to compute for each planar partition σ the probability $\Pr(\sigma)$ of it occurring in a random grove, as a function of the electrical properties of the graph $\mathcal{G}$, when $\mathcal{G}$ is viewed as a resistor network with conductances equal to the edge weights.

We let $R_{i,j}$ denote the effective electrical resistance between nodes i and j; i.e., the voltage at node i which, when node j is held at 0 volts, causes a unit current to flow through the circuit from node i to node j. (A good reference for basic electric circuit theory is [DS84].) Let $L_{i,j}$ denote the current that would flow into node j if node i were set to one volt and the remaining nodes set to zero volts. Though it is not obvious from this definition, $L_{i,j} = L_{j,i}$ (see Appendix A). The $L_{i,j}$ are the negatives of the entries of the "response matrix" (Dirichlet-to-Neumann matrix) of $(\mathcal{G},\mathbf{N})$, on which we provide further background in Appendix A; see also [CdV98]. We prove

Theorem 1.1. *For any planar partition σ,*

$$\frac{\Pr(\sigma)}{\Pr(\text{tree})} = \begin{array}{c}\textit{integer-coefficient homogeneous polynomial in the } R_{i,j}/2\textit{'s}\\ \textit{where the degree is } -1+\#\textit{parts of } \sigma,\end{array}$$

and

$$\frac{\Pr(\sigma)}{\Pr(\text{uncrossing})} = \begin{array}{c}\textit{integer-coefficient homogeneous polynomial in the } L_{i,j}\textit{'s}\\ \textit{where the degree is } n - \#\textit{parts of } \sigma.\end{array}$$

This theorem is a corollary to our main theorem on groves, Theorem 1.2 in Section 1.2, which gives explicit formulas. These formulas are in effect a multinode generalization (for planar graphs) of Kirchhoff's formula $\Pr(1|2)/\Pr(12) = R_{1,2} = 1/L_{1,2}$ (which holds whether or not $\mathcal{G}$ is planar) [Kir90] (see [Pem95, Thm 3.7] for a good exposition). (When $n = 2$ we have $L_{1,2} = 1/R_{1,2}$, but this does not hold for $n > 2$.) We illustrate Theorem 1.1 by giving the polynomials for $n = 2$ and $n = 3$, and in Appendix B we give the polynomials for $n = 4$ nodes. For notational convenience we write

$$\overline{\Pr}(\sigma) := \Pr(\sigma)/\Pr(\text{tree}) \qquad \text{and} \qquad \ddot{\Pr}(\sigma) := \Pr(\sigma)/\Pr(\text{uncrossing}).$$

(The dots in $\ddot{\Pr}()$ remind us that in the normalization the nodes are disconnected.) The polynomials for $n = 2$ come from Kirchhoff's formula:

$$\overline{\Pr}(12) = 1, \quad \overline{\Pr}(1|2) = R_{1,2}, \qquad \text{and} \qquad \ddot{\Pr}(12) = L_{1,2}, \quad \ddot{\Pr}(1|2) = 1.$$

When $n = 3$ the polynomials start to become more interesting:

$$\overline{\Pr}(123) = 1, \qquad \overline{\Pr}(1|23) = \tfrac{1}{2}R_{1,3} + \tfrac{1}{2}R_{1,2} - \tfrac{1}{2}R_{2,3},$$
$$\overline{\Pr}(2|13) = \tfrac{1}{2}R_{2,3} + \tfrac{1}{2}R_{1,2} - \tfrac{1}{2}R_{1,3}, \qquad \overline{\Pr}(3|12) = \tfrac{1}{2}R_{1,3} + \tfrac{1}{2}R_{2,3} - \tfrac{1}{2}R_{1,2},$$
$$\overline{\Pr}(1|2|3) = \tfrac{1}{2}R_{1,2}R_{1,3} + \tfrac{1}{2}R_{1,2}R_{2,3} + \tfrac{1}{2}R_{1,3}R_{2,3} - \tfrac{1}{4}R_{1,2}^2 - \tfrac{1}{4}R_{2,3}^2 - \tfrac{1}{4}R_{1,3}^2,$$

and

$$\ddot{\Pr}(123) = L_{1,2}L_{1,3} + L_{1,2}L_{2,3} + L_{1,3}L_{2,3},$$
$$\ddot{\Pr}(1|23) = L_{2,3}, \qquad \ddot{\Pr}(2|13) = L_{1,3}, \qquad \ddot{\Pr}(3|12) = L_{1,2}, \qquad \ddot{\Pr}(1|2|3) = 1.$$

The $n = 3$ formulas also hold whether or not the graph is planar, but when $n \geq 4$ planarity becomes important.

When n gets large these polynomials can have many terms, but for certain special classes of partitions, most notably the "parallel crossing" $1, n|2, n-1|3, n-2|\cdots$, the polynomials can be expressed in terms of determinants [CIM98], [Fom01]. We discuss these and other determinant/Pfaffian formulas in article [KW08].

1.2. **The partition projection matrix and explicit formulas.** The formulas in Theorem 1.1 in terms of the $L_{i,j}$'s are the simplest ones to explain. For a partition τ on $1, \ldots, n$ we define

$$L_\tau = \sum_F \prod_{\{i,j\} \in F} L_{i,j}, \tag{1}$$

where the sum is over those spanning forests F of the complete graph on n vertices $1, \ldots, n$ for which trees of F span the parts of τ, and where the product is over edges $\{i, j\}$ of forest F.

This definition makes sense whether or not the partition τ is planar. For example, $L_{1|234} = L_{2,3}L_{3,4} + L_{2,3}L_{2,4} + L_{2,4}L_{3,4}$ and $L_{13|24} = L_{1,3}L_{2,4}$. As we shall see, the "$L$ polynomials" of Theorem 1.1 are in fact integer linear combinations of the L_τ's:

$$\frac{\Pr(\sigma)}{\Pr(1|2|\cdots|n)} = \sum_\tau \mathcal{P}^{(\mathrm{t})}_{\sigma,\tau} L_\tau.$$

(We write the superscript (t) to distinguish these coefficients from ones that arise in the double-dimer model in Section 1.4.) The rows of the matrix $\mathcal{P}^{(\mathsf{t})}$ are indexed by planar partitions, and the columns are indexed by all partitions. In the case of $n = 4$ nodes, the matrix $\mathcal{P}^{(\mathsf{t})}$ is

	1|2|3|4	12|3|4	13|2|4	14|2|3	23|1|4	24|1|3	34|1|2	12|34	14|23	1|234	2|134	3|124	4|123	1234	13|24
1|2|3|4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12|3|4	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
13|2|4	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
14|2|3	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
23|1|4	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
24|1|3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
34|1|2	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
12|34	0	0	0	0	0	0	0	1	0	0	0	0	0	0	−1
114|23	0	0	0	0	0	0	0	0	1	0	0	0	0	0	−1
1|234	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
2|134	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
3|124	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
4|123	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
1234	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0.

For example, the row for 1|234 tells us

$$\frac{\Pr(1|234)}{\Pr(1|2|3|4)} = L_{1|234} + L_{13|24} = L_{2,3}L_{3,4} + L_{2,3}L_{2,4} + L_{2,4}L_{3,4} + L_{1,3}L_{2,4}$$

and the row for 12|34 tells us

$$\frac{\Pr(12|34)}{\Pr(1|2|3|4)} = L_{12|34} - L_{13|24} = L_{1,2}L_{3,4} - L_{1,3}L_{2,4}.$$

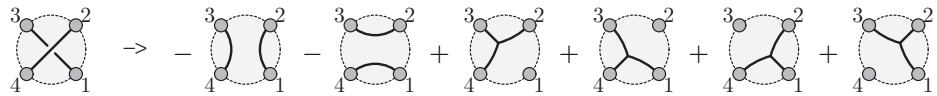
We call this matrix $\mathcal{P}^{(\mathsf{t})}$ the **projection matrix from partitions to planar partitions**, since it can be interpreted as a map from the vector space whose basis vectors are indexed by all partitions to the vector space whose basis vectors are indexed by planar partitions, and the map is the identity on planar partitions. For example, the column for 13|24 tells us

$$13|24 \quad \text{projects to} \quad -12|34 - 14|23 + 1|234 + 2|134 + 3|124 + 4|123.$$

(We could have written the right-hand side as $-e_{12|34} - e_{14|23} + e_{1|234} + e_{2|134} + e_{3|124} + e_{4|123}$, where the e's are basis vectors, but it is convenient to suppress the vector notation and instead write formal linear combinations of partitions.)

The projection matrix may be computed using some simple combinatorial transformations of partitions. Given a partition τ, the τth column of $\mathcal{P}^{(\mathsf{t})}$ may be computed by repeated application of the following transformation rule, until the resulting formal linear combination of partitions only involves planar partitions. The

rule generalizes the transformation



$$13|24 \to -12|34 - 14|23 + 1|234 + 2|134 + 3|124 + 4|123$$

(which is derived in Lemma 2.3) to partitions τ containing additional items and parts. If partition τ is nonplanar, then there will exist items $a < b < c < d$ such that a and c belong to one part, and b and d belong to another part. Arbitrarily subdivide the part containing a and c into two sets A and C such that $a \in A$ and $c \in C$, and similarly subdivide the part containing b and d into $B \ni b$ and $D \ni d$. Let the remaining parts of partition τ (if any) be denoted by "rest." Then the transformation rule is

(Rule 1)

$$\begin{aligned} AC|BD|\,\text{rest} \to A|BCD|\,\text{rest} + B|ACD|\,\text{rest} + C|ABD|\,\text{rest} + D|ABC|\,\text{rest} \\ - \; AB|CD|\,\text{rest} - AD|BC|\,\text{rest}\,. \end{aligned}$$

In Section 2 we prove

Theorem 1.2. *Any partition τ may be transformed into a formal linear combination of planar partitions by repeated application of* Rule 1*, and the resulting linear combination does not depend on the choices made when applying* Rule 1*, so that we may write*

$$\tau \to \sum_{\textit{planar partitions } \sigma} \mathcal{P}^{(\mathsf{t})}_{\sigma,\tau}\sigma.$$

For any planar partition σ, these same coefficients $\mathcal{P}^{(\mathsf{t})}_{\sigma,\tau}$ satisfy the equation

$$\frac{\Pr(\sigma)}{\Pr(1|2|\cdots|n)} = \sum_{\textit{partitions } \tau} \mathcal{P}^{(\mathsf{t})}_{\sigma,\tau} L_\tau$$

for circular planar graphs. More generally, for any graph these coefficients satisfy

$$\sum_{\textit{partitions } \tau} \mathcal{P}^{(\mathsf{t})}_{\sigma,\tau} \frac{\Pr(\tau)}{\Pr(1|2|\cdots|n)} = \sum_{\textit{partitions } \tau} \mathcal{P}^{(\mathsf{t})}_{\sigma,\tau} L_\tau.$$

(The last equation specializes to the previous one because $\mathcal{P}^{(\mathsf{t})}_{\sigma,\sigma} = 1$; for planar $\tau \neq \sigma$ we have $\mathcal{P}^{(\mathsf{t})}_{\sigma,\tau} = 0$, and for nonplanar τ we have $\Pr(\tau) = 0$ for circular planar graphs.)

1.3. **Double-dimer pairings.** An analogous computation can be done for the double-dimer model. Let $\mathcal{G}$ be a finite edge-weighted bipartite planar graph with a set $\mathbf{N}$ of $2n$ special vertices called nodes on the outer face, which we label $1, \ldots, 2n$ in counterclockwise order along the outer face. A double-dimer configuration on $(\mathcal{G}, \mathbf{N})$ is by definition a configuration of disjoint loops (i.e. simple cycles of length more than two), doubled edges, and simple paths on $\mathcal{G}$ that connect all the nodes in pairs, which collectively cover the vertices of $\mathcal{G}$. See Figure 2 for an example. We weight each configuration by the product of its edge weights times 2^ℓ, where ℓ is the number of loops (a doubled edge does not count as a loop). The number of possible ways that the $2n$ nodes may be paired up with one another is the nth Catalan number (see [Sta99, ex. 6.19(n), pg. 222]), and we show how to compute the

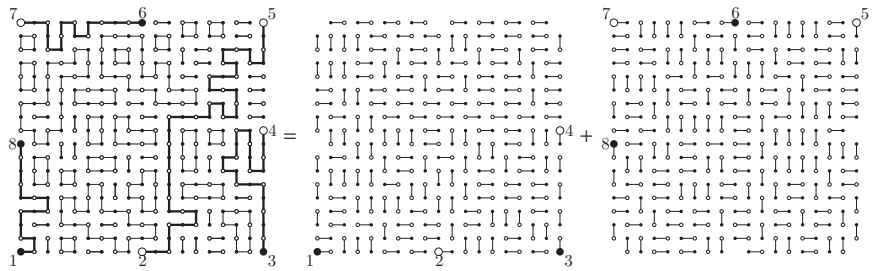


FIGURE 2. At the left is a double-dimer configuration on a rectangular grid with 8 nodes. In this configuration, the pairing of the nodes is $\{\{1,8\},\{3,4\},\{5,2\},\{7,6\}\}$, which we write as $\frac{1}{8} \mid \frac{3}{4} \mid \frac{5}{2} \mid \frac{7}{6}$ (odd-numbered nodes are always paired with even-numbered nodes). A double-dimer configuration is formed from the union of two dimer configurations, one on the graph $\mathcal{G}^{\mathsf{BW}} \subseteq \mathcal{G}$ (defined in Subsection 1.3 and shown in the middle) and the other on the graph $\mathcal{G}^{\mathsf{WB}} \subseteq \mathcal{G}$ (shown on the right).

probabilities of each of these pairings when a random double-dimer configuration is chosen according to these weights. We will show how to express these pairing probabilities in terms of polynomials in a set of variables that are analogous to the $L_{i,j}$ variables for random groves.

Let $Z^{\mathsf{DD}}(\mathcal{G}, \mathbf{N})$ be the weighted sum of all double-dimer configurations. Since the graph $\mathcal{G}$ is bipartite, we view the vertices as being colored black and white so that each edge of $\mathcal{G}$ connects a black and white vertex, and we define the parity of a node to be the parity of its numerical label. Let $\mathcal{G}^{\mathsf{BW}}$ be the subgraph of $\mathcal{G}$ formed by deleting the nodes except the ones that are black and odd or white and even, and let $\mathcal{G}^{\mathsf{BW}}_{i,j}$ be defined as $\mathcal{G}^{\mathsf{BW}}$, but with node i included in $\mathcal{G}^{\mathsf{BW}}_{i,j}$ if and only if it was not included in $\mathcal{G}^{\mathsf{BW}}$, and similarly for node j. Let Z^{BW} and $Z^{\mathsf{BW}}_{i,j}$ be the weighted sum of dimer configurations of $\mathcal{G}^{\mathsf{BW}}$ and $\mathcal{G}^{\mathsf{BW}}_{i,j}$, respectively, and define Z^{WB} and $Z^{\mathsf{WB}}_{i,j}$ similarly but with the roles of black and white reversed. Each of these quantities can be computed via determinants; see [Kas67] and Section 3. It turns out that $Z^{\mathsf{DD}} = Z^{\mathsf{BW}} Z^{\mathsf{WB}}$; this is essentially Ciucu's graph factorization theorem [Ciu97, Thm. 1.2] (except that in the factorization theorem any edges that connect two nodes are reweighted by 1/2, and here they are not); see Section 3. (The two dimer configurations in Figure 2 are on the graphs $\mathcal{G}^{\mathsf{BW}}$ and $\mathcal{G}^{\mathsf{WB}}$.) The variables that play the role of $L_{i,j}$ in groves are defined by

$$X_{i,j} = Z^{\mathsf{BW}}_{i,j} / Z^{\mathsf{BW}}.$$

For each planar matching of the nodes σ, let $\Pr(\sigma)$ be the probability that a random double-dimer configuration has this set of connections. In Section 3 we prove

Theorem 1.3. *For any planar pairing σ on $2n$ nodes,*

$$\Pr(\sigma)\frac{Z^{\mathsf{WB}}}{Z^{\mathsf{BW}}} = \textit{an integer-coefficient homogeneous polynomial of degree } n \textit{ in the quantities } X_{i,j}.$$

This theorem is a corollary to our main theorem on double-dimer pairings, Theorem 1.4 in Section 1.4, which gives the polynomials explicitly. To illustrate this

theorem, we give a few examples. For notational simplicity let us define

$$\widehat{\Pr}(\sigma) = \Pr(\sigma) Z^{\mathsf{WB}}/Z^{\mathsf{BW}}.$$

In the simplest nontrivial case there are $2n = 4$ nodes and two possible planar pairings, $\{\{1,2\},\{3,4\}\}$ and $\{\{1,4\},\{2,3\}\}$, which we write as $\frac{1}{2} \mid \frac{3}{4}$ and $\frac{1}{4} \mid \frac{3}{2}$. Their normalized probabilities are

$$\widehat{\Pr}(\tfrac{1}{2}\,|\,\tfrac{3}{4}) = X_{1,2}X_{3,4}, \qquad \widehat{\Pr}(\tfrac{1}{4}\,|\,\tfrac{3}{2}) = X_{1,4}X_{2,3}.$$

These two formulas are essentially equivalent to a formula of Kuo [Kuo04, Thms 2.1 and 2.3]. In the case of $2n = 6$ nodes, we have

$$\begin{aligned}\widehat{\Pr}(\tfrac{1}{2}\,|\,\tfrac{3}{6}\,|\,\tfrac{5}{4}) &= X_{3,6}(X_{1,2}X_{4,5} - X_{1,4}X_{2,5}),\\ \widehat{\Pr}(\tfrac{1}{2}\,|\,\tfrac{3}{4}\,|\,\tfrac{5}{6}) &= X_{1,4}X_{2,5}X_{3,6} + X_{1,2}X_{3,4}X_{5,6}\end{aligned}$$

and similarly for cyclic permutations of the indices. These cover the five possibilities.

In the case of $2n = 8$ nodes, we have

$$\begin{aligned}\widehat{\Pr}(\tfrac{1}{2}\,|\,\tfrac{3}{8}\,|\,\tfrac{5}{6}\,|\,\tfrac{7}{4}) &= (X_{1,2}X_{7,4} - X_{1,4}X_{7,2})(X_{3,8}X_{5,6} - X_{5,8}X_{3,6})\\ &= -\det\begin{bmatrix} X_{1,2} & X_{1,4} & 0 & 0\\ 0 & 0 & X_{3,6} & X_{3,8}\\ 0 & 0 & X_{5,6} & X_{5,8}\\ X_{7,2} & X_{7,4} & 0 & 0\end{bmatrix},\\ \widehat{\Pr}(\tfrac{1}{2}\,|\,\tfrac{3}{4}\,|\,\tfrac{5}{6}\,|\,\tfrac{7}{8}) &= X_{1,2}X_{3,4}X_{5,6}X_{7,8} + X_{1,4}X_{3,8}X_{5,6}X_{7,2} + X_{1,6}X_{3,4}X_{5,8}X_{7,2}\\ &\quad + X_{1,6}X_{3,8}X_{5,2}X_{7,4} + X_{1,2}X_{3,6}X_{5,8}X_{7,4} + X_{1,4}X_{3,6}X_{5,2}X_{7,8}\\ &\quad - 2X_{1,4}X_{3,6}X_{5,8}X_{7,2},\\ \widehat{\Pr}(\tfrac{1}{2}\,|\,\tfrac{3}{8}\,|\,\tfrac{5}{4}\,|\,\tfrac{7}{6}) &= \det\begin{bmatrix} X_{1,2} & X_{1,4} & X_{1,6} & 0\\ 0 & 0 & X_{3,6} & X_{3,8}\\ X_{5,2} & X_{5,4} & 0 & X_{5,8}\\ X_{7,2} & X_{7,4} & X_{7,6} & 0\end{bmatrix}\end{aligned}$$

and similarly for cyclic permutations of the indices. These cover the fourteen possibilities.

As can be seen from the way we have written these polynomials, for some pairings they are expressible as determinants. We discuss such formulas further in [KW08].

