

## THE MODULI OF CURVES OF GENUS SIX AND K3 SURFACES

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ABSTRACT. We prove that the coarse moduli space of curves of genus six is birational to an arithmetic quotient of a bounded symmetric domain of type IV by giving a period map to the moduli space of some lattice-polarized K3 surfaces.

### INTRODUCTION

This paper gives a birational period map between the coarse moduli space  $\mathcal{M}_6$  of curves of genus six and a bounded symmetric domain of type IV, parametrizing some lattice-polarized K3 surfaces. A similar correspondence, between a moduli space of curves and a period space of K3 surfaces, was given by the second author for curves of genus three and genus four in [Ko2] and [Ko3]. Some of the results in this paper were announced in [Ko3].

Realizing a moduli space as an arithmetic quotient of a bounded symmetric domain is interesting for several reasons.

The first reason is that this gives a connection with the theory of automorphic forms. Recently Borcherds [Bor] gave a systematic method to construct automorphic forms on a bounded symmetric domain of type IV. Allcock and Freitag [AF] and the second author [Ko4], [Ko5] used his theory to obtain explicit embeddings in a projective space for the moduli spaces of cubic surfaces, Enriques surfaces and curves of genus three, respectively.

Secondly, an arithmetic quotient of a bounded symmetric domain has several compactifications, that is, Satake-Baily-Borel's, Mumford's toroidal and Looijenga's compactifications [Lo1], [Lo2]. A general theory explaining the relations between these types of compactifications and those of geometric nature (i.e. given by GIT) has been developed by Looijenga in loc. cit. (see §8, [Lo2]). In case of moduli spaces of polarized K3 surfaces, the geometric compactification is known only in the case of degree two and four (J. Shah [Sh1], [Sh2]).

Third, this paper is related to the study of the birational geometry of the moduli space of curves. Hassett [Ha] gave a general program to describe the canonical models of the moduli space of stable curves. In case of genus three curves, Hyeon and Lee gave a complete description of Hassett's program in [HL]. As mentioned above, the moduli space of curves of genus three can be written as the arithmetic

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quotient of a complex ball [Ko2]. Thus we hope that the present paper shall give a first step to obtain a new example in these related topics.

We now present the main results of this paper. Let  $C$  be a general curve of genus six, then its canonical model is a quadratic section of a unique quintic del Pezzo surface  $Y \subset \mathbb{P}^5$  (e.g. [SB]). The double cover of  $Y$  branched along  $C$  is a K3 surface  $X$  whose Picard lattice contains a rank 5 lattice  $S$ . We denote by  $T$  the orthogonal complement of  $S$  in  $H^2(X, \mathbb{Z})$ . By taking the period point of  $X$  we define a period map  $\mathcal{P}$  from an open dense subset of the coarse moduli space  $\mathcal{M}_6$  of curves of genus six to an arithmetic quotient

$$\mathcal{P} : \mathcal{M}_6 \dashrightarrow \mathcal{D}/\Gamma,$$

where  $\mathcal{D} = \{\omega \in \mathbb{P}(T \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}$  is a union of two copies of a 15-dimensional bounded symmetric domain of type IV and  $\Gamma$  is the orthogonal group  $O(T)$  of  $T$ . The same construction defines rational period maps

$$\mathcal{P}^* : \mathcal{W}_6^2 \dashrightarrow \mathcal{D}/\Gamma^*, \quad \mathcal{P}^{**} : \widetilde{\mathcal{W}}_6^2 \dashrightarrow \mathcal{D}/\Gamma^{**}.$$

Here the moduli space  $\mathcal{W}_6^2$  parametrizes pairs  $(C, D)$  where  $C$  is a curve of genus six and  $D$  is a  $g_2^6$  on  $C$ , while  $\widetilde{\mathcal{W}}_6^2$  is the moduli space of smooth non-special genus six curves with five ordered  $g_2^6$ 's. Here “special” means either hyperelliptic, trigonal, bi-elliptic or isomorphic to a smooth plane quintic curve.

The group  $\Gamma^*$  is a subgroup of  $\Gamma$  of index 5 and  $\Gamma^{**}$  is a normal subgroup of  $\Gamma$  with  $\Gamma/\Gamma^{**} \cong S_5$ . The purpose of this paper is to prove the following three theorems (see Theorem 2.7, Proposition 3.3, Corollaries 3.6, 3.8, 3.10, 4.2 and Theorem 4.5).

**Theorem 0.1.**  $\mathcal{P}, \mathcal{P}^*, \mathcal{P}^{**}$  are birational maps, in particular  $\mathcal{P}$  induces an isomorphism

$$\mathcal{M}_6 \setminus \{\text{special curves}\} \cong (\mathcal{D} \setminus \mathcal{H})/\Gamma,$$

where  $\mathcal{H}$  is an arithmetic arrangement of hyperplanes with respect to  $\Gamma$ .

The divisor  $\mathcal{H}$  is the union of countably many hyperplanes of  $\mathcal{D}$ , each of which is orthogonal to a vector in  $T$  with self-intersection  $-2$  (see Subsection 2.3). In literature, its image in  $\mathcal{D}/\Gamma$  is also called *discriminant divisor*.

**Theorem 0.2.**  $\mathcal{H}/\Gamma$  has three irreducible components  $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ . A generic point in  $\mathcal{H}_1, \mathcal{H}_2$ , or  $\mathcal{H}_3$  is a nodal curve of genus six, the union of a smooth plane quintic and a line, or a trigonal curve  $C$  of genus six plus a section in  $L \in |K_C - 2g_3^1|$ .

The third theorem gives a description of the Satake-Baily-Borel compactification  $\overline{\mathcal{D}/\Gamma}$  of  $\mathcal{D}/\Gamma$ . In the case of a bounded symmetric domain of type IV it is known that the boundary components are at most 1-dimensional.

**Theorem 0.3.** The boundary of  $\overline{\mathcal{D}/\Gamma}$  has two zero dimensional components and 14 one dimensional components.

According to [Lo2], the Satake-Baily-Borel compactification can be described as a modification of a GIT compactification of the moduli space. In case of genus three curves, the birational map between the two compactifications has been explicitly described by the first author [A]: the Baily-Borel compactification can be obtained from the GIT moduli space of smooth plane quartics by blowing up a point and contracting a rational curve. At the moment, we cannot give a similar description for the case of genus six curves.

An interesting problem would also be to describe the behavior of the period map  $\mathcal{P}$  with respect to the boundary components of the moduli space  $\overline{\mathcal{M}}_6$  of stable curves. By Theorem 0.2 we know that the boundary divisor  $\Delta_1$  of  $\overline{\mathcal{M}}_6$ , consisting of nodal curves, is birational to a component of the discriminant divisor in our model. The other two components of the discriminant divisor are birational to blowing ups of the loci of plane quintics or trigonal curves (which have codimension 3 and 2, respectively, in  $\mathcal{M}_6$ ).

Here is a brief plan of the paper. In the first section we review some classical properties of curves of genus six, in particular we recall the structure of the space  $\mathcal{W}_6^2$ . The natural projection map  $\mathcal{W}_6^2 \rightarrow \mathcal{M}_6$  is surjective by Brill-Noether theory. Its fiber over the general curve  $C$  of genus six is a finite set of cardinality 5 and any of its points gives a birational map from  $C$  to a plane sextic with 4 nodes. The fiber is known to be positive dimensional if and only if the curve of genus six is *special*, i.e. it is either trigonal, hyperelliptic, bi-elliptic or isomorphic to a plane quintic curve.

In section 2 we define the period maps  $\mathcal{P}, \mathcal{P}^*, \mathcal{P}^{**}$ . In fact, we show that the map  $\mathcal{P}^{**}$  is equivariant with respect to the natural actions of  $S_5$  and the maps  $\mathcal{P}, \mathcal{P}^*$  are obtained by taking the quotient for the action of subgroups. Afterwards, we shall prove Theorem 0.1.

In section 3 we study the discriminant divisor  $\mathcal{H}$  and its geometric meaning. In particular we give a proof of Theorem 0.2.

In the final section we give a proof of Theorem 0.3 and we compare this compactification with the GIT compactification of the space of plane sextics.

*Notation.* A lattice  $L$  is a free abelian group of finite rank equipped with a non-degenerate bilinear form, which will be denoted by  $(, )$ .

- The discriminant group of  $L$  is the finite abelian group  $A_L = L^*/L$ , where  $L^* = \text{Hom}(L, \mathbb{Z})$ , equipped with the quadratic form  $q_L : A_L \rightarrow \mathbb{Q}/2\mathbb{Z}$  defined by  $q_L(x + L) = (x, x) \pmod{2\mathbb{Z}}$ .

