1-POINT GROMOV-WITTEN INVARINTS
OF THE MODULI SPACES OF SHEAVES
OVER THE PROJECTIVE PLANE

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Abstract. The Gieseker-Uhlenbeck morphism maps the Gieseker moduli space of stable rank-2 sheaves on a smooth projective surface to the Uhlenbeck compactification and is a generalization of the Hilbert-Chow morphism for Hilbert schemes of points. When the surface is the complex projective plane, we determine all the 1-point genus-0 Gromov-Witten invariants extremal with respect to the Gieseker-Uhlenbeck morphism. The main idea is to understand the virtual fundamental class of the moduli space of stable maps by studying the obstruction sheaf and using a meromorphic 2-form on the Gieseker moduli space.

1. Introduction

Recently there has been intensive interest in studying the quantum cohomology and Gromov-Witten theory of Hilbert schemes of points on algebraic surfaces. Two main reasons are the connections with the Donaldson-Thomas theory of 3-folds and with Ruan’s Cohomological Crepant Resolution Conjecture. Roughly speaking, the Crepant Resolution Conjecture asserts that the quantum cohomology of an orbifold Z coincides with the quantum cohomology of a crepant resolution Y of Z after analytic continuation and specialization of quantum parameters. For an algebraic surface X, let $X^{[n]}$ be the Hilbert scheme of $n$ points on X and $\text{Sym}^n(X)$ be the $n$-th symmetric product of X. It is well known that $X^{[n]}$ is smooth of dimension $2n$ and the Hilbert-Chow morphism $\Phi : X^{[n]} \to \text{Sym}^n(X)$ is a crepant resolution of the global orbifold $\text{Sym}^n(X)$.

A natural generalization of the Hilbert-Chow morphism $\Phi$ is the Gieseker-Uhlenbeck morphism $\Psi$ from the moduli space of Gieseker semistable rank-2 torsion-free sheaves on X to the Uhlenbeck compactification space. This morphism was constructed in [LJ1, Mor], and was shown to be crepant [LJ2, Q-Z] when the Gieseker moduli space is smooth. For the projective plane $X = \mathbb{P}^2$, the moduli space $\mathfrak{M}(n)$ of Gieseker semistable sheaves $V$ on $X$ with $c_1(V) = -1$ and $c_2(V) = n$ is a smooth irreducible projective variety of dimension $4n - 4$ when $n \geq 1$. In [Q-Z], it is proved that there is exactly one primitive integral class $f \in H_2(\mathfrak{M}(n); \mathbb{Z})$ contracted by the Gieseker-Uhlenbeck morphism $\Psi : \mathfrak{M}(n) \to \mathfrak{U}(n)$. 

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The goal of this paper is to determine all the 1-point genus-0 Gromov-Witten invariants \( \langle \alpha \rangle_{0,d} \), \( \alpha \in H^{6n-12}(\overline{M}(n); \mathbb{C}) \) extremal with respect to \( \Psi \) for \( n \geq 3 \). When \( n \geq 3 \), the homology group \( H_4(\overline{M}(n); \mathbb{C}) \) is of rank 6, and a basis is given by \( \{ \Xi_1, \ldots, \Xi_6 \} \) (see [4] for details). The Poincaré duals \( \text{PD}(\Xi_1), \ldots, \text{PD}(\Xi_6) \) form a basis of \( H^{6n-12}(\overline{M}(n); \mathbb{C}) \).

**Theorem 1.1.** Let \( d \geq 1 \) and \( n \geq 3 \). The Gromov-Witten invariants \( \langle \alpha \rangle_{0,d} \) for the classes \( \alpha = \text{PD}(\Xi_1), \ldots, \text{PD}(\Xi_6) \in H^{6n-12}(\overline{M}(n); \mathbb{C}) \) are respectively equal to
\[
-6/d^2, \quad 12/d^2, \quad 0, \quad -6/d^2, \quad 0, \quad 0.
\]

When \( n = 1 \), the moduli space \( \overline{M}(n) \) is a point. When \( n = 2 \), the fourth Betti number \( b_4 \) of the moduli space \( \overline{M}(n) \) is equal to 3, which is different from the case \( n \geq 3 \). The result for \( n = 2 \) will appear elsewhere via a different method (see Remark 3.2).

An interesting observation is that the 1-point genus-0 Gromov-Witten invariants \( \langle \alpha \rangle_{0,d} \) are independent of the second Chern class \( n \).

**Conjecture 1.2.** Let \( d \geq 1 \) and \( n \geq 3 \). Then the extremal genus-0 Gromov-Witten invariants \( \langle \alpha_1, \ldots, \alpha_k \rangle_{0,d} \) of the moduli space \( \overline{M}(n) \) are independent of \( n \).

There are two main ideas in our proof of Theorem 1.1. The first one is to determine the restriction of the obstruction sheaf of the Gromov-Witten theory for \( \overline{M}(n) \) to certain open subsets of the moduli space \( \overline{M}_{0,1}(\overline{M}(n), d) \) of stable maps. This enables us to determine the 1-point invariants \( \langle \alpha \rangle_{0,d} \) for the first four cohomology classes \( \alpha = \text{PD}(\Xi_1), \ldots, \text{PD}(\Xi_4) \).

The second one is to study the support of the virtual fundamental class
\[
\langle \overline{M}_{0,1}(\overline{M}(n), d) \rangle^\vir \in A_{4n-6}(\overline{M}_{0,1}(\overline{M}(n), d))
\]
using the techniques developed in [K-L] [L-L]. By introducing a suitable meromorphic 2-form \( \Theta \) on the Gieseker moduli space \( \overline{M}(n) \), we show that
\[
e v_1(\text{Supp}([\overline{M}_{0,1}(\overline{M}(n), d)]^\vir)) \subset \Sigma_{C_0}(n) \coprod C_0(n), \tag{1.1}
\]
where \( ev_1 : \overline{M}_{0,1}(\overline{M}(n), d) \to \overline{M}(n) \) is the evaluation map, and \( \Sigma_{C_0}(n) \) (respectively, \( C_0(n) \)) is the subset of \( \overline{M}(n) \) consisting of all the nonlocally free sheaves \( V \) such that \( V|_{C_0} \) contains torsion (respectively, \( V|_{C_0} \) is torsion-free and unstable). This allows us to show that \( \langle \alpha \rangle_{0,d} = 0 \) for \( \alpha = \text{PD}(\Xi_5), \text{PD}(\Xi_6) \).

This paper is organized as follows. In [2], the Gromov-Witten theory is reviewed. In [3], we recall some properties of the Gieseker moduli space \( \overline{M}(n) \) and the Gieseker-Uhlenbeck morphism \( \Psi \). We study the boundary divisor of \( \overline{M}(n) \) consisting of nonlocally free sheaves in \( \overline{M}(n) \). In [4], the basis \( \{ \Xi_1, \ldots, \Xi_6 \} \) for \( H_4(\overline{M}(n); \mathbb{C}) \) is constructed. In [5], we analyze the obstruction sheaf of the Gromov-Witten theory for \( \overline{M}(n) \). In [6], (1.1) is proved. In [7], we verify Theorem 1.1.

2. Stable maps and Gromov-Witten invariants

Let \( Y \) be a smooth projective variety. A \( k \)-pointed stable map to \( Y \) consists of a complete nodal curve \( D \) with \( k \) distinct ordered smooth points \( p_1, \ldots, p_k \) and a morphism \( \mu : D \to Y \) such that the data \( (\mu, D, p_1, \ldots, p_k) \) has only finitely many automorphisms. In this case, the stable map is denoted by \( [\mu : (D, p_1, \ldots, p_k) \to Y] \).

For a fixed homology class \( \beta \in H_2(Y, \mathbb{Z}) \), let \( \overline{M}_{g,k}(Y, \beta) \) be the coarse moduli space
parameterizing all the stable maps $[\mu : (D; p_1, \ldots, p_k) \to Y]$ such that $\mu_\ast [D] = \beta$ and the arithmetic genus of $D$ is $g$. Then, we have the evaluation map:

\[(2.1)\]
\[ev_k : \mathcal{M}_{g,k}(Y, \beta) \to Y^k\]

defined by $ev_k([\mu : (D; p_1, \ldots, p_k) \to Y]) = (\mu(p_1), \ldots, \mu(p_k))$. It is known [F-P, LT1, LT2] that the coarse moduli space $\mathcal{M}_{g,k}(Y, \beta)$ is projective and has a virtual fundamental class $[\mathcal{M}_{g,k}(Y, \beta)]^{vir} \in A_0(\mathcal{M}_{g,k}(Y, \beta))$, where

\[(2.2)\]
\[\mathfrak{d} = -(KY \cdot \beta) + (\dim(Y) - 3)(1 - g) + k\]

is the expected complex dimension of $\mathcal{M}_{g,k}(Y, \beta)$, and $A_0(\mathcal{M}_{g,k}(Y, \beta))$ is the Chow group of $\mathfrak{d}$-dimensional cycles in the moduli space $\mathcal{M}_{g,k}(Y, \beta)$.

The Gromov-Witten invariants are defined by using the virtual fundamental class $[\mathcal{M}_{g,k}(Y, \beta)]^{vir}$. Recall that an element $\alpha \in H^\ast(Y, \mathbb{C}) = \bigoplus_{j=0}^{2\dim(Y)} H^j(Y, \mathbb{C})$ is homogeneous if $\alpha \in H^j(Y, \mathbb{C})$ for some $j$; in this case, we take $|\alpha| = j$. Let $\alpha_1, \ldots, \alpha_k \in H^\ast(Y, \mathbb{C})$ such that every $\alpha_i$ is homogeneous and $\sum_{i=1}^{k} |\alpha_i| = 2\mathfrak{d}$. Then, we have the $k$-point Gromov-Witten invariant defined by

\[(2.3)\]
\[\langle \alpha_1, \ldots, \alpha_k \rangle_{g, \beta} = \int_{[\mathcal{M}_{g,k}(Y, \beta)]^{vir}} ev_k^\ast (\alpha_1 \otimes \ldots \otimes \alpha_k).\]

Next, we summarize certain properties concerning the virtual fundamental class. To begin with, we recall that the \textit{excess dimension} is the difference between the dimension of $\mathcal{M}_{g,k}(Y, \beta)$ and the expected dimension $\mathfrak{d}$ in (2.2). For $0 \leq i < k$, use

\[(2.4)\]
\[f_{k,i} : \mathcal{M}_{g,k}(Y, \beta) \to \mathcal{M}_{g,i}(Y, \beta)\]

to stand for the forgetful map obtained by forgetting the last $k - i$ marked points and contracting all the unstable components. It is known that $f_{k,i}$ is flat when $\beta \neq 0$ and $0 \leq i < k$. The following can be found in [LT1, Beh, Get, C-K].

\begin{proposition}
Let $\beta \in H_2(Y, \mathbb{Z})$ and $\beta \neq 0$. Let $e$ be the excess dimension of $\mathcal{M}_{g,k}(Y, \beta)$, and $\mathfrak{m} \subset \mathcal{M}_{g,k}(Y, \beta)$ be a closed subscheme. Then,

(i) $[\mathcal{M}_{g,k}(Y, \beta)]^{vir} = (f_{k,0})^\ast [\mathcal{M}_{g,0}(Y, \beta)]^{vir}$;

(ii) $[\mathcal{M}_{g,k}(Y, \beta)]^{vir} = c_e(R^1(f_{k+1,k})_\ast (ev_{k+1})^\ast T_Y)$ if $R^1(f_{k+1,k})_\ast (ev_{k+1})^\ast T_Y$ is a rank-$e$ locally free sheaf over the moduli space $\mathcal{M}_{g,k}(Y, \beta)$;