Of course we could also express the pairing probabilities in terms of the variables $X^*_{i,j} = Z^{\mathsf{WB}}_{i,j}/Z^{\mathsf{WB}}$. The polynomials are exactly the same, although the underlying variables represent different quantities.

1.4. **The odd-even pairing projection matrix and explicit formulas.** The explicit computation of the double-dimer pairing probability formulas is quite analogous to the computation of the grove partition probability formulas. The role of the L_τ variables for groves is replaced by variables that are indexed by the $n!$ pairings between the odd nodes and the even nodes ("odd-even pairings"). If τ is such an odd-even pairing, then we define

$$X_\tau = X(\tau) = \prod_{i \text{ odd}} X_{i,\tau(i)},$$

and it turns out to be more convenient to work with

$$X'_\tau = X'(\tau) = (-1)^{\#\text{crosses in } \tau} \prod_{i \text{ odd}} X_{i,\tau(i)},$$

where a cross in a pairing τ is a set of four nodes $a < b < c < d$ such that a and c are paired with one another and b and d are paired with one another. The "X polynomials" are in fact integer linear combinations of the X'_τ's:

$$\widehat{\Pr}(\sigma) = \sum_{\textit{odd-even pairings } \tau} \mathcal{P}^{(\mathsf{DD})}_{\sigma,\tau} X'_\tau.$$

As with grove partitions, we construct a projection matrix, but with double-dimer pairings the matrix projects a vector space with basis vectors indexed by odd-even pairings to a vector space whose basis vectors are indexed by planar pairings. Recall that for groves the projection matrix $\mathcal{P}^{(\mathsf{t})}$ has dimensions $C_n \times B_n$, the nth Catalan number by the nth Bell number; for double-dimers the projection matrix $\mathcal{P}^{(\mathsf{DD})}$ has dimensions $C_n \times n!$. When $n = 4$ the projection matrix $\mathcal{P}^{(\mathsf{DD})}$ is

	$\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}$	$\frac{1}{2}\frac{3}{4}\frac{5}{8}\frac{7}{6}$	$\frac{1}{2}\frac{3}{6}\frac{5}{4}\frac{7}{8}$	$\frac{1}{2}\frac{3}{8}\frac{5}{4}\frac{7}{6}$	$\frac{1}{2}\frac{3}{8}\frac{5}{6}\frac{7}{4}$	$\frac{1}{4}\frac{3}{2}\frac{5}{6}\frac{7}{8}$	$\frac{1}{4}\frac{3}{2}\frac{5}{8}\frac{7}{6}$	$\frac{1}{4}\frac{3}{2}\frac{5}{6}\frac{7}{8}$	$\frac{1}{4}\frac{3}{2}\frac{5}{8}\frac{7}{6}$	$\frac{1}{6}\frac{3}{4}\frac{5}{8}\frac{7}{2}$	$\frac{1}{6}\frac{3}{4}\frac{5}{2}\frac{7}{8}$	$\frac{1}{8}\frac{3}{2}\frac{5}{4}\frac{7}{6}$	$\frac{1}{8}\frac{3}{2}\frac{5}{6}\frac{7}{4}$	$\frac{1}{8}\frac{3}{4}\frac{5}{2}\frac{7}{6}$	$\frac{1}{8}\frac{3}{4}\frac{5}{6}\frac{7}{2}$	$\frac{1}{8}\frac{3}{6}\frac{5}{4}\frac{7}{2}$	$\frac{1}{2}\frac{3}{6}\frac{5}{8}\frac{7}{4}$	$\frac{1}{4}\frac{3}{6}\frac{5}{2}\frac{7}{8}$	$\frac{1}{4}\frac{3}{6}\frac{5}{8}\frac{7}{2}$	$\frac{1}{4}\frac{3}{8}\frac{5}{2}\frac{7}{6}$	$\frac{1}{4}\frac{3}{8}\frac{5}{6}\frac{7}{2}$	$\frac{1}{6}\frac{3}{2}\frac{5}{8}\frac{7}{4}$	$\frac{1}{6}\frac{3}{4}\frac{5}{8}\frac{7}{2}$	$\frac{1}{6}\frac{3}{8}\frac{5}{4}\frac{7}{2}$	$\frac{1}{8}\frac{3}{6}\frac{5}{2}\frac{7}{4}$	$\frac{1}{4}\frac{3}{6}\frac{5}{8}\frac{7}{2}$	$\frac{1}{6}\frac{3}{8}\frac{5}{2}\frac{7}{4}$
$\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	−1	0	−1	0	−1	0	0	−2	1
$\frac{1}{2}\frac{3}{4}\frac{5}{8}\frac{7}{6}$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	−1	0	0	1	0	0	1	−1
$\frac{1}{2}\frac{3}{6}\frac{5}{4}\frac{7}{8}$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	−1	0	1	−1
$\frac{1}{2}\frac{3}{8}\frac{5}{4}\frac{7}{6}$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	−1	0	1	0	0	0	1	0	−1	1
$\frac{1}{2}\frac{3}{8}\frac{5}{6}\frac{7}{4}$	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0
$\frac{1}{4}\frac{3}{2}\frac{5}{6}\frac{7}{8}$	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	−1	0	0	0	1	−1
$\frac{1}{4}\frac{3}{2}\frac{5}{8}\frac{7}{6}$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	1
$\frac{1}{6}\frac{3}{2}\frac{5}{4}\frac{7}{8}$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	−1	0	0	1	0	1	0	−1	1
$\frac{1}{6}\frac{3}{4}\frac{5}{2}\frac{7}{8}$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	1	0
$\frac{1}{8}\frac{3}{2}\frac{5}{4}\frac{7}{6}$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	−1	0	−1	0	−1	−1	1	−2
$\frac{1}{8}\frac{3}{2}\frac{5}{6}\frac{7}{4}$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	−1	1	0	0	1	−1	1
$\frac{1}{8}\frac{3}{4}\frac{5}{2}\frac{7}{6}$	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	−1	0	1	−1	1
$\frac{1}{8}\frac{3}{4}\frac{5}{6}\frac{7}{2}$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	−1	1	−1
$\frac{1}{8}\frac{3}{6}\frac{5}{4}\frac{7}{2}$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	1

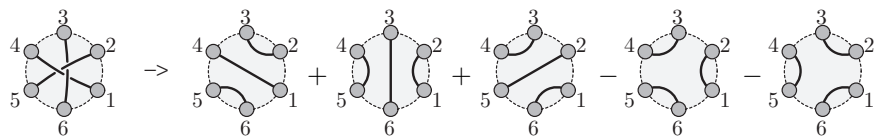
We call this matrix $\mathcal{P}^{(\mathsf{DD})}$ the **projection matrix from odd-even pairings to planar pairings**.

For example, the first row tells us that

$$\begin{aligned}
\widehat{\Pr}\left(\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}\right) = {} & X'\left(\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}\right) - X'\left(\tfrac{1}{2}\tfrac{3}{6}\tfrac{5}{8}\tfrac{7}{4}\right) - X'\left(\tfrac{1}{4}\tfrac{3}{6}\tfrac{5}{2}\tfrac{7}{8}\right) - X'\left(\tfrac{1}{4}\tfrac{3}{8}\tfrac{5}{6}\tfrac{7}{2}\right) \\
& - X'\left(\tfrac{1}{6}\tfrac{3}{4}\tfrac{5}{8}\tfrac{7}{2}\right) - 2X'\left(\tfrac{1}{4}\tfrac{3}{6}\tfrac{5}{8}\tfrac{7}{2}\right) + X'\left(\tfrac{1}{6}\tfrac{3}{8}\tfrac{5}{2}\tfrac{7}{4}\right) \\
= {} & X_{1,2}X_{3,4}X_{5,6}X_{7,8} + X_{1,2}X_{3,6}X_{5,8}X_{7,4} + X_{1,4}X_{3,6}X_{5,2}X_{7,8} \\
& + X_{1,4}X_{3,8}X_{5,6}X_{7,2} \\
& + X_{1,6}X_{3,4}X_{5,8}X_{7,2} - 2X_{1,4}X_{3,6}X_{5,8}X_{7,2} \\
& + X_{1,6}X_{3,8}X_{5,2}X_{7,4}.
\end{aligned}$$

As with the matrix $\mathcal{P}^{(\mathsf{t})}$, we may compute the τth column of the matrix $\mathcal{P}^{(\mathsf{DD})}$ via a sequence of simple combinatorial transformations on the odd-even pairings.

The prototypical example of the transformation rule is



$$\begin{smallmatrix}1\\4\end{smallmatrix}\,|\,\begin{smallmatrix}3\\6\end{smallmatrix}\,|\,\begin{smallmatrix}5\\2\end{smallmatrix} \to \begin{smallmatrix}1\\4\end{smallmatrix}\,|\,\begin{smallmatrix}3\\2\end{smallmatrix}\,|\,\begin{smallmatrix}5\\6\end{smallmatrix} + \begin{smallmatrix}1\\2\end{smallmatrix}\,|\,\begin{smallmatrix}3\\6\end{smallmatrix}\,|\,\begin{smallmatrix}5\\4\end{smallmatrix} + \begin{smallmatrix}1\\6\end{smallmatrix}\,|\,\begin{smallmatrix}3\\4\end{smallmatrix}\,|\,\begin{smallmatrix}5\\2\end{smallmatrix} - \begin{smallmatrix}1\\2\end{smallmatrix}\,|\,\begin{smallmatrix}3\\4\end{smallmatrix}\,|\,\begin{smallmatrix}5\\6\end{smallmatrix} - \begin{smallmatrix}1\\6\end{smallmatrix}\,|\,\begin{smallmatrix}3\\2\end{smallmatrix}\,|\,\begin{smallmatrix}5\\4\end{smallmatrix}$$

(which is derived in Lemma 3.7). Compare with the 16th column, column $\begin{smallmatrix}1\\4\end{smallmatrix}\,|\,\begin{smallmatrix}3\\6\end{smallmatrix}\,|\,\begin{smallmatrix}5\\2\end{smallmatrix}\,|\,\begin{smallmatrix}7\\8\end{smallmatrix}$, in the above matrix. More generally, suppose that $n \geq 3$ and the odd-even pairing is $\begin{smallmatrix}a_1\\b_1\end{smallmatrix}\,|\,\begin{smallmatrix}a_2\\b_2\end{smallmatrix}\,|\cdots|\,\begin{smallmatrix}a_n\\b_n\end{smallmatrix}$, where the a_i's are odd and the b_i's are even. Then the transformation rule is

$$\text{(Rule 2)}\quad \begin{smallmatrix}a_1\\b_1\end{smallmatrix}\,|\,\begin{smallmatrix}a_2\\b_2\end{smallmatrix}\,|\,\begin{smallmatrix}a_3\\b_3\end{smallmatrix}\,|\,\text{rest} \to \begin{smallmatrix}a_1\\b_2\end{smallmatrix}\,|\,\begin{smallmatrix}a_2\\b_1\end{smallmatrix}\,|\,\begin{smallmatrix}a_3\\b_3\end{smallmatrix}\,|\,\text{rest} + \begin{smallmatrix}a_1\\b_1\end{smallmatrix}\,|\,\begin{smallmatrix}a_2\\b_3\end{smallmatrix}\,|\,\begin{smallmatrix}a_3\\b_2\end{smallmatrix}\,|\,\text{rest} + \begin{smallmatrix}a_1\\b_3\end{smallmatrix}\,|\,\begin{smallmatrix}a_2\\b_2\end{smallmatrix}\,|\,\begin{smallmatrix}a_3\\b_1\end{smallmatrix}\,|\,\text{rest} - \begin{smallmatrix}a_1\\b_2\end{smallmatrix}\,|\,\begin{smallmatrix}a_2\\b_3\end{smallmatrix}\,|\,\begin{smallmatrix}a_3\\b_1\end{smallmatrix}\,|\,\text{rest} - \begin{smallmatrix}a_1\\b_3\end{smallmatrix}\,|\,\begin{smallmatrix}a_2\\b_1\end{smallmatrix}\,|\,\begin{smallmatrix}a_3\\b_2\end{smallmatrix}\,|\,\text{rest},$$

where in the above the "rest" represents $\begin{smallmatrix}a_4\\b_4\end{smallmatrix}\,|\cdots|\,\begin{smallmatrix}a_n\\b_n\end{smallmatrix}$ with unchanged pairings.

For example, when transforming $\begin{smallmatrix}1\\4\end{smallmatrix}\,|\,\begin{smallmatrix}3\\6\end{smallmatrix}\,|\,\begin{smallmatrix}5\\8\end{smallmatrix}\,|\,\begin{smallmatrix}7\\2\end{smallmatrix}$ we can hold the pair $\{7,2\}$ fixed:

$$\begin{smallmatrix}1\\4\end{smallmatrix}\,|\,\begin{smallmatrix}3\\6\end{smallmatrix}\,|\,\begin{smallmatrix}5\\8\end{smallmatrix}\,|\,\begin{smallmatrix}7\\2\end{smallmatrix} \to \begin{smallmatrix}1\\6\end{smallmatrix}\,|\,\begin{smallmatrix}3\\4\end{smallmatrix}\,|\,\begin{smallmatrix}5\\8\end{smallmatrix}\,|\,\begin{smallmatrix}7\\2\end{smallmatrix} + \begin{smallmatrix}1\\4\end{smallmatrix}\,|\,\begin{smallmatrix}3\\8\end{smallmatrix}\,|\,\begin{smallmatrix}5\\6\end{smallmatrix}\,|\,\begin{smallmatrix}7\\2\end{smallmatrix} + \begin{smallmatrix}1\\8\end{smallmatrix}\,|\,\begin{smallmatrix}3\\6\end{smallmatrix}\,|\,\begin{smallmatrix}5\\4\end{smallmatrix}\,|\,\begin{smallmatrix}7\\2\end{smallmatrix} - \begin{smallmatrix}1\\6\end{smallmatrix}\,|\,\begin{smallmatrix}3\\8\end{smallmatrix}\,|\,\begin{smallmatrix}5\\4\end{smallmatrix}\,|\,\begin{smallmatrix}7\\2\end{smallmatrix} - \begin{smallmatrix}1\\8\end{smallmatrix}\,|\,\begin{smallmatrix}3\\4\end{smallmatrix}\,|\,\begin{smallmatrix}5\\6\end{smallmatrix}\,|\,\begin{smallmatrix}7\\2\end{smallmatrix}.$$

Of these odd-even pairings, the third and fifth ones are planar, but the others require additional applications of Rule 2. When transforming the first of these terms, if we hold the pair $\{3,4\}$ fixed,

$$\begin{smallmatrix}1\\6\end{smallmatrix}\,|\,\begin{smallmatrix}3\\4\end{smallmatrix}\,|\,\begin{smallmatrix}5\\8\end{smallmatrix}\,|\,\begin{smallmatrix}7\\2\end{smallmatrix} \to \begin{smallmatrix}1\\8\end{smallmatrix}\,|\,\begin{smallmatrix}3\\4\end{smallmatrix}\,|\,\begin{smallmatrix}5\\6\end{smallmatrix}\,|\,\begin{smallmatrix}7\\2\end{smallmatrix} + \begin{smallmatrix}1\\6\end{smallmatrix}\,|\,\begin{smallmatrix}3\\4\end{smallmatrix}\,|\,\begin{smallmatrix}5\\2\end{smallmatrix}\,|\,\begin{smallmatrix}7\\8\end{smallmatrix} + \begin{smallmatrix}1\\2\end{smallmatrix}\,|\,\begin{smallmatrix}3\\4\end{smallmatrix}\,|\,\begin{smallmatrix}5\\8\end{smallmatrix}\,|\,\begin{smallmatrix}7\\6\end{smallmatrix} - \begin{smallmatrix}1\\8\end{smallmatrix}\,|\,\begin{smallmatrix}3\\4\end{smallmatrix}\,|\,\begin{smallmatrix}5\\2\end{smallmatrix}\,|\,\begin{smallmatrix}7\\6\end{smallmatrix} - \begin{smallmatrix}1\\2\end{smallmatrix}\,|\,\begin{smallmatrix}3\\4\end{smallmatrix}\,|\,\begin{smallmatrix}5\\6\end{smallmatrix}\,|\,\begin{smallmatrix}7\\8\end{smallmatrix},$$

then the resulting odd-even pairings are all planar. The other terms above may be similarly transformed into linear combinations of planar pairings, and when they are added up, the result is summarized in column $\begin{smallmatrix}1\\4\end{smallmatrix}\,|\,\begin{smallmatrix}3\\6\end{smallmatrix}\,|\,\begin{smallmatrix}5\\8\end{smallmatrix}\,|\,\begin{smallmatrix}7\\2\end{smallmatrix}$ of the projection matrix $\mathcal{P}^{(\mathsf{DD})}$.

In Section 3 we prove

Theorem 1.4. *Any odd-even pairing τ may be transformed into a formal linear combination of planar pairings by repeated application of* Rule 2*, and the resulting linear combination does not depend on the choices made when applying* Rule 2*, so that we may write*

$$\tau \to \sum_{\textit{planar pairings } \sigma} \mathcal{P}^{(\mathsf{DD})}_{\sigma,\tau}\sigma.$$

For any planar pairing σ, these same coefficients $\mathcal{P}^{(\mathsf{DD})}_{\sigma,\tau}$ satisfy the equation

$$\Pr(\sigma)\frac{Z^{\mathsf{WB}}}{Z^{\mathsf{BW}}} = \sum_{\textit{odd-even pairings } \tau} \mathcal{P}^{(\mathsf{DD})}_{\sigma,\tau} X'(\tau)$$

for bipartite circular planar graphs.

In Section 4 we show that the $\mathcal{P}^{(\mathsf{DD})}$ projection matrix of order n is up to signs embedded in the $\mathcal{P}^{(\mathsf{t})}$ projection matrix of order $2n$:

Theorem 1.5. *We have*

$$\mathcal{P}^{(\mathsf{DD})}_{\sigma,\tau} = (-1)^{\sigma^{-1}\tau}\mathcal{P}^{(\mathsf{t})}_{\sigma,\tau},$$

where on the left σ and τ denote odd-even pairings, and on the right they are interpreted as partitions consisting of parts of size 2*, and in the sign they are interpreted as maps from odd nodes to even nodes, so that $\sigma^{-1}\tau$ is a permutation on odd nodes and $(-1)^{\sigma^{-1}\tau}$ is its signature.*

1.5. **Multichordal SLE connection probabilities.** In Section 5 we consider the connection probabilities in the scaling limits of the spanning tree and double-dimer models, and also of the contour lines in the scaling limit of the discrete Gaussian free field with certain boundary conditions.

Connection probabilities of this sort were first studied by Cardy, who gave (a physics derivation for) an explicit formula for the probability (in the scaling limit) of a percolation crossing from one segment of the boundary of a domain to another segment of the boundary [Car92]. Carleson noticed that Cardy's formula is especially nice when the domain is an equilateral triangle, and this was one of the insights that led to Smirnov's proof that the percolation interface converges to SLE_6 [Smi01] (see also [CN07]). (The SLE_κ process, which we do not define here, was introduced by Schramm [Sch00] and describes the scaling limits of random curves arising in statistical physics; see [Sch07].) Arguin and Saint-Aubin [ASA02, § 3] gave (a physics derivation of) the corresponding crossing probabilities for spins of the critical Ising model, shown in Figure 3. As Figure 3 shows, there are two spin interfaces that connect the four boundary locations where the boundary conditions change, there are two ways in which the interfaces may pair up these boundary points, and the pairing probabilities are computed in [ASA02, § 3]. Each spin interface, conditional on the location of the other interface, is distributed according to a chordal SLE_3 curve [Smi07] within the domain cut by the other interface, so Arguin and Saint-Aubin effectively studied a bichordal version of SLE_3. Kirchhoff's formula $\Pr(1|2)/\Pr(12) = R_{1,2}$ can, in view of [LSW04], be interpreted as giving the connection probabilities in a bichordal version of SLE_8 (see § 5.2). The corresponding connection probabilities for other values of κ were given in [BBK05, §§ 8.2 and 8.3] (see also [Dub06, § 4.1]).

