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UNIQUENESS OF GINZBURG-RALLIS MODELS: THE ARCHIMEDEAN CASE

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Abstract. In this paper we prove the uniqueness of Ginzburg-Rallis models in the Archimedean case. As a key ingredient, we introduce a new descent argument based on two geometric notions attached to submanifolds, which we call metrical properness and unipotent χ-incompatibility.

1. Introduction and main results

In the year 2000 Ginzburg and Rallis formulated a conjecture to characterize the nonvanishing of central values of partial exterior cube L-functions attached to irreducible cuspidal automorphic representations of GL_6 in terms of certain periods ([GR00]). This is analogous to the Jacquet conjecture for the triple product L-functions for GL_2 (established in full by Harris and Kudla in [HK04]) and to the Gross-Prasad conjecture for classical groups ([GP92, GP94, GJR04, GJR05, GJR09]).

To be precise, let $\mathbb{A}$ be the ring of adeles of a number field $\mathbf{k}$. Fix a nontrivial unitary character $\psi_{\mathbb{A}}$ of $\mathbf{k}\backslash\mathbb{A}$ and a (nonnecessarily unitary) character $\chi_{\mathbb{A}^\times}$ of $\mathbf{k}^\times\backslash\mathbb{A}^\times$. For any quaternion algebra $\mathbf{D}$ over $\mathbf{k}$, denote $G_{\mathbf{D}} = \mathrm{GL}_3(\mathbf{D})$, and let $S_{\mathbf{D}}$ be its subgroup consisting of elements of the form

$$\begin{bmatrix} a & b & d \\ 0 & a & c \\ 0 & 0 & a \end{bmatrix}. \tag{1.1}$$

Define a character $\chi_{S_{\mathbf{D}}}$ of $S_{\mathbf{D}}(\mathbb{A})$ by

$$\chi_{S_{\mathbf{D}}}\left(\begin{bmatrix} 1 & b & d \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}\right) = \chi_{\mathbb{A}^\times}(\det(a))\,\psi_{\mathbb{A}}(\mathrm{tr}(b+c)), \tag{1.2}$$

where det and tr stand for the reduced norm and the reduced trace, respectively.

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Let $\varphi_{\mathbf{D}}$ be an automorphic form on $G_{\mathbf{D}}(\mathbf{k})\backslash G_{\mathbf{D}}(\mathbb{A})$. The Ginzburg-Rallis period $\mathcal{P}_{\chi_{S_{\mathbf{D}}}}(\varphi_{\mathbf{D}})$ of $\varphi_{\mathbf{D}}$ is defined by the integral

$$\mathcal{P}_{\chi_{S_{\mathbf{D}}}}(\varphi_{\mathbf{D}}) = \int_{\mathbb{A}^\times S_{\mathbf{D}}(\mathbf{k})\backslash \mathbf{S}_{\mathbf{D}}(\mathbb{A})} \varphi_{\mathbf{D}}(s)(\chi_{S_{\mathbf{D}}}(s))^{-1} ds,$$

where $\mathbb{A}^\times$ is identified with the center of $G_{\mathbf{D}}(\mathbb{A})$. The Ginzburg-Rallis conjecture can then be stated as follows.

Conjecture 1.1 (Ginzburg-Rallis, [GR00])**.** *Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_6(\mathbb{A})$ with central character $\chi^2_{\mathbb{A}^\times}$. For any quaternion algebra $\mathbf{D}$ over $\mathbf{k}$, denote by $\pi_{\mathbf{D}}$ the generalized Jacquet-Langlands correspondence of π, which is either zero or an irreducible cuspidal automorphic representation of $G_{\mathbf{D}}(\mathbb{A})$. Consider the irreducible representation $\Lambda^3 \otimes \mathbb{C}^1$ of the L-group $\mathrm{GL}_6(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$, where Λ^3 is the exterior cube product of the standard representation of $\mathrm{GL}_6(\mathbb{C})$ and $\mathbb{C}^1$ is the standard representation of $\mathrm{GL}_1(\mathbb{C})$. The partial L-function $L^S(s, \pi \otimes \chi^{-1}_{\mathbb{A}^\times}, \Lambda^3 \otimes \mathbb{C}^1)$ does not vanish at $s = \frac{1}{2}$ if and only if there exists a unique quaternion algebra $\mathbf{D}$ such that*

(a) *the period $\mathcal{P}_{\chi_{S_{\mathbf{D}}}}(\varphi_{\mathbf{D}})$ is nonzero for some $\varphi_{\mathbf{D}} \in \pi_{\mathbf{D}}$, and*
(b) *for any quaternion algebra $\mathbf{D}'$ which is not isomorphic to $\mathbf{D}$, the period $\mathcal{P}_{\chi_{S_{\mathbf{D}'}}}(\varphi_{\mathbf{D}'})$ is zero for every $\varphi_{\mathbf{D}'} \in \pi_{\mathbf{D}'}$.*

See [GR00] and [GJ] for some partial results on the conjecture.

We consider the corresponding local theory. Let $\mathbb{K}$ be a local field of characteristic zero. Fix a nontrivial unitary character $\psi_{\mathbb{K}}$ of $\mathbb{K}$ and an arbitrary character $\chi_{\mathbb{K}^\times}$ of $\mathbb{K}^\times$. For any quaternion algebra $\mathbb{D}$ over $\mathbb{K}$, denote $G_{\mathbb{D}} = \mathrm{GL}_3(\mathbb{D})$ and define its subgroup $S_{\mathbb{D}}$ as in the number field case. We also define the local analogy $\chi_{S_{\mathbb{D}}}$ of $\chi_{S_{\mathbf{D}}}$ by the same formula in terms of the characters $\psi_{\mathbb{K}}$ and $\chi_{\mathbb{K}^\times}$.

If $\mathbb{K}$ is non-Archimedean, we let $V_{\mathbb{D}}$ be an irreducible smooth representation of $G_{\mathbb{D}}$, and if $\mathbb{K}$ is Archimedean, let $V_{\mathbb{D}}$ be an irreducible representation of $G_{\mathbb{D}}$ in the class $\mathcal{FH}$. The notion of representations in the class $\mathcal{FH}$ will be explained in Section 10.

As in the proof of the Jacquet conjecture, in order to tackle the Ginzburg-Rallis Conjecture, the first basic property that we should establish is

Conjecture 1.2. *The Ginzburg-Rallis models on $V_{\mathbb{D}}$ is unique up to scalar, i.e.,*

$$\dim \mathrm{Hom}_{S_{\mathbb{D}}}(V_{\mathbb{D}}, \mathbb{C}_{\chi_{S_{\mathbb{D}}}}) \leq 1,$$

where $\mathbb{C}_{\chi_{S_{\mathbb{D}}}}$ is the one dimensional representation of $S_{\mathbb{D}}$ given by the character $\chi_{S_{\mathbb{D}}}$.

This conjecture has been expected since the work [GR00] and was first discussed with details in [J08]. In her Minnesota thesis (directed by the first-named author), Nien proved Conjecture 1.2 in the non-Archimedean case ([N06]). We remark that there is a generalization of the Ginzburg-Rallis models to GL_{3n}, which may be viewed as the "three block" version of the Whittaker models for GL_n. As noted in [N06], the local uniqueness property is not expected to hold for the generalized Ginzburg-Rallis models for GL_{3n} with $n > 2$.

The first main purpose of this paper is to prove the Archimedean case of Conjecture 1.2, which requires substantially more delicate analysis than the non-Archimedean case.

From now on, we will assume that $\mathbb{K}$ is the Archimedean local field $\mathbb{R}$ or $\mathbb{C}$.

Theorem 1.3. *Let $V_{\mathbb{D}}$ be an irreducible representation of $G_{\mathbb{D}}$ in the class $\mathcal{FH}$. Then*

$$\dim \operatorname{Hom}_{S_{\mathbb{D}}}(V_{\mathbb{D}}, \mathbb{C}_{\chi_{S_{\mathbb{D}}}}) \leq 1.$$

Note that the notion of representations in the class $\mathcal{FH}$ includes the requirement of moderate growth. This has the implication that

$$\operatorname{Hom}_{S_{\mathbb{D}}}(V_{\mathbb{D}}, \mathbb{C}_{\chi_{S_{\mathbb{D}}}}) = 0$$

if one replaces the additive character $\psi_{\mathbb{K}}$ with one which is not unitary.

Ginzburg-Rallis models are so called "mixed models", as the group $S_{\mathbb{D}}$ is neither unipotent nor reductive. On the other hand, we have the Whittaker models and linear models, where the subgroup involved is unipotent or reductive, respectively. By now we know that uniqueness of Whittaker models is relatively easy to establish (see Section 11.4 for a short proof). The study of uniqueness of linear models was initiated by Jacquet-Rallis in [JR96], and there have been a number of recent advances in this direction (see [AGRS, AG3, SZ2], for example). We remark that in each case a good understanding of algebraic and geometric structure of the orbital decomposition is required. (The task is made easier by geometric invariant theory; see [AG2].) Although in some special cases one may reduce uniqueness of mixed models to that of linear models (cf. [JR96, AGJ09] and Remark 1.6 of this section), there is still a lack of general techniques to treat the mixed model problems (save for a few low rank cases; see for example [BR07]). Besides a proof of Theorem 1.3, another main purpose of this paper is to introduce a descent method in the Archimedean case that reduces uniqueness of mixed models to that of linear models. We carry out the descent process for the Ginzburg-Rallis model, which is considered as an exceptional model and which is also sufficiently complicated to reveal difficulties in general Archimedean mixed model problems.

We introduce some notation. For any natural number n, denote by $\mathfrak{gl}_n(\mathbb{K})$ the space of $n \times n$ matrices with entries in $\mathbb{K}$. When the quaternion algebra $\mathbb{D}$ is split, we fix an identification of $\mathbb{D}$ with $\mathfrak{gl}_2(\mathbb{K})$, and then $G_{\mathbb{D}}$ is identified with $\mathrm{GL}_6(\mathbb{K})$. For a square matrix x, if its entries are from $\mathbb{K}$, denote by x^τ its transpose. If $\mathbb{D}$ is not split and $x \in G_{\mathbb{D}} = \mathrm{GL}_3(\mathbb{D})$, set

$$x^\tau = \text{the transpose of } \bar{x},$$

where "$^-$" denotes the (element-wise) quaternionic conjugation.

Define the real trace form $\langle\, ,\, \rangle_{\mathbb{R}}$ on the Lie algebra $\mathfrak{gl}_3(\mathbb{D})$ of $G_{\mathbb{D}}$ by

$$\langle x, y\rangle_{\mathbb{R}} = \begin{cases} \text{the real part of the trace of } xy & \text{if } \mathbb{D} \text{ is split,} \\ \text{the reduced trace of } xy & \text{otherwise.} \end{cases} \tag{1.3}$$

Denote by $\Delta_{\mathbb{D}}$ the Casimir element with respect to $\langle\, ,\, \rangle_{\mathbb{R}}$, which is viewed as a bi-invariant differential operator on $G_{\mathbb{D}}$.

We will see in Section 10 that by (a general form of) the Gelfand-Kazhdan criterion, Theorem 1.3 is implied by the following.

Theorem 1.4. *Let f be a tempered generalized function on $G_{\mathbb{D}}$, which is an eigenvector of $\Delta_{\mathbb{D}}$. If f satisfies*

$$f(sx) = f(xs^\tau) = \chi_{S_{\mathbb{D}}}(s) f(x), \quad \textit{for all } s \in S_{\mathbb{D}}, \tag{1.4}$$

then

$$f(x) = f(x^\tau).$$

The notion of tempered generalized functions will be explained in Section 2.3. We remark that the equalities in the theorem are to be understood as equalities of generalized functions, and $f(sx)$ denotes the left translate of f by s^{-1}. Similar notation apply throughout the article.

Assume now that $\mathbb{D}$ is split. Thus $G = \mathrm{GL}_6$. (We drop the subscript $\mathbb{D}$ and the coefficient field $\mathbb{K}$ in all notation.) The nonsplit case, which is simpler, will be investigated at the end of Section 9.

Following a well-known scheme of Bruhat, we first decompose

$$G = \bigsqcup_R G_R$$

into P-P^τ double cosets, where

$$P = \left\{ \begin{bmatrix} a_1 & b & d \\ 0 & a_2 & c \\ 0 & 0 & a_3 \end{bmatrix} \in G \right\}$$

is a parabolic subgroup of G containing S.

The proof of Theorem 1.4 will consist of three steps and will involve three types of arguments:

(a) The transversality of certain vector fields to all except four G_R's among the twenty-one P-P^τ double cosets of G. The technique is due to Shalika [S74]. This allows us to focus our attention on the open submanifold G' of G consisting of the four exceptional double cosets.

(b) A descent argument based on two new notions attached to submanifolds, which we call metrical properness (Definition 3.1) and unipotent χ-incompatibility (Definition 3.3), as well as a synthesis of these two notions which we call $\mathrm{U}_\chi\,\mathrm{M}$ property (Definition 3.6). This lies at the heart of our approach and forms the main part of our argument. It eventually leads us to two linear model problems: the uniqueness of trilinear models for GL_2 and the multiplicity one property for the pair $(\mathrm{GL}_2, \mathrm{GL}_1)$.