(iii) $[\mathcal{M}_{g,k}(Y, \beta)]^{vir}|_{\mathfrak{m}} = c_e((R^1(f_{k+1,k})_\ast (ev_{k+1})^\ast T_Y)|_{\mathfrak{m}})$ if there exists an open subset $\mathfrak{D}$ of $\mathcal{M}_{g,k}(Y, \beta)$ such that $\mathfrak{m} \subset \mathfrak{D}$ (i.e., $\mathfrak{D}$ is an open neighborhood of $\mathfrak{m}$) and the restriction $(R^1(f_{k+1,k})_\ast (ev_{k+1})^\ast T_Y)|_{\mathfrak{D}}$ is a rank-$e$ locally free sheaf over $\mathfrak{D}$.
\end{proposition}

3. The moduli space of stable rank-2 sheaves on $\mathbb{P}^2$

3.1. Some basic facts of the moduli space. Throughout the rest of this paper, let $X = \mathbb{P}^2$ be the projective plane, and let $E$ be a line in $X$. For an integer $n$, let $\mathcal{M}(n)$ be the moduli space parametrizing all Gieseker-semistable rank-2 sheaves $V$ over $X$ with $c_1(V) = -E, c_2(V) = n$. Note that every such sheaf $V$ is actually slope-stable and hence is Gieseker-stable. It is well known that, when $n \geq 1$, $\mathcal{M}(n)$ is nonempty, smooth, irreducible and rational with the expected dimension $4n - 4$; in addition, a universal sheaf over $\mathcal{M}(n) \times X$ exists. By Theorem 1 in [Mar], the cohomology groups $H^i(\mathcal{M}(n); \mathbb{Z})$ are torsion-free for all $i$ and vanish for odd $i$. The
same is true of the homology groups $H_i(\mathcal{M}(n); \mathbb{Z})$. Let $b_i(\mathcal{M}(n))$ be the $i$-th Betti number of $\mathcal{M}(n)$, and put

$$p(\mathcal{M}(n); q) = \sum_{i=0}^{8n-8} b_i(\mathcal{M}(n)) q^i.$$  

By Theorem 0.1 of [Yao], $p(\mathcal{M}(n); q)$ equals the coefficient of $t^n$ of the series

$$\frac{1}{(q^2 - 1) \cdot \sum_{n \in \mathbb{Z}} q^{2n(2n-1)t^2}} \cdot \sum_{b \geq 0} \left( \frac{q^{2(b+1)(2b+1)}}{1 - q^{8(b+1)t^{2b+1}}} - \frac{q^{2b(2b+5)}}{1 - q^{8b(2b+1)}} \right) t^{(b+1)^2} \prod_{d \geq 1} \left( 1 - q^{4d-2d^2}(1 - q^{4dt}d^2)(1 - q^{4dt+2dt^2}) \right)^{2b}.$$  

(3.1)

In the rest of the subsection, we review Stremme’s work in [Str] and give a basis of $H^4(\mathcal{M}(n), \mathbb{Z})$ in terms of the classes from [Str]. A basis of $H^4(\mathcal{M}(n), \mathbb{Z})$ with geometric flavors will be constructed in [H].

Fix $n \geq 2$. Let $\mathcal{E}$ be a universal sheaf over $\mathcal{M}(n) \times X$, and let $\pi_1$ and $\pi_2$ be the two natural projections on $\mathcal{M}(n) \times X$. For $0 \leq k \leq 2$, define

$$A_k = R^1 \pi_1_*(\mathcal{E} \otimes \pi_2^* \mathcal{O}_X(-k\ell)).$$

For $V \in \mathcal{M}(n)$, denote $V \otimes \mathcal{O}_X(k\ell)$ by $V \otimes \mathcal{O}_X(k)$ or $V(k)$. Then,

$$h^0(X, V(-k)) = h^2(X, V(-k)) = 0 \quad \text{for } 0 \leq k \leq 2,$$

$$h^1(X, V(-2)) = n - 1, \quad h^1(X, V(-1)) = n$$

by Proposition 1.5 in [Str]. It follows that the three sheaves $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ over $\mathcal{M}(n)$ are locally free of rank $n - 1, n, n - 1$ respectively.

**Definition 3.1.** Let $0 \leq k \leq 2$ and $r_k$ be the rank of $A_k$. Define

$$\epsilon = c_1(\mathcal{A}_0) - c_1(\mathcal{A}_2),$$

$$\delta = n \cdot c_1(\mathcal{A}_0) - (n - 1) \cdot c_1(\mathcal{A}_1),$$

$$\tau_k = 2r_k \cdot c_2(A_k) - (r_k - 1) \cdot c_1(A_k)^2.$$  

Note that these classes $\epsilon, \delta, \tau_k$ are independent of the choices of the universal sheaf $\mathcal{E}$ over $\mathcal{M}(n) \times X$. Let $K_{\mathcal{M}(n)}$ be the canonical class of $\mathcal{M}(n)$, $\mathcal{M}(n)$ be the open subset of $\mathcal{M}(n)$ parametrizing stable bundles, and

$$\mathcal{B} = \mathcal{M}(n) - \mathcal{M}(n).$$

(3.4)

By the theorem in [Str], Pic($\mathcal{M}(n)$) is freely generated by $\epsilon$ and $\delta$, and

$$K_{\mathcal{M}(n)} = -3\epsilon, \quad \mathcal{B} = n\epsilon - 2\delta.$$  

Also, $a\epsilon + b\delta$ is ample if and only if $a, b > 0$, and $\mathcal{B}$ is irreducible and reduced.

It is known from [Bea, E-S, Mar] that the cohomology ring $H^*(\mathcal{M}(n); \mathbb{Z})$ is generated by the Chern classes of the bundles $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$. It follows that $H^4(\mathcal{M}(n); \mathbb{Z})$ is the $\mathbb{Z}$-linear span of the six integral classes:

$$\epsilon^2, \epsilon\delta, \delta^2, \tau_0, \tau_1, \tau_2.$$  

(3.6)

By (3.6), the rank of $H^4(\mathcal{M}(n); \mathbb{Z})$ is 3 when $n = 2$, and is 6 when $n \geq 3$. Therefore, if $n \geq 3$, then a linear basis of $H^4(\mathcal{M}(n); \mathbb{Z})$ is given by the six classes in (3.6).
Remark 3.2. When $n = 2$, the rank of $H_4(\overline{\mathcal{M}}(n); \mathbb{Z})$ is different from that of the case $n \geq 3$. Therefore the construction of a basis of $H_4(\overline{\mathcal{M}}(n); \mathbb{Z})$ in \[1\] for $n \geq 3$ needs to be modified. However, there is a method to describe the moduli space $\mathcal{M}_n$ using the moduli spaces of stable sheaves on the Hirzebruch surface $\mathbb{F}_1$ and chamber structures, which enable us to compute all the Gromov-Witten invariants of $\mathcal{M}(2)$ (instead of a special kind considered in this paper for $n \geq 3$). Since the method is different, the result for $n = 2$ will appear elsewhere.

3.2. The boundary and the Uhlenbeck compactification.

The quasi-projective variety $\mathcal{M}(n)$ has a Uhlenbeck compactification

$$\overline{\mathcal{M}}(n) = \bigcup_{0 \leq i \leq n-1} \mathcal{M}(n-i) \times \text{Sym}^i(X)$$

according to [Uhl, LJ1, Mor]. Moreover, there exists a birational morphism, called the Gieseker-Uhlenbeck morphism,

$$\Psi : \mathcal{M}(n) \to \overline{\mathcal{M}}(n)$$

sending $V \in \mathcal{M}(n)$ to the pair $(V^{**}, \eta)$, where $V^{**}$ is the double-dual of $V$ and

$$\eta = \sum_{x \in X} h^0(X, (V^{**})_x) x.$$ 

Then the boundary divisor $B$ in \[2\] is contracted by $\Psi$ to the codimension-2 subset $\bigcup_{1 \leq i \leq n-1} \mathcal{M}(n-i) \times \text{Sym}^i(X)$ in $\overline{\mathcal{M}}(n)$.

Let $\mathcal{B}_* = \Psi^{-1}(\mathcal{M}(n-1) \times X) \subset \overline{\mathcal{M}}(n)$ which parametrizes all $V \in \mathcal{M}(n)$ sitting in exact sequences $0 \to V \to V_1 \to \mathcal{O}_x \to 0$ for some bundle $V_1 \in \mathcal{M}(n-1)$ and some point $x \in X$. It is an open dense subset of the boundary divisor $\mathcal{B}$.

To construct a universal sheaf over $\mathcal{B}_* \times X$, let $\mathcal{E}_{n-1}^0$ be a universal sheaf over $\mathcal{M}(n-1) \times X$. By [Q-Z], $\mathcal{B}_* \cong \mathcal{P}(\mathcal{E}_{n-1}^0)$. For simplicity, write $\mathcal{B}_* = \mathcal{P}(\mathcal{E}_{n-1}^0)$. Let

$$\pi : \mathcal{B}_* = \mathcal{P}(\mathcal{E}_{n-1}^0) \to \mathcal{M}(n-1) \times X$$

be the natural projection. Let $\Delta_X$ be the diagonal of $X \times X$. Consider the obvious isomorphism $\alpha : \mathcal{M}(n-1) \times \Delta_X \to \mathcal{M}(n-1) \times X$. Then, we have the isomorphisms:

$$(\pi \times \text{Id}_X)^{-1}(\mathcal{M}(n-1) \times \Delta_X) \cong \mathcal{P}(\alpha^* \mathcal{E}_{n-1}^0),$$

$$\tilde{\pi}^* \mathcal{E}_{n-1}^0 | (\pi \times \text{Id}_X)^{-1}(\mathcal{M}(n-1) \times \Delta_X) \cong \check{\alpha}^* \mathcal{E}_{n-1}^0,$$

where $\tilde{\pi} : \mathcal{B}_* \times X \to \mathcal{M}(n-1) \times X$ is the composition of $\pi \times \text{Id}_X$ and the map $\mathcal{M}(n-1) \times X \times X \to \mathcal{M}(n-1) \times X$ which denotes the projection to the product of the first and third factors, and $\check{\alpha} : \mathcal{P}(\alpha^* \mathcal{E}_{n-1}^0) \to \mathcal{M}(n-1) \times X$ is the composition of the natural projection $\mathcal{P}(\alpha^* \mathcal{E}_{n-1}^0) \to \mathcal{M}(n-1) \times \Delta_X$ and $\alpha$. A universal sheaf $\mathcal{E}'$ over $\mathcal{B}_* \times X$ sits in the exact sequence

$$0 \to \mathcal{E}' \to \tilde{\pi}^* \mathcal{E}_{n-1}^0 \to \mathcal{O}_{(\pi \times \text{Id}_X)^{-1}(\mathcal{M}(n-1) \times \Delta_X)}(1) \to 0.$$ 

The following lemma will be used in later sections.