-  $O(L)$  and  $O(q_L)$  will denote the groups of isometries of  $L$  and  $A_L$ , respectively.

- A lattice is unimodular if  $|A_L| = |\det L| = 1$ .

- If  $M$  is the orthogonal complement of  $L$  in a unimodular lattice, then  $A_L \cong A_M$  and  $q_M = -q_L$ .

- We will denote by  $U$  the hyperbolic plane and by  $A_n, D_n, E_n$  the negative definite lattices of rank  $n$  associated to the Dynkin diagrams of the corresponding types.

- The lattice  $L(\alpha)$  is obtained multiplying by  $\alpha$  the form on  $L$ .

- The lattice  $L^m$  is the orthogonal direct sum of  $m$  copies of the lattice  $L$ .

We will refer the reader to [N1] for basic facts about lattices.

### 1. CURVES OF GENUS SIX AND QUINTIC DEL PEZZO SURFACES

We start by recalling some well-known properties of curves of genus six. By Brill-Noether theory any smooth curve of genus six  $C$  has a special divisor  $D$  with  $\deg(D) = 6$  and  $h^0(C, D) = 3$ . Let  $\varphi_D$  be the morphism associated to  $D$ :

$$\varphi_D : C \longrightarrow \mathbb{P}^2.$$

**Definition 1.** A smooth genus six curve will be called *special* if it is either hyperelliptic, trigonal, bi-elliptic or isomorphic to a smooth plane quintic curve.

The following is given, for example, in section A, Ch.V in [ACGH].

**Proposition 1.1.** *Let  $C$  be a smooth curve of genus six. Then one of the following holds:*

- a)  $\varphi_D$  is birational and  $\varphi_D(C)$  is an irreducible plane sextic having only double points.
- b)  $C$  is special.

*Case a.* Assume first that  $\varphi_D(C)$  is an irreducible plane sextic with 4 nodes  $p_1, \dots, p_4$  in general position. The blowing up of  $\mathbb{P}^2$  in these points is a quintic del Pezzo surface  $Y$  and  $C \in |-2K_Y|$ . In fact, the embedding  $C \subset Y \subset \mathbb{P}^5$  is the canonical embedding of  $C$  and  $Y$  is the unique quintic del Pezzo surface containing  $C$  (see e.g. [SB]). Let  $e_0$  be the class of the pull back of a line and let  $e_i$  be the classes of exceptional divisors over the points  $p_i$ . The surface  $Y$  admits exactly five birational morphisms to  $\mathbb{P}^2$ , called *blowing down maps*, induced by the linear systems:

$$e_0, 2e_0 - \sum_{i=1}^4 e_i + e_j, \quad j = 1, \dots, 4.$$

Note that any such morphism maps  $C$  to a plane sextic with 4 double points. In fact, the converse also holds i.e. there is a one-to-one correspondence between the set of blowing down maps for  $Y$  and the set  $W_6^2(C)$  of  $g_6^2$  on  $C$ . In particular, the generic curve of genus six has exactly five  $g_6^2$ .

The automorphisms group of  $Y$  is known to be isomorphic to  $S_5$  and to be generated by the automorphisms induced by projectivities permuting  $p_1, \dots, p_4$  and by an automorphism of order five which is realized by a quadratic transformation  $\alpha$  with fundamental points at  $p_1, p_2, p_3$  (see [Do, Theorem 10.2.2]).

If the double points  $p_1, \dots, p_4$  of  $\varphi_D(C)$  are not in general position or are infinitely near, then the blowing up of  $\mathbb{P}^2$  at these points is still a *nodal* del Pezzo surface, i.e.  $-K_Y$  is nef and big or, equivalently, the anti-canonical model of  $Y$  has at most rational double points. Moreover, as before, there is a unique nodal del Pezzo surface  $Y$  containing  $C$  (see [AH, 5.14]). In this case the surface  $Y$  has less than five blowing down maps and  $C$  has less than five  $g_6^2$ .

*Case b.* The following characterization holds:

**Proposition 1.2.** *A curve of genus six  $C$  is special if and only if  $\dim W_6^2(C) > 0$ .*

*Proof.* We have seen that if  $C$  is not special, then  $\dim W_6^2(C) = 0$  and contains at most five points. We now see what happens for special curves ([ACGH]).

- If  $C$  is *trigonal*, then it has two types of  $g_6^2$ :

$$D = 2g_3^1 \quad \text{and} \quad D(p) = K_C - g_3^1 - p, \quad p \in C.$$

Hence  $W_6^2(C)$  is one dimensional and has two irreducible components. The plane model  $\varphi_D(C)$  is a triple conic and  $\varphi_{D(p)}(C)$  is a plane sextic with a triple point and a node.

- If  $C$  is isomorphic to a *plane quintic*, then any  $g_6^2$  on  $C$  is of type:  $D(p) = g_5^2 + p, p \in C$ . Hence  $W_6^2(C) \cong C$ . The plane model  $\varphi_{D(p)}(C)$  is a plane quintic.
- If  $C$  is *bi-elliptic* i.e. there exists  $\pi : C \rightarrow E$ , where  $E$  is an elliptic curve, then any  $g_6^2$  corresponds to  $\phi \circ \pi$  where  $\phi$  is a  $g_2^1$  on  $E$ . The plane model of  $C$  is a double cubic.

- If  $C$  is hyperelliptic, then any  $g_6^2$  is of type:

$$D(p, q) = K_C - g_2^1 - p - q, \quad p, q \in C.$$

Hence  $W_6^2(C) \cong \text{Sym}^2(C)$ . In fact  $D = K_C - 2g_2^1$  is a singular point of  $W_6^2(C)$ . The plane model  $\varphi_D(C)$  is a double rational cubic and  $\varphi_{D(p,q)}(C)$  is a double conic.  $\square$

Let  $p_1, \dots, p_4$  be the fundamental points of  $\mathbb{P}^2$  and let  $U$  be the space of plane sextics with double points at  $p_1, \dots, p_4$ . The group  $G \cong S_5$  generated by the projectivities permuting the  $p_i$ 's and the quadratic transformation  $\alpha$  acts on  $U$ .

We will denote by  $W_6^2$  the moduli space of pairs  $(C, D)$ , where  $C$  is a smooth curve of genus six and  $D$  is a  $g_6^2$  of  $C$  (see [ACGH]), and by  $\widetilde{W}_6^2$  the moduli space of sets  $(C, D_1, \dots, D_5)$ , where  $C$  is a non-special smooth curve of genus six and  $D_i$  are the  $g_6^2$  of  $C$ . Observe that there are natural projections

$$\widetilde{W}_6^2 \xrightarrow{\pi_1} W_6^2 \xrightarrow{\pi_2} \mathcal{M}_6 \quad (C, D_1, \dots, D_5) \longmapsto (C, D_1) \longmapsto C.$$

**Corollary 1.3.** *Let  $G_1 \cong S_4$  be the subgroup of  $G$  generated by projectivities permuting the  $p_i$ 's, then we have the following diagram:*

$$\begin{array}{ccccc} \widetilde{W}_6^2 & \xrightarrow{\pi_1} & W_6^2 & \xrightarrow{\pi_2} & \mathcal{M}_6 \\ \uparrow & & \uparrow & & \uparrow \\ U & \longrightarrow & U/G_1 & \longrightarrow & U/G, \end{array}$$

where the vertical arrows are birational maps.

*Proof.* The first vertical arrow on the left is defined as follows. If  $S$  is a plane sextic in  $U$ , then its normalization is a smooth curve  $C$  of genus six which is not special and carries five ordered  $g_6^2$ 's: the linear system of hyperplane sections of  $S$  and the linear systems of conics through all the fundamental points except  $p_i$ , for  $i = 1, \dots, 4$ . Taking the quotient for the action of  $G_1$  is equivalent to forgetting the order of the last 4 linear systems, thus to a class  $[S] \in U/G_1$  we can associate a genus six curve and the linear system of hyperplane sections of  $S$ . Finally, taking the quotient for  $G$ , is equivalent to forgetting the order of the  $g_6^2$ 's on  $C$ .  $\square$

## 2. K3 SURFACES ASSOCIATED TO CURVES OF GENUS SIX

**2.1. The geometric construction.** Let  $C \subset \mathbb{P}^5$  be the canonical model of a non-special smooth curve of genus six. By the remarks in the previous section, there is a unique nodal del Pezzo surface  $Y$  such that  $C$  lies in the anti-canonical model of  $Y$  in  $\mathbb{P}^5$ . Let  $Y' \rightarrow Y$  be the canonical resolution of rational double points of  $Y$ . Since  $C \in |-2K_{Y'}|$ , there exists a double cover

$$\pi : X \longrightarrow Y'$$

branched along  $C$  and  $X$  is a K3 surface. It is well known that  $H^2(X, \mathbb{Z})$ , together with the cup product, is an even unimodular lattice of signature  $(3, 19)$ . The covering involution  $\sigma$  of  $\pi$  acts on this lattice with eigenspaces

$$H^2(X, \mathbb{Z})^\pm = \{x \in H^2(X, \mathbb{Z}) : \sigma^*(x) = \pm x\}.$$

**Lemma 2.1.**  $H^2(X, \mathbb{Z})^+ \cong A_1(-1) \oplus A_1^4$ ,  $H^2(X, \mathbb{Z})^- \cong U \oplus U \oplus E_8 \oplus A_1^5$ .

