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GENERALISED MORPHISMS OF k-GRAPHS: k-MORPHS

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ABSTRACT. In a number of recent papers, $(k+l)$-graphs have been constructed from k-graphs by inserting new edges in the last l dimensions. These constructions have been motivated by C^*-algebraic considerations, so they have not been treated systematically at the level of higher-rank graphs themselves. Here we introduce k-morphs, which provide a systematic unifying framework for these various constructions. We think of k-morphs as the analogue, at the level of k-graphs, of C^*-correspondences between C^*-algebras. To make this analogy explicit, we introduce a category whose objects are k-graphs and whose morphisms are isomorphism classes of k-morphs. We show how to extend the assignment $\Lambda \mapsto C^*(\Lambda)$ to a functor from this category to the category whose objects are C^*-algebras and whose morphisms are isomorphism classes of C^*-correspondences.

1. INTRODUCTION

Over the last ten years, graph C^*-algebras and their analogues have been the subject of intense research interest (see for example [5, 7, 8, 13, 16, 20, 25, 38], or see [30] for a good overview). In particular, the higher-rank graphs and associated C^*-algebras introduced in [18] have recently been widely studied [10, 12, 19, 28]. Higher-rank graphs generalise directed graphs, so there are many points of similarity between the two theories, especially at the level of fundamental existence and uniqueness results. However, as both fields progress, the two sets of results are diverging more and more rapidly.

One reason for this is the relatively involved combinatorial structure of higher-rank graphs as opposed to "ordinary" one-dimensional graphs. It is fairly straightforward to modify an ordinary graph by simply adding vertices or edges because these are local operations. By contrast, adding vertices and edges to a higher-rank graph is quite complicated because the combinatorial peculiarities of higher-rank graphs mean that the addition of an edge at some vertex typically necessitates similar changes throughout a large portion of the higher-rank graph. A good illustration of this is the contrast between the straightforward process of "adding tails" to a directed graph [3] and the analogous but vastly more complicated "removing sources" construction for higher-rank graphs [11].

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It has become clear recently, however, that if higher-rank graphs are not well-suited to constructions which involve localised modifications, they are amenable to a somewhat different style of construction which is not available in the one-dimensional setting and which is proving very profitable from a C^*-algebraic standpoint. Specifically, k-graphs lend themselves to constructions whereby one increases the rank of a graph or graphs by adding edges in new dimensions [15, 21, 27]. The resulting $(k+l)$-graph C^*-algebras have been analysed as direct limits [21, 27] and as crossed-products by group actions [15].

So far these constructions of $(k+l)$-graphs from k-graphs have been ad hoc: in each of [15, 21, 27], given a k-graph with natural symmetry or a pair of k-graphs with structural similarities, the authors have constructed a $(k+l)$-graph with bare hands. In each case, the $(k+l)$-graph contains copies of the original k-graph or graphs in the first k-dimensions, and encodes the additional symmetry or structural similarities in the remaining l dimensions.

The purpose of this article is to replace these ad hoc methods with a unifying construction which is functorial with respect to the assignment of C^*-algebras to higher-rank graphs. More specifically, in Section 3 we axiomatise the data required to insert a set X of edges in a $(k+1)^{\mathrm{st}}$ dimension between vertices in a k-graph Γ and those in a k-graph Λ so as to obtain a $(k+1)$-graph. We call a set X endowed with such data a Λ–Γ morph, or a k-morph from Γ to Λ. Given a Λ_0–Λ_1 morph X_1 and a Λ_1–Λ_2 morph X_2, we define a fibred product $X_1 *_{\Lambda_1^0} X_2$ which is a Λ_0–Λ_2 morph. We show in Theorem 3.10 that there is a category $\mathcal{M}_k$ whose objects are k-graphs and whose morphisms are isomorphism classes of k-morphs.

In Sections 4 and 5 we discuss how k-morphs can be used as a model for constructions such as those of [15, 21, 27]. Given k-graphs Λ and Γ, and a Λ–Γ morph X, we define in Section 4 what we call a *linking graph* for X. Roughly speaking, a linking graph for X is a $(k+1)$-graph Σ containing disjoint copies of Λ and Γ connected in the $(k+1)^{\mathrm{st}}$ dimension by a copy of X. We show in Proposition 4.5 that a linking graph always exists, is unique up to isomorphism, and is determined up to isomorphism by the isomorphism class of X.

The constructions set out in [15, 21, 27] typically involve a system of linking graphs which are glued together in some systematic way. For example, we can think of the covering systems of [21] as a system, organised by an underlying Bratteli diagram, of linking graphs for k-morphs determined by k-graph coverings. To capture this idea, we introduce in Section 5 the notion of a Γ-system of k-morphs and the notion of a Γ-bundle for a Γ-system. Given an l-graph Γ, a Γ system consists of a collection $\{\Lambda_v : v \in \Gamma^0\}$ of k-graphs connected by k-morphs $\{X_\gamma : \gamma \in \Gamma\}$ so that composition in Γ corresponds in a consistent way to the fibred-product operation on the associated k-morphs. A Γ-bundle for this system is then a $(k+l)$-graph Σ together with a map $f : \Sigma \to \Gamma$ such that $f^{-1}(v) \cong \Lambda_v$ for each $v \in \Gamma^0$ and such that $f^{-1}(\{r(\gamma), \gamma, s(\gamma)\})$ is a linking graph for X_γ for each $\gamma \in \Gamma$. We call the map f the bundle map for the Γ-bundle Σ. We show in Theorem 5.9 that every Γ-system admits a Γ-bundle and that the Γ-bundle is unique up to isomorphism and depends only on the isomorphism class of the Γ-system. We indicate how to realise the k-graphs constructed in [15, 21, 27, 28] as Γ-bundles in a natural way.

A Γ-system X of k-morphs determines a functor from Γ into the category $\mathcal{M}_k$ via the assignments $v \mapsto \Lambda_v$ and $\gamma \mapsto [X_\gamma]$ (where $[X_\gamma]$ is the isomorphism class of X_γ). One might initially hope that the isomorphism class of a Γ-bundle for the system would be determined by this functor so that we could replace Γ-systems with functors. We show in Proposition 5.14 that when Γ is a 1-graph each Γ-system is indeed determined up to isomorphism by the functor $\gamma \mapsto [X_\gamma]$. However, this is the best we can hope for: Example 5.15(i) shows that if Γ has rank 2 or more there may be nonisomorphic Γ-systems which determine the same functor from Γ to $\mathcal{M}_k$; and an example of Spielberg's, which we present as Example 5.15(ii), shows that there exists a 3-graph Γ and a functor $\Gamma \to \mathcal{M}_0$ which is not the functor determined by any Γ-system of 0-graphs. In particular, Γ-systems cannot be replaced with functors.

An example of a Γ system is the following. Let X be a Λ–Λ morph (we refer to these as Λ endomorphs). Then X gives rise to a T_1-system of k-morphs, where T_1 is the 1-graph with a single vertex and a single edge. We call the T_1-bundle for such a system the endomorph skew graph of Λ by X, and we denote it $\Lambda \times_X \mathbb{N}$. When X arises from an automorphism of Λ, we recover the crossed product graph of [15].

In Section 6 we discuss how our constructions behave at the level of C^*-algebras. The category $\mathcal{M}_k$ is reminiscent of the category (which we shall denote by $\mathcal{C}$) of [9, 23, 37], whose objects are C^*-algebras and whose morphisms are isomorphism classes of C^*-correspondences (also known as Hilbert bimodules). To simplify arguments, we restrict our attention to a subcategory $\mathcal{M}_k^{\maltese}$ of $\mathcal{M}_k$. We construct a C^*-correspondence $\mathcal{H}(X)$ for each k-morph X in such a way that the isomorphism class of $\mathcal{H}(X)$ depends only on that of X. We show in Theorem 6.6 that the assignments $\Lambda \mapsto C^*(\Lambda)$ and $[X] \mapsto [\mathcal{H}(X)]$ determine a contravariant functor between $\mathcal{M}_k^{\maltese}$ and $\mathcal{C}$. In the special case where X is a Λ-endomorph, so that $[X] \in \operatorname{End}_{\mathcal{M}_k^{\maltese}}(\Lambda)$, Theorem 6.8 shows that the C^*-algebra of the endomorph skew graph is canonically isomorphic to the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}(X)}$.

2. Preliminaries

2.1. **Higher-rank graphs.** In this paper, unlike previous treatments of k-graphs [10, 12, 18, 31], we allow 0-graphs. To make sense of this, we take the convention that $\mathbb{N}^0$ is the trivial semigroup $\{0\}$. We will also insist that all k-graphs are nonempty.

Modulo the minor differences mentioned above, we will adopt the conventions of [18, 26] for k-graphs. Given a nonnegative integer k, a *k-graph* is a nonempty countable small category Λ equipped with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorisation property*: for all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$, there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$. When $d(\lambda) = n$ we say λ has *degree* n. By abuse of notation, we will use d to denote the degree functor in every k-graph in this paper; the domain of d is always clear from context.

For $k \geq 1$, the standard generators of $\mathbb{N}^k$ are denoted $e_1, \ldots, e_k$, and for $n \in \mathbb{N}^k$ and $1 \leq i \leq k$ we write n_i for the i^{th} coordinate of n.

For $n \in \mathbb{N}^k$, we write Λ^n for $d^{-1}(n)$. In particular, Λ^0 is the vertex set. The *vertices* of Λ are the elements of Λ^0. The factorisation property implies that $o \mapsto \mathrm{id}_o$ is a bijection from the objects of Λ to Λ^0. We will frequently use this bijection to silently identify $\mathrm{Obj}(\Lambda)$ with Λ^0. The domain and codomain maps in the category Λ therefore become maps $s, r : \Lambda \to \Lambda^0$. More precisely, for $\alpha \in \Lambda$, the *source* $s(\alpha)$ is the identity morphism associated with the object $\mathrm{dom}(\alpha)$ and, similarly, $r(\alpha) = \mathrm{id}_{\mathrm{cod}(\alpha)}$.

Note that a 0-graph is then a countable category whose only morphisms are the identity morphisms; we think of them as a collection of isolated vertices.

For $u, v \in \Lambda^0$ and $E \subset \Lambda$, we write uE for $E \cap r^{-1}(u)$ and Ev for $E \cap s^{-1}(v)$. For $n \in \mathbb{N}^k$, we denote by $\Lambda^{\leq n}$ the set

$$\Lambda^{\leq n} = \{\lambda \in \Lambda : d(\lambda) \leq n \text{ and } s(\lambda)\Lambda^{e_i} = \emptyset \text{ whenever } d(\lambda) + e_i \leq n\}.$$

We say that Λ is *row-finite* if $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. We say that Λ is *locally convex* if whenever $1 \leq i < j \leq k$, $e \in \Lambda^{e_i}$, $f \in \Lambda^{e_j}$ and $r(e) = r(f)$, we can extend both e and f to paths ee' and ff' in $\Lambda^{e_i + e_j}$.

2.2. **Maps between higher-rank graphs.** A *k-graph morphism* is a degree-preserving functor. More generally, if $\omega : \mathbb{N}^k \to \mathbb{N}^l$ is a homomorphism, Λ is a k-graph and Γ is an l-graph, we say that a functor $f : \Lambda \to \Gamma$ is an *ω-quasimorphism* if $d_\Gamma(f(\lambda)) = \omega(d_\Lambda(\lambda))$ for all $\lambda \in \Lambda$. A k-graph morphism is then an id_k-quasimorphism.

Let $\omega : \mathbb{N}^k \to \mathbb{N}^l$ be a homomorphism, and let Γ be an l-graph. The pullback $\omega^*\Gamma$ is the k-graph $\omega^*\Gamma = \{(\gamma, n) \in \Gamma \times \mathbb{N}^k : \omega(n) = d(\gamma)\}$ with degree map $d_{\omega^*\Gamma}(\gamma, n) = n$ [18, Definition 1.9]. In the case where ω is injective, it will also sometimes be convenient to regard the subcategory $\Gamma^\omega := \bigcup_{n \in \mathbb{N}^k} \Gamma^{\omega(n)}$ of Γ as a k-graph as follows. We define the degree functor d_ω on Γ^ω by $d_\omega(\gamma) = n$ when $\omega(n) = d_\Gamma(\gamma)$. Of course Γ^ω and $\omega^*\Gamma$ are isomorphic, but the former is a subset of Γ, whereas the latter is formally disjoint from Γ.

As in [26], a *covering* of a k-graph Λ by a k-graph Γ is a surjective k-graph morphism $p : \Gamma \to \Lambda$ such that for all $v \in \Gamma^0$, p restricts to bijections between $v\Gamma$ and $p(v)\Lambda$ and between Γv and $\Lambda p(v)$. The covering $p : \Gamma \to \Lambda$ is *finite* if $p^{-1}(v)$ is finite for all $v \in \Lambda^0$. Every covering $p : \Gamma \to \Lambda$ has the unique path lifting property: for every $\lambda \in \Lambda$ and $v \in \Gamma^0$ with $p(v) = s(\lambda)$ there is a unique $\gamma \in \Gamma$ such that $p(\gamma) = \lambda$ and $s(\gamma) = v$; and similarly at $r(\lambda)$.

2.3. **C^*-algebras associated to higher-rank graphs.** Given a row-finite, locally convex k-graph (Λ, d), a Cuntz-Krieger Λ-family is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying the Cuntz-Krieger relations:

- $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
- $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;
- $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
- $t_v = \sum_{\lambda \in v\Lambda^{\leq n}} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The k-graph C^*-algebra $C^*(\Lambda)$ is the universal C^*-algebra generated by a Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$. That is, for every Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ there is a homomorphism π_t of $C^*(\Lambda)$ satisfying $\pi_t(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$.

A k-graph with no sources is automatically locally convex with $\Lambda^{\leq n} = \Lambda^n$ for all $n \in \mathbb{N}^k$. Hence the definition of $C^*(\Lambda)$ above reduces in this case to [18, Definition 1.5].

By [31, Theorem 3.15], the generating partial isometries $\{s_\lambda : \lambda \in \Lambda\} \subset C^*(\Lambda)$ are all nonzero.

If Λ is a 0-graph, then it trivially has no sources, and the last three Cuntz-Krieger relations follow from the first one. So $C^*(\Lambda)$ is the universal C^*-algebra generated by mutually orthogonal projections $\{s_v : v \in \Lambda^0\}$; that is, $C^*(\Lambda) \cong c_0(\Lambda^0)$.

Let Λ be a k-graph. There is a strongly continuous action γ of $\mathbb{T}^k$ on $C^*(\Lambda)$, called the *gauge-action*, such that $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$ for all $z \in \mathbb{T}^k$ and $\lambda \in \Lambda$.

2.4. C^***-correspondences.** We define Hilbert modules following [22] and [4, §II.7]. Let B be a C^*-algebra and let $\mathcal{H}$ be a right B-module. Then a *B-valued inner product* on $\mathcal{H}$ is a function $\langle\cdot,\cdot\rangle_B : \mathcal{H}\times\mathcal{H}\to B$ satisfying the following conditions for all $\xi,\eta,\zeta\in\mathcal{H}$, $b\in B$ and $\alpha,\beta\in\mathbb{C}$:

- $\langle\xi,\alpha\eta+\beta\zeta\rangle_B = \alpha\langle\xi,\eta\rangle_B + \beta\langle\xi,\zeta\rangle_B$,
- $\langle\xi,\eta b\rangle_B = \langle\xi,\eta\rangle_B b$,
- $\langle\xi,\eta\rangle_B = \langle\eta,\xi\rangle_B^*$,
- $\langle\xi,\xi\rangle_B \geq 0$ and $\langle\xi,\xi\rangle_B = 0$ if and only if $\xi = 0$.