In a related vein, there has been much study of the physics of multiple-strand networks of self-avoiding walks (polymers), loop-erased random walks, and other random paths that are either known or believed to be related to SLE_κ. For example,

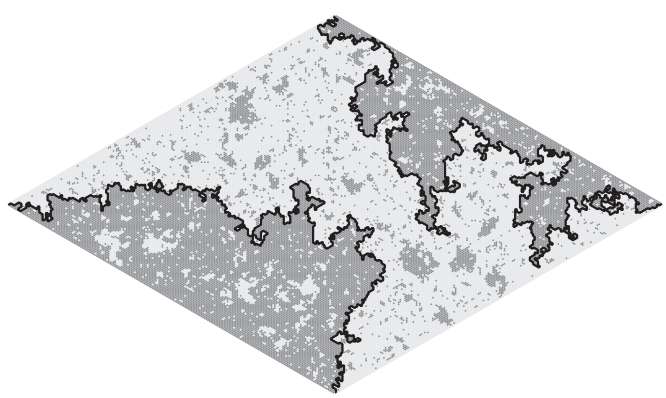


FIGURE 3. The critical 2D Ising model with mixed spin-up and spin-down boundary conditions, where the spins are shown as black and white hexagons. In this configuration there is a connection within the white spins connecting the two white boundary segments. The two spin interfaces are shown in bold and in the scaling limit converge to bichordal SLE_3.

the "watermelon exponents" describe the scaling behavior of the partition function when multiple strands meet at a point in the interior or on the boundary of a domain [Sal86, DS87, Dup92, Dup87]. The boundary watermelon exponents for L strands can be extracted from the limiting behavior of the connection probabilities of L strands of (multichordal) SLE connecting $2L$ boundary points when L of the boundary points are close to each other and the other L boundary points are close to each other, though of course these connection probabilities contain more information than is summarized in the exponents. This approach does not yield the bulk watermelon exponents, since to obtain the bulk exponents one would need the endpoints of the strands to be in the interior of the domain. The scaling exponents associated with more complicated multiple-strand networks (for $2 \leq \kappa \leq 8$) have also been derived [Dup87, Dup89, OB88]. See [Dup06] for up-to-date lecture notes on the physics of networks of polymers and other types of strands and their relation to SLE.

It is only natural to consider the connection probabilities of multichordal SLE_κ, which is defined in [KL07] and has the property that, conditional on one curve, the remaining curves are distributed according to multichordal SLE_κ with fewer curves within the domain cut by the conditioned curve. Kozdron and Lawler [KL07] studied multichordal SLE_κ for a fixed connection topology (when $\kappa \leq 4$). Cardy exhibited several discrete models with multiple interfaces that arise naturally in the context of conformal field theory, for which the interfaces have the same joint distribution (whose scaling limit is thought to be multichordal SLE), conditional on a particular connection topology, but for which the connection topologies have different probabilities [Car07]. (Thus to discuss connection topology probabilities of multichordal SLE, one must specify a discrete model whose scaling limit is being taken.) Comparing two such models, the connection topology probabilities in one model have different weights compared to another such model, but these weights do not depend on the domain or on the location of the boundary points where the boundary conditions change, so that the set of all connection topology probabilities in one such model determines the connection topology probabilities in another such model. Dubédat [Dub06] analyzed the general multichordal SLE_κ connection probabilities (and included special discussion of the cases $\kappa = 2, 6, 8$), but there still remain open problems regarding these connection probabilities when the number of curves is large.

The scaling limit calculations in Section 5 yield explicit formulas for the multichordal SLE_κ connection probabilities in the cases $\kappa = 2, 4, 8$ (for any number of curves, for any connection topology). There is a scaling limit relation [LSW04] between branches of uniform spanning trees on periodic planar graphs and SLE_2. Essentially, the scaling limit, as the lattice spacing tends to zero, of a branch of the uniform spanning tree on a bounded domain tends to a random simple curve which is equal in law to SLE_2. Similarly, the curve which winds between the uniform spanning tree and its dual spanning tree (Figure 9) converges in the scaling limit to an SLE_8 [LSW04]. The double-dimer paths are thought to have a scaling limit that is given by SLE_4, but this has not been proved. However, the contour lines in the discrete Gaussian free field with certain boundary conditions have been proved to converge to SLE_4 [SS06] in the scaling limit, and using the results of [SS06] we prove in Theorem 5.1 that the probability distribution of the pairings of these contour lines coincides with the pairing distribution for the double-dimer model.

2. Grove partitions

2.1. **The meander matrix.** Associated to a planar partition σ on n nodes is a **planar chord diagram** consisting of n disjoint chords, winding between the components and the dual components of σ; see Figure 4 for an example. The chords have in total $2n$ endpoints, one between each node and adjacent dual node. A planar chord diagram determines σ, and vice versa.

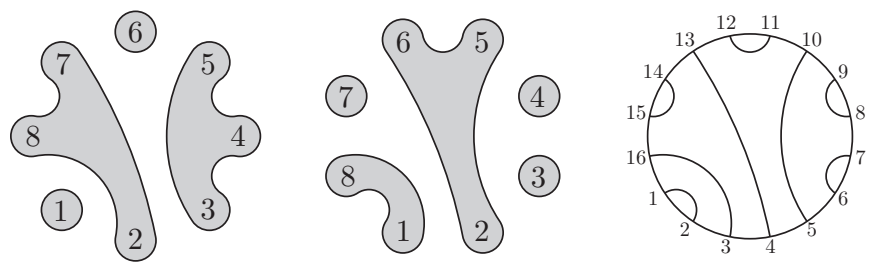


Figure 4. The planar partition 1|278|345|6, its dual planar partition 18|256|3|4|7, and their planar chord diagram $\begin{smallmatrix}1\\2\end{smallmatrix}\,|\,\begin{smallmatrix}3\\16\end{smallmatrix}\,|\,\begin{smallmatrix}5\\10\end{smallmatrix}\,|\,\begin{smallmatrix}7\\6\end{smallmatrix}\,|\,\begin{smallmatrix}9\\8\end{smallmatrix}\,|\,\begin{smallmatrix}11\\12\end{smallmatrix}\,|\,\begin{smallmatrix}13\\4\end{smallmatrix}\,|\,\begin{smallmatrix}15\\14\end{smallmatrix}$.

There is a natural bilinear form on the space of formal linear combinations of chord diagrams that is defined as follows. Given two chord diagrams C_1 and C_2, we can draw them on a sphere, one in the upper hemisphere and one in the lower hemisphere, so their common boundary consists of $2n$ points around the equator. In the resulting figure the chords join up to form simple closed loops, each crossing the equator (at least twice) and such that the set of all loops crosses the equator a total of $2n$ times. Such an object is a **meander** of order n. The bilinear form $\langle C_1, C_2\rangle_q$ is defined by

$$\langle C_1, C_2\rangle_q = q^{\#\text{ loops in meander formed from } C_1 \text{ and } C_2}. \tag{2}$$

This definition can then be extended linearly to formal linear combinations of chord diagrams.

The matrix $\mathcal{M}_q$ which has rows and columns indexed by chord diagrams and has entries $(\mathcal{M}_q)_{C_1,C_2} = \langle C_1, C_2\rangle_q$ is known as the Gram matrix of the Temperley-Lieb algebra (also called the meander matrix). Jones [Jon83] determined the values of q for which $\mathcal{M}_q$ is positive definite and positive semidefinite, Ko and Smolinsky [KS91] determined when $\mathcal{M}_q$ is nonsingular, and Di Francesco, Golinelli, and Guitter [DFGG97] gave an explicit diagonalization of $\mathcal{M}_q$. For our results on groves we shall use the fact that $\lim_{q\to 0} \mathcal{M}_q/q$ is nonsingular [DFGG97, Eqn (5.18)], and for our results on the double-dimer model we shall use the fact that $\mathcal{M}_2$ is nonsingular [DFGG97, Eqn (5.6)].

2.2. **The "L" polynomials.** In this subsection we show that $\ddot{\Pr}(\sigma)$ is a polynomial in the $L_{i,j}$ for each planar partition σ.

For a planar partition σ let C_σ be its associated chord diagram. We define a bilinear form $\langle,\rangle_{\mathtt{t}}$ on the vector space generated by planar partitions by

$$\langle \sigma, \tau\rangle_{\mathtt{t}} = \lim_{q\to 0} \frac{\langle C_\sigma, C_\tau\rangle_q}{q} = \begin{cases} 1, & \text{if meander formed from } C_\sigma \text{ and } C_\tau \text{ has one loop,} \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

We let $\mathcal{M}_{\mathtt{t}} = \lim_{q\to 0} \mathcal{M}_q/q$ denote the Gram matrix of $\langle,\rangle_{\mathtt{t}}$, which is nonsingular.

For a partition σ on $\{1,\dots,n\}$ and graph $\mathcal{G}$ with nodes $\{1,\dots,n\}$, let $\mathcal{G}_\sigma$ be the graph $\mathcal{G}$ with nodes identified according to σ; that is, $\mathcal{G}_\sigma = \mathcal{G}/\sim$, where $v \sim v'$ if and only if v and v' are in the same component of σ.

If σ and τ are planar partitions and $\mathcal{G}$ is a circular planar graph, it is not hard to see that

$$\langle\sigma,\tau\rangle_{\mathtt{t}} = \begin{cases} 1, & \text{if a grove of type } \tau \text{ in } \mathcal{G} \text{ gives a spanning tree in } \mathcal{G}_\sigma \\ & \text{when the vertices are identified according to } \sigma, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Indeed, given a grove of $\mathcal{G}$ of type τ, and a "grove" of type σ living within the outer face of $\mathcal{G}$, if their union is a spanning tree, the path winding around the outside of this spanning tree is a meander. We can contract to points the components of the "grove" of type σ, without changing the topology of the meander. Another way to say this is that $\langle\sigma,\tau\rangle_{\mathtt{t}} = 1$ if and only if the number of parts of σ and τ add up to $n+1$ and the transitive closure of the relation defined by the union of σ and τ has a single equivalence class.

Let $Z(\tau)$ be the weighted sum of groves of type τ on $\mathcal{G}$ and $Z_{\mathcal{G}_\sigma}(\tau)$ be the corresponding weighted sum on $\mathcal{G}_\sigma$. In particular, $Z_{\mathcal{G}_\sigma}(\text{tree})$ is the weighted sum of spanning trees of $\mathcal{G}_\sigma$.

Lemma 2.1. *For a planar partition σ and circular planar graph $\mathcal{G}$,*

$$Z_{\mathcal{G}_\sigma}(\text{tree}) = \sum_\tau \langle\sigma,\tau\rangle_{\mathtt{t}} Z(\tau),$$

where the sum is over all planar partitions.

Proof. If $\langle\sigma,\tau\rangle_{\mathtt{t}} = 1$, then every grove of $\mathcal{G}$ of type τ, when we identify vertices in σ, becomes a spanning tree of $\mathcal{G}_\sigma$. Conversely, any spanning tree of $\mathcal{G}_\sigma$ arises from a unique grove of $\mathcal{G}$, and this grove has partition τ satisfying $\langle\sigma,\tau\rangle_{\mathtt{t}} = 1$. □

Note also that $Z_{\mathcal{G}_\sigma}(\text{uncrossing}) = Z(\text{uncrossing})$, since an uncrossing of $\mathcal{G}$ is the same as an uncrossing of $\mathcal{G}_\sigma$.

Lemma 2.2. *For any graph $\mathcal{G}$ (not necessarily planar) and partition σ (not necessarily planar), $Z_{\mathcal{G}_\sigma}(\text{tree})/Z(\text{uncrossing})$ is a polynomial in the $L_{i,j}$.*

Proof. Let S be the matrix whose rows index the equivalence classes of the relation σ and whose columns index the nodes $1,\dots,n$, and whose i,j-entry is $S_{i,j} = 1$ if node j is in class i, and $s_{i,j} = 0$ otherwise.

The response matrix $\Lambda(\mathcal{G}_\sigma)$ for $\mathcal{G}_\sigma$ (see Appendix A) is obtained from the response matrix Λ for $\mathcal{G}$ simply as

$$\Lambda(\mathcal{G}_\sigma) = S\Lambda S^T. \tag{5}$$

That is, putting potential v_i on vertex i of $\mathcal{G}_\sigma$ is the same as putting potential v_i on each node of $\mathcal{G}$ in the equivalence class i. The resulting current out of node i of $\mathcal{G}_\sigma$ is the sum of the currents out of nodes of $\mathcal{G}$ in the ith equivalence class. In particular, entries in $\Lambda(\mathcal{G}_\sigma)$ are sums of entries in Λ. From Lemma A.1 we have

$$\det \tilde{\Lambda}(\mathcal{G}_\sigma) = Z_{\mathcal{G}_\sigma}(\text{tree})/Z_{\mathcal{G}_\sigma}(\text{uncrossing}),$$

where $\tilde{\Lambda}(\mathcal{G}_\sigma)$ is $\Lambda(\mathcal{G}_\sigma)$ with one row and column removed. The left-hand side is a polynomial in the $L_{i,j}$, by (5), and since $Z_{\mathcal{G}_\sigma}(\text{uncrossing}) = Z(\text{uncrossing})$, this concludes the proof. □

Using Lemma 2.1, we can invert matrix $\mathcal{M}_{\mathsf{t}}$ (since it is nonsingular) to write $Z(\tau)$ as a rational linear combination of the $Z_{\mathcal{G}_\sigma}(\text{tree})$ as σ varies. Dividing both sides by $Z(\text{uncrossing})$ we see that $\ddot{\Pr}(\tau)$ is a polynomial in the $L_{i,j}$ with rational coefficients. In the next subsection we prove that (even though $\mathcal{M}_{\mathsf{t}}^{-1}$ has noninteger entries) the coefficients of these polynomials are actually integers.

2.3. **Integrality of the coefficients.** We can extend the bilinear form $\langle,\rangle_{\mathsf{t}}$ to work on any pair of partitions (not necessarily planar), simply by taking the characterization in Equation (4) and dropping the requirement that the partitions be planar. Define an "extended meander matrix" $\mathcal{E}_{\mathsf{t}}$ with rows indexed by planar partitions and columns indexed by all partitions as above, so that $(\mathcal{E}_{\mathsf{t}})_{\sigma,\tau} = \langle\sigma,\tau\rangle_{\mathsf{t}}$. The matrix $\mathcal{E}_{\mathsf{t}}$ has dimensions $C_n \times B_n$, where C_n is the nth Catalan number and B_n is the nth Bell number (the number of partitions on n items). The matrix $\mathcal{M}_{\mathsf{t}}$ is the submatrix of $\mathcal{E}_{\mathsf{t}}$ containing only the columns for planar partitions.

Note that with this extended definition of $\langle,\rangle_{\mathsf{t}}$, Lemma 2.1 holds for general, nonplanar graphs, provided we sum over all (not necessarily planar) partitions. We let $\vec{G}$ be the column vector of "glue variables" whose entries are $\vec{G}_\sigma = Z_{\mathcal{G}_\sigma}(\text{tree})$ for σ a planar partition of $1,\ldots,n$. Let $\vec{Z}$ be the column vector of partition variables whose entries are $\vec{Z}_\tau = Z(\tau)$, where τ runs over all partitions. The extension of Lemma 2.1 to the nonplanar setting gives

$$\vec{G} = \mathcal{E}_{\mathsf{t}}\vec{Z}. \tag{6}$$

For any not necessarily planar graph $\mathcal{G}$ on n nodes there is an electrically equivalent complete graph $\mathcal{K}$ on n vertices (in which every vertex is a node): it is the graph whose edge $\{i,j\}$ has conductance $L_{i,j}(\mathcal{G},\mathbf{N})$. (The graphs $\mathcal{G}$ and $\mathcal{K}$ are electrically equivalent in the sense that, when the same voltages are applied to the nodes of $\mathcal{G}$ and $\mathcal{K}$, the current responses will be the same, i.e., they have the same Dirichlet-to-Neumann matrix.) Note that each $Z_{\mathcal{K}}(\tau)$ is trivially a polynomial in the $L_{i,j}$ since each crossing of $\mathcal{K}$ has weight which is a monomial in the $L_{i,j}$. In fact, we can write this polynomial explicitly: for each part λ of τ, we count the weighted sum of spanning trees of the complete graph on the vertices in λ, and then take the product over the different parts of τ. Recalling our definition of L_τ in Equation (1), we have that $Z_{\mathcal{K}}(\tau) = L_\tau$. For example,

$$Z_{\mathcal{K}}(1|23|456) = 1 \cdot L_{2,3} \cdot (L_{4,5}L_{5,6} + L_{4,5}L_{4,6} + L_{4,6}L_{5,6}) = L_{1|23|456}.$$

We let $\vec{L}$ be the column vector of partition variables for $\mathcal{K}$: $\vec{L}_\tau = Z_{\mathcal{K}}(\tau) = L_\tau$.

By the preceding equation, for the graph $\mathcal{K}$, Equation (6) specializes to $\vec{G} = \mathcal{E}_{\mathsf{t}}\vec{L}$. By Lemma 2.2, for any graph, $\vec{G}$ is determined by the L's, so we have

$$\vec{G} = \mathcal{E}_{\mathsf{t}}\vec{Z} = \mathcal{E}_{\mathsf{t}}\vec{L}$$

for any graph. For planar graphs, the entries of $\vec{Z}$ corresponding to nonplanar partitions are 0, so $\mathcal{E}_{\mathsf{t}}\vec{Z} = \mathcal{M}_{\mathsf{t}}\vec{Z}$, and hence $\mathcal{M}_{\mathsf{t}}\vec{Z} = \mathcal{E}_{\mathsf{t}}\vec{L}$. Since $\mathcal{M}_{\mathsf{t}}$ is invertible,

$$\vec{Z} = \mathcal{M}_{\mathsf{t}}^{-1}\mathcal{E}_{\mathsf{t}}\vec{L}.$$

We define

$$\mathcal{P}^{(\mathsf{t})} = \mathcal{M}_{\mathsf{t}}^{-1}\mathcal{E}_{\mathsf{t}}.$$

We shall see how to compute $\mathcal{P}^{(\mathsf{t})}$ directly, i.e. without inverting $\mathcal{M}_{\mathsf{t}}$. The direct computation of $\mathcal{P}^{(\mathsf{t})}$ involves only integer operations, from which it will follow that the "L polynomials" have integer coefficients.

What is the matrix $\mathcal{P}^{(\mathsf{t})}$? For each planar partition σ, the row σ tells us $Z_\sigma = \sum_\tau \mathcal{P}^{(\mathsf{t})}_{\sigma,\tau} L_\tau$. For each partition τ, the τth column gives us a linear combination of planar partitions $\sum_\sigma \mathcal{P}^{(\mathsf{t})}_{\sigma,\tau}\sigma$ that is equivalent to τ in the sense that for any *planar* partition ρ,

$$\left\langle \rho, \sum_\sigma \mathcal{P}^{(\mathsf{t})}_{\sigma,\tau}\sigma \right\rangle_{\mathsf{t}} = \sum_\sigma \mathcal{P}^{(\mathsf{t})}_{\sigma,\tau}(\mathcal{M}_{\mathsf{t}})_{\rho,\sigma} = (\mathcal{M}_{\mathsf{t}}\mathcal{P}^{(\mathsf{t})})_{\rho,\tau} = (\mathcal{E}_{\mathsf{t}})_{\rho,\tau} = \langle \rho, \tau\rangle_{\mathsf{t}}.$$

(We shall see that $\left\langle \rho, \sum_\sigma \mathcal{P}^{(\mathsf{t})}_{\sigma,\tau}\sigma \right\rangle_{\mathsf{t}} = \langle \rho, \tau\rangle_{\mathsf{t}}$ for nonplanar partitions ρ, too.)

Let us say that two linear combinations of partitions on n items $\sum_\tau \alpha_\tau \tau$ and $\sum_\tau \beta_\tau \tau$ are equivalent ($\overset{\mathsf{t}}{\equiv}$) if for any (possibly nonplanar) partition ρ on n items $\sum_\tau \alpha_\tau \langle \rho, \tau\rangle_{\mathsf{t}} = \sum_\tau \beta_\tau \langle \rho, \tau\rangle_{\mathsf{t}}$. For example,

Lemma 2.3. $1|234 + 2|134 + 3|124 + 4|123 \overset{\mathsf{t}}{\equiv} 12|34 + 13|24 + 14|23$.