*Proof.* By definition, the lattices  $H^\pm = H^2(X, \mathbb{Z})^\pm$  are 2-elementary, i.e. their discriminant groups are 2-elementary abelian groups. By [N1, Theorem 3.6.2] the isomorphism class of a 2-elementary even indefinite lattice  $L$  is determined uniquely by the triple  $(s, \ell, \delta)$ , where  $s$  is the signature,  $\ell$  is the minimal number of generators of  $A_L$  and  $\delta$  is 0 (resp. 1) if the quadratic form on  $A_L$  always assumes integer values (resp. otherwise). On the other hand, [N2, Theorem 4.2.2] shows that  $H^+$  has  $s = (1, 4)$ ,  $\ell = 5$ ,  $\delta = 1$ . Since  $H^-$  is the orthogonal complement of  $H^+$  in the unimodular lattice  $H^2(X, \mathbb{Z})$ , it has  $s = (2, 15)$ ,  $\ell = 5$ ,  $\delta = 1$ . Hence it is enough to check that the lattices in the right hand sides have the same triple of invariants.  $\square$

Let  $S_X$  be the Picard lattice of  $X$  and let  $T_X$  be its transcendental lattice:

$$S_X = H^2(X, \mathbb{Z}) \cap \omega_X^\perp, \quad T_X = S_X^\perp.$$

Note that the invariant lattice  $H^2(X, \mathbb{Z})^+$  coincides with the pull-back of the Picard lattice of  $Y$ , hence

$$H^2(X, \mathbb{Z})^+ \subset S_X, \quad T_X \subset H^2(X, \mathbb{Z})^-.$$

If  $\omega_X$  is a nowhere vanishing holomorphic 2-form on  $X$ , then  $\omega_X \in T_X \otimes \mathbb{C}$ , hence  $\sigma^*(\omega_X) = -\omega_X$ .

**Lemma 2.2.** *There are no  $(-2)$ -vectors in  $S_X \cap H^2(X, \mathbb{Z})^-$ .*

*Proof.* Assume that  $r$  is such a vector. By the Riemann-Roch theorem we may assume that  $r$  is effective. Then  $\sigma^*(r) = -r$  is also effective. This is a contradiction.  $\square$

**2.2. Lattices.** We will denote by  $L_{K3}$  an even unimodular lattice of signature  $(3, 19)$ . This is known to be unique up to isomorphisms (see e.g. [N1, Theorem 1.1.1]), hence the lattice  $H^2(X, \mathbb{Z})$  is isomorphic to  $L_{K3}$ . Let  $\bar{e}_0, \bar{e}_1, \dots, \bar{e}_4$  be the pull-backs of the classes  $e_0, e_1, \dots, e_4$  under  $\pi^*$ . These generate a sublattice of  $S_X$  isometric to  $A_1(-1) \oplus A_1^4$ . Let

$$S = A_1(-1) \oplus A_1^4, \quad T = U \oplus U \oplus E_8 \oplus A_1^5.$$

Denote by  $s_0, s_1, \dots, s_4$  an orthogonal basis for  $S$  with  $s_0^2 = 2, s_i^2 = -2, i = 1, \dots, 4$  and denote by  $r_1, \dots, r_5$  an orthogonal basis for the  $A_1^5$  component of  $T$ .

**Lemma 2.3.** *Let  $\xi_i = r_i/2$ , then the discriminant group  $A_T$  consists of the following vectors:*

$$\begin{aligned} q(x) = 0 : & \quad 0, \sum_{i \neq j} \xi_i, \quad 1 \leq j \leq 5, \\ q(x) = 1 : & \quad \xi_i + \xi_j, \quad 1 \leq i < j \leq 5, \\ q(x) = -1/2 : & \quad \xi_i, \quad 1 \leq i \leq 5, \sum_{i=1}^5 \xi_i, \\ q(x) = -3/2 : & \quad \sum_{i \neq j, k} \xi_i, \quad 1 \leq j < k \leq 5. \end{aligned}$$

It follows from [N1, Theorem 1.14.4] that  $S$  can be embedded uniquely in  $L_{K3}$  and  $T$  is isomorphic to its orthogonal complement. Since  $L_{K3}$  is unimodular,

$$A_S \cong A_T \cong \mathbb{F}_2^5, \quad q_S \cong -q_T$$

and an isomorphism from  $A_S$  to  $A_T$  is given by

$$s_0/2 \mapsto \xi_1, \quad (2s_0 - \sum_{i=1}^4 s_i + s_j)/2 \mapsto \xi_{j+1}, \quad j = 1, \dots, 4.$$

**Lemma 2.4.** *There are isomorphisms  $O(q_S) \cong O(q_T) \cong S_5$  and the natural maps*

$$O(T) \rightarrow O(q_T), \quad O(S) \rightarrow O(q_S)$$

*are surjective.*

*Proof.* The first statement follows from [MS]. Note that  $O(q_T)$  acts on  $A_T$  by permuting the  $\xi_i$ 's. The surjectivity statement for  $T$  is obvious, since clearly isometries of  $T$  permuting the  $r_i$ 's exist. On the other hand, the automorphism group  $S_5$  of  $Y$  acts on  $S$  as isometries. These isometries act on  $A_S$  as  $S_5$ . More concretely, the isometries of  $S$  permuting the  $s_i$ 's ( $1 \leq i \leq 4$ ) and the isometry

$$s_0 \mapsto 2s_0 - s_1 - s_2 - s_3, \quad s_1 \mapsto s_0 - s_1 - s_3, \quad s_2 \mapsto s_4, \quad s_3 \mapsto s_0 - s_2 - s_3, \quad s_4 \mapsto s_0 - s_1 - s_2$$

generate  $O(q_S)$ . □

In the following we will consider three arithmetic groups acting on  $T$ :

$$\Gamma = O(T), \quad \Gamma^* = \{\gamma \in O(T) : \gamma(\xi_1) = \xi_1\}, \quad \Gamma^{**} = \{\gamma \in O(T) : \gamma|_{A_T} = 1\}.$$

Note that  $\Gamma/\Gamma^{**} \cong O(q_T) \cong S_5$ .

**Lemma 2.5.** *Let  $O_T = \{\gamma \in O(L_{K3}) : \gamma(T) = T\}$ . Then the restriction homomorphisms*

$$O_T \rightarrow \Gamma, \quad \{\gamma \in O_T : \gamma(s_0) = s_0\} \rightarrow \Gamma^* \quad \text{and} \quad \{\gamma \in O_T : \gamma|_S = 1_S\} \rightarrow \Gamma^{**}$$

*are surjective.*

*Proof.* Let  $\gamma \in \Gamma$ . By Lemma 2.4 there exists  $\beta \in O(S)$  such that  $\beta = \gamma$  on  $A_S \cong A_T$ . Then the isometry  $\beta \oplus \gamma$  on  $S \oplus T$  lifts to an isometry in  $O_T$ . If  $\gamma \in \Gamma^*$  or  $\Gamma^{**}$ , then  $\beta$  can be chosen such that  $\beta(s_0) = s_0$  or  $\beta = 1_S$ , respectively (see the proof of Lemma 2.4). □

*Remark 2.6.* There are two orbits of vectors with  $q(x) = -1/2$  under the action of  $O(q_T)$ :

$$O_1 = \left\{ \sum_{i=1}^5 \xi_i \right\}, \quad O_2 = \{\xi_1, \dots, \xi_5\}.$$