Lemma 3.3. Let $\tilde{\pi}_i$ be the $i$-th projection on $\mathcal{M}(n-1) \times X$, and let

$$\mathcal{D}_{n-1} = 2c_1(A_{n-1,1}) - c_1(A_{n-1,2}) - c_1(A_{n-1,0}),$$

where $A_{n-1,k}^0 = R^1\tilde{\pi}_{1,*}(\mathcal{E}_{n-1}^0 \otimes \tilde{\pi}_2^* \mathcal{O}_X(-kt\ell))$. Then, as divisors in $\mathcal{B}_*$,

$$\mathcal{B}_*|_{\mathcal{B}_*} = -2c_1(\mathcal{O}_{\mathcal{B}_*}(1)) + (\tilde{\pi}_1 \circ \pi)^* \mathcal{D}_{n-1} + 2(\tilde{\pi}_2 \circ \pi)^* \ell.$$
Proof: Since \( \mathcal{B}_* \) is open and dense in \( \mathcal{B} \), we see from (3.3) and Definition 3.1 that
\[
\mathcal{B}_*|_{\mathcal{B}_*} = (2(n-1)c_1(A_1) - nc_1(A_2) - nc_1(A_0))|_{\mathcal{B}_*}.
\]

Next, let \( 0 \leq k \leq 2 \), and let \( \pi_i \) be the \( i \)-th projection on \( \mathcal{B}_* \times X \). Then the restriction of \( \pi_i \) to the subset \((\pi \times \text{Id}_X)^{-1}(\mathcal{M}(n-1) \times \Delta_X)\) is an isomorphism from \((\pi \times \text{Id}_X)^{-1}(\mathcal{M}(n-1) \times \Delta_X)\) to \( \mathcal{B}_* \). Thus tensoring (3.11) by \( \pi_2^*\mathcal{O}_X(-k\ell) \) and then applying the functor \( \pi_1^* \), we get the exact sequence
\[
0 \rightarrow \mathcal{O}_{\mathcal{B}_*}(1) \otimes (\tilde{\pi}_2 \circ \pi)^*\mathcal{O}_X(-k\ell) \rightarrow R^1\pi_1^*(\mathcal{E}' \otimes \pi_2^*\mathcal{O}_X(-k\ell)) \rightarrow R^1\pi_1^*(\tilde{\pi}_2^* \mathcal{E}_{n-1}^0 \otimes \pi_2^*\mathcal{O}_X(-k\ell)) \rightarrow 0.
\]
Note that \( \pi_2 = \tilde{\pi}_2 \circ \pi \). Also, \( R^1\pi_1^* \tilde{\pi}_2^* \cong (\tilde{\pi}_1 \circ \pi)^* R^1\tilde{\pi}_{1*} \) via the trivial base change:
\[
\mathcal{B}_* \times X \xrightarrow{\pi_1} \mathcal{B}_* \\
\downarrow \tilde{\pi} \quad \quad \downarrow \tilde{\pi}_1 \circ \pi \\
\mathcal{M}(n-1) \times X \xrightarrow{\pi_1} \mathcal{M}(n-1).
\]
Therefore, rewriting the 3rd term in the above exact sequence, we obtain
\[
0 \rightarrow \mathcal{O}_{\mathcal{B}_*}(1) \otimes (\tilde{\pi}_2 \circ \pi)^*\mathcal{O}_X(-k\ell) \rightarrow R^1\pi_1^*(\mathcal{E}' \otimes \pi_2^*\mathcal{O}_X(-k\ell)) \rightarrow (\tilde{\pi}_1 \circ \pi)^*A_{n-1,k} \rightarrow 0.
\]
So the first Chern class \( c_1(R^1\pi_1^*(\mathcal{E}' \otimes \pi_2^*\mathcal{O}_X(-k\ell))) \) equals
\[
c_1(\mathcal{O}_{\mathcal{B}_*}(1)) - k(\tilde{\pi}_2 \circ \pi)^*\ell + (\tilde{\pi}_1 \circ \pi)^*c_1(A_{n-1,k}^0),
\]
and the second Chern class \( c_2(R^1\pi_1^*(\mathcal{E}' \otimes \pi_2^*\mathcal{O}_X(-k\ell))) \) is equal to
\[
(\tilde{\pi}_1 \circ \pi)^* c_1(A_{n-1,k}^0) + (\tilde{\pi}_1 \circ \pi)^* c_2(A_{n-1,k}^0).
\]

Finally, we conclude from (3.13) and (3.14) that
\[
\mathcal{B}_*|_{\mathcal{B}_*} = -2c_1(\mathcal{O}_{\mathcal{B}_*}(1)) + 2(\tilde{\pi}_2 \circ \pi)^*\ell + (\tilde{\pi}_1 \circ \pi)^* \mathcal{D},
\]
where
\[
\mathcal{D} = 2(n-1)c_1(A_{n-1,1}^0) - nc_1(A_{n-1,2}^0) - nc_1(A_{n-1,0}^0).
\]
Let \( \epsilon, \delta \in \text{Pic}(\mathfrak{M}(n-1)) \) be the counterparts of \( \epsilon, \delta \in \text{Pic}(\mathfrak{M}(n)) \). By (3.3) and Definition 3.1
\[
0 = (\mathfrak{M}(n-1) - \mathfrak{M}(n-1))|_{\mathfrak{M}(n-1)} = [(n-1)c_1 - 2\delta]|_{\mathfrak{M}(n-1)} = 2(n-2)c_1(A_{n-1,1}^0) - (n-1)c_1(A_{n-1,2}^0) - (n-1)c_1(A_{n-1,0}^0).
\]
So \( \mathcal{B}_*|_{\mathcal{B}_*} = 2c_1(\mathcal{O}_{\mathcal{B}_*}(1)) + 2(\tilde{\pi}_2 \circ \pi)^*\ell + (\tilde{\pi}_1 \circ \pi)^* \mathcal{D}_{n-1} \).

3.3. Curves in \( \mathfrak{M}(n) \).
We shall construct two curves in the Gieseker moduli space \( \mathfrak{M}(n) \) which freely generate the homology group \( H_2(\mathfrak{M}(n), \mathbb{Z}) \). One such curve is a fiber \( f \) of the morphism \( \pi \) from (3.3). The following is Lemma 3.2 in [Q-Z].

**Lemma 3.4.** Let \( N_{f|\mathfrak{M}(n)} \) be the normal bundle of \( f \) in \( \mathfrak{M}(n) \). Then,
\[
(i) \quad f \cdot K_{\mathfrak{M}(n)} = 0 \quad \text{and} \quad f \cdot \mathcal{B} = -2;
(ii) \quad N_{f|\mathfrak{M}(n)} \cong \mathcal{O}_f^\oplus(4n-6) \oplus \mathcal{O}_f(-2);
(iii) \quad T_{\mathfrak{M}(n)}|_f \cong \mathcal{O}_f^\oplus(4n-6) \oplus \mathcal{O}_f(-2) \oplus \mathcal{O}_f(2).
\]

\[\square\]
Next, we shall construct the other curve. Let \( n \geq 3 \), and let \( \xi \) consist of \( n \) distinct points in general position in \( X = \mathbb{P}^2 \). If \( V \) sits in a nontrivial extension

\[(3.16) \quad 0 \to \mathcal{O}_X(-1) \to V \to I_\xi \to 0,\]

then \( V \) is stable and hence \( V \in \mathfrak{M}(n) \). Moreover, since \( H^0(X, I_\xi \otimes \mathcal{O}_X(1)) = 0 \), the injection \( \mathcal{O}_X(-1) \to V \) is unique up to scalars. It follows that

\[(3.17) \quad \mathfrak{E}_n = \mathbb{P}(\text{Ext}^1(I_\xi, \mathcal{O}_X(-1))) \cong \mathbb{P}^{n-1}\]

can be regarded as the subset of \( \mathfrak{M}(n) \) parametrizing all the sheaves \( V \in \mathfrak{M}(n) \) sitting in nontrivial extensions \( (3.16) \). A universal sheaf \( \mathcal{E}' \) over \( \mathfrak{E}_n \times X \) sits in

\[(3.18) \quad 0 \to \pi_2^* \mathcal{O}_X(-\ell) \to \mathcal{E}' \to \pi_1^* \mathcal{O}_{\mathfrak{E}_n}(-1) \otimes \pi_2^* I_\xi \to 0.\]

Tensoring by \( \pi_2^* \mathcal{O}_X(-k\ell) \) and applying \( \pi_1^* \) lead to the exact sequence:

\[0 \to R^1\pi_{1*}(\mathcal{E}' \otimes \pi_2^* \mathcal{O}_X(-k\ell)) \to \mathcal{O}_{\mathfrak{E}_n}(-1)^{\oplus \text{h}^1(X,I_\xi(-k))} \to \mathcal{O}_{\mathfrak{E}_n}^\vee \to 0,\]

where \( 0 \leq k \leq 2 \). An easy computation gives rise to the following:

\[c_1(R^1\pi_{1*}\mathcal{E}') = -(n-1) \cdot c_1(\mathcal{O}_{\mathfrak{E}_n}(1)),\]

\[c_1(R^1\pi_{1*}(\mathcal{E}' \otimes \pi_2^* \mathcal{O}_X(-k\ell))) = -n \cdot c_1(\mathcal{O}_{\mathfrak{E}_n}(1)),\]

where \( k = 1, 2 \). It follows immediately from Definition 3.1 that

\[(3.19) \quad \ell|_{\mathfrak{E}_n} = c_1(\mathcal{O}_{\mathfrak{E}_n}(1)), \quad \delta|_{\mathfrak{E}_n} = 0.\]

**Lemma 3.5.** Let \( n \geq 3 \) and \( l \) be a line in the projective space \( \mathfrak{E}_n \).

(i) The homology group \( H_2(\mathfrak{M}(n); \mathbb{Z}) \) is freely generated by \( f \) and \( l \).

(ii) The class \( af + bl \in H_2(\mathfrak{M}(n); \mathbb{Z}) \) is effective if and only if \( a, b \geq 0 \).

(iii) If \( C \subset \mathfrak{M}(n) \) is an irreducible curve contracted by the Gieseker-Uhlenbeck morphism \( \Psi \), then \( C = df + fl \in H_2(\mathfrak{M}(n); \mathbb{Z}) \) for some positive integer \( d \).

**Proof.** (i) Since \( \mathfrak{M}(n) \) is rational, \( H^i(\mathfrak{M}(n), \mathcal{O}_{\mathfrak{M}(n)}) = 0 \) for all \( i \geq 1 \). Hence \( H^2(\mathfrak{M}(n); \mathbb{Z}) \cong \text{Pic}(\mathfrak{M}(n)) \), and \( H^2(\mathfrak{M}(n); \mathbb{Z}) \) is freely generated by \( \epsilon \) and \( \delta \). By (3.19), \( l \cdot \epsilon = 1 \) and \( l \cdot \delta = 0 \). By Definition 3.1 and (3.14), \( f \cdot \epsilon = 0 \) and \( f \cdot \delta = 1 \). Since \( H_2(\mathfrak{M}(n); \mathbb{Z}) \) is torsion-free, \( H_2(\mathfrak{M}(n); \mathbb{Z}) \) is freely generated by \( f \) and \( l \).

(ii) Since \( f \) and \( l \) are effective, \( af + bl \) is effective if \( a, b \geq 0 \). Conversely, if \( af + bl \) is effective, then \( a, b \geq 0 \) since the divisor \( ce + d\delta \) is ample if and only if \( c, d > 0 \).

(iii) By (ii), \( C = df + bl \in H_2(\mathfrak{M}(n); \mathbb{Z}) \), where \( d \) and \( b \) are nonnegative integers not both zero. Let \( L \) be a very ample divisor on \( \mathfrak{M}(n) \). Then, \( f \cdot \Psi^* L = 0 \) and \( l \cdot \Psi^* L \geq 0 \). Since \( \Psi^* L \) is a nonzero divisor and \( H_2(\mathfrak{M}(n); \mathbb{Z}) \) is freely generated by \( f \) and \( l \), we must have \( l \cdot \Psi^* L > 0 \). Thus, \( C \cdot \Psi^* L = 0 \) forces \( b = 0 \).

4. A Basis of \( H_4(\mathfrak{M}(n); \mathbb{C}) \)

In this section, we assume \( n \geq 3 \). Then the integral homology group \( H_4(\mathfrak{M}(n); \mathbb{Z}) \) is free of rank 6. In the following, we construct a basis \( \{\Xi_1, \ldots, \Xi_6\} \) for \( H_4(\mathfrak{M}(n); \mathbb{C}) \). This construction makes use of a result due to Hirschowitz and Hulek.