Proof. For any partition ρ with three parts, $\langle \rho, \mathrm{LHS}\rangle_{\mathsf{t}} = 2 = \langle \rho, \mathrm{RHS}\rangle_{\mathsf{t}}$. By symmetry considerations, we need only consider one such partition, say $12|3|4$, and

$$\langle 12|3|4, \mathrm{LHS}\rangle_{\mathsf{t}} = 1 + 1 + 0 + 0 = 0 + 1 + 1 = \langle 12|3|4, \mathrm{RHS}\rangle_{\mathsf{t}}.$$

For partitions ρ with other numbers of parts, $\langle \rho, \mathrm{LHS}\rangle_{\mathsf{t}} = 0 = \langle \rho, \mathrm{RHS}\rangle_{\mathsf{t}}$, since from Equation (4) $\langle \rho, \tau\rangle_{\mathsf{t}} = 0$ whenever $\#\{\text{parts of } \rho\} + \#\{\text{parts of } \tau\} \neq \#\text{items} + 1$. $\square$

As we shall see, this lemma, together with the following two lemmas, which show how to adjoin new parts and new items to the partitions of the left- and right-hand sides of an equivalence ($\overset{\mathsf{t}}{\equiv}$), will allow us to write any partition as an equivalent ($\overset{\mathsf{t}}{\equiv}$) sum of planar partitions.

Lemma 2.4. *Suppose $n \geq 2$, τ is a partition of $1, \ldots, n-1$, and $\tau \overset{\mathsf{t}}{\equiv} \sum_\sigma \alpha_\sigma \sigma$. Then*

$$\tau|n \overset{\mathsf{t}}{\equiv} \sum_\sigma \alpha_\sigma \sigma|n.$$

Proof. If $\{n\}$ is a part of ρ, then $\langle \rho, \tau|n\rangle_{\mathsf{t}} = 0 = \langle \rho, \mathrm{RHS}\rangle_{\mathsf{t}}$. Otherwise $\langle \rho, \tau|n\rangle_{\mathsf{t}} = \langle \rho \smallsetminus n, \tau\rangle_{\mathsf{t}} = \sum_\sigma \alpha_\sigma \langle \rho \smallsetminus n, \sigma\rangle_{\mathsf{t}} = \sum_\sigma \alpha_\sigma \langle \rho, \sigma|n\rangle_{\mathsf{t}}$. $\square$

If τ is a partition of $1, \ldots, n-1$ and $j \in \{1, \ldots, n-1\}$, we can insert a new item n into the part of τ that contains item j. We refer to the resulting partition on $1, \ldots, n$ as "τ with n inserted into j's part."

Lemma 2.5. *Suppose $n \geq 2$, τ is a partition of $1, \ldots, n-1$, $j \in \{1, \ldots, n-1\}$, and $\tau \overset{\mathsf{t}}{\equiv} \sum_\sigma \alpha_\sigma \sigma$. Then*

$$[\tau \text{ with } n \text{ inserted into } j\text{'s part}] \overset{\mathsf{t}}{\equiv} \sum_\sigma \alpha_\sigma [\sigma \text{ with } n \text{ inserted into } j\text{'s part}].$$

Proof. If j and n are in the same part of partition ρ, as well as being in the same part of partition π, then by (4) $\langle \rho, \pi\rangle_{\mathsf{t}} = 0$. Thus $\langle \rho, [\tau \text{ with } n \text{ inserted into } j\text{'s part}]\rangle_{\mathsf{t}} = 0 = \langle \rho, \mathrm{RHS}\rangle_{\mathsf{t}}$. If j and n are in separate parts of ρ, then let ρ' denote the partition obtained from ρ by merging the two parts containing j and n, and then deleting n. By (4) we have

$$\langle \rho, [\tau \text{ with } n \text{ inserted into } j\text{'s part}]\rangle_{\mathsf{t}} = \langle \rho', \tau\rangle_{\mathsf{t}} = \sum_\sigma \alpha_\sigma \langle \rho', \sigma\rangle_{\mathsf{t}} = \langle \rho, \mathrm{RHS}\rangle_{\mathsf{t}}. \quad \square$$

Lemmas 2.3, 2.4, and 2.5 imply that the left-hand side and right-hand side of transformation Rule 1 are equivalent ($\overset{\mathsf{t}}{\equiv}$).

Theorem 2.6. *For any partition τ, there is an equivalent ($\overset{\mathsf{t}}{\equiv}$) integer linear combination of planar partitions $\sum_\sigma \alpha_\sigma \sigma$ (i.e., where each $\alpha_\sigma \in \mathbb{Z}$).*

Proof. We prove this by induction on the number of items in the partition. The theorem is true for $n \le 3$ since each such partition τ is already planar (and with Lemma 2.3, we see that it is also true for $n = 4$). Suppose τ contains more items. If $\{n\}$ is a part of τ, then we may use the induction hypothesis, together with Lemma 2.4, to find the desired linear combination of planar partitions. Otherwise, item n is in the same part as some other item j (if there is more than one choice of j, it does not matter which one we pick). By the induction hypothesis, we may write $\tau \setminus n \overset{\mathsf{t}}{\equiv}$ an integer linear combination of planar partitions, and by Lemma 2.5, we may write τ as an integer linear combination of "almost planar" partitions, by which we mean partitions that would be planar if the item n were deleted from them.

Next we use Lemmas 2.3, 2.4, 2.5 to express an almost-planar partition μ as an equivalent integer linear combination of planar partitions. We shall use induction on the number k of parts of μ that cross the chord from j to n. There is nothing to show if $k = 0$, and otherwise we consider the part S of μ closest to j that crosses the chord from j to n. We let $a = \{i \in S : i < j\}$ and $c = \{i \in S : i > j\}$, both of which are nonempty, $b =$ the part of μ containing j, and $d = \{n\}$. Let $a_0 \in a$ and $c_0 \in c$. From Lemma 2.3, upon relabeling $1 \to a_0$, $2 \to j$, $3 \to c_0$, and $4 \to n$, we get

$$a_0, c_0|j, n \overset{\mathsf{t}}{\equiv} a_0|j, c_0, n + j|a_0, c_0, n + c_0|a_0, j, n + n|a_0, j, c_0 - a_0, j|c_0, n - a_0, n|j, c_0.$$

Using Lemma 2.5 we may insert the rest of a into the parts containing a_0, the rest of c into the parts containing c_0, and the rest of b into the parts containing j, to obtain

$$\begin{aligned} a \cup c|b \cup d \overset{\mathsf{t}}{\equiv} a|b \cup c \cup d + b|a \cup c \cup d + c|a \cup b \cup d \\ + d|a \cup b \cup c - a \cup b|c \cup d - a \cup d|b \cup c. \end{aligned}$$

By further application of Lemmas 2.4 and 2.5, we may adjoin each of the remaining parts of μ to the left-hand side (thereby obtaining μ) and to each partition on the right-hand side. Of the resulting partitions on the right-hand side, the fourth one is planar, and the other partitions are almost planar with $k - 1$ parts crossing the chords from n to the rest of n's part. □

For a partition τ, let $\sum_\sigma \alpha_\sigma \sigma$ be any linear combination of planar partitions equivalent ($\overset{\mathsf{t}}{\equiv}$) to τ. Now $\sum_\sigma (\mathcal{P}^{(\mathsf{t})}_{\sigma,\tau} - \alpha_\sigma)\sigma$ lies in the null-space of $\mathcal{M}_\mathsf{t}$, but $\mathcal{M}_\mathsf{t}$ is nonsingular, so $\mathcal{P}^{(\mathsf{t})}_{\sigma,\tau} = \alpha_\sigma$ for each σ. In particular, the linear combination promised by Theorem 2.6 is unique and gives the τth column of $\mathcal{P}^{(\mathsf{t})}$. This linear combination was obtained by repeated application of Rule 1, which completes the proof of Theorem 1.2, which in turn implies the second part of Theorem 1.1 as a corollary.

We remark that the entries of the matrix $\mathcal{P}^{(\mathsf{t})}$ are all 0 or ± 1 for $n \le 7$ nodes, but that when $n \ge 8$ other integers appear.

2.4. **The R polynomials.** In this subsection we prove the first part of Theorem 1.1, i.e., we show that $\overline{\mathrm{Pr}}(\sigma)$ is a certain polynomial in the $R_{i,j}$'s for each planar partition σ. We start with some elementary characterizations of the $R_{i,j}$ and $L_{i,j}$ variables in terms of groves.

Proposition 2.7. *The resistance between i and j is*

$$R_{i,j} = \sum_{A: i \in A, j \in A^c} \overline{\mathrm{Pr}}(A|A^c).$$

Proof. If nodes i and j were the only two nodes, then we know from Kirchhoff's formula (the case $n = 2$) that $R_{i,j}$ is the ratio of the probability that i and j are in different components of a 2-component random grove, to the probability of a 1-component grove. When there are other nodes, these 2-component groves necessarily take the form $A|A^c$ where $i \in A$ and $j \in A^c$. (See the top half of Figure 6.) □

The following proposition may be derived from Lemma 4.1 of [CIM98], but for the reader's convenience we provide a short proof.

Proposition 2.8. *For $i \neq j$,*

$$L_{i,j} = \ddot{\mathrm{Pr}}(i, j|\text{rest singletons}),$$

i.e., $L_{i,j} = \ddot{\mathrm{Pr}}(\sigma)$, where σ is the partition in which every part is a singleton except for the part $\{i, j\}$.

Proof. Recall from the construction in Theorem 2.6 that whenever a partition τ is expressed as an equivalent sum of planar partitions, each of the partitions has the same number of parts as τ. Since σ has $n-1$ parts and any partition τ with $n-1$ parts is already planar, the σth row of $\mathcal{P}^{(\mathsf{t})}$ is nonzero only in the σth column, so $\ddot{\mathrm{Pr}}(\sigma) = \vec{L}_\sigma = L_{i,j}$. □

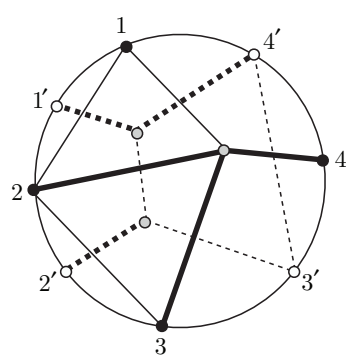


FIGURE 5. Shown here is a circular planar graph $\mathcal{G}$ with four nodes (shown in black) and one inner vertex, with edges shown as solid lines. The dual circular planar graph $\mathcal{G}^*$ has four nodes (shown in white) and two inner vertices, with edges shown as dashed lines. Also shown is a grove of $\mathcal{G}$ of type $1|234$ (the edges of the grove are shown in bold) and its dual grove of $\mathcal{G}^*$ (with edges also shown in bold), which has type $14|2|3$.

Any circular planar graph $\mathcal{G}$ has a dual circular planar graph $\mathcal{G}^*$, as shown in Figure 5. The dual $\mathcal{G}^*$ has an inner vertex for every bounded face of $\mathcal{G}$ and a node numbered i between consecutive nodes $i, i+1 \bmod n$ of $\mathcal{G}$. For each edge of $\mathcal{G}$ there is a dual edge of $\mathcal{G}^*$ whose conductance is, by definition, the reciprocal of the conductance on the corresponding edge of $\mathcal{G}$. For each grove of $\mathcal{G}$ there is a dual grove of $\mathcal{G}^*$ formed from the duals of the edges of $\mathcal{G}$ not contained in the grove (see Figure 5). A grove in $\mathcal{G}$ has weight equal to the weight of its dual grove in $\mathcal{G}^*$, times the product of the conductances of all edges in $\mathcal{G}$.

Proposition 2.9. *For the dual graph we have*

$$L^*_{i,j} = \tfrac{1}{2}R_{i,j} + \tfrac{1}{2}R_{i+1,j+1} - \tfrac{1}{2}R_{i,j+1} - \tfrac{1}{2}R_{i+1,j} \tag{7}$$

and

$$R^*_{i,j} = \sum_{\substack{1\le i'<j'\le n \\ \textit{chord } i',j' \textit{ crosses dual chord } i,j}}\sum L_{i',j'}. \tag{8}$$

Proof. By Proposition 2.7, $R^*_{i,j} = \sum_{A: i\in A, j\in A^c} \overline{\Pr}^*(A|A^c)$. But dual nodes i and j are in opposite parts of $A|A^c$ if and only if the partition dual to $A|A^c$ is $i', j'|$rest singleton for some chord i', j' crossing dual chord i, j (see Figure 6); applying Proposition 2.8 then yields (8). Expanding the right-hand side of (7) using (8) yields the left-hand side of (7). □

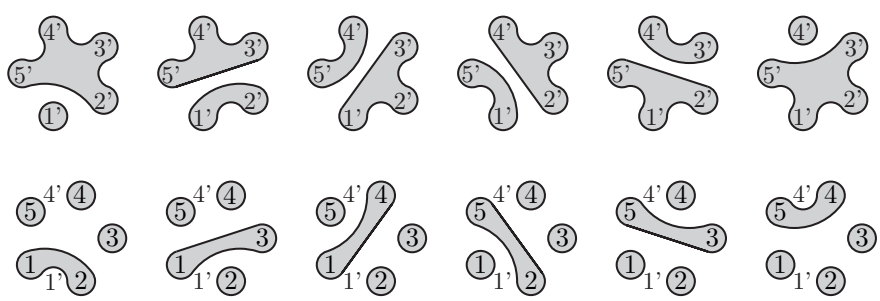


FIGURE 6. $R_{1,4}$ can be expressed as a sum of 2-part partitions in which 1 and 4 are in different parts. For the dual resistance $R^*_{1,4}$ in the dual graph, the sum over dual partitions becomes a sum over partitions consisting of a doubleton part and the rest singletons, where the doubleton part separates dual nodes 1 and 4.

Proof of first part of Theorem 1.1. Since the dual of a partition σ is a partition σ^* on the dual graph and since each grove has the same weight as its dual grove, times a constant, we have

$$\frac{\Pr(\sigma)}{\Pr(\text{tree})} = \frac{\Pr(\sigma^*)}{\Pr(\text{dual uncrossing})},$$

which by the second part of Theorem 1.1 is an integer-coefficient polynomial in the $L^*_{i,j}$'s, which by Proposition 2.9 is an integer-coefficient polynomial in the $R_{i,j}/2$'s. □

3. Double-dimer pairings

Recall that for our double-dimer results we assume the circular planar graph $\mathcal{G}$ is bipartite, so we may color its vertices black and white so that each edge connects a black vertex to a white vertex. It is convenient to assume that the nodes on the outer face alternate in color, so that the odd-numbered nodes are black and the even-numbered nodes are white. If the graph $\mathcal{G}$ does not satisfy this property, we can extend $\mathcal{G}$ by adjoining an extra vertex and edge with weight 1 for each node that has the wrong color (refer to Figure 7), and the double-dimer configurations of the extended graph are in one-to-one weight-preserving and connection-preserving correspondence with the double-dimer configurations of the original graph. Furthermore, the quantities $Z^{\mathsf{WB}}, Z^{\mathsf{BW}}$ as well as the variables $X_{i,j}$ are the same on this new graph as they were on $\mathcal{G}$. Henceforth we assume without loss of generality that the node colors of $\mathcal{G}$ alternate between black and white.

3.1. **Polynomiality.** Since the nodes alternate in color, Z^{BW} is the weighted sum of dimer covers of $\mathcal{G}$. Recall that Z^{DD} is the weighted sum of double-dimer covers of $(\mathcal{G}, \mathbf{N})$, where each of the terminal nodes is included in only one of the dimer coverings. Let S be a balanced subset of nodes, that is, a subset containing an equal number k of white and black nodes. Let $Z^{\mathsf{D}}(S) = Z^{\mathsf{D}}(\mathcal{G} \setminus S)$ be the weighted sum of dimer covers of $\mathcal{G} \setminus S$, and let $Z^{\mathsf{D}} = Z^{\mathsf{D}}(\emptyset) = Z^{\mathsf{BW}}$. The superposition of a dimer cover of $\mathcal{G} \setminus S$ and a dimer cover of $\mathcal{G} \setminus S^c$, where $S^c = \mathbf{N} \setminus S$, is a double-dimer cover of $(\mathcal{G}, \mathbf{N})$. In fact, we have

Lemma 3.1. *$Z^{\mathsf{D}}(S)Z^{\mathsf{D}}(S^c)$ is a sum of double-dimer configurations for all connection topologies π for which π connects no element of S to an element of S^c. That*

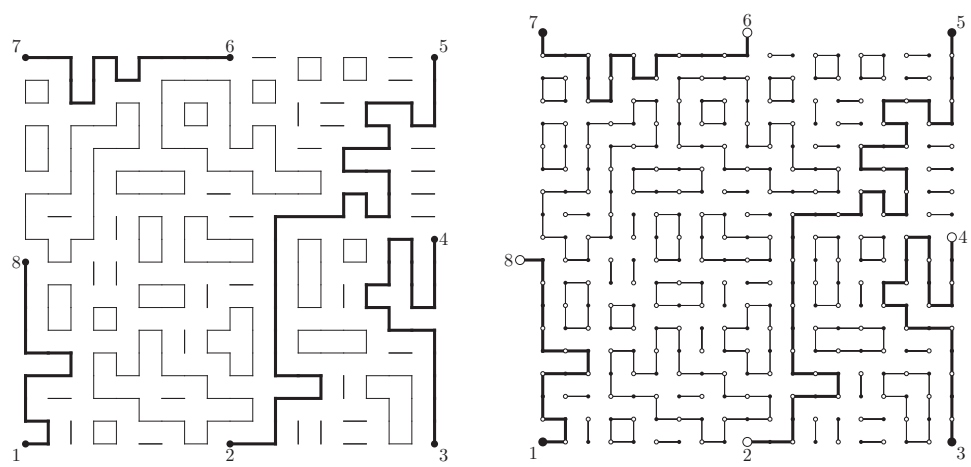


Figure 7. Double-dimer configuration on a rectangular grid with 8 terminal nodes. In this configuration the pairing of the nodes is $\{\{1,8\},\{3,4\},\{5,2\},\{7,6\}\}$, which we write as $\frac{1}{8}\,|\,\frac{3}{4}\,|\,\frac{5}{2}\,|\,\frac{7}{6}$. The double-dimer configurations on the graph on the left are in one-to-one correspondence with the double-dimer configurations of the extended graph on the right, for which the odd terminals are colored black and the even terminals are colored white.

is,

$$Z^{\mathsf{D}}(S)Z^{\mathsf{D}}(S^c) = Z^{\mathsf{DD}} \sum_{\pi} M_{S,\pi} \Pr(\pi), \tag{9}$$

where $M_{S,\pi}$ is 0 or 1 according to whether or not π connects nodes in S to S^c.

As a special case, when $S = \emptyset$ we have $Z^{\mathsf{D}}(\emptyset)Z^{\mathsf{D}}(\mathbf{N}) = Z^{\mathsf{DD}}$, which is closely related to Ciucu's graph factorization theorem [Ciu97, Thm. 1.2]. Ciucu showed how to enumerate dimer coverings in a bipartite graph with bilateral symmetry. Dimer coverings of the whole graph correspond to double-dimer configurations of one half of the graph, with vertices on the symmetry axis corresponding to nodes, except that there is an extra factor of 2 in weight for each path connecting a pair of nodes, unless the path consists of a single edge. So the graph factorization theorem contains a power of 2 for each pair of nodes and a weight of 1/2 for each edge on the symmetry axis, neither of which appear in Lemma 3.1.

Our remaining double-dimer results only make sense if there are double-dimer configurations of $\mathcal{G}$, which implies that there are dimer configurations of $\mathcal{G}$, so that $\mathcal{G}$ has equal numbers of black and white vertices. After proving this lemma, we shall henceforth make this assumption.

Proof of Lemma 3.1. Each double-dimer path connecting a pair of nodes separates an even number of nodes on its two sides, and since the white and black nodes alternate around the boundary, each double-dimer path must go from a node to a node of the opposite color.