**2.3. Moduli spaces.** Since both  $S$  and  $T$  are 2-elementary lattices, the isometry  $(1_S, -1_T)$  on  $S \oplus T$  can be extended to an isometry  $\iota$  of  $L_{K3}$ . Let  $\alpha : H^2(X, \mathbb{Z}) \rightarrow L_{K3}$  be an isometry satisfying  $\alpha(H^2(X, \mathbb{Z})^+) = S$ . Then  $\iota \circ \alpha = \alpha \circ \sigma^*$ . Since  $\sigma^*(\omega_X) = -\omega_X$ , then the *period*

$$p_X(\alpha) = \alpha_{\mathbb{C}}(\omega_X)$$

belongs to the set

$$\mathcal{D} = \{\omega \in \mathbb{P}(T \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\},$$

called the *period domain of  $S$ -polarized K3 surfaces*. By Lemma 2.2, there are no  $(-2)$ -vectors orthogonal to the period, hence  $p_X(\alpha)$  belongs to the complement of the divisor

$$\mathcal{H} = \bigcup_{r \in T, r^2 = -2} \mathcal{H}_r \quad \text{where} \quad \mathcal{H}_r = \{\omega \in \mathcal{D} : (r, \omega) = 0\}.$$

Consider the orbit spaces

$$\mathcal{M} = \mathcal{D}/\Gamma, \quad \mathcal{M}^* = \mathcal{D}/\Gamma^*, \quad \mathcal{M}^{**} = \mathcal{D}/\Gamma^{**}.$$

**Theorem 2.7.** *The geometric construction in Subsection 2.1 defines the following diagram:*

$$\begin{array}{ccccc}
 \widetilde{\mathcal{W}}_6^2 & \xrightarrow{\pi_1} & \mathcal{W}_6^2 & \xrightarrow{\pi_2} & \mathcal{M}_6 \\
 \downarrow \mathcal{P}^{**} & & \downarrow \mathcal{P}^* & & \downarrow \mathcal{P} \\
 \mathcal{M}^{**} & \xrightarrow{q_1} & \mathcal{M}^* & \xrightarrow{q_2} & \mathcal{M},
 \end{array}$$

where  $\mathcal{P}^{**}, \mathcal{P}^*, \mathcal{P}$  are birational maps,  $\pi_1, \pi_2$  are defined as in Section 1 and  $q_1, q_2 \circ q_1$  are the quotients for the action of  $\Gamma^*/\Gamma^{**}$  and  $\Gamma/\Gamma^{**}$ , respectively. In fact,  $\mathcal{P}$  induces an isomorphism

$$\mathcal{M}_6 \setminus \{\text{special curves}\} \cong \mathcal{M} \setminus (\mathcal{H}/\Gamma).$$

*Proof.* Let  $(C, D_1, \dots, D_5) \in \widetilde{\mathcal{W}}_6^2$ , i.e.  $C$  is a non-special smooth curve of genus six and  $D_i$  are its  $g_6^2$ 's. Let  $Y$  be the unique del Pezzo surface containing the canonical model of  $C$ . Since the  $g_6^2$  on  $C$  are in bijection with the blowing down maps on  $Y$ , then  $Y$  comes with an ordered basis  $e_0, e_1, \dots, e_4$  for its Picard lattice (see Section 1).

As explained in Subsection 2.1, the double cover of  $Y$  branched along  $C$  is a K3 surface  $X$ . If  $C$  is general, then  $S_X = H^2(X, \mathbb{Z})^+$  is the pull-back of the Picard lattice of  $Y$  and  $\{\bar{e}_0, \bar{e}_1, \dots, \bar{e}_4\}$  is an ordered basis of  $S_X$ .

As observed before, the embedding of  $S$  in  $L_{K3}$  is unique by [N1, Theorem 1.14.4]. This implies, together with [N1, Corollary 1.5.2] and Lemma 2.4, that there exists a marking  $\alpha : H^2(X, \mathbb{Z}) \rightarrow L_{K3}$  such that  $\alpha(H^2(X, \mathbb{Z})^+) = S$  and  $\alpha(\bar{e}_i) = s_i, 0 \leq i \leq 4$ . Then, by Lemma 2.2,  $\alpha_C(\omega_X) \in \mathcal{D} \setminus \mathcal{H}$ . Moreover, if  $\alpha_1, \alpha_2$  are two markings of this type, then  $\alpha_2 \circ \alpha_1^{-1}$  preserves the ordered basis  $\{s_i\}$ , hence its restriction to  $T$  belongs to  $\Gamma^{**}$ . Thus we can associate to  $(C, D_1, \dots, D_5)$  a point in  $\mathcal{D}/\Gamma^{**}$ , i.e. we defined a rational map  $\mathcal{P}^{**} : \widetilde{\mathcal{W}}_6^2 \dashrightarrow \mathcal{M}^{**}$ .

Conversely, let  $\omega \in \mathcal{D} \setminus \mathcal{H}$ . By the surjectivity theorem of the period map ([Ku, PP]) there exists a marked K3 surface  $(X, \alpha)$  such that  $\alpha_C(\omega_X) = \omega$ . Then  $\iota(\omega) = -\omega$  and there exist no  $(-2)$ -vectors in  $T \cap \omega^\perp$  since  $\omega \notin \mathcal{H}$ , hence  $\iota$  preserves an ample class. It now follows from the Torelli theorem [Na, Theorem 3.10] that  $\iota$  is induced by an automorphism  $\sigma$  on  $X$ .

By [N2, Theorem 4.2.2] the fixed locus of  $\sigma$  is a smooth curve  $C$  of genus six. The quotient surface  $Y = X/(\sigma)$  is smooth and the image of  $C$  belongs to  $|-2K_Y|$ . Hence  $-K_Y$  is nef and big with  $K_Y^2 = 5$ , i.e.  $Y$  is a nodal quintic del Pezzo surface. In fact, the pull back of  $\text{Pic}(Y)$  is exactly  $\alpha^{-1}(S) \subset S_X$ .

If we choose  $\omega \in \mathcal{D} \setminus \mathcal{H}$  up to the action of  $\Gamma^{**}$ , then, by Lemma 2.5, we get  $\alpha$  up to an isometry in  $O_T$  which preserves an ordered basis  $\{s_i\}$ . Thus the K3 surface  $X$  comes with a basis  $\alpha^{-1}(s_i), 0 \leq i \leq 4$  for  $\alpha^{-1}(S)$ . This gives an ordered basis for the Picard lattice of  $Y$  and induces an order on its blowing down maps. Thus  $C$  is equipped with an order on its  $g_6^2$ 's and  $\mathcal{P}^{**}$  is birational.

The quotients  $\Gamma/\Gamma^{**} \cong S_5$  and  $\Gamma^*/\Gamma^{**} \cong S_4$  act on  $\mathcal{M}^{**}$  and by Lemma 2.5 they lift to isometries of  $L_{K3}$  which preserve  $T$  and  $S$ . By taking the quotient for these actions, we get birational maps  $\mathcal{P}$  and  $\mathcal{P}^*$ .

The previous construction gives an isomorphism between the moduli space of non-special smooth curves of genus six (not necessarily carrying five distinct  $g_6^2$ 's) and the quotient  $\mathcal{D} \setminus \mathcal{H}$ .  $\square$

*Remark 2.8.* As observed in Section 1, if  $C$  carries less than five  $g_6^2$ , then the del Pezzo surface  $Y$  is nodal, i.e. its anti-canonical model has rational double points. In this case the K3 surface  $X$  is obtained by taking the minimal resolution of the double cover of  $Y$  branched along  $C$ .

If  $C$  has exactly four  $g_6^2$ , then it can easily be proved that  $Y$  is the blowing up of  $\mathbb{P}^2$  at four points  $p_1, \dots, p_4$  such that either 3 are collinear or two are infinitely near. Assume that  $p_1, p_2, p_3$  lie on a line  $L$ . The pencil of lines through  $p_4$  induces an elliptic fibration on  $X$  with a general fiber  $f$ , three singular fibers of Kodaira type  $I_2$  and two sections  $\sigma_1, \sigma_2$ , given by the two (disjoint) inverse images of the line  $L$ . In particular, the Picard lattice of  $X$  contains the sublattice  $S' = U \oplus A_1^3 \oplus \langle -4 \rangle$ , where  $U$  is generated by the fiber  $f$  and  $\sigma_1, A_1^3$  by the reducible components in each fiber and  $\langle -4 \rangle$  by  $2f + \sigma_1 - \sigma_2$ . Conversely, let  $r \in T$  be a primitive vector with  $r^2 = -4$  such that  $r/2 \in A_T$ , then its orthogonal complement in  $T$  is isomorphic to  $T' = U \oplus U \oplus E_8 \oplus A_1^3 \oplus \langle -4 \rangle$  and  $(T')^\perp \cong S'$ .

By choosing a different blowing down map for  $Y$ , we get two infinitely near points in the image of the exceptional divisors. In this case, the image of  $C$  by this map is a plane sextic with a tacnode and two nodes. The elliptic fibration described above is induced by the pencil of lines through the tacnode.