We review the results in [H-H], where complete rational curves were found in \( \mathfrak{M}(n) \). Let \( n \geq 2 \) and \( \Gamma = \mathbb{P}^1 \). Fix lines \( \ell_1, \ldots, \ell_n \subset X = \mathbb{P}^2 \) in general position. For \( 1 \leq i \leq n \), let \( \phi_i : \Gamma \to \ell_i \) be an isomorphism, and define \( Y_i \subset \Gamma \times X \) to be the graph of \( \phi_i \). For generic choices of \( \phi_1, \ldots, \phi_n \), it was proved in [H-H] that \( Y_1, \ldots, Y_n \)}
are disjoint. Moreover, if $N_{Y/G \times X}$ denotes the normal bundle of $Y \overset{\text{def}}{=} \coprod_{i=1}^{n} Y_i$ in $G \times X$, then $\det(N_{Y/G \times X}) \cong \mathcal{O}_{G \times X}(2,1)|_{Y}$. Therefore, the element

$$1 \in H^0(Y, \mathcal{O}_Y) \cong \text{Ext}^1(\mathcal{O}_{G \times X}(2,0) \otimes \mathcal{I}_Y, \mathcal{O}_{G \times X}(0, -1))$$

defines a rank-2 bundle $\tilde{E}_n$ over $G \times X$ sitting in an exact sequence

$$(4.1) \quad 0 \to \mathcal{O}_{G \times X}(0, -1) \to \tilde{E}_n \to \mathcal{O}_{G \times X}(2,0) \otimes \mathcal{I}_Y \to 0$$

such that $\tilde{E}_n|_{(p) \times X} \in \mathcal{M}(n)$ for all $p \in G$, and $\tilde{E}_n$ induces a nonconstant morphism

$$\iota : G \to \mathcal{M}(n).$$

Let $\tilde{\pi}_1$ and $\tilde{\pi}_2$ be the natural projections on $G \times X$. Let

$$(4.2) \quad \tilde{A}_{n,k} = R^3\tilde{\pi}_1*(\tilde{E} \otimes \pi_2^*\mathcal{O}_X(-k\ell)).$$

By Lemma 3.5 in [H-H], the degrees of the bundles $\tilde{A}_{n,k}$ are

$$(4.3) \quad a_{n,k} = \begin{cases} 2n - 2 & \text{if } k = 0, \\ n & \text{if } k = 1, \\ 0 & \text{if } k = 2. \end{cases}$$

Remark 4.1. (i) Let $n \geq 2$. By (4.3), the condition in Corollarie (6.9.9) of [Gro] is satisfied. Hence the base-change theorem of the first direct image holds for every projective morphism to $\mathcal{M}(n)$ and for the sheaf $\mathcal{E} \otimes \pi_2^*\mathcal{O}_X(-k\ell)$ with $0 \leq k \leq 2$. For a finite morphism $\varphi$ from $Y$ onto a subvariety of $\mathcal{M}(n)$, the intersection numbers of $\varphi(Y)$ with $e^2, e, \delta, \delta$, $\tau_0, \tau_1, \tau_2$ can be computed on $Y$, via the projection formula, by pulling back the Chern classes of the first direct images of the sheaf $\mathcal{E} \otimes \pi_2^*\mathcal{O}_X(-k\ell)$ with $0 \leq k \leq 2$.

(ii) Let $n \geq 3$. Then $(n-1) \geq 2$. In Subsection 4.2 we will construct a surface $Y = \pi^{-1}(G \times \{y\})$ admitting a finite morphism onto a surface $\Xi_3$ of $\mathcal{M}(n)$. This finite morphism factors through the nonconstant morphism $\iota : G \to \mathcal{M}(n-1)$. Applying (i), we may assume for convenience that $G \subset \mathcal{M}(n-1)$ and consequently $Y = \Xi_3 \subset \mathcal{M}(n)$ (in fact, it can be proved that when $n \geq 5$, the morphism $\iota : G \to \mathcal{M}(n-1)$ is injective). A similar discussion works for $\Xi_4$.

4.1. The homology classes $\Xi_1, \Xi_2 \in H_4(\mathcal{M}(n); \mathbb{C})$. Let $n \geq 3$, and assume $G \subset \mathcal{M}(n-1)$ as pointed out in Remark 4.1 (ii). Let $\mathbb{P} = \mathbb{P}(\mathcal{E}_{n-1}) = \pi^{-1}(G \times X) \subset \mathbb{B}_s$. Then $\mathbb{P}$ parametrizes all the sheaves $V \in \mathcal{M}(n)$ sitting in

$$0 \to V \to \tilde{E}_{n-1}|_{(p) \times X} \to \mathcal{O}_x \to 0$$

for some $p \in G$ and $x \in X$. We still use $\pi$ to denote the natural projection

$$\pi : \mathbb{P} \to G \times X.$$
where \( \tilde{A}_{n-1,k} \) is defined in (4.2), and \( c_2\left(R^3\pi_{1,*}(\bar{E} \otimes \pi_{2,*}O_X(-k\ell)) \right) \) equals

\[
(4.6) \quad (c_1(O_Y(1)) - k(\bar{E} \o \pi) \cdot (\bar{E} \o \pi)) c_1(\tilde{A}_{n-1,k}).
\]

In addition, we conclude from (4.1) that

\[
\begin{align*}
(4.7) \quad c_1(O_Y(1))^2 &= \pi^* c_1(\tilde{E}_{n-1}) \cdot c_1(O_Y(1)) - \pi^* c_2(\tilde{E}_{n-1}) \\
&= \pi^* c_1(O_Y(1)) - \pi^* \left[ (2,0) \cdot (0,-1) + \sum_{i=1}^{n-1} Y_i \right].
\end{align*}
\]

Fix a point \( p \in \Gamma \) and let \( V_i = \tilde{E}_{n-1}(p) \times X \). Consider \( \mathbb{P}(V_i) \subset \mathbb{P} \subset \mathcal{M}(n) \). By Definition 3.1, (4.3), (4.5) and (4.6), we obtain

\[
\begin{align*}
\epsilon|_{\mathbb{P}(V_1)} &= 2\pi^* \ell, \\
\delta|_{\mathbb{P}(V_1)} &= c_1(O_{\mathbb{P}(V_1)}(1)) + (n-1) \pi^* \ell, \\
\tau_k|_{\mathbb{P}(V_1)} &= -(r_k - 1)(c_1(O_{\mathbb{P}(V_1)}(1)) - k \pi^* \ell)^2,
\end{align*}
\]

(4.8) \( c_1(O_{\mathbb{P}(V_1)}(1))^2 = -\pi^* \ell \cdot c_1(O_{\mathbb{P}(V_1)}(1)) - (n-1) \pi^* \ell \),

where by abusing notation, \( \pi \) denotes the natural projection \( \mathbb{P}(V_i) \to X \).

Define \( \Xi_1 \in H_2(\mathcal{M}(n);\mathbb{C}) \) to be the surface \( \pi^{-1}(\{p\} \times \ell) \subset \mathbb{P}(V_i) \subset \mathbb{P} \subset \mathcal{M}(n) \), where \( \ell \subset X = \mathbb{P}^2 \) is a fixed line. Then we have

\[
\begin{align*}
\epsilon^2 \cdot \Xi_1 &= 0, \quad \epsilon \cdot \delta \cdot \Xi_1 = 2, \quad \delta^2 \cdot \Xi_1 = 2n - 3, \\
\tau_0 \cdot \Xi_1 &= n - 2, \quad \tau_1 \cdot \Xi_1 = 3n - 3, \quad \tau_2 \cdot \Xi_1 = 5n - 10.
\end{align*}
\]

Next, define \( \Xi_2 \in H_4(\mathcal{M}(n);\mathbb{C}) \) to be \( c_1(O_{\mathbb{P}(V_1)}(1)) \) regarded as a 2-dimensional cycle in \( \mathcal{M}(n) \) via the inclusion \( \mathbb{P}(V_1) \subset \mathcal{M}(n) \). Then

\[
\begin{align*}
\epsilon^2 \cdot \Xi_2 &= 4, \quad \epsilon \cdot \delta \cdot \Xi_2 = 2n - 4, \quad \delta^2 \cdot \Xi_2 = n^2 - 5n + 5, \\
\tau_0 \cdot \Xi_2 &= (n - 2)^2, \quad \tau_1 \cdot \Xi_2 = (n - 1)(n - 5), \quad \tau_2 \cdot \Xi_2 = (n - 2)(n - 10).
\end{align*}
\]

4.2. The homology classes \( \Xi_3, \Xi_4 \in H_4(\mathcal{M}(n);\mathbb{C}) \).

Fix a line \( \ell \subset X = \mathbb{P}^2 \). Let \( W = \pi^{-1}(\Gamma \times \ell) \subset \mathbb{P} \subset \mathcal{M}(n) \), where \( \Gamma, \mathbb{P} \) and \( \pi \) are from the previous subsection. By Definition 3.1, (4.3), (4.5) and (4.6),

\[
\begin{align*}
\epsilon|_W &= \pi^*(2n - 4, 2), \\
\delta|_W &= c_1(O_W(1)) + \pi^*(2n^2 - 2n - 1, 0, 0), \\
\tau_k|_W &= \pi^*(2\sigma_{n-1,k} - 2(r_k - 1), 0, 0) c_1(O_W(1)) \\
&\quad + ((r_k - 1)(n - 3) - 2k\sigma_{n-1,k}) \pi^* x,
\end{align*}
\]

where \( x \in \Gamma \times \ell \) is a fixed point and by abusing notation, \( \pi: W \to \Gamma \times \ell = \mathbb{P}^1 \times \mathbb{P}^1 \) stands for the natural projection. In addition, by (4.7), we conclude that

\[
\begin{align*}
(4.11) \quad c_1(O_W(1))^2 &= \pi^*(2, -1) c_1(O_W(1)) - (n - 3) \pi^* x, \\
\end{align*}
\]

Define \( \Xi_3 \in H_4(\mathcal{M}(n);\mathbb{C}) \) to be the surface \( \pi^{-1}(\{p\} \times \{y\}) \subset W \subset \mathcal{M}(n) \), where \( y \in \ell \) is a fixed point. A straightforward computation shows that

\[
\begin{align*}
\epsilon^2 \cdot \Xi_3 &= 0, \quad \epsilon \cdot \delta \cdot \Xi_3 = 2n - 4, \quad \delta^2 \cdot \Xi_3 = 2n^2 - 4n, \\
\tau_0 \cdot \Xi_3 &= 2n - 4, \quad \tau_1 \cdot \Xi_3 = 0, \quad \tau_2 \cdot \Xi_3 = -2n + 4.
\end{align*}
\]
Define $\Xi_4 \in H_4(\mathcal{M}(n); \mathbb{C})$ to be $c_1(O_W(1))$ regarded as a 2-dimensional cycle in $\mathcal{M}(n)$ via the inclusion $W \subset \mathcal{M}(n)$. Then we have

$$\epsilon^2 \cdot \Xi_4 = 8(n-2), \quad \epsilon \cdot \delta \cdot \Xi_4 = 4n^2 - 12n + 10, \quad \delta^2 \cdot \Xi_4 = 2n^3 - 8n^2 + 9n - 1, $$

(4.13) $\tau_0 \cdot \Xi_4 = n^2 - 5n + 6, \quad \tau_1 \cdot \Xi_4 = n^2 - 1, \quad \tau_2 \cdot \Xi_4 = (n-2)(n+9)$. 