If $\mathcal{H}$ is complete with respect to the norm given by $\|\xi\|^2 = \|\langle\xi,\xi\rangle_B\|$, then $\mathcal{H}$ is said to be a (right-) *Hilbert B-module.* If the range of the inner product is not contained in any proper ideal in B, $\mathcal{H}$ is said to be *full.* Note that B may be endowed with the structure of a full Hilbert B-module by taking $\langle\xi,\eta\rangle_B = \xi^*\eta$ for all $\xi,\eta\in B$. A map $T:\mathcal{H}\to\mathcal{H}$ is an *adjointable operator* if there is a map $T^*:\mathcal{H}\to\mathcal{H}$ such that $\langle T\xi,\eta\rangle_B = \langle\xi,T^*\eta\rangle_B$ for all $\xi,\eta\in\mathcal{H}$. Such an operator is necessarily linear and bounded, and the collection $\mathcal{L}(\mathcal{H})$ of all adjointable operators on $\mathcal{H}$ is a C^*-algebra. Each pair $\xi,\eta\in\mathcal{H}$ determines a *rank-one operator* $\theta_{\xi,\eta}$ (with adjoint $\theta_{\eta,\xi}$) given by $\theta_{\xi,\eta}\zeta = \xi\langle\eta,\zeta\rangle_B$ for $\zeta\in\mathcal{H}$. The C^*-subalgebra $\mathcal{K}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ generated by the $\theta_{\xi,\eta}$ is called the algebra of compact operators on $\mathcal{H}$. Note that $\mathcal{L}(\mathcal{H})$ may be identified with the multiplier algebra of $\mathcal{K}(\mathcal{H})$.

Let A and B be C^*-algebras; then a C^*-correspondence from A to B, or more briefly an A–B C^*-correspondence, is a Hilbert B-module $\mathcal{H}$ together with a $*$-homomorphism $\varphi : A\to\mathcal{L}(\mathcal{H})$. Given a homomorphism $\varphi : A\to B$, we may endow B with the structure of A–B C^*-correspondence in a canonical way. So it is natural to think of an A–B C^*-correspondence as a generalised homomorphism from A to B. A C^*-correspondence $\mathcal{H}$ is said to be *nondegenerate* if span $\{\varphi(a)\xi : a\in A, \xi\in\mathcal{H}\}$ is dense in $\mathcal{H}$ (some authors have also called such C^*-correspondences *essential*). We often suppress φ by writing $a\cdot\xi$ for $\varphi(a)\xi$.

As discussed in [4, 9, 23, 37], there is a category $\mathcal{C}$ with $\mathrm{Obj}(\mathcal{C})$ the class of C^*-algebras and with $\mathrm{Hom}_{\mathcal{C}}(A,B)$ the collection of isomorphism classes of A–B C^*-correspondences with identity morphisms $[A]$. The composition

$$\mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B)\to\mathrm{Hom}_{\mathcal{C}}(A,C)$$

is defined by $([\mathcal{H}_1],[\mathcal{H}_2])\mapsto[\mathcal{H}_2\otimes_B\mathcal{H}_1]$, where $\mathcal{H}_2\otimes_B\mathcal{H}_1$ denotes the tensor product of C^*-correspondences. The C^*-correspondence $\mathcal{H}_1\otimes\mathcal{H}_2$ is called the *internal tensor product* of $\mathcal{H}_1$ and $\mathcal{H}_2$ by Blackadar and the *interior tensor product* by Lance (see [4, II.7.4.1] and [22, Prop. 4.5] and the following discussion).

2.5. **Representations of** C^***-correspondences.** Let $\mathcal{H}$ be a C^*-correspondence over A. Recall from [29] that a representation of $\mathcal{H}$ in a C^*-algebra B is a pair (t,π) where $\pi : A\to B$ is a homomorphism, $t:\mathcal{H}\to B$ is linear, and such that for all $a\in A$ and $\xi,\eta\in\mathcal{H}$, we have $t(a\cdot\xi) = \pi(a)t(\xi)$, $t(\xi\cdot a) = t(\xi)\pi(a)$, and $\pi(\langle\xi,\eta\rangle_A) = t(\xi)^*t(\eta)$.

Given a C^*-correspondence $\mathcal{H}$ over A and a representation (t,π) of $\mathcal{H}$ on B, there is a homomorphism $t^{(1)} : \mathcal{K}(\mathcal{H})\to B$ satisfying $t^{(1)}(\theta_{\xi,\eta}) = t(\xi)t(\eta)^*$ for all $\xi,\eta\in\mathcal{H}$ (Pimsner denotes this homomorphism $\pi^{(1)}$ in [29]). In the cases of interest later in this paper, the left action of A on $\mathcal{H}$ is by elements of $\mathcal{K}(\mathcal{H})$ (that is, $\varphi : A\to\mathcal{L}(\mathcal{H})$ in fact takes values in $\mathcal{K}(\mathcal{H})$), so $t^{(1)}\circ\varphi$ is a homomorphism from A to B. In this case, the pair (t,π) is said to be *Cuntz-Pimsner covariant* if $t^{(1)}\circ\varphi = \pi$.

Given a C^*-correspondence $\mathcal{H}$ over A, there is a representation $(j_{\mathcal{H}}, j_A)$ in a C^*-algebra $\mathcal{O}_{\mathcal{H}}$ which is universal in the sense that given another representation (t, π) of $\mathcal{H}$ in B there is a homomorphism $t \times \pi : \mathcal{O}_{\mathcal{H}} \to B$ satisfying $(t \times \pi) \circ j_{\mathcal{H}} = t$ and $(t \times \pi) \circ j_A = \pi$.

3. k-morphs

In this section we define k-morphs, provide some motivating examples, and show how isomorphism classes of k-morphs can be regarded as the morphisms of a category whose objects are k-graphs. Conceptually, a k-morph may be thought of as a bridge between two k-graphs Λ and Γ. It consists of a set X and some structure maps which are precisely what is needed to build a $(k+1)$-graph that contains disjoint copies of Λ and Γ and in which elements of X become edges of degree e_{k+1} from vertices in the copy of Γ to vertices in the copy of Λ. We now give the formal definition.

Definition 3.1. Let Λ and Γ be k-graphs, let X be a countable set, and fix functions $r : X \to \Lambda^0$ and $s : X \to \Gamma^0$. We will write $X *_{\Gamma^0} \Gamma$ for the fibred product $\{(x, \gamma) : x \in X, \gamma \in \Gamma, s(x) = r(\gamma)\}$. Likewise, we will write $\Lambda *_{\Lambda^0} X$ for the fibred product $\{(\lambda, x) : \lambda \in \Lambda, x \in X, s(\lambda) = r(x)\}$. Fix a bijection $\phi : X *_{\Gamma^0} \Gamma \to \Lambda *_{\Lambda^0} X$, and suppose that whenever $\phi(x_1, \gamma_1) = (\lambda_1, x_2)$, we have

(1) $d(\gamma_1) = d(\lambda_1)$,
(2) $s(\gamma_1) = s(x_2)$, and
(3) $r(\lambda_1) = r(x_1)$.

Suppose further that whenever $\phi(x_1, \gamma_1) = (\lambda_1, x_2)$ and $\phi(x_2, \gamma_2) = (\lambda_2, x_3)$[†], we have

(4) $\phi(x_1, \gamma_1\gamma_2) = (\lambda_1\lambda_2, x_3)$.

Then we call X a Λ–Γ *morph*, or simply a k-morph. If $\Lambda = \Gamma$, then we call X a Λ *endomorph*.

Remark 3.2. Technically a Λ–Γ morph is a quadruple (X, r, s, ϕ), but by the usual abuse of notation, we will say "X is a Λ–Γ morph" without reference to the additional structure.

Examples 3.3. We now present a series of examples of k-morphs. In each case we shall describe the set X and the structure maps. It is straightforward to check in each case that the resulting data define a k-morph.

(i) Let Λ and Γ be k-graphs, and let $\alpha : \Gamma \to \Lambda$ be an isomorphism. Let $X(\alpha) = \Gamma^0$, and define structure maps $r_\alpha = \alpha$, $s_\alpha = \mathrm{id}_{\Gamma^0}$ and $\phi(r(\gamma), \gamma) = (\alpha(\gamma), s(\gamma))$. Then $X(\alpha)$ is a Λ–Γ morph. If $\Lambda = \Gamma$ so that α is an automorphism, then $X(\alpha)$ is a Λ endomorph. In the special case where α is the identity isomorphism id_Λ, we refer to $X(\mathrm{id}_\Lambda)$ as the *identity endomorph* on Λ. When it is useful to highlight its dependence on Λ, we will denote it as I_Λ.

(ii) Let $p : \Gamma \to \Lambda$ be a covering map. Let ${}_pX = \Gamma^0$, and define structure maps by $r = p$, $s = \mathrm{id}_{\Gamma^0}$ and $\phi(r(\gamma), \gamma) = (p(\gamma), s(\gamma))$. Then ${}_pX$ is a Λ–Γ morph. Such a k-morph is called a *covering k-morph*. Note that if $p = \alpha$ is

[†]Note that the conditions $\phi(x_1, \gamma_1) = (\lambda_1, x_2)$ and $\phi(x_2, \gamma_2) = (\lambda_2, x_3)$, together with (2) and (3), imply that $s(\gamma_1) = s(x_2) = r(\gamma_2)$ and $s(\lambda_1) = r(x_2) = r(\lambda_2)$. It then follows that $(\lambda_1\lambda_2, x_3) \in \Lambda *_{\Lambda^0} X$ and $(x_1, \gamma_1\gamma_2) \in X *_{\Gamma^0} \Gamma$.

an isomorphism, then ${}_pX$ is equal to the k-morph $X(\alpha)$ of the preceding example.

(iii) We may reverse the "direction" of the elements of X in the preceding example to get a Γ–Λ morph. Let $p : \Gamma \to \Lambda$ be a covering of k-graphs. Let $X_p := \Gamma^0$, and define $r = \mathrm{id}_{\Gamma^0}$, $s = p$ and $\phi(r(\gamma), p(\gamma)) = (\gamma, s(\gamma))$ (where we are using the unique path lifting property to recover γ from $p(\gamma)$ and $r(\gamma)$). Then X_p is a Γ–Λ morph.

(iv) Let $\Lambda_1, \Lambda_2, \Gamma$ be k-graphs and $p : \Gamma \to \Lambda_1$, $q : \Gamma \to \Lambda_2$ be coverings. Let ${}_pX_q = \Gamma^0$, and define structure maps by $r = p$, $s = q$, and $\phi(r(\gamma), q(\gamma)) = (p(\gamma), s(\gamma))$. Then ${}_pX_q$ is a Λ_1–Λ_2 morph. This generalises the preceding two examples: if $\Lambda_2 = \Gamma$ and $q = \mathrm{id}_\Gamma$, then ${}_pX_{\mathrm{id}_\Gamma} = {}_pX$, and similarly if $\Lambda_1 = \Gamma$, then ${}_{\mathrm{id}_\Lambda}X_q = X_q$.

(v) Number (iv) (hence also numbers (ii) and (iii)) above can be enriched with multiple "edges" as in [21]. Let $p : \Gamma \to \Lambda_1$ and $q : \Gamma \to \Lambda_2$ be covering maps. Write S_m for the group of permutations of $\{1, \ldots, m\}$; let c be a cocycle from Γ to S_m (that is, $c(\alpha)c(\beta) = c(\alpha\beta)$ whenever α and β are composable in Γ). Set ${}^c_pX_q = \Gamma^0 \times \{1, \ldots, m\}$, and define structure maps by $r(v, i) = p(v)$, $s(v, i) = q(v)$ and $\phi((r(\gamma), i), q(\gamma)) = (p(\gamma), (s(\gamma), c(\gamma)^{-1}i))$. Then ${}^c_pX_q$ is a Λ_1–Λ_2 morph.

(vi) Let (Σ, d) be a $(k+1)$-graph and $\iota : \mathbb{N}^k \to \mathbb{N}^{k+1}$ be the homomorphism $n \to (n, 0)$. Recall from Section 2.2 that we can regard $\Sigma^\iota := \{\lambda \in \Sigma : d(\lambda) \in \iota(\mathbb{N}^k)\}$ as a k-graph. Let $X = \Sigma^{e_{k+1}}$, and define $r_X, s_X : X \to \Sigma^0$ to be the range and source maps r, s inherited from Σ. The bijection ϕ is obtained from the factorisation property in Σ: $\phi(x, \lambda) = (\lambda', x')$, where $x' \in X$ and $\lambda' \in \Sigma^\iota$ are the unique elements satisfying $x\lambda = \lambda' x'$ in Σ. Then X is a Σ^ι endomorph.

(vii) Let Λ and Γ be k-graphs, and let X_1 and X_2 be Λ–Γ morphs. Then $X := X_1 \sqcup X_2$ is a Λ–Γ morph with the inherited structure maps.

We next define a kind of fibred product of k-morphs. This fibred product, like tensor products, is not quite associative on k-morphs. However, Proposition 3.6 and Lemma 3.8 show that it does determine an associative binary operation on isomorphism classes of k-morphs. Of course, we must first say exactly what we mean by an isomorphism of k-morphs.

Definition 3.4. Fix k-graphs Λ and Γ. Let X and Y be Λ–Γ morphs. We say that X and Y are *isomorphic* if there is a bijection $\theta : X \to Y$ which respects the structure maps; that is, θ intertwines the range and source maps and satisfies

$$(\mathrm{id}_\Lambda \times \theta) \circ \phi_X = \phi_Y \circ (\theta \times \mathrm{id}_\Gamma).$$

We call such a bijection θ an *isomorphism* and write $X \cong Y$; we denote the isomorphism class of a k-morph X by $[X]$.

We now introduce the notation associated with fibred products of k-morphs, and we then show in Proposition 3.6 that the resulting object is itself a k-morph.

Notation 3.5. Let Λ_0, Λ_1 and Λ_2 be k-graphs, and let X_i be a Λ_{i-1}–Λ_i morph with structure maps r_i, s_i and ϕ_i for $i = 1, 2$. Let

$$X_1 *_{\Lambda_1^0} X_2 = \{(x_1, x_2) \in X_1 \times X_2 : s(x_1) = r(x_2)\}.$$

Define $r : X_1 *_{\Lambda_1^0} X_2 \to \Lambda_0^0$ and $s : X_1 *_{\Lambda_1^0} X_2 \to \Lambda_2^0$ by

$$r(x_1, x_2) = r_1(x_1) \qquad \text{and} \qquad s(x_1, x_2) = s_2(x_2).$$

To define $\phi : (X_1 *_{\Lambda_1^0} X_2) *_{\Lambda_2^0} \Lambda_2 \to \Lambda_0 *_{\Lambda_0^0} (X_1 *_{\Lambda_1^0} X_2)$, fix $((x_1, x_2), \lambda_2) \in (X_1 *_{\Lambda_1^0} X_2) *_{\Lambda_2^0} \Lambda_2$. Then $s_2(x_2) = r(\lambda_2)$, so $\phi_2(x_2, \lambda_2) = (\lambda_1, x_2')$ for some $\lambda_1 \in \Lambda_1$ and $x_2' \in X_2$. Moreover, $r(\lambda_1) = r_2(x_2) = s_1(x_1)$, so $\phi_1(x_1, \lambda_1) = (\lambda_0, x_1')$ for some $\lambda_0 \in \Lambda_0$ and $x_1' \in X_1$ with $s_1(x_1') = r_2(x_2')$. We define

$$\phi((x_1, x_2), \lambda_2) = (\lambda_0, (x_1', x_2')). \tag{3.1}$$

Proposition 3.6. *With the notation above, $X_1 *_{\Lambda_1^0} X_2$ is a Λ_0–Λ_2 morph. Moreover, the isomorphism class $[X_1 *_{\Lambda_1^0} X_2]$ depends only on the isomorphism classes $[X_1]$ and $[X_2]$.*

Proof. Conditions (1)–(3) of Definition 3.1 are easily checked using (3.1) and that X_1 and X_2 are k-morphs, so we need only check (4).

