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BISPECTRAL COMMUTING DIFFERENCE OPERATORS FOR MULTIVARIABLE ASKEY-WILSON POLYNOMIALS

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Abstract. We construct a commutative algebra $\mathcal{A}_z$, generated by d algebraically independent q-difference operators acting on variables $z_1, z_2, \ldots, z_d$, which is diagonalized by the multivariable Askey-Wilson polynomials $P_n(z)$ considered by Gasper and Rahman (2005). Iterating Sears' ${}_4\phi_3$ transformation formula, we show that the polynomials $P_n(z)$ possess a certain duality between z and n. Analytic continuation allows us to obtain another commutative algebra $\mathcal{A}_n$, generated by d algebraically independent difference operators acting on the discrete variables $n_1, n_2, \ldots, n_d$, which is also diagonalized by $P_n(z)$. This leads to a multivariable q-Askey-scheme of bispectral orthogonal polynomials which parallels the theory of symmetric functions.

Contents

1. Introduction

In [1], Askey and Wilson introduced a remarkable family $\{p_n(x) : n \in \mathbb{N}_0\}$ of orthogonal polynomials on the real line depending on four parameters that satisfy

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a second-order q-difference equation

$$A(z)E_{q,z}(p_n(x)) + B(z)p_n(x) + C(z)E_{q,z}^{-1}(p_n(x)) = \lambda_n p_n(x), \tag{1.1}$$

where $x = \frac{1}{2}(z + \frac{1}{z})$, $E_{q,z}$ is the q-shift operator acting on functions of z by $E_{q,z}^{\pm 1}(g(z)) = g(zq^{\pm 1})$, $A(z)$, $B(z)$, $C(z)$ are independent of n, and λ_n is independent of z. The orthogonality implies that the polynomials $p_n(x)$ also satisfy a three-term recursion relation

$$a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) = x p_n(x), \tag{1.2}$$

where a_n, b_n and c_n are independent of x and $c_0 = 0$.

If we set $L^z = A(z)E_{q,z} + B(z)\mathrm{Id} + C(z)E_{q,z}^{-1}$, then the operator L^z acts naturally on the vector space consisting of all polynomials in the variable x and equation (1.1) means that the polynomials $p_n(x)$ are eigenvectors of this operator with eigenvalues λ_n. Similarly, let E_n denote the shift operator acting on an arbitrary function $f_n = f(n)$ of a discrete variable n by $E_n(f_n) = f_{n+1}$ and let $L^n = a_n E_n + b_n \mathrm{Id} + c_n E_n^{-1}$. Then the operator L^n acts on the vector space of complex-valued functions of a discrete variable $n \in \mathbb{N}_0$, and equation (1.2) means that the $p_n(x)$ are eigenvectors of the operator L^n with eigenvalue x. Following [3] we can say that the Askey-Wilson polynomials solve a *q-difference-difference bispectral problem*. Moreover, this bispectral property can be used to characterize the Askey-Wilson polynomials [9, 10].

As the title of the paper [1] states, the Askey-Wilson polynomials generalize those of Jacobi. In fact most of the polynomials used in analysis and mathematical physics can be obtained as special or limiting cases of these polynomials; see for instance [14]. For that reason, the Askey-Wilson polynomials are at the very top of the q-Askey scheme. For the terminating branch of the Askey scheme, the bispectral property has a natural interpretation within the framework of Leonard pairs [20].

An interesting question arising from the above discussion is whether there is a multivariable q-Askey scheme and if the polynomials in this scheme possess the bispectral property. We refer the reader to the book [4] for a very nice account of the general theory of orthogonal polynomials of several variables, which also indicates the two possible ways to proceed.

One possible extension is linked to the theory of symmetric functions. In this setting, polynomials which are eigenfunctions of q-difference operators were proposed by Macdonald [17] and Koornwinder [15]; see also [22] for connections with the bispectral problem mentioned above and [23] for multivariable q-Racah polynomials. These polynomial systems are associated with root systems, and the corresponding polynomials are invariant under the action of the Weyl group.

In a different vein, we can look for multivariable extensions, within the usual theory of polynomials of several variables; i.e., we consider an inner product in $\mathbb{R}^d$ and we apply the Gram-Schmidt process to all monomials, respecting the total degree ordering. Within this context, multivariable Askey-Wilson polynomials $\{P_d(n;x;\alpha) : n \in \mathbb{N}_0^d,\ x \in \mathbb{R}^d\}$ depending on $d+3$ parameters $\alpha_0, \alpha_2, \dots, \alpha_{d+2}$ were discovered by Gasper and Rahman [6]. These polynomials represent q-analogs of the multivariable Wilson polynomials defined by Tratnik [21].

In the present paper we prove that the polynomials of Gasper and Rahman are also bispectral. More precisely, we define two commutative algebras $\mathcal{A}_z$ and $\mathcal{A}_n$ of operators which are simultaneously diagonalized by the polynomials $P_d(n;x;\alpha)$. The algebra $\mathcal{A}_z$ is generated by d algebraically independent q-difference operators

$\mathfrak{L}_1^z, \dots, \mathfrak{L}_d^z$ acting on the variables $z_1, z_2, \dots, z_d$, where $x_j = \frac{1}{2}(z_j + z_j^{-1})$. Thus the operators $\mathfrak{L}_j^z$ act naturally on the vector space $\mathbb{C}[x_1, \dots, x_d]$ and are diagonalized by the polynomials $P_d(n; x; \alpha)$. Likewise, the algebra $\mathcal{A}_n$ is generated by d algebraically independent difference operators $\mathfrak{L}_1^n, \dots, \mathfrak{L}_d^n$ acting on the variables $n_1, \dots, n_d$, and the polynomials $P_d(n; x; \alpha)$ are eigenfunctions of these operators (considered as elements of the vector space of complex-valued functions defined on $\mathbb{N}_0^d$).

The paper extends the results in our joint work with Geronimo [8] and it raises a series of interesting questions. For instance, in the one-dimensional case, for specific values of the free parameters, the Askey-Wilson recurrence operator L^n commutes with a difference operator of odd order. Jointly with Haine [12] we explored this fact to connect the above theory to algebraic curves, having specific singularities, or equivalently, to specific soliton solutions of the Toda lattice hierarchy, using the correspondence established by van Moerbeke-Mumford [18, 19] and Krichever [16]. In particular, this led to a different explanation of the bispectral property using algebro-geometric considerations [11]. Moreover, this approach suggested using techniques from integrable systems (such as the Darboux transformation) to construct extensions of the Askey-Wilson polynomials which satisfy higher-order q-difference equations [12]. In the multivariable case, algebro-geometric methods were used by Chalykh [2], within the context of symmetric functions, to give more elementary proofs of several of Macdonald's conjectures. It is a challenging problem to construct a Baker-Akhiezer type function for the operators discussed in this paper and prove the duality using algebro-geometric tools. Another interesting question is to classify the multivariable orthogonal polynomials that satisfy q-difference equations. Recently discrete multivariable orthogonal polynomials satisfying second-order difference equations were classified in our joint work with Xu [13].

The paper is organized as follows. In Section 2 we recall the basic definitions and orthogonal properties of the one-dimensional Askey-Wilson polynomials as well as the multivariable extension proposed by Gasper and Rahman. We also show that a change of variables leads to the multivariable q-Racah polynomials discussed in [7]. In Section 3 we define a q-difference operator $\mathcal{L}_d$ acting on the variables $z_1, z_2, \dots, z_d$. We prove that it preserves the ring of polynomials in $x_1, x_2, \dots, x_d$ and we establish its triangular structure.[1] In Section 4 we show that $\mathcal{L}_d$ is self-adjoint with respect to the inner product defined by the multivariable Askey-Wilson measure. This leads to the construction of the commutative algebra $\mathcal{A}_z$ diagonalized by the multivariable Askey-Wilson polynomials. In Section 5 we iterate an identity of Sears to show that, appropriately normalized, the polynomials $P_d(n; x; \alpha)$ possess a certain duality between the discrete variables $(n_1, n_2, \dots, n_d)$ and the continuous variables $(z_1, z_2, \dots, z_d)$. Using this duality we obtain the commutative algebra $\mathcal{A}_n$ of difference operators acting on the variables $n_1, n_2, \dots, n_d$ which is also diagonalized by $P_d(n; x; \alpha)$, thus proving the bispectrality.

2. Multivariable Askey-Wilson polynomials

2.1. **Notation and one-dimensional theory.** Throughout the paper we assume that q is a real number in the open interval $(0, 1)$. We shall use the standard

[1] With respect to the total degree ordering.

q-shifted factorials

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \text{ and } (a_1,a_2,\dots,a_k;q)_n = \prod_{j=1}^{k}(a_j;q)_n$$

and the corresponding basic hypergeometric series; see [5] for more details. We denote by $p_n(x;a,b,c,d)$ the Askey-Wilson polynomials [1]

$$p_n(x;a,b,c,d) = \frac{(ab,ac,ad;q)_n}{a^n}\,{}_4\phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right], \tag{2.1}$$

where $x = \frac{1}{2}(z+\frac{1}{z})$. When a,b,c,d are such that $\max(|a|,|b|,|c|,|d|) < 1$ the polynomials defined by (2.1) satisfy the orthogonality relation

$$\frac{1}{2\pi}\int_{-1}^{1} p_n(x;a,b,c,d)p_m(x;a,b,c,d)\frac{w(z;a,b,c,d)}{\sqrt{1-x^2}}dx = \delta_{n,m}h_n, \tag{2.2}$$

where

$$w(z;a,b,c,d) = \frac{(z^2,z^{-2};q)_\infty}{(az,az^{-1},bz,bz^{-1},cz,cz^{-1},dz,dz^{-1};q)_\infty} \tag{2.3}$$

and

$$h_n = \frac{(abcdq^{n-1};q)_n(abcdq^{2n};q)_\infty}{(q^{n+1},abq^n,acq^n,adq^n,bcq^n,bdq^n,cdq^n;q)_\infty}. \tag{2.4}$$

2.2. **Multivariable extensions.** Consider $x = (x_1,x_2,\dots,x_d) \in \mathbb{R}^d$. For $|x_j| \le 1$ we put

$$x_j = \cos(\theta_j) = \frac{1}{2}(z_j + z_j^{-1}), \tag{2.5}$$

where $z_j = e^{i\theta_j}$ for $j = 1,2,\dots,d$. We define a weight function $w(z) = w_d(z;\alpha)$ depending on $d+3$ nonzero real parameters $\alpha_0,\alpha_1,\dots,\alpha_{d+2}$ by

$$w(z) = \frac{\prod_{j=1}^{d}(z_j^2,z_j^{-2};q)_\infty}{\prod_{k=0}^{d}\prod_{\varepsilon_1,\varepsilon_2\in\{-1,1\}}(\alpha_{k+1}\alpha_k^{-1}z_{k+1}^{\varepsilon_1}z_k^{\varepsilon_2};q)_\infty}, \tag{2.6}$$

with the convention that $z_0 = \alpha_0$ and $z_{d+1} = \alpha_{d+2}$. We shall assume that the parameters α_k are such that if $|z_j| = 1$ for $j = 1,2,\dots,d$, then

$$|\alpha_{k+1}\alpha_k^{-1}z_{k+1}^{\pm1}z_k^{\pm1}| < 1 \quad \text{for} \quad k = 0,1,\dots,d.$$

It is easy to see that this is equivalent to the constraints

$$\begin{gathered} 0 < |\alpha_{d+1}| < |\alpha_d| < \cdots < |\alpha_1| < \min(1,|\alpha_0|^2), \\ \frac{|\alpha_{d+1}|}{|\alpha_d|} < |\alpha_{d+2}| < \frac{|\alpha_d|}{|\alpha_{d+1}|}. \end{gathered} \tag{2.7}$$

Let us consider the following measure on $[-1,1]^d$:

$$d\mu(x) = \frac{w(z)}{(2\pi)^d}\prod_{j=1}^{d}\frac{dx_j}{\sqrt{1-x_j^2}} \tag{2.8a}$$

and the corresponding inner product

$$\langle f,g\rangle = \int_{[-1,1]^d} f(x)g(x)d\mu(x). \tag{2.8b}$$

The following theorem established in [6] constructs an explicit basis of polynomials mutually orthogonal with respect to the inner product defined by (2.6)-(2.8).