Consider the double-dimer cover of $(\mathcal{G}, \mathbf{N})$ formed by the superposition of a dimer cover of $\mathcal{G} \setminus S$ and a dimer cover of $\mathcal{G} \setminus S^c$. On the double-dimer path starting from a white node of S, every dimer from a white vertex to a black vertex is from the second dimer cover and every dimer from a black vertex to a white vertex is from the first cover. So such a path necessarily ends at a black vertex in S. Similarly a path from a black vertex in S necessarily ends at a white vertex in S.

Conversely, any double-dimer configuration with no connections from S to S^c can be decomposed into a dimer cover of $\mathcal{G} \setminus S$ and a dimer cover of $\mathcal{G} \setminus S^c$. There are 2^ℓ choices of such a decomposition, 2 choices for each of the ℓ closed loops. □

Before proceeding with the proofs of Theorems 1.4 and 1.3, we recall some basic facts about Kasteleyn matrices. Kasteleyn matrices may be used to enumerate dimer coverings of any planar graph [Kas67], but here we review just the bipartite case. A Kasteleyn matrix of an edge-weighted bipartite planar graph (with a given embedding in the plane) is defined to be a signed adjacency matrix, $K = (K_{w,b})$, with rows indexed by the white vertices and columns indexed by the black vertices, satisfying the following: $K_{w,b}$ is $\pm$ the weight on edge wb and 0 if there is no edge. The signs of the edges are chosen so that around each face there are an odd number of $-$ signs if the face has 0 mod 4 edges and an even number of $-$ signs if the face has 2 mod 4 edges. Kasteleyn [Kas67] showed that every bipartite plane graph with an even number of vertices has a Kasteleyn matrix, and if there are equal numbers of black and white vertices, then $|\det K|$ is the weighted sum of all dimer coverings, where the weight of a dimer covering is the product of its edge weights.

From Kasteleyn theory [Kas67], [Ken97] it is straightforward to compute $Z^{\mathsf{D}}(S)$, though some work is needed to get all the signs right. But the signs for $Z^{\mathsf{D}}(S)Z^{\mathsf{D}}(S^c)$

are simpler, and this product is all we need anyway. Recall that $X_{i,j} = Z^{\mathsf{D}}(\{i,j\})/Z^{\mathsf{D}}$.

Lemma 3.2. *Let S be a balanced subset of $\{1,\dots,2n\}$. Then*

(10)
$$Z^{\mathsf{D}}(S)Z^{\mathsf{D}}(S^c)/(Z^{\mathsf{D}})^2 = \det\left[(1_{i,j\in S} + 1_{i,j\notin S}) \times (-1)^{(|i-j|-1)/2} X_{i,j}\right]_{j=2,4,\dots,2n}^{i=1,3,\dots,2n-1}.$$

Proof. For convenience we adjoin to the graph $\mathcal{G}$ $2n$ edges along the outer face connecting adjacent terminal nodes and give these edges weight 0 (or weight ε and then take the limit $\varepsilon \downarrow 0$). Given a Kasteleyn matrix of a graph, the signs of edges incident to a vertex may be reversed, and each face will still have a correct number of minus signs. For each $i = 1,\dots,2n-1$, in order, if the edge from node i to node $i+1$ has a minus sign, let us reverse the signs of all edges incident to node $i+1$. Doing this ensures that for $1 \le i \le 2n-1$, the sign of the edge from node i to node $i+1$ is positive. The sign of the remaining adjoined edge, from node $2n$ to 1, will necessarily be $-(-1)^n$ for the outer face to have a correct number of minus signs.

Let $(w_1,b_1),\dots,(w_k,b_k)$ be any noncrossing pairing of the nodes of S, where $w_1,\dots,w_k$ are the white nodes of S and $b_1,\dots,b_k$ are the black nodes of S. Let us adjoin edges of weight W to the outer face connecting w_i to b_i for $1 \le i \le k$. To retain the Kasteleyn sign condition, the sign of a new edge connecting black node b and white node w will be $(-1)^{(|b-w|-1)/2}$. Let K_W be the Kasteleyn matrix of the resulting graph, with the rows and columns ordered so that $w_1,\dots,w_k$ are the first k rows and $b_1,\dots,b_k$ are the first k columns, and let $K = K_0$ be the corresponding Kasteleyn matrix when $W = 0$. Recall that $[x^\alpha]p(x)$ denotes the coefficient of x^α in the polynomial $p(x)$. Then $Z^{\mathsf{D}}(S) = \pm[W^k]\det K_W$ and $Z^{\mathsf{D}} = \pm\det K_0$. But $\det K_W$ enumerates the weighted matchings of the enlarged graph with all the same sign, so

$$\begin{aligned}\frac{Z^{\mathsf{D}}(S)}{Z^{\mathsf{D}}} &= \frac{[W^k]\det K_W}{[W^0]\det K_W} = (-1)^{\sum_{\ell=1}^k \frac{|b_\ell - w_\ell|-1}{2}} \frac{\det K_{\setminus S}}{\det K} \\ &= (-1)^{\sum_{\ell=1}^k \frac{|b_\ell - w_\ell|-1}{2}} \det[K^{-1}_{b,w}]_{w=w_1,\dots,w_k}^{b=b_1,\dots,b_k},\end{aligned} \tag{11}$$

where $K_{\setminus S}$ denotes the submatrix of K formed by deleting the rows and columns from S, and the last equality is Jacobi's determinant identity. The special case $S = \{b,w\}$ yields

$$X_{b,w} = Z^{\mathsf{D}}(\{b,w\})/Z^{\mathsf{D}} = (-1)^{(|b-w|-1)/2} K^{-1}_{b,w}.$$

From this equation and (11) we get (10), except possibly for a global sign change.

Next we compare the sign of $Z^{\mathsf{D}}(S)Z^{\mathsf{D}}(S^c)/(Z^{\mathsf{D}})^2$ that we get from (11) to the sign in (10). In the event that $S = \{1,\dots,2k\}$, we can take $b_1,\dots,b_k = 1,3,\dots,2k-1$ and $w_1,\dots,w_k = 2,4,\dots,2k$, and for S^c we can take $b_1,\dots,b_{n-k} = 2k+1,2k+3,\dots,2n-1$ and $w_1,\dots,w_{n-k} = 2k+2,2k+4,\dots,2n$, so we see that the sign for $Z^{\mathsf{D}}(S)Z^{\mathsf{D}}(S^c)/(Z^{\mathsf{D}})^2$ from (11) agrees with (10). Now suppose that for some S the signs from (11) agree with (10), and we replace one of the nodes $s \in S$ with $s+2 \notin S$ to get S'. If $s+1 \notin S$, the power of -1 in (11) changes by 1. If the $s+1 \in S$, we may assume s was paired with $s+1$, and we see that the power of -1 in (11) does not change. Thus the power of -1 from (11) is the opposite for $Z^{\mathsf{D}}(S')Z^{\mathsf{D}}(S'^c)/(Z^{\mathsf{D}})^2$ compared to $Z^{\mathsf{D}}(S)Z^{\mathsf{D}}(S^c)/(Z^{\mathsf{D}})^2$. But the product of determinants from (11) is $\pm$ the determinant from (10), and the choice of sign is

the opposite for (S, S^c) and (S', S'^c). Thus (10) has the correct global sign for S', and hence by induction for any balanced set of nodes. □

By Lemma 3.2 the quantities $Z^{\mathsf{D}}(S)Z^{\mathsf{D}}(S^c)/(Z^{\mathsf{D}})^2$ are homogeneous polynomials of degree n (in fact determinants) in the $X_{i,j}$, so the matrix M from (9) maps the quantities $\Pr(\pi)Z^{\mathsf{DD}}/(Z^{\mathsf{D}})^2$ to homogeneous polynomials of degree n in the $X_{i,j}$. We need to show that the matrix M has full rank; that is, the rank of M is C_n, the nth Catalan number, which is the number of different planar partitions π.

Lemma 3.3. *$M^TM = \mathcal{M}_2$, the meander matrix $\mathcal{M}_q$ from* (2) [DFGG97] *evaluated at $q = 2$.*

Proof. Given two planar matchings π, σ, the integer $\delta_\sigma M^T M \delta_\pi$ is the number of subsets S with the property that neither π nor σ connects a node in S to a node in S^c.

We draw π and σ on the sphere, with π in the upper hemisphere and σ in the lower hemisphere, so that their union is a meander crossing the equator $2n$ times. Suppose this meander has k components. For each component, the nodes alternate black and white, and so the component has the same number of black nodes as white nodes. For each component we choose to put all of its nodes in S or all of its nodes in S^c. We can construct 2^k possible sets S in this way, and these are exactly the sets S for which neither π nor σ connects a node in S to a node in S^c.

Thus an entry in M^TM which corresponds to a meander with k components is 2^k. □

By [DFGG97, Eqn (5.6)] the determinant of the meander matrix at $q = 2$ is

$$\prod_{i=1}^{n}(1+i)^{a_{n,i}},$$

where $a_{n,i}$ are certain integers. In particular it is nonzero, so M has full rank, and we can solve (9) for $\Pr(\pi)Z^{\mathsf{DD}}/(Z^{\mathsf{D}})^2$ in terms of the quantities $Z^{\mathsf{D}}(S)Z^{\mathsf{D}}(S^c)/(Z^{\mathsf{D}})^2$ which by Lemma 3.2 are homogeneous polynomials in the $X_{i,j}$. This effectively proves Theorem 1.3, except for the part about the coefficients being integers.

3.2. **Integrality of the coefficients.** We start by collecting the vectors and matrices we need. Extend the Gram matrix M^TM when $q = 2$ to a $C_n \times n!$ matrix $\mathcal{E}_2$ whose rows are indexed by planar pairings and whose columns are indexed by all (not necessarily planar)n pairings connecting odd nodes to even nodes. We have the following matrices and vectors.

$$\begin{aligned}
M &= \text{matrix from balanced subsets } S \subseteq \{1,\dots,2n\} \text{ to planar pairings } \pi, \text{ from (9)},\\
\mathcal{M}_2 &= M^TM = \text{the } C_n \times C_n \text{ meander matrix (with } q=2) \text{ for planar pairings},\\
\mathcal{E}_2 &= C_n \times n! \text{ extended meander matrix (with } q=2) \text{ for odd-even pairings},\\
P &= \text{vector of normalized pairing probabilities } P_\pi = \widehat{\Pr}(\pi)\\
&= \Pr(\pi)Z^{\mathsf{D}}(\{1,\dots,2n\})/Z^{\mathsf{D}} \text{ indexed by planar pairings } \pi,\\
D &= MP = \text{vector of products } D_S = Z^{\mathsf{D}}(S)Z^{\mathsf{D}}(S^c)/(Z^{\mathsf{D}})^2\\
&\qquad \text{indexed by balanced subsets } S \subseteq \{1,\dots,2n\},\\
X &= \text{vector of } X\text{-monomials indexed by odd-even pairings; } X_\rho = \textstyle\prod_{\{i,j\}\in\rho} X_{i,j},\\
X' &= \text{vector of } X\text{-monomials as above with sign } (-1)^{\#\text{ crosses of }\rho}.
\end{aligned}$$

Recall that we defined a cross of a pairing ρ to be a set of two parts $\{a, c\}$ and $\{b, d\}$ of ρ such that $a < b < c < d$. We define the parity of an odd-even pairing $\rho = {}^{1}_{w_1} \,|\, {}^{3}_{w_2} \,|\, \cdots \,|\, {}^{2n-1}_{w_n}$ to be the parity of the permutation $(w_1/2)(w_2/2)\dots(w_n/2)$. We shall use the following fact.

Lemma 3.4. *For odd-even pairings ρ,*

$$(-1)^{\textit{parity of } \rho} \prod_{(i,j)\in\rho} (-1)^{(|i-j|-1)/2} = (-1)^{\#\textit{ crosses of } \rho}. \tag{12}$$

Proof. When $\rho = {}^{1}_{2} \,|\, {}^{3}_{4} \,|\, \cdots$ both sides of the above equation are $+1$. Now suppose we do a transposition, swapping the locations of w and $w+2$. (Such swaps are enough to connect the set of odd-even pairings.) In the event that one of w or $w+2$ is paired with $w+1$, such a swap will not change the sign of the left-hand side (it changes the sign of exactly one term in the product and also the parity of ρ), nor does it change the number of chords that cross. Otherwise the sign of the left-hand side does change (exactly two terms of the product change sign, and the parity of ρ also changes). Also, the number of chords crossing $w+1$'s chord changes by 0 or ± 2, and the chords containing w and $w+2$ now cross if they didn't before, and vice versa. □

Lemma 3.5. $M^T D = \mathcal{E}_2 X'$.

Proof. Recall from Lemma 3.2 that

$$D_S = \frac{Z^{\mathsf{D}}(S) Z^{\mathsf{D}}(S^c)}{(Z^{\mathsf{D}})^2} = \det\left[(1_{i,j\in S} + 1_{i,j\notin S}) \times (-1)^{(|i-j|-1)/2} X_{i,j}\right]_{j=2,4,\dots,2n}^{i=1,3\dots,2n-1}.$$

When we expand the determinant, we get

$$D_S = \sum_{\substack{\textit{odd-even pairings } \rho \\ \rho \textit{ does not bridge } S \textit{ to } S^c}} \underbrace{(-1)^{\textit{parity of } \rho} \prod_{(i,j)\in\rho} (-1)^{(|i-j|-1)/2} X_{i,j}}_{= X'_\rho \text{ by (12)}}.$$

Let π be a planar pairing. Upon summing over sets S that are not bridged by π we get

$$\begin{aligned}
\sum_{\substack{S\subseteq\{1,\dots,2n\} \\ \pi \textit{ does not bridge } S \textit{ to } S^c}} D_S &= \sum_{\substack{\textit{odd-even pairings } \rho, \\ S\colon \pi,\rho \textit{ do not bridge } S, S^c}} X'_\rho \\
&= \sum_{\substack{\textit{odd-even} \\ \textit{pairings } \rho}} \sum_{\substack{S:\pi,\rho \textit{ do not} \\ \textit{bridge } S}} X'_\rho \\
&= \sum_{\substack{\textit{odd-even} \\ \text{pairings } \rho}} 2^{\#\textit{ cycles in } \rho\cup\pi} X'_\rho.
\end{aligned}$$

The left-hand side is the πth row of $M^T D$, and the right-hand side is the πth row of $\mathcal{E}_2 X'$. □

Theorem 3.6. $\mathcal{M}_2 P = \mathcal{E}_2 X'$.

Proof. Since $MP = D$, we have $\mathcal{M}_2 P = M^T M P = M^T D = \mathcal{E}_2 X'$. □

Since $\mathcal{M}_2$ is invertible, we may define

$$\mathcal{P}^{(\mathsf{DD})} = \mathcal{M}_2^{-1}\mathcal{E}_2.$$

Since $P = \mathcal{P}^{(\mathsf{DD})}X'$, we can interpret $\mathcal{P}^{(\mathsf{DD})}$ as the matrix of coefficients of the X' polynomials: for a given planar pairing σ, the σth row of $\mathcal{P}^{(\mathsf{DD})}$ gives the polynomial $\widehat{\Pr}(\sigma)$. The τth column of $\mathcal{P}^{(\mathsf{DD})}$ gives a linear combination of planar pairings that is in a sense equivalent under $\langle,\rangle_2$ to τ: for any *planar* pairing ρ,

$$\left\langle \rho, \sum_\sigma \mathcal{P}^{(\mathsf{DD})}_{\sigma,\tau}\sigma \right\rangle_2 = \sum_\sigma \mathcal{P}^{(\mathsf{DD})}_{\sigma,\tau}(\mathcal{M}_2)_{\rho,\sigma} = (\mathcal{M}_2\mathcal{P}^{(\mathsf{DD})})_{\rho,\tau} = (\mathcal{E}_2)_{\rho,\tau} = \langle \rho,\tau\rangle_2.$$

(We shall see $\left\langle \rho, \sum_\sigma \mathcal{P}^{(\mathsf{DD})}_{\sigma,\tau}\sigma \right\rangle_2 = \langle \rho,\tau\rangle_2$ for nonplanar odd-even pairings ρ, too.)

We say that two linear combinations of odd-even pairings $\sum_\tau \alpha_\tau \tau$ and $\sum_\tau \beta_\tau \tau$ are equivalent ($\overset{q}{\equiv}$) if for any (not necessarily planar) odd-even pairing ρ, we have $\sum_\tau \alpha_\tau \langle \rho,\tau\rangle_q = \sum_\tau \beta_\tau \langle \rho,\tau\rangle_q$. Following the approach we used in Section 2.3 for groves, we will show how to transform an odd-even pairing τ into an equivalent ($\overset{2}{\equiv}$) linear combination of planar pairings, and thereby compute $\mathcal{P}^{(\mathsf{DD})}$ using only integer operations, i.e. without inverting $\mathcal{M}_2$.

The following key lemma is analogous to Lemma 2.3:

Lemma 3.7. $\begin{smallmatrix}1\\2\end{smallmatrix}\,|\,\begin{smallmatrix}3\\4\end{smallmatrix}\,|\,\begin{smallmatrix}5\\6\end{smallmatrix} + \begin{smallmatrix}1\\4\end{smallmatrix}\,|\,\begin{smallmatrix}3\\6\end{smallmatrix}\,|\,\begin{smallmatrix}5\\2\end{smallmatrix} + \begin{smallmatrix}1\\6\end{smallmatrix}\,|\,\begin{smallmatrix}3\\2\end{smallmatrix}\,|\,\begin{smallmatrix}5\\4\end{smallmatrix} \overset{2}{\equiv} \begin{smallmatrix}1\\4\end{smallmatrix}\,|\,\begin{smallmatrix}3\\2\end{smallmatrix}\,|\,\begin{smallmatrix}5\\6\end{smallmatrix} + \begin{smallmatrix}1\\2\end{smallmatrix}\,|\,\begin{smallmatrix}3\\6\end{smallmatrix}\,|\,\begin{smallmatrix}5\\4\end{smallmatrix} + \begin{smallmatrix}1\\6\end{smallmatrix}\,|\,\begin{smallmatrix}3\\4\end{smallmatrix}\,|\,\begin{smallmatrix}5\\2\end{smallmatrix}$.

Proof. For any odd-even pairing ρ, we have $\langle \rho, \mathrm{LHS}\rangle_2 = 12 = \langle \rho, \mathrm{RHS}\rangle_2$. For example, if $\rho = \begin{smallmatrix}1\\4\end{smallmatrix}\,|\,\begin{smallmatrix}3\\2\end{smallmatrix}\,|\,\begin{smallmatrix}5\\6\end{smallmatrix}$, then $\langle \rho, \mathrm{LHS}\rangle_2 = 2^2 + 2^2 + 2^2 = 12$, while $\langle \rho, \mathrm{RHS}\rangle_2 = 2^3 + 2^1 + 2^1 = 12$. For the other odd-even pairings ρ on $\{1,\dots,6\}$, it is similarly straightforward to verify $\langle \rho, \mathrm{LHS}\rangle_2 = \langle \rho, \mathrm{RHS}\rangle_2$ because $2^1 + 2^1 + 2^3 = 2^2 + 2^2 + 2^2$. □

Lemma 3.7 is a special case of Lemma 3.8 below. Though we do not need the extra generality of Lemma 3.8 for our results, some readers may find its proof more instructive.

Lemma 3.8. *If n is a positive integer, then*

$$\sum_{\textit{odd-even pairings } \sigma \textit{ on } 2n \textit{ items}} (-1)^\sigma \sigma \overset{q}{\equiv} 0$$

if and only if q is an integer satisfying $0 \le q < n$.