Fix a composable pair $\mu_2, \nu_2 \in \Lambda_2$ and $(x_1, x_2) \in X_1 *_{\Lambda_1^0} X_2$ such that $s(x_1, x_2) = r(\mu_2)$. Let $x_i', x_i'' \in X_i$ for $i = 1, 2$ and $\mu_i, \nu_i \in \Lambda_i$ for $i = 0, 1$ be the unique elements such that

$$\phi_i(x_i, \mu_i) = (\mu_{i-1}, x_i'), \tag{3.2}$$

$$\phi_i(x_i', \nu_i) = (\nu_{i-1}, x_i''), \tag{3.3}$$

so that by (3.1),

$$\phi((x_1, x_2), \mu_2) = (\mu_0, (x_1', x_2')), \tag{3.4}$$

$$\phi((x_1', x_2'), \nu_2) = (\nu_0, (x_1'', x_2'')). \tag{3.5}$$

By definition, $\phi((x_1, x_2), \mu_2\nu_2)$ is calculated as follows: we write $\phi_2(x_2, \mu_2\nu_2) = (\lambda, y)$ and then write $\phi_1(x_1, \lambda) = (\lambda', y')$; then $\phi((x_1, x_2), \mu_2\nu_2) = (\lambda', (y', y))$. By Definition 3.1(4) for the k-morph X_2 and by equations (3.2) and (3.3), we have $\lambda = \mu_1\nu_1$ and $y = x_2''$. Now Definition 3.1(4) for the k-morph X_1, and equations (3.2) and (3.3) force $\lambda' = \mu_0\nu_0$ and $y' = x_1''$. That is, $\phi((x_1, x_2), \mu_2\nu_2) = (\mu_0\nu_0, (x_1'', x_2''))$. Combining this with equations (3.4) and (3.5) shows that $X_1 *_{\Lambda_1^0} X_2$ satisfies Definition 3.1(4), and therefore is a k-morph.

For the last statement, one checks that if $\theta_1 : X_1 \to X_1'$ and $\theta_2 : X_2 \to X_2'$ are isomorphisms, then $\theta_1 \times \theta_2$ is an isomorphism of $X_1 *_{\Lambda_1^0} X_2$ onto $X_1' *_{\Lambda_1^0} X_2'$. $\square$

Remarks 3.7.

(i) Let Λ_0, Λ_1 and Λ_2 be k-graphs, and let $q : \Lambda_2 \to \Lambda_1$ and $p : \Lambda_1 \to \Lambda_0$ be coverings. Then $p \circ q$ is a covering of Λ_0 by Λ_2. Furthermore, ${}_pX *_{\Lambda_1^0} {}_qX = \{(q(v), v) : v \in \Lambda_2^0\}$. One can easily check that $\theta : v \mapsto (q(v), v)$ determines an isomorphism of k-morphs ${}_{p\circ q}X \cong {}_pX *_{\Lambda_1^0} {}_qX$.

(ii) Fix coverings $p : \Gamma \to \Lambda_1$ and $q : \Gamma \to \Lambda_2$. Let ${}_pX$, X_q and ${}_pX_q$ be as in parts (ii), (iii) and (iv) respectively of Examples 3.3. Then we have ${}_pX_q \cong {}_pX *_{\Gamma^0} X_q$: the isomorphism θ is defined by $\theta(x) = (x, x)$.

(iii) Let Λ and Γ be k-graphs, and let X be a Λ–Γ morph. Then there are isomorphisms

$$I_\Lambda *_{\Lambda^0} X \cong X \cong X *_{\Gamma^0} I_\Gamma,$$

determined by $(r(x), x) \mapsto x$ and $(x, s(x)) \mapsto x$.

To state the next lemma, we describe the fibred product of n k-morphs.

Let $\Lambda_0, \Lambda_1, \dots, \Lambda_n$ be k-graphs, and let X_i be a Λ_{i-1}–Λ_i morph for $i = 1, \dots, n$. Let

$$X_1 *_{\Lambda_1^0} \cdots *_{\Lambda_{n-1}^0} X_n = \{(x_1, \cdots, x_n) \in X_1 \times \cdots \times X_n : s(x_{i-1}) = r(x_i) \text{ for } 1 \le i \le n\}.$$

Define structure maps r, s, ϕ associated to $X = X_1 *_{\Lambda_1^0} \cdots *_{\Lambda_{n-1}^0} X_n$ as follows. We set $r(x_1, \dots, x_n) = r(x_1)$ and $s(x_1, \dots, x_n) = s(x_n)$. Given $((x_1, \dots, x_n), \lambda_n) \in X *_{\Lambda_n^0} \Lambda_n$, let $\lambda_i \in \Lambda_i$ and $x_i' \in X_i$ be the unique elements such that

$$\phi_i(x_i, \lambda_i) = (\lambda_{i-1}, x_i') \text{ for } 1 \le i \le n.$$

Since each X_i is a k-morph, we have $s(x_i') = s(\lambda_i) = r(x_{i+1}')$ for $1 \le i \le n-1$. So we define ϕ by $\phi((x_1, \dots, x_n), \lambda_n) = (\lambda_0, (x_1', \dots, x_n'))$.

Lemma 3.8. *With notation as above, $X = X_1 *_{\Lambda_1^0} \cdots *_{\Lambda_{n-1}^0} X_n$ is a Λ_0–Λ_n morph. For $1 < m \le n$, there is an isomorphism*

$$(X_1 *_{\Lambda_1^0} \cdots *_{\Lambda_{m-2}^0} X_{m-1}) *_{\Lambda_{m-1}^0} (X_m *_{\Lambda_m^0} \cdots *_{\Lambda_{n-1}^0} X_n) \cong X$$

implemented by $\theta_{m,n-m}((x_1, \dots, x_{m-1}), (x_m, \dots, x_n)) = (x_1, \dots, x_n)$.

Proof. We first show that X is a k-morph: properties (1)–(3) of Definition 3.1 are clear from the definition of ϕ_X, and Definition 3.1(4) is established by iterating an argument similar to that of Proposition 3.6.

It is easy to see that the bijection $\theta_{m,n-m}$ determines an isomorphism. $\square$

Notation 3.9. Let X be a Λ endomorph. For $n \ge 2$, we write X^{*n} for the k-morph

$$X^{*n} := \overbrace{X *_{\Lambda^0} X *_{\Lambda^0} \cdots *_{\Lambda^0} X}^{n \text{ terms}}.$$

By X^{*1} we mean X, and by X^{*0} we mean I_Λ.

Theorem 3.10. *There is a category $\mathcal{M}_k$ such that:* $\mathrm{Obj}(\mathcal{M}_k)$ *is the class of k-graphs;* $\mathrm{Hom}_{\mathcal{M}_k}(\Gamma, \Lambda)$ *is the set of isomorphism classes of Λ–Γ morphs; the identity morphism associated to $\Lambda \in \mathrm{Obj}(\mathcal{M}_k)$ is $[I_\Lambda]$; and the composition map*

$$\mathrm{Hom}_{\mathcal{M}_k}(\Lambda_1, \Lambda_0) \times \mathrm{Hom}_{\mathcal{M}_k}(\Lambda_2, \Lambda_1) \to \mathrm{Hom}_{\mathcal{M}_k}(\Lambda_2, \Lambda_0)$$

is defined by $([X_1], [X_2]) \mapsto [X_1 *_{\Lambda_1^0} X_2]$.

Proof. Remark 3.7(iii) shows that the $[I_\Lambda]$ act as identity morphisms. Proposition 3.6 shows that the composition map is well-defined. Since Lemma 3.8 gives

$$(X_1 *_{\Lambda_1^0} X_2) *_{\Lambda_2^0} X_3 \cong X \cong X_1 *_{\Lambda_1^0} (X_2 *_{\Lambda_2^0} X_3)$$

whenever the expressions make sense, the composition map is also associative. $\square$

Remark 3.11. In light of the preceding theorem, it is natural to ask when the isomorphism class of a k-morph is an invertible morphism of $\mathcal{M}_k$.

By an abuse of terminology, we will say that a Λ–Γ morph X is *invertible* if $[X]$ is invertible in $\mathcal{M}_k$; that is, if there is a Γ–Λ morph Y such that $X *_{\Gamma^0} Y \cong I_\Lambda$ and $Y *_{\Lambda^0} X \cong I_\Gamma$. We claim that X is invertible if and only if X is isomorphic to $X(\alpha)$ for some isomorphism $\alpha : \Gamma \to \Lambda$.

To see this, we first show that the range and source maps on X are bijections. Certainly the range and source maps on X and Y are surjective. In particular, $s : X \to \Gamma^0$ and $r : Y \to \Gamma^0$ are surjective, and hence the projections from $X *_{\Gamma^0} Y$ to X and Y are surjective. As $X *_{\Gamma^0} Y$ is isomorphic to I_Λ, the range and source maps on $X *_{\Gamma^0} Y$ are bijections. Since the range map on $X *_{\Gamma^0} Y$ is defined by first projecting onto the first coordinate in $X *_{\Gamma^0} Y$ and then applying the range map on X, it follows that the range map on X is bijective. A similar argument shows that the source map on Y is bijective. Applying the same argument to $Y *_{\Lambda^0} X \cong I_\Lambda$ shows that the range map on Y and the source map on X are both bijective.

We may now define $\alpha : \Gamma \to \Lambda$ as follows. Given $\gamma \in \Gamma$, there is a unique $x \in X$ with $s(x) = r(\gamma)$. We then have $\phi(x, \gamma) = (\lambda, x')$ for some $\lambda \in \Lambda$ and $x' \in X$, and we define $\alpha(\gamma) = \lambda$. The properties of ϕ can be used to show that α is an isomorphism of k-graphs, and it is straightforward to check that $X \cong X(\alpha)$.

4. Linking graphs

In this section we define the notion of a linking graph Σ for a Λ–Γ morph X. Linking graphs generalise the $(k+1)$-graphs $\Lambda \overset{p}{\leftarrow} \Gamma$ built from covering maps p in [21].

We begin by showing how appropriate inclusions of k-graphs Λ and Γ in a $(k+1)$-graph Σ can be used to manufacture a Λ–Γ morph X. This will provide a template for linking graphs (see Definition 4.3).

Notation 4.1. Let Σ be a $(k+1)$-graph, and let Λ and Γ be k-graphs. Let $\iota : \mathbb{N}^k \to \mathbb{N}^{k+1}$ be the inclusion $\iota(n) = (n, 0)$. Suppose that $i : \Lambda \sqcup \Gamma \to \Sigma$ is an ι-quasimorphism (where $\sqcup$ denotes a disjoint union) such that i induces a k-graph isomorphism $\Lambda \sqcup \Gamma \cong \Sigma^\iota$. Suppose further that for all $\alpha \in \Sigma^{e_{k+1}}$, $s(\alpha) \in i(\Gamma^0)$ and $r(\alpha) \in i(\Lambda^0)$.

Let $X(\Lambda, \Gamma, \Sigma, i) = \Sigma^{e_{k+1}}$. Define structure maps on $X = X(\Lambda, \Gamma, \Sigma, i)$ as follows. For $\alpha \in X$, $r_X(\alpha)$ is the unique vertex $v \in \Lambda^0$ such that $i(v) = r(\alpha)$ and, similarly, $s_X(\alpha)$ is the unique $w \in \Gamma^0$ satisfying $i(w) = s(\alpha)$. For $(\alpha, \gamma) \in X *_{\Gamma^0} \Gamma$, the factorisation property in Σ ensures that $\alpha i(\gamma) = i(\lambda)\alpha'$ for some unique $\lambda \in \Lambda^{d(\gamma)}$ and $\alpha' \in X$, and we define $\phi_X(\alpha, \gamma) = (\lambda, \alpha')$.

Lemma 4.2. *With the notation just established, $X(\Lambda, \Gamma, \Sigma, i)$ is a Λ–Γ morph.*

Proof. Properties (1)–(3) of Definition 3.1 are clear because Σ is a $(k+1)$-graph and i is a quasimorphism. Property (4) follows from the associativity of composition in Σ. □

Definition 4.3. Let Λ, Γ be k-graphs and let X be a Λ–Γ morph. Suppose that Σ, ι and i are as in Notation 4.1. We say that the pair (Σ, i) is a *linking graph* for X if the Λ–Γ morph $X(\Lambda, \Gamma, \Sigma, i)$ is isomorphic to X.

In practice, we will just say that Σ is a linking graph for X, leaving i implicit.

Example 4.4. Let $p : \Gamma \to \Lambda$ be a covering of k-graphs, and let ${}_pX$ be the k-morph described in Examples 3.3(ii). Then the $(k+1)$-graph $\Lambda \overset{p}{\leftarrow} \Gamma$ of [21, Proposition 2.6] is a linking graph for ${}_pX$.

Note that if Σ is a linking graph for a k-morph, then necessarily $d_\Sigma(\Sigma) \subset \mathbb{N}^k \times \{0, 1\}$.

Proposition 4.5. *Let Λ, Γ be k-graphs and let X be a Λ–Γ morph. Then there exists a linking graph for X, and this linking graph is unique up to isomorphism.*

Proof. As a set, we define $\Sigma = \Lambda \sqcup \Gamma \sqcup (\Lambda *_{\Lambda^0} X)$. We endow Σ with the structure of a $(k+1)$-graph as follows. First set $\Sigma^0 = \Lambda^0 \sqcup \Gamma^0$. The restrictions of r_Σ and s_Σ to $\Lambda \sqcup \Gamma$ are inherited from the range and source maps on Λ and Γ. For $\sigma \in \Lambda \sqcup \Gamma$, set $d_\Sigma(\sigma) = (d(\sigma), 0)$. For $(\lambda, x) \in \Lambda *_{\Lambda^0} X$, let $r_\Sigma(\lambda, x) = r(\lambda)$, $s_\Sigma(\lambda, x) = s_X(x)$ and $d_\Sigma(\lambda, x) = (d(\lambda), 1)$. Now fix $\sigma_1, \sigma_2 \in \Sigma$ such that $s_\Sigma(\sigma_1) = r_\Sigma(\sigma_2)$. We must define the composition $\sigma_1\sigma_2$. There are three cases to consider.