Theorem 2.1. *For* $n = (n_1, n_2, \dots, n_d) \in \mathbb{N}_0^d$ *define*

$$P_d(n; x; \alpha) = \prod_{j=1}^{d} p_{n_j}\left(x_j; \alpha_j q^{N_{j-1}}, \frac{\alpha_j}{\alpha_0^2} q^{N_{j-1}}, \frac{\alpha_{j+1}}{\alpha_j} z_{j+1}, \frac{\alpha_{j+1}}{\alpha_j} z_{j+1}^{-1}\right), \tag{2.9}$$

where $N_k = n_1 + n_2 + \cdots + n_k$ *and* $N_0 = 0$*. If the parameters* $\{\alpha_j\}_{j=0}^{d+2}$ *satisfy* (2.7)*, then for every* $n, m \in \mathbb{N}_0^d$ *we have*

$$\langle P_d(n; x; \alpha), P_d(m; x; \alpha)\rangle = \delta_{n,m} H_n, \tag{2.10}$$

where the inner product is defined by (2.6) *and* (2.8)*, and*

$$\begin{aligned} H_n = &\prod_{k=1}^{d} \frac{\left(\frac{\alpha_{k+1}^2}{\alpha_0^2} q^{N_{k-1}+N_k-1}; q\right)_{n_k} \left(\frac{\alpha_{k+1}^2}{\alpha_0^2} q^{2N_k}; q\right)_\infty}{\left(q^{n_k+1}, \frac{\alpha_k^2}{\alpha_0^2} q^{N_{k-1}+N_k}, \frac{\alpha_{k+1}^2}{\alpha_k^2} q^{n_k}; q\right)_\infty} \\ &\times \prod_{\varepsilon\in\{-1,1\}} \frac{1}{(\alpha_{d+1}\alpha_{d+2}^{\varepsilon} q^{N_d}, \alpha_{d+1}\alpha_{d+2}^{\varepsilon}\alpha_0^{-2} q^{N_d}; q)_\infty}. \end{aligned} \tag{2.11}$$

For the convenience of the reader we sketch the proof.

Proof. We can write the weight (2.6) as $w(z) = w_1(z)w_2(z)$, where

$$w_1(z) = \frac{(z_1^2, z_1^{-2}; q)_\infty}{\prod_{k=0}^{1} \prod_{\varepsilon_1,\varepsilon_2\in\{-1,1\}} (\alpha_{k+1}\alpha_k^{-1} z_{k+1}^{\varepsilon_1} z_k^{\varepsilon_2}; q)_\infty} \tag{2.12a}$$

is the one-dimensional Askey-Wilson weight for z_1, depending also on z_2, and

$$w_2(z) = \frac{\prod_{j=2}^{d} (z_j^2, z_j^{-2}; q)_\infty}{\prod_{k=2}^{d} \prod_{\varepsilon_1,\varepsilon_2\in\{-1,1\}} (\alpha_{k+1}\alpha_k^{-1} z_{k+1}^{\varepsilon_1} z_k^{\varepsilon_2}; q)_\infty} \tag{2.12b}$$

is independent of z_1. Notice also that in the right-hand side of equation (2.9) only the first polynomial p_{n_1} depends on z_1. Writing the integral representation of the inner product in (2.10) we see that the only terms depending on x_1 are p_{n_1}, p_{m_1} and $w_1(z)/\sqrt{1-x_1^2}$. Using (2.2) we obtain

$$\frac{1}{2\pi}\int_{-1}^{1} p_{n_1} p_{m_1} \frac{w_1(z)}{\sqrt{1-x_1^2}} dx_1 = \delta_{n_1,m_1} h_{n_1} \tilde{w}_1(z_2),$$

where

$$h_{n_1} = \frac{\left(\frac{\alpha_2^2}{\alpha_0^2} q^{n_1-1}; q\right)_{n_1} \left(\frac{\alpha_2^2}{\alpha_0^2} q^{2n_1}; q\right)_\infty}{\left(q^{n_1+1}, \frac{\alpha_1^2}{\alpha_0^2} q^{n_1}, \frac{\alpha_2^2}{\alpha_1^2} q^{n_1}; q\right)_\infty}$$

and

$$\tilde{w}_1(z_2) = \prod_{\varepsilon\in\{-1,1\}} \frac{1}{\left(\alpha_2 q^{n_1} z_2^{\varepsilon}, \frac{\alpha_2}{\alpha_0^2} q^{n_1} z_2^{\varepsilon}; q\right)_\infty}.$$

It easy to see that $\tilde{w}_1(z_2)w_2(z)$ coincides with the weight $w_{d-1}(\tilde{z}, \tilde{\alpha})$ given in (2.6) for the variables $\tilde{z}_1 = z_2, \dots, \tilde{z}_{d-1} = z_d$ and parameters $\tilde{\alpha}_0 = \alpha_0$, $\tilde{\alpha}_1 = \alpha_2 q^{n_1}$, $\tilde{\alpha}_2 = \alpha_3 q^{n_1}, \dots, \tilde{\alpha}_d = \alpha_{d+1} q^{n_1}$, $\tilde{\alpha}_{d+1} = \alpha_{d+2}$. Moreover, if we denote $\tilde{n} = (n_2, n_3, \dots, n_d) \in \mathbb{N}_0^{d-1}$ one can check that $P_{d-1}(\tilde{n}; \tilde{x}; \tilde{\alpha})$ coincides with the product

$\prod_{j=2}^{d} p_{n_j}$ consisting of the last $(d-1)$ polynomials for $P_d(n;x;\alpha)$. The proof now follows by induction. □

Remark 2.2. One can easily obtain the parametrization used by Gasper and Rahman [6], by introducing parameters $a, b, c, d,\ a_2, a_3, \dots, a_d$ related to α_j by the formulas $a = \alpha_1,\ b = \alpha_1/\alpha_0^2,\ c = \alpha_{d+1}\alpha_{d+2}/\alpha_d,\ d = \alpha_{d+1}/(\alpha_d\alpha_{d+2})$ and $a_{k+1} = \alpha_{k+1}/\alpha_k$ for $k = 1, 2, \dots, d-1$.

Remark 2.3. The multivariable q-Racah polynomials discussed in [7] can be obtained as follows. Suppose

$$\frac{\alpha_{d+2}}{\alpha_{d+1}} = q^N, \text{ where } N \in \mathbb{N}.$$

Making the substitution

$$z_k = \alpha_k q^{y_k} \text{ for } k = 0, 1, \dots, d+1,$$

we see that polynomials in (2.9) become the multivariable q-Racah polynomials (2.13)

$$R_n = \prod_{k=1}^{d} r_{n_k}\left(y_k - N_{k-1}; \frac{\alpha_k^2}{\alpha_0^2} q^{2N_{k-1}-1}, \frac{\alpha_{k+1}^2}{q\alpha_k^2}, \alpha_k^2 q^{y_{k+1}+N_{k-1}}, y_{k+1} - N_{k-1}\right),$$

where the r_k are the one-dimensional q-Racah polynomials defined by

$$r_k(y; a, b, c, N) = (aq, bcq, q^{-N}; q)_k (q^N/c)^{k/2} {}_4\phi_3 \left[\begin{matrix} q^{-k}, abq^{k+1}, q^{-y}, cq^{y-N} \\ aq, bcq, q^{-N} \end{matrix}; q, q\right].$$

The polynomials R_n are orthogonal on $0 = y_0 \leq y_1 \leq y_2 \leq \cdots \leq y_d \leq y_{d+1} = N$ with respect to the weight

$$\rho(y) = \prod_{k=0}^{d} \frac{(\alpha_{k+1}^2/\alpha_k^2; q)_{y_{k+1}-y_k} (\alpha_{k+1}^2; q)_{y_{k+1}+y_k}}{(q;q)_{y_{k+1}-y_k} (q\alpha_k^2; q)_{y_{k+1}+y_k}} \prod_{k=1}^{d} (1 - \alpha_k^2 q^{2y_k}) \left(\frac{\alpha_{k-1}}{\alpha_k}\right)^{2y_k}.$$

The parametrization in [7] can be obtained if we replace $\alpha_0, \dots, \alpha_{d+1}$ by $a_1, \dots, a_{d+1}, b$ related via the formulas $a_1 = \alpha_1^2,\ a_k = \alpha_k^2/\alpha_{k-1}^2$ for $k = 2, \dots, d+1$ and $b = \alpha_1^2/(q\alpha_0^2)$.

Thus, all difference equations derived for the multivariable Askey-Wilson polynomials later in the paper can be translated to the multivariable q-Racah polynomials, using the change of variables and parameters above.

3. q-Difference operators in $\mathbb{C}^d$

In this section we construct a q-difference operator $\mathcal{L}_d$ which is triangular and selfadjoint with respect to the inner product (2.6)-(2.8).

3.1. **Basic definitions.** Consider the ring $\mathcal{P}_{z^{\pm 1}} = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_d^{\pm 1}]$ of Laurent polynomials in the variables $z_1, z_2, \dots, z_d$ with complex coefficients. We denote by $\mathcal{P}_x = \mathbb{C}[x_1, x_2, \dots, x_d]$ the subring of $\mathcal{P}_{z^{\pm 1}}$ consisting of polynomials in the variables $x_1, x_2, \dots, x_d$, where

$$x_j = \frac{1}{2}(z_j + z_j^{-1}) \qquad \text{for } j = 1, 2, \dots, d. \tag{3.1}$$

For $j = 1, 2, \dots, d$ we define an automorphism I_j of $\mathcal{P}_{z^{\pm 1}}$ by

$$I_j(z_j) = z_j^{-1} \text{ and } I_j(z_k) = z_k \text{ for } j \neq k. \tag{3.2}$$

Clearly, I_j is an involution (i.e. $I_j \circ I_j = \mathrm{Id}$) which preserves $\mathcal{P}_x$. Conversely, if a polynomial $p \in \mathcal{P}_{z^{\pm 1}}$ is preserved by the involutions I_j for $j = 1, 2, \ldots, d$, then $p \in \mathcal{P}_x$.