4.3. The homology class $\Xi_5 \in H_4(\mathcal{M}(n); \mathbb{C})$.

Since $n \geq 3$, the moduli space $\mathcal{M}(n-2)$ is nonempty. Fix a vector bundle $V_2 \in \mathcal{M}(n-2)$ and two distinct points $x_1, x_2 \in X$. Let $\Xi_5 \subset \mathcal{M}(n)$ parametrize all the sheaves $V \in \mathcal{M}(n)$ sitting in exact sequences:

$$0 \rightarrow V \rightarrow V_2 \rightarrow O_{x_1} \oplus O_{x_2}. $$

Then we have the isomorphisms $\Xi_5$ via the inclusion

$$0 \rightarrow \mathcal{E} \rightarrow \pi_1^* O_{\mathbb{P}^1} \oplus \pi_2^* O_{\mathbb{P}^1} \rightarrow 0,$$

where $\pi_1$ and $\pi_2$ denote the $i$-th projection on $\Xi_5 \times X$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times X$, respectively. Tensoring the above exact sequence by $\pi_1^* O_X(-k\ell)$ and applying $\pi_1^*$ yield

$$0 \rightarrow O_{\Xi_5}(1,0) \oplus O_{\Xi_5}(0,1) \rightarrow R^1\pi_1^*(\mathcal{E} \otimes \pi_2^* O_X(-k\ell)) \rightarrow O_{\Xi_5}^{\oplus h^1(X,V_2(-k))} \rightarrow 0,$$

where $0 \leq k \leq 2$. Therefore $c_1(R^1\pi_1^*(\mathcal{E} \otimes \pi_2^* O_X(-k\ell))) = c_1(O_{\Xi_5}(1,1))$ and $c_2(R^1\pi_1^*(\mathcal{E} \otimes \pi_2^* O_X(-k\ell))) = 1$ for $0 \leq k \leq 2$. It follows from Definition 3.1 that $c_1|_{\Xi_5} = 0, \quad \delta|_{\Xi_5} = c_1(O_{\Xi_5}(1,1)), \quad c_2|_{\Xi_5} = c_2(O_{\Xi_5}(1,1)).$

Regarding $\Xi_5 \in H_4(\mathcal{M}(n); \mathbb{C})$, we obtain the intersection numbers on $\mathcal{M}(n)$:

$$\epsilon^2 \cdot \Xi_5 = \epsilon \cdot \delta \cdot \Xi_5 = 0, \quad \delta^2 \cdot \Xi_5 = 2, \quad \tau_0 \cdot \Xi_5 = \tau_1 \cdot \Xi_5 = \tau_2 \cdot \Xi_5 = 2. $$

4.4. The surface $\Xi_6$ in $\mathcal{M}(n)$.

Fix a vector bundle $V_2 \in \mathcal{M}(n-2)$ and a point $x \in X$. Fix a trivialization of $V_2$ in an open neighborhood $O_x$ of $x$. Let $X^{[2]}$ be the Hilbert scheme parametrizing the length-2 closed subschemes of $X$, and let

$$M_2(x) = \{ \xi \in X^{[2]} | \text{Supp}(\xi) = \{ x \} \} \cong \mathbb{P}^1.$$

For $\xi \in M_2(x)$, let $\iota_\xi : O_X \rightarrow O_\xi$ be the natural quotient morphism.

Let $\Xi_6 = \mathbb{P}^1 \times M_2(x)$ be the subset of $\mathcal{M}(n)$ parametrizing all the sheaves $V \in \mathcal{M}(n)$ sitting in extensions of the form

$$0 \rightarrow V \rightarrow V_2 \xrightarrow{(a,b)} O_\xi \rightarrow 0,$$

where $\xi \in M_2(x)$, and for $(a, b) \in \mathbb{P}^1$, the map $V_2 \xrightarrow{(a,b)} O_\xi$ denotes the composition:

$$V_2 \rightarrow V_2|_{O_x} \cong O_{\Xi_6}^{\oplus 2} \xrightarrow{(a_{i_1}, b_{i_1})} O_\xi.$$

Next, we construct a universal sheaf over $\Xi_6 \times X = \mathbb{P}^1 \times M_2(x) \times X$. Let $\pi_{i_1}, \ldots, \pi_{i_m}$ denote the projection of $\mathbb{P}^1 \times M_2(x) \times X$ to the product of the $i_1$-th, $\ldots$, $i_m$-th factors. Over $\Xi_6 = \mathbb{P}^1 \times M_2(x)$, there is a tautological surjection $O_{\Xi_6}^{\oplus 2} \rightarrow O_{\Xi_6}(1,0) \rightarrow 0$. This pulls back to a surjection over $\mathbb{P}^1 \times M_2(x) \times X$:

$$O_{\Xi_6 \times X}^{\oplus 2} \rightarrow \pi_1^* O_{\mathbb{P}^1}(1) \rightarrow 0.$$
Let $Z_2(x)$ be the universal codimension-2 subscheme in $M_2(x) \times X$, and let $O_{M_2(x) \times X} \to O_{Z_2(x)} \to 0$ be the natural surjection. Pulling back to $\mathbb{P}^1 \times M_2(x) \times X$ yields a surjection:

$$O_{\Xi_6 \times X} \to \pi_2^* O_{Z_2(x)} \to 0. \tag{4.16}$$

Tensoring (4.15) and (4.16), we obtain a surjection

$$O_{\Xi_6 \times X}^\oplus 2 \to \pi_1^* O_{\mathbb{P}^1(1)} \otimes \pi_2^* O_{Z_2(x)} \to 0. \tag{4.17}$$

Since $Z_2(x)$ is supported on $M_2(x) \times \{x\}$, in view of the trivialization of $V_2$ near $x$, (4.17) induces a surjection $\pi_2^* V_2 \to \pi_1^* O_{\mathbb{P}^1(1)} \otimes \pi_2^* O_{Z_2(x)} \to 0$. The kernel $\mathcal{E}'$ of the map $\pi_2^* V_2 \to \pi_1^* O_{\mathbb{P}^1(1)} \otimes \pi_2^* O_{Z_2(x)}$ is a universal sheaf over $\Xi_6 \times X$:

$$0 \to \mathcal{E}' \to \pi_2^* V_2 \to \pi_1^* O_{\mathbb{P}^1(1)} \otimes \pi_2^* O_{Z_2(x)} \to 0.$$

It follows that for $0 \leq k \leq 2$, we have the exact sequence over $\Xi_6$:

$$0 \to \hat{\pi}_1^* O_{\mathbb{P}^1(1)} \otimes \hat{\pi}_2^* (O_X^{[2]}|_{M_2(x)}) \to R^1(\pi_{1,2})_* (\mathcal{E}' \otimes \pi_2^* O_X(-k\ell)) \to O_{\Xi_6}^\oplus H^1(X, \mathcal{E}_V(-k)) \to 0,$$

where $\hat{\pi}_1$ and $\hat{\pi}_2$ are the two natural projections on $\Xi_6 = \mathbb{P}^1 \times M_2(x)$, and $O_X^{[2]}$ is the tautological rank-2 bundle over the Hilbert scheme $X^{[2]}$ whose fiber at a point $\xi \in X^{[2]}$ is the space $H^0(X, \mathcal{O}_\xi)$. It is well known that $-2c_1(O_X^{[2]}) = M_2(X)$, where $M_2(X) \subset X^{[2]}$ consists of all the elements $\xi \in X^{[2]}$ such that $|\text{Supp}(\xi)| = 1$. Since $M_2(X) \cdot M_2(x) = -2$, we conclude from the above exact sequence that

$$c_1(R^1(\pi_{1,2})_* (\mathcal{E}' \otimes \pi_2^* O_X(-k\ell))) = (2, 1) \in \text{Pic}(\Xi_6) \cong \text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1),$$

$$c_2(R^1(\pi_{1,2})_* (\mathcal{E}' \otimes \pi_2^* O_X(-k\ell))) = 1.$$

By Definition 3.1, we get the six intersection numbers on $\overline{M}(n)$:

$$\epsilon^2 \cdot \Xi_6 = 0, \quad \epsilon \cdot \delta \cdot \Xi_6 = 0, \quad \delta^2 \cdot \Xi_6 = 4,$$

$$\tau_0 \cdot \Xi_6 = -2n + 6, \quad \tau_1 \cdot \Xi_6 = -2n + 4, \quad \tau_2 \cdot \Xi_6 = -2n + 6. \tag{4.18}$$

Now we can summarize the above in the following proposition.

**Proposition 4.2.** Let $n \geq 3$. Then $\{\Xi_1, \ldots, \Xi_6\}$ is a linear basis of $H_4(\overline{M}(n); \mathbb{C})$.

**Proof.** Note from (1.9), (1.11), (1.12), (1.13), (1.14) and (1.18) that the intersection matrix between the classes $\Xi_1, \ldots, \Xi_6$ and the classes $\epsilon^2, \epsilon \cdot \delta, \delta^2, \tau_0, \tau_1, \tau_2$ has a nonzero determinant. Since $H_4(\overline{M}(n); \mathbb{C})$ has dimension 6, our result follows. \qed

### 5. The restriction of the obstruction sheaf on certain open subsets

From the previous section, we see that, if we let $\overline{M}_*(n) = \mathcal{B}_* \cup \overline{M}(n)$, then the classes $\Xi_1, \Xi_2, \Xi_3, \Xi_4$ lie in $\overline{M}_*(n)$ while $\Xi_5$ and $\Xi_6$ lie in the complement $\overline{M}(n) - \overline{M}_*(n)$. Since a stable map $[\mu: (D, p) \to \overline{M}(n)]$ in $ev^{-1}_*(\Xi_i) \subset \overline{M}_{0,1}(\overline{M}(n), df)$ has $\mu(p) \in \Xi_i$ for $1 \leq i \leq 4$, we have $\mu(D) \subset \mathcal{B}_*$. In this section, we use the geometric construction of $\mathcal{B}_*$ in Subsect. 5.2 to effectively compute the virtual cycle restricted to $ev^{-1}_*(\mathcal{B}_*)$. The result will be used to compute the Gromov-Witten invariants $\langle \alpha \rangle_{0,0}$ when $\alpha$ is dual to the classes $\Xi_1, \Xi_2, \Xi_3, \Xi_4$.

Fix $d \geq 1$. Consider the open subset $\Omega_0$ of $\overline{M}_{0,0}(\overline{M}(n), df)$ consisting of stable maps $[\mu: D \to \overline{M}(n)]$ such that $\mu(D) \subset \overline{M}_*(n)$. Similarly, take the open subset $\Omega_1$
of \( \mathfrak{M}_{0,1}(\mathfrak{M}(n), df) \) consisting of stable maps \([\mu: (D; p) \to \mathfrak{M}(n)]\) such that \( \mu(D) \subset \mathfrak{M}_*(n) \). Clearly \( \mathfrak{O}_1 = f_{1,0}^*\mathfrak{O}_0 \). Let \([\mu: (D; p) \to \mathfrak{M}(n)] \in \mathfrak{O}_1 \). Since \( \mu(D) = df \) in \( H_2(\mathfrak{M}(n); \mathbb{Z}) \), \( \mu(D) \subset \mathfrak{B}_* \) and \( \mu(D) \) is a fiber of the projection \( \pi \). Moreover, the composition \( \Psi \circ ev_1 \) sends the stable map \([\mu: (D; p) \to \mathfrak{M}(n)]\) to a point in \( \mathfrak{M}(n-1) \times X \subset \mathfrak{M}(n) \), which is independent of the marked point \( p \) on \( D \). Hence \( ev_1 \) induces a morphism \( \phi \) from \( \mathfrak{O}_0 \) to \( \mathfrak{M}(n-1) \times X \). Putting \( \tilde{e}v_1 = ev_1|_{\mathfrak{O}_1} \) and \( f_{1,0} = f_{1,0}|_{\mathfrak{O}_1} \), we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{O}_1 & \xrightarrow{\tilde{e}v_1} & \mathfrak{B}_* \\
\downarrow f_{1,0} & & \downarrow \pi \\
\mathfrak{O}_0 & \xrightarrow{\phi} & \mathfrak{M}(n-1) \times X & \xrightarrow{\tilde{e}v_1} & X.
\end{array}
\]

(5.1)

Note that the fiber \( \phi^{-1}(V_1, x) \) over a point \((V_1, x) \in \mathfrak{M}(n-1) \times X\) is simply

\[ \mathfrak{M}_{0,0}(\pi^{-1}(V_1, x), df[\pi^{-1}(V_1, x)]) , \]

which is isomorphic to \( \mathfrak{M}_{0,0}(\mathbb{P}^1, df[\mathbb{P}^1]) \) via the isomorphism \( \pi^{-1}(V_1, x) \cong \mathbb{P}^1 \). Hence the complex dimension of the open subset \( \mathfrak{O}_0 \subset \mathfrak{M}_{0,0}(\mathfrak{M}(n), df) \) is equal to

\[
\dim \mathfrak{M}_{0,0}(\mathbb{P}^1, df[\mathbb{P}^1]) + \dim \mathfrak{M}(n-1) + \dim X = (2d - 2) + [4(n - 1) - 4] + 2 = 2d + 4n - 8.
\]

Since \( K_{\mathfrak{M}(n)} \cdot df = 0 \), the expected dimension of \( \mathfrak{O}_0 \subset \mathfrak{M}_{0,0}(\mathfrak{M}(n), df) \) is \( 4n - 7 \) by (2.2). Hence the excess dimension of \( \mathfrak{O}_0 \) is \( e = 2d - 1 \).