### 3. THE DISCRIMINANT DIVISOR

In the previous section we introduced a divisor  $\mathcal{H}$  in  $\mathcal{D}$ . The image of this divisor in  $\mathcal{M}$  or  $\mathcal{M}^*$  will be called *discriminant divisor*. We now describe its structure and its geometric meaning. We will show that  $\mathcal{H}/\Gamma$  has 3 irreducible components corresponding to nodal curves, plane quintics with a line, and trigonal curves with a section in  $L \in |K_C - 2g_3^1|$ , respectively, (see Proposition 3.3).

#### 3.1. Irreducible components.

**Lemma 3.1.** *Let  $\Delta$  be the set of vectors  $r \in T$  with  $r^2 = -2$ . Then*

- *the group  $\Gamma$  has three orbits in  $\Delta$  :*

$$\Delta_1 = \{r \in \Delta : r/2 \notin T^*\}, \quad \Delta_2 = \{r \in \Delta : r/2 \in O_1\}, \quad \Delta_3 = \{r \in \Delta : r/2 \in O_2\};$$

- *the group  $\Gamma^*$  has 4 orbits in  $\Delta$  :  $\Delta_1, \Delta_2$  and two orbits decomposing  $\Delta_3$*

$$\Delta_{3a} = \{r \in \Delta : r/2 = \xi_2\}, \quad \Delta_{3b} = \{r \in \Delta : r/2 = \xi_1\}.$$

*Proof.* Given a vector  $r \in \Delta$  we will classify the embeddings of  $\Lambda = \langle r \rangle$  in  $T$  up to the action of  $\Gamma$  by applying [N1, Proposition 1.15.1]. We first need to give an isometry  $\alpha$  between a subgroup of  $A_\Lambda$  and a subgroup of  $A_T \cong \mathbb{F}_2^5$ . If  $H$  is such a subgroup, then either  $H = 0$  or  $H = \mathbb{F}_2$ . Note that  $H = \mathbb{F}_2$  if and only if  $r/2 \in T^*$ .

In case  $H = 0$ , since there is a unique lattice  $K$  with  $q_K = q_\Lambda \oplus (-q_T)$  and  $O(K) \rightarrow O(q_K)$  is surjective by [N1, Theorem 1.14.2], then by [N1, Proposition 1.15.1] there is a unique embedding of  $\Lambda$  in  $T$  such that  $\Lambda \oplus \Lambda^\perp = T$ .

In case  $H = \mathbb{F}_2$  there are two different embeddings of  $\Lambda$ , according to the choice of  $\alpha(r/2)$  in  $O_1$  or  $O_2$ . This gives the first assertion.

The second assertion can be proved in a similar way, by observing that  $\Gamma^*$  has three orbits on the set of vectors  $x \in A_T$  with  $q(x) = -1/2$ . □

For  $r \in \Delta_i$ , let  $T_i = \{x \in T : (x, r) = 0\}$  and denote by  $S_i$  the orthogonal complement of  $T_i$  in  $L_{K3}$ . Then we have the following.

**Lemma 3.2.**

$$\begin{aligned} S_1 &\cong A_1(-1) \oplus A_1^5, & T_1 &\cong U \oplus U \oplus E_7 \oplus A_1^5, \\ S_2 &\cong U(2) \oplus D_4, & T_2 &\cong U \oplus U(2) \oplus E_8 \oplus D_4, \\ S_3 &\cong U \oplus A_1^4, & T_3 &\cong U \oplus U \oplus E_8 \oplus A_1^4. \end{aligned}$$

*Proof.* Because of Lemma 3.1 the isomorphism class of  $T_i$  does not depend on the choice of  $r \in \Delta_i$ . If  $r \in \Delta_1$  or  $\Delta_3$ , then we can assume  $r$  to be one generator of  $E_8$  or, respectively, one generator of  $A_1$  in a decomposition  $T = U \oplus U \oplus E_8 \oplus A_1^5$ . If  $r \in \Delta_2$ , we can assume  $r$  to be a generator of  $A_1$  in a decomposition  $T = U \oplus U(2) \oplus E_8 \oplus D_4 \oplus A_1$ . In all these cases the orthogonal complement of  $r$  in  $T$  can be computed easily. The lattices  $S_i$  can be computed by applying [N1, Theorem 3.6.2].  $\square$

**Proposition 3.3.** *The divisor  $\mathcal{H}/\Gamma$  has 3 irreducible components  $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$  and  $\mathcal{H}/\Gamma^*$  has 4 irreducible components  $\mathcal{H}_1^*, \mathcal{H}_2^*, \mathcal{H}_{3a}^*, \mathcal{H}_{3b}^*$  such that*

- $\mathcal{H}_i^* \rightarrow \mathcal{H}_i, \quad i = 1, 2$  have degree 5,
- $\mathcal{H}_{3a}^* \rightarrow \mathcal{H}_3, \quad \mathcal{H}_{3b}^* \rightarrow \mathcal{H}_3$  have degree 4 and 1, respectively.

Let  $\iota_i$  be the isometry of  $L_{K3}$  defined by  $\iota_i|_{S_i} = 1_{S_i}$  and  $\iota_i|_{T_i} = -1_{T_i}$ . The following can be proved by means of the Torelli theorem, as in the proof of Theorem 2.7.

**Lemma 3.4.** *There exists a K3 surface  $X_i$  such that  $S_{X_i} \cong S_i$  and carrying an involution  $\sigma_i$  of  $X_i$  with  $\sigma_i^* = \iota_i$ .*

**3.2. Curves of genus six with a node.** Let  $C_1$  be a generic plane sextic with five nodes. The blowing up of the projective plane at the nodes is a quartic del Pezzo surface  $Y_1$  and its double cover branched along the strict transform of  $C_1$  is a K3 surface  $X$ . Alternatively, if we blow up the plane at four nodes, we get a quintic del Pezzo surface on which the strict transform of  $C_1$  is a curve of genus six with a node. The pull-back of  $\text{Pic}(Y_1)$  is a sublattice of the Picard lattice of  $X$  isomorphic to  $S_1$ . We now show that the converse is also true.

**Proposition 3.5.** *The K3 surface  $X_1$  is birational to the double cover of a quintic del Pezzo surface branched along a generic curve of genus six with a node or, equivalently, to a double plane branched along a generic sextic with 5 nodes.*

*Proof.* Consider the involution  $\sigma_1$  on  $X_1$  as in Lemma 3.4. By [N2, Theorem 4.2.2] the fixed locus of  $\sigma_1$  is a smooth curve  $C_1$  of genus 5. The quotient of  $X_1$  by  $\sigma_1$  is a smooth rational surface  $Y_1$  and the image of  $C_1$  belongs to  $|-2K_{Y_1}|$ , hence  $Y_1$  is a del Pezzo surface of degree 4.

Any  $(-1)$ -curve  $e$  on  $Y_1$  intersects the image of  $C_1$  at two points since  $(-K_{Y_1}, e) = 1$ . Hence, contracting one  $(-1)$ -curve of  $Y_1$  we get a quintic del Pezzo surface where the image of  $C_1$  is a curve of genus six with a node, and contracting five disjoint  $(-1)$ -curves  $C_1$  is mapped to a plane sextic with five nodes.  $\square$

**Corollary 3.6.** *The divisor  $\mathcal{H}_1$  is birational to the moduli space of curves of genus six with one node and  $\mathcal{H}_1^*$  to the moduli space of plane sextics with 5 nodes, with one marked.*

*Proof.* Taking the quotient of  $\mathcal{H}_1$  for the action of  $\Gamma$ , we identify two markings on  $X_1$  which give the same embedding of  $\alpha^{-1}(S)$  in  $\text{Pic}(X_1)$ . This data identifies a  $(-1)$ -curve on  $Y_1$ , whose contraction gives a quintic del Pezzo surface and a curve

of genus six with a node. The group  $\Gamma^*$ , instead, identifies two markings on  $X_1$  if they also give the same embedding of  $\alpha^{-1}(h)$  in the Picard lattice. This class gives a blowing down map on  $Y_1$  with a distinguished exceptional divisor.

Using these remarks and Proposition 3.5, the result follows as in the proof of Theorem 2.7. □

**3.3. Plane quintics.** Let  $C_2$  be a smooth plane quintic and let  $L$  be a line transversal to  $C_2$ . The minimal resolution of the double plane branched along  $C_2 \cup L$  is a K3 surface  $X$ . The Picard lattice of  $X$  contains five disjoint  $(-2)$ -curves, coming from the resolution of singularities, and a  $(-2)$ -curve which is the proper transform of  $L$ . These rational curves generate a lattice which is isomorphic to  $S_2$ .