Let $\{e_1, e_2, \ldots, e_d\}$ be the standard basis for $\mathbb{C}^d$. We denote by $E_{q,\,z_j}$, $\triangle_{q,\,z_j}$ and $\nabla_{q,\,z_j}$, respectively, the q-shift, forward and backward difference operators in the j-th coordinate acting on functions $f(z)$ as follows:

$$\begin{aligned} E_{q,\,z_j} f(z) &= f(z_1, z_2, \ldots, z_j q, \ldots, z_d),\\ \triangle_{q,\,z_j} f(z) &= (E_{q,\,z_j} - 1) f(z),\\ \nabla_{q,\,z_j} f(z) &= (1 - E^{-1}_{q,\,z_j}) f(z). \end{aligned}$$

Throughout the paper we use the standard multi-index notation. For instance, if $\nu = (\nu_1, \nu_2, \ldots, \nu_d) \in \mathbb{Z}^d$, then

$$z^\nu = z_1^{\nu_1} z_2^{\nu_2} \cdots z_d^{\nu_d}, \quad E^\nu_{q,\,z} = E^{\nu_1}_{q,\,z_1} E^{\nu_2}_{q,\,z_2} \cdots E^{\nu_d}_{q,\,z_d}, \quad zq^\nu = (z_1 q^{\nu_1}, z_2 q^{\nu_2}, \ldots, z_d q^{\nu_d})$$

and $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_d$.

We denote by $\mathcal{D}_z = \mathbb{C}(z_1, \ldots, z_d)[E^{\pm 1}_{q,\,z_1}, \ldots, E^{\pm 1}_{q,\,z_d}]$ the associative algebra of q-difference operators with rational in z coefficients; i.e., the elements of $\mathcal{D}_z$ are operators L of the form

$$L = \sum_{\nu \in S} l_\nu(z) E^\nu_{q,\,z},$$

where S is a finite subset of $\mathbb{Z}^d$ and $l_\nu(z)$ are rational functions of z. Thus, the algebra $\mathcal{D}_z$ is generated by rational functions of z, the shift operators $E_{q,\,z_1}, E_{q,\,z_2}, \ldots, E_{q,\,z_d}$ and their inverses $E^{-1}_{q,\,z_1}, E^{-1}_{q,\,z_2}, \ldots, E^{-1}_{q,\,z_d}$ subject to the relations

$$E_{q,\,z_j} \cdot g(z) = g(zq^{e_j}) E_{q,\,z_j}, \tag{3.3}$$

for all rational functions $g(z)$ and for $j = 1, 2, \ldots, d$. For every $k \in \{1, 2, \ldots, d\}$ the involution I_k can be naturally extended to $\mathcal{D}_z$, by defining

$$I_k(E_{q,\,z_k}) = E^{-1}_{q,\,z_k} \quad \text{and} \quad I_k(E_{q,\,z_j}) = E_{q,\,z_j} \text{ for } j \neq k. \tag{3.4}$$

Indeed, it is easy to see that I_k is correctly defined because the relations (3.3) are preserved under the action of I_k; i.e., we have

$$I_k(E_{q,\,z_j}) \cdot I_k(g(z)) = I_k(g(zq^{e_j})) I_k(E_{q,\,z_j}),$$

for $k, j \in \{1, 2, \ldots, d\}$.

For the sake of brevity we say that $L \in \mathcal{D}_z$ is I-invariant if $I_j(L) = L$ for all $j \in \{1, 2, \ldots, d\}$.

For $k \in \mathbb{N}_0$ we denote by $\mathcal{P}^k_x$ the space of polynomials in $\mathcal{P}_x$ of (total) degree at most k in the variables $x_1, x_2, \ldots, x_d$, with the convention that $\mathcal{P}^{-1}_x = \{0\}$.

Definition 3.1. We say that a linear operator L on $\mathcal{P}_x$ is triangular if for every $k \in \mathbb{N}_0$ there is $c_k \in \mathbb{C}$ such that

$$L(p) = c_k p \mod \mathcal{P}^{k-1}_x \text{ for all } p \in \mathcal{P}^k_x.$$

3.2. **The triangular operator $\mathcal{L}_d$.** Below we define an operator which is I-invariant and triangular. Clearly, an operator which is I-invariant is uniquely determined by its coefficient of $E^\nu_{q,\,z}$ (or equivalently $\triangle^\nu_{q,\,z}$) with $\nu_i \geq 0$ for $i = 1, 2, \ldots, d$.

Let $\nu=(\nu_1,\nu_2,\dots,\nu_d)\in\{0,1\}^d\setminus\{0\}^d$, and let $\{\nu_{i_1},\nu_{i_2},\dots,\nu_{i_s}\}$ be the nonzero components of ν with $1\le i_1<i_2<\dots<i_s\le d$ and $s\ge 1$. Denote

$$A_\nu=(1-\alpha_{i_1}z_{i_1})(1-\alpha_{i_1}z_{i_1}/\alpha_0^2)\times\frac{\prod_{k=2}^{s}(1-\alpha_{i_k}z_{i_k}z_{i_{k-1}}/\alpha_{i_{k-1}})(1-q\alpha_{i_k}z_{i_k}z_{i_{k-1}}/\alpha_{i_{k-1}})}{\prod_{k=1}^{s}(1-z_{i_k}^2)(1-qz_{i_k}^2)}\times(1-\alpha_{d+1}\alpha_{d+2}z_{i_s}/\alpha_{i_s})(1-\alpha_{d+1}z_{i_s}/(\alpha_{i_s}\alpha_{d+2})). \tag{3.5a}$$

An arbitrary $\nu\in\mathbb{Z}^d$ can be decomposed as $\nu=\nu^+-\nu^-$, where $\nu^\pm\in\mathbb{N}_0^d$ with components $\nu_j^+=\max(\nu_j,0)$ and $\nu_j^-=-\min(\nu_j,0)$. For $\nu\in\{-1,0,1\}^d\setminus\{0,1\}^d$ we define

$$A_\nu=I^{\nu^-}(A_{\nu^++\nu^-}). \tag{3.5b}$$

Here I^{ν^-} is the composition of the involutions corresponding to the positive coordinates of ν^-.

Finally, we define the operator

$$\mathcal{L}_d=\mathcal{L}_d(z_1,\dots,z_d;\alpha_0,\alpha_1,\dots,\alpha_{d+2})=\sum_{\nu\in\{-1,0,1\}^d\setminus\{0\}^d}(-1)^{|\nu^-|}A_\nu\triangle_{q,z}^{\nu^+}\nabla_{q,z}^{\nu^-}. \tag{3.6}$$

Since $I_i(\triangle_{q,z_i})=-\nabla_{q,z_i}$ and $I_i(\triangle_{q,z_j})=\triangle_{q,z_j}$ for $i\ne j$, the operator defined above is I-invariant.

Lemma 3.2. *Let $L\in\mathcal{D}_z$ be an I-invariant q-difference operator. If*

$$\prod_{j=1}^{d}(1-z_j^2)L(p)\in\mathcal{P}_{z^{\pm1}}\ \textit{for every}\ p\in\mathcal{P}_x, \tag{3.7}$$

then L preserves $\mathcal{P}_x$, i.e. $L(\mathcal{P}_x)\subset\mathcal{P}_x$. In particular, the operator $\mathcal{L}_d$ defined by (3.5)-(3.6) *preserves $\mathcal{P}_x$.*

Proof. Let $p(x)\in\mathcal{P}_x$. Since both L and p are I-invariant, it follows that $L(p)$ is also I-invariant, and therefore to show that $L(p)\in\mathcal{P}_x$ is enough to prove that $L(p)\in\mathcal{P}_{z^{\pm1}}$. From (3.7) it follows that we can write $L(p)$ as a ratio

$$L(p)=\frac{p_1(z)}{\prod_{j=1}^{d}(z_j-z_j^{-1})},\ \text{where}\ p_1(z)\in\mathcal{P}_{z^{\pm1}}.$$

Notice that $I_j(z_j-z_j^{-1})=-(z_j-z_j^{-1})$, and therefore $I_j(p_1(z))=-p_1(z)$ for every $j=1,2,\dots,d$. This implies that $p_1(z)=0$ when $z_j=\pm1$ and therefore $p_1(z)$ is divisible (in the ring $\mathcal{P}_{z^{\pm1}}$) by $\prod_{j=1}^{d}(z_j-z_j^{-1})$, proving that $L(p)\in\mathcal{P}_{z^{\pm1}}$.

It remains to show that $\mathcal{L}_d$ satisfies the conditions of the lemma. This follows easily from formulas (3.5)-(3.6) combined with the fact that $\triangle_{q,z_j}(x_j^k)$ is divisible by $\triangle_{q,z_j}(x_j)=\frac{(z_j^2q-1)(q-1)}{2z_jq}$, and $\nabla_{q,z_j}(x_j^k)$ is divisible by $\nabla_{q,z_j}(x_j)=\frac{(z_j^2-q)(q-1)}{2z_jq}$ for every $k\in\mathbb{N}$. ∎

Proposition 3.3. *The operator $\mathcal{L}_d$ defined by* (3.5)-(3.6) *is triangular. More precisely, we have*

$$\mathcal{L}_d(p(x))=-(1-q^{-k})\left(1-\frac{\alpha_{d+1}^2}{\alpha_0^2}q^{k-1}\right)p(x)\mod\mathcal{P}_x^{k-1}\ \textit{for every}\ p(x)\in\mathcal{P}_x^k. \tag{3.8}$$

Proof. From Lemma 3.2 we know that $\mathcal{L}_d(p(x)) \in \mathcal{P}_x$ for every $p(x) \in \mathcal{P}_x$. It is enough to prove that (3.8) holds when $p(x) = x^n = x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d}$. Clearly the highest (total) power of x in $\mathcal{L}_d(x^n)$ can be uniquely determined by the highest power of z. Thus it is enough to see that, after canceling the denominators of the coefficients A_ν of the operator $\mathcal{L}_d$, we have

$$\mathcal{L}_d(x^n) = -\frac{1}{2^{|n|}}(1 - q^{-|n|})\left(1 - \frac{\alpha_{d+1}^2}{\alpha_0^2} q^{|n|-1}\right) z^n$$
$$+ \text{ a linear combination of } z^k \text{ with } |k| < |n|.$$

Let $0 \neq \nu \in \{-1, 0, 1\}^d$. Notice that

$$\triangle_{q,\,z_j}(x_j^{n_j}) = \frac{1}{2^{n_j}}(q^{n_j} - 1) z_j^{n_j} + O(z_j^{n_j - 1}), \tag{3.9a}$$

$$\nabla_{q,\,z_j}(x_j^{n_j}) = \frac{1}{2^{n_j}}(1 - q^{-n_j}) z_j^{n_j} + O(z_j^{n_j - 1}), \tag{3.9b}$$

and therefore $\triangle_{q,\,z}^{\nu^+} \nabla_{q,\,z}^{\nu^-}(z^n)$ is a linear combination of z^n and z^k, where $|k| < |n|$. From (3.5) it follows that the denominator of A_ν is a polynomial of degree $4s$ in $z_1, z_2, \ldots, z_d$, where s denotes the number of nonzero components of ν. However, the numerator is of degree $4s$ if and only if all the nonzero coordinates of ν have the same sign (i.e., they are all positive, or all negative). If at least two of the coordinates of ν have different signs, then the degree of the numerator is less than $4s$ in the variables $z_1, z_2, \ldots, z_d$. Thus we conclude that $\mathcal{L}_d(x^n)$ is a linear combination of z^n and z^k, where $|k| < |n|$. Moreover, the coefficient of z^n can be computed by extracting the coefficient of z^n from the terms $(-1)^{|\nu^-|} A_\nu \triangle_{q,\,z}^{\nu^+} \nabla_{q,\,z}^{\nu^-}(x^n)$ for $0 \neq \nu \in \{0, 1\}^d \cup \{-1, 0\}^d$ in the representation (3.6) of the operator $\mathcal{L}_d$. Using formulas (3.9) we deduce that

$$\mathcal{L}_d(x^n) = \frac{c_n}{2^{|n|}} z^n + \text{ a linear combination of } z^k \text{ with } |k| < |n|,$$

where $c_n = c_n^+ + c_n^-$ with

$$c_n^+ = \frac{\alpha_{d+1}^2}{q\alpha_0^2} \sum_{0 \neq \nu \in \{0,1\}^d} \prod_{\substack{s \in \{1,\ldots,d\} \\ \text{such that } \nu_s = 1}} (q^{n_s} - 1), \tag{3.10a}$$

$$c_n^- = \sum_{0 \neq \nu \in \{0,-1\}^d} \prod_{\substack{s \in \{1,\ldots,d\} \\ \text{such that } \nu_s = -1}} (q^{-n_s} - 1). \tag{3.10b}$$