Let \( \mathcal{V} \) be the restriction of \( R^1(f_{1,0})_* (ev_1)^* \mathcal{O}_{\mathfrak{B}_*}(\mathfrak{B}_*) \) to \( \mathfrak{O}_0 \).

**Lemma 5.1.** (i) The sheaf \( \mathcal{V} \) is locally free of rank \( 2d - 1 \);

(ii) \( \mathcal{V} \cong R^1(f_{1,0})_* (\tilde{ev}_1)^* \mathcal{O}_{\mathfrak{B}_*}(\mathfrak{B}_*) \).

**Proof.** (i) Take a stable map \( u = [\mu: D \to \mathfrak{M}(n)] \) in \( \mathfrak{O}_0 \), and consider

\[
H^1(f_{1,0}^{-1}(u), (ev_1)^* \mathcal{O}_{\mathfrak{M}(n)}|_{f_{1,0}^{-1}(u)}) \cong H^1(D, \mu^* \mathcal{T}_{\mathfrak{M}(n)}).
\]

Since \( \mu(D) \cong \mathbb{P}^1 \) is a fiber of the projection \( \pi \), we see from Lemma 3.4 (iii) that

\[
\mathcal{T}_{\mathfrak{M}(n)}|_{\mu(D)} = \mathcal{O}_{\mu(D)}(4n - 6) \oplus \mathcal{O}_{\mu(D)}(-2) \oplus \mathcal{O}_{\mu(D)}(2).
\]

Thus \( H^1(D, \mu^* \mathcal{T}_{\mathfrak{M}(n)}) \cong H^1(D, \mathcal{O}_{\mu(D)}(-2)) \) whose dimension equals the excess dimension \( e = 2d - 1 \). Therefore, the restriction \( \mathcal{V} \) of \( R^1(f_{1,0})_* (ev_1)^* \mathcal{T}_{\mathfrak{M}(n)} \) to \( \mathfrak{O}_0 \) is a locally free sheaf of rank \( 2d - 1 \).

(ii) Since \( ev_1(\mathfrak{O}_1) \subset \mathfrak{B}_* \), we have \( ((ev_1)^* \mathcal{T}_{\mathfrak{M}(n)})|_{\mathfrak{O}_1} = (\tilde{ev}_1)^* (\mathcal{T}_{\mathfrak{M}(n)}|_{\mathfrak{B}_*}) \) and

\[
\mathcal{V} = R^1(f_{1,0})_* (ev_1)^* (\mathcal{T}_{\mathfrak{M}(n)})|_{\mathfrak{O}_1} = R^1(f_{1,0})_* ((ev_1)^* \mathcal{T}_{\mathfrak{M}(n)})|_{\mathfrak{O}_1} = R^1(f_{1,0})_* (\tilde{ev}_1)^* (\mathcal{T}_{\mathfrak{M}(n)}|_{\mathfrak{B}_*}).
\]

Since \( \mathfrak{B}_* \) is smooth of codimension 1 in \( \mathfrak{M}(n) \), we obtain the exact sequence

\[
(5.2) \quad 0 \to \mathcal{T}_{\mathfrak{B}_*} \to \mathcal{T}_{\mathfrak{M}(n)}|_{\mathfrak{B}_*} \to \mathcal{O}_{\mathfrak{B}_*}(\mathfrak{B}_*) \to 0.
\]

Applying \( (\tilde{ev}_1)^* \) and \( (f_{1,0})_* \) to the exact sequence (5.2), we get

\[
(5.3) \quad R^1(f_{1,0})_* (\tilde{ev}_1)^* \mathcal{T}_{\mathfrak{B}_*} \to \mathcal{V} \to R^1(f_{1,0})_* (ev_1)^* \mathcal{O}_{\mathfrak{B}_*}(\mathfrak{B}_*) \to 0,
\]

where we have used \( R^2(f_{1,0})_* (ev_1)^* \mathcal{T}_{\mathfrak{B}_*} = 0 \) since \( f_{1,0} \) is of relative dimension 1.
If $[\mu : D \to \mathfrak{M}(n)]$ is a stable map in $\mathfrak{O}_0$, then $\mu(D)$ is a fiber of the projection $\pi$ in (5.3). Hence the normal bundle of $\mu(D)$ in $\mathfrak{B}_*$ is trivial. Therefore we have

$$T_{\mathfrak{B}_*}|_{\mu(D)} \cong \mathcal{O}_{\mu(D)}^{(4n-6)} \oplus \mathcal{O}_{\mu(D)}(2),$$

and

$$H^1(D, \mu^* T_{\mathfrak{B}_*}) \cong H^1(D, \mu^*(\mathcal{O}_{\mu(D)}^{(4n-6)} \oplus \mathcal{O}_{\mu(D)}(2))) = 0.$$

It follows that $R^1(\tilde{f}_{1,0})_* (\tilde{e}v_1)^* T_{\mathfrak{B}_*} = 0$. \hfill \Box

**Proposition 5.2.** Put $\mathcal{L} = \det(\mathcal{E}_{n-1}^0)$. Then $\mathcal{V}$ sits in the exact sequence

$$0 \to \phi^* (\mathcal{L}^{-1} \otimes \hat{\pi}_1^* \mathcal{O}_{\mathfrak{M}(n-1)}(\mathcal{O}_{n-1}) \otimes \hat{\pi}_2^* \mathcal{O}_X(2\ell)) \to \mathcal{V} \to R^1(\tilde{f}_{1,0})_* \tilde{e}v_1^* (\pi^*(\mathcal{E}_{n-1}^0)^* \otimes \mathcal{O}_{\mathfrak{B}_*}(-1)) \otimes \phi^* (\hat{\pi}_1^* \mathcal{O}_{\mathfrak{M}(n-1)}(\mathcal{O}_{n-1}) \otimes \hat{\pi}_2^* \mathcal{O}_X(2\ell)) \to 0.$$

**Proof.** Recall that $\mathfrak{B}_* = \mathbb{P}(\mathcal{E}_{n-1}^0)$. The kernel of the tautological surjection $\pi^* \mathcal{E}_{n-1}^0 \to \mathcal{O}_{\mathfrak{B}_*}(1) \to 0$ is a fiber bundle. By comparing the first Chern classes, we get

$$0 \to \pi^* \mathcal{L} \otimes \mathcal{O}_{\mathfrak{B}_*}(-1) \to \pi^* \mathcal{E}_{n-1}^0 \to \mathcal{O}_{\mathfrak{B}_*}(1) \to 0.$$ 

Tensoring with $\pi^* \mathcal{L}^{-1} \otimes \mathcal{O}_{\mathfrak{B}_*}(-1)$, we obtain the exact sequence

$$0 \to \mathcal{O}_{\mathfrak{B}_*}(-2) \to \pi^* (\mathcal{E}_{n-1}^0 \otimes \mathcal{L}^{-1}) \otimes \mathcal{O}_{\mathfrak{B}_*}(-1) \to \pi^* \mathcal{L}^{-1} \to 0.$$

Note that $\mathcal{E}_{n-1}^0 \otimes \mathcal{L}^{-1} \cong (\mathcal{E}_{n-1}^0)^*$. Applying $\tilde{e}v_1^*$ to the above exact sequence yields

$$0 \to \tilde{e}v_1^* \mathcal{O}_{\mathfrak{B}_*}(-2) \to \tilde{e}v_1^* (\pi^* (\mathcal{E}_{n-1}^0)^* \otimes \mathcal{O}_{\mathfrak{B}_*}(-1)) \to (\pi \circ \tilde{e}v_1)^* \mathcal{L}^{-1} \to 0.$$

By (5.1), $\pi \circ \tilde{e}v_1 = \phi \circ \tilde{f}_{1,0}$. Rewriting the 3rd term, we have

$$0 \to \tilde{e}v_1^* \mathcal{O}_{\mathfrak{B}_*}(-2) \to \tilde{e}v_1^* (\pi^* (\mathcal{E}_{n-1}^0)^* \otimes \mathcal{O}_{\mathfrak{B}_*}(-1)) \to \tilde{f}_{1,0}^* \mathcal{L}^{-1} \to 0.$$

Applying the functor $(\tilde{f}_{1,0})_*$ to (5.4), we get the exact sequence

$$0 \to \phi^* \mathcal{L}^{-1} \to R^1(\tilde{f}_{1,0})_* (\tilde{e}v_1)^* \mathcal{O}_{\mathfrak{B}_*}(-2) \to R^1(\tilde{f}_{1,0})_* \tilde{e}v_1^* (\pi^* (\mathcal{E}_{n-1}^0)^* \otimes \mathcal{O}_{\mathfrak{B}_*}(-1)) \to 0,$$

where we have used the projection formula, $(\tilde{f}_{1,0})_* \mathcal{O}_{\mathfrak{D}_1} \cong \mathcal{O}_{\mathfrak{D}_0}$, and

$$R^1(\tilde{f}_{1,0})_* \mathcal{O}_{\mathfrak{D}_1} = 0, \quad (\tilde{f}_{1,0})_* \tilde{e}v_1^* (\pi^* (\mathcal{E}_{n-1}^0)^* \otimes \mathcal{O}_{\mathfrak{B}_*}(-1)) = 0.$$

By Lemma 5.1 and Lemma 5.3 we obtain the desired exact sequence for $\mathcal{V}$. \hfill \Box

**Remark 5.3.** Fix a point $(V_1, x) \in \mathfrak{M}(n-1) \times X$. Via $\phi^{-1}(V_1, x) \cong \mathfrak{M}_{0,0}(\mathbb{P}^1, d[\mathbb{P}^1])$, the restriction of $R^1(\tilde{f}_{1,0})_* \tilde{e}v_1^* (\pi^* (\mathcal{E}_{n-1}^0)^* \otimes \mathcal{O}_{\mathfrak{B}_*}(-1))$ to $\phi^{-1}(V_1, x)$ is isomorphic to

$$R^1(\tilde{f}_{1,0})_* (ev_1)^* (\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)),$$

where by abusing notation, we still use $\tilde{f}_{1,0}$ and $ev_1$ to denote the forgetful map and the evaluation map from $\mathfrak{M}_{0,1}(\mathbb{P}^1, d[\mathbb{P}^1])$ to $\mathfrak{M}_{0,0}(\mathbb{P}^1, d[\mathbb{P}^1])$ and $\mathbb{P}^1$, respectively.
6. The virtual fundamental class $\{\mathcal{M}_{0,1}(\mathcal{M}(n), df)\}^{vir}$

As we saw in the construction of the classes $\Xi_1$, $\Xi_5$ and $\Xi_6$ don’t lie in $\mathcal{B}_s$. The method to compute the virtual cycle restricted to $e_{i=1}^{vir}(\Xi_i)$ in the previous section won’t work for $i = 5, 6$. In this section, we shall employ the localization method of Kiem-Li to find a sufficiently small closed subset of $\mathcal{M}(n)$ containing the image of the virtual cycle under the evaluation map $e_1$. The result will be used to show the vanishing of the Gromov-Witten invariants $\langle \alpha \rangle_{0,d} \otimes \eta$ when $\alpha$ is dual to $\Xi_5, \Xi_6$.

Let $X = \mathbb{P}^2$, and let $C_0 \subset X$ be a smooth cubic curve. Recall that the Zariski tangent space of $\mathcal{M}(n)$ at $V \in \mathcal{M}(n)$ is canonically $\text{Ext}^1(V, V)$. Therefore, the natural map $K_X = \mathcal{O}_X(-C_0) \to \mathcal{O}_X$ induces a meromorphic 2-form $\Theta$ on the moduli space $\mathcal{M}(n)$ given point-wisely by

$$\text{Ext}^1(V, V)^* \cong \text{Ext}^1(V, V \otimes K_X) \to \text{Ext}^1(V, V).$$

Note that $\Theta$ is holomorphic at $V \in \mathcal{M}(n)$ if $\langle 0, 1 \rangle$ is an isomorphism.