(1) If $\sigma_1, \sigma_2 \in \Gamma \sqcup \Lambda$, their composition as elements of Σ is computed in $\Lambda \sqcup \Gamma$.
(2) If $\sigma_1 = \mu \in \Lambda$ and $\sigma_2 = (\nu, x) \in \Lambda *_{\Lambda^0} X$, we define $\sigma_1\sigma_2 = (\mu\nu, x) \in \Lambda *_{\Lambda^0} X$.
(3) If $\sigma_1 = (\mu, x) \in \Lambda *_{\Lambda^0} X$ and $\sigma_2 = \nu \in \Gamma$, we write $\phi(x, \nu) = (\nu', x')$ and define $\sigma_1\sigma_2 = (\mu\nu', x') \in \Lambda *_{\Lambda^0} X$.

Associativity follows from Definition 3.1(4).

It is straightforward to check that $d_\Sigma : \Sigma \to \mathbb{N}^{k+1}$ is a functor. To show that Σ has the factorisation property, fix $\sigma \in \Sigma$, and suppose $d_\Sigma(\sigma) = m + n$. We must show that there exist unique paths $\tau \in \Sigma^m$ and $\rho \in \Sigma^n$ with $\sigma = \tau\rho$. By definition of d_Σ, we have $d_\Sigma(\sigma)_{k+1} \le 1$. If $m_{k+1} = n_{k+1} = 0$, then $\sigma \in \Lambda \sqcup \Gamma$, and the factorisation property in $\Lambda \sqcup \Gamma$ produces the desired paths τ and ρ. If $m_{k+1} = 0$ and $n_{k+1} = 1$, then $\sigma \in \Lambda *_{\Lambda^0} X$, say $\sigma = (\lambda, x)$. By the factorisation property in Λ, there is a unique factorisation $\lambda = \mu\nu$ where $d(\mu) = (m_1, \ldots, m_k)$, and then $\tau = \mu$ and $\rho = (\nu, x)$ are the desired paths. Finally, suppose that $m_{k+1} = 1$ and $n_{k+1} = 0$, and let $p \in \mathbb{N}^k$ be the element such that $m = (p, 1)$. Again, write $\sigma = (\lambda, x) \in \Lambda *_{\Lambda^0} X$. Use the factorisation property in Λ to write $\lambda = \mu\nu$, where $d(\mu) = p$. We have $(\nu, x) \in \Lambda *_{\Lambda^0} X$, so $\phi(\nu, x) = (x', \nu')$ for some $x' \in X$ and $\nu' \in \Gamma$. One checks that $\tau = (\mu, x')$ and $\rho = \nu'$ are the desired paths.

For uniqueness, let Σ' be a linking graph for X. Then there are a quasimorphism $i' : \Lambda \sqcup \Gamma \to \Sigma'$ satisfying the conditions set forth in Notation 4.1 and an isomorphism $\psi : X \to X(\Lambda, \Gamma, \Sigma', i')$. We define $\tilde\psi : \Sigma \to \Sigma'$ as follows. For $\sigma \in \Lambda \sqcup \Gamma \subset \Sigma$ we set $\tilde\psi(\sigma) = i'(\sigma)$, and for $\sigma = (\lambda, x) \in \Lambda *_{\Lambda^0} X$ we set $\tilde\psi(\lambda, x) = i'(\lambda)\psi(x)$. One then checks that $\tilde\psi$ is an isomorphism. □

Remark 4.6. If X and X' are isomorphic k-morphs, then any linking graph for X is by definition also a linking graph for X'. Hence Proposition 4.5 implies that, up to isomorphism, there is a unique linking graph for each isomorphism class of k-morphs.

5. Γ-systems and Γ-bundles

In this section we describe a generalisation, based on k-morphs and linking graphs, of the $(k + 1)$-graphs $\varprojlim(\Lambda_n; p_n)$ constructed in [21] from a sequence of coverings $p_n : \Lambda_{n+1} \to \Lambda_n$ of k-graphs. The idea is that the sequence $(p_n)_{n=0}^\infty$ of coverings determines a consistent collection of k-morphs

$$X_{(m,n)} = {}_{p_m}X *_{\Lambda^0_{m+1}} \cdots *_{\Lambda^0_{n-1}} {}_{p_{n-1}}X$$

indexed by pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$ such that $m \le n$. Recall from [18] that such pairs are morphisms in the 1-graph Ω_1 whose vertices are identified with $\mathbb{N}$. That is, a sequence of coverings as in [21] gives rise to a consistent collection of k-morphs indexed by a 1-graph.

We generalise this situation by replacing Ω_1 with an l-graph Γ. Given a consistent collection (which we call a Γ-system) of k-morphs indexed by paths in Γ, we construct a $(k + l)$-graph Σ which we call a Γ-bundle. As we shall see, each Γ-system X determines a functor F_X from Γ to $\mathcal{M}_k$. Naïvely, one might expect to be able to recover Σ from F_X, thus obviating the need to discuss Γ-systems at all. It turns out, however, that this is not the case: not only do there exist functors from which we may build two nonisomorphic $(k+l)$-graphs, but there also exist functors from which no $(k+l)$-graph may be built (see Examples 5.15).

Definition 5.1. Let Γ be an l-graph, and let $k \geq 0$. Fix

- for each vertex $v \in \Gamma^0$ a k-graph Λ_v,
- for each $\gamma \in \Gamma$ a $\Lambda_{r(\gamma)}$–$\Lambda_{s(\gamma)}$ morph X_γ, and
- for each composable pair α, β in Γ an isomorphism $\theta_{\alpha,\beta} : X_\alpha *_{\Lambda^0_{s(\alpha)}} X_\beta \to X_{\alpha\beta}$.

Suppose that Λ, X and θ have the following properties:

(1) for each $v \in \Gamma^0$, $X_v = I_{\Lambda_v}$,
(2) for each $\gamma \in \Gamma$, the isomorphisms $\theta_{r(\gamma),\gamma}$ and $\theta_{\gamma,s(\gamma)}$ are those of Remark 3.7(iii), and
(3) for each composable triple $\alpha, \beta, \gamma \in \Gamma$, the following diagram commutes:

$$\begin{tikzcd}
X_\alpha *_{\Lambda^0_{s(\alpha)}} X_\beta *_{\Lambda^0_{s(\beta)}} X_\gamma \arrow[r, "\theta_{\alpha,\beta}\times \mathrm{id}_{X_\gamma}"] \arrow[d, "\mathrm{id}_{X_\alpha}\times\theta_{\beta,\gamma}"'] & X_{\alpha\beta} *_{\Lambda^0_{s(\beta)}} X_\gamma \arrow[d, "\theta_{\alpha\beta,\gamma}"'] \\
X_\alpha *_{\Lambda^0_{s(\alpha)}} X_{\beta\gamma} \arrow[r, "\theta_{\alpha,\beta\gamma}"] & X_{\alpha\beta\gamma}.
\end{tikzcd}$$

Then we say that X *is a* Γ*-system of* k*-morphs with data* Λ, θ or just that X is a Γ-system, in which case Λ and θ are implicit.

Remark 5.2. Given a Γ-system X of k-morphs, there is a functor F_X from Γ to $\mathcal{M}_k$ determined by $F_X(\gamma) = [X_\gamma]$; in particular, the object map satisfies $F^0_X(v) = \Lambda_v$. However, the Γ-system X contains more information than the functor F_X: the Γ-system picks out a concrete representative X_γ of each isomorphism class $F_X(\gamma)$ and a compatible system of concrete isomorphisms $X_\alpha *_{\Lambda^0_{s(\alpha)}} X_\beta \cong X_{\alpha\beta}$ implementing the compositions $F_X(\alpha)F_X(\beta) = F_X(\alpha\beta)$. We show that this distinction is important in Examples 5.15.

Examples 5.3.

(i) Fix k-graphs Λ and Γ and a Λ–Γ morph X. Let E be the unique 1-graph with a single edge e and two vertices $r(e)$ and $s(e)$. Setting $\Lambda_{r(e)} = \Lambda$, $\Lambda_{s(e)} = \Gamma$, $X_{r(e)} = I_\Lambda$, $X_{s(e)} = I_\Gamma$, and $X_e = X$, we obtain an E-system of k-morphs.
(ii) Fix a k-graph Λ and a Λ endomorph X. Recall from [18, Examples 1.7(iii)] that T_1 denotes the unique 1-graph with a single edge e and a single vertex $v = r(e) = s(e)$. The degree functor is an isomorphism of T_1 onto $\mathbb{N}$, and we use it to identify the two. Let $\Lambda_0 = \Lambda$, and for $n \geq 0$ let $X_n = X^{*n}$ as in Notation 3.9. Then the isomorphisms $\theta_{m,n} : X_m *_{\Lambda^0} X_n \to X_{m+n}$ of Lemma 3.8 give this collection of k-morphs the structure of a T_1-system.
(iii) Fix an l-graph Γ, a countable discrete group G, and a functor $c : \Gamma \to G$. For each $v \in \Gamma^0$, let Λ_v be the 0-graph such that $\Lambda^0_v = G$. For $\gamma \in \Gamma$, there is an automorphism $\alpha_{c(\gamma)} : \Lambda_{s(\gamma)} \to \Lambda_{r(\gamma)}$ determined by $\alpha_{c(\gamma)}(g) = c(\gamma)g$ for $g \in \Lambda^0_{s(\gamma)}$. For each $\gamma \in \Gamma$, let X_γ be the 0-morph $X(\alpha_{c(\gamma)})$ arising from $\alpha_{c(\gamma)}$ as in Examples 3.3(i). The isomorphisms $X(\alpha_{c(\gamma\gamma')}) \cong X(\alpha_{c(\gamma)}) *_G X(\alpha_{c(\gamma')})$

described in Remark 3.7(i) give this collection of 0-morphs the structure of a Γ-system, denoted $X(c)$, of 0-morphs.

(iv) Let Cov_k denote the category whose objects are k-graphs and whose morphisms are k-graph coverings, and let Γ be an l-graph. Let F be a functor from Γ to Cov_k. For each $v \in \Gamma^0$ let Λ_v be the k-graph $F^0(v)$, and for each $\gamma \in \Gamma$ let $X_\gamma = {}_{F(\gamma)}X$. Then the isomorphisms $\theta_{\gamma,\gamma'} : X_\gamma *_{\Lambda^0_{s(\gamma)}} X_{\gamma'} \to X_{\gamma\gamma'}$ of Remark 3.7(i) give this collection of k-morphs the structure of a Γ-system. For instance, the covering systems of [21] give rise to Ω_1-systems.

(v) Let Σ be a 2-graph satisfying the hypotheses of [28, Theorem 3.1]. That is, each vertex lies on a unique simple cycle in the graph whose edges are Σ^{e_2}, and the graph with edge-set Σ^{e_1} contains no cycles. The simple cycles in edges in Σ^{e_2} determine an equivalence relation on vertices in Σ by $v \sim w$ if and only if $v\Sigma^{ne_2}w \neq \emptyset$ for some $n \in \mathbb{N}$. We write $[v]$ for the equivalence class of $v \in \Sigma^0$ under this relation. There is a 1-graph Γ with

$$\Gamma^0 = \{[v] : v \in \Sigma^0\} \quad \text{and} \quad \Gamma^1 = \{([v],[w]) : [v],[w] \in \Sigma^0/\sim, [v]\Sigma^{e_1}[w] \neq \emptyset\},$$

where $r([v],[w]) = [v]$ and $s([v],[w]) = [w]$. For $[v] \in \Gamma^0$, we denote by $\Lambda_{[v]}$ the sub-1-graph of Σ such that $\Lambda^1_{[v]} = [v]\Sigma^{e_2}[v]$; so each $\Lambda_{[v]}$ is the path-category of a simple cycle. For each path $\gamma \in \Gamma$ we define $X_\gamma = r(\gamma)\Sigma^{d(\gamma)e_1}s(\gamma)$, and we endow it with the range and source maps $r : X_\gamma \to \Lambda^0_{r(\gamma)}$ and $s : X_\gamma \to \Lambda^0_{s(\gamma)}$ inherited from Σ, and with the map $\phi : X_\gamma *_{s(\gamma)} \Lambda_{s(\gamma)} \to \Lambda_{r(\gamma)} *_{r(\gamma)} X_\gamma$ determined by the factorisation property in Σ. Then each X_γ is a $\Lambda_{r(\gamma)}$–$\Lambda_{s(\gamma)}$ morph, composition in Σ defines isomorphisms $\theta_{\alpha,\beta} : X_\alpha *_{s(\alpha)} X_\beta \to X_{\alpha\beta}$ and from this structure we obtain a Γ-system of 1-morphs which we denote by $X(\Sigma)$. If Σ is a rank-2 Bratteli diagram as in [28, Section 4], then Γ is the path-category of a Bratteli diagram.

The next step is to show how to assemble a $(k+l)$-graph from the data contained in a Γ-system of k-graphs. This construction should simultaneously generalise the linking graphs of the previous section and the construction of the skew-product of a k-graph by a group.

The model for this construction is the following prototypical Γ-system which generalises the construction of Examples 5.3(v).

Notation 5.4. Let Σ be a $(k+l)$-graph, and let Γ be an l-graph. Let $\pi : \mathbb{N}^{k+l} \to \mathbb{N}^l$ denote the projection onto the last l coordinates; that is, $\pi(m,n) = n$. Suppose $f : \Sigma \to \Gamma$ is a π-quasimorphism which restricts to a surjection of Σ^0 onto Γ^0. Let $\imath : \mathbb{N}^k \to \mathbb{N}^{(k+l)}$ be the inclusion $\imath(m) = (m,0)$ and let $\jmath : \mathbb{N}^l \to \mathbb{N}^{k+l}$ be the inclusion $\jmath(n) = (0,n)$. For each $v \in \Gamma^0$, we define $\Lambda_v = f^{-1}(v)$ which is a sub-k-graph of $\Sigma^\imath$. For $\gamma \in \Gamma$, we define $X(f)_\gamma = f^{-1}(\gamma) \cap \Sigma^\jmath$ (note that for each $x \in X(f)_\gamma$, $d_\Sigma(x) = (0, d(\gamma))$). Each $X(f)_\gamma$ becomes a $\Lambda_{r(\gamma)}$–$\Lambda_{s(\gamma)}$ morph under the range, source and factorisation maps inherited from Σ. Composition in Σ defines maps

$$\theta_{\gamma_1,\gamma_2} : X(f)_{\gamma_1} *_{\Lambda^0_{s(\gamma_1)}} X(f)_{\gamma_2} \to X(f)_{\gamma_1\gamma_2}$$

for each composable pair (γ_1,γ_2) in Γ. Moreover, under these structure maps, $X(f)$ becomes a Γ-system of k-graphs: conditions (1) and (2) of Definition 5.1 are satisfied by definition, and condition (3) follows from associativity of composition in Σ.

We will show that every Γ-system is isomorphic to one of the form $X(f)$ for some π-quasimorphism $f : \Sigma \to \Gamma$. We must first make clear what we mean by an isomorphism of Γ-systems.

Definition 5.5. Let Γ be a l-graph. Suppose that (Λ, X, θ) and (Ξ, Y, ψ) are Γ-systems of k-graphs. An isomorphism from X to Y consists of k-graph isomorphisms $h^0_v : \Lambda_v \to \Xi_v$ and bijections $h_\gamma : X_\gamma \to Y_\gamma$ which intertwine all the structure maps. That is:

(1) for each $\gamma \in \Gamma$, $s \circ h_\gamma = h^0_{s(\gamma)} \circ s$, $r \circ h_\gamma = h^0_{r(\gamma)} \circ r$ and
$$(h^0_{r(\gamma)} \times h_\gamma) \circ \phi_{X_\gamma} = \phi_{Y_\gamma} \circ (h_\gamma \times h^0_{s(\gamma)}), \text{ and}$$

(2) for every composable pair $\alpha, \beta \in \Gamma$,
$$h_{\alpha\beta} \circ \theta_{\alpha,\beta} = \psi_{\alpha,\beta} \circ (h_\alpha \times h_\beta).$$

We can now say what we mean by a Γ-bundle for a Γ-system.