It is easy to see (for instance by induction on d) that

$$\sum_{0 \neq \nu \in \{0,1\}^d} \prod_{\substack{s \in \{1,\ldots,d\} \\ \text{such that } \nu_s = 1}} (q^{n_s} - 1) = q^{|n|} - 1,$$

which combined with (3.10) gives

$$c_n^+ = \frac{\alpha_{d+1}^2}{q\alpha_0^2}(q^{|n|} - 1) \quad \text{and} \quad c_n^- = q^{-|n|} - 1.$$

Thus

$$c_n = c_n^+ + c_n^- = -(1 - q^{-|n|})\left(1 - \frac{\alpha_{d+1}^2}{\alpha_0^2} q^{|n|-1}\right),$$

completing the proof. □

4. The commutative algebra $\mathcal{A}_z$

In this section we show that the operator $\mathcal{L}_d$ defined in Section 3 is selfadjoint with respect to the inner product (2.6)-(2.8). This allows us to construct a commutative subalgebra $\mathcal{A}_z$ of $\mathcal{D}_z$, generated by d algebraically independent q-difference operators, which is diagonalized by the polynomials (2.9).

4.1. **Selfadjointness of $\mathcal{L}_d$.** We start with a simple lemma.

Lemma 4.1. *Consider an inner product defined by equations* (2.8) *and let* $L = \sum_{\nu\in S} C_\nu(z)E^{\nu}_{q,z} \in \mathcal{D}_z$. *If for every* $\nu \in S$ *the following conditions are satisfied,*

(i) $C_\nu(z)w(z) = C_{-\nu}(zq^\nu)w(zq^\nu)$,
(ii) *the function* $C_\nu(z)w(z)$ *is holomorphic on the* d*-dimensional torus* $\mathbb{T}^\nu_t = \{z \in \mathbb{C}^d : |z_j| = q^{-t\nu_j}$ *for* $j = 1, 2, \dots, d\}$ *for every* $t \in [0, 1]$,

then L *is selfadjoint with respect to the inner product* (2.8).

Proof. Using the change of variables (2.5) we can write the inner product as an integral over the d-dimensional torus $\mathbb{T} = \{z \in \mathbb{C}^d : |z_j| = 1$ for $j = 1, 2, \dots, d\}$,

$$\langle f, g\rangle = \frac{1}{(4\pi i)^d}\int_{\mathbb{T}} f(z)g(z)w(z)\prod_{j=1}^{d}\frac{dz_j}{z_j}. \tag{4.1}$$

Denote $L_\nu = C_\nu(z)E^\nu_{q,z}$. Then from (4.1) we obtain

$$\langle f, L_{-\nu}g\rangle = \frac{1}{(4\pi i)^d}\int_{\mathbb{T}} f(z)C_{-\nu}(z)g(zq^{-\nu})w(z)\prod_{j=1}^{d}\frac{dz_j}{z_j}$$

replacing z by zq^ν

$$= \frac{1}{(4\pi i)^d}\int_{\mathbb{T}^\nu_1} C_{-\nu}(zq^\nu)f(zq^\nu)g(z)w(zq^\nu)\prod_{j=1}^{d}\frac{dz_j}{z_j}$$

using (i)

$$= \frac{1}{(4\pi i)^d}\int_{\mathbb{T}^\nu_1} C_{\nu}(z)f(zq^\nu)g(z)w(z)\prod_{j=1}^{d}\frac{dz_j}{z_j}$$

using (ii) we can replace $\mathbb{T}^\nu_1$ by $\mathbb{T}^\nu_0 = \mathbb{T}$

$$\begin{aligned} &= \frac{1}{(4\pi i)^d}\int_{\mathbb{T}} C_{\nu}(z)f(zq^\nu)g(z)w(z)\prod_{j=1}^{d}\frac{dz_j}{z_j} \\ &= \langle L_\nu f, g\rangle. \end{aligned}$$

The proof follows immediately by writing L as a sum of L_ν's. □

In order to apply Lemma 4.1 we need to rewrite the operator $\mathcal{L}_d$ as a linear combination of the q-shift operators $E^\nu_{q,z}$.

For $j \in \{0,1,\dots,d\}$ and $(k,l) \in \{0,1\}^2$ we define $B_j^{k,l}(z)$ as follows:

$$B_j^{0,0}(z) = 1 + \frac{\alpha_{j+1}^2}{q\alpha_j^2} - \frac{4\alpha_{j+1}x_jx_{j+1}}{(q+1)\alpha_j}, \tag{4.2a}$$

$$B_j^{0,1}(z) = \left(1 - \frac{\alpha_{j+1}z_jz_{j+1}}{\alpha_j}\right)\left(1 - \frac{\alpha_{j+1}z_{j+1}}{\alpha_jz_j}\right), \tag{4.2b}$$

$$B_j^{1,0}(z) = \left(1 - \frac{\alpha_{j+1}z_jz_{j+1}}{\alpha_j}\right)\left(1 - \frac{\alpha_{j+1}z_j}{\alpha_jz_{j+1}}\right), \tag{4.2c}$$

$$B_j^{1,1}(z) = \left(1 - \frac{\alpha_{j+1}z_jz_{j+1}}{\alpha_j}\right)\left(1 - \frac{q\alpha_{j+1}z_{j+1}z_j}{\alpha_j}\right). \tag{4.2d}$$

In the above formulas x_j is related to z_j by (2.5), with the convention $z_0 = \alpha_0$ and $z_{d+1} = \alpha_{d+2}$. We extend the definition of $B_j^{k,l}(z)$ for $(k,l) \in \{-1,0,1\}^2$ by defining

$$B_j^{-1,l}(z) = I_j(B_j^{1,l}(z)) \text{ for } l = 0,1, \tag{4.2e}$$

$$B_j^{k,-1}(z) = I_{j+1}(B_j^{k,1}(z)) \text{ for } k = 0,1, \tag{4.2f}$$

$$B_j^{-1,-1}(z) = I_j(I_{j+1}(B_j^{1,1}(z))). \tag{4.2g}$$

Next, for $j \in \{1,\dots,d\}$ we denote

$$b_j^0(z) = (1-qz_j^2)(1-qz_j^{-2}), \tag{4.3a}$$

$$b_j^1(z) = (1-z_j^2)(1-qz_j^2), \tag{4.3b}$$

$$b_j^{-1}(z) = I_j(b_j^1(z)). \tag{4.3c}$$

Finally, for $\nu \in \{-1,0,1\}^d$ we put

$$C_\nu(z) = (q(q+1))^{d-|\nu^+|-|\nu^-|}\,\frac{\prod_{k=0}^d B_k^{\nu_k,\nu_{k+1}}(z)}{\prod_{k=1}^d b_k^{\nu_k}(z)}, \tag{4.4}$$

with the convention $\nu_0 = \nu_{d+1} = 0$.

Proposition 4.2. *The operator $\mathcal{L}_d = \mathcal{L}_d(z;\alpha)$ defined by* (3.5)-(3.6) *can be written as*

$$\mathcal{L}_d(z;\alpha) = \sum_{\nu\in\{-1,0,1\}^d} C_\nu(z)E_{q,z}^\nu - \left(1 + \frac{\alpha_{d+1}^2}{q\alpha_0^2} - \frac{4\alpha_{d+1}x_0x_{d+1}}{(q+1)\alpha_0}\right), \tag{4.5}$$

where $C_\nu(z)$ are given by (4.2), (4.3) *and* (4.4).

Proof. We shall prove the statement by induction on d. For $d = 1$ formulas (3.5) give

$$A_{e_1} = \frac{(1-\alpha_1z_1)\left(1-\frac{\alpha_1z_1}{\alpha_0^2}\right)\left(1-\frac{\alpha_2\alpha_3z_1}{\alpha_1}\right)\left(1-\frac{\alpha_2z_1}{\alpha_1\alpha_3}\right)}{(1-z_1^2)(1-qz_1^2)},$$

$$A_{-e_1} = I_1(A_{e_1}) = \frac{(z_1-\alpha_1)\left(z_1-\frac{\alpha_1}{\alpha_0^2}\right)\left(z_1-\frac{\alpha_2\alpha_3}{\alpha_1}\right)\left(z_1-\frac{\alpha_2}{\alpha_1\alpha_3}\right)}{(1-z_1^2)(q-z_1^2)}.$$

Using (4.2), (4.3) and (4.4) we get

$$\begin{aligned}
C_{e_1} &= \frac{B_0^{0,1}(z)B_1^{1,0}(z)}{b_1^1(z)} = \frac{(1-\alpha_1 z_1)\left(1-\frac{\alpha_1 z_1}{\alpha_0^2}\right)\left(1-\frac{\alpha_2\alpha_3 z_1}{\alpha_1}\right)\left(1-\frac{\alpha_2 z_1}{\alpha_1\alpha_3}\right)}{(1-z_1^2)(1-qz_1^2)} = A_{e_1},\\
C_{-e_1} &= I_1(C_{e_1}(z)) = A_{-e_1},\\
C_0 &= q(q+1)\frac{B_0^{0,0}(z)B_1^{0,0}(z)}{b_1^0(z)}\\
&= q(q+1)\frac{\left(1+\frac{\alpha_1^2}{q\alpha_0^2}-\frac{4\alpha_1}{(q+1)\alpha_0}x_0x_1\right)\left(1+\frac{\alpha_2^2}{q\alpha_1^2}-\frac{4\alpha_2}{(q+1)\alpha_1}x_1x_2\right)}{(1-qz_1^2)(1-qz_1^{-2})},
\end{aligned}$$

where $x_0 = \frac{1}{2}(\alpha_0+\alpha_0^{-1})$, $x_1 = \frac{1}{2}(z_1+z_1^{-1})$ and $x_2 = \frac{1}{2}(\alpha_3+\alpha_3^{-1})$. We need to check that

$$\begin{aligned}
\mathcal{L}_1 &= A_{e_1}(E_{q,z_1}-1) - A_{-e_1}(1-E_{q,z_1}^{-1})\\
&= C_{e_1}E_{q,z_1} + C_{-e_1}E_{q,z_1}^{-1} + C_0 - \left(1+\frac{\alpha_2^2}{q\alpha_0^2}-\frac{4\alpha_2 x_0 x_2}{(q+1)\alpha_0}\right),
\end{aligned}$$

which amounts to checking that

$$-A_{e_1} - A_{-e_1} = C_0 - \left(1+\frac{\alpha_2^2}{q\alpha_0^2}-\frac{4\alpha_2 x_0 x_2}{(q+1)\alpha_0}\right).$$

This equality can be verified by a straightforward computation using the explicit formulas for A_{e_1}, A_{-e_1} and C_0 above.