The constructions in $[K-L, L-L]$ show that the meromorphic 2-form $\Theta$ on the moduli space $\mathcal{M}(n)$ induces a meromorphic homomorphism:

$$\eta : \mathcal{E} \to \mathcal{O}$$

over $\mathcal{M}_{0,1}(\mathcal{M}(n), df)$. Here $\mathcal{E}$ is a suitable bundle on $\mathcal{M}_{0,1}(\mathcal{M}(n), df)$ such that $[\mathcal{M}_{0,1}(\mathcal{M}(n), df)]^{vir} \in H_{2(4n-6)}(\Lambda)$, where $\Lambda \subset \mathcal{M}_{0,1}(\mathcal{M}(n), df)$ is the degeneracy loci consisting of points at which either the map $\eta$ is undefined or not surjective.

Next, we analyze the degeneracy loci $\Lambda$. Let $[\mu : (D; p) \to \mathcal{M}(n)] \in \Lambda$. Since $\mu_*(\mathcal{D}) = df$, $\mu_*(\mathcal{D})$ is contracted to a point by the Gieseker-Uhlenbeck morphism $\Psi$.

Lemma 6.1. Let $[\mu : (D; p) \to \mathcal{M}(n)] \in \Lambda$, and put

$$(\Psi \circ \mu)(D) = (V_k; \zeta) \in \mathcal{M}(n-k) \times \text{Sym}^k(X) \subset \mathcal{M}(n)$$

for some $1 \leq k \leq (n-1)$. Then, either $\text{Supp}(\zeta) \cap C_0 \neq \emptyset$ or $V_k|_{C_0}$ is not stable.

Proof. Assume that $\text{Supp}(\zeta) \cap C_0 = \emptyset$ and $V_k|_{C_0}$ is stable. We will draw a contradiction by showing that $\eta$ is both defined and surjective at $[\mu : (D; p) \to \mathcal{M}(n)]$.

Note that $\eta$ is defined at $[\mu : (D; p) \to \mathcal{M}(n)]$ if $\Theta$ is holomorphic along $\mu(D)$. Let $V \in \mu(D)$. Then, $V^{**} = V_k$ and we have $0 \to V \to V_k \to Q \to 0$, where $Q$ is a torsion sheaf with $\sum_{x \in X} h^0(X, Q_x) x = \zeta$. So $V|_{C_0} \cong V_k|_{C_0}$ is locally free and stable. From $0 \to K_X \to \mathcal{O}_X \to \mathcal{O}_{C_0} \to 0$, we get $0 \to V \otimes K_X \to V \to V|_{C_0} \to 0$. Applying the functor $\text{Hom}(V, \cdot)$, we obtain a long exact sequence:

$$0 \to \text{Hom}(V, V \otimes K_X) \to \text{Hom}(V, V) \to \text{Hom}(V, V|_{C_0})$$

$$\to \text{Ext}^1(V, V \otimes K_X) \to \text{Ext}^1(V, V).$$

Since $V$ and $V_k|_{C_0}$ are stable, we have $\text{Hom}(V, V) \cong \mathbb{C}$, $\text{Hom}(V, V \otimes K_X) = 0$ and $\text{Hom}(V, V|_{C_0}) \cong \text{Hom}(V_k|_{C_0}, V_k|_{C_0}) \cong \mathbb{C}$. The above exact sequence is simplified to

$$0 \to \text{Ext}^1(V, V \otimes K_X) \to \text{Ext}^1(V, V).$$

Since $\dim \text{Ext}^1(V, V \otimes K_X) = \dim \text{Ext}^1(V, V)$, we obtain an isomorphism

$$\text{Ext}^1(V, V \otimes K_X) \cong \text{Ext}^1(V, V).$$

Hence the meromorphic 2-form $\Theta$ is defined at $V \in \mu(D)$. This proves that $\Theta$ is holomorphic along $\mu(D)$. So $\eta$ is defined at $[\mu : (D; p) \to \mathcal{M}(n)]$. 

The above argument also shows that $\Theta|_{\mu(D)}$ is an isomorphism. Since $\mu$ is not a constant map, the image of $\mu_* : T_{D,reg} \to T_{\mathfrak{M}(n)}$ does not lie in the null space of $\Theta : T_{\mathfrak{M}(n)} \to \left(T_{\mathfrak{M}(n)}\right)^*$. By the vanishing criterion in [K-L], $\eta$ is surjective at $[\mu : (D;p) \to \mathfrak{M}(n)]$. □

**Lemma 6.2.** Let $C_0 \subset X = \mathbb{P}^2$ be a smooth cubic curve. Let $n \geq 1$, and let $V \in \mathfrak{M}(n)$ be generic. Then, the restriction $V|_{C_0}$ is stable.

**Proof.** It is well known that the cotangent bundle $\Omega_X$ is stable. So $\Omega_X \otimes \mathcal{O}_X(1) \in \mathfrak{M}(1)$. Let $\xi$ consist of $n-1$ distinct points away from $C_0$. Choose a surjection $\Omega_X \otimes \mathcal{O}_X(1) \rightarrow \mathcal{O}_\xi$, and let $V_0$ be the kernel. Then $V_0 \in \mathfrak{M}(n)$ and $V_0|_{C_0} \cong (\Omega_X \otimes \mathcal{O}_X(1))|_{C_0}$. Since $\mathfrak{M}(n)$ is irreducible and the open subset $\mathfrak{M}(n)$ is nonempty, our lemma will follow if we can prove that $(\Omega_X \otimes \mathcal{O}_X(1))|_{C_0}$ is stable.

Let $\mathcal{O}_{C_0}(D)$ be any sub-line-bundle of $(\Omega_X \otimes \mathcal{O}_X(1))|_{C_0}$. Note that the degree of $(\Omega_X \otimes \mathcal{O}_X(1))|_{C_0}$ is $-3$, and there is an exact sequence

$$(6.3) \quad 0 \rightarrow (\Omega_X \otimes \mathcal{O}_X(1))|_{C_0} \rightarrow \mathcal{O}^{\oplus 3}_{C_0} \rightarrow \mathcal{O}_X(1)|_{C_0} \rightarrow 0$$

induced from the exact sequence $0 \rightarrow \Omega_X \rightarrow \mathcal{O}_X(-1)^{\oplus 3} \rightarrow \mathcal{O}_X \rightarrow 0$. Thus

$$(6.4) \quad \text{deg}(D) \leq 0.$$}

If $\text{deg}(D) = 0$, then $D$ must be the trivial divisor and $\mathcal{O}_{C_0}(D) = \mathcal{O}_{C_0}$. So $H^0(C_0, (\Omega_X \otimes \mathcal{O}_X(1))|_{C_0}) \neq 0$. On the other hand, $(6.3)$ induces an exact sequence

$$0 \rightarrow H^0(C_0, (\Omega_X \otimes \mathcal{O}_X(1))|_{C_0}) \rightarrow H^0(C_0, \mathcal{O}^{\oplus 3}_{C_0}) \rightarrow H^0(C_0, \mathcal{O}_X(1)|_{C_0}).$$

The image of $f_0$ is $H^0(C_0, \mathcal{O}_X(1)|_{C_0}) = H^0(C_0, \mathcal{O}_X(1)|_{C_0})$. So $f_0$ is surjective. Since $H^0(C_0, \mathcal{O}^{\oplus 3}_{C_0})$ and $H^0(C_0, \mathcal{O}_X(1)|_{C_0})$ have the same dimension, $f_0$ is an isomorphism and $H^0(C_0, (\Omega_X \otimes \mathcal{O}_X(1))|_{C_0}) = 0$. Hence we obtain a contradiction.

If $\text{deg}(D) = -1$, then $\mathcal{O}_{C_0}(D) = \mathcal{O}_{C_0}(-x)$ for a unique $x \in C_0$. So the map

$$H^0(C_0, \mathcal{O}_{C_0}(x)|^{\oplus 3}) \rightarrow H^0(C_0, \mathcal{O}_X(1)|_{C_0} \otimes \mathcal{O}_{C_0}(x))$$

induced from $(6.3)$ is not injective. On the other hand, since $H^0(C_0, \mathcal{O}_{C_0}(x)) \cong \mathbb{C}$,

$$\text{im}(f_1) = H^0(C_0, \mathcal{O}_X(1)|_{C_0} \otimes H^0(C_0, \mathcal{O}_{C_0}(x)))$$

$$= H^0(C_0, \mathcal{O}_X(1)|_{C_0}) \otimes H^0(C_0, \mathcal{O}_{C_0}(x)) \subset H^0(C_0, \mathcal{O}_X(1)|_{C_0} \otimes \mathcal{O}_{C_0}(x)).$$

So the linear system corresponding to $\text{im}(f_1)$ consists of all the elements $\ell|_{C_0} + x$, where $\ell$ denotes lines in $X$. In particular, the dimension of $\text{im}(f_1)$ is $3$. Thus $f_1$ must be injective. Again, we obtain a contradiction.

By $(6.4)$, $\text{deg}(D) \leq -2$. Therefore, $(\Omega_X \otimes \mathcal{O}_X(1))|_{C_0}$ is stable. □

**Definition 6.3.** For $n \geq 2$, we define $\mathfrak{X}_{C_0}(n)$ (respectively, $\mathfrak{U}_{C_0}(n)$) to be the subset of $\mathfrak{M}(n)$ consisting of all the nonlocally free sheaves $V$ such that $V|_{C_0}$ contains torsion (respectively, $V|_{C_0}$ is torsion-free and unstable).

**Lemma 6.4.** Let $V \in \mathfrak{M}(n) - \mathfrak{M}(n)$ and $\Psi(V) = (V_k; \zeta)$. Then,

(i) $V \in \mathfrak{X}_{C_0}(n)$ if and only if $\text{supp}(\zeta) \cap C_0 \neq \emptyset$;

(ii) $V \in \mathfrak{U}_{C_0}(n)$ if and only if $\text{supp}(\zeta) \cap C_0 = \emptyset$ and $V_k|_{C_0}$ is unstable. □

**Lemma 6.5.** (i) $(\text{ev})(\Lambda) \subset \mathfrak{X}_{C_0}(n) \bigsqcup \mathfrak{U}_{C_0}(n)$.

(ii) Both $\mathfrak{X}_{C_0}(n)$ and $\mathfrak{U}_{C_0}(n)$ are closed subsets of $\mathfrak{M}(n)$. □
Proof. By Lemma 6.1, the image of a point in \((ev_1)(\Lambda)\) under \(\Psi\) is of the form
\[
(V_k; \zeta) \in \mathcal{M}(n - k) \times \text{Sym}^k(X) \subset \mathcal{M}(n),
\]
where \(1 \leq k \leq n - 1\), and either \(\text{Supp}(\zeta) \cap C_0 \neq \emptyset\) or \(V_k|_{C_0}\) is not stable. So we see from Lemma 6.4 that \((ev_1)(\Lambda) \subset \mathcal{T}_{C_0}(n) \bigcup \mathcal{U}_{C_0}(n)\). This proves (i).

Since being torsion-free and being stable are open conditions, both \(\mathcal{T}_{C_0}(n)\) and \(\mathcal{U}_{C_0}(n)\) are closed subsets of \(\mathcal{M}(n)\). This proves (ii). \(\square\)

7. The 1-point Gromov-Witten invariants

Now we are ready to compute the 1-point Gromov-Witten invariants

\[
(\alpha)_{0,d} = \int_{\mathcal{M}_0,1(\mathcal{M}(n),d)}^{vir} \text{ev}_1^* \alpha,
\]

where \(\alpha \in H^{8n-12}(\mathcal{M}(n); \mathbb{C})\) denotes the Poincaré duals of the classes \(\Xi_1, \ldots, \Xi_6 \in H_1(\mathcal{M}(n); \mathbb{C})\). By abusing notation, we use \(\Xi_i\) to stand for both the class in \(H_1(\mathcal{M}(n); \mathbb{C})\) and its Poincaré dual in \(H^{8n-12}(\mathcal{M}(n); \mathbb{C})\).