Proof. If σ is a permutation on $\{1,2,\dots,n\}$, we can interpret σ as the odd-even pairing $\begin{smallmatrix}1\\2\sigma_1\end{smallmatrix}\,|\,\begin{smallmatrix}3\\2\sigma_2\end{smallmatrix}\,|\cdots\begin{smallmatrix}2n-1\\2\sigma_n\end{smallmatrix}$, and vice versa. Note that if ρ and σ are two odd-even pairings, then the cycles in the union of ρ and σ are the cycles of $\rho^{-1}\sigma$ when ρ and σ are interpreted as permutations, i.e., $\langle \rho,\sigma\rangle_q = q^{\#\text{ cycles in }\rho^{-1}\sigma}$. Note also that $(-1)^\sigma = (-1)^n(-1)^\rho(-1)^{\#\text{ cycles in }\rho^{-1}\sigma}$. Thus

$$\left\langle \rho, \sum_{\textit{odd-even pairings } \sigma \textit{ on } 2n \textit{ items}} (-1)^\sigma \sigma \right\rangle_q = (-1)^n(-1)^\rho \sum_{k=1}^n c(n,k)(-q)^k,$$

where the $c(n,k)$ are the unsigned Stirling numbers of the first kind, which count the number of permutations on n letters that have k cycles. It is well known (see e.g. [Sta86, Proposition 1.3.4]) that $\sum_k c(n,k)x^k = x(x+1)(x+2)\cdots(x+n-1)$. □

The next lemma is analogous to Lemmas 2.4 and 2.5.

Lemma 3.9. *Suppose $n \geq 2$, τ is an odd-even pairing of $1, 2, \ldots, 2(n-1)$, and $\tau \overset{q}{\equiv} \sum_\sigma \alpha_\sigma \sigma$. Then*

$$\tau \,|\, {}^{2n-1}_{2n} \overset{q}{\equiv} \sum_\sigma \alpha_\sigma \sigma \,|\, {}^{2n-1}_{2n}.$$

Proof. If $\{2n-1, 2n\}$ is a part of both odd-even pairings ρ and π, then let ρ' and π' denote the odd-even pairings obtained by deleting this part. Then $\langle \rho, \pi \rangle_q = q\langle \rho', \pi' \rangle_q$, so $\langle \rho, \mathrm{LHS} \rangle_q = \langle \rho, \mathrm{RHS} \rangle_q$. Now suppose that $\{2n-1, 2n\}$ is a part of π but not ρ, and that instead $\{2n-1, w\}$ and $\{b, 2n\}$ are parts of ρ. Let ρ' denote the odd-even pairing obtained from ρ by deleting these two parts and replacing them with $\{b, w\}$, and let π' be as above. Then $\langle \rho', \pi' \rangle_q = \langle \rho, \pi \rangle_q$, so $\langle \rho, \mathrm{LHS} \rangle_q = \langle \rho, \mathrm{RHS} \rangle_q$. □

Lemmas 3.7 and 3.9 imply that the left-hand side and right-hand side of transformation Rule 2 are equivalent ($\overset{2}{\equiv}$).

Theorem 3.10. *For any odd-even pairing τ, there is an equivalent integer-linear combination of planar pairings $\tau \overset{2}{\equiv} \sum_\sigma \alpha_\sigma \sigma$, with $\alpha_\sigma \in \mathbb{Z}$.*

Proof. We start with a particular planar pairing, say $\pi = \frac{1}{2} \,|\, \frac{3}{4} \,|\, \frac{5}{6} \,|\, \cdots$. We have the equivalence $\pi \overset{2}{\equiv} \pi$. We shall then make modifications on the left side, doing adjacent transpositions of even labels and making the same modification on the right, and then retransform the right side into a combination of planar pairings. Eventually we have converted the LHS into τ, and the RHS is an integer linear combination of planar pairings.

Suppose we swap the labels b and $b+2$. Assuming that the RHS was a linear combination of planar pairings, we need to check that after the swap the RHS can be transformed so as to again be a linear combination of planar pairings. Consider one such planar pairing of the RHS. If $b+1$ were paired with one of b or $b+2$, then after the swap the result is still planar. Otherwise the three chords containing b, $b+1$, and $b+2$ form three parallel crossings. These divide the remaining vertices into four contiguous regions so that remaining chords only connect vertices in the same region. After the $b, b+2$ swap, the three parallel crossings form an asterisk. The asterisk can be transformed using Rule 2 (Lemmas 3.7 and 3.9). None of the remaining chords cross the newly transformed chords, so once they are added back in, the results are planar. □

For an odd-even pairing τ, let $\sum_\sigma \alpha_\sigma \sigma$ be any linear combination of planar pairings equivalent ($\overset{2}{\equiv}$) to τ. Now $\sum_\sigma (\mathcal{P}^{(\mathsf{DD})}_{\sigma,\tau} - \alpha_\sigma)\sigma$ lies in the null-space of $\mathcal{M}_2$, but $\mathcal{M}_2$ is nonsingular, so $\mathcal{P}^{(\mathsf{DD})}_{\sigma,\tau} = \alpha_\sigma$ for each σ. In particular, the linear combination promised by Theorem 3.10 is unique and gives the τth column of $\mathcal{P}^{(\mathsf{DD})}$. This linear combination was obtained by the repeated application of Rule 2, which completes the proof of Theorem 1.4, which implies Theorem 1.3 as a corollary.

4. Comparing grove and double-dimer polynomials

As we shall see in this section, the grove partition "L" polynomials and double-dimer pairing "X" polynomials are closely related to each other. In fact, the "X" polynomials are a specialization of the "L" polynomials. We start with a lemma about groves.

Lemma 4.1. *If a planar partition σ contains only singleton and doubleton parts, and σ' is the partition obtained from σ by deleting all the singleton parts, then the "L" polynomials for σ and σ' are lexicographically equal. (We say "lexicographically equal" rather than "equal" because the formulas look the same even though the underlying "L variables" represent different electrical quantities.)*

For example, $\ddot{\mathrm{Pr}}(13|2|4|5|68|7|9) = L_{1,3}L_{6,8} - L_{1,6}L_{3,8}$ and $\ddot{\mathrm{Pr}}(13|68) = L_{1,3}L_{6,8} - L_{1,6}L_{3,8}$, but the L's mean different things in these two equations, since they are the current responses when different numbers of nodes are held at zero volts. Note that for general partitions we cannot simply drop the singleton parts. For example, $\ddot{\mathrm{Pr}}(123) = L_{1,2}L_{2,3} + L_{1,2}L_{1,3} + L_{1,3}L_{2,3}$, while $\ddot{\mathrm{Pr}}(123|4)$ has these three terms together with a fourth term, namely $L_{1,3}L_{2,4}$.

Proof. Recall the algorithm that computes columns of the projection matrix by finding for a given partition τ an equivalent linear combination of planar partitions (the proof of Theorem 2.6). Each partition in the result will have the same number of parts as τ, and every singleton of τ is a singleton of each partition in the combination. If we are only interested in the σth row of the projection matrix, where σ has k parts of size 2 and the rest singletons, then we need only consider columns τ for which τ has $n-k$ parts. If τ has any part of size 3 or more, then τ has more singletons than σ, and it does not contribute to σ's row. The only partitions τ that contribute have k parts of size 2 and the same singletons as σ. For such partitions τ, we may drop the singleton parts, find the equivalent linear combination of planar partitions, and then re-adjoin the singleton parts. Thus the computation of $\ddot{\mathrm{Pr}}(\sigma)$ precisely mirrors the computation of $\ddot{\mathrm{Pr}}(\sigma')$. □

The following theorem is a reformulation of Theorem 1.5.

Theorem 4.2. *If a planar partition σ contains only doubleton parts and we make the following substitutions to the grove partition polynomial $\ddot{\mathrm{Pr}}(\sigma)$:*

$$L_{i,j} \to \begin{cases} 0, & \text{if } i \text{ and } j \text{ have the same parity,} \\ (-1)^{(|i-j|-1)/2} X_{i,j}, & \text{otherwise,} \end{cases}$$

then the result is $(-1)^\sigma$ times the double-dimer pairing polynomial $\widehat{\mathrm{Pr}}(\sigma)$ when we interpret σ as a pairing, and $(-1)^\sigma$ is the signature of the permutation $\sigma_1, \sigma_3, \ldots, \sigma_{2n-1}$.

Proof. Consider the computation of $\ddot{\mathrm{Pr}}(\sigma)$ when the planar partition σ contains only doubleton parts. When we express a partition τ as a linear combination of planar partitions, any singleton parts of τ show up in each planar partition (with nonzero coefficient), so if τ contains singleton parts, $\mathcal{P}^{(\mathsf{t})}_{\sigma,\tau} = 0$. When we transform partitions according to Rule 1, the number of parts is conserved, so partitions that end up contributing to the σth row of the projection matrix will contain only doubleton parts. Rule 1 transforms a partition into a combination of six partitions, but the first four of them have the wrong part sizes to contribute to $\mathcal{P}^{(\mathsf{t})}_{\sigma,\tau}$, so we may keep only the last two. In other words, we may use the abbreviated transformation rule

$$13|24 \to -12|34 - 14|23.$$

Let us transform 14|25|36 using this abbreviated rule:

$$\begin{aligned} 14|25|36 &\to -12|45|36 - 15|24|36 \\ &\to -12|45|36 + 15|23|46 + 15|26|34 \\ &\to -12|45|36 - 14|23|56 - 16|23|45 - 16|25|34 - 12|56|34. \end{aligned}$$

Compare this to the corresponding rule for double-dimer pairings:

$$14|25|36 \to +12|45|36 + 14|23|56 - 16|23|45 + 16|25|34 - 12|56|34.$$

The transformation rules are the same except for a sign factor equal to the signature of the pairing. Thus $\mathcal{P}^{(\mathsf{DD})}_{\sigma,\tau} = (-1)^{\sigma^{-1}\tau}\mathcal{P}^{(\mathsf{t})}_{\sigma,\tau}$. Recall now that the double-dimer projection matrix gave coefficients for monomials weighted by $(-1)^{\#\text{ pairs that cross}}$, and that this sign was by (12) equal to the signature of the pairing times the product over pairs (i,j) in the pairing of $(-1)^{(|i-j|-1)/2}$. □

5. Scaling limits and multichordal SLE

The results in the preceding sections are all exact results that hold for any planar graph. In this section we study the asymptotic behavior of the partition probabilities when the graphs approximate a fine lattice restricted to a domain in the plane. We consider two different limits for groves; the first one gives a multichordal loop-erased random walk, which in the limit converges to multichordal SLE_2, while the second one in the limit converges to multichordal SLE_8. We also obtain the limiting pairing probabilities of the paths in the double-dimer model, the limiting connection probabilities for the contour lines of the discrete Gaussian free field with certain boundary conditions, and in particular show that they are equal. The latter of these two models converges to multichordal SLE_4 [SS06], [SS07].

Dubédat [Dub06] has also studied the pairing probabilities in multichordal SLE_κ. We only treat the cases $\kappa = 2, 4, 8$, while Dubédat's calculations are relevant for the continuous range $0 \le \kappa \le 8$. On the other hand, our calculations are more explicit. For example, Dubédat's solution is actually a multiparameter family of solutions that solve a certain PDE. (It is not known *a priori* that the solutions in [Dub06] span the solution space, though for a given number of strands this can be verified *a posteriori*.) Each of these solutions is relevant to multichordal SLE_κ, but when there are more than two strands, it is not clear which solution to the PDE is the "canonical solution" that describes the behavior of a given discrete model such as spanning trees. Our calculations avoid that issue entirely by working directly with the discrete models.

5.1. **Multichordal loop-erased random walk and SLE(2).** We consider multichordal SLE_2 on a domain U, with m chords connecting $2m$ points. To get SLE_2 as a limit of groves, we consider groves on finer and finer planar grids approximating U, with $n = 4m$ nodes, where each of the $2m$ SLE-endpoints is associated to a node, and each of the $2m$ intervals between the SLE-endpoints is wired together and associated to a node. (See Figure 8.) The "interval" nodes are required to be in singleton parts of the corresponding grove partition, and the point nodes are required to be in doubleton parts. By Lemma 4.1, to compute the SLE_2 connection probabilities it is sufficient to compute the current responses amongst the point nodes (when the interval nodes are grounded) and substitute these values into the "L" polynomials associated with pairings of these $2m$ point nodes.

By conformal invariance, we may without loss of generality fix U to be the upper half-plane, with point nodes at $x_1, \ldots, x_{2m} \in \mathbb{R}$, and we assume that no point node is at ∞. We approximate the upper half-plane with the upper half cartesian lattice $\varepsilon\mathbb{Z} \times \varepsilon\mathbb{N}$, round x_i to the nearest lattice point $x_i^{(\varepsilon)} \in \varepsilon\mathbb{Z} \times \varepsilon\mathbb{N}$, and let each edge have unit resistance. The current response between distinct nodes $x_i^{(\varepsilon)}$ and $x_j^{(\varepsilon)}$ is (assuming the nodes are nonadjacent) the voltage at the vertex one lattice spacing above $x_j^{(\varepsilon)}$ when $x_i^{(\varepsilon)}$ is at one volt and the remainder of the real axis is at zero volts, which is the probability that a random walk starting one lattice spacing above $x_j^{(\varepsilon)}$ ends up at $x_i^{(\varepsilon)}$ when it first hits the real axis, which is

$$L_{i,j} = (1 + o(1)) \frac{1}{\pi} \frac{\varepsilon^2}{(x_i - x_j)^2}. \tag{13}$$

See e.g. [Spi76, Chapter III §15], where the $o(1)$ term goes to 0 as ε goes to 0.

When we consider the ratios of these probabilities, the ε^2/π factors drop out and we can then take the limit $\varepsilon \to 0$. We then find that $L_{i,j}$ is inversely proportional to the square of the distance between points x_i and x_j.

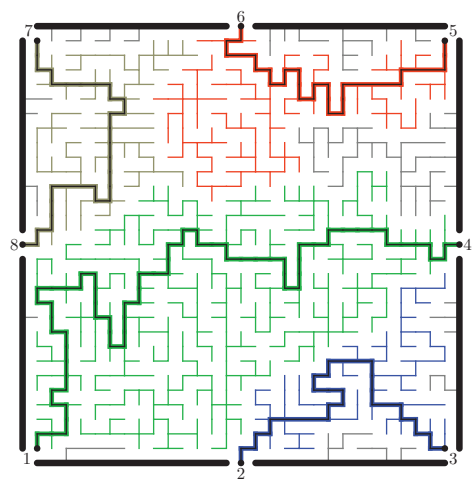


FIGURE 8. Multichordal loop-erased random walk: Shown here is a random grove with 16 nodes, conditioned to have the long extended (and unnumbered) nodes in singleton parts and the short (numbered) nodes each in a part of size 2. In this case the pairing is $\begin{smallmatrix}1\\4\end{smallmatrix}|\begin{smallmatrix}3\\2\end{smallmatrix}|\begin{smallmatrix}5\\6\end{smallmatrix}|\begin{smallmatrix}7\\8\end{smallmatrix}$, and in the scaling limit for this geometry this pairing occurs with probability $\frac{48777}{965776} - \frac{1135\sqrt{2}}{60361} \doteq 0.0239$. Within the grove the paths connecting the numbered nodes are shown in bold. Each of these connecting paths is, conditional upon the other paths, the loop-erasure of a random walk starting at one endpoint and conditioned to hit the other endpoint before hitting either the boundary or the other paths.

In the bichordal case, the normalized probabilities are

$$\begin{aligned}\ddddot{\Pr}(12|34) =& L_{12|34} - L_{13|24},\\ \ddddot{\Pr}(14|23) =& L_{14|23} - L_{13|24}.\end{aligned}$$

In the $\varepsilon \to 0$ limit, the unnormalized probability (conditional on there being two chords) is

$$\begin{aligned}\Pr(14|23) &= \frac{\ddddot{\Pr}(14|23)}{\ddddot{\Pr}(14|23) + \ddddot{\Pr}(12|34)}\\ &\to \frac{\frac{1}{(x_1-x_4)^2}\frac{1}{(x_2-x_3)^2} - \frac{1}{(x_1-x_3)^2}\frac{1}{(x_2-x_4)^2}}{\frac{1}{(x_1-x_2)^2}\frac{1}{(x_3-x_4)^2} + \frac{1}{(x_1-x_4)^2}\frac{1}{(x_2-x_3)^2} - 2\frac{1}{(x_1-x_3)^2}\frac{1}{(x_2-x_4)^2}}.\end{aligned}$$

If we let s denote the cross ratio $s = (x_4 - x_3)(x_2 - x_1)/[(x_4 - x_2)(x_3 - x_1)]$, this limiting probability can be written as

$$= \frac{2s^3 - s^4}{1 - 2s + 4s^3 - 2s^4},$$

which agrees with the known formula for bichordal SLE_2; see [BBK05, § 8.2 and 8.3] [SW05] [Dub06, § 4.1 and 4.2].

For trichordal SLE_2, the relevant polynomials are

$$\begin{aligned}\ddddot{\Pr}(12|34|56) &= -L_{15|26|34} + L_{12|56|34} + L_{14|26|35} + L_{15|24|36}\\ &\quad - L_{14|25|36} + L_{13|25|46} - L_{12|35|46} - L_{13|24|56},\\ \ddddot{\Pr}(16|23|45) &= -L_{16|24|35} + L_{14|26|35} + L_{15|24|36} - L_{14|25|36}\\ &\quad + L_{16|23|45} - L_{13|26|45} - L_{15|23|46} + L_{13|25|46},\\ \ddddot{\Pr}(14|23|56) &= L_{15|24|36} - L_{14|25|36} - L_{15|23|46} + L_{13|25|46} + L_{14|23|56} - L_{13|24|56},\\ \ddddot{\Pr}(25|16|34) &= L_{16|25|34} - L_{15|26|34} - L_{16|24|35} + L_{14|26|35} + L_{15|24|36} - L_{14|25|36},\\ \ddddot{\Pr}(36|12|45) &= L_{14|26|35} - L_{12|46|35} - L_{14|25|36} - L_{13|26|45} + L_{12|36|45} + L_{13|25|46}.\end{aligned}$$

5.2. **Grove Peano curves and multichordal SLE_8.** We can do a similar calculation for multichordal SLE_8 containing m chords connecting $2m$ points. Lawler, Schramm, and Werner [LSW04] showed that on a fine grid in which part of the boundary is wired and part is free, the path separating a random spanning tree from its dual spanning tree converges to chordal SLE_8 connecting the two boundary points where the boundary conditions change from wired to free. If the boundary conditions alternate between wired and free and back m times, there will be m paths separating a random grove from its dual grove which connect the $2m$ points where the boundary conditions change from wired to free, and it is not difficult to see that each such path, conditional upon the other paths, converges to SLE_8. Here the groves have $n = m$ nodes, which correspond to "interval" nodes between the $(2i-1)^{\text{th}}$ and $(2i)^{\text{th}}$ endpoints of the SLE_8 strands. (See Figure 9.) We again consider U to be the upper half-plane and $x_1, \ldots, x_{2m}$ the endpoints of the m SLE_8 strands. Let I_i be the interval node whose endpoints are x_{2i-1} and x_{2i}. Now $L_{i,j}$ is the current into I_j for the harmonic function which is 1 on I_i, zero on the other $I_{j'}$, and has Neumann boundary conditions elsewhere.