(c) Use of the oscillator representation to conclude the uniqueness of the two afore-mentioned linear models.

For step (c), which is relatively easy, we just appeal to the following.

Proposition 1.5 ([Pr89, Theorem C.7])**.** *Let E be a finite dimensional non-degenerate quadratic space over $\mathbb{K}$, and let the orthogonal group $\mathrm{O}(E)$ act on E^k diagonally, where k is a positive integer. If $k < \dim E$ and if a tempered generalized function f on E^k is $\mathrm{SO}(E)$-invariant, then f is $\mathrm{O}(E)$-invariant.*

The above proposition may also be stated as: the determinant character of $\mathrm{O}(E)$ does not occur in Howe duality correspondence of $(\mathrm{O}(E), \mathrm{Sp}(2k))$ if $k < \dim E$. In fact the determinant character occurs if and only if $k \geq \dim E$. See [LZ97, Theorem 2.2].

The descent process reveals a very interesting interplay between the Ginzburg-Rallis model and other (smaller) models. The first model occurring is as follows. Take the maximal Levi subgroup $G_{4,2} = \mathrm{GL}_4 \times \mathrm{GL}_2$ of G, and write

$$S_{4,2} = G_{4,2} \cap S = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a \in \mathrm{GL}_2,\, b \in \mathfrak{gl}_2 \right\}. \tag{1.5}$$

In the course of proof of Theorem 1.4, we find that for any irreducible representation π of $G_{4,2}$ in the class $\mathcal{FH}$,

$$\dim \operatorname{Hom}_{S_{4,2}}(\pi, \chi_{S_{4,2}}) \leq 1, \tag{1.6}$$

where $\chi_{S_{4,2}}$ is the restriction of the character χ_S to $S_{4,2}$. A proof of (1.6) will be given in Section 11.3.

Remark 1.6. This model may be viewed as the Bessel model for the orthogonal group pair $(\mathrm{O}_6, \mathrm{O}_3)$, via the (incidental) identification of low rank algebraic groups. In the p-adic case, for a general pair $(\mathrm{O}_m, \mathrm{O}_n)$ with $m > n$ and having different parity (and its analog for unitary groups), Gan, Gross and Prasad reduce the uniqueness of Bessel models to the Multiplicity One Theorems proved by Aizenbud, Gourevitch, Rallis and Schiffmann ([AGRS, GGP09]). In the Archimedean case, the uniqueness of Bessel models for general linear groups, unitary groups and orthogonal groups was proved by the authors ([JSZ09]), using a different reduction technique (from the p-adic case) and the Archimedean Multiplicity One Theorems proved in [SZ2]. Note that the latter for general linear groups is independently due to Aizenbud and Gourevitch ([AG3]).

To examine the case $(G_{4,2}, S_{4,2})$, we perform a further descent. Consider the maximal Levi subgroup

$$G_{2,2,2} = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$$

of $G_{4,2}$ and the intersection

$$S_{2,2,2} = G_{2,2,2} \cap S_{4,2} = \mathrm{GL}_2^{\Delta},$$

which embeds diagonally into $G_{2,2,2}$. This is the well-known case of the trilinear model for GL_2. See Section 11.1.

An interesting phenomenon here is that in order to complete the proof for the case $(G_{4,2}, S_{4,2})$, one must also consider the maximal Levi subgroup

$$G_{3,1,2} = \mathrm{GL}_3 \times \mathrm{GL}_1 \times \mathrm{GL}_2$$

of $G_{4,2} = \mathrm{GL}_4 \times \mathrm{GL}_2$. This case reduces essentially to the case (GL_3, S_3), where

$$S_3 = \left\{ s(c,d,a) = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & & \\ & 1 & \\ & & a \end{pmatrix} \mid c, d \in \mathbb{K}, a \in \mathbb{K}^\times \right\}.$$

The corresponding character of S_3 is given by

$$\chi_{S_3}(s(c,d,a)) = \chi_{\mathbb{K}^\times}(a)\psi_{\mathbb{K}}(d).$$

This is a mixed model. It should come as no surprise that the pair (GL_3, S_3) is a special case of the model introduced by Jacquet-Shalika ([JS90]) to construct the exterior square L-functions for GL_{2n+1}. The uniqueness for this case was not known for any n. In the course of our proof for Theorem 1.4, we shall prove the uniqueness for the pair (GL_3, S_3) over Archimedean local fields. (The p-adic case follows similarly.) See Section 11.2.

We now describe the contents and the organization of this paper. In Section 2, we review some generalities on differential operators, generalized and invariant generalized functions, basics of Nash manifolds and the associated notion of temperedness. In Section 3, we define the notions of metrical properness, unipotent χ-incompatibility, and their synthesis, $\mathrm{U}_\chi\,\mathrm{M}$ property. Based on these three new notions, we give respectively three vanishing results on certain spaces of generalized functions (Lemmas 3.2, 3.4, 3.7). In Section 4, we prove the transversality of certain vector fields to all but four of the P-P^τ double cosets, which as mentioned allows us to focus our attention on an open submanifold G' only. In Section 5 and Section 6, we show (through lengthy but straightforward computations) that a certain submanifold Z_4 of $G_{4,2}$ and a certain submanifold Z_6 of G' has $\mathrm{U}_\chi\mathrm{M}$ properties, respectively. This eventually reduces our problem to the submanifolds $\mathrm{GL}_2 \times \mathrm{GL}_2$ and $\mathrm{GL}_3 \times \mathrm{GL}_1$ of G. In Sections 7 and 8, we show that certain spaces of quasi-invariant tempered generalized functions on $\mathrm{GL}_2 \times \mathrm{GL}_2$ and $\mathrm{GL}_3 \times \mathrm{GL}_1$ vanish.

The complete proof of Theorem 1.4 will be given in Section 9. In Section 10, we derive Theorem 1.3 from Theorem 1.4. Finally, in Section 11, we record uniqueness of models occurring in the process of descent. In addition and as further evidence for the relevance of the notion of unipotent χ-incompatibility (for mixed models, as opposed to linear models), we give a quick proof of the uniqueness of the Whittaker models based on this notion.

2. Generalities

We emphasize that materials of this section are all known. In particular, nothing is due to the authors.

2.1. **Generalized functions and differential operators.** Let M be a smooth manifold. Denote by $C_0^\infty(M)$ the space of compactly supported (complex valued) smooth functions on M, which is a complete locally convex topological vector space under the usual inductive smooth topology. Denote by $D^{-\infty}(M)$ the strong dual of $C_0^\infty(M)$, whose members are called distributions on M. A distribution on M is called a smooth density if under a local coordinate it is the multiple of a smooth function with the Lebesgue measure. Under the inductive smooth topology, the space $D_0^\infty(M)$ of compactly supported smooth densities is again a complete locally convex topological vector space, which is (noncanonically) isomorphic to $C_0^\infty(M)$. Denote by $C^{-\infty}(M)$ the strong dual of $D_0^\infty(M)$, whose members are called generalized functions on M. The space $C^\infty(M)$ of smooth functions is canonically and continuously embedded in $C^{-\infty}(M)$, with a dense image.

If $\phi : M \to M'$ is a smooth map of smooth manifolds, then the pushing forward sends compactly supported distributions on M to compactly supported distributions on M'. If furthermore ϕ is a submersion, then the pushing forward induces a continuous linear map

$$\phi_* : D_0^\infty(M) \to D_0^\infty(M').$$

We define the pulling back

$$\phi^* : C^{-\infty}(M') \to C^{-\infty}(M) \tag{2.1}$$

as the transpose of ϕ_*, which extends the usual pulling back of smooth functions. The map ϕ^* is injective if ϕ is a surjective submersion.

Remark. Pulling back is not canonically defined for distributions. For this reason, we work with generalized functions instead of distributions.

For $k \in \mathbb{Z}$, denote by $\mathrm{DO}(M)_k$ the Fréchet space of differential operators on M of order at most k, which by convention is 0 if $k < 0$. It is well known that every differential operator $D : C^\infty(M) \to C^\infty(M)$ may be continuously extended to $D : C^{-\infty}(M) \to C^{-\infty}(M)$.

Recall that we have the principal symbol map

$$\sigma_k : \mathrm{DO}(M)_k \to \Gamma^\infty(M, \mathrm{S}^k(\mathrm{T}(M) \otimes_{\mathbb{R}} \mathbb{C})),$$

where $\mathrm{T}(M)$ is the real tangent bundle of M, S^k stands for the k-th symmetric power, and Γ^∞ stands for smooth sections. The continuous linear map σ_k is specified by the following rule:

$$\sigma_k(X_1 X_2 \cdots X_k)(x) = X_1(x) X_2(x) \cdots X_k(x) \quad \text{and}$$
$$\sigma_k|_{\mathrm{DO}(M)_{k-1}} = 0,$$

for all $x \in M$ and all (smooth real) vector fields $X_1, X_2, \cdots, X_k$ on M.

Let Z be a (locally closed) submanifold of M. Write

$$\mathrm{N}_Z(M) = \mathrm{T}(M)|_Z / \mathrm{T}(Z)$$

for the normal bundle of Z in M. Denote by

$$\sigma_{k,Z} : \mathrm{DO}(M)_k \to \Gamma^\infty(Z, \mathrm{S}^k(\mathrm{N}_Z(M) \otimes_{\mathbb{R}} \mathbb{C}))$$

the map formed by composing σ_k with the restriction map to Z and followed by the quotient map

$$\Gamma^\infty(Z, \mathrm{S}^k(\mathrm{T}(M)|_Z \otimes_{\mathbb{R}} \mathbb{C})) \to \Gamma^\infty(Z, \mathrm{S}^k(\mathrm{N}_Z(M) \otimes_{\mathbb{R}} \mathbb{C})).$$

Definition 2.1. (a) A vector field X on M is said to be tangential to Z if $X(z)$ is in the tangent space $\mathrm{T}_z(Z)$ for all $z \in Z$, and transversal to Z if $X(z) \notin \mathrm{T}_z(Z)$ for all $z \in Z$; more generally

(b) a differential operator D is said to be tangential to Z if for every point $z \in Z$ there is an open neighborhood U_z in M such that $D|_{U_z}$ is a finite sum of differential operators of the form $\varphi X_1 X_2 \cdots X_r$, where φ is a smooth function on U_z, $r \geq 0$, and $X_1, X_2, \cdots, X_r$ are vector fields on U_z which are tangential to $U_z \cap Z$. For $D \in \mathrm{DO}(M)_k$, it is said to be transversal to Z if $\sigma_{k,Z}(D)$ does not vanish at any point of Z.

We introduce some notation. For a locally closed subset Z of M, denote

$$C^{-\infty}(M; Z) = \{f \in C^{-\infty}(U) \mid \mathrm{supp}(f) \subseteq Z\}, \tag{2.2}$$

where U is any open subset of M containing Z as a closed subset. This definition is independent of U. For any differential operator D on M, denote

$$C^{-\infty}(M; Z; D) = \{f \in C^{-\infty}(M; Z) \mid Df = 0\}. \tag{2.3}$$

We record the following lemma, which is due to Shalika (cf. the proof of Proposition 2.10 in [S74]).

Lemma 2.2. *Let D_1 be a differential operator on M of order $k \geq 1$, which is transversal to a submanifold Z of M. Let D_2 be a differential operator on M which is tangential to Z. Then*

$$C^{-\infty}(M; Z; D_1 + D_2) = 0.$$

2.2. **Invariant generalized functions.** Let H be a Lie group, acting smoothly on a manifold M. Fix a character χ on H. Denote by

$$C_{\chi}^{-\infty}(M) = \{f \in C^{-\infty}(M) \mid f(hx) = \chi(h)f(x), \text{ for } h \in H\} \tag{2.4}$$

the space of χ-equivariant generalized functions.

Let $\mathfrak{M}$ be a submanifold of M and denote

$$\rho_{\mathfrak{M}} : H \times \mathfrak{M} \to M$$

as the action map.

Definition 2.3. (a) We say that $\mathfrak{M}$ is a local H slice of M if $\rho_{\mathfrak{M}}$ is a submersion and an H slice of M if $\rho_{\mathfrak{M}}$ is a surjective submersion.

(b) Given two submanifolds $\mathfrak{Z} \subset \mathfrak{M}$ of M, we say that $\mathfrak{Z}$ is relatively H stable in $\mathfrak{M}$ if

$$\mathfrak{M} \cap H\mathfrak{Z} = \mathfrak{Z}.$$

Note that the relative stable condition amounts to saying that $H \times \mathfrak{Z}$ is a union of fibres of the action map $\rho_{\mathfrak{M}}$. We first prove the following two lemmas in a general setting.

Lemma 2.4. *Let Z be a subset of $\mathbb{R}^m$. Assume that $Z \times \mathbb{R}^n$ is a submanifold of $\mathbb{R}^{m+n}$. Then Z is a submanifold of $\mathbb{R}^m$.*

Proof. Let $z_0 \in Z$. Then there is an open neighborhood $U \times V$ of $(z_0, 0)$ in $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ and a submersion

$$\phi : U \times V \to \mathbb{R}^d$$

such that

$$(U \times V) \cap (Z \times \mathbb{R}^n) = \phi^{-1}(0).$$

Denote by

$$\phi' : U \to \mathbb{R}^d$$

the restriction of ϕ to $U = U \times \{0\}$. Then

$$Z \cap U = \phi'^{-1}(0),$$

and therefore it suffices to show that ϕ' is submersive at every point $z \in Z \cap U$.