Now let $d>1$ and assume that the statement is true for $d-1$. We can write $\mathcal{L}_d$ as follows:

$$\begin{aligned}
\mathcal{L}_d &= \mathcal{L}'\,\frac{\left(1-\frac{\alpha_{d+1}\alpha_{d+2}z_d}{\alpha_d}\right)\left(1-\frac{\alpha_{d+1}z_d}{\alpha_d\alpha_{d+2}}\right)}{(1-z_d^2)(1-qz_d^2)}\triangle_{q,z_d}\\
&\quad+\mathcal{L}''\,\frac{\left(1-\frac{\alpha_{d+1}\alpha_{d+2}}{\alpha_d z_d}\right)\left(1-\frac{\alpha_{d+1}}{\alpha_d\alpha_{d+2}z_d}\right)}{(1-z_d^{-2})(1-qz_d^{-2})}(-\nabla_{q,z_d}) + \mathcal{L}''',
\end{aligned} \tag{4.6}$$

where $\mathcal{L}'$, $\mathcal{L}''$, $\mathcal{L}'''$ are q-difference operators in the variables $z_1, z_2, \dots, z_{d-1}$ with coefficients depending on $z_1,\dots,z_d$ and the parameters $\alpha_0,\alpha_1,\dots,\alpha_{d+2}$. Clearly, the operators $\mathcal{L}'$, $\mathcal{L}''$, $\mathcal{L}'''$ are uniquely determined from $\mathcal{L}_d$. This implies that they are $I_1, I_2, \dots, I_{d-1}$ invariant and therefore they are characterized by the coefficients of $\triangle_{q,\bar{z}}^{\bar{\nu}} = \triangle_{q,z_1}^{\nu_1}\triangle_{q,z_2}^{\nu_2}\cdots\triangle_{q,z_{d-1}}^{\nu_{d-1}}$, where $\bar{z} = (z_1, z_2, \dots, z_{d-1})$ and $\bar{\nu} = (\nu_1,\nu_2,\dots,\nu_{d-1}) \in \{0,1\}^{d-1}$. The coefficient of $\triangle_{q,\bar{z}}^{\bar{0}}$ in $\mathcal{L}'$ is equal to $(1-\alpha_d z_d)(1-\alpha_d z_d/\alpha_0^2)$ (it comes from the term $A_{e_d}\triangle_{q,z_d}$ in $\mathcal{L}_d$). Using equations (3.5) one can see that the coefficients of $\triangle_{q,\bar{z}}^{\bar{\nu}}$ for $\bar{\nu}\in\{0,1\}^{d-1}\backslash\{\bar{0}\}$ are the same as the coefficients of the operator $\mathcal{L}_{d-1}(z_1,\dots,z_{d-1};\alpha_0,\dots,\alpha_{d-1},\alpha_d',\alpha_{d+1}')$, where $\alpha_d' = \alpha_d z_d q^{1/2}$ and $\alpha_{d+1}' = q^{-1/2}$. Notice that

$$1+\frac{(\alpha_d')^2}{q\alpha_0^2}-\frac{4\alpha_d' x_0 x_d'}{(q+1)\alpha_0} = (1-\alpha_d z_d)(1-\alpha_d z_d/\alpha_0^2).$$

Thus, using the induction hypothesis, we deduce that

$$\begin{aligned}\mathcal{L}' &= \mathcal{L}_{d-1}(z_1,\dots,z_{d-1};\alpha_0,\dots,\alpha_{d-1},\alpha_d z_d q^{1/2},q^{-1/2}) \\ &\qquad + (1-\alpha_d z_d)(1-\alpha_d z_d/\alpha_0^2) \\ &= \sum_{\nu\in\{-1,0,1\}^{d-1}} C'_\nu E^\nu_{q,\bar{z}},\end{aligned} \tag{4.7a}$$

where C'_ν are computed from (4.2)-(4.4) for the operator $\mathcal{L}_{d-1}$ with parameters given in (4.7a). For $\mathcal{L}''$ we have

$$\mathcal{L}'' = I_d(\mathcal{L}') = \sum_{\nu\in\{-1,0,1\}^{d-1}} I_d(C'_\nu)E^\nu_{q,\bar{z}} = \sum_{\nu\in\{-1,0,1\}^{d-1}} C''_\nu E^\nu_{q,\bar{z}}. \tag{4.7b}$$

For the last operator $\mathcal{L}'''$ we obtain

$$\begin{aligned}\mathcal{L}''' &= \mathcal{L}_{d-1}(z_1,\dots,z_{d-1};\alpha_0,\dots,\alpha_{d-1},\alpha_{d+1},\alpha_{d+2}) \\ &= \sum_{\nu\in\{-1,0,1\}^{d-1}} C'''_\nu E^\nu_{q,\bar{z}} - \left(1+\frac{\alpha_{d+1}^2}{q\alpha_0^2} - \frac{4\alpha_{d+1}x_0x_{d+1}}{(q+1)\alpha_0}\right),\end{aligned} \tag{4.7c}$$

where C'''_ν are computed from (4.2)-(4.4) for the operator $\mathcal{L}_{d-1}$ with parameters given in (4.7c).

Notice that the last term in (4.7c) gives the last term in the right-hand side of (4.5). Thus, it remains to show that we get the stated formulas (4.4) for the coefficients C_ν in (4.5), using the decomposition (4.6). Since the operator $\mathcal{L}_d$ is I-invariant, it is enough to prove the formulas for C_ν when ν has nonnegative coordinates. We have two possible cases depending on whether $\nu_d = 0$ or $\nu_d = 1$.

Case 1 ($\nu_d = 1$). Write $\nu = (\nu', 1)$ with $\nu' \in \{0,1\}^{d-1}$. From (4.6) it is clear that $E^\nu_{q,z}$ appears only in the first term on the right-hand side and we have

$$C_\nu = C'_{\nu'}\frac{\left(1-\frac{\alpha_{d+1}\alpha_{d+2}z_d}{\alpha_d}\right)\left(1-\frac{\alpha_{d+1}z_d}{\alpha_d\alpha_{d+2}}\right)}{(1-z_d^2)(1-qz_d^2)}.$$

Notice that the factors $(B_k^{\nu_k,\nu_{k+1}})'$ in formula (4.4) for $C'_{\nu'}$ are the same as the factors $B_k^{\nu_k,\nu_{k+1}}$ in formula (4.4) for C_ν when $k = 0,1,\dots,d-2$. Similarly, the factors $(b_k^{\nu_k})'$ in formula (4.4) for $C'_{\nu'}$ are the same as the factors $b_k^{\nu_k}$ in formula (4.4) for C_ν when $k = 1,\dots,d-1$. This combined with the last formula above and

$$b_d^1 = (1-z_d^2)(1-qz_d^2), \qquad B_d^{1,0} = \left(1-\frac{\alpha_{d+1}\alpha_{d+2}z_d}{\alpha_d}\right)\left(1-\frac{\alpha_{d+1}z_d}{\alpha_d\alpha_{d+2}}\right)$$

show that in order to complete the proof we need to check that $(B_{d-1}^{\nu_{d-1},0})' = B_{d-1}^{\nu_{d-1},1}$. This can be easily verified from the defining relations (4.2) by considering the two possible cases $\nu_{d-1} = 0$ and $\nu_{d-1} = 1$.

Case 2 ($\nu_d = 0$). Let us again write $\nu = (\nu', 0)$ with $\nu' \in \{0,1\}^{d-1}$. Then from (4.6) we deduce

$$\begin{aligned}C_\nu = &-C'_{\nu'}\frac{\left(1-\frac{\alpha_{d+1}\alpha_{d+2}z_d}{\alpha_d}\right)\left(1-\frac{\alpha_{d+1}z_d}{\alpha_d\alpha_{d+2}}\right)}{(1-z_d^2)(1-qz_d^2)} \\ &- C''_{\nu'}\frac{\left(1-\frac{\alpha_{d+1}\alpha_{d+2}}{\alpha_d z_d}\right)\left(1-\frac{\alpha_{d+1}}{\alpha_d\alpha_{d+2}z_d}\right)}{(1-z_d^{-2})(1-qz_d^{-2})} + C'''_{\nu'}.\end{aligned}$$

We need to check formula (4.4) for C_ν. Again the factors $B_k^{\nu_k,\nu_{k+1}}$ for $k = 0, 1, \dots, d-2$ and the denominator factors $b_k^{\nu_k}$ for $k = 1, 2, \dots, d-1$ are common for C_ν, $C'_{\nu'}$, $C''_{\nu'}$ and $C'''_{\nu'}$. Thus we need to verify that

$$q(q+1)\frac{B_{d-1}^{\nu_{d-1},0}B_d^{0,0}}{(1-qz_d^2)(1-qz_d^{-2})} = -(B_{d-1}^{\nu_{d-1},0})'\frac{\left(1-\frac{\alpha_{d+1}\alpha_{d+2}z_d}{\alpha_d}\right)\left(1-\frac{\alpha_{d+1}z_d}{\alpha_d\alpha_{d+2}}\right)}{(1-z_d^2)(1-qz_d^2)}$$
$$-(B_{d-1}^{\nu_{d-1},0})''\frac{\left(1-\frac{\alpha_{d+1}\alpha_{d+2}}{\alpha_d z_d}\right)\left(1-\frac{\alpha_{d+1}}{\alpha_d\alpha_{d+2}z_d}\right)}{(1-z_d^{-2})(1-qz_d^{-2})} + (B_{d-1}^{\nu_{d-1},0})'''.$$

Using the explicit formulas (4.2) and considering separately the two possible cases $\nu_{d-1} = 0$ and $\nu_{d-1} = 1$ one can check that the above equality holds, thus completing the proof. □

Proposition 4.3. *The operator* $\mathcal{L}_d$ *defined by* (3.5)-(3.6) *is selfadjoint with respect to the inner product* (2.6)-(2.8).