Definition 5.6. Let Γ be an l-graph, and let X be a Γ-system of k-graphs. Let $\pi : \mathbb{N}^{k+l} \to \mathbb{N}^l$ be the projection onto the last l coordinates. A Γ-*bundle* for X is a $(k+l)$-graph Σ endowed with a π-quasimorphism $f : \Sigma \to \Gamma$ which restricts to a surjection of Σ^0 onto Γ^0 such that the Γ-system $X(f)$ of Notation 5.4 is isomorphic to X. We call the π-quasimorphism f the *bundle map* for the Γ-system.

Remark 5.7. Our formulation of Γ-bundles emphasises that the construction of Notation 5.4 is prototypical: given a π-quasimorphism $f : \Sigma \to \Gamma$, the $(k+l)$-graph Σ is automatically a Γ-bundle (with bundle map f) for the resulting Γ-system $X(f)$.

Remark 5.8. Let X be a Γ-system of k-graphs, and suppose that Σ is a Γ-bundle for X. The maps h^0_v and h_γ of Definition 5.5 determine injective quasimorphisms $h^0_v : \Lambda_v \to \Sigma$ for each $v \in \Gamma^0$, and inclusions $h_\gamma : X_\gamma \to \Sigma^{\jmath(d(\gamma))}$. Moreover, the factorisation property implies that every element of $\sigma \in \Sigma$ can be expressed as $\sigma = h_{f(\sigma)}(x) h^0_{s(f(\sigma))}(\lambda)$ for unique elements, $x \in X_{f(\sigma)}$ and $\lambda \in \Lambda_{s(f(\sigma))}$.

We now show that every Γ-system admits a Γ-bundle.

Theorem 5.9. *Let Γ be an l-graph and let X be a Γ-system of k-morphs. Then there exists a Γ-bundle Σ for X. Moreover, Σ is unique up to isomorphism. That is, if Ψ is another Γ-bundle for X, then there is an isomorphism $\Sigma \cong \Psi$ of $(k+l)$-graphs which intertwines the bundle maps on Σ and Ψ.*

Proof. Throughout this proof, for $\gamma \in \Gamma$ we will write ϕ_γ for the isomorphism $\phi_{X_\gamma} : X_\gamma *_{\Lambda^0_{s(\gamma)}} \Lambda_{s(\gamma)} \to \Lambda_{r(\gamma)} *_{\Lambda^0_{r(\gamma)}} X_\gamma$ associated with the k-morph X_γ.

We must first construct a Γ-bundle for X, and then show that it is unique. We begin by constructing the $(k+l)$-graph Σ. Define a set Σ by
$$\Sigma = \{(\lambda, \gamma, x) : \gamma \in \Gamma, \lambda \in \Lambda_{r(\gamma)}, x \in X_\gamma\}.$$
Define $d : \Sigma \to \mathbb{N}^{k+l}$ by $d(\lambda, \gamma, x) = (d(\lambda), d(\gamma))$. We write Σ^p for the set $d^{-1}(p) \subset \Sigma$ for each $p \in \mathbb{N}^{k+l}$. By condition (1) of Definition 5.1, for $v \in \Gamma^0$, $X_v = I(\Lambda_v)$ is

equal as a set to Λ_v^0. Hence $\Sigma^0 = \{(u,v,u) : v \in \Gamma^0, u \in \Lambda_v^0\}$. To simplify notation and to help distinguish vertices from arbitrary paths, we will discard the redundant u and write (u,v) for the element (u,v,u) of Σ^0.

Define $r, s : \Sigma \to \Sigma^0$ by

$$r(\lambda,\gamma,x) = (r(\lambda), r(\gamma)) \quad \text{and} \quad s(\lambda,\gamma,x) = (s(x), s(\gamma)).$$

Suppose that $s(\lambda_0,\gamma_0,x_0) = r(\lambda_1,\gamma_1,x_1)$, and define λ_1' and x_0' by $\phi_{\gamma_0}(x_0,\lambda_1) = (\lambda_1', x_0')$. We define

$$(\lambda_0,\gamma_0,x_0)(\lambda_1,\gamma_1,x_1) = (\lambda_0\lambda_1', \gamma_0\gamma_1, \theta_{\gamma_0,\gamma_1}(x_0', x_1)).$$

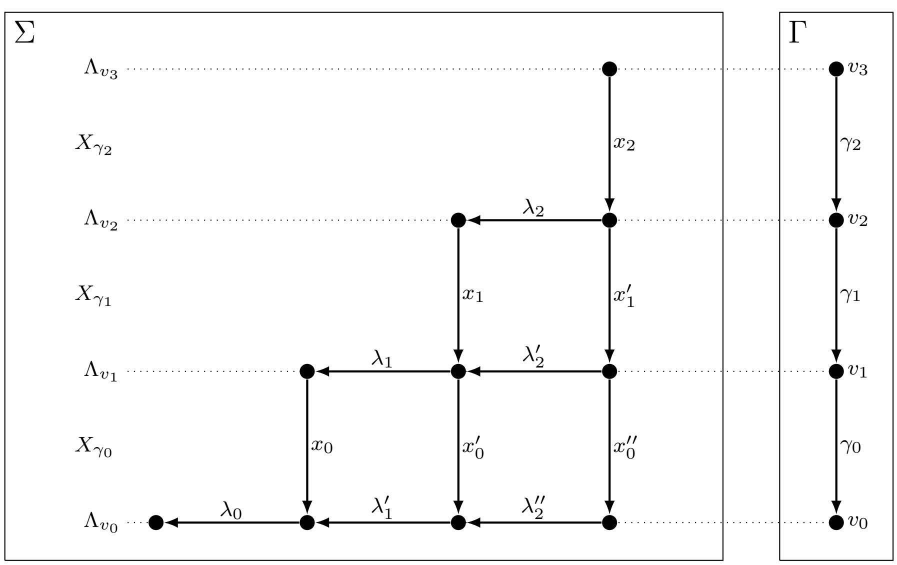
It is easy to check that the triple on the right lies in Σ and that

$$d(\lambda_0\lambda_1', \gamma_0\gamma_1, \theta_{\gamma_0,\gamma_1}(x_0', x_1)) = d(\lambda_0,\gamma_0,x_0) + d(\lambda_1,\gamma_1,x_1).$$

We aim to show that Σ is a $(k+l)$-graph when endowed with these structure maps. We must check that the composition we have defined is associative and that under this composition, (Σ, d) satisfies the factorisation property. To see that composition is associative, fix a composable triple $(\lambda_0,\gamma_0,x_0), (\lambda_1,\gamma_1,x_1), (\lambda_2,\gamma_2,x_2)$ of elements of Σ. Let $v_i = r(\gamma_i)$ and $v_{i+1} = s(\gamma_i)$ for $i = 0,1,2$. Define $x_0', x_1', x_1'', \lambda_1', \lambda_2'$ and λ_2'' by

$$\begin{aligned} \phi_{\gamma_0}(x_0,\lambda_1) &= (\lambda_1', x_0'), \\ \phi_{\gamma_1}(x_1,\lambda_2) &= (\lambda_2', x_1'), \quad \text{and} \\ \phi_{\gamma_0}(x_0',\lambda_2') &= (\lambda_2'', x_0''). \end{aligned}$$

We may visualise the situation as follows:



To prove associativity, we must show that the products

$$\big((\lambda_0,\gamma_0,x_0)(\lambda_1,\gamma_1,x_1)\big)(\lambda_2,\gamma_2,x_2) \quad \text{and} \quad (\lambda_0,\gamma_0,x_0)\big((\lambda_1,\gamma_1,x_1)(\lambda_2,\gamma_2,x_2)\big)$$

coincide. We begin by calculating the first of these. First, notice that the pair

$((x_0', x_1), \lambda_2)$ belongs to $(X_{\gamma_0} *_{\Lambda^0_{v_1}} X_{\gamma_1}) *_{\Lambda^0_{v_2}} \Lambda_{v_2}$. We have

$$\phi_{X_{\gamma_0} *_{\Lambda^0_{v_1}} X_{\gamma_1}}((x_0', x_1), \lambda_2) = (\lambda_2'', (x_0'', x_1')) \tag{5.1}$$

by definition of $\phi_{X_{\gamma_0} *_{\Lambda^0_{v_1}} X_{\gamma_1}}$ and of the elements λ_2'', x_0'' and x_1' given above. Since $\theta_{\gamma_0,\gamma_1} : X_{\gamma_0} *_{\Lambda^0_{v_1}} X_{\gamma_1} \to X_{\gamma_0\gamma_1}$ is an isomorphism of k-morphs,

$$\begin{aligned}\phi_{\gamma_0\gamma_1}(\theta_{\gamma_0,\gamma_1}(x_0', x_1), \lambda_2) &= (\mathrm{id}_{\Lambda_{v_0}} \times \theta_{\gamma_0,\gamma_1})(\phi_{X_{\gamma_0} *_{\Lambda^0_{v_1}} X_{\gamma_1}}((x_0', x_1), \lambda_2)) \\ &= (\lambda_2'', \theta_{\gamma_0,\gamma_1}(x_0'', x_1')) \qquad (5.2)\end{aligned}$$

by (5.1). Therefore,

$$\begin{aligned}&\big((\lambda_0, \gamma_0, x_0)(\lambda_1, \gamma_1, x_1)\big)(\lambda_2, \gamma_2, x_2) \\ &\quad= (\lambda_0\lambda_1', \gamma_0\gamma_0, \theta_{\gamma_0,\gamma_1}(x_0', x_1))(\lambda_2, \gamma_2, x_2) \\ &\quad= \big((\lambda_0\lambda_1')\lambda_2'',\, (\gamma_0\gamma_1)\gamma_2,\, \theta_{\gamma_0\gamma_1,\gamma_2}(\theta_{\gamma_0,\gamma_1}(x_0'', x_1'), x_2)\big),\end{aligned}$$

where the second step uses (5.2). Similar calculations show that

$$\begin{aligned}&(\lambda_0, \gamma_0, x_0)\big((\lambda_1, \gamma_1, x_1)(\lambda_2, \gamma_2, x_2)\big) \\ &\quad= \big(\lambda_0(\lambda_1'\lambda_2''), \gamma_0(\gamma_1\gamma_2), \theta_{\gamma_0,\gamma_1\gamma_2}(x_0'', \theta_{\gamma_1,\gamma_2}(x_1', x_2))\big).\end{aligned}$$

Associativity in Σ now follows from associativity in Λ_{v_0} and Γ and from property (3) of Definition 5.1.

To establish the factorisation property in Σ, fix $m, p \in \mathbb{N}^k$ and $n, q \in \mathbb{N}^l$ and an element $(\lambda, \gamma, x) \in \Sigma^{(m+p,n+q)}$. By the factorisation properties in $\Lambda_{r(\gamma)}$ and in Γ, there are unique factorisations $\lambda = \lambda_0\lambda_1$ and $\gamma = \gamma_0\gamma_1$ where $d(\lambda_0) = m$, $d(\lambda_1) = p$, $d(\gamma_0) = n$ and $d(\gamma_1) = q$. Since $\theta_{\gamma_0,\gamma_1}$ is an isomorphism, there are unique elements $x_0 \in X_{\gamma_0}$ and $x_1 \in X_{\gamma_1}$ such that $x = \theta_{\gamma_0,\gamma_1}(x_0, x_1)$. As ϕ_{γ_0} is also a bijection, there are unique elements $\lambda_1' \in \Lambda_{s(\gamma_0)}$ and $x_0' \in X_{\gamma_0}$ such that $\phi_{\gamma_0}(x_0', \lambda_1') = (\lambda_1, x_0)$. We then have $d(\lambda_0, \gamma_0, x_0') = (m, p)$ and $d(\lambda_1', \gamma_1, x_1) = (n, q)$, and

$$(\lambda, \gamma, x) = (\lambda_0, \gamma_0, x_0')(\lambda_1', \gamma_1, x_1)$$

by definition. Uniqueness is clear. We have now established that Σ is a $(k+l)$-graph.

The formula $f(\lambda, \gamma, x) = \gamma$ defines a π-quasimorphism from Σ onto Γ. Let $\imath : \mathbb{N}^k \to \mathbb{N}^{k+l}$ and $\jmath : \mathbb{N}^l \to \mathbb{N}^{k+l}$ be as in Notation 5.4. For each $v \in \Gamma^0$, $f^{-1}(v) = \{(\lambda, v, s(\lambda)) : \lambda \in \Lambda_v\}$, and $h^0_v : f^{-1}(v) \to \Lambda_v$ defined by $h^0_v(\lambda, v, s(\lambda)) = \lambda$ is an isomorphism of k-graphs. For each $\gamma \in \Gamma$, $f^{-1}(\gamma) \cap \Sigma^\jmath = \{(r(x), \gamma, x) : x \in X_\gamma\}$, and $h_\gamma : f^{-1}(\gamma) \cap \Sigma^\jmath \to X_\gamma$ defined by $(r(x), \gamma, x) = x$ is a bijection. Routine calculations show that these maps satisfy conditions (1) and (2) of Definition 5.5. Hence Σ is a Γ-bundle for X as claimed when equipped with the bundle map $f : \Sigma \to \Gamma$.

It remains to establish the uniqueness of the Γ-bundle. Suppose that Ψ is another Γ-bundle for X with bundle map g. So we have isomorphisms $h^0_v : \Lambda_v \to g^{-1}(v)$ for each $v \in \Gamma^0$ and bijections $h_\gamma : X_\gamma \to g^{-1}(\gamma) \cap \Psi^\jmath$ determining an isomorphism of Γ-systems. Define $H : \Sigma \to \Psi$ by

$$H(\lambda, \gamma, x) = h^0_{r(\gamma)}(\lambda)h_\gamma(x) \quad \text{ for all } (\lambda, x, \gamma) \in \Sigma.$$

The factorisation property in Ψ ensures that each $\psi \in \Psi$ with $d(\psi) = (m, n)$ can be written uniquely as $\psi = \psi_m\psi_n$ where $d(\psi_m) = (m, 0)$ and $d(\psi_n) = (0, n)$. We then have $\psi_m \in g^{-1}(g(r(\psi)))$ and $\psi_n \in g^{-1}(g(\psi)) \cap \Psi^{\jmath}$. Bijectivity of H follows from this. It is clear that H respects the degree map and intertwines the range and source maps. A straightforward calculation shows that it also respects composition, and hence it is an isomorphism of $(k+l)$-graphs. $\square$

Remark 5.10. Theorem 5.9 implies that it makes sense to talk about *the* Γ-bundle for a Γ-system X, and we shall frequently do so. Unless specified otherwise, the bundle is denoted Σ and the bundle map is denoted f.

Examples 5.11.

(i) Let Λ and Γ be k-graphs, and let X be a Λ–Γ morph. As noted in Examples 5.3(i), this corresponds to an E-system of k-graphs where E is the 1-graph with a single edge e and two vertices $r(e)$ and $s(e)$. An E-bundle for this E-system amounts to a linking graph for X.