Since ϕ is submersive, we have that

$$d\phi|_{z,0}(\mathbb{R}^m \oplus \mathbb{R}^n) = \mathbb{R}^d.$$

Since ϕ is constant on $\{z\} \times V$, we have that

$$d\phi|_{z,0}(\mathbb{R}^n) = 0.$$

Therefore,

$$d\phi|_{z,0}(\mathbb{R}^m) = \mathbb{R}^d,$$

which implies that ϕ' is submersive at z. $\square$

Lemma 2.5. *Let $\rho : M_1 \to M_2$ be a surjective submersion of smooth manifolds. Let Z_1 be a submanifold of M_1 which is a union of fibres of ρ. Then $Z_2 := \rho(Z_1)$ is a submanifold of M_2, and the restriction $\rho_0 : Z_1 \to Z_2$ of ρ is also a surjective submersion. Furthermore, if Z_1 is closed in M_1, then Z_2 is closed in M_2.*

Proof. Write

$$n_1 := \dim M_1 \geq n_2 := \dim M_2.$$

Take two open embeddings $i_1 : \mathbb{R}^{n_1} \hookrightarrow M_1$ and $i_2 : \mathbb{R}^{n_2} \hookrightarrow M_2$ such that the diagram

$$\begin{array}{ccc} \mathbb{R}^{n_1} & \xrightarrow{i_1} & M_1 \\ {\scriptstyle \rho'}\downarrow & & \downarrow{\scriptstyle \rho} \\ \mathbb{R}^{n_2} & \xrightarrow{i_2} & M_2 \end{array}$$

commutes, where ρ' is the projection to the first n_2 coordinates.

Write

$$Z_1' := i_1^{-1}(Z_1) \subset \mathbb{R}^{n_1} \quad \text{and} \quad Z_2' := i_2^{-1}(Z_2) \subset \mathbb{R}^{n_2}.$$

Then Z_1' is a submanifold of $\mathbb{R}^{n_1}$ which is a union of fibres of ρ'. The condition that Z_1 is a union of fibres of ρ implies that

$$Z_2' = \rho'(Z_1').$$

By the local triviality of submersions, it suffices to prove the lemma for ρ' and Z_1'. The latter is now immediate in view of Lemma 2.4 and the fact that

$$Z_1' = Z_2' \times \mathbb{R}^{n_1 - n_2}.$$

□

By setting

$$(\rho : M_1 \to M_2) = (\rho_{\mathfrak{M}} : H \times \mathfrak{M} \to M)$$

and

$$Z_1 = H \times \mathfrak{Z}$$

in Lemma 2.5, we have the following.

Lemma 2.6. *Let $\mathfrak{M}$ be an H slice of M, and let $\mathfrak{Z}$ be a relatively H stable submanifold of $\mathfrak{M}$. Then $Z = H\mathfrak{Z}$ is a submanifold of M, and $\mathfrak{Z}$ is an H slice of Z. Furthermore, if $\mathfrak{Z}$ is closed in $\mathfrak{M}$, then Z is closed in M.*

Lemma 2.6 will be used extensively in Sections 5 and 6.

Now assume that $\mathfrak{M}$ is a local H slice of M and that $H_{\mathfrak{M}}$ is a closed subgroup of H which leaves $\mathfrak{M}$ stable. Let H act on $H \times \mathfrak{M}$ by left multiplication on the first factor, and let $H_{\mathfrak{M}}$ act on $H \times \mathfrak{M}$ by

$$g(h, x) = (ghg^{-1}, gx), \quad g \in H_{\mathfrak{M}}, h \in H, x \in \mathfrak{M}.$$

Then the submersion $\rho_{\mathfrak{M}}$ is H intertwining as well as $H_{\mathfrak{M}}$ intertwining. Therefore the pulling back yields a linear map

$$\rho_{\mathfrak{M}}^* : C_\chi^{-\infty}(M) \to C_\chi^{-\infty}(H \times \mathfrak{M}) \cap C_{\chi_{\mathfrak{M}}}^{-\infty}(H \times \mathfrak{M}),$$

where $\chi_{\mathfrak{M}} = \chi|_{H_{\mathfrak{M}}}$. By the Schwartz Kernel Theorem and the fact that every invariant distribution on a Lie group is a scalar multiple of the Haar measure ([W88, 8.A]), we have

$$C_\chi^{-\infty}(H \times \mathfrak{M}) = \chi \otimes C^{-\infty}(\mathfrak{M}).$$

Consequently,

$$C_\chi^{-\infty}(H \times \mathfrak{M}) \cap C_{\chi_{\mathfrak{M}}}^{-\infty}(H \times \mathfrak{M}) = \chi \otimes C_{\chi_{\mathfrak{M}}}^{-\infty}(\mathfrak{M}).$$

We shall record this as

Lemma 2.7. *There is a well-defined map which is called the restriction to $\mathfrak{M}$:*

$$C^{-\infty}_{\chi}(M) \to C^{-\infty}_{\chi_{\mathfrak{M}}}(\mathfrak{M}), \quad f \mapsto f|_{\mathfrak{M}}$$

by requiring that

$$\rho^*_{\mathfrak{M}}(f) = \chi \otimes f|_{\mathfrak{M}}.$$

The map is injective when $\mathfrak{M}$ is an H slice.

2.3. **Nash manifolds and tempered generalized functions.** We begin with a review of basic concepts and properties of Nash manifolds, in which the notion of tempered generalized functions is defined. Our main reference on Nash manifolds is [S87], and temperedness is discussed in [C91, AG1].

Remark. We will use Fourier transforms implicitly in Section 7, and explicitly in Section 8. Fourier transforms are only defined for tempered generalized functions. This is the main reason that we work with tempered generalized functions instead of arbitrary generalized functions.

Recall that the collection $\mathcal{SA}_n$ of semialgebraic subsets of $\mathbb{R}^n$ is the smallest set with the following properties:

(a) every element of $\mathcal{SA}_n$ is a subset of $\mathbb{R}^n$;
(b) for every real polynomial function p on $\mathbb{R}^n$, we have

$$\{x \in \mathbb{R}^n \mid p(x) > 0\} \in \mathcal{SA}_n;$$

(c) $\mathcal{SA}_n$ is closed under the operation of taking intersection, and taking complement in $\mathbb{R}^n$.

A Nash manifold of dimension n is a manifold M, together with a collection $\mathcal{N}$, whose members are called Nash charts, such that the following hold:

(a) every Nash chart has the form (ϕ, U, U'), where U is an open semialgebraic subset of $\mathbb{R}^n$, U' is an open subset of M, and $\phi : U \to U'$ is a diffeomorphism;
(b) every two Nash charts (ϕ_1, U_1, U_1') and (ϕ_2, U_2, U_2') are Nash compatible, i.e., the graph of the diffeomorphism

$$\phi_2^{-1} \circ \phi_1 : \phi_1^{-1}(U_1' \cap U_2') \to \phi_2^{-1}(U_1' \cap U_2')$$

is semialgebraic;
(c) for every triple (ϕ, U, U') as in (a), if it is Nash compatible with all Nash charts, then itself is a Nash chart;
(d) there are finitely many Nash charts (ϕ_i, U_i, U_i'), $i = 1, 2, \cdots r$, such that

$$M = U_1' \cup U_2' \cup \cdots \cup U_r'.$$

A subset Z of M is called semialgebraic if

$$\phi^{-1}(Z \cap U') \text{ is semialgebraic in } \mathbb{R}^n$$

for all Nash charts (ϕ, U, U') of M. A Nash manifold is either the empty set or a nonempty Nash manifold of dimension $n \geq 0$. A submanifold of a Nash manifold which is semialgebraic is called a Nash submanifold, which is automatically a Nash manifold. The product of two Nash manifolds is again a Nash manifold. A smooth map $\phi : M_1 \to M_2$ of Nash manifolds is called a Nash map if its graph is semialgebraic in $M_1 \times M_2$. (A Nash map always sends a semialgebraic set to a semialgebraic set.) A Nash function on a Nash manifold M is a Nash map from M to $\mathbb{C}$, and a differential operator D on M is called Nash if $D(f)$ is Nash for every Nash function f on every Nash open submanifold of M.

A Nash group is a group as well as a Nash manifold such that the group operations are Nash maps. A Nash action of a Nash group on a Nash manifold is defined similarly.

We proceed to our discussion on the notion of tempered generalized functions on a Nash manifold. A smooth function f on a semialgebraic open subset U of $\mathbb{R}^n$ is called a Schwartz function if $D(f)$ is bounded for every Nash differential operator D on U. Denote by $\mathcal{S}(U)$ the Fréchet space of Schwartz functions on U. Now let M be a Nash manifold of dimension n. Pick a covering of M by Nash charts (ϕ_i, U_i, U_i'), $i = 1, 2, \cdots, r$. By extending to zero outside U_i', ϕ_i induces a continuous linear map

$$(\phi_i)_* : \mathcal{S}(U_i) \to C^\infty(M).$$

The Fréchet space of Schwartz functions on M, denoted by $\mathcal{S}(M)$, is then defined to be the image of the map

$$\bigoplus (\phi_i)_* : \bigoplus_{i=1}^{r} \mathcal{S}(U_i) \to C^\infty(M),$$

equipped with the quotient topology of $\bigoplus_{i=1}^r \mathcal{S}(U_i)$. This definition is independent of the covering we choose. One may similarly define the Fréchet space of Schwartz densities. Denote by $C^{-\xi}(M)$ its strong dual whose members are called tempered generalized functions. All tempered generalized functions are generalized functions.

Now let H be a Nash group with a Nash action on a Nash manifold M. For any character χ on H, we set

$$C_\chi^{-\xi}(M) = C_\chi^{-\infty}(M) \cap C^{-\xi}(M). \tag{2.5}$$

Let N be a Nash manifold, and let $\phi : M \to N$ be an H invariant Nash map. We record the following obvious fact as a lemma.

Lemma 2.8. *If $C_\chi^{-\xi}(M) = 0$, then $C_\chi^{-\xi}(\phi^{-1}(N')) = 0$ for all Nash open submanifolds N' of N.*

Let $\mathfrak{M}$, $H_{\mathfrak{M}}$ and $\chi_{\mathfrak{M}}$ be as in Lemma 2.7. Furthermore, if $\mathfrak{M}$ is a Nash submanifold of M and $H_{\mathfrak{M}}$ is a Nash subgroup of H, then the restriction map sends $C_\chi^{-\xi}(M)$ into $C_{\chi_{\mathfrak{M}}}^{-\xi}(\mathfrak{M})$.

3. Metrical properness and unipotent χ-incompatibility

3.1. **Metrical properness.** This notion requires that the manifold M is pseudo-Riemannian, i.e., the tangent spaces are equipped with a smoothly varying family $\{\langle\, ,\, \rangle_x : x \in M\}$ of nondegenerate symmetric bilinear forms.

Definition 3.1. (a) A submanifold Z of a pseudo-Riemannian manifold M is said to be metrically proper if for all $z \in Z$ the tangent space $\mathrm{T}_z(Z)$ is contained in a proper nondegenerate subspace of $\mathrm{T}_z(M)$.

(b) A differential operator $D \in \mathrm{DO}(M)_2$ is said to be of Laplacian type if for all $x \in M$ the principal symbol

$$\sigma_2(D)(x) = u_1 v_1 + u_2 v_2 + \cdots + u_m v_m,$$

where $u_1, u_2, \cdots, u_m$ is a basis of the tangent space $\mathrm{T}_x(M)$, and $v_1, v_2, \cdots, v_m$ is the dual basis in $\mathrm{T}_x(M)$ with respect to $\langle\, ,\, \rangle_x$.

Note that a Laplacian type differential operator is transversal to any metrically proper submanifold from its very definition. Therefore the following is a special case of Lemma 2.2.

Lemma 3.2. *Let Z be a metrically proper submanifold of M, and let D be a Laplacian type differential operator on M. Then*

$$C^{-\infty}(M; Z; D) = 0.$$

3.2. **Unipotent χ-incompatibility.** As in Section 2.2, let H be a Lie group with a character χ on it, acting smoothly on a manifold M. If a locally closed subset Z of M is H stable, denote by $C_\chi^{-\infty}(M; Z)$ the space of all f in $C^{-\infty}(M; Z)$ which are χ-equivariant. We shall use similar notation (such as $C_\chi^{-\infty}(M; D)$ and $C_\chi^{-\infty}(M; Z; D)$) without further explanation.

Definition 3.3. An H stable submanifold Z of M is said to be unipotently χ-incompatible if for every $z_0 \in Z$ there is a local H slice $\mathfrak{Z}$ of Z, containing z_0, and a smooth map $\phi : \mathfrak{Z} \to H$ such that the following hold for all $z \in \mathfrak{Z}$:

(a) $\phi(z)z = z$ and
(b) the linear map

$$\mathrm{T}_z(M)/\,\mathrm{T}_z(Z) \to \mathrm{T}_z(M)/\,\mathrm{T}_z(Z)$$

induced by the action of $\phi(z)$ on M is unipotent;
(c) $\chi(\phi(z)) \neq 1$.