Proof. We shall use Proposition 4.2 to check that the conditions in Lemma 4.1 are satisfied. We can represent the weight $w(z)$ in (2.6) as

$$w(z) = \frac{\prod_{k=1}^d w'_k(z)}{\prod_{k=0}^d w''_k(z)},$$

where

$$w'_k(z) = (z_k^2, z_k^{-2}; q)_\infty, \tag{4.8a}$$

$$w''_k(z) = \left(\frac{\alpha_{k+1}}{\alpha_k}z_{k+1}z_k, \frac{\alpha_{k+1}}{\alpha_k}\frac{z_{k+1}}{z_k}, \frac{\alpha_{k+1}}{\alpha_k}\frac{z_k}{z_{k+1}}, \frac{\alpha_{k+1}}{\alpha_k}\frac{1}{z_k z_{k+1}}; q\right)_\infty. \tag{4.8b}$$

Using Proposition 4.2 and formula (4.4) for the coefficients of the operator $\mathcal{L}_d(z;\alpha)$ we see that condition (i) of Lemma 4.1 will be satisfied if we can show that for every $\nu \in \{-1, 0, 1\}^d$ we have

$$\frac{b_k^{\nu_k}(z)}{b_k^{-\nu_k}(zq^\nu)} = \frac{w'_k(z)}{w'_k(zq^\nu)} \tag{4.9a}$$

and

$$\frac{B_k^{\nu_k,\nu_{k+1}}(z)}{B_k^{-\nu_k,-\nu_{k+1}}(zq^\nu)} = \frac{w''_k(z)}{w''_k(zq^\nu)}. \tag{4.9b}$$

It is easy to see that if these formulas are true for some ν, then they are also true for $-\nu$. Since $b_k^{\nu_k}(z)$ depends only on z_k, it is enough to check (4.9a) for $\nu_k = 0$ and $\nu_k = 1$. The case $\nu_k = 0$ is trivial because clearly both sides of equation (4.9a) are equal to 1. If $\nu_k = 1$, then the right-hand side of (4.9a) is

$$\frac{w'_k(z)}{w'_k(zq^\nu)} = \frac{(z_k^2, z_k^{-2}; q)_\infty}{(q^2z_k^2, z_k^{-2}q^{-2}; q)_\infty} = \frac{(1-z_k^2)(1-qz_k^2)}{\left(1-\frac{1}{q^2z_k^2}\right)\left(1-\frac{1}{qz_k^2}\right)}.$$

Using (4.3b)-(4.3c) we see at once that the last expression equals $b_k^1(z)/b_k^{-1}(zq^{e_k})$, completing the proof of (4.9a). Similarly, $B_k^{\nu_k,\nu_{k+1}}(z)$ depends only on z_k and z_{k+1}. Thus it is enough to verify (4.9b) for $(\nu_k, \nu_{k+1}) = \{(1,0), (0,1), (1,1), (1,-1)\}$, which can be done in the same manner by a straightforward verification using formulas (4.2).

Condition (ii) in Lemma 4.1 will follow if we show that for every $\nu \in \{-1,0,1\}^d$, the functions

$$\frac{w'_k(z)}{b_k^{\nu_k}(z)} \quad \text{for} \quad k=1,2,\dots,d \tag{4.10a}$$

and

$$\frac{B_k^{\nu_k,\nu_{k+1}}(z)}{w''_k(z)} \quad \text{for} \quad k=0,1,2,\dots,d \tag{4.10b}$$

are holomorphic on $\mathbb{T}_t^\nu$ for $t \in [0,1]$. Notice that

$$w'_k(z) = (1-z_k^2)(1-qz_k^2)(1-z_k^{-2})(1-qz_k^{-2})(z_k^2q^2, z_k^{-2}q^2; q)_\infty,$$

which combined with (4.3) shows that all the factors in the denominator in (4.10a) cancel, leading to a holomorphic function on $\mathbb{T}_t^\nu$. Next we show that the function in (4.10b) is holomorphic on $\mathbb{T}_t^\nu$. From (4.9b) it follows that if this is true for ν, then it will also be true for $-\nu$. Thus we can consider only the cases: $(\nu_k,\nu_{k+1}) = \{(0,0),(1,0),(0,1),(1,1),(1,-1)\}$. The case $(\nu_k,\nu_{k+1}) = (0,0)$ is trivial since equations (2.7) guarantee that $w''_k(z)$ has no zeros on the the d-dimensional torus $\mathbb{T} = \{z \in \mathbb{C}^d : |z_j| = 1 \text{ for } j=1,2,\dots,d\}$. Let us consider for instance the case $(\nu_k,\nu_{k+1}) = (1,1)$. Then for $z \in \mathbb{T}_t^\nu$ we have $1 \le |z_k z_{k+1}| \le q^{-2}$ and $|z_k/z_{k+1}| = 1$. From (2.7) and (4.8b), it follows that the only factors in $w''_k(z)$ that can vanish during the homotopy are

$$\left(1-\frac{\alpha_{k+1}z_kz_{k+1}}{\alpha_k}\right)\left(1-\frac{q\alpha_{k+1}z_{k+1}z_k}{\alpha_k}\right).$$

From (4.2d) we see that these two factors cancel with $B_k^{1,1}$ showing that the function in (4.10b) is holomorphic on $\mathbb{T}_t^\nu$. The remaining three cases follow along the same lines. □

As an immediate corollary of Proposition 3.3 and Proposition 4.3 we obtain the following theorem.

Theorem 4.4. *Let $\alpha_0,\alpha_1,\dots,\alpha_{d+2}$ be real parameters satisfying* (2.7) *and let $\{Q(n;x;\alpha) : n \in \mathbb{N}_0^d\}$ be a set of orthogonal polynomials with respect to the inner product* (2.8)*; i.e., $Q(n;x;\alpha)$ is a polynomial of total degree $|n|$, orthogonal to all polynomials of degree at most $|n|-1$. Then $Q(n;x;\alpha)$ is an eigenfunction of the operator $\mathcal{L}_d$ defined by* (3.5)-(3.6) *with eigenvalue $\mu = -(1-q^{-|n|})\left(1-\frac{\alpha_{d+1}^2}{\alpha_0^2}q^{|n|-1}\right)$.*

4.2. **Construction of $\mathcal{A}_z$.** Next we focus on the polynomials $P_d(n;x;\alpha)$ defined by (2.1) and (2.9).

Proposition 4.5. *Let $\alpha \in (\mathbb{C}^*)^{d+3}$. For $j \in \{1,2,\dots,d\}$ define*

$$\mathfrak{L}_j^z = \mathcal{L}_j(z_1,\dots,z_j;\alpha_0,\dots,\alpha_{j+1},z_{j+1}), \tag{4.11a}$$

$$\mu_j = -(1-q^{-(n_1+n_2+\cdots+n_j)})\left(1-\frac{\alpha_{j+1}^2}{\alpha_0^2}q^{n_1+n_2+\cdots+n_j-1}\right). \tag{4.11b}$$

Then the polynomials $P_d(n;x;\alpha)$ defined by (2.1) *and* (2.9) *satisfy the spectral equations*

$$\mathfrak{L}_j^z P_d(n;x;\alpha) = \mu_j P_d(n;x;\alpha) \tag{4.12}$$

for every $j \in \{1,2,\dots,d\}$, and the operators $\mathfrak{L}_j^z$ commute with each other; i.e., $\mathcal{A}_z = \mathbb{C}[\mathfrak{L}_1^z, \mathfrak{L}_2^z, \dots, \mathfrak{L}_d^z]$ is a commutative subalgebra of $\mathcal{D}_z$.

Proof. If the α_j are real satisfying (2.7), then (4.12) for $j = d$ follows immediately from Theorem 4.4. Notice that for fixed $n \in \mathbb{N}_0^d$ both sides of (4.12) depend rationally on the parameters α_j. Thus when $j = d$, equation (4.12) holds for arbitrary values of the parameters α_j. From (2.9) it is clear that for $j < d$ the product of the first j terms on the right-hand side is precisely

$$P_j(n_1,\dots,n_j;x_1,\dots,x_j;\alpha_0,\dots,\alpha_{j+1},z_{j+1}),$$

while the remaining terms $\prod_{l=j+1}^{d} p_{n_l}$ do not depend on the variables $x_1,\dots,x_j$. This shows that (4.12) also holds for $j < d$.

It remains to prove that the operators $\mathfrak{L}_j^z$ commute with each other. Fix $i \neq j \in \{1,2,\dots,d\}$. From (4.12) it follows that

$$[\mathfrak{L}_i^z, \mathfrak{L}_j^z] P_d(n;x;\alpha) = 0, \text{ for all } n \in \mathbb{N}_0^d.$$

This means that the operator $L = [\mathfrak{L}_i^z, \mathfrak{L}_j^z] \in \mathcal{D}_z$ vanishes on $\mathcal{P}_x$, and we want to show that it is identically equal to zero, i.e. that its coefficients are identically equal to zero. Equivalently, it is enough to show that for fixed $k = (k_1,\dots,k_d) \in \mathbb{N}_0^d$ the operator $L' = E_{q,z}^k L$ (which also vanishes on $\mathcal{P}_x$) is identically equal to zero. We can assume that the components k_j are large enough, and therefore the operator L' will contain only nonnegative powers of $E_{q,z}$. If we denote

$$D_{q,z_j} = \frac{1}{z_j}(1 - E_{q,z_j}),$$

then the operator L' can be uniquely written as

$$L' = \sum_{\substack{\nu \in \mathbb{N}_0^d \\ |\nu| \le M}} l_\nu(z) D_{q,z}^\nu,$$

where $M \in \mathbb{N}$ is large enough and $l_\nu(z)$ are rational functions of z. Now we use the fact that $L'(x^n) = 0$ for all $n \in \mathbb{N}_0^d$ such that $|n| \le M$. We obtain a homogeneous linear system for $l_\nu(z)$ with determinant

$$\mathrm{Det} = \det_{\substack{\nu,n \in \mathbb{N}_0^d \\ |\nu|,|n| \le M}} (D_{q,z}^\nu x^n).$$

In the above determinant we order in the same way ν (the rows) and n (the columns) respecting the total degree, i.e. $\nu < \mu$ when $|\nu| < |\mu|$. It remains to show that Det is not identically equal to 0. Notice that

$$D_{q,z_j}^{\nu_j} x_j^{n_j} = \frac{1}{2^{n_j}} (q^{n_j-\nu_j+1};q)_{\nu_j} z^{n_j-\nu_j} + O(z^{n_j-\nu_j-1}),$$

and therefore if we put

$$g_{\nu,n} = \frac{\prod_{j=1}^{d} (q^{n_j-\nu_j+1};q)_{\nu_j}}{2^{|n|}},$$

then

$$D_{q,z}^\nu x^n = g_{\nu,n} z^{n-\nu} + \text{ terms involving } z^k \text{ with } |k| < |n| - |\nu|.$$

From this formula it follows that

$$\mathrm{Det} = \det_{\substack{\nu,n\in\mathbb{N}_0^d \\ |\nu|,|n|\leq M}} (g_{\nu,n}) + \text{ a linear combination of } z^k \text{ with } |k|<0.$$

Clearly, $g_{\nu,n}\neq 0$ if and only if $\nu_j \leq n_j$ for all j. This shows that the matrix $(g_{\nu,n})$ is upper triangular with nonzero entries on the main diagonal; hence $\det_{\nu,n}(g_{\nu,n})\neq 0$, completing the proof. □

5. Bispectrality

5.1. **Duality of the multivariable Askey-Wilson polynomials.** Since q^{n_j} determines n_j uniquely modulo $\frac{2\pi i}{\log(q)}\mathbb{Z}$, we consider in this section complex variables $n=(n_1,\dots,n_d)\in\left(\mathbb{C} \mod \frac{2\pi i}{\log(q)}\mathbb{Z}\right)^d$, complex variables $z=(z_1,z_2,\dots,z_d)\in(\mathbb{C}^*)^d$ with nonzero components, and nonzero parameters $\alpha=(\alpha_0,\dots,\alpha_{d+2})\in(\mathbb{C}^*)^{d+3}$.