**Proposition 3.7.** *The surface  $X_2$  is birational to a double plane branched along the union of a plane quintic and a line.*

*Proof.* This was proved in [La, Ch.6]. □

**Corollary 3.8.** *The divisor  $\mathcal{H}_2$  is birational to the moduli space of pairs  $(C, L)$  where  $C$  is a plane quintic and  $L$  is a line, while  $\mathcal{H}_2^*$  parametrizes triples  $(C, L, p)$  where  $p \in C \cap L$ .*

*Proof.* The first statement is [La, Corollary 6.21]. The second statement can be proved similarly to Corollary 3.6. □

**3.4. Trigonal curves of genus six.** Let  $C \subset \mathbb{P}^5$  be the canonical model of a trigonal curve of genus six. Any 3 points in the  $g_1^3$  lie on a line by Riemann-Roch theorem and the closure of the union of all these lines is a quadric  $Q$  such that the curve  $C$  belongs to  $|4f + 3e|$ , where  $e, f$  are the rulings of  $Q$ . The minimal resolution of the double cover of  $Q$  branched along the union of  $C$  with a line  $L \in |e|$  is a K3 surface  $X$ . The ruling  $f$ , the proper transform of  $L$  and the exceptional divisors over the four points in  $C \cap L$  generate a sublattice of  $S_X$  isomorphic to  $S_3$ .

As before, we now prove a converse statement.

**Proposition 3.9.** *The surface  $X_3$  is birational to:*

- *the double cover of a quadric  $Q$  branched along a line and a trigonal curve of genus six.*
- *the double cover of a Hirzebruch surface  $\mathbb{F}_4$  branched along a curve with 4 nodes in  $|3h|$  and the rational curve in  $|s|$ , where  $h^2 = 4, s^2 = -4, (h, s) = 0$ .*

*Proof.* By [N2, Theorem 4.2.2] the set of fixed points of  $\sigma_3$  on  $X_3$  is the disjoint union of a smooth curve  $C$  of genus six and a smooth rational curve  $L$ . Since  $S_3 \subset \text{Pic}(X_3)$ ,  $X_3$  admits an elliptic fibration  $\pi$  with a section and four singular fibers of Kodaira type  $I_2$  or type  $III$ . Since any fiber of  $\pi$  is preserved by  $\sigma_3$ , then  $L$  is a section of  $\pi$  and  $C$  intersects each fiber in 3 points. Hence  $C$  has a triple cover to  $\mathbb{P}^1$  and its ramification points are the singular points of irreducible fibers of  $\pi$ .

We will denote by  $F_1, \dots, F_4$  the singular fibers of  $\pi$  of type  $I_2$  or  $III$ , by  $E_i$  the component of  $F_i$  meeting  $L$  and by  $E'_i$  the other component. Let  $p : X_3 \rightarrow Y_3$  be the quotient by the involution  $\sigma_3$ . Note that  $p(E_i)$  and  $p(E'_i)$  are  $(-1)$ -curves.

By contracting the curves  $p(E_i)$ , we get a smooth quadric surface. This gives the first assertion.

On the other hand, contracting the curves  $p(E'_i)$ , we get a Hirzebruch surface  $\mathbb{F}_4$  (note that the image of  $L$  has self-intersection  $-4$ ). Since  $C$  intersects the ruling in 3 points, each  $E'_i$  at two points and it does not intersect  $L$ , then its image in  $\mathbb{F}_4$  has 4 nodes and belongs to the class  $3h$ . This gives the second assertion.  $\square$

**Corollary 3.10.**  $\bullet$  *The divisor  $\mathcal{H}_3$  is birational to the moduli space of pairs  $(C, L)$  where  $C$  is a trigonal curve of genus six and  $L \in |K_C - 2g_3^1|$ .*

- $\bullet$  *The divisor  $\mathcal{H}_{3a}^*$  parametrizes pairs  $(C, p)$  where  $C$  is trigonal and  $p \in C$  or, equivalently, plane sextics with a node and a triple point.*
- $\bullet$  *The divisor  $\mathcal{H}_{3b}^*$  is birational to the moduli space of curves in  $|3h|$  of  $\mathbb{F}_4$  with 4 nodes.*

*Proof.* The first statement follows from Proposition 3.9 and the remarks at the beginning of this subsection since, by adjunction formula, the restriction of  $e$  to  $C$  coincides with  $K_C - 2g_3^1$ .

Given a trigonal curve  $C \subset Q$  of genus six and  $p \in C$ , there exists a unique line  $L \in |e|$  through  $p$ . This determines a K3 surface  $X$  with  $S_3 \subset S_X$  as before. Moreover, the projection of  $C$  from  $p$  is a plane sextic with a triple point and a double point. The hyperplane class of  $\mathbb{P}^2$  induces the linear system  $K_C - g_3^1 - p$  on  $C$  and its pull-back to  $X$  is a nef class  $h$  with  $h^2 = 2$ .

Conversely, a generic point in  $\mathcal{H}_{3a}^* \cup \mathcal{H}_{3b}^*$  gives a K3 surface  $X$  with  $S_X \cong S_3 = U \oplus A_1^4$  and a degree two polarization  $h$ . Let  $e, f$  be a basis of  $U$  and  $e_1, \dots, e_4$  an orthogonal basis of  $A_1^4$ . Up to an isometry of  $S_3$  we can assume that  $r = e - f$  and that  $f$  gives an elliptic fibration on  $X$ . The orthogonal complement  $S_3 \cap r^\perp \cong S$  has two types of degree two polarizations:  $h_j = 2(e + f) - \sum_{i=1}^4 e_i + e_j$  or  $h = e + f$ .

A point in  $\mathcal{H}_{3b}^*$  gives a polarization  $h_b$  such that  $h_b/2 = r/2$  in  $A_T \cong A_S$ , hence  $h_b = h$ . The class  $h_b$  contains  $r$  in the base locus and  $2h_b$  maps  $X$  onto a cone over a rational normal quartic. In fact, the morphism associated to  $2h_b$  is exactly the contraction of the curves  $p(E'_i)$  and the image of the curve  $L$  described in the proof of Proposition 3.9.

A point in  $\mathcal{H}_{3a}^*$  gives a polarization  $h_a = h_j$  for some  $j = 1, \dots, 4$ . In this case  $h_a$  has no base locus and gives a generically 2:1 map  $X \rightarrow \mathbb{P}^2$ . The branch locus of this map is a plane sextic with a triple point (in the image of  $r$ ) and a node (in the image of  $e_j$ ). The line through the two singular points intersects the sextic in one more point  $p$ . Hence this gives a pair  $(C, p)$ , where  $C$  is trigonal and  $p \in C$ .  $\square$

*Remark 3.11.* The two irreducible components in  $\mathcal{M}^*$  over  $\mathcal{H}_3$  correspond to the components in  $\mathcal{W}_6^2$  over the trigonal divisor in  $\mathcal{M}_6$ . With the notation in the proof of Proposition 1.2: the divisor  $\mathcal{H}_{3a}^*$  corresponds to pairs  $(C, D(p))$  and  $\mathcal{H}_{3b}^*$  to  $(C, 2g_3^1)$ . This agrees with [Sh1], where it is proved that the triple conic, which is the plane model of  $C$  associated to  $2g_3^1$  (Proposition 1.2), “represents” K3 surfaces with a degree two polarization with a fixed component.

#### 4. COMPACTIFICATIONS

As mentioned in the Introduction, an arithmetic quotient of a bounded symmetric domain carries natural compactifications of arithmetic and geometric nature. In particular, it is an interesting problem to investigate the relation between the Satake-Baily-Borel (SBB) compactification of the period space and the compactifications coming from the GIT construction. In the case of degree two K3 surfaces,

it is known an explicit bijection between boundary components of the SBB compactification and boundary components of the GIT moduli space of plane sextics (see Table 2).

In this section we will determine the boundary components of the SBB compactification of the moduli spaces  $\mathcal{M}$  and  $\mathcal{M}^*$ . Also, we will give a conjectural correspondence between the boundary components of this compactification of  $\mathcal{M}$  and the GIT boundary components of the moduli space of plane sextics with some “extra” structure.

**4.1. Satake-Baily-Borel compactification.** The moduli spaces  $\mathcal{M}$ ,  $\mathcal{M}^*$  are quasi-projective algebraic varieties. Since they are arithmetic quotients of a symmetric bounded domain, we can consider their Satake-Baily-Borel (SBB) compactifications  $\overline{\mathcal{M}}$  and  $\overline{\mathcal{M}^*}$  (see [BB] and [Sc], §2).