The following lemma will be important for later considerations.

Lemma 3.4. *Let Z be an H stable submanifold of M which is unipotently χ-incompatible. Then $C_\chi^{-\infty}(M; Z) = 0$.*

By using a well-known result of L. Schwartz on the filtration of the sheaf of generalized functions with supports in a submanifold, Lemma 3.4 is implied by the following.

Sublemma 3.5. *Let $\mathfrak{Z}$ be an H slice of an H manifold Z. Let E be an H equivariant smooth complex vector bundle over Z, of finite rank. Assume that there is a smooth map $\phi : \mathfrak{Z} \to H$ such that for all $z \in \mathfrak{Z}$,*

(a) *$\phi(z)z = z$ and*
(b) *the linear map*

$$\phi(z) : E_z \to E_z$$

is unipotent, where E_z is the fibre of E at z;
(c) $\chi(\phi(z)) \neq 1$.

Then

$$\Gamma_\chi^{-\infty}(E) = 0.$$

Here and as usual, "$\Gamma^{-\infty}$" stands for the space of generalized sections. (We omit its definition since it is a straightforward generalization of the notion of generalized functions in Section 2.1.) The space $\Gamma_\chi^{-\infty}(E)$ consists of all $f \in \Gamma^{-\infty}(E)$ such that

$$f(hx) = \chi(h)h(f(x)), \quad \text{for all } h \in H. \tag{3.1}$$

The meaning of (3.1) will be made clear in the following proof.

Proof. As in the case of generalized functions, define the pulling back

$$\rho_{\mathfrak{z}}^*: \Gamma^{-\infty}(E) \to \Gamma^{-\infty}(\tilde{E})$$

of the action map

$$\rho_{\mathfrak{z}}: H \times \mathfrak{Z} \to Z,$$

which continuously extends the usual pulling back of smooth sections. Here $\tilde{E}$ is the pulling back of E via $\rho_{\mathfrak{z}}$, which is obviously an H equivariant vector bundle over $H \times \mathfrak{Z}$. Note that the bundle $\tilde{E}|_{\{e\}\times\mathfrak{Z}}$ is identified with $E|_{\mathfrak{Z}}$. The restriction $f|_{\mathfrak{Z}} \in \Gamma^{-\infty}(E|_{\mathfrak{Z}})$ of an element $f \in \Gamma_\chi^{-\infty}(E)$ is then specified by

$$\tilde{f}(h, z) = \chi(h)hf|_{\mathfrak{Z}}(z), \quad \text{where } \tilde{f} = \rho_{\mathfrak{z}}^*(f). \tag{3.2}$$

Cf. Lemma 2.7. Here we caution the reader due to the fact that we are dealing with generalized (as opposed to smooth) sections. The formula (3.2) is to be understood as an equality in $\Gamma^{-\infty}(\tilde{E})$. The right-hand side makes sense since the map

$$\Gamma^{\infty}(E|_{\mathfrak{Z}}) \to \Gamma^{\infty}(\tilde{E}), \quad f'(z) \mapsto \chi(h)hf'(z)$$

of smooth sections extends continuously to a (well-defined) map

$$\Gamma^{-\infty}(E|_{\mathfrak{Z}}) \to \Gamma^{-\infty}(\tilde{E}), \quad f'(z) \mapsto \chi(h)hf'(z) \tag{3.3}$$

of generalized sections. Similarly, all the equalities below, which are obvious when $f|_{\mathfrak{Z}}$ is a smooth section, make sense and hold true by a continuity argument.

Condition (a) implies

$$\tilde{f}(h\phi(z), z) = \tilde{f}(h, z),$$

and (3.2) implies

$$\tilde{f}(h\phi(z), z) = \chi(h\phi(z))h\phi(z)f|_{\mathfrak{Z}}(z).$$

Therefore

$$\chi(h\phi(z))h\phi(z)f|_{\mathfrak{Z}}(z) = \chi(h)hf|_{\mathfrak{Z}}(z). \tag{3.4}$$

Since the map (3.3) is injective, (3.4) implies that

$$(\chi(\phi(z))\phi(z) - 1_{E_z})f|_{\mathfrak{Z}}(z) = 0,$$

where $\phi(z)$ is viewed as a linear automorphism of E_z and 1_{E_z} is the identity map of E_z. Conditions (b) and (c) imply that $\chi(\phi(z))\phi(z) - 1_{E_z}$ is invertible on E_z, and so

$$f|_{\mathfrak{Z}}(z) = (\chi(\phi(z))\phi(z) - 1_{E_z})^{-1}(\chi(\phi(z))\phi(z) - 1_{E_z})f|_{\mathfrak{Z}}(z) = 0,$$

which implies that $f = 0$. □

Recall the notion of a Nash group from Section 2.3. It is said to be unipotent if it is Nash isomorphic to a connected closed subgroup of some U_n, where U_n is the Nash group of unipotent upper triangular real matrices of size n. An element of a Nash group is said to be (Nash) unipotent if it is contained in a unipotent Nash closed subgroup. We note that the general linear group $\mathrm{GL}_n(\mathbb{K})$ is Nash and an element of $\mathrm{GL}_n(\mathbb{K})$ is (Nash) unipotent if and only if it is unipotent in the usual sense, i.e., is a unipotent linear transformation.

If H, M and the action of H on M are all Nash, then an H stable submanifold Z of M is unipotently χ-incompatible if the following holds: for every point $z_0 \in Z$, there is a local H slice $\mathfrak{Z}$ of Z, containing z_0, and a smooth map $\phi : \mathfrak{Z} \to H$ such that, for all $z \in \mathfrak{Z}$,

(a) $\phi(z)z = z$ and
(b) $\phi(z)$ is (Nash) unipotent;
(c) $\chi(\phi(z)) \neq 1$.

The reason for this is that the hypothesis of Nash action ensures that the map

$$\mathrm{T}_z(M) \to \mathrm{T}_z(M),$$

induced by the action of the unipotent element $\phi(z)$, is unipotent. This implies condition (b) in Definition 3.3.

3.3. $\mathrm{U}_\chi\mathrm{M}$ **property.** As before, let H be a Lie group acting smoothly on a manifold M, and let χ be a character on H. We further assume that M is a pseudo-Riemannian manifold.

Definition 3.6. We say that an H stable locally closed subset Z of M has $\mathrm{U}_\chi\mathrm{M}$ property if there is a finite filtration

$$Z = Z_0 \supset Z_1 \supset \cdots \supset Z_k \supset Z_{k+1} = \emptyset$$

of Z by H stable closed subsets of Z such that each $Z_i \setminus Z_{i+1}$ is a submanifold of M which is either unipotently χ-incompatible or metrically proper in M.

As a combination of Lemma 3.2 and Lemma 3.4, we have

Lemma 3.7. *Let D be a differential operator on M of Laplacian type. Let Z be an H stable closed subset of M having $\mathrm{U}_\chi\mathrm{M}$ property. Then*

$$C_\chi^{-\infty}(M; Z; D) = 0.$$

4. Small submanifolds of GL_6

We return to the group $G = \mathrm{GL}_6(\mathbb{K})$. Recall from the introduction the subgroup S and its character χ_S. From now on, we set

$$H = S \times S \quad \text{and} \quad \chi = \chi_S \otimes \chi_S. \tag{4.1}$$

Let H act on G by

$$(g_1, g_2)x = g_1 x g_2^\tau.$$

Our main object of concern is the space $C_\chi^{-\infty}(G)$.

For $x \in G$, define its rank matrix

$$\mathrm{R}(x) = \begin{bmatrix} \mathrm{rank}_{4\times 4}(x) & \mathrm{rank}_{4\times 2}(x) \\ \mathrm{rank}_{2\times 4}(x) & \mathrm{rank}_{2\times 2}(x) \end{bmatrix},$$

where $\mathrm{rank}_{i\times j}(x)$ is the rank of the lower right $i \times j$ block of x. Then $R(x)$ takes the following 21 possible values [N06]:

$$\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

For R one of the above, denote

$$G_R = \{x \in G \mid \mathrm{R}(x) = R\}. \tag{4.2}$$

Then

$$G = \bigsqcup_R G_R$$

is the decomposition of G into P-P^τ double cosets. Define an open submanifold as

$$\begin{aligned} G' = \{x \in G \mid \mathrm{rank}_{2\times 4}(x) = \mathrm{rank}_{4\times 2}(x) = 2,\\ \mathrm{rank}_{2\times 2}(x) \geq 1,\ \mathrm{rank}_{4\times 4}(x) \geq 3\}. \end{aligned}$$

Then we have

$$G' = \bigsqcup G_R, \tag{4.3}$$

where R in the union runs through the following four matrices:

$$\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}. \tag{4.4}$$

Let Δ be the Casimir operator on G, as in the introduction. The goal of this section is to prove the following.

Proposition 4.1. *Let $f \in C^{-\infty}_\chi(G)$. If f is an eigenvector of Δ and f vanishes on G', then $f = 0$.*

Set

$$x_{\mathrm{left}} = \begin{bmatrix} 0 & I_2 & 0 \\ 0 & 0 & I_2 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{gl}_6(\mathbb{K}) \quad \text{and } x_{\mathrm{right}} = x_{\mathrm{left}}^\tau. \tag{4.5}$$

Denote by X_{left} the left invariant vector field on G whose tangent vector at x is xx_{left} and by X_{right} the right invariant vector field on G whose tangent vector at x is $x_{\mathrm{right}}x$.

The key to Proposition 4.1 is the following transversality result. We shall divide it into a number of lemmas (Lemmas 4.3, 4.5, 4.6, 4.7).

Proposition 4.2. *Assume that R is not one of the four matrices in* (4.4). *Then either X_{left} or X_{right} is transversal to the double coset G_R.*

Lemma 4.3. *If the lower right entry of R is zero, then X_{left} is transversal to G_R.*

Proof. Assume that there is an

$$x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & 0 \end{bmatrix} \in G_R$$

such that

$$X_{\mathrm{left}}(x) \in \mathrm{T}_x(G_R),$$

i.e.,

$$\begin{bmatrix} 0 & x_{11} & x_{12} \\ 0 & x_{21} & x_{22} \\ 0 & x_{31} & x_{32} \end{bmatrix} \in \mathrm{Lie}(P)x + x\mathrm{Lie}(P^\tau).$$

Note that the lower right 2×2 block of very element of $\mathrm{Lie}(P)x + x\mathrm{Lie}(P^\tau)$ is 0. Therefore $x_{32} = 0$, which further implies that the lower right 2×4 block of very element of $\mathrm{Lie}(P)x + x\mathrm{Lie}(P^\tau)$ is 0. Therefore $x_{31} = 0$. This contradicts the fact that x is invertible. $\square$

The following lemma provides a technical simplification.

Lemma 4.4. *Let x, y be two matrices in G_R such that $PxS^\tau = PyS^\tau$. Then*

$$X_{\mathrm{left}}(x) \in \mathrm{T}_x(G_R) \quad \textit{if and only if} \quad X_{\mathrm{left}}(y) \in \mathrm{T}_y(G_R).$$

Proof. Write

$$y = pxq, \quad p \in P, q \in S^\tau,$$

and assume that $X_{\mathrm{left}}(x) \in \mathrm{T}_x(G_R)$, i.e.,

$$xx_{\mathrm{left}} \in \mathrm{Lie}(P)x + x\mathrm{Lie}(P^\tau).$$

One easily checks that

$$x_{\mathrm{left}}q - qx_{\mathrm{left}} \in \mathrm{Lie}(P^\tau).$$

Therefore

$$\begin{aligned} yx_{\mathrm{left}} &= pxqx_{\mathrm{left}} \\ &\in pxx_{\mathrm{left}}q + px\mathrm{Lie}(P^\tau) \\ &\subset p(\mathrm{Lie}(P)x + x\mathrm{Lie}(P^\tau))q + px\mathrm{Lie}(P^\tau) \\ &= \mathrm{Lie}(P)y + y\mathrm{Lie}(P^\tau). \end{aligned}$$

The last equality holds because

$$p\mathrm{Lie}(P) = \mathrm{Lie}(P)p = \mathrm{Lie}(P), \quad q\mathrm{Lie}(P^\tau) = \mathrm{Lie}(P^\tau)q = \mathrm{Lie}(P^\tau).$$

$\square$

Lemma 4.5. *If the second row of R is $[1 \;\; 1]$, then X_{left} is transversal to G_R.*

Proof. Let R be as in the lemma. Then every matrix in G_R is in the same P-S^τ double coset with a matrix of the form

$$x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & 0 & \delta_2 \end{bmatrix} \in G_R,$$

where

$$\delta_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Assume that

$$X_{\mathrm{left}}(x) \in \mathrm{T}_x(G_R),$$

i.e.,

$$\begin{bmatrix} 0 & x_{11} & x_{12} \\ 0 & x_{21} & x_{22} \\ 0 & x_{31} & 0 \end{bmatrix} \in \mathrm{Lie}(P)x + x\mathrm{Lie}(P^\tau).$$

Note that the middle 2×2 block of the last two rows of every matrix in $\mathrm{Lie}(P)x + x\mathrm{Lie}(P^\tau)$ has the form

$$\delta_2 u, \quad u \in \mathfrak{gl}_2(\mathbb{K}).$$

This implies that the first row of x_{31} is zero and, consequently, the fifth row of x is zero, which contradicts the fact that x is invertible. $\square$

Similarly, we have

Lemma 4.6. *If the second column of R is* $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, *then $X_{\rm right}$ is transversal to G_R.*

Lemma 4.7. *If*

$$R = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix},$$

then $X_{\rm left}$ is transversal to G_R.