We define dual variables $\tilde{n}\in\left(\mathbb{C} \mod \frac{2\pi i}{\log(q)}\mathbb{Z}\right)^d$, $\tilde{z}\in(\mathbb{C}^*)^d$ and dual parameters $\tilde{\alpha}_j\in(\mathbb{C}^*)^{d+3}$ by

$$q^{\tilde{n}_j} = \frac{\alpha_{d+1-j}z_{d+1-j}}{\alpha_{d+2-j}z_{d+2-j}} \qquad \text{for } j=1,2,\dots,d, \tag{5.1a}$$

$$\tilde{z}_j = \frac{\alpha_{d+2-j}}{\alpha_0}q^{N_{d+1-j}-1/2} \qquad \text{for } j=1,2,\dots,d, \tag{5.1b}$$

$$\tilde{\alpha}_0=\alpha_0, \tag{5.1c}$$

$$\tilde{\alpha}_j = \frac{\alpha_0\alpha_{d+1}\alpha_{d+2}}{\alpha_{d+2-j}}q^{1/2} \qquad \text{for } j=1,2,\dots,d+1, \tag{5.1d}$$

$$\tilde{\alpha}_{d+2} = \frac{\alpha_1}{\alpha_0}q^{-1/2}, \tag{5.1e}$$

where as before we have set $z_{d+1}=\alpha_{d+2}$ and $N_k=n_1+\cdots+n_k$. In analogy with (2.5) we put $\tilde{x}_j=\frac{1}{2}(\tilde{z}_j+\tilde{z}_j^{-1})$.

Lemma 5.1. *The mapping* $\mathfrak{f}:(n,z,\alpha)\to(\tilde{n},\tilde{z},\tilde{\alpha})$ *given by* (5.1) *defines an involution on* $\left(\mathbb{C} \mod \frac{2\pi i}{\log(q)}\mathbb{Z}\right)^d\times(\mathbb{C}^*)^d\times(\mathbb{C}^*)^{d+3}$.

Proof. Using formulas (5.1) one can easily check that $\mathfrak{f}\circ\mathfrak{f}=\mathrm{Id}$. □

Iterating a formula of Sears we obtain the following lemma.

Lemma 5.2. *If* $k\in\mathbb{N}_0$ *and* $abc=defq^{k-1}$, *then*

$$\begin{aligned}(d,e,f;q)_k\, {}_4\phi_3\left[\begin{matrix}q^{-k},a,b,c\\ d,e,f\end{matrix};q,q\right] &= c^k(b,aq^{1-k}/e,e/c;q)_k\\ &\times {}_4\phi_3\left[\begin{matrix}q^{-k},q^{1-k}/e,d/b,f/b\\ q^{1-k}/b,aq^{1-k}/e,cq^{1-k}/e\end{matrix};q,q\right].\end{aligned} \tag{5.2}$$

Proof. Sears' formula (see for instance [5, page 49, formula (2.10.4)]) gives

$$\begin{aligned}(d,e,f;q)_k\, {}_4\phi_3\left[\begin{matrix}q^{-k},a,b,c\\ d,e,f\end{matrix};q,q\right] &= a^k(d,e/a,f/a,;q)_k\\ &\times {}_4\phi_3\left[\begin{matrix}q^{-k},a,d/b,d/c\\ d,aq^{1-k}/e,aq^{1-k}/f\end{matrix};q,q\right].\end{aligned} \tag{5.3}$$

Applying the same formula on the right-hand side of (5.3) with a, b, c, d, e, f replaced by $d/b, a, d/c, aq^{1-k}/e, d, aq^{1-k}/f$, respectively, and using the identity

$$(s;q)_k = (-s)^k q^{(k-1)k/2}(q^{1-k}/s;q)_k$$

we obtain (5.2). □

Let us normalize the multivariable Askey-Wilson polynomials (2.9) as follows:

$$\hat{P}_d(n;x;\alpha) = \frac{(\alpha_{d+1}\alpha_{d+2})^{|n|}P_d(n;x;\alpha)}{(\alpha_{d+1}\alpha_{d+2}, \alpha_{d+1}\alpha_{d+2}/\alpha_0^2;q)_{|n|}\prod_{j=1}^d \alpha_j^{n_j}(\alpha_{j+1}^2/\alpha_j^2;q)_{n_j}}. \tag{5.4}$$

The main result in this subsection is the following theorem.

Theorem 5.3. *If* (n, z, α) *and* $(\tilde{n}, \tilde{z}, \tilde{\alpha})$ *are related by* (5.1) *and if* $n, \tilde{n} \in \mathbb{N}_0^d$, *then*

$$\hat{P}_d(n;x;\alpha) = \hat{P}_d(\tilde{n};\tilde{x};\tilde{\alpha}). \tag{5.5}$$

Proof. Using (2.1) and applying formula (5.2) with k, a, b, c, d, e, f replaced by n_j, $\alpha_{j+1}^2 q^{N_{j-1}+N_j-1}/\alpha_0^2$, $\alpha_j q^{N_{j-1}} z_j$, $\alpha_j q^{N_{j-1}} z_j^{-1}$, $\alpha_j^2 q^{2N_{j-1}}/\alpha_0^2$, $\alpha_{j+1} q^{N_{j-1}} z_{j+1}^{-1}$, $\alpha_{j+1} q^{N_{j-1}} z_{j+1}$ we can write the polynomial p_{n_j} in (2.9) as follows:

$$\begin{aligned} p_{n_j} &= (z_j)^{-n_j}(\alpha_j q^{N_{j-1}} z_j, \alpha_{j+1} q^{N_{j-1}} z_{j+1}/\alpha_0^2, \alpha_{j+1}\alpha_j^{-1} z_j z_{j+1}^{-1};q)_{n_j} \\ &\quad \times {}_4\phi_3\left[\begin{matrix} q^{-n_j}, z_{j+1}q^{1-N_j}\alpha_{j+1}^{-1}, \alpha_0^{-2}\alpha_j q^{N_{j-1}} z_j^{-1}, \alpha_{j+1}\alpha_j^{-1} z_j^{-1} z_{j+1} \\ q^{1-N_j}\alpha_j^{-1} z_j^{-1}, \alpha_0^{-2}\alpha_{j+1} q^{N_{j-1}} z_{j+1}, q^{1-n_j}\alpha_j\alpha_{j+1}^{-1} z_j^{-1} z_{j+1} \end{matrix}; q, q\right]. \end{aligned} \tag{5.6}$$

Using the above formula for p_{n_j} and (5.1) one can check that the ${}_4\phi_3$ term of p_{n_j} in the variables (n, z, α) coincides with the ${}_4\phi_3$ term of $p_{\tilde{n}_{d+1-j}}$ in the variables $(\tilde{n}; \tilde{z}; \tilde{\alpha})$ for $j = 1, 2, \ldots, d$. Thus if we compute the ratio $P_d(n;x;\alpha)/P_d(\tilde{n};\tilde{x};\tilde{\alpha})$, all ${}_4\phi_3$ terms cancel and we get

$$\frac{P_d(n;x;\alpha)}{P_d(\tilde{n};\tilde{x};\tilde{\alpha})} = \prod_{j=1}^d \frac{(z_j)^{-n_j}(\alpha_j q^{N_{j-1}} z_j, \alpha_{j+1} q^{N_{j-1}} z_{j+1}/\alpha_0^2, \alpha_{j+1}\alpha_j^{-1} z_j z_{j+1}^{-1};q)_{n_j}}{(\tilde{z}_j)^{-\tilde{n}_j}(\tilde{\alpha}_j q^{\tilde{N}_{j-1}} \tilde{z}_j, \tilde{\alpha}_{j+1} q^{\tilde{N}_{j-1}} \tilde{z}_{j+1}/\tilde{\alpha}_0^2, \tilde{\alpha}_{j+1}\tilde{\alpha}_j^{-1} \tilde{z}_j \tilde{z}_{j+1}^{-1};q)_{\tilde{n}_j}}. \tag{5.7}$$

Using (5.1) we can eliminate z_j and $\tilde{z}_j$ in the right-hand side of (5.7). From (5.1d)-(5.1e) it follows that $\alpha_{d+1}\alpha_{d+2} = \tilde{\alpha}_{d+1}\tilde{\alpha}_{d+2}$. Next, notice that $\alpha_j z_j = \alpha_{d+1}\alpha_{d+2} q^{\tilde{N}_{d+1-j}}$. Thus we have

$$\prod_{j=1}^d \frac{z_j^{-n_j}}{\tilde{z}_j^{-\tilde{n}_j}} = \prod_{j=1}^d \frac{\left(\alpha_j\alpha_{d+1}^{-1}\alpha_{d+2}^{-1}\right)^{n_j}}{\left(\tilde{\alpha}_j\tilde{\alpha}_{d+1}^{-1}\tilde{\alpha}_{d+2}^{-1}\right)^{\tilde{n}_j}} \prod_{j=1}^d \frac{q^{-\tilde{N}_{d+1-j}n_j}}{q^{-N_{d+1-j}\tilde{n}_j}}.$$

It is easy to see that the second product on the right-hand side of the above formula is equal to one, by replacing $n_j = N_j - N_{j-1}$ and $\tilde{n}_j = \tilde{N}_j - \tilde{N}_{j-1}$. Therefore

$$\prod_{j=1}^d \frac{z_j^{-n_j}}{\tilde{z}_j^{-\tilde{n}_j}} = \prod_{j=1}^d \frac{\left(\alpha_j\alpha_{d+1}^{-1}\alpha_{d+2}^{-1}\right)^{n_j}}{\left(\tilde{\alpha}_j\tilde{\alpha}_{d+1}^{-1}\tilde{\alpha}_{d+2}^{-1}\right)^{\tilde{n}_j}}. \tag{5.8}$$

Using the identity

$$(aq^l;q)_k = \frac{(a;q)_{l+k}}{(a;q)_l} \quad \text{for} \quad k, l \in \mathbb{N}_0,$$

we obtain

$$\prod_{j=1}^{d}\frac{(\alpha_j q^{N_{j-1}}z_j;q)_{n_j}}{(\tilde{\alpha}_j q^{\tilde{N}_{j-1}}\tilde{z}_j;q)_{\tilde{n}_j}}=\prod_{j=1}^{d}\frac{(\alpha_{d+1}\alpha_{d+2}q^{N_{j-1}+\tilde{N}_{d+1-j}};q)_{n_j}}{(\alpha_{d+1}\alpha_{d+2}q^{\tilde{N}_{j-1}+N_{d+1-j}};q)_{\tilde{n}_j}}$$

$$=\prod_{j=1}^{d}\frac{(\alpha_{d+1}\alpha_{d+2};q)_{N_j+\tilde{N}_{d+1-j}}}{(\alpha_{d+1}\alpha_{d+2};q)_{\tilde{N}_j+N_{d+1-j}}}\prod_{j=1}^{d}\frac{(\alpha_{d+1}\alpha_{d+2};q)_{\tilde{N}_{j-1}+N_{d+1-j}}}{(\alpha_{d+1}\alpha_{d+2};q)_{N_{j-1}+\tilde{N}_{d+1-j}}}.$$