Lemma 7.1. \((\Xi_i)_{0,d} = -6/d^2\).

Proof. Let \(\alpha = \Xi_i\). By Subsection 5.1 \(\Xi_i = \pi^{-1}([V_1] \times \ell) = P(V_1|\ell)\), where the stable vector bundle \(V_1 \in \mathcal{M}(n - 1)\) and the line \(\ell \subset X\) are fixed. Let \(\mathcal{M} = (ev_1)^{-1}(\Xi_i)\). Then \(\mathcal{M} \subset \mathcal{D}_i\). By Proposition 5.2 and Proposition 2.1 for \(k = 1\) and \(\mathcal{D} = \mathcal{D}_i\), we obtain

\[
(\alpha)_{0,d} = \int_{\mathcal{M}_0,1(\mathcal{M}(n),d)}^{vir} \text{ev}_1^* \alpha = \int_{(\tilde{f}_{1,0})^* c_{2d-1}(V)} (\text{ev}_1)^* \alpha
\]

(7.2)

\[
= \int_{(\tilde{f}_{1,0})^* (\pi^* (\mathbb{L}) + 2\hat{\pi}^* \ell) \cdot (\tilde{f}_{1,0})^* c_{2d-2}(E)} (\text{ev}_1)^* \alpha,
\]

where \(E = R^1(\tilde{f}_{1,0})_* \text{ev}_1^* (\pi^* (\mathcal{L}) \otimes \mathcal{O}_{\mathcal{M}_i} (-1)) \otimes \phi^* (\hat{\pi}_1^* \mathcal{O}_{\mathcal{M}(n-1)}(\mathcal{D}_{n-1}) \otimes \hat{\pi}_2^* \mathcal{O}_X(2\ell))\).

Note that \(\Xi_1 \subset \mathcal{B}_s\) and \(\phi \circ f_{1,0} = \pi \circ ev_1\). So we obtain

\[
(\alpha)_{0,d} = (\text{ev}_1)^* \pi^* (\mathbb{L}) \cdot (\tilde{f}_{1,0})^* c_{2d-2}(E) \cdot (\text{ev}_1)^* (\Xi_1 \cdot c_1(\mathcal{O}_{\mathcal{B}_s})/(\mathcal{B}_s))
\]

(7.3)

\[
= (\tilde{f}_{1,0})^* c_{2d-2}(E) \cdot (\text{ev}_1)^* (\pi^* (\mathbb{L}) + 2\hat{\pi}^* \ell) \cdot (\Xi_1 \cdot c_1(\mathcal{O}_{\mathcal{B}_s})/(\mathcal{B}_s))
\]

(7.4)

Recall the definitions of \(\mathbb{L}\) and \(\mathcal{D}_{n-1}\) in Proposition 5.2 and Lemma 5.3. We have

\[
\mathbb{L} = \text{det}(\mathcal{E}_{n-1}) = (2, -1) \in \text{Pic}(\Gamma \times X),
\]

(7.5)

\[
\mathcal{D}_{n-1} = 2a_{n-1} - a_{n-1,2} - a_{n-1,0} = 2 \in \text{Pic}(\Gamma),
\]

(7.6)

\[
c_1(\mathcal{O}_{\mathcal{B}_s})|_{\mathbb{P}} = -2c_1(\mathcal{O}_{\mathbb{P}}(1)) + \pi^*(2, 2) \in \text{Pic}(\mathbb{P})
\]

in view of the exact sequence (4.1), the degrees in (4.3) and Lemma 5.3. Thus,

\[
(\alpha)_{0,d} = (\tilde{f}_{1,0})^* c_{2d-2}(E) \cdot (\text{ev}_1)^* (\pi^*(0, 3) \cdot [\Xi_1] \cdot (-2c_1(\mathcal{O}_{\mathbb{P}}(1)) + \pi^*(2, 2))).
\]

(7.7)

Since \([\xi] = \pi^*([V_1] \times \ell)\), it follows immediately that

\[
(\alpha)_{0,d} = -6 \cdot (\tilde{f}_{1,0})^* c_{2d-2}(E) \cdot (\text{ev}_1)^* [\xi],
\]

where \([\xi]\) denotes the cycle of a fixed point \(\xi \in \mathbb{P} \subset \mathcal{B}_s\).

Let \(\mathcal{M}'_1 = (ev_1)^{-1}(\xi) = (ev_1)^{-1}(\xi)\). If \(\mu : (D, p) \rightarrow \mathcal{M}(n) \in \mathcal{M}'_1\), then \(\mu(p) = \xi\) and \(\pi(\mu(D)) = \pi(\mu(p)) = \pi(\xi) = (V_1, x) \in \mathcal{M}(n-1) \times X\). So \(\mu(D) = f\), which
denotes the unique fiber of $\pi$ from $\mathfrak{X}$ containing the point $\xi \in \mathcal{B}_*$. Thus the restriction of the forgetful map $\tilde{f}_{1,0}$ to $\mathfrak{M}'_1$ gives a degree-$d$ morphism from $\mathfrak{M}'_1$ to

$$\mathfrak{M}'_0 \overset{\text{def}}{=} \tilde{f}_{1,0}(\mathfrak{M}'_1) = \phi^{-1}(V_1, x).$$

Hence, as algebraic cycles, we have $(\tilde{f}_{1,0})_*[\mathfrak{M}'_1] = d[\mathfrak{M}'_0] = d \cdot \phi^*([V_1, x])$. By (7.7),

$$\langle \alpha \rangle_{0, df} = -6 \cdot c_{2d-2}(E) \cdot (\tilde{f}_{1,0})_*[\mathfrak{M}'_1] = -6d \cdot c_{2d-2}(E) \cdot \phi^*([V_1, x])$$

(7.8)

By Remark 5.3, $(\mathfrak{X}|_{\phi^{-1}(V_1, x)}) \cong R^1(f_{1,0})_*((\mathcal{E}_1)_*\mathfrak{O}_P(-1) \oplus \mathfrak{O}_P(-1))$, where $f_{1,0}$ and $\mathcal{E}_1$ denote the forgetful map and the evaluation map from the moduli space $\mathfrak{M}_{0,1}(\mathbb{P}^1, d[\mathbb{P}^1])$ to $\mathfrak{M}_{0,0}(\mathbb{P}^1, d[\mathbb{P}^1])$ and $\mathbb{P}^1$, respectively. We have

$$c_{2d-2}(R^1(f_{1,0})_*((\mathcal{E}_1)_*\mathfrak{O}_P(-1) \oplus \mathfrak{O}_P(-1))) = \frac{1}{d^3}$$

d by Theorem 9.2.3 in [C.K]. Therefore, $\langle \alpha \rangle_{0, df} = -6/d^2$ by (7.8). \hfill $\square$

For $i = 2, 3$ or 4, we may assume that the classes $\Xi_2, \Xi_3$ and $\Xi_4$ are represented by complex surfaces in $\mathcal{P} \subset \mathcal{B}_*$. The same proofs of (7.6) and (7.7) show that

$$\langle [\xi] \rangle_{0, df} = a_i \cdot (\tilde{f}_{1,0})^*c_{2d-2}(E) \cdot (\tilde{e}v_1)^*[\xi],$$

where $[\xi]$ denotes the cycle of a fixed point $\xi$ in $\mathcal{P} \subset \mathcal{B}_*$, and

$$a_i = \pi^*(0, 3) \cdot [\Xi_i] \cdot (-2c_1(\mathfrak{O}_\ell(1)) + \pi^*(2, 2))$$

is the intersection number in $\mathcal{P}$. Note from the last two paragraphs in the proof of Lemma 7.1 that $(f_{1,0})^*c_{2d-2}(E) \cdot (\tilde{e}v_1)^*[\xi] = 1/d^2$. Therefore,

$$\langle [\xi] \rangle_{0, df} = \frac{a_i}{d^2}$$

(7.10)

**Theorem 7.2.** Let $d \geq 1$ and $n \geq 3$. The Gromov-Witten invariants $\langle \alpha \rangle_{0, df}$ for the classes $\alpha = PD(\Xi_1), \ldots, PD(\Xi_6) \in H^{8n-12}(\overline{\mathfrak{M}}(n); \mathbb{C})$ are respectively equal to $-6/d^2$, $12/d^2$, $0$, $-6/d^2$, $0$, $0$.

**Proof.** First of all, $\langle [\xi] \rangle_{0, df} = -6/d^2$ is Lemma 7.1.

Next, $\langle [\xi] \rangle_{0, df} = 12/d^2$ follows from the computation of the number in (7.4):

$$a_2 = 3\pi^*\ell \cdot c_1(\mathfrak{O}_\ell(V_1)(1)) \cdot (-2c_1(\mathfrak{O}_\ell(V_1)(1)) + 2\pi^*\ell) = 12$$

by (4.3), where $\pi : \mathcal{P}(V_1) \to X$ denotes the tautological projection.

Since $\Xi_3 = \pi^{-1}(\Gamma \times \{y\})$ with $y \in X$, $\langle [\xi] \rangle_{0, df} = 0$ comes from the computation

$$a_3 = \pi^*(0, 3) \cdot [\Xi_3] \cdot (-2c_1(\mathfrak{O}_\ell(1)) + \pi^*(2, 2)) = 0.$$  

(7.11)

Similarly, $\langle [\xi] \rangle_{0, df} = 0$ follows from the computation

$$a_4 = \pi^*(0, 3) \cdot c_1(\mathfrak{O}_W(1)) \cdot (-2c_1(\mathfrak{O}_W(1)) + \pi^*(2, 2)) = -6$$

by (4.11), where $W = \pi^{-1}(\Gamma \times \ell) \subset \mathcal{P} \subset \overline{\mathfrak{M}}(n)$ for a fixed line $\ell \subset X$.

To prove $\langle [\xi] \rangle_{0, df} = 0$, choose the vector bundle $V_2 \in \mathfrak{M}(n-2)$ and the distinct points $x_1, x_2 \in X$ in Subsection 4.3 such that $V_2|_{C_{0}}$ is stable and $x_1, x_2 \not\in C_0$. By Lemma 5.4 $\Xi_5 \cap (\mathfrak{U}_C(n) \cup \mathfrak{U}_C(n)) = \emptyset$. Hence $\Xi_5 \cap (ev_1(\Lambda)) = \emptyset$ by Lemma 6.5(i).

Since $[\overline{\mathfrak{M}}_{0,1}^{vir}(\mathfrak{M}(n), df)]^{vir} \in H_{2(4n-6)}(\Lambda)$, we get

$$\langle [\xi] \rangle_{0, df} = \int_{[\overline{\mathfrak{M}}_{0,1}^{vir}(\mathfrak{M}(n), df)]^{vir}} ev_1^*\Xi_5 = ev_1^*\left[\overline{\mathfrak{M}}_{0,1}^{vir}(\mathfrak{M}(n), df)\right]^{vir} \cdot \Xi_5 = 0.$$  

(7.12)
Finally, to prove $\langle \Xi_0 \rangle_{0,d} = 0$, choose the vector bundle $V_2 \in \mathfrak{M}(n - 2)$ and the point $x \in X$ in Subsection 4.3 such that $V_2|C_0$ is stable and $x \not\in C_0$. Now our result follows from the same proof in the previous paragraph.

Remark 7.3. Let $d \geq 1$ and $n \geq 3$. Using the theorem above, one can show that

\[
ev_{1*}[\mathfrak{M}_{0,1}(\mathfrak{M}(n), df)]^{\text{vir}} = \frac{1}{d^2} \Xi_{C_0}(n).
\]

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