Proof. Every matrix in G_R is in the same P-S^τ double coset with a matrix of the form

$$x = \begin{bmatrix} x_{11} & x_{12} & 0 \\ x_{21} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in G_R.$$

Assume that

$$X_{\rm left}(x) \in \mathrm{T}_x(G_R),$$

i.e.,

$$\begin{bmatrix} 0 & x_{11} & x_{12} \\ 0 & x_{21} & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathrm{Lie}(P)x + x\mathrm{Lie}(P^\tau).$$

Note that the central 2×2 block of every matrix in $\mathrm{Lie}(P)x + x\mathrm{Lie}(P^\tau)$ is zero, which implies that $x_{21} = 0$. This contradicts the fact that x is invertible. □

The proof of Proposition 4.2 is now finished.

Lemma 4.8. *There exists a nonzero number c, an element $\lambda \in \mathbb{K}^\times$, and a differential operator $D_{\rm left}$ on G, which is tangential to every P-P^τ double coset of G, such that*

$$\Delta f = (cX_{\rm left}(\lambda) + D_{\rm left})f$$

for all $f \in C_\chi^{-\infty}(G)$. Here $X_{\rm left}(\lambda)$ is the left invariant vector field on G whose tangent vector at $x \in G$ is $\lambda x x_{\rm left}$, and $x_{\rm left}$ is given in (4.5). *The same is true if one replaces "left" by "right" everywhere.*

Proof. The Lie algebra $\mathfrak{g}$ of G has a decomposition

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{l} + \mathfrak{n}^\tau,$$

where $\mathfrak{n}$ is the Lie algebra of the unipotent radical N of P and $\mathfrak{l}$ is the Lie algebra of the Levi factor $\mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$. Recall that $\mathfrak{g}$ is equipped with the real trace form. Let $X_1, X_2, \cdots, X_r$ be a basis of $\mathfrak{n}$, and write

$$\Delta_1 = X_1 X_1' + X_2 X_2' + \cdots + X_r X_r' \in \mathrm{U}(\mathfrak{g}),$$

where $X_1', X_2', \cdots, X_r'$ is the dual basis of $X_1, X_2, \cdots, X_r$ in $\mathfrak{n}^\tau$. Note that Δ_1 is independent of the choice of basis of $\mathfrak{n}$. We identify elements of $\mathrm{U}(\mathfrak{g})$ with left invariant (real) differential operators on G as usual. It is then easy to see that

$$\Delta - 2\Delta_1 \in \mathrm{U}(\mathfrak{l}). \tag{4.6}$$

Let $d\chi_S$ be the differential of the character χ_S. Write

$$\chi_{\mathfrak{n}^\tau}(X) = d\chi_S(-X^\tau), \quad X \in \mathfrak{n}^\tau,$$

which defines a character of $\mathfrak{n}^\tau$. Then every generalized function $f \in C_\chi^{-\infty}(G)$ satisfies

$$Xf = -\chi_{\mathfrak{n}^\tau}(X)f, \quad \text{for all } X \in \mathfrak{n}^\tau.$$

Now choose X_1 to be perpendicular to the kernel of $\chi_{\mathfrak{n}^\tau}$. This is unique up to a multiple in $\mathbb{R}^\times$ and has the form $X_{\mathrm{left}}(\lambda)$ for some $\lambda \in \mathbb{K}^\times$. This choice of X_1 also implies that

$$\chi_{\mathfrak{n}^\tau}(X_2') = \chi_{\mathfrak{n}^\tau}(X_3') = \cdots = \chi_{\mathfrak{n}^\tau}(X_r') = 0,$$

and $\chi_{\mathfrak{n}^\tau}(X_1')$ is a nonzero number.

Therefore

$$\Delta_1 f = -\chi_{\mathfrak{n}^\tau}(X_1')X_{\mathrm{left}}(\lambda)f \quad \text{for all } f \in C_\chi^{-\infty}(G). \tag{4.7}$$

Equations (4.6) and (4.7) will now imply the lemma, in view of the fact that a differential operator in $\mathrm{U}(\mathfrak{l})$ is tangential to every P-P^τ double coset. □

Now we are ready to prove Proposition 4.1. Take a sequence

$$G' = G^4_{\mathrm{open}} \subset G^5_{\mathrm{open}} \subset \cdots \subset G^{20}_{\mathrm{open}} \subset G^{21}_{\mathrm{open}} = G$$

of open subsets of G so that every difference $G^i_{\mathrm{open}} \setminus G^{i-1}_{\mathrm{open}}$ is a P-P^τ double coset, $i = 5, 6, \cdots, 21$. Denote by f_i the restriction of $f \in C_\chi^{-\infty}(G)$ to G^i_{open}. We shall use induction to show that all f_i's are zero. Thus assume that $f_{i-1} = 0$.

By Proposition 4.2, either X_{left} or X_{right} is transversal to $G^i_{\mathrm{open}} \setminus G^{i-1}_{\mathrm{open}}$. Without loss of generality assume that X_{left} is transversal to $G^i_{\mathrm{open}} \setminus G^{i-1}_{\mathrm{open}}$. Lemma 4.8 implies that

$$(X_{\mathrm{left}}(\lambda) + D)f_i = 0,$$

where D is a differential operator on G^i_{open} which is tangential to $G^i_{\mathrm{open}} \setminus G^{i-1}_{\mathrm{open}}$. It is clear that X_{left} is transversal to $G^i_{\mathrm{open}} \setminus G^{i-1}_{\mathrm{open}}$ will imply the same for $X_{\mathrm{left}}(\lambda)$. Invoking Lemma 2.2, we see that $f_i = 0$.

5. A submanifold Z_4 of $\mathrm{GL}_4 \times \mathrm{GL}_2$

As always, we equip $G = \mathrm{GL}_6(\mathbb{K})$ with the bi-invariant pseudo-Riemannian metric whose restriction to $\mathrm{T}_e(G) = \mathfrak{gl}_6(\mathbb{K})$ is the real trace form $\langle\,,\,\rangle_{\mathbb{R}}$, given in (1.3).

As in the introduction, write $G_{4,2} = \mathrm{GL}_4(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$, which embeds into G in the usual way. Then $G_{4,2}$ is a nondegenerate submanifold of G, with $\mathrm{T}_e(M) = \mathfrak{gl}_4(\mathbb{K}) \times \mathfrak{gl}_2(\mathbb{K})$. Thus $G_{4,2}$ is itself a pseudo-Riemannian manifold.

Denote

$$\begin{aligned} S_{4,2} &= (\mathrm{GL}_4(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})) \cap S = \left\{ \begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in G \right\}, \\ H_{4,2} &= S_{4,2} \times S_{4,2} \subset H = S \times S, \end{aligned} \tag{5.1}$$

and the character $\chi_{4,2} = \chi|_{H_{4,2}}$.

Let Z_4 be the following $H_{4,2}$ stable submanifold of $G_{4,2}$:

$$Z_4 = \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & y \end{bmatrix} \in G_{4,2} \mid y^{-1}a_{22} \text{ is nilpotent and nonzero} \right\}. \tag{5.2}$$

The purpose of this section is to prove the following proposition. This will take a number of steps.

Proposition 5.1. *As an $H_{4,2}$ submanifold of $G_{4,2}$, Z_4 has* $\mathrm{U}_{\chi_{4,2}}\mathrm{M}$ *property.*

Denote by $\mathfrak{Z}_4$ all matrices in $G_{4,2}$ of the form

$$x = \begin{bmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 0 & 0 \\ x_{31} & x_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{5.3}$$

Lemma 5.2. *The submanifold $\mathfrak{Z}_4$ is an $H_{4,2}$ slice of Z_4.*

Proof. Let $x \in Z_4$ be as in (5.2). Define

$$\bar{\phi}(x) = \begin{bmatrix} a_{22} & 0 \\ 0 & y \end{bmatrix} \in \mathfrak{gl}_4(\mathbb{K}).$$

Note that $\bar{\phi}(\mathfrak{Z}_4)$ consists of a single matrix

$$\bar{x}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The action of $H_{4,2}$ on Z_4 descents to a transitive action on the quotient manifold

$$\bar{Z}_4 = \{\bar{\phi}(x) \mid x \in Z_4\}.$$

Therefore to show that the $H_{4,2}$ equivariant action map

$$\rho_{\mathfrak{Z}_4} : H_{4,2} \times \mathfrak{Z}_4 \to Z_4$$

is a surjective submersion, it suffices to show the same for its restriction map

$$(\bar{\phi} \circ \rho_{\mathfrak{Z}_4})^{-1}(\bar{x}_0) \to \bar{\phi}^{-1}(\bar{x}_0).$$

Denote by $N_{4,2}$ the unipotent radical of $S_{4,2}$. Then

$$(N_{4,2} \times N_{4,2}) \times \mathfrak{Z}_4 \subset (\bar{\phi} \circ \rho_{\mathfrak{Z}_4})^{-1}(\bar{x}_0),$$

and hence it suffices to show that the action map

$$(N_{4,2} \times N_{4,2}) \times \mathfrak{Z}_4 \to \bar{\phi}^{-1}(\bar{x}_0) \tag{5.4}$$

is a surjective submersion.

Now let

$$x = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & 0 & 0 \\ x_{21} & x_{22} & x_{23} & x_{24} & 0 & 0 \\ x_{31} & x_{32} & 0 & 0 & 0 & 0 \\ x_{41} & x_{42} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \bar{\phi}^{-1}(\bar{x}_0).$$

Then $u(x)xv(x) \in \mathfrak{Z}_4$, with

$$u(x) = \begin{bmatrix} 1 & 0 & 0 & -x_{14} & 0 & 0 \\ 0 & 1 & 0 & -x_{24} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$v(x)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -x_{41} & -x_{42} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which proves that the map (5.4) is surjective. One shows similarly that the differential of the map (5.4) is also surjective. □

Let

$$\mathfrak{z}_{4,1}=\{\, x\in \mathfrak{z}_4 \text{ of the form (5.3) with } x_{13}=x_{31}\,\}.$$

Lemma 5.3. *The closed submanifold $\mathfrak{z}_{4,1}$ is relatively $H_{4,2}$ stable in $\mathfrak{z}_4$.*

Proof. Let $x\in \mathfrak{z}_{4,1}$,

$$g=\begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}\in S_{4,2} \quad \text{and} \quad g'=\begin{bmatrix} a' & 0 & 0 \\ b' & a' & 0 \\ 0 & 0 & a' \end{bmatrix}\in S_{4,2}^{\tau}.$$

We need to show that $gxg'\in \mathfrak{z}_{4,1}$, provided that $gxg'\in \mathfrak{z}_4$. The condition $gxg'\in \mathfrak{z}_4$ implies that

$$a\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}a'=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad a\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}a'=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

which is equivalent to

$$a=\alpha\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \quad \text{and} \quad a'=\alpha^{-1}\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix},$$

for some $\alpha\in \mathbb{K}^\times$ and $t\in \mathbb{K}$. It is now straightforward to check that $gxg'\in \mathfrak{z}_{4,1}$. □

Remark. In the sequel, we will skip the verification when we assert that a submanifold is relatively stable or is a slice with respect to a certain group action.

Write

$$Z_{4,1}=H_{4,2}\mathfrak{z}_{4,1},$$

which is a closed submanifold of Z_4, by Lemma 2.6.

Lemma 5.4. *The submanifold $Z_4\setminus Z_{4,1}$ is unipotently $\chi_{4,2}$-incompatible.*

Proof. Let $x\in \mathfrak{z}_4\setminus \mathfrak{z}_{4,1}$ be as in (5.3) and write

$$u(x,t)=\begin{bmatrix} 1 & 0 & x_{13}t & 0 & 0 & 0 \\ 0 & 1 & x_{23}t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$v(x,t)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ tx_{31} & tx_{32} & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$u(x,t)x = xv(t,x), \quad \text{i.e.,} \quad (u(x,t), v(x,t)^{-\tau})x = x.$$

Since $x_{13} \neq x_{31}$,

$$\chi_{4,2}(u(x,t), v(x,t)^{-\tau}) = \psi_{\mathbb{K}}(x_{13}t - x_{31}t) \neq 1$$

for a suitably chosen $t \in \mathbb{K}$. This proves the lemma. □

Write

$$\mathfrak{Z}_{4,2} = \{\, x \in \mathfrak{Z}_{4,1} \text{ of the form (5.3) with } x_{13} = x_{31} = 0 \,\},$$

which is a relatively $H_{4,2}$ stable closed submanifold of $\mathfrak{Z}_{4,1}$. Therefore

$$Z_{4,2} = H_{4,2}\mathfrak{Z}_{4,2}$$

is a closed submanifold of $Z_{4,1}$.