(ii) Let Λ be a k-graph and X be a Λ endomorph. As noted in Examples 5.3(ii), this corresponds to a T_1-system of k-graphs. We shall denote the T_1-bundle for this system by $\Lambda \times_X \mathbb{N}$. We call $\Lambda \times_X \mathbb{N}$ the *endomorph skew-graph for* X. Every $(k+1)$-graph arises this way: given a $(k+1)$-graph Σ, with $\Lambda = \Sigma^{\iota}$ and $X = \Sigma^{e_{k+1}}$ as in Examples 3.3(vi), Σ is isomorphic to $\Lambda \times_X \mathbb{N}$.

(iii) Let Λ be a k-graph, and let α be an automorphism of Λ. Let $X = X(\alpha)$ be the associated Λ endomorph. In this case, the endomorph skew-graph $\Lambda \times_X \mathbb{N}$ discussed in the preceding example is the same as the crossed-product $(k+1)$-graph $\Lambda \times_\alpha \mathbb{Z}$ of [15]. More generally, let T_l denote the l-graph isomorphic to $\mathbb{N}^l$, and suppose that α is an action of $\mathbb{Z}^l$ by automorphisms of Λ. Let $X_n = X(\alpha_n)$ for each $n \in T_l$, and let $\theta_{m,n} : X_m *_{\Lambda^0} X_n \to X_{m+n}$ be the isomorphism of Remark 3.7(i). Then the X_n form a T_l-system $X(\alpha)$, and the crossed product $(k+l)$-graph $\Lambda \times_\alpha \mathbb{Z}^l$ described in [15] is a T_l-bundle for $X(\alpha)$.

(iv) Fix a functor c from an l-graph Γ to a group G, and construct from this a Γ-system $X(c)$ of 0-morphs as in Examples 5.3(iii). Then the skew-product k-graph $G \times_c \Gamma$ of [18] is a Γ-bundle for the Γ-system $X(c)$; the bundle map is the functor $f(g, \gamma) := \gamma$.

(v) Let Σ be a 2-graph satisfying the hypotheses of [28, Theorem 3.1], and let Γ and X be as in Examples 5.3(v). Then Σ is a Γ-bundle for X and the bundle map is the natural quotient map $\Lambda \to \Sigma$. In particular, we may regard a rank-2 Bratteli diagram Σ as a bundle of cycle-graphs over the path-category of a conventional Bratteli diagram.

Our Γ-bundle construction is quite general: the next proposition shows that every k-graph Λ is a T_k-bundle for some T_k-system. It is a strong point of our formulation of Γ-bundles that the proof of this result is almost trivial (see Remark 5.7).

Proposition 5.12. *Let Λ be a k-graph. Let T_k denote the k-graph isomorphic to $\mathbb{N}^k$. Then the degree map on Λ determines an* id_k*-quasimorphism (also denoted d) from Λ onto T_k. In particular Λ is a T_k-bundle (with bundle map d) for the T_k-system $X(d)$ of* 0*-morphs described in Notation* 5.4.

Remark 5.13. If $p : \Gamma \to \Lambda$ is a covering of k-graphs, then p is an id_k-quasimorphism. Hence p induces a Λ-system of 0-morphs as in Notation 5.4. Moreover, these 0-morphs are all invertible (see Remark 3.11).

Conversely, given a k-graph Λ and a Λ-system of invertible 0-morphs, the bundle map associated to a Λ-bundle for the system is a covering map.

We conclude the section by investigating the relationship between Γ-systems and functors from Γ into $\mathcal{M}_k$. If Γ is a 1-graph, the two are essentially the same thing.

Proposition 5.14. *Let Γ be a 1-graph, and let $F : \Gamma \to \mathcal{M}_k$ be a functor. Then there is, up to isomorphism, exactly one Γ-system X of k-graphs such that $F(\gamma) = [X_\gamma]$ for all $\gamma \in \Gamma$.*

Proof. For each $v \in \Gamma^0$, set $\Lambda_v = F^0(v)$ and $X_v = I(\Lambda_v)$. For each edge $e \in \Gamma^1$, fix a k-morph X_e such that $[X_e] = F(e)$. For $n \geq 2$ and a path $\alpha = e_1 \cdots e_n \in \Gamma^n$, let $X_\alpha = X_{e_1} *_{\Lambda^0_{s(e_1)}} X_{e_2} *_{\Lambda^0_{s(e_2)}} \cdots *_{\Lambda^0_{s(e_{n-1})}} X_{e_n}$ as in Lemma 3.8, and for composable α, β, let $\theta_{\alpha,\beta}$ be the isomorphism described in the same lemma. It is easy to verify that this data determines a Γ-system of k-morphs which induces the functor F.

Now suppose that Y is another Γ system of k-morphs (with data Λ, ψ) which induces F. Let $\psi_{\alpha,\beta} : Y_\alpha *_{\Lambda^0_{s(\alpha)}} Y_\beta \to Y_{\alpha\beta}$ denote the isomorphisms in the Γ-system Y. In particular, each Y_α is a $\Lambda_{r(\alpha)}$–$\Lambda_{s(\alpha)}$ morph which is isomorphic to X_α. For $v \in \Gamma^0$, let h^0_v denote the identity map on Λ_v. For each $e \in \Gamma^1$, we may fix an isomorphism $h_e : X_e \to Y_e$. By induction on n, for $\alpha \in \Gamma^n$ and $f \in \Gamma^1$ with $s(\alpha) = r(f)$, we may define an isomorphism $h_{\alpha f}$ from $X_{\alpha f}$ to $Y_{\alpha f}$ by

$$X_{\alpha f} \cong X_\alpha *_{\Lambda^0_{s(\alpha)}} X_f \xrightarrow{h_\alpha \times h_f} Y_\alpha *_{\Lambda^0_{s(\alpha)}} Y_f \xrightarrow{\psi_{\alpha,h_f}} Y_{\alpha f}.$$

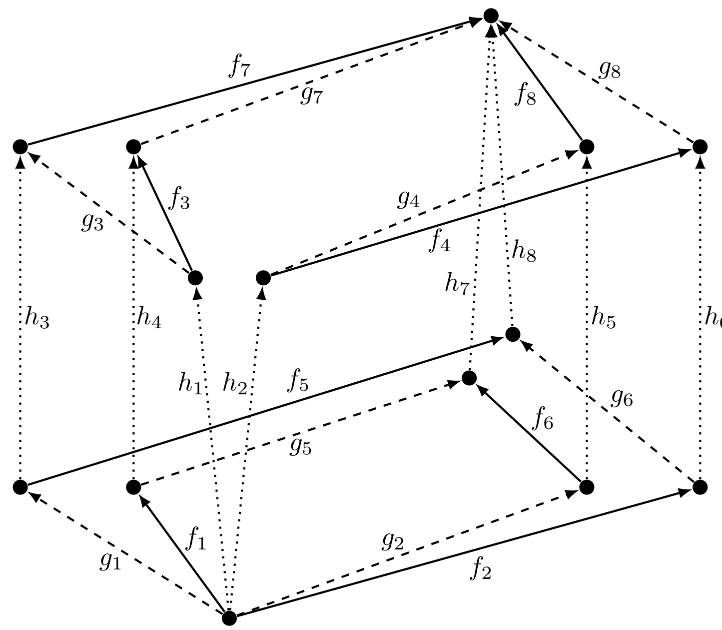
As the isomorphisms h_α are defined using the structure maps in X and Y, it is easy to check that they determine an isomorphism of Γ-systems. $\square$

Examples 5.15. We cannot expect an analogue of Proposition 5.14 to hold if Γ is an l-graph with $l > 1$, as the following two examples show.

(i) Let $\Gamma = T_2$ (the 2-graph isomorphic to $\mathbb{N}^2$), and let Λ be the 0-graph consisting of a single vertex v. Each finite set is a Λ endomorph when endowed with the only possible structure maps. In particular, each multiplicative map $x : \mathbb{N}^2 \to \mathbb{N}$ determines a functor from T_2 to $\mathcal{M}_0$: the image of $n \in \mathbb{N}^2$ is (the isomorphism class of) the Λ endomorph X_n with $x(n)$ elements.

Example 6.1 of [18] describes two nonisomorphic 2-graphs Λ and Λ', each with a single vertex, two edges of degree e_1 and two edges of degree e_2. As in Notation 5.4, Λ and Λ' determine T_2-systems X and X' such that Λ is a T_2-bundle for X and Λ' is a T_2-bundle for X'. By Theorem 5.9, X and X' are nonisomorphic. However, X and X' determine the same functor $F : T_2 \to \mathcal{M}_0$ with $F^0(0) = \Lambda$, namely the one corresponding to the multiplicative map $x : \mathbb{N}^2 \to \mathbb{N}$ given by $x(n) = 2^{n_1+n_2}$.

(ii) The following example is due to Jack Spielberg [39]. We thank Jack for allowing us to reproduce it here. The following diagram represents a 3-coloured graph where the edges have colours c_1, c_2 and c_3; we draw c_1-coloured edges as solid lines, c_2-coloured edges as dashed lines, and c_3-coloured edges as dotted lines.



For distinct $1 \leq i, j \leq 3$, there is a unique range- and source-preserving bijection $\theta_{i,j}$ between c_ic_j-coloured paths and c_jc_i-coloured paths. For example, the only c_2c_1-coloured path with the same range and source as the c_1c_2-coloured path f_5g_1 is g_6f_2, so $\theta_{1,2}(f_5g_1) = g_6f_2$. Thus, the factorisation rules in any 3-graph with the skeleton pictured above must be implemented by the $\theta_{i,j}$. To see that no such 3-graph exists, we consider the two possible ways of reversing the colouring of the path $h_8g_6f_2$ using the $\theta_{i,j}$:

$$\begin{aligned} &h_8g_6f_2 \to h_8f_5g_1 \to f_7h_3g_1 \to f_7g_3h_1 \quad \text{and} \\ &h_8g_6f_2 \to g_8h_6f_2 \to g_8f_4h_2 \to f_8g_4h_2. \end{aligned}$$

Since $f_8 \neq f_7$, $g_3 \neq g_4$ and $h_1 \neq h_2$, the $\theta_{i,j}$ do not specify a valid collection of factorisation rules (see [31, Section 2]).

Let $\Gamma = T_3$ (the 3-graph isomorphic to $\mathbb{N}^3$), and let Λ be the 0-graph whose vertices are those in the diagram above. The sets $X_1 := \{f_1, \ldots, f_8\}$, $X_2 := \{g_1, \ldots, g_8\}$ and $X_3 = \{h_1, \ldots, h_8\}$ are Λ-endomorphs when endowed with the obvious structure maps. For $i \neq j$, $\theta_{i,j}$ determines an isomorphism $X_i *_{\Lambda^0} X_j \cong X_j *_{\Lambda^0} X_i$, so there is a unique functor $F : \Gamma \cong \mathbb{N}^3 \to \mathcal{M}_0$ such that $F^0(v) = \Lambda$ and $F(e_i) = [X_i]$ for $i = 1, 2, 3$. However, this functor is not determined by any T_3-system of 0-morphs: the T_3-bundle for such a system would be a 3-graph whose skeleton was the 3-coloured graph we started with.

6. C^*-correspondences and functoriality

In this section we consider how the constructions of the preceding sections behave with respect to higher-rank graph C^*-algebras. To keep the length of the paper down, we restrict our attention to Λ–Γ morphs such that

(✠) Λ and Γ are row-finite k-graphs with no sources, $s : X \to \Gamma^0$ is surjective, and $r : X \to \Lambda^0$ is surjective and finite-to-one.

This simplifying assumption ensures that the Γ-bundles we construct are covered by the results of [31].

To each k-morph X satisfying (✠) we associate a C^*-correspondence $\mathcal{H}(X)$. The germ of this construction, at least for k-morphs arising from covering maps, is present in the proof of [21, Proposition 3.2]. However, here we make it explicit and extend it to arbitrary k-morphs.

In Theorem 6.6 we show that the assignment $[X] \mapsto [\mathcal{H}(X)]$ determines a contravariant functor to the category $\mathcal{C}$ whose objects are C^*-algebras and whose morphisms are isomorphism classes of C^*-correspondences. Theorem 6.8 shows that when X is a Λ endomorph satisfying (✠), the Cuntz-Pimsner algebra of $\mathcal{H}(X)$ is isomorphic to the $(k+1)$-graph C^*-algebra $C^*(\Lambda \times_X \mathbb{N})$.

Proposition 6.1. *There is a subcategory $\mathcal{M}_k^{✠}$ of $\mathcal{M}_k$ whose objects are row-finite k-graphs with no sources and whose morphisms are isomorphism classes of k-morphs X satisfying (✠).*

Proof. This follows from the observation that if k-morphs X_1 and X_2 satisfy (✠), then so does their product. □

Lemma 6.2. *Let Λ and Γ be k-graphs and let X be a Λ–Γ morph satisfying (✠). Let (Σ, i) be a linking graph for X. Then Σ is row-finite and locally convex, and there are injective homomorphisms*

$$i_\Lambda^* : C^*(\Lambda) \to C^*(\Sigma) \text{ such that } i_\Lambda^*(s_\lambda) = s_{i(\lambda)} \text{ and}$$
$$i_\Gamma^* : C^*(\Gamma) \to C^*(\Sigma) \text{ such that } i_\Gamma^*(s_\gamma) = s_{i(\gamma)}.$$

The series $\sum_{v \in \Lambda^0} s_{i(v)}$ and $\sum_{w \in \Gamma^0} s_{i(w)}$ converge strictly to complementary full projections P_Λ and P_Γ in $MC^*(\Sigma)$. The homomorphism i_Γ^* induces an isomorphism $C^*(\Gamma) \cong P_\Gamma C^*(\Sigma) P_\Gamma$. The homomorphism i_Λ^* induces an embedding $C^*(\Lambda) \hookrightarrow P_\Lambda C^*(\Sigma) P_\Lambda$ which takes an approximate identity for $C^*(\Lambda)$ to an approximate identity for $P_\Lambda C^*(\Sigma) P_\Lambda$.

Proof. The linking graph Σ is row-finite because (✠) ensures that Λ and Γ are both row-finite and the range map on X is finite-to-one. To see that Σ is locally convex, suppose that $e, f \in \Sigma$ satisfy $r(e) = r(f)$, $d(e) = e_i$ and $d(f) = e_j$, where $1 \le i < j \le k+1$. If $j \le k$, then $s(e)\Sigma^{e_j}$ and $s(f)\Sigma^{e_i}$ are nonempty because Λ and Γ have no sources. If $d(f) = e_{k+1}$, then $s(e)\Sigma^{e_{k+1}}$ is nonempty because $r : X \to \Lambda^0$ is surjective by (✠), and $s(f)\Sigma^{e_i}$ is nonempty because Γ has no sources.

The existence of homomorphisms i_Λ^* and i_Γ^* satisfying the required formulae follows from the universal properties of $C^*(\Lambda)$ and $C^*(\Gamma)$, and their injectivity follows from the gauge-invariant uniqueness theorem [18, Theorem 3.4].