For $m = 2$, the event where x_1, x_4 are connected and x_2, x_3 are connected is the event where the grove has one tree connecting nodes 1 (the interval $I_1 = [x_1, x_2]$)

and 2 (the interval $I_2 = [x_3, x_4]$), which has probability $\ddot{\Pr}(12)/(\ddot{\Pr}(12)+\ddot{\Pr}(1|2)) = L_{1,2}/(L_{1,2}+1)$, where $L_{1,2}$ is the modulus of the rectangle which is the conformal image of U and whose vertices are the images of x_1, x_2, x_3, x_4. By the Schwarz-Christoffel formula, the Riemann map from U to the rectangle is

$$f(z) = \int_0^z \frac{dw}{i(w-x_1)^{1/2}(w-x_2)^{1/2}(w-x_3)^{1/2}(w-x_4)^{1/2}}.$$

Using $f(z)$ we may express

$$L_{1,2} = \frac{f(x_2)-f(x_1)}{-i(f(x_3)-f(x_2))}.$$

For $m = 3$, an interesting special case is where the domain is the regular hexagon and the three nodes are three nonadjacent sides of the hexagon. Recall now the dual electric network with its dual grove, which was illustrated in Figure 5. Here the dual electric network also converges to the regular hexagon, but with extended nodes on the other three nonadjacent sides of the hexagon. Thus the probability that the primal grove has type $1|2|3$ equals the probability that the dual grove has type $1|2|3$, but this latter event is the event where the primal grove has type 123, so $\Pr[1|2|3] = \Pr[123]$. From this we see $1 = \ddot{\Pr}(1|2|3) = \ddot{\Pr}(123) = L_{1,2}L_{2,3} + L_{1,2}L_{1,3} + L_{1,3}L_{2,3}$, and hence by symmetry $L_{1,2} = L_{1,3} = L_{2,3} = 1/\sqrt{3}$. We can now compute the other normalized probabilities $\ddot{\Pr}(12|3) = \ddot{\Pr}(13|2) = \ddot{\Pr}(23|1) = L_{2,3} = 1/\sqrt{3}$. Thus the unnormalized partition probabilities are $\Pr[1|2|3] = \Pr[123] = 1/(2\times 1 + 3\times 1/\sqrt{3}) = 2-\sqrt{3} \doteq 0.268$ and $\Pr[12|3] = \Pr[13|2] = \Pr[1|23] = 2/\sqrt{3} - 1 \doteq 0.154$. (The corresponding calculation for the connection probabilities for percolation (SLE_6) on a regular hexagon was carried out by Dubédat [Dub06, § 4.4], but while the SLE_8 calculations here

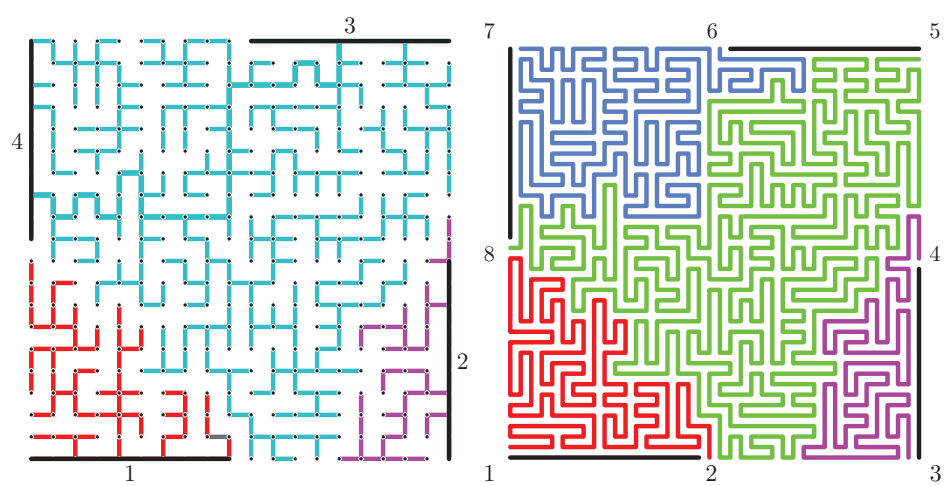


FIGURE 9. The multichordal Peano curves (which converge to SLE_8) associated to a grove with four nodes (the extended black segments). In this case the grove partition is $1|2|34$, which gives the Peano curve pairing $\frac{1}{2} \mid \frac{3}{4} \mid \frac{5}{8} \mid \frac{7}{6}$. In the scaling limit for this geometry these connections occur with probability $(2-\sqrt{2})/8 \doteq 0.0732$.

are almost trivial in retrospect, the corresponding SLE_6 calculations are much more intricate.) For the example in Figure 9 the scaling limit of the L's can be similarly worked out using duality $L_{i,i+1} = 1/2$ and $L_{i,i+2} = 1/\sqrt{2} - 1/2$. The response matrix for the regular $2n$-gon with sides that are alternately free and wired has a nice formula that is given in [BSW07].

When $m > 2$, the quantities $L_{i,j}$ for more general domains can be represented geometrically as the moduli of images of maps of U onto various vertical slit rectangles; see Figure 10. (In the remainder of this subsection $i = \sqrt{-1}$.) Here we explain how to compute the $L_{j,k}$'s when the domain U is the upper half-plane. Consider the nodes formed from the intervals $[x_1, x_2], [x_3, x_4], \ldots, [x_{2m-1}, x_{2m}]$, where $x_1 < x_2 < \cdots < x_{2m}$. Consider the analytic function f defined in the upper half-plane by

$$f(z) = \int_0^z \frac{\prod_{\ell=1}^{m-2}(w - \alpha_\ell)\, dw}{i \prod_{\ell=1}^{2m}(w - x_\ell)^{1/2}},$$

where $\alpha_1, \ldots, \alpha_{m-2}$ are parameters. By the Schwarz-Christoffel formula, if the α's are chosen judiciously, then f will map the upper half-plane to a rectangle with vertical slits, such that one of the intervals $[x_{2j-1}, x_{2j}]$ $(j \neq 1)$ is mapped to the top of the rectangle, the remaining intervals are mapped to the bottom of the rectangle, and the spaces between the intervals are mapped to the two sides of the rectangle and the vertical slits on the bottom of the rectangle. (The example in Figure 10 is for $j = 2$. The case $j = 1$ is similar, except that "top" and "bottom" are reversed.) The quantity $L_{j,k}$ is then given by the length of the image of $[x_{2k-1}, x_{2k}]$ divided by the height of the rectangle:

$$L_{j,k} = i \frac{f(x_{2k}) - f(x_{2k-1})}{f(x_{2j-1}) - f(x_{2j-2})}.$$

(Here the choice of the α's, and hence f, depends upon j.)

Define the analytic function g_p in the upper half-plane by

$$g_p(z) = \int_0^z \frac{w^p dw}{i \prod_{j=1}^{2m}(w - x_j)^{1/2}}.$$

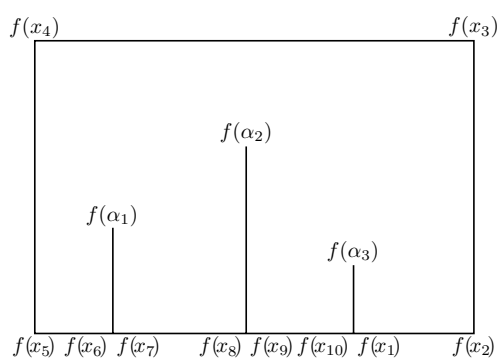


FIGURE 10. The function $f(z) = \int_0^z \frac{\prod_{\ell=1}^{m-2}(w-\alpha_\ell)\, dw}{i \prod_{\ell=1}^{2m}(w-x_\ell)^{1/2}}$ maps the upper half-plane to the slit rectangle, for suitable choices of the α_ℓ's. In this figure $L_{[x_3,x_4],[x_7,x_8]} = i\frac{f(x_8)-f(x_7)}{f(x_3)-f(x_2)}$.

Each of the above analytic functions $f = f_j$ is a linear combination of $g_0, g_1, \ldots, g_{m-2}$. Let us define two $m \times (m-1)$ matrices V' and I' by

$$V'_{k,p} = \mathrm{Im}(g_p(x_2) - g_p(x_{2k-1})) \quad \text{and} \quad I'_{k,p} = g_p(x_{2j}) - g_p(x_{2j-1}).$$

Since for any $j \neq 1$ we can express f_j as a linear combination of the g_p's, this same linear combination of the columns of V' will be nonzero (the height of the slit rectangle) at row j and zero elsewhere, and the same linear combination of the columns of I' gives the lengths of the images of the intervals in the slit rectangle. Thus V' has rank $m-1$, and since the first row is all 0, we may adjoin an mth column which is all 1 to get a nonsingular matrix V, and adjoin an mth column to I' which is all 0 to get a matrix I. Then for any vector $\vec{v}$, we have $LV\vec{v} = I\vec{v}$, which allows us to compute L by

$$L = IV^{-1}.$$

Thus we may compute the limiting connection probabilities completely mechanically.

5.3. **Double-dimer crossings—multichordal SLE$_4$.** There is a lot of evidence to support the hypothesis that the scaling limit of chains in the double-dimer model is given by SLE$_4$, but this has not yet been rigorously proved. The formulas given here, which are for double-dimer path scaling limits, agree with the formulas for SLE$_4$ (see also Section 5.4), and so provide further evidence supporting this hypothesis.

As in Section 5.1, we approximate the upper half-plane with the upper half cartesian lattice $\varepsilon\mathbb{Z} \times \varepsilon\mathbb{N}$ and consider a set of black nodes near the points $x_1, x_3, \ldots, x_{2m-1}$ and white nodes near the points $x_2, x_4, \ldots, x_{2m}$, letting node i be the nearest lattice point $x_i^{(\varepsilon)} \in \varepsilon\mathbb{Z} \times \varepsilon\mathbb{N}$ to x_i which has the appropriate color. From [Ken00, § 5.2] we have

$$X_{i,j} = (1 + o(1))\frac{2}{\pi}\frac{\varepsilon}{|x_i - x_j|}. \tag{14}$$

For the double-dimer model we can easily recover the cross-ratio formula that is known for bichordal SLE$_4$ [BBK05, § 8.2 and 8.3], [SW05], [Dub06, § 4.1] as follows:

$$\begin{aligned}\widehat{\Pr}(12|34) =& X_{1,2}X_{3,4},\\ \widehat{\Pr}(14|23) =& X_{1,4}X_{3,2}.\end{aligned}$$

When $x_1 < x_2 < x_3 < x_4$, in the scaling limit $\varepsilon \to 0$ we get

$$\begin{aligned}\Pr(14|23) = \frac{\widehat{\Pr}(14|23)}{\widehat{\Pr}(12|34) + \widehat{\Pr}(14|23)} &\to \frac{\frac{1}{|x_1-x_4|}\frac{1}{|x_2-x_3|}}{\frac{1}{|x_1-x_2|}\frac{1}{|x_4-x_3|} + \frac{1}{|x_1-x_4|}\frac{1}{|x_2-x_3|}}\\ &= \frac{(x_2-x_1)(x_4-x_3)}{(x_3-x_1)(x_4-x_2)},\end{aligned}$$

which is the formula for the cross-ratio.

For the trichordal case the relevant polynomials are

$$\begin{aligned}
\widehat{\Pr}(12|34|56) =& X_{1,2}X_{3,4}X_{5,6} + X_{1,4}X_{3,6}X_{5,2},\\
\widehat{\Pr}(16|23|45) =& X_{1,6}X_{3,2}X_{5,4} + X_{1,4}X_{3,6}X_{5,2},\\
\widehat{\Pr}(14|23|56) =& X_{1,4}X_{3,2}X_{5,6} - X_{1,4}X_{3,6}X_{5,2},\\
\widehat{\Pr}(25|16|34) =& X_{1,6}X_{3,4}X_{5,2} - X_{1,4}X_{3,6}X_{5,2},\\
\widehat{\Pr}(36|12|45) =& X_{1,2}X_{3,6}X_{5,4} - X_{1,4}X_{3,6}X_{5,2}.
\end{aligned}$$

5.4. **Gaussian free-field contour lines and multichordal** SLE_4**.** Schramm and Sheffield studied the contour lines of the discrete Gaussian free field on a planar domain with boundary conditions $+\lambda$ on one segment of the boundary and $-\lambda$ on the rest of the boundary [SS06], and they showed that for a certain value of λ the contour line of 0 height connecting the two special boundary points is in the scaling limit given by chordal SLE_4. In general there can be multiple boundary points between which the boundary heights alternate between $+\lambda$ and $-\lambda$, and the contour lines will form a random pairing of these boundary points. When there are multiple boundary points, at which the height alternates between $+\lambda$ and $-\lambda$, each of these boundary points induces a drift term on the Brownian motion driving the SLE_4 process [SS06], [SS07]. Given the conjecture that the scaling limit of the double-dimer model yields SLE_4, and the fact that we know the scaling limit of the connection probabilities in the double-dimer model with multiple strands, it is natural to compare these probabilities to the connection probabilities of the contour lines in the discrete Gaussian free field. We already have a natural guess for these contour line connection probabilities (namely that they equal the connection probabilities in the double-dimer model), and it is not difficult to verify that this guess is in fact correct using the SLE formulation. In this subsection we prove

Theorem 5.1. *Consider the scaling limit of the discrete Gaussian free field on a planar lattice in the upper half-plane, with boundary heights that alternate between* $+\lambda$ *and* $-\lambda$ *at the points* $x_1 < \cdots < x_{2n}$*, for the value of* λ *determined in* [SS06]*. Then the probability distribution of the manner in which the height-*0 *contour lines connect up with one another coincides with the connection probability distribution for the scaling limit of the double-dimer model in the upper half-plane with boundary nodes that alternate between black and white at locations* $x_1, \ldots, x_{2n}$*.*

Recall from Section 3.1 and Section 3.2 that the probability of any double-dimer pairing is a linear combination of $D_S/D_\emptyset$'s, where S ranges over balanced subsets of nodes. (An expression for D_S is recalled below.) We therefore start with a lemma giving the scaling limit of $D_S/D_\emptyset$.

Lemma 5.2. *In the scaling limit of the double-dimer model in the upper half-plane, if the nodes are located at* $x_1 < \cdots < x_{2n}$ *and the node colors alternate between black and white, and if* S *is a balanced subset of the nodes, then*

$$\frac{D_S}{D_\emptyset} = \prod_{i\in S, j\notin S} |x_j - x_i|^{(-1)^{1+i+j}}.$$

Proof. Recall that for a set S of nodes with equal numbers of odd and even nodes, we defined

$$D_S = \det[(1_{i,j\in S} + 1_{i,j\notin S}) \times (-1)^{(|i-j|-1)/2} X_{i,j}]_{j=2,4\ldots,2n,}^{i=1,3,\ldots,2n-1}$$

which, in the scaling limit, using Equation (14), when $x_1 < x_2 < \cdots < x_{2n}$, is given by

$$= \det \left[(1_{i,j\in S} + 1_{i,j\notin S}) \times \frac{(-1)^{(|i-j|-1)/2}}{x_j - x_i} \right]_{j=2,4,\ldots,2n}^{i=1,3,\ldots,2n-1}$$
$$= \pm \det \left[\frac{1}{x_j - x_i} \right]_{j \text{ even},\, j\in S}^{i \text{ odd},\, i\in S} \det \left[\frac{1}{x_j - x_i} \right]_{j \text{ even},\, j\notin S,}^{i \text{ odd},\, i\notin S}$$

which is a product of two Cauchy determinants

$$= \pm \frac{\displaystyle\prod_{\substack{i<j \\ i+j \text{ even} \\ i,j\in S \text{ or } i,j\notin S}} (x_j - x_i)}{\displaystyle\prod_{\substack{i<j \\ i+j \text{ odd} \\ i,j\in S \text{ or } i,j\notin S}} (x_j - x_i)}$$

where the $i, j \in S$ factors are from the first determinant and the $i, j \notin S$ factors are from the second determinant. Since each multiplicand is positive and by Lemma 3.2 D_S is a ratio of positive quantities, the global sign is $+$.

We may evaluate the ratio

$$\frac{D_S}{D_\emptyset} = \prod_{i<j} (x_j - x_i)^{\alpha_{i,j}},$$

where

$$\alpha_{i,j} = \begin{cases} 0, & \text{if } i,j \in S \text{ or } i,j \notin S, \\ (-1)^{1+i+j}, & \text{if } i \in S \text{ and } j \notin S \text{ or vice versa.} \end{cases}$$

□

We remark that products of the form $\prod_{i<j}(x_j - x_i)^{*_{i,j}}$ also arise in [Dub06].

Proof of Theorem 5.1. The strategy is to show that each $D_S/D_\emptyset$ is a martingale for the multichordal SLE_4 diffusion associated with the contour lines of the discrete Gaussian free field [SS06], [SS07]. Then each of the double-dimer crossing probabilities will be a martingale for the diffusion, and since each of them has the right boundary conditions, this will show that the contour line crossing probabilities are equal to the double-dimer crossing probabilities. By symmetry considerations, it suffices to consider the SLE_4 process started at x_1 where the other x_i's are force points. To check that M is a local martingale we need to verify that

$$0 \stackrel{?}{=} \partial_t M = \sum_{i>1} \frac{2}{x_i - x_1} \frac{\partial M}{\partial x_i} + \frac{\partial M}{\partial x_1} \sum_{i>1} \frac{(-1)^i 2}{x_i - x_1} + \frac{4}{2} \frac{\partial^2 M}{\partial x_1^2}.$$

The terms in the first sum come from the Loewner evolution of the points x_i $(i \neq 1)$ when the SLE_4 at x_1 is developed, the terms in the second sum arise from the drifts that the points at x_i $(i \neq 1)$ induce on the Brownian motion driving the SLE_4 at x_1, and the last term comes from the quadratic variation of the Brownian motion driving the SLE_4 process at x_1. Upon dividing by $2M$ the equation becomes

$$0 \stackrel{?}{=} \frac{\partial_t M}{2M} = \sum_{i>1} \frac{1}{x_i - x_1} \frac{\partial \log M}{\partial x_i} + \frac{\partial \log M}{\partial x_1} \sum_{i>1} \frac{(-1)^i}{x_i - x_1} + \frac{\partial^2 \log M}{\partial x_1^2} + \Big(\frac{\partial \log M}{\partial x_1} \Big)^2.$$

If $M = D_S/D_\emptyset$, then $\log M = \sum_{i<j} \alpha_{i,j} \log(x_j - x_i)$, so the first term is

$$\sum_{j<k} \frac{\alpha_{j,k}}{(x_k - x_j)(x_k - x_1)} + \sum_{1<j<k} \frac{-\alpha_{j,k}}{(x_k - x_j)(x_j - x_1)}$$
$$= \sum_{1<j<k} \frac{-\alpha_{j,k}}{(x_j - x_1)(x_k - x_1)} + \sum_{1<k} \frac{\alpha_{1,k}}{(x_k - x_1)^2},$$

the second term is

$$\left(-\sum_{1<j} \frac{\alpha_{1,j}}{x_j - x_1}\right)\left(\sum_{1<k} \frac{(-1)^k}{x_k - x_1}\right),$$

the third term is

$$\sum_{1<j} \frac{-\alpha_{1,j}}{(x_j - x_1)^2},$$

and the fourth term is

$$\left(\sum_{1<j} \frac{-\alpha_{1,j}}{x_j - x_1}\right)^2.$$

Adding these up we get

$$\sum_{1<j} \frac{\alpha_{1,j} - \alpha_{1,j}(-1)^j - \alpha_{1,j} + \alpha_{1,j}^2}{(x_j - x_1)^2}$$
$$+ \sum_{1<j<k} \frac{-\alpha_{j,k} - \alpha_{1,j}(-1)^k - \alpha_{1,k}(-1)^j + 2\alpha_{1,j}\alpha_{1,k}}{(x_j - x_1)(x_k - x_1)}.$$

If $\alpha_{1,j} \neq 0$, then $\alpha_{1,j} = (-1)^j$, so the numerators in the first sum are all 0. For the second sum, there are a few cases to consider. If $\alpha_{j,k} \neq 0$, then $\alpha_{j,k} = (-1)^{1+j+k}$, and either $\alpha_{1,j} = 0$ or else $\alpha_{1,k} = 0$ but not both, so the numerator simplifies to 0. If $\alpha_{j,k} = 0$, then either $\alpha_{1,j} = \alpha_{1,k} = 0$ (in which case the numerator simplifies to 0), or else $\alpha_{1,j} = (-1)^j$ and $\alpha_{1,k} = (-1)^k$, and the numerator again simplifies to 0. Thus $D_S/D_\emptyset$ is a local martingale. Since $D_S/D_\emptyset$ is the probability that no node in S is paired to a node in S^c (in the double-dimer model), it is bounded, and therefore it is actually a martingale. □

Appendix A. The response matrix

To a pair $(\mathcal{G}, \mathbf{N})$ consisting of a graph and a subset $\mathbf{N}$ of its vertices is associated an $|\mathbf{N}| \times |\mathbf{N}|$ matrix Λ, the *response* matrix, or *Dirichlet-to-Neumann* matrix. It is a real symmetric matrix defined from the Laplacian matrix as follows. The variables $L_{i,j}$ used in this article are negatives of the entries of this matrix: $L_{i,j} = -\Lambda_{i,j}$.