Lemma 5.5. *The submanifold $Z_{4,1} \setminus Z_{4,2}$ is metrically proper in $G_{4,2}$.*

Proof. Denote by $\mathfrak{Z}'_{4,1}$ all matrices in $G_{4,2}$ of the form

$$x=\begin{bmatrix} 0 & 0 & a & 0 & 0 & 0\\ 0 & x_{22} & 0 & 0 & 0 & 0\\ a & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & b \end{bmatrix}, \tag{5.5}$$

which forms an $H_{4,2}$ slice of $Z_{4,1} \setminus Z_{4,2}$.

Let x be as in (5.5). Then one checks that

$$\mathrm{T}_x(Z_{4,1}) = \mathrm{T}_x(\mathfrak{Z}'_{4,1}) + (\mathrm{Lie}S_{4,2})x + x(\mathrm{Lie}S_{4,2})^{\tau} \subset \mathfrak{gl}_4(\mathbb{K})_{13=31} \times \mathfrak{gl}_2(\mathbb{K}),$$

where $\mathfrak{gl}_4(\mathbb{K})_{13=31}$ is the set of matrices in $\mathfrak{gl}_4(\mathbb{K})$ whose $(1,3)$ entry equals its $(3,1)$ entry. We shall adopt similar notation in the sequel.

Let

$$x' = e_{13} - e_{31} \in \mathfrak{gl}_6(\mathbb{K}),$$

where e_{ij} denotes the matrix with 1 at the (i,j) entry and 0 elsewhere. Then

$$\mathfrak{gl}_4(\mathbb{K})_{13=31} \times \mathfrak{gl}_2(\mathbb{K}) \subset (\mathbb{K}x')^{\perp},$$

where $\perp$ denotes the orthogonal complement with respect to the real trace form. Consequently,

$$x^{-1}\,\mathrm{T}_x(Z_{4,1}) \subset x^{-1}(\mathbb{K}x')^{\perp} = (\mathbb{K}xx')^{\perp}.$$

Note that

$$xx' = a(e_{11} - e_{33}),$$

which spans a nondegenerate $\mathbb{K}$ subspace of $\mathrm{T}_e(G_{4,2})$. This implies that $x^{-1}\,\mathrm{T}_x(Z_{4,1})$ is contained in a proper nondegenerate subspace of $\mathrm{T}_e(G_{4,2})$. Therefore, by invariance of the metric, $\mathrm{T}_x(Z_{4,1})$ is contained in a nondegenerate proper subspace of $\mathrm{T}_x(G_{4,2})$, for any $x \in Z_{4,1} \setminus Z_{4,2}$. □

Denote by $\mathfrak{z}_{4,2}{}'$ all matrices in $\mathfrak{z}_{4,2}$ of the form

$$x = \begin{bmatrix} x_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{23} & 0 & 0 & 0 \\ 0 & x_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \tag{5.6}$$

which also forms an $H_{4,2}$ slice of $Z_{4,2}$.

Write

$$\mathfrak{z}_{4,3}^1{}' = \{\, x \in \mathfrak{z}_{4,2}{}' \text{ of the form (5.6) with } x_{23} = x_{32} \,\}$$

and

$$\mathfrak{z}_{4,3}^2{}' = \{\, x \in \mathfrak{z}_{4,2}{}' \text{ of the form (5.6) with } x_{23}x_{32} + x_{11} = 0 \,\}.$$

They are both relatively $H_{4,2}$ stable closed submanifolds of $\mathfrak{z}_{4,2}{}'$. Therefore both

$$Z_{4,3}^1 = H_{4,2}\mathfrak{z}_{4,3}^1{}' \quad \text{and} \quad Z_{4,3}^2 = H_{4,2}\mathfrak{z}_{4,3}^2{}'$$

are closed submanifolds of $Z_{4,2}$.

Lemma 5.6. *The manifold $Z_{4,2} \setminus (Z_{4,3}^1 \cup Z_{4,3}^2)$ is unipotently $\chi_{4,2}$-incompatible.*

Proof. Let $x \in \mathfrak{z}_{4,2}{}' \setminus (\mathfrak{z}_{4,3}^1{}' \cup \mathfrak{z}_{4,3}^2{}')$ be as in (5.6). Set

$$u(x,t) = \begin{bmatrix} 1 & 0 & x_{11}x_{32}^{-1}t & 0 & 0 & 0 \\ t & 1 & 0 & x_{23}t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & t & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & t & 1 \end{bmatrix}$$

and

$$v(x,t) = \begin{bmatrix} 1 & t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ x_{11}x_{23}^{-1}t & 0 & 1 & t & 0 & 0 \\ 0 & x_{32}t & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $u(x,t)x = xv(x,t)$ and

$$\chi_{4,2}(u(x,t), v(x,t)^{-\tau}) = \psi_{\mathbb{K}}((x_{32}^{-1} - x_{23}^{-1})t(x_{11} + x_{23}x_{32})) \neq 1$$

for a suitably chosen t. The lemma follows. □

Lemma 5.7. *The submanifold $Z_{4,3}^1$ is metrically proper in $G_{4,2}$.*

Proof. Let $x \in \mathfrak{z}_{4,3}^1{}'$ as in (5.6). Write $a = x_{23} = x_{32}$ and

$$x' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & -a & 0 \end{bmatrix}.$$

Then one checks that

$$\mathrm{T}_x(Z^1_{4,3}) = \mathrm{T}_x({\mathfrak{Z}^1_{4,3}}') + (\mathrm{Lie}S_{4,2})x + x(\mathrm{Lie}S_{4,2})^\tau \subset (\mathbb{K}x')^\perp.$$

Note that

$$x'x = a(\mathrm{diag}(0,1,-1,0,1,-1)),$$

which spans a nondegenerate $\mathbb{K}$ subspace of $\mathrm{T}_e(G_{4,2})$. Here and as usual, diag represents a diagonal matrix (with the obvious diagonal entries). The lemma follows, as in the proof of Lemma 5.5. □

Lemma 5.8. *The submanifold $Z^2_{4,3}$ is metrically proper in $G_{4,2}$.*

Proof. Let $x \in {\mathfrak{Z}^2_{4,3}}'$ as in (5.6). Write

$$x' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{23} & 0 & 0 & 0 \\ 0 & x_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_{23}x_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then one checks that

$$\mathrm{T}_x(Z^2_{4,3}) = \mathrm{T}_x({\mathfrak{Z}^2_{4,3}}') + (\mathrm{Lie}S_{4,2})x + x(\mathrm{Lie}S_{4,2})^\tau \subset (\mathbb{K}x')^\perp$$

and

$$x'x = x_{23}x_{32}(\mathrm{diag}(-1,1,1,-1,0,0)).$$

The lemma follows, as before. □

We now consider the $H_{4,2}$ stable filtration

$$Z_4 \supset Z_{4,1} \supset Z_{4,2} \supset Z^1_{4,3} \cup Z^2_{4,3} \supset Z^1_{4,3} \supset \emptyset.$$

In view of the preceding lemmas, the proof of Proposition 5.1 is complete.

6. A submanifold Z_6 of GL_6

Recall from Section 4 the $P - P^\tau$ double coset G_R indexed by a rank matrix R. Set

$$Z_6 = G_R, \quad \text{with } R = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}. \tag{6.1}$$

Clearly Z_6 is an $H = S \times S$ stable submanifold of G, as with each G_R.

The purpose of this section is to prove the following proposition. Again it will take a number of steps.

Proposition 6.1. *As an H submanifold of G, Z_6 has $\mathrm{U}_\chi\mathrm{M}$ property.*

Denote by $\mathfrak{Z}_6$ all matrices in Z_6 of the form

$$x = \begin{bmatrix} * & * & * & * & x_{15} & 0 \\ * & * & * & * & x_{25} & 0 \\ * & * & 0 & 0 & x_{35} & 0 \\ * & * & 0 & 0 & x_{45} & 0 \\ x_{51} & x_{52} & x_{53} & x_{54} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{6.2}$$

which forms an H slice of Z_6. Write

$$\mathfrak{z}_{6,1} = \{\, x \in \mathfrak{z}_6 \text{ of the form (6.2) with } x_{35} = x_{53} \,\}$$

and

$$\mathfrak{z}_{6,2} = \{\, x \in \mathfrak{z}_6 \text{ of the form (6.2) with } x_{35} = x_{53} = 0 \,\}.$$

They are both relatively H stable closed submanifolds of $\mathfrak{z}_6$. Therefore both

$$Z_{6,1} = H\mathfrak{z}_{6,1} \quad \text{and } Z_{6,2} = H\mathfrak{z}_{6,2}$$

are closed submanifolds of Z_6.

Lemma 6.2. *The submanifold $Z_6 \setminus Z_{6,1}$ is unipotently χ-incompatible.*

Proof. Let $x \in \mathfrak{z}_6 \setminus \mathfrak{z}_{6,1}$ be as in (6.2). Write

$$u(x,t) = \begin{bmatrix} 1 & 0 & 0 & 0 & x_{15}t & 0 \\ 0 & 1 & 0 & 0 & x_{25}t & 0 \\ 0 & 0 & 1 & 0 & x_{35}t & 0 \\ 0 & 0 & 0 & 1 & x_{45}t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$v(x,t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ tx_{51} & tx_{52} & tx_{53} & tx_{54} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $u(x,t)x = xv(x,t)$, and the lemma follows, as before. $\square$

Lemma 6.3. *The submanifold $Z_{6,1} \setminus Z_{6,2}$ is metrically proper in G.*

Proof. Every element of $\mathfrak{z}_{6,1} \setminus \mathfrak{z}_{6,2}$ is in the same H-orbit as an element of the form

$$x = \begin{bmatrix} * & * & 0 & * & 0 & 0 \\ * & * & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Fix such an x. Then

$$\mathrm{T}_x(Z_{6,1}) = \mathrm{T}_x(\mathfrak{z}_{6,1}) + \mathrm{Lie}(S)x + x\mathrm{Lie}(S^\tau) \subset \mathfrak{gl}_6(\mathbb{K})_{35=53} = (\mathbb{K}x')^\perp,$$

where $x' = e_{35} - e_{53}$. Now $x'x = a(e_{33} - e_{55})$, and we finish the proof, as before. $\square$

Denote by $\mathfrak{z}'_{6,2}$ all matrices in $\mathfrak{z}_{6,2}$ of the form

$$x = \begin{bmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 0 & 0 \\ x_{31} & x_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{6.3}$$

which also forms an H slice of $Z_{6,2}$. Set

$$\mathfrak{Z}'_{6,3} = \{\, x \in \mathfrak{Z}'_{6,2} \text{ of the form (6.3) with } x_{13} = x_{31} \,\}$$

and

$$\mathfrak{Z}'_{6,4} = \{\, x \in \mathfrak{Z}'_{6,2} \text{ of the form (6.3) with } x_{13} = x_{31} = 0 \,\}.$$

They are both relatively H stable closed submanifolds of $\mathfrak{Z}'_{6,2}$. Therefore both

$$Z_{6,3} = H\mathfrak{Z}'_{6,3} \quad \text{and } Z_{6,4} = H\mathfrak{Z}'_{6,4}$$

are closed submanifolds of $Z_{6,2}$.

Lemma 6.4. *The manifold $Z_{6,2} \setminus Z_{6,3}$ is unipotently χ-incompatible.*

Proof. This is identical to the proof of Lemma 5.4 in Section 5. We omit the details. □

Lemma 6.5. *The submanifold $Z_{6,3} \setminus Z_{6,4}$ is metrically proper in G.*

Proof. Every matrix in $\mathfrak{Z}'_{6,3} \setminus \mathfrak{Z}'_{6,4}$ is in the same H orbit as a matrix of the form

$$x = \begin{bmatrix} 0 & 0 & a & 0 & 0 & 0 \\ 0 & x_{22} & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Fix such an x. Then

$$\mathrm{T}_x(Z_{6,3}) = \mathrm{T}_x(\mathfrak{Z}'_{6,3}) + \mathrm{Lie}(S)x + x\mathrm{Lie}(S^\tau) \subset \mathfrak{gl}_6(\mathbb{K})_{13=31} = (\mathbb{K}x')^\perp,$$

where $x' = e_{13} - e_{31}$. Now $x'x = a(e_{11} - e_{33})$, and we finish the proof, as before. □

Denote by $\mathfrak{Z}''_{6,4}$ all matrices in $\mathfrak{Z}'_{6,4}$ of the form

$$x = \begin{bmatrix} x_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{23} & 0 & 0 & 0 \\ 0 & x_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{6.4}$$

which also forms an H slice of $Z_{6,4}$. Set

$$\mathfrak{Z}''_{6,5} = \{\, x \in \mathfrak{Z}''_{6,4} \text{ of the form (6.4) with } x_{23} = x_{32} \,\},$$

which is a relatively H stable closed submanifold of $\mathfrak{Z}''_{6,4}$. Therefore

$$Z_{6,5} = H\mathfrak{Z}''_{6,5}$$

is a closed submanifold of $Z_{6,4}$.