It is easy to see now that the first product above is equal to 1, while the second simplifies to $\frac{(\alpha_{d+1}\alpha_{d+2};q)_{N_d}}{(\alpha_{d+1}\alpha_{d+2};q)_{\tilde{N}_d}}$. Thus we have

$$\prod_{j=1}^{d}\frac{(\alpha_j q^{N_{j-1}}z_j;q)_{n_j}}{(\tilde{\alpha}_j q^{\tilde{N}_{j-1}}\tilde{z}_j;q)_{\tilde{n}_j}}=\frac{(\alpha_{d+1}\alpha_{d+2};q)_{|n|}}{(\tilde{\alpha}_{d+1}\tilde{\alpha}_{d+2};q)_{|\tilde{n}|}}. \tag{5.9}$$

Similar manipulations show that

$$\begin{aligned}&\prod_{j=1}^{d}\frac{(\alpha_{j+1}q^{N_{j-1}}z_{j+1}/\alpha_0^2,\alpha_{j+1}\alpha_j^{-1}z_jz_{j+1}^{-1};q)_{n_j}}{(\tilde{\alpha}_{j+1}q^{\tilde{N}_{j-1}}\tilde{z}_{j+1}/\tilde{\alpha}_0^2,\tilde{\alpha}_{j+1}\tilde{\alpha}_j^{-1}\tilde{z}_j\tilde{z}_{j+1}^{-1};q)_{\tilde{n}_j}}\\&\qquad=\frac{(\alpha_{d+1}\alpha_{d+2}/\alpha_0^2;q)_{|n|}}{(\tilde{\alpha}_{d+1}\tilde{\alpha}_{d+2}/\tilde{\alpha}_0^2;q)_{|\tilde{n}|}}\prod_{j=1}^{d}\frac{(\alpha_{j+1}^2/\alpha_j^2;q)_{n_j}}{(\tilde{\alpha}_{j+1}^2/\tilde{\alpha}_j^2;q)_{\tilde{n}_j}}.\end{aligned} \tag{5.10}$$

The proof follows from equations (5.7), (5.8), (5.9) and (5.10). □

5.2. **The commutative algebra $\mathcal{A}_n$.** We denote by $\mathcal{D}_z^{\alpha}$ the associative subalgebra of $\mathcal{D}_z$ of difference operators with coefficients depending rationally on the parameters α_j. Clearly, the commutative algebra $\mathcal{A}_z$ defined in Proposition 4.5 is contained in $\mathcal{D}_z^{\alpha}$. Similarly, we denote by $\mathcal{D}_n^{\alpha}$ the associative algebra of difference operators in the variables $n=(n_1,n_2,\dots,n_d)$ with coefficients depending rationally on $q^{n_1},q^{n_2},\dots,q^{n_d}$ and the parameters α_j. We use E_{n_k} to denote the forward shift in the variable n_k; i.e., for every function $f(n)=f(n_1,\dots,n_d)$ we have $E_{n_k}f(n)=f(n+e_k)$.

Replacing j by $d+1-k$ in (5.1a) we see that

$$q^{\tilde{n}_{d+1-k}}=\frac{\alpha_k z_k}{\alpha_{k+1}z_{k+1}}.$$

From this equation it follows that a forward q-shift in the variable z_k will correspond to a forward shift in $\tilde{n}_{d+1-k}$ and a backward shift in $\tilde{n}_{d+2-k}$ for $k=2,3,\dots,d$ and to a forward shift in $\tilde{n}_d$ when $k=1$. Thus, in view of the duality established in Theorem 5.3, we define a function $\mathfrak{b}$ as follows:

$$\mathfrak{b}(\alpha_0)=\alpha_0, \tag{5.11a}$$

$$\mathfrak{b}(\alpha_j)=\frac{\alpha_0\alpha_{d+1}\alpha_{d+2}}{\alpha_{d+2-j}}q^{1/2}\qquad\text{for } j=1,2,\dots,d+1, \tag{5.11b}$$

$$\mathfrak{b}(\alpha_{d+2})=\frac{\alpha_1}{\alpha_0}q^{-1/2}, \tag{5.11c}$$

$$\mathfrak{b}(z_j)=\frac{\alpha_{d+2-j}}{\alpha_0}q^{n_1+n_2+\cdots+n_{d+1-j}-1/2}\qquad\text{for } j=1,2,\dots,d, \tag{5.11d}$$

$$\mathfrak{b}(E_{q,z_j})=E_{n_{d+1-j}}E_{n_{d+2-j}}^{-1},\ \text{for } j=1,2,\dots,d \tag{5.11e}$$

with the convention that $E_{n_{d+1}}$ is the identity operator.

Lemma 5.4. *The mapping* (5.11) *extends to an isomorphism* $\mathfrak{b} : \mathcal{D}_z^{\alpha} \to \mathcal{D}_n^{\alpha}$. *In particular, the operators* $\mathfrak{L}_k^n$ *defined by*

$$\mathfrak{L}_k^n = \mathfrak{b}(\mathfrak{L}_k^z) \quad \text{for } k = 1, 2, \dots, d \tag{5.12}$$

commute with each other, and therefore

$$\mathcal{A}_n = \mathfrak{b}(\mathcal{A}_z) = \mathbb{C}[\mathfrak{L}_1^n, \mathfrak{L}_2^n, \dots, \mathfrak{L}_d^n] \tag{5.13}$$

is a commutative subalgebra of $\mathcal{D}_n^{\alpha}$.

Proof. Using (5.11) one can check that

$$\mathfrak{b}(E_{q,\,z_k}) \cdot \mathfrak{b}(f(z)) = \mathfrak{b}(f(zq^{e_k}))\mathfrak{b}(E_{q,\,z_k})$$

holds for every $k = 1, 2, \dots, d$, which shows that $\mathfrak{b}$ is a well-defined homomorphism from $\mathcal{D}_z^{\alpha}$ to $\mathcal{D}_n^{\alpha}$. The fact that $\mathfrak{b}$ is one-to-one and onto follows easily. □

Recall that the eigenvalues μ_j of the operators $\mathfrak{L}_j^z$ are given in (4.11b). We denote

$$\begin{aligned} \kappa_j = \mathfrak{b}^{-1}(\mu_j) &= -\left(1 - \frac{\alpha_{d+1}\alpha_{d+2}}{\alpha_{d+1-j}z_{d+1-j}}\right)\left(1 - \frac{\alpha_{d+1}\alpha_{d+2}z_{d+1-j}}{\alpha_{d+1-j}}\right) \\ &= -1 - \frac{\alpha_{d+1}^2\alpha_{d+2}^2}{\alpha_{d+1-j}^2} + \frac{2\alpha_{d+1}\alpha_{d+2}}{\alpha_{d+1-j}}x_{d+1-j}. \end{aligned} \tag{5.14}$$

With this notation we can formulate the main result.

Theorem 5.5. *The polynomials* $\hat{P}_d(n; x; \alpha)$ *defined by equations* (2.9) *and* (5.4) *diagonalize the algebras* $\mathcal{A}_z$ *and* $\mathcal{A}_n$. *More precisely, the following spectral equations hold:*

$$\mathfrak{L}_j^z \hat{P}_d(n; x; \alpha) = \mu_j \hat{P}_d(n; x; \alpha), \tag{5.15a}$$

$$\mathfrak{L}_j^n \hat{P}_d(n; x; \alpha) = \kappa_j \hat{P}_d(n; x; \alpha), \tag{5.15b}$$

for $j = 1, 2, \dots, d$, *where* $\mathfrak{L}_j^z, \mu_j$ *are given in* (4.11) *and* $\mathfrak{L}_j^n, \kappa_j$ *are given in* (5.12) *and* (5.14).

Remark 5.6 (Boundary conditions). Since $\mathfrak{L}_k^n$ contains backward shift operators, and since $\hat{P}_d(n; x; \alpha)$ is defined only for $n \in \mathbb{N}_0^d$ it is natural to ask what happens when $\mathfrak{L}_k^n$ produces a term with a negative n_j for some j. From (5.11e) it follows that $\mathfrak{b}(E_{q,\,z}^{\nu})$ with $\nu \in \{-1, 0, 1\}^d$ will contain a negative power of $E_{n_{d+2-j}}$ $(j \geq 2)$ in one of the following two cases:

Case 1 ($\nu_j = 1$, $\nu_{j-1} = 0$). In this case, $\mathfrak{b}(E_{q,\,z}^{\nu})$ will contain $E_{n_{d+2-j}}^{-1}$. Notice that the coefficient of $E_{q,\,z}^{\nu}$ is C_ν which has the factor $\left(1 - \frac{\alpha_j z_j}{\alpha_{j-1}z_{j-1}}\right)$ (in $B_{j-1}^{0,1}$; see formula (4.2b)). Since $\mathfrak{b}\left(1 - \frac{\alpha_j z_j}{\alpha_{j-1}z_{j-1}}\right) = (1 - q^{-n_{d+2-j}})$ we see that this term is 0 when $n_{d+2-j} = 0$.

Case 2 ($\nu_j = 1$, $\nu_{j-1} = -1$). This time $\mathfrak{b}(E_{q,\,z}^{\nu})$ contains $E_{n_{d+2-j}}^{-2}$. The coefficient C_ν has the factor $B_{j-1}^{-1,1} = \left(1 - \frac{\alpha_j z_j}{\alpha_{j-1}z_{j-1}}\right)\left(1 - \frac{q\alpha_j z_j}{\alpha_{j-1}z_{j-1}}\right)$ (see (4.2d) and (4.2e)). Since $\mathfrak{b}(B_{j-1}^{-1,1}) = (1 - q^{-n_{d+2-j}})(1 - q^{1-n_{d+2-j}})$ we see that this coefficient is 0 when $n_{d+2-j} = 0$ or 1.

Proof of Theorem 5.5. Equation (5.15a) follows immediately from Proposition 4.5 and the fact that $\hat{P}_d(n;x;\alpha)$ and $P_d(n;x;\alpha)$ differ by a factor independent of z. It remains to prove (5.15b). Let us fix n, α. By Lemma 5.1 for $z \in (\mathbb{C}^*)^d$ we can find $(\tilde{n}, \tilde{z}, \tilde{\alpha})$ such that equations (5.1) hold. If $\tilde{n} \in \mathbb{N}_0^d$, then we can use Theorem 5.3 and replace $\hat{P}_d(n;x;\alpha)$ by $\hat{P}_d(\tilde{n};\tilde{x};\tilde{\alpha})$. Equation (5.15b) follows from the fact that the operator $\mathfrak{L}_j^n$ in the variables n with parameters α_j coincides with the operator $\mathfrak{L}_j^z$ in the variables $\tilde{z}$ with parameters $\tilde{\alpha}_j$. We can use analytic continuation and complete the proof as follows. Fix n, α and write z in terms of the dual variables $\tilde{n}$; i.e., we put

$$z_k = \frac{\alpha_{d+1}\alpha_{d+2}}{\alpha_k} q^{\tilde{n}_1+\tilde{n}_2+\cdots+\tilde{n}_{d+1-k}} \text{ for } k = 1, 2, \ldots, d.$$

Both sides of equation (5.15b) become Laurent polynomials in the variables $q^{\tilde{n}_1}$, $q^{\tilde{n}_2}$, …, $q^{\tilde{n}_d}$. Since (5.15b) is true for every $\tilde{n} \in \mathbb{N}_0^d$, we conclude that it must be true for arbitrary $\tilde{n} \in \mathbb{C}^d$, or equivalently, for arbitrary $z \in (\mathbb{C}^*)^d$. □

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