A.1. **Definition.** Let Δ be the Laplacian matrix of $\mathcal{G}$, defined on functions on the vertices of $\mathcal{G}$ by $\Delta(f)(v) = \sum_{w \sim v} c_{v,w}(f(v) - f(w))$, where the sum is over neighbors w of v and $c_{v,w}$ is the edge weight. Let $\mathbf{N}$ be the set of nodes of $\mathcal{G}$ and $\mathbf{I} = \mathbf{V} \smallsetminus \mathbf{N}$ the inner vertices. Then $\mathbb{R}^{\mathbf{V}} = \mathbb{R}^{\mathbf{N}} \oplus \mathbb{R}^{\mathbf{I}}$. In this ordering, we can write

$$\Delta = \begin{pmatrix} F & G \\ H & K \end{pmatrix},$$

where K is an $|\mathbf{I}| \times |\mathbf{I}|$ matrix.

Let $V \in \mathbb{R}^{\mathbf{N}}$ be a choice of potentials at the nodes. If we hold the nodes at these potentials, the resulting current flow in the network will have divergence zero except at the nodes; that is,

$$\begin{pmatrix} F & G \\ H & K \end{pmatrix} \begin{pmatrix} v_{\mathbf{N}} \\ v_{\mathbf{I}} \end{pmatrix} = \begin{pmatrix} c_{\mathbf{N}} \\ 0 \end{pmatrix},$$

where $c_{\mathbf{N}} \in \mathbb{R}^{\mathbf{N}}$ is the vector of currents exiting the network at the nodes.

In particular we must have $v_{\mathbf{I}} = -K^{-1}Hv_{\mathbf{N}}$, and so $(F - GK^{-1}H)v_{\mathbf{N}} = c_{\mathbf{N}}$. (The matrix K is invertible since it is the Dirichlet Laplacian on $\mathbf{I}$ with nonempty boundary $\mathbf{N}$.) The matrix $\Lambda = F - GK^{-1}H$ satisfies

$$\Lambda v_{\mathbf{N}} = c_{\mathbf{N}}$$

and is called the *response matrix,* or *Dirichlet-to-Neumann matrix* of the pair $(\mathcal{G}, \mathbf{N})$, and is also known as the Schur complement Δ/K. It is a symmetric positive semidefinite matrix from $\mathbb{R}^{\mathbf{N}}$ to $\mathbb{R}^{\mathbf{N}}$ with kernel consisting of the "all ones" vector [CdV98].

The following lemma is a special case of a more general determinant formula, but we need this special case in Section 2, so we prove it here.

Lemma A.1. $\Pr(\text{tree})/\Pr(\text{uncrossing}) = \det \tilde{\Lambda}$, *where* $\tilde{\Lambda}$ *is obtained from* Λ *by removing any row and column.*

Proof. Since K is the Dirichlet Laplacian for the graph $\mathcal{G}/\mathbf{N}$ in which all vertices in $\mathbf{N}$ have been identified, by the matrix-tree theorem (often attributed to Kirchhoff; see [vLW92, Chapter 34]), $\det K$ is the weighted sum of spanning trees of $\mathcal{G}/\mathbf{N}$. These lift to completely disconnected groves of $\mathcal{G}$, so $\det K = Z(\text{uncrossing})$, where $Z(\tau)$ denotes the weighted sum of groves of partition type τ, in this case $1|2|\cdots|n$.

Let $\Delta_\varepsilon = \Delta + \varepsilon I_{\mathbf{N}}$. Recall that this is the Dirichlet Laplacian for the graph $\mathcal{G}'$ obtained from $\mathcal{G}$ by adjoining an external vertex connected to every vertex in $\mathbf{N}$ by an edge of weight ε, i.e., Δ_ε is obtained from $\Delta(\mathcal{G}')$ by striking out the row and column corresponding to the adjoined external vertex so that

$$\lim_{\varepsilon \to 0} \det(\Delta + \varepsilon I_{\mathbf{N}})/\varepsilon = nZ(\text{tree}).$$

Let $\Lambda_\varepsilon = \Delta_\varepsilon/K = F + \varepsilon I_{\mathbf{N}} - GK^{-1}H$. Then

$$\Delta_\varepsilon = \begin{pmatrix} \Lambda_\varepsilon & GK^{-1} \\ 0 & I_{\mathbf{I}} \end{pmatrix} \begin{pmatrix} I_{\mathbf{N}} & 0 \\ H & K \end{pmatrix},$$

so $\det \Delta_\varepsilon = \det \Lambda_\varepsilon \det K$ (a standard fact about Schur complements).

But if $\tilde{\Lambda}$ denotes Λ with a row and column removed, then

$$\det \tilde{\Lambda} = \frac{1}{n} \lim_{\varepsilon \to 0} (\det \Lambda_\varepsilon)/\varepsilon = \frac{1}{n} \lim_{\varepsilon \to 0} (\det \Delta_\varepsilon)/\varepsilon/\det K = Z(\text{tree})/Z(\text{uncrossing}).$$

□

A.2. **The relation between the $L_{i,j}$'s and the $R_{i,j}$'s.** The Λ matrix is related to the matrix of pairwise resistances between the nodes as follows. We assume here for convenience that the electrical network is connected.

Proposition A.2. *We have*

$$R_{i,j} = (\delta_i - \delta_j)^T \Lambda^{-1} (\delta_i - \delta_j).$$

Note that the right-hand side is well defined since $\delta_i - \delta_j$ is in the image of Λ and that the inverse image of this vector is well defined up to addition of an element of the kernel of Λ, which is perpendicular to $\delta_i - \delta_j$.

Proof. Find $v \in \mathbb{R}^V$ such that $\Lambda v = -\delta_i + \delta_j$. This represents the (unique up to an additive constant) choice of potentials for which there is one unit of current flowing into the circuit at i and out at j. By definition, the resistance between i and j is the difference in potentials at i and j for this current flow. □

The equation of Proposition A.2 can be given in a computationally simpler form as follows. When finding $v \in \mathbb{R}^V$ such that $\Lambda v = -\delta_i + \delta_j$, we may choose v so that $v_n = 0$, and let $\tilde{v}$ denote v with v_n dropped. Let $\tilde{\Lambda}$ be the matrix obtained from Λ by deleting the last row and column. Then $\tilde{\Lambda}\tilde{v} = -\delta_i + \delta_j$, where δ_n is interpreted as 0. By the matrix-tree theorem, $\tilde{\Lambda}$ is invertible since $\det \tilde{\Lambda}$ is the weighted sum of spanning trees with edge weights given by $-\Lambda_{i,j}$. Thus $R_{i,n} = \tilde{\Lambda}^{-1}_{i,i}$, and if $i, j \neq n$, then

$$R_{i,j} = (\delta_i - \delta_j)^T \tilde{\Lambda}^{-1} (\delta_i - \delta_j). \tag{15}$$

Reciprocally to Proposition A.2 we can write Λ in terms of R as follows:

Proposition A.3. *We have* $\tilde{\Lambda}^{-1}_{i,i} = R_{i,n}$ *and*

$$\tilde{\Lambda}^{-1}_{i,j} = \frac{1}{2}(R_{i,n} + R_{j,n} - R_{i,j}).$$

Proof. This is just the inverse of the transformation in (15). □

Appendix B. Complete list of grove probabilities on 4 nodes

Here we give the formulas for the planar partition probabilities when there are 4 nodes. There are 14 planar partitions on 4 nodes, but for each class of planar partitions that may be obtained from one another by cyclically rotating and/or reversing the indices, we need only give the formula for one representative planar partition in the class.

$$\begin{aligned}
\overline{\mathrm{Pr}}(1234) &= 1,\\
\overline{\mathrm{Pr}}(2|143) &= \tfrac{1}{2}R_{2,3} + \tfrac{1}{2}R_{1,2} - \tfrac{1}{2}R_{1,3},\\
\overline{\mathrm{Pr}}(14|23) &= \tfrac{1}{2}R_{1,3} + \tfrac{1}{2}R_{2,4} - \tfrac{1}{2}R_{1,4} - \tfrac{1}{2}R_{2,3},\\
\overline{\mathrm{Pr}}(1|2|34) &= (+R_{1,2}R_{1,3} + R_{1,2}R_{2,3} + R_{1,2}R_{1,4} + R_{1,2}R_{2,4} - 2R_{1,2}R_{3,4} - R_{1,2}^2\\
&\qquad + R_{1,3}R_{2,4} + R_{1,4}R_{2,3} - R_{1,3}R_{1,4} - R_{2,3}R_{2,4})/4,\\
\overline{\mathrm{Pr}}(1|3|24) &= (+R_{1,3}R_{1,2} + R_{1,3}R_{2,3} + R_{1,3}R_{1,4} + R_{1,3}R_{3,4} - 2R_{1,3}R_{2,4} - R_{1,3}^2\\
&\qquad + R_{1,2}R_{3,4} + R_{1,4}R_{2,3} - R_{1,2}R_{1,4} - R_{2,3}R_{3,4})/4,\\
\overline{\mathrm{Pr}}(1|2|3|4) &= (+R_{1,2}R_{2,3}R_{3,4} + R_{1,2}R_{2,4}R_{4,3} + R_{1,3}R_{3,2}R_{2,4} + R_{1,3}R_{3,4}R_{4,2}\\
&\quad + R_{1,4}R_{4,2}R_{2,3} + R_{1,4}R_{4,3}R_{3,2} + R_{2,1}R_{1,3}R_{3,4} + R_{2,1}R_{1,4}R_{4,3}\\
&\quad + R_{2,3}R_{3,1}R_{1,4} + R_{2,4}R_{4,1}R_{1,3} + R_{3,1}R_{1,2}R_{2,4} + R_{3,2}R_{2,1}R_{1,4}\\
&\qquad - R_{1,2}R_{2,3}R_{3,1} - R_{1,2}R_{2,4}R_{4,1} - R_{1,3}R_{3,4}R_{4,1} - R_{2,3}R_{3,4}R_{4,2}\\
&\qquad - R_{1,2}R_{3,4}^2 - R_{1,2}^2R_{3,4} - R_{1,3}R_{2,4}^2 - R_{1,3}^2R_{2,4} - R_{1,4}R_{2,3}^2 - R_{1,4}^2R_{2,3})/4.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\ddot{\Pr}(1234) = L_{1234} \quad &= L_{1,2}L_{1,3}L_{1,4} + L_{1,2}L_{2,3}L_{2,4} + L_{1,3}L_{2,3}L_{3,4} \\
&\quad + L_{1,4}L_{2,4}L_{3,4} + L_{1,2}L_{2,3}L_{3,4} + L_{1,3}L_{2,3}L_{2,4} \\
&\quad + L_{1,2}L_{2,4}L_{3,4} + L_{1,4}L_{2,3}L_{2,4} + L_{1,2}L_{1,3}L_{2,4} \\
&\quad + L_{1,2}L_{1,4}L_{2,3} + L_{1,3}L_{2,4}L_{3,4} + L_{1,4}L_{2,3}L_{3,4} \\
&\quad + L_{1,3}L_{1,4}L_{2,3} + L_{1,2}L_{1,3}L_{3,4} + L_{1,3}L_{1,4}L_{2,4} \\
&\quad + L_{1,2}L_{1,4}L_{3,4}, \\
\ddot{\Pr}(2|143) = L_{2|143} + L_{13|24} &= L_{1,3}L_{1,4} + L_{1,4}L_{3,4} + L_{1,3}L_{3,4} + L_{1,3}L_{2,4}, \\
\ddot{\Pr}(14|23) = L_{14|23} - L_{13|24} &= L_{1,4}L_{2,3} - L_{1,3}L_{2,4}, \\
\ddot{\Pr}(1|2|34) = L_{1|2|34} \quad &= L_{3,4}, \\
\ddot{\Pr}(1|3|24) = L_{1|3|24} \quad &= L_{2,4}, \\
\ddot{\Pr}(1|2|3|4) = L_{1|2|3|4} \quad &= 1.
\end{aligned}$$

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References

[ASA02] Louis-Pierre Arguin and Yvan Saint-Aubin. Non-unitary observables in the 2d critical Ising model. *Physics Letters B*, 541(3-4):384–389, 2002. MR1928682 (2003h:82016)

[BBK05] Michel Bauer, Denis Bernard, and Kalle Kytölä. Multiple Schramm-Loewner evolutions and statistical mechanics martingales. *J. Stat. Phys.*, 120(5-6):1125–1163, 2005. MR2187598 (2007d:82032)

[BSW07] Nathaniel D. Blair-Stahn and David B. Wilson. Electrical response matrix of a regular $2n$-gon, *Proc. Amer. Math. Soc.*, 137(6):2015–2025, 2009. MR2480283

[Car92] John L. Cardy. Critical percolation in finite geometries. *J. Phys. A*, 25(4):L201–L206, 1992. MR1151081 (92m:82048)

[Car07] John Cardy. ADE and SLE. *J. Phys. A*, 40(7):1427–1438, 2007. MR2303259 (2008k:82023)

[CdV98] Yves Colin de Verdière. *Spectres de Graphes*, volume 4 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 1998. MR1652692 (99k:05108)

[CIM98] E. B. Curtis, D. Ingerman, and J. A. Morrow. Circular planar graphs and resistor networks. *Linear Algebra Appl.*, 283(1-3):115–150, 1998. MR1657214 (99k:05096)

[Ciu97] Mihai Ciucu. Enumeration of perfect matchings in graphs with reflective symmetry. *J. Combin. Theory Ser. A*, 77(1):67–97, 1997. MR1426739 (98a:05112)

[CN07] Federico Camia and Charles M. Newman. Critical percolation exploration path and SLE_6: a proof of convergence. *Probab. Theory Related Fields*, 139(3-4):473–519, 2007. MR2322705 (2008k:82040)

[CS04] Gabriel D. Carroll and David Speyer. The cube recurrence. *Electron. J. Combin.*, 11(1):Research Paper 73, 31 pp., 2004. MR2097339 (2005f:05007)

[DFGG97] P. Di Francesco, O. Golinelli, and E. Guitter. Meanders and the Temperley-Lieb algebra. *Comm. Math. Phys.*, 186(1):1–59, 1997. MR1462755 (99f:82028)

[DS84] Peter G. Doyle and J. Laurie Snell. *Random Walks and Electric Networks*, volume 22 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington, DC, 1984. MR920811 (89a:94023)

[DS87] B. Duplantier and H. Saleur. Exact critical properties of two-dimensional dense self-avoiding walks. *Nuclear Phys. B*, 290(3):291–326, 1987. MR919521 (89g:82049)

[Dub06] Julien Dubédat. Euler integrals for commuting SLEs. *J. Stat. Phys.*, 123(6):1183–1218, 2006. MR2253875 (2007g:82027)

[Dup87] Bertrand Duplantier. Critical exponents of Manhattan Hamiltonian walks in two dimensions, from Potts and O(n) models. *J. Statist. Phys.*, 49(3-4):411–431, 1987. MR926190 (89b:82025)

[Dup89] Bertrand Duplantier. Statistical mechanics of polymer networks of any topology. *J. Statist. Phys.*, 54(3-4):581–680, 1989. MR988554 (90e:82061)

[Dup92] Bertrand Duplantier. Loop-erased self-avoiding walks in two dimensions: Exact critical exponents and winding numbers. *Physica A*, 191:516–522, 1992.

[Dup06] Bertrand Duplantier. Conformal random geometry. In A. Bovier, F. Dunlop, A. van Enter, F. den Hollander, and J. Dalibard, editors, *Mathematical Statistical Physics*, Lecture Notes of the Les Houches Summer School Session LXXXIII, 2005, pages 101–217. Elsevier, 2006,

[Fom01] Sergey Fomin. Loop-erased walks and total positivity. *Trans. Amer. Math. Soc.*, 353(9):3563–3583 (electronic), 2001. MR1837248 (2002f:15030)

[Jon83] V. F. R. Jones. Index for subfactors. *Invent. Math.*, 72(1):1–25, 1983. MR696688 (84d:46097)

[Kas67] P. W. Kasteleyn. Graph theory and crystal physics. In *Graph Theory and Theoretical Physics*, pages 43–110. Academic Press, London, 1967. MR0253689 (40:6903)

[Ken97] Richard Kenyon. Local statistics of lattice dimers. *Ann. Inst. H. Poincaré Probab. Statist.*, 33(5):591–618, 1997. MR1473567 (99b:82039)

[Ken00] Richard Kenyon. Conformal invariance of domino tiling. *Ann. Probab.*, 28(2):759–795, 2000. MR1782431 (2002e:52022)

[Kir90] Gustav Kirchhoff. Beweis der Existenz des Potentials das an der Grenze des betrachteten Raumes gegebene Werthe hat für den Fall dass diese Grenze eine überall convexe Fläche ist. *Acta Math.*, 14(1):179–183, 1890. MR1554794

[KL07] Michael J. Kozdron and Gregory F. Lawler. The configurational measure on mutually avoiding SLE paths. In *Universality and Renormalization*, volume 50 of *Fields Inst. Commun.*, pages 199–224. Amer. Math. Soc., Providence, RI, 2007. MR2310306 (2008m:60079)

[KS91] Ki Hyoung Ko and Lawrence Smolinsky. A combinatorial matrix in 3-manifold theory. *Pacific J. Math.*, 149(2):319–336, 1991. MR1105701 (92d:57008)

[Kuo04] Eric H. Kuo. Applications of graphical condensation for enumerating matchings and tilings. *Theoretical Computer Science*, 319(1-3):29–57, 2004. MR2074946 (2005f:05038)

[KW08] Richard W. Kenyon and David B. Wilson. Combinatorics of tripartite boundary connections for trees and dimers, *Elec. J. Comb.*, 16 (2009), no. 1, Research Paper 112, 28 pp. MR2539354 (2010j:05239)

[LSW04] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, 32(1B):939–995, 2004. MR2044671 (2005f:82043)

[OB88] Kaoru Ohno and Kurt Binder. Scaling theory of star polymers and general polymer networks in bulk and semi-infinite good solvents. *J. Physique*, 49(8):1329–1351, 1988. MR966320 (89j:82053)

[Pem95] Robin Pemantle. Uniform random spanning trees. In J. Laurie Snell, editor, *Topics in Contemporary Probability and its Applications*, Probab. Stochastics Ser., pages 1–54. CRC, Boca Raton, FL, 1995. MR1410532 (97i:60013)

[PS05] T. Kyle Petersen and David Speyer. An arctic circle theorem for Groves. *J. Combin. Theory Ser. A*, 111(1):137–164, 2005. MR2144860 (2006m:05055)

[Sal86] H. Saleur. New exact exponents for two-dimensional self-avoiding walks. *J. Phys. A*, 19:L807–L810, 1986.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000. MR1776084 (2001m:60227)

[Sch07] Oded Schramm. Conformally invariant scaling limits (an overview and a collection of problems). In *International Congress of Mathematicians. Vol. I*, pages 513–543. Eur. Math. Soc., Zürich, 2007. MR2334202 (2008j:60237)

[Smi01] Stanislav Smirnov. Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(3):239–244, 2001. MR1851632 (2002f:60193)

[Smi07] Stanislav Smirnov. Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model, 2007, arXiv:0708.0039.

[Spi76] Frank Spitzer. *Principles of Random Walk*. Springer-Verlag, New York, second edition, 1976. Graduate Texts in Mathematics, Vol. 34. MR0388547 (52:9383)

[SS06] Oded Schramm and Scott Sheffield. Contour lines of the two-dimensional discrete Gaussian free field, *Acta Math.* 202 (2009), no. 1, 21–137. MR2486487 (2010f:60238)

[SS07] Oded Schramm and Scott Sheffield, 2007. Personal communication.

[Sta86] Richard P. Stanley. *Enumerative Combinatorics, Volume 1*. Wadsworth & Brooks/Cole Advanced Books & Software, 1986. MR847717 (87j:05003)

[Sta99] Richard P. Stanley. *Enumerative Combinatorics, Volume 2*. Cambridge Studies in Advanced Mathematics #62. Cambridge University Press, Cambridge, 1999. With an appendix by Sergey Fomin. MR1676282 (2000k:05026)

[SW05] Oded Schramm and David B. Wilson. SLE, quadrangles, and curvilinear triangles, 2005.

[vLW92] J. H. van Lint and R. M. Wilson. *A Course in Combinatorics*. Cambridge University Press, Cambridge, 1992. MR1207813 (94g:05003)

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