Lemma 6.6. *The manifold $Z_{6,4} \setminus Z_{6,5}$ is unipotently χ-incompatible.*

Proof. Let $x \in \mathfrak{Z}''_{6,4} \setminus \mathfrak{Z}''_{6,5}$ be as in (6.4). Set

$$u(x,t) = \begin{bmatrix} 1 & 0 & x_{11}x_{32}^{-1}t & 0 & 0 & 0 \\ t & 1 & 0 & 0 & x_{23}t & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & t & 1 & 0 & t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & t & 1 \end{bmatrix}$$

and

$$v(x,t) = \begin{bmatrix} 1 & t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ x_{23}^{-1}tx_{11} & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & tx_{32} & 0 & 0 & 1 & t \\ 0 & 0 & 0 & t & 0 & 1 \end{bmatrix}.$$

Then

$$u(x,t)x = \begin{bmatrix} x_{11} & x_{11}t & 0 & 0 & 0 & 0 \\ tx_{11} & 0 & x_{23} & x_{23}t & 0 & 0 \\ 0 & x_{32} & 0 & 0 & 0 & 0 \\ 0 & tx_{32} & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 1 \end{bmatrix} = xv(x,t),$$

and the lemma follows, as before. □

Lemma 6.7. *The submanifold $Z_{6,5}$ is metrically proper in G.*

Proof. Let $x \in \mathfrak{Z}''_{6,5}$ be as in (6.4), with $x_{23} = x_{32} = a$. Write

$$x' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.5}$$

Then

$$\mathrm{T}_x(Z_{6,5}) = \mathrm{T}_x(Z_{6,5}) + \mathrm{Lie}(S)x + x\mathrm{Lie}(S^\tau) \subset (\mathbb{K}x')^\perp$$

and $x'x = a(\mathrm{diag}(0,1,-1,-1,1,0))$. The lemma follows, as before. □

We now consider the H stable filtration

$$Z_6 \supset Z_{6,1} \supset Z_{6,2} \supset Z_{6,3} \supset Z_{6,4} \supset Z_{6,5} \supset \emptyset.$$

In view of the preceding lemmas, the proof of Proposition 6.1 is complete.

7. The manifold $\mathrm{GL}_2 \times \mathrm{GL}_2$

Set

$$M_2 = \mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K}) = \mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K}) \times \{I_2\} \subset G,$$

which is stable under the subgroup

$$H_2 = \mathrm{GL}_2(\mathbb{K}) = \{(x, x^{-\tau}) \mid x \in \mathrm{GL}_2^\Delta(\mathbb{K})\} \subset H = S \times S.$$

It will be slightly more convenient to work with the following:

$$\tilde{H}_2 = \{1, \tau\} \ltimes \mathrm{GL}_2(\mathbb{K}),$$

where the semidirect product is given by the action

$$\tau(g) = g^{-\tau}.$$

Denote by $\tilde{\chi}_2$ the character of $\tilde{H}_2$ such that

$$\tilde{\chi}_2|_{\mathrm{GL}_2(\mathbb{K})} = 1 \quad \text{and} \quad \tilde{\chi}_2(\tau) = -1.$$

Proposition 7.1. *Let $\tilde{H}_2$ act on $M_2 = \mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$ by*

$$g(x, y) = (gxg^{-1}, gyg^{-1}), \quad g \in \mathrm{GL}_2(\mathbb{K}),$$

and

$$\tau(x, y) = (x^\tau, y^\tau).$$

Then

$$C^{-\xi}_{\tilde{\chi}_2}(M_2) = 0.$$

Proof. Using the same formula, we may extend the action of $\tilde{H}_2$ on $\mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$ to the larger space $\mathfrak{gl}_2(\mathbb{K}) \times \mathfrak{gl}_2(\mathbb{K})$. By Lemma 2.8, it suffices to prove that

$$C^{-\xi}_{\tilde{\chi}_2}(\mathfrak{gl}_2(\mathbb{K}) \times \mathfrak{gl}_2(\mathbb{K})) = 0.$$

Identify $\mathbb{K}$ with the center of $\mathfrak{gl}_2(\mathbb{K})$. We have

$$\mathfrak{gl}_2(\mathbb{K}) \times \mathfrak{gl}_2(\mathbb{K}) = (\mathfrak{sl}_2(\mathbb{K}) \times \mathfrak{sl}_2(\mathbb{K})) \oplus (\mathbb{K} \times \mathbb{K})$$

as a $\mathbb{K}$ linear representation of $\tilde{H}_2$, where $\tilde{H}_2$ acts on $\mathbb{K} \times \mathbb{K}$ trivially. Therefore it suffices to prove that

$$C^{-\xi}_{\tilde{\chi}_2}(\mathfrak{sl}_2(\mathbb{K}) \times \mathfrak{sl}_2(\mathbb{K})) = 0.$$

We view $\mathfrak{sl}_2(\mathbb{K})$ as a three dimensional quadratic space under the trace form. Under this identification, the action of $\tilde{H}_2$ yields the diagonal action of $\mathrm{O}(\mathfrak{sl}_2(\mathbb{K}))$ on $\mathfrak{sl}_2(\mathbb{K}) \times \mathfrak{sl}_2(\mathbb{K})$, with $\tilde{\chi}_2$ corresponding to the determinant character. So the required vanishing result is a special case of Proposition 1.5. □

8. The manifold $\mathrm{GL}_3 \times \mathrm{GL}_1$

Set

$$M_3 = \left\{ \begin{bmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in G \mid x_{33} \neq x_{44} \right\},$$

which is stable under the subgroup H_3 of $H = S \times S$ consisting of elements of the form

$$\left(\begin{bmatrix} a & 0 & * & 0 & 0 & 0 \\ 0 & 1 & * & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} a^{-1} & 0 & * & 0 & 0 & 0 \\ 0 & 1 & * & 0 & 0 & 0 \\ 0 & 0 & a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right).$$

Write

$$L_3 = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} \mid a \in \mathbb{K}^\times \right\}$$

and

$$N_3 = \left\{ \begin{bmatrix} 1 & 0 & d \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid c, d \in \mathbb{K} \right\}.$$

Then

$$H_3 = L_3 \ltimes (N_3 \times N_3),$$

where the semidirect product is defined by the action

$$l(g_1, g_2) = (l g_1 l^{-1}, l^{-1} g_2 l).$$

Define

$$\tilde{H}_3 = \{1, \tau\} \ltimes H_3,$$

with the semidirect product given by the action

$$\tau(l, g_1, g_2) = (l^{-1}, g_2, g_1).$$

Write

$$\chi_{N_3}\left(\begin{bmatrix} 1 & 0 & d \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right) = \psi_{\mathbb{K}}(d),$$

and let $\tilde{\chi}_3$ be the character of $\tilde{H}_3$ such that

$$\tilde{\chi}_3(l, g_1, g_2) = \chi_{N_3}(g_1)\chi_{N_3}(g_2), \quad (l, g_1, g_2) \in H_3,$$

and

$$\tilde{\chi}_3(\tau) = -1.$$

Proposition 8.1. *Let $\tilde{H}_3$ act on*

$$M_3 = \{(x, y) \in \mathrm{GL}_3(\mathbb{K}) \times \mathbb{K}^\times \mid y \neq \textit{the } (3,3) \textit{ entry of } x\} \tag{8.1}$$

by

$$(l, g_1, g_2)(x, y) = (l g_1 x g_2^\tau l^{-1}, y)$$

and

$$\tau(x, y) = (x^\tau, y).$$

Then

$$C^{-\xi}_{\tilde{\chi}_3}(M_3) = 0.$$

Proof. First we note that the (3, 3) entry of x is invariant under $\tilde{H}_3$. Denote by $\mathrm{GL}_3(\mathbb{K})'$ the set of matrices in $\mathrm{GL}_3(\mathbb{K})$ whose (3, 3) entry is not 1. Let $\tilde{H}_3$ act on $\mathrm{GL}_3(\mathbb{K})' \times \mathbb{K}^\times$ by the same formula as its action on M_3. Then the map

$$\begin{array}{rcl} \mathrm{GL}_3(\mathbb{K})' \times \mathbb{K}^\times & \to & M_3, \\ (x, y) & \mapsto & (yx, y) \end{array}$$

is an $\tilde{H}_3$-equivariant Nash diffeomorphism. Therefore

$$C^{-\xi}_{\tilde{\chi}_3}(M_3) \cong C^{-\xi}_{\tilde{\chi}_3}(\mathrm{GL}_3(\mathbb{K})' \times \mathbb{K}^\times).$$

As the action of $\tilde{H}_3$ on $\mathbb{K}^\times$ is trivial, it suffices to show that

$$C^{-\xi}_{\tilde{\chi}_3}(\mathrm{GL}_3(\mathbb{K})') = 0.$$

This will be implied by Lemma 2.8 and Proposition 8.2 below. □

The rest of this section is devoted to the proof of

Proposition 8.2. *Let $\tilde{H}_3$ act on $\mathfrak{gl}_3(\mathbb{K})$ by*

$$(l, g_1, g_2)x = lg_1xg_2^\tau l^{-1}$$

and

$$\tau x = x^\tau.$$

Then

$$C^{-\xi}_{\tilde{\chi}_3}(\mathfrak{gl}_3(\mathbb{K})) = 0.$$

Write

$$Z_{3,1} = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & 0 \end{bmatrix} \in \mathfrak{gl}_3(\mathbb{K}) \right\}$$

and

$$Z_{3,2} = \left\{ \begin{bmatrix} * & * & a \\ * & * & * \\ a & * & 0 \end{bmatrix} \in \mathfrak{gl}_3(\mathbb{K}) \right\}.$$

Lemma 8.3. *One has that $C^{-\xi}_{\tilde{\chi}_3}(\mathfrak{gl}_3(\mathbb{K}) \setminus Z_{3,1}) = 0$.*

Proof. The Nash submanifold $\mathfrak{gl}_2(\mathbb{K}) \times \mathbb{K}^\times$ is an $\tilde{H}_3$-slice of $\mathfrak{gl}_3(\mathbb{K}) \setminus Z_{3,1}$, which is stable under the subgroup

$$\tilde{H}_{3,1} = \{1, \tau\} \ltimes L_3 = \{1, \tau\} \ltimes \mathbb{K}^\times.$$

Denote by $\tilde{\chi}_{3,1}$ the restriction of $\tilde{\chi}_3$ to $\tilde{H}_{3,1}$. Then we have the injective restriction map

$$C^{-\xi}_{\tilde{\chi}_3}(\mathfrak{gl}_3(\mathbb{K}) \setminus Z_{3,1}) \hookrightarrow C^{-\xi}_{\tilde{\chi}_{3,1}}(\mathfrak{gl}_2(\mathbb{K}) \times \mathbb{K}^\times).$$

Let $\tilde{H}_{3,1}$ act on $\mathbb{K}_3 = \mathbb{K} \times \mathbb{K} \times \mathbb{K}^\times$ trivially and act on $\mathbb{K} \times \mathbb{K}$ by

$$a(x, y) = (ax, a^{-1}y), \ a \in \mathbb{K}^\times, \quad \text{and} \quad \tau(x, y) = (y, x).$$

Then

$$\mathfrak{gl}_2(\mathbb{K}) \times \mathbb{K}^\times \cong (\mathbb{K} \times \mathbb{K}) \times \mathbb{K}_3$$

as Nash manifolds with $\tilde{H}_{3,1}$ actions. It thus suffices to show that

$$C^{-\xi}_{\tilde{\chi}_{3,1}}(\mathbb{K} \times \mathbb{K}) = 0. \tag{8.2}$$

We view $\mathbb{K} \times \mathbb{K}$ as a split two dimensional quadratic space so that both $\mathbb{K} \times \{0\}$ and $\{0\} \times \mathbb{K}$ are isotropic. Then $\tilde{H}_{3,1}$ is identified with the orthogonal group $\mathrm{O}(\mathbb{K} \times \mathbb{K})$, with $\tilde{\chi}_{3,1}$ corresponding to the determinant character. So (8.2) is a special case of Proposition 1.5. □

Lemma 8.4. *The H_3 stable manifold $Z_{3,1} \setminus Z_{3,2}$ is unipotently χ_3-incompatible, where $\chi_3 = \tilde{\chi}_3|_{H_3}$.*

Proof. For

$$x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & 0 \end{bmatrix} \in Z_{3,1} \setminus Z_{3,2} \quad \text{and} \quad t \in \mathbb{K},$$

write

$$u(x,t) = \begin{bmatrix} 1 & 0 & x_{13}t \\ 0 & 1 & x_{23}t \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad v(x,t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tx_{31} & tx_{32} & 1 \end{bmatrix}.$$

Then

$$u(x,t)x = xv(t,x),$$

and the lemma follows, as in the proof of Lemma 5.4. □

Lemma 8.3, Lemma 8.4 and Lemma 3.4 now imply the following

Lemma 8.5. *Every generalized function in $C^{-\xi}_{\tilde{\chi}_3}(\mathfrak{gl}_3(\mathbb{K}))$ is supported in $Z_{3,2}$.*

We shall employ a Fourier transform to finish the proof of Proposition 8.2. In general, let E be a finite dimensional real vector space, equipped with a nondegenerate symmetric bilinear form $\langle\, , \,\rangle_E$. The Fourier transform is a topological linear isomorphism

$$\widehat{\ } : \mathcal{S}(E) \to \mathcal{S}(E)$$

of the space of Schwartz functions, given by

$$\widehat{f}(x) = \int_E f(y) e^{-2\pi\sqrt{-1}\,\langle x,y\rangle_E}\, dy,$$

where dy is the Lebesgue measure on E, normalized such that the volume of the cube

$$\{t_1 v_1 + t_2 v_2 + \cdots + t_r v_r \mid 0 \leq t_1, t_2, \cdots, t_r \leq 1\}$$

is 1, for any orthogonal basis $v_1, v_2, \cdots, v_r$ of E such that $\langle v_i, v_i\rangle_E = \pm 1$, $i = 1, 2, \cdots, r$. The Fourier transform extends continuously to a topological linear isomorphism

$$\widehat{\ } : C^{-\xi}(E) \to C^{-\xi}(E),$$

which is still called the Fourier transform.