A standard argument (see for example [21, Proposition 3.2]) shows that P_Λ and P_Γ make sense and are complementary projections. To see that P_Λ is full, we fix a generator s_σ of $C^*(\Sigma)$ and show that $s_\sigma \in C^*(\Sigma) P_\Lambda C^*(\Sigma)$. If $r(\sigma) \in i_\Lambda(\Lambda^0)$, then $s_\sigma = P_\Lambda s_\sigma$; and if $r(\sigma) \in i_\Gamma(\Gamma^0)$, then since the source map on X is surjective by (✠), we have $s_\sigma = s_\alpha^* P_\Lambda s_\alpha s_\sigma$ for some $\alpha \in \Sigma^{e_{k+1}}$. To see that P_Γ is full, we fix a generator s_σ of $C^*(\Sigma)$ and show that $s_\sigma \in C^*(\Sigma)P_\Gamma C^*(\Sigma)$. If $s(\sigma) \in i_\Gamma(\Gamma^0)$, then $s_\sigma = s_\sigma P_\Gamma$; and if $s(\sigma) \in i_\Lambda(\Lambda^0)$, then since the range map on X is surjective and finite-to-one by (✠), we have

$$s_\sigma = \sum_{\alpha \in s(\sigma)\Sigma^{e_{k+1}}} s_\sigma s_\alpha P_\Gamma s_\alpha^*.$$

We have

$$\begin{aligned} P_\Gamma C^*(\Sigma) P_\Gamma &= \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in \Sigma, r(\alpha), r(\beta) \in i(\Gamma^0)\} \\ &= \overline{\operatorname{span}}\{i_\Gamma^*(s_{\gamma_1} s_{\gamma_2}^*) : \gamma_i \in \Gamma, s(\gamma_1) = s(\gamma_2)\}, \end{aligned}$$

and it follows from an application of the gauge-invariant uniqueness theorem [31, Theorem 4.1] that i_Γ^* implements the desired isomorphism $C^*(\Gamma) \cong P_\Gamma C^*(\Sigma) P_\Gamma$. Since Σ is row-finite and locally convex, the Cuntz-Krieger relations in $C^*(\Sigma)$ ensure that the partial isometries $\{s_{i_\Lambda(\lambda)} : \lambda \in \Lambda\}$ form a Cuntz-Krieger Λ-family. Another application of [31, Theorem 4.1] then shows that $i_\Lambda^* : C^*(\Lambda) \to P_\Lambda C^*(\Sigma) P_\Lambda$ is injective. For each finite subset $F \subset \Lambda^0$, let p_F denote the projection $\sum_{v \in F} s_v$. Then the net $(p_F)_{F \subset \Lambda^0 \text{ finite}}$ is an approximate identity for $C^*(\Lambda)$ and $(i_\Lambda^*(p_F))_{F \subset \Lambda^0 \text{ finite}}$ converges strictly to P_Λ by definition of P_Λ. □

Definition 6.3. Resume the hypotheses of Lemma 6.2. Let $\mathcal{H}(X)$ denote the vector space $P_\Lambda C^*(\Sigma) P_\Gamma$. Define a left action of $C^*(\Lambda)$ and a right action of $C^*(\Gamma)$ on $\mathcal{H}(X)$ by

$$a \cdot \xi \cdot b = i_\Lambda^*(a)\, \xi\, i_\Gamma^*(b) \quad \text{ for } a \in C^*(\Lambda),\ b \in C^*(\Gamma) \text{ and } \xi \in \mathcal{H}(X),$$

where the product is taken in $C^*(\Sigma)$. Define $\langle \cdot, \cdot \rangle_{C^*(\Gamma)} : \mathcal{H}(X) \times \mathcal{H}(X) \to C^*(\Gamma)$ as follows: $\langle \xi, \eta \rangle_{C^*(\Gamma)}$ is the unique element of $C^*(\Gamma)$ such that

$$\xi^* \eta = i_\Gamma^*(\langle \xi, \eta \rangle_{C^*(\Gamma)}),$$

where $\xi^*\eta$ is calculated in $C^*(\Sigma)$.

Proposition 6.4. *Let Λ and Γ be k-graphs and let X be a Λ–Γ morph satisfying (✠). The space $\mathcal{H}(X)$ defined above satisfies*

$$\mathcal{H}(X) = \overline{\operatorname{span}}\{s_{xi(\alpha)} s_{i(\beta)}^* : x \in X, \alpha, \beta \in \Gamma, s(x) = r(\alpha), s(\alpha) = s(\beta)\}. \tag{6.1}$$

Under the operations defined above, $\mathcal{H}(X)$ is a full nondegenerate $C^*(\Lambda)$–$C^*(\Gamma)$ C^*-correspondence and the left-action is implemented by an injective homomorphism of $C^*(\Lambda)$ into $\mathcal{K}(\mathcal{H}(X))$. Moreover, the isomorphism class of $\mathcal{H}(X)$ depends only on the isomorphism class of X.

Remark 6.5. If we do not insist on (✠), but assume only that Σ is finitely aligned (so that $C^*(\Sigma)$ makes sense), then Definition 6.3 still specifies a $C^*(\Lambda)$–$C^*(\Gamma)$ correspondence $\mathcal{H}(X)$ satisfying (6.1). However, our proofs of the remaining assertions of Proposition 6.4 and of Theorems 6.6 and 6.8 all rely on (✠) via their dependence on Lemma 6.2.

Proof of Proposition 6.4. Fix a nonzero spanning element $s_\mu s_\nu^*$ of $C^*(\Sigma)$. Then $s_\mu s_\nu^* \in P_\Lambda C^*(\Sigma) P_\Gamma$ only if $r(\mu) \in i(\Lambda^0)$, $r(\nu) \in i(\Gamma^0)$, and $s(\mu) = s(\nu)$. Since $r(\nu) \in i(\Gamma^0)$ implies $s(\nu) \in \Gamma^0$, we have $\nu \in i(\Gamma)$. Since $s(\mu) = s(\nu)$, we also have $\mu \in i(\Lambda^0)\Sigma i(\Gamma^0)$, and the factorisation property in Σ forces $\mu = xi(\alpha)$ for some $x \in X$ and $\alpha \in \Gamma$ with $s(x) = r(\alpha)$. This establishes (6.1).

Since P_Λ and P_Γ are complementary full projections in $MC^*(\Sigma)$, Theorem 3.19 of [33] shows that $\mathcal{H}(X)$ is an imprimitivity bimodule for the corners $P_\Lambda C^*(\Sigma) P_\Lambda \cong \mathcal{K}(\mathcal{H}(X))$ and $P_\Gamma C^*(\Sigma) P_\Gamma \cong C^*(\Gamma)$. That $\mathcal{H}(X)$ is full follows from the definition of an imprimitivity bimodule (see [33, Definition 3.1]). The injective homomorphism $C^*(\Lambda) \to \mathcal{K}(\mathcal{H}(X))$ comes from the embedding $C^*(\Lambda) \hookrightarrow P_\Lambda C^*(\Sigma) P_\Lambda$ induced by i_Λ^* and the identification $P_\Lambda C^*(\Sigma) P_\Lambda \cong \mathcal{K}(\mathcal{H}(X))$. Since i_Λ^* maps an approximate identity for $C^*(\Lambda)$ to an approximate identity for $P_\Lambda C^*(\Sigma) P_\Lambda$, $\mathcal{H}(X)$ is nondegenerate.

The final statement follows from Remark 4.6. $\square$

For the following, recall from Section 2.4 that $\mathcal{C}$ denotes the category whose objects are C^*-algebras and whose morphisms are isomorphism classes of C^*-correspondences.

Theorem 6.6. *For each $k \geq 0$, the assignments $\Lambda \mapsto C^*(\Lambda)$ and $[X] \mapsto [\mathcal{H}(X)]$ determine a contravariant functor $\mathcal{H}_k$ from $\mathcal{M}_k^{✠}$ to $\mathcal{C}$.*

Proof. It suffices to show that for any $\Lambda_0, \Lambda_1, \Lambda_2 \in \mathrm{Obj}(\mathcal{M}_k^{✠})$ and for any $[X_i] \in \mathrm{Hom}_{\mathcal{M}_k^{✠}}(\Lambda_i, \Lambda_{i-1})$, there is an isomorphism of C^*-correspondences

$$\mathcal{H}(X_1) \otimes_{C^*(\Lambda_1)} \mathcal{H}(X_2) \cong \mathcal{H}(X_1 *_{\Lambda_1^0} X_2).$$

For $i = 1, 2$, let Σ_i be a linking graph for X_i, and let Σ_{12} be a linking graph for $X_1 *_{\Lambda_1^0} X_2$. Let Γ be the 1-graph with two edges a_1, a_2 and three distinct vertices $v_0 = r(a_1)$, $v_1 = s(a_1) = r(a_2)$ and $v_2 = s(a_2)$. Then $\Lambda_{v_i} := \Lambda_i$, $X_{a_i} := X_i$ and $X_{a_1 a_2} := X_1 *_{\Lambda_1^0} X_2$ defines a Γ-system of k-morphs (θ_{X_1, X_2} is the identity on $X_1 *_{\Lambda_1^0} X_2$). Let Σ be a Γ-bundle for this system, and let $f : \Sigma \to \Gamma$ be the bundle map. For $i = 0, 1, 2$, let $P_i = \sum_{w \in f^{-1}(v_i) \cap \Sigma^0} s_w \subset MC^*(\Sigma)$.

By applications of the gauge-invariant uniqueness theorem [31, Theorem 4.1], there are canonical isomorphisms

$$\begin{aligned} C^*(\Sigma_1) &\cong (P_0 + P_1) C^*(\Sigma)(P_0 + P_1), \\ C^*(\Sigma_2) &\cong (P_1 + P_2) C^*(\Sigma)(P_1 + P_2), \text{ and} \\ C^*(\Sigma_{12}) &\cong (P_0 + P_2) C^*(\Sigma)(P_0 + P_2) \end{aligned}$$

(to establish the third of these isomorphisms, we must slightly modify the gauge action on $(P_0 + P_2) C^*(\Sigma)(P_0 + P_2)$). In particular, it follows that

$$\begin{aligned} \mathcal{H}(X_1) &\cong P_0 C^*(\Sigma) P_1, \\ \mathcal{H}(X_2) &\cong P_1 C^*(\Sigma) P_2, \text{ and} \\ \mathcal{H}(X_1 *_{\Lambda_1^0} X_2) &\cong P_0 C^*(\Sigma) P_2. \end{aligned}$$

As in Lemma 6.2, the P_i are all full projections in $MC^*(\Sigma)$, so

$$\mathcal{H}(X_1 *_{\Lambda_1^0} X_2) \cong P_0 C^*(\Sigma) P_2 = (P_0 C^*(\Sigma) P_1)(P_1 C^*(\Sigma) P_2),$$

and hence, if we identify $\mathcal{H}(X_1) \otimes_{C^*(\Lambda_1)} \mathcal{H}(X_2)$ with $(P_0 C^*(\Sigma) P_1) \otimes_{P_1 C^*(\Sigma) P_1} (P_1 C^*(\Sigma) P_2)$, multiplication in $C^*(\Sigma)$ induces an isomorphism

$$\mathcal{H}(X_1) \otimes_{C^*(\Lambda_1)} \mathcal{H}(X_2) \cong (P_0 C^*(\Sigma) P_1)(P_1 C^*(\Sigma) P_2).$$

This completes the proof. $\square$

We now present an alternative construction of the C^*-correspondence $\mathcal{H}(X)$ (see [6] for a similar construction). In the following, given a k-morph X, we denote the point-mass function at $x \in X$ by $\delta_x \in C_c(X)$. We regard $C_c(X)$ as a right pre-Hilbert $C_0(\Gamma^0)$ module with pointwise operations.

Proposition 6.7. *Let Λ and Γ be k-graphs, and let X be a Λ–Γ morph satisfying (✠). Let (Σ, i) be a linking graph for X, and identify X with $\Sigma^{e_{k+1}}$. Let $\mathcal{H}(X)$ be the C^*-correspondence obtained from Proposition* 6.4. *Then there is an isomorphism of right-Hilbert $C^*(\Gamma)$-modules*

$$\mathcal{H}(X) \cong \overline{C_c(X) \otimes_{C_0(\Gamma^0)} C^*(\Gamma)}$$

determined by

$$s_{xi(\alpha)}s_{i(\beta)}^* \mapsto \delta_x \otimes s_\alpha s_\beta^* \qquad \text{for } x \in \Sigma^{e_{k+1}},\ \alpha, \beta \in \Gamma. \tag{6.2}$$

This isomorphism carries the left action of $s_\lambda \in C^*(\Lambda)$ on $\mathcal{H}(X)$ to

$$s_\lambda \cdot (\delta_x \otimes s_\alpha s_\beta^*) = \delta_{x'} \otimes s_\gamma s_\alpha s_\beta^* \qquad \text{for } x \in \Sigma^{e_{k+1}} \text{ and } \alpha, \beta \in \Gamma,$$

where $\phi_X(x', \gamma) = (\lambda, x)$.

Proof. For the first statement, we just need to check that the formula (6.2) extends to an inner-product-preserving map. For all $x, x' \in \Sigma^{e_{k+1}}$, $\alpha \in s(x)\Gamma$, $\alpha' \in s(x')\Gamma$, $\beta \in \Gamma s(\alpha)$ and $\beta' \in \Gamma s(\alpha')$, we have

$$\begin{aligned} \langle \delta_x \otimes s_\alpha s_\beta^*, \delta_{x'} \otimes s_{\alpha'} s_{\beta'}^* \rangle &= \langle s_\alpha s_\beta^*, \langle \delta_x, \delta_{x'} \rangle \cdot s_{\alpha'} s_{\beta'}^* \rangle \\ &= \begin{cases} s_\beta s_\alpha^* s_{\alpha'} s_{\beta'}^* & \text{if } x = x', \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{6.3}$$

Since $s_{xi(\alpha)}^* s_{x'i(\alpha')} = s_{i(\alpha)}^*(s_x^* s_{x'}) s_{i(\alpha')}$, the third Cuntz-Krieger relation in $C^*(\Sigma)$ forces

$$s_{xi(\alpha)}^* s_{x'i(\alpha')} = \begin{cases} s_{i(\alpha)}^* s_{i(\alpha')} & \text{if } x = x', \\ 0 & \text{otherwise.} \end{cases}$$

Then equations (6.3) and (6.2) and the definition of the inner product on $\mathcal{H}(X)$ imply that

$$\langle \delta_x \otimes s_\alpha s_\beta^*, \delta_{x'} \otimes s_{\alpha'} s_{\beta'}^* \rangle = \langle s_{xi(\alpha)} s_\beta^*, s_{x'i(\alpha')} s_{\beta'}^* \rangle.$$

By linearity and continuity, it follows that (6.2) is inner-product preserving.

The last assertion follows from a direct calculation. □

We now consider the case where X is a Λ endomorph. We show that the $(k+1)$-graph C^*-algebra $C^*(\Lambda \times_X \mathbb{N})$ coincides with the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}(X)}$.

Let X be a Λ endomorph satisfying (✠). Let $\Lambda \times_X \mathbb{N}$ be the T_1-bundle for the system induced by X as in Examples 5.11(ii). Because T_1 has just one object, we may simplify the notation of Remark 5.8 as follows: there are injective maps $h_\Lambda : \Lambda \to \Lambda \times_X \mathbb{N}$ and $h_n : X^{*n} \to \Lambda \times_X \mathbb{N}$ (where h_0 is the identity map on vertices), and every element of $\Lambda \times_X \mathbb{N}$ is of the form $h_n(x)h_\Lambda(\lambda)$ for some $n \in \mathbb{N}$, $x \in X^{*n}$ and $\lambda \in \Lambda$.