It is known that boundary components of the SBB compactification are in bijection with primitive isotropic sublattices of  $T$  up to  $\Gamma$  and  $\Gamma^*$  respectively, such that  $k$ -dimensional boundary components correspond to rank  $k + 1$  isotropic sublattices. Since  $T$  has signature  $(2, 15)$ , the boundary components will be either 0 or 1 dimensional.

**Lemma 4.1.** *Let  $\mathcal{I}$  be the set of primitive isotropic vectors in  $T$ . There are two orbits in  $\mathcal{I}$  with respect to the action of  $\Gamma$  :*

$$\mathcal{I}_1 = \{v \in \mathcal{I} : (v, T) = \mathbb{Z}\} \quad \mathcal{I}_2 = \{v \in \mathcal{I} : (v, T) = 2\mathbb{Z}\}.$$

*There are three orbits with respect to  $\Gamma^*$  :  $\mathcal{I}_1$  and two orbits decomposing  $\mathcal{I}_2$ .*

*Proof.* By Proposition 4.1.3 in [Sc] there is a bijection between orbits of isotropic vectors in  $T$  modulo  $\Gamma$  ( $\Gamma^*$ ) and isotropic vectors in  $A_T$  modulo the induced action of  $\Gamma$  ( $\Gamma^*$ ). By Lemma 2.4 the map  $\Gamma \rightarrow O(q_T)$  is surjective and clearly the image of  $\Gamma^*$  is given by elements of  $O(q_T)$  fixing  $\xi_1$ . Then it follows from Lemma 2.3 that there are exactly two orbits of isotropic vectors in  $A_T$  for the action of  $\Gamma$  and three for the action induced by  $\Gamma^*$ . □

**Corollary 4.2.** *The boundaries of  $\overline{\mathcal{M}}$  and  $\overline{\mathcal{M}^*}$  contain two and three zero-dimensional components respectively.*

We will denote by  $p, q$  the zero-dimensional boundary components of  $\overline{\mathcal{M}}$  corresponding to the orbits  $\mathcal{I}_1, \mathcal{I}_2$  in Lemma 4.1 respectively and with  $q_1, q_2$  the zero-dimensional boundary components of  $\overline{\mathcal{M}^*}$  corresponding to the orbits of  $\Gamma^*$  decomposing  $\mathcal{I}_2$ .

*Remark 4.3.* By [N1, Theorem 3.6.2] there is also an isomorphism

$$T \cong U \oplus U(2) \oplus A_1 \oplus D_4 \oplus E_8.$$

In the following we will denote by  $e, f$  and  $e', f'$  the standard bases of  $U$  and  $U(2)$ , by  $\beta$  a generator of  $A_1$ , and by  $\gamma_1, \dots, \gamma_4$  and  $\alpha_1, \dots, \alpha_8$  the standard root bases of  $D_4$  and  $E_8$ . Note that  $e, f \in \mathcal{I}_1$  and  $e', f' \in \mathcal{I}_2$ .

We now classify one dimensional boundary components in  $\overline{\mathcal{M}}$  by studying  $\Gamma$ -orbits of primitive isotropic planes in  $T$ . We will say that such a plane is of *type*  $(i, j)$ ,  $i, j = 1, 2$  if it is generated by a vector in  $\mathcal{I}_i$  and one in  $\mathcal{I}_j$ .

Let  $\mathcal{G}_1$  be the genus of  $E_8 \oplus A_1^5$  and let  $\mathcal{G}_2$  be the genus of  $E_8 \oplus A_1 \oplus D_4$ . If  $N$  is a lattice in  $\mathcal{G}_1$ , then  $T \cong U \oplus U \oplus N$  by [N1, Theorem 3.6.2]. By taking two isotropic vectors, each in one copy of  $U$ , we get an isotropic plane in  $T$  of type

(1, 1). Similarly, if  $N_2 \in \mathcal{G}_2$ , then  $T \cong U \oplus U(2) \oplus N_2$  and the plane generated by a generator of  $U$  and one of  $U(2)$  is isotropic of type (1, 2).

**Lemma 4.4.** *The isomorphism classes of lattices in  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are given in the following table.*

TABLE 1. One dimensional boundary components

	$R$	$\mathcal{G}_1$	$\mathcal{G}_2$
a	$E_8^3$	$E_7 \oplus D_4 \oplus A_1^2, D_6^2 \oplus A_1, E_8 \oplus A_1^5$	$A_1 \oplus E_8 \oplus D_4$
b	$E_7^2 \oplus D_{10}$	$\overline{E_7 \oplus A_1^6}, \overline{D_6 \oplus D_4 \oplus A_1^3}, D_8 \oplus D_4 \oplus A_1,$ $\overline{D_8 \oplus A_1^5}, D_{10} \oplus A_1^3$	$E_7 \oplus D_6, \overline{D_{10} \oplus A_1^3}$
c	$D_{16} \oplus E_8$	$\overline{D_8 \oplus A_1^5}$	$D_{12} \oplus A_1$
d	$A_{17} \oplus E_7$	$(A_1^4)^\perp$ in $A_{17}$	

*Proof.* The orthogonal complements of  $E_8 \oplus A_1^5$  and  $E_8 \oplus A_1 \oplus D_4$  in  $E_8^3$  are isomorphic to  $R_1 = E_7 \oplus A_1^4$  and  $R_2 = E_7 \oplus D_4$ , respectively. By Proposition 6.1.1, [Sc] the isomorphism classes in  $\mathcal{G}_1$  and  $\mathcal{G}_2$  can be obtained by taking the orthogonal complements of primitive embeddings of  $R_1$  and, respectively,  $R_2$ , into even negative definite unimodular lattices of rank 24, i.e. Niemeier lattices. These lattices are uniquely determined by their root sublattice  $R$ , hence they are denoted by  $N(R)$  (see [CS], Chap. 18). In order to determine all lattices in the  $\mathcal{G}_i$  we first classify all primitive embeddings of  $R_1, R_2$  into  $R$  and take their orthogonal complements  $R_i^\perp$  in  $R$ . Then we take the primitive overlattice  $\overline{R_i^\perp}$  of  $R_i^\perp$  in  $N(R)$  which contains  $R_i^\perp$  as a subgroup of index at most 2. Here we have used the classification of embeddings between root lattices due to Nishiyama [Ni]. This gives isomorphism classes  $\overline{R_i^\perp}$  in  $\mathcal{G}_i$ . In Table 1 all root lattices  $R$  appear such that  $R_i$  can be embedded in  $N(R)$  and the corresponding lattices in  $\mathcal{G}_1$  and  $\mathcal{G}_2$ . If  $R_i^\perp$  is primitive in  $N(R)$ , then we omit the overline.  $\square$

**Theorem 4.5.** *The boundary of  $\overline{\mathcal{M}}$  contains 14 one dimensional components  $B_1, \dots, B_{14}$  where the closure of  $B_i, i = 1, \dots, 10$  contains only  $p$  and the closure of  $B_j, j = 11, \dots, 14$  contains both  $p$  and  $q$ .*

*Proof.* As remarked before, to the lattices in  $\mathcal{G}_1$  we can associate isotropic planes of type (1, 1) in  $T$  which are not  $\Gamma$ -equivalent. Conversely, by Lemma 5.2 in [Sc], any isotropic plane  $E$  of type (1, 1) can be embedded in  $U \oplus U$  and  $T \cong U \oplus U \oplus E^\perp/E$  where  $E^\perp/E \in \mathcal{G}_1$ . Hence, boundary components containing only  $p$  are in one-to-one correspondence with lattices in  $\mathcal{G}_1$ .

The proof is more subtle for isotropic planes of type (1, 2). Note that if  $v \in T$  is a primitive isotropic vector of type 2 and  $E$  is an isotropic plane containing  $v$ , then  $E$  determines a primitive vector in  $M_v = v^\perp/\mathbb{Z}v$ . Hence, isotropic planes of type (1, 2) correspond to orbits of isotropic vectors in  $M_v$ . In this case  $M_v \cong U \oplus E_8 \oplus D_4 \oplus A_1$  and orbits of isotropic vectors can be determined by Vinberg’s algorithm (see §1.4 [V] or §4.3 [St]).