Theorem 6.8. *Let Λ be a k-graph and let X be a Λ endomorph satisfying (✠). Let $\mathcal{H}(X)$ be the associated $C^*(\Lambda)$–$C^*(\Lambda)$ correspondence and let $\Lambda \times_X \mathbb{N}$ be the T_1-bundle associated to X regarded as a Λ endomorph. There are a homomorphism $\pi : C^*(\Lambda) \to C^*(\Lambda \times_X \mathbb{N})$ and a linear map $t : \mathcal{H}(X) \to C^*(\Lambda \times_X \mathbb{N})$ determined by*

$$\pi(s_\lambda) = s_{h_\Lambda(\lambda)} \qquad \textit{and} \qquad t(s_{xi(\alpha)} s_{i(\beta)}^*) = s_{h_1(x)} s_{h_\Lambda(\alpha)} s_{h_\Lambda(\beta)}^*.$$

The pair (t, π) is a Cuntz-Pimsner covariant representation of $\mathcal{H}(X)$, and the induced C^*-homomorphism $t \times \pi : \mathcal{O}_{\mathcal{H}(X)} \to C^*(\Lambda \times_X \mathbb{N})$ is an isomorphism.

Proof. The universal property of $C^*(\Lambda)$ shows that there is a homomorphism $\pi : C^*(\Lambda) \to C^*(\Lambda \times_X \mathbb{N})$ satisfying $\pi(s_\lambda) = s_{h_\Lambda(\lambda)}$ for all $\lambda \in \Lambda$. This homomorphism π is equivariant for the gauge action on $C^*(\Lambda)$ and the restriction of the gauge action on $C^*(\Lambda \times_X \mathbb{N})$ to the first k coordinates of $\mathbb{T}^{k+1}$. Hence an application of the gauge-invariant uniqueness theorem for $C^*(\Lambda)$ [18, Theorem 3.4] shows that π is injective.

To see that the formula given for t determines a well-defined linear map, we will show that for any finite linear combination of the form $\sum_{j=1}^n a_j s_{x_j i(\alpha_j)} s_{i(\beta_j)}^*$ in $\mathcal{H}(X)$, we have

$$\Big\| \sum_{j=1}^n a_j s_{x_j i(\alpha_j)} s_{i(\beta_j)}^* \Big\|_{\mathcal{H}(X)} = \Big\| \sum_{j=1}^n a_j s_{h_1(x_j)} s_{h_\Lambda(\alpha_j)} s_{h_\Lambda(\beta_j)}^* \Big\|_{C^*(\Lambda \times_X \mathbb{N})}.$$

We have already shown that π is injective, so by the C^*-identity for $C^*(\Lambda \times_X \mathbb{N})$ and the definition of the norm on $\mathcal{H}(X)$ it suffices to show that for spanning elements $s_{xi(\alpha)} s_{i(\beta)}^*$ and $s_{x'i(\alpha')} s_{i(\beta')}^*$ of $\mathcal{H}(X)$, we have

$$\pi(\langle s_{xi(\alpha)} s_{i(\beta)}^*, s_{x'i(\alpha')} s_{i(\beta')}^* \rangle_{\mathcal{H}(X)}) = t(s_{xi(\alpha)} s_{i(\beta)}^*)^* t(s_{x'i(\alpha')} s_{i(\beta')}^*).$$

This follows from a routine calculation such as (6.3) above.

That $\pi(\langle \xi, \eta \rangle_{C^*(\Lambda)}) = t(\xi)^* t(\eta)$ for all $\xi, \eta \in \mathcal{H}(X)$ follows by linearity from the preceding paragraph. To see that (t, π) is a representation, it therefore suffices to show that for a generator s_λ of $C^*(\Lambda)$ and spanning elements $s_{xi(\alpha)} s_{i(\beta)}^*$ and $s_{x'i(\alpha')} s_{i(\beta')}^*$ of $\mathcal{H}(X)$,

$$\pi(s_\lambda) t(s_{xi(\alpha)} s_{i(\beta)}^*) = t(s_\lambda \cdot (s_{xi(\alpha)} s_{i(\beta)}^*)) \text{ and}$$
$$t(s_{xi(\alpha)} s_{i(\beta)}^*) \pi(s_\lambda) = t((s_{xi(\alpha)} s_{i(\beta)}^*) \cdot s_\lambda).$$

One verifies these identities with short calculations using the definitions of t and π and the structure of $\mathcal{H}(X)$. We give the first of these calculations, as it is the least elementary. Fix s_λ and $s_{xi(\alpha)} s_{i(\beta)}^*$ as above, and let x' and λ' be the elements such that $\phi_X(x', \lambda') = (\lambda, x)$. Then $s_\lambda \cdot (s_{(xi(\alpha))} s_{i(\beta)}^*) = s_{(x'i(\lambda'\alpha))} s_{i(\beta)}^*$, and we have

$$\begin{aligned} \pi(s_\lambda) t(s_{xi(\alpha)} s_{i(\beta)}^*) &= s_{h_\Lambda(\lambda)} s_{h_1(x)} s_{h_\Lambda(\alpha)} s_{h_\Lambda(\beta)}^* \\ &= s_{h_1(x')} s_{h_\Lambda(\lambda')} s_{h_\Lambda(\alpha)} s_{h_\Lambda(\beta)}^* = t(s_\lambda \cdot (s_{(xi(\alpha))} s_{i(\beta)}^*)). \end{aligned}$$

To check that (t, π) is Cuntz-Pimsner covariant, fix $\lambda \in \Lambda$, and note that the left action of s_λ on $\mathcal{H}(X)$ is implemented by

$$\sum_{\substack{r(x)=s(\lambda) \\ \phi_X(x',\lambda')=(\lambda,x)}} \Theta_{s_{x'i(\lambda')}, s_x} \in \mathcal{K}(\mathcal{H}(X)).$$

Hence if φ denotes the homomorphism which implements the left action, we have

$$t^{(1)}(\varphi(s_\lambda)) = \sum_{\substack{r(x)=s(\lambda) \\ \phi_X(x',\lambda')=(\lambda,x)}} t(s_{x'i(\lambda')}) t(s_x)^* = \sum_{r(x)=s(\lambda)} \pi(s_\lambda) t(s_x) t(s_x)^*,$$

and this is equal to $\pi(s_\lambda)$ by the fourth Cuntz-Krieger relation in $C^*(\Lambda \times_X \mathbb{N})$.

The restriction of the gauge-action on $C^*(\Lambda \times_X \mathbb{N})$ to the last coordinate in $\mathbb{T}^{k+1}$ is compatible with the gauge action on $\mathcal{O}_{\mathcal{H}(X)}$. The gauge-invariant uniqueness theorem [14, Theorem 4.1] for $\mathcal{O}_{\mathcal{H}(X)}$ (see also [17, Theorem 6.4]) now implies that

$t \times \pi$ is injective. It remains only to observe that for $h_n(x)h_\Lambda(\lambda) \in \Lambda \times_X \mathbb{N}$, we can write $x = (x_1, \ldots, x_n)$ where each $x_i \in X$, and $r(x_{i+1}) = s(x_i)$, and then

$$s_{h_n(x)h_\Lambda(\lambda)} = t(s_{x_1}) \cdots t(s_{x_n})\pi(s_\lambda)$$

(if $n = 0$, then $x = r(\lambda) \in I(\Lambda)$, so $s_{h_n(x)h_\Lambda(\lambda)} = \pi(s_\lambda)$). Hence $t \times \pi$ is surjective. $\square$

Remark 6.9. Let Σ be a row-finite $(k+1)$-graph with no sources such that $\Sigma^n v \neq \emptyset$ for all $n \in \mathbb{N}^{k+1}$ and $v \in \Sigma^0$. Let Σ^ι and X be the k-graph and Σ^ι endomorph discussed in Examples 3.3(vi). Then X satisfies (✠). As in Example 5.11(iii), we have that Σ is isomorphic to the endomorph skew graph $\Sigma^\iota \times_X \mathbb{N}$. Hence Theorem 6.8 implies that $C^*(\Sigma) \cong \mathcal{O}_{\mathcal{H}(X)}$; that is, the C^*-algebra of Σ can be realised as the Cuntz-Pimsner algebra of a C^*-correspondence over $C^*(\Sigma^\iota)$.

Remark 6.10. Let $\Lambda \in \mathrm{Obj}(\mathcal{M}_k^{✠})$, let α be an automorphism of Λ, and let $X = X(\alpha)$ be the associated endomorph; clearly X satisfies (✠). Let $\widetilde{\alpha}$ denote the induced automorphism of $C^*(\Lambda)$. As in Example 5.11(iii), the endomorph skew graph $\Lambda \times_X \mathbb{N}$ is isomorphic to the crossed product $(k+1)$-graph $\Lambda \times_\alpha \mathbb{Z}$ constructed in [15].

In this situation, the bimodule $\mathcal{H}(X)$ constructed above is isomorphic to the bimodule constructed by Pimsner in [29, Example 3, p.193] with $A = C^*(\Lambda)$ and $\pi = \widetilde{\alpha}$. We therefore recover the isomorphism $C^*(\Lambda \times_\alpha \mathbb{Z}) \cong C^*(\Lambda) \times_{\widetilde{\alpha}} \mathbb{Z}$ of [15, Theorem 3.5] from Theorem 6.8 and Pimsner's result.

Corollary 6.11. *Let Λ, Γ be k-graphs, let R be a Λ–Γ morph and let S be a Γ–Λ morph. Suppose R and S satisfy (✠). Let X be the Λ endomorph $S *_{\Gamma^0} R$, and let Y be the Γ-endomorph $R *_{\Lambda^0} S$. Then the $(k+1)$-graph algebras $C^*(\Lambda \times_X \mathbb{N})$ and $C^*(\Gamma \times_Y \mathbb{N})$ are Morita equivalent.*

In particular, the above Morita equivalence holds if $X = {}_pX_q$ and $Y = X_q *_{\Lambda^0} {}_pX$, where $p, q : \Gamma \to \Lambda$ are finite coverings of row-finite k-graphs.

Proof. Theorem 6.6 implies that

$$\mathcal{H}(X) \cong \mathcal{H}(S) \otimes_{C^*(\Gamma)} \mathcal{H}(R) \quad \text{and} \quad \mathcal{H}(Y) \cong \mathcal{H}(R) \otimes_{C^*(\Lambda)} \mathcal{H}(S).$$

By Proposition 6.7, $E := \mathcal{H}(X)$ and $F := \mathcal{H}(Y)$ satisfy the hypotheses of [24, Theorem 3.14] (the C^*-correspondences $\mathcal{H}(R)$ and $\mathcal{H}(S)$ implement the elementary strong shift equivalence). Hence $\mathcal{O}_{\mathcal{H}(X)}$ and $\mathcal{O}_{\mathcal{H}(Y)}$ are Morita equivalent, and our result follows from Theorem 6.8. $\square$

Remark 6.12. When $k = 0$, the first statement of the above corollary reduces to Bates' results on shift-equivalence for 1-graphs [2].

Example 6.13. For $n \in \mathbb{N} \setminus \{0\}$, let D_n be the directed graph with n vertices $\{v_0, \ldots, v_{n-1}\}$ and edges $\{x_i, y_i : 0 \leq i \leq n-1\}$, where $r(x_i) = v_i = s(y_i)$ and $s(x_i) = v_{i+1} = r(y_i)$ (see [21, Section 6.2]). In particular, D_1 is equal to the bouquet of two loops whose C^*-algebra is canonically isomorphic to $\mathcal{O}_2$. We will consider a D_1 endomorph ${}_pX_q$ constructed from coverings $p, q : D_2 \to D_1$. To reduce confusion, we will denote $x_0, y_0 \in D_1$ by x and y.

The covering maps $p, q : D_2 \to D_1$ are defined as follows:

$$
\begin{aligned}
p(x_i) &= x, & p(y_i) &= y,\\
q(x_0) &= x, & q(y_0) &= y,\\
q(x_1) &= y, & q(y_1) &= x.
\end{aligned}
$$

Construct $X = {}_pX_q$ as in Examples 3.3(iv) (so, as a set, $X = D_2^0$). The endomorph skew-graph $\Lambda = D_1 \times_X \mathbb{N}$ is a 2-graph whose skeleton and factorisation rules can be described as follows: $\Lambda^0 = v$, $\Lambda^{e_1} = \{x, y\}$, $\Lambda^{e_2} = X = \{v_0, v_1\}$, and

$$v_0 x = y v_1, \quad v_0 y = x v_1, \quad v_1 x = x v_0, \quad \text{and} \quad v_1 y = y v_0.$$

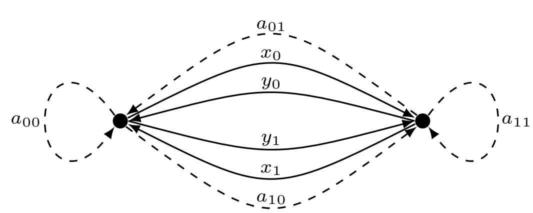
Results of [1, 35, 36] can be used to see that $C^*(\Lambda)$ is a Kirchberg algebra (the details appear in an unpublished manuscript of D. Robertson [34]) and has trivial K-theory. Hence $C^*(\Lambda)$ is isomorphic $\mathcal{O}_2$ by the Kirchberg-Phillips theorem.

Using Proposition 6.7, we see that $\mathcal{H}(X) \cong \mathbb{C}^2 \otimes_{\mathbb{C}} \mathcal{O}_2$ as a right-Hilbert $\mathcal{O}_2$-module (where $\mathbb{C}^2$ is the two-dimensional Hilbert space with orthonormal basis $\{\delta_0, \delta_1\}$). The left action of $\mathcal{O}_2 = C^*(\{S_0, S_1\})$ is determined by

$$
\begin{aligned}
S_0 \cdot (\delta_0 \otimes_{\mathbb{C}} 1_{\mathcal{O}_2}) &= \delta_1 \otimes_{\mathbb{C}} S_0, & S_0 \cdot (\delta_1 \otimes_{\mathbb{C}} 1_{\mathcal{O}_2}) &= \delta_0 \otimes_{\mathbb{C}} S_1,\\
S_1 \cdot (\delta_0 \otimes_{\mathbb{C}} 1_{\mathcal{O}_2}) &= \delta_1 \otimes_{\mathbb{C}} S_1, & S_1 \cdot (\delta_1 \otimes_{\mathbb{C}} 1_{\mathcal{O}_2}) &= \delta_0 \otimes_{\mathbb{C}} S_0.
\end{aligned}
$$

By Theorem 6.8, the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}(X)}$ of this C^*-correspondence is isomorphic to $C^*(\Lambda)$ which, as we saw above, is isomorphic to $\mathcal{O}_2$.

Let $Y = X_q *_{\Lambda^0} {}_pX$. Note that Y satisfies $(\maltese)$. The endomorph skew-graph $D_2 \times_Y \mathbb{N}$ has skeleton



with factorisation rules

$$
\begin{aligned}
a_{00} y_0 &= y_0 a_{11} & a_{00} x_1 &= x_1 a_{11} & a_{10} y_0 &= x_0 a_{01} & a_{10} x_1 &= y_1 a_{01}\\
a_{11} x_0 &= y_1 a_{01} & a_{11} y_1 &= x_0 a_{01} & a_{01} x_0 &= x_0 a_{10} & a_{01} y_1 &= y_1 a_{10}.
\end{aligned}
$$

Corollary 6.11 shows that $C^*(D_2 \times_Y \mathbb{N})$ is Morita equivalent to $C^*(\Lambda) \cong \mathcal{O}_2$, and, as it is also unital, it is in fact isomorphic to $\mathcal{O}_2$.

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