By [N2, Theorem 0.2.3], the Weyl group  $W(M_v)$  has finite index in  $O(M_v)$ . This implies that the algorithm will finish in a finite number of steps. To start the algorithm we fix the vector  $\bar{x} = e + f$ . Then at each step we have to choose roots  $x \in M_v$  such that the *height*

$$h = \frac{(x, \bar{x})}{\sqrt{-x^2}}$$

is minimal and  $(x_i, x_j) \geq 0$  for  $j = 1, \dots, i - 1$ . In our case we get:

- i)  $(x, \bar{x}) = 0$ :  $u := e - f, \alpha_1, \dots, \alpha_8, \gamma_1, \dots, \gamma_4, \beta$ .
- ii)  $(x, \bar{x}) = 1$ :  $\alpha := f + \bar{\alpha}_8, \gamma := f + \bar{\gamma}_1, \beta' := f - \beta$ .
- iii)  $(x, \bar{x}) = 4$ :  $\delta_i := 2(e + f) - \beta + \bar{\alpha}_1 + \bar{\gamma}_i, j = 2, 3, 4$ .
- iv)  $(x, \bar{x}) = 12$ :  $\alpha' := 6(e + f) - 3\beta + 2\bar{\alpha}_4 + \bar{\gamma}_2 + \bar{\gamma}_3 + \bar{\gamma}_4$

where  $\bar{\alpha}_1, \dots, \bar{\alpha}_8$  and  $\bar{\gamma}_1, \dots, \bar{\gamma}_4$  are the dual bases of  $E_8$  and  $D_4$ . We now draw the Dynkin diagram associated to these roots. Let  $g_{ij} = (e_i, e_j) / \sqrt{e_i^2 e_j^2}$ . Then two vertices  $i, j$  corresponding to vectors  $e_i, e_j$  are connected by

- •      if       $g_{ij} = 0$ ,
- ——— •      if       $g_{ij} = 1/2$ ,
- - - - •      if       $g_{ij} = 1$ ,
- —||— •      if       $g_{ij} > 1$ .

The diagram in our case is given in Figure 1 (see also Figure 5, [Ko1]). Note that the symmetry group of the diagram is  $\mathbb{Z}_2 \times S_3$  and it can be easily seen that all symmetries can be realized by isometries in  $\Gamma$ . The maximal parabolic subdiagrams of rank 13 are of four types :

$$\begin{aligned} \tilde{E}_8 \oplus \tilde{D}_4 \oplus \tilde{A}_1 &= \langle \alpha_i, \alpha, \beta', \beta, \gamma, \gamma_j \rangle \quad i = 1, \dots, 8; \quad j = 1, \dots, 4, \\ \tilde{D}_{12} \oplus \tilde{A}_1 &= \langle \alpha_i, \alpha, u, \gamma, \gamma_j, \beta, \delta_4 \rangle \quad i = 2, \dots, 8; \quad j = 2, 3, \\ \tilde{E}_7 \oplus \tilde{D}_6 &= \langle \alpha_i, \delta_2, \alpha, u, \beta', \gamma, \gamma_j \rangle \quad i = 1, \dots, 7; \quad j = 3, 4, \\ \tilde{D}_{10} \oplus \tilde{A}_1^3 &= \langle \alpha_i, \alpha, u, \gamma, \beta', \gamma_j, \delta_j \rangle \quad i = 2, \dots, 8; \quad j = 2, 3, 4. \end{aligned}$$

Note that each type is an orbit for the action of  $\Gamma$ . These subdiagrams correspond to non-equivalent isotropic vectors in  $M_v$ . Hence, we get 4 isotropic planes in  $T$  containing a vector in  $\mathcal{I}_2$  and a direct analysis shows that all of them are of type (1, 2). □

It follows from the proof of Theorem 4.5 that the boundary components of  $\overline{\mathcal{M}}$  are in one-to-one correspondence with the lattices in  $\mathcal{G}_1$  and  $\mathcal{G}_2$ . These lattices appear in connection to degenerations of K3 surfaces, as explained for example in [Sc]. This allows us to compare the SBB compactification with more geometrically meaningful compactifications, as the ones obtained by means of geometric invariant theory.

In case of K3 surfaces with a degree two polarization this is well understood ([Sh1], [F], [Lo3]). Table 2 describes the correspondence between type II boundary components of the GIT compactification of plane sextics and one dimensional boundary components of the Baily-Borel compactification for degree two K3 surfaces. The lattice appearing in the SBB column is  $E^\perp/E$ , where  $E$  is the isotropic lattice associated to the boundary component.

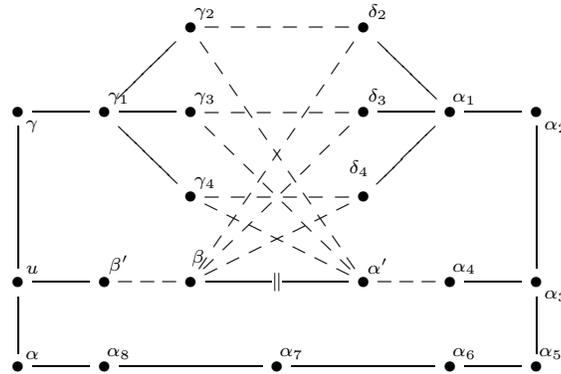


FIGURE 1. The Dynkin diagram of  $W(M_v)$

TABLE 2. GIT and SBB of plane sextics

	GIT	SBB
IIa:	$(x_0x_2 + a_1x_1^2)(x_0x_2 + a_2x_1^2)(x_0x_2 + a_3x_1^2) = 0$	$E_8 \oplus E_8 \oplus A_1$
IIb:	$x_2^2f_4(x_0, x_1) = 0.$	$E_7 \oplus D_{10}$
IIc:	$(x_0x_2 + x_1^2)^2f_2(x_0, x_1, x_2) = 0.$	$D_{16} \oplus A_1$
IId:	$f_3(x_0, x_1, x_2)^2 = 0.$	$A_{17}$

*Remark 4.6.* We expect that any one dimensional boundary component of  $\overline{\mathcal{M}}$  corresponds to a boundary component in Shah’s list plus an additional condition about the distribution of the four points on the semistable sextics. More precisely, we conjecture that a one dimensional boundary component  $B$  of  $\overline{\mathcal{M}}$  of type a, b, c or d (see Table 1) corresponds to a boundary component of type IIa, IIb, IIc or IId respectively with

- 4 marked nodes (eventually collapsing) if  $q \notin B$ ,
- a marked triple point if  $q \in B$ .

In fact, in the proof of Theorem 4.5, we showed that boundary components of  $\overline{\mathcal{M}}$  containing only  $p$  in their closure correspond to primitive embeddings of the lattice  $E_7 \oplus A_1^4$  into Neimeier lattices. Equivalently, they correspond to primitive embeddings of the lattice  $A_1^4$  in the root lattices  $E_8 \oplus E_8, E_7 \oplus D_{10}, D_{16}, A_{17}$ . Note that a double cover branched over a node has an  $A_1$  singularity hence, embedding  $A_1^4$  in the root lattices is equivalent to choosing a distribution of the 4 nodes on the corresponding configurations in Table 2 (where more than one node can “collapse” to the same singular point of the configuration).

For example, let  $q_1, q_2$  be the two singular points in the IIa configuration. We can either embed one node in  $q_1$  and 3 nodes in  $q_2$  (this gives the root lattice  $E_7 \oplus D_1 \oplus A_1$ ), two nodes in  $q_1$  and two in  $q_2$  (this gives the root lattice  $D_6^2 \oplus A_1$ ) or 4 nodes in  $q_1$  (this gives the root lattice  $E_8 \oplus A_1^5$ ).

Similarly, boundary components containing both  $p$  and  $q$  in their closure correspond to embeddings of the lattice  $D_4$  into the previous root lattices (a double cover branched over a triple point has a  $D_4$  singularity).

Note that the configuration II*d* has no triple points, in fact there is no one dimensional boundary component of type  $d$  containing  $q$  in its closure.

*Remark 4.7.* By Corollaries 3.8 and 3.10 the moduli space  $\mathcal{M}$  contains two divisors  $\mathcal{H}_2, \mathcal{H}_3$  parametrizing plane quintics and trigonal curves, respectively, with some extra structure. Forgetting a line, the divisor  $\mathcal{H}_2$  is birationally a  $\mathbb{P}^2$  fibration over the locus of plane quintics in  $\mathcal{M}_6$ . Similarly, forgetting  $L \in |K_C - 2g_3^1|$ ,  $\mathcal{H}_3$  is birationally a  $\mathbb{P}^1$  fibration over the locus of trigonal curves of genus six. This suggests that we need to blow up the moduli space of curves of genus six in order to extend the period map to these loci.

Bi-elliptic and hyperelliptic curves of genus six are mapped to one dimensional boundary components of  $\overline{\mathcal{M}}$ . In fact, the configuration II*c* is a plane model for hyperelliptic curves and case II*d* is the plane model of a bi-elliptic curve of genus six (see Section 1).

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