The following lemma is a form of the uncertainty principle.

Lemma 8.6. *Let $f \in C^{-\xi}(E)$. If both f and $\widehat{f}$ are supported in a common nondegenerate proper subspace of E, then $f = 0$.*

Proof. Let $v \in E$ be a nondegenerate vector such that both f and $\widehat{f}$ are supported in its perpendicular space. Denote by v^* the function

$$E \to \mathbb{R}, \quad u \mapsto \langle u, v\rangle_E.$$

Due to temperedness, $\widehat{f}$ has a finite order, and therefore

$$(v^*)^k\, \widehat{f} = 0 \quad \text{for some } k \geq 1.$$

Consequently $(\partial/\partial v)^k f = 0$, and we finish the proof by applying Lemma 2.2. □

We continue with the proof of Proposition 8.2. Let $\mathfrak{gl}_3(\mathbb{K})$ be equipped with the real trace from as in the introduction and define the Fourier transform accordingly. Given $f \in C^{-\xi}_{\tilde{\chi}_3}(\mathfrak{gl}_3(\mathbb{K}))$, it is easy to check that its Fourier transform $\widehat{f} \in C^{-\xi}(\mathfrak{gl}_3(\mathbb{K}))$ satisfies the following:

$$\begin{cases} \text{(a) } \widehat{f}(lxl^{-1}) = \widehat{f}(x), \quad l \in L_3, \\ \\ \text{(b) } \widehat{f}(g_1^\tau x g_2) = \chi_{N_3}(g_1)^{-1}\chi_{N_3}(g_2)^{-1}\widehat{f}(x), \quad g_1, g_2 \in N_3, \text{ and,} \\ \\ \text{(c) } \widehat{f}(x^\tau) = -\widehat{f}(x). \end{cases}$$

Then as in Lemma 8.5, we conclude that $\widehat{f}$ is supported in

$$Z'_{3,2} = \left\{ \begin{bmatrix} 0 & * & a \\ * & * & * \\ a & * & * \end{bmatrix} \in \mathfrak{gl}_3(\mathbb{K}) \right\}.$$

Therefore both f and $\widehat{f}$ are supported in the proper nondegenerate subspace

$$Z_{3,2} + Z'_{3,2} = \left\{ \begin{bmatrix} * & * & a \\ * & * & * \\ a & * & * \end{bmatrix} \in \mathfrak{gl}_3(\mathbb{K}) \right\}.$$

Lemma 8.6 then implies that $f = 0$. The proof of Proposition 8.2 is now complete.

Remark. We may view the Fourier transform argument of this section as a variation of the metrical properness argument of Sections 5 and 6. In view of Lemma 8.4 on unipotent χ_3-incompatibility, we have in some sense used $\mathrm{U}_\chi\,\mathrm{M}$ property to reduce Proposition 8.1 to the vanishing of (8.2). The latter is closely related to the multiplicity one property of the pair $(\mathrm{GL}_2(\mathbb{K}), \mathrm{GL}_1(\mathbb{K}))$.

9. Proof of Theorem 1.4

We will first examine the case where the quaternion algebra $\mathbb{D}$ is split, namely $G = \mathrm{GL}_6(\mathbb{K})$. We start with the following

Lemma 9.1. *Recall the notation of Section* 5.

(a) *If $\mathfrak{Z}$ is a unipotently $\chi_{4,2}$-incompatible $H_{4,2}$ stable submanifold of $G_{4,2}$, then $Z = H\mathfrak{Z}$ is a unipotently χ-incompatible submanifold of G.*
(b) *If $\mathfrak{Z}$ is a metrically proper $H_{4,2}$ stable submanifold of $G_{4,2}$, then $Z = H\mathfrak{Z}$ is a metrically proper submanifold of G.*

Proof. Part (a) is clear. For part (b), we note that

$$Z = H\mathfrak{Z} = U_{4,2}\mathfrak{Z}\, U_{4,2}^\tau,$$

where

$$U_{4,2} = \left\{ \begin{bmatrix} I_2 & 0 & d \\ 0 & I_2 & c \\ 0 & 0 & I_2 \end{bmatrix} \mid c, d \in \mathfrak{gl}_2(\mathbb{K}) \right\}.$$

By invariance of the metric, we only need to show that Z is metrically proper at every point $z \in \mathfrak{Z}$, i.e., the tangent space $\mathrm{T}_z(Z)$ is contained in a nondegenerate proper subspace of $\mathrm{T}_z(G)$.

First we assume that z is the identity matrix e. Then

$$\mathrm{T}_e(Z) = \mathrm{T}_e(\mathfrak{Z}) + (\mathrm{Lie}(U_{4,2}) + \mathrm{Lie}(U_{4,2}^\tau))$$

is metrically proper since

$$\mathrm{T}_e(\mathfrak{Z}) \text{ is metrically proper in } \mathrm{T}_e(\mathrm{GL}_4(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K}))$$

and

$$\mathrm{T}_e(G) = \mathrm{T}_e(\mathrm{GL}_4(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})) \oplus (\mathrm{Lie}(U_{4,2}) + \mathrm{Lie}(U_{4,2}^\tau))$$

is an orthogonal decomposition.

Now let $z \in \mathfrak{Z}$. Note that

$$z^{-1}Z = U_{4,2}(z^{-1}\mathfrak{Z})U_{4,2}^\tau$$

and

$$z^{-1}\mathfrak{Z} \text{ is metrically proper in } \mathrm{GL}_4(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K}).$$

Therefore the above argument implies that $z^{-1}Z$ is metrically proper at e. Using the left multiplication by z,

$$l_z : (G, z^{-1}Z, e) \to (G, Z, z),$$

we conclude that Z is metrically proper at z. $\square$

Recall the open submanifold G' of G from Section 4. Set

$$G'_{4,2} = (\mathrm{GL}_4 \times \mathrm{GL}_2) \cap G',$$

which is stable under $H_{4,2} = S_{4,2} \times S_{4,2}$. Define $G'_{2,4}$ and $H_{2,4}$ similarly.

Recall also the submanifolds M_2 and M_3 from Sections 7 and 8. Also define the following symmetric counterpart of M_3:

$$\check{M}_3 = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 \\ 0 & 0 & * & * & * & 0 \\ 0 & 0 & * & * & y_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & y_{44} \end{bmatrix} \in G \mid y_{33} \neq y_{44} \right\}.$$

Note that

$$M_2 \subset G'_{4,2} \cap G'_{2,4}, \quad M_3 \subset G'_{4,2}, \quad \check{M}_3 \subset G'_{2,4}.$$

We have

$$G'_{4,2} \setminus (H_{4,2}M_2 \cup H_{4,2}M_3) = Z_4,$$
$$G'_{2,4} \setminus (H_{2,4}M_2 \cup H_{2,4}\check{M}_3) = W_4,$$

where Z_4 is given in (5.2) and W_4 is given similarly by

$$W_4 = \left\{ \begin{bmatrix} y & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{bmatrix} \in G_{2,4} \mid y^{-1}a_{22} \text{ is nilpotent and nonzero} \right\}.$$

Let

$$G'' = HM_2 \cup HM_3 \cup H\check{M}_3 \subset G'. \tag{9.1}$$

Proposition 9.2. *As an H manifold, $G' \setminus G''$ has $\mathrm{U}_\chi\mathrm{M}$ property. Consequently, if $f \in C_\chi^{-\infty}(G')$ is an eigenvector of Δ and f vanishes on G'', then $f = 0$.*

Proof. It is easy to check that

- $G' \setminus G'' = Z_6 \sqcup HZ_4 \sqcup HW_4$.
- Z_6 is closed in $G' \setminus G''$.
- Both HZ_4 and HW_4 are closed in $HZ_4 \sqcup HW_4$.

By Proposition 6.1, the submanifold Z_6 has $\mathrm{U}_\chi\,\mathrm{M}$ property. By Proposition 5.1 and Lemma 9.1, the submanifold HZ_4 has $\mathrm{U}_\chi\,\mathrm{M}$ property. Similarly, HW_4 also has $\mathrm{U}_\chi\,\mathrm{M}$ property. Therefore the H stable closed subset $Z_6 \sqcup HZ_4 \sqcup HW_4$ of G' has $\mathrm{U}_\chi\,\mathrm{M}$ property. The assertion follows. □

Now set

$$\tilde{H} = \{1, \tau\} \ltimes H = \{1, \tau\} \ltimes (S \times S),$$

where the semidirect product is defined by the action

$$\tau(g_1, g_2) = (g_2, g_1), \quad g_1, g_2 \in S.$$

Extend χ to a character $\tilde{\chi}$ of $\tilde{H}$ by requiring

$$\widetilde{\chi}(\tau) = -1,$$

and extend the action on G of H to $\tilde{H}$ by requiring

$$\tau x = x^\tau.$$

Proposition 9.3. *One has that* $C_{\tilde{\chi}}^{-\xi}(G'') = 0$.

Proof. By using the restriction map, Proposition 7.1 implies that

$$C_{\tilde{\chi}}^{-\xi}(HM_2) = 0.$$

Similarly, Proposition 8.1 implies that

$$C_{\tilde{\chi}}^{-\xi}(HM_3) = 0$$

and, likewise,

$$C_{\tilde{\chi}}^{-\xi}(H\check{M}_3) = 0.$$

The proposition follows from the above three vanishing results. □

We are now ready to prove Theorem 1.4 for the split case. Let f be as in the theorem. Write

$$f^\tau(x) = f(x^\tau).$$

Then f^τ still satisfies (1.4), which implies that

$$f - f^\tau \in C_{\tilde{\chi}}^{-\xi}(G).$$

From Proposition 9.3, we know that $f - f^\tau = 0$ on G''. Note that τ commutes with the differential operator Δ on G. So f^τ is an eigenvector of Δ, with the same eigenvalue as that of f. Therefore $f - f^\tau$ is again an eigenvector of Δ. Proposition 9.2 implies that $f - f^\tau = 0$ on G'. By Proposition 4.1, we finally conclude that

$$f - f^\tau = 0.$$

In the rest of the section, we sketch the proof of Theorem 1.4 for the case $\mathbb{D} = \mathbb{H}$ (the real quaternion division algebra), which is much simpler than the split case of $\mathrm{GL}_6(\mathbb{K})$. As in the split case, define a parabolic subgroup $P_{\mathbb{H}}$ containing $S_{\mathbb{H}}$ and the rank matrix $\mathrm{R}(x)$ (for $x \in G_{\mathbb{H}}$) in the obvious way. Then $\mathrm{R}(x)$ takes the following 6 possible values:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = R_{\mathrm{open}}, \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which gives rise to 6 P-P^τ double cosets $\{G_{\mathbb{H},R}\}$.

Let f be as in the theorem. If we replace $\mathrm{GL}_2(\mathbb{K})$ by $\mathbb{H}^\times$, the analog of Proposition 7.1 still holds. This will imply that $f - f^\tau$ vanishes on $G_{\mathbb{H},R_{\mathrm{open}}}$. As in the split case, we define a left invariant vector field X_{left} on $G_{\mathbb{H}}$ using $x_{\mathrm{left}} = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right] \in \mathfrak{gl}_3(\mathbb{H})$. Then as in Section 4 one checks that X_{left} is transversal to every double coset $G_{\mathbb{H},R}$ for $R \neq R_{\mathrm{open}}$. We conclude as in the split case that $f - f^\tau = 0$.

Remarks. (a) Theorem 1.4 in fact holds without the temperedness condition on f. But we shall not prove or exploit this fact.

(b) We also expect Theorem 1.4 to hold without the assumption that f is an eigenvector of $\Delta_{\mathbb{D}}$.

10. Proof of Theorem 1.3

The argument of this section is standard, and it works for a more general real reductive group G.

By a representation of G we mean a continuous linear action of G on a complete, locally convex, Hausdorff complex topological vector space. We say that a representation V of G is in the class $\mathcal{FH}$ if it is Fréchet, smooth, of moderate growth, admissible and $\mathrm{Z}(\mathfrak{g}_{\mathbb{C}})$ finite. Here and as usual, $\mathrm{Z}(\mathfrak{g}_{\mathbb{C}})$ is the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. The reader may consult [C89, W92] for more details about representations in the class $\mathcal{FH}$.

Let V_1 and V_2 be two representations of G in the class $\mathcal{FH}$. We say that they are contragredient to each other if there exists a nondegenerate continuous G invariant bilinear form

$$\langle\, , \rangle : V_1 \times V_2 \to \mathbb{C}.$$

If V_1 and V_2 are contragredient to each other, then V_1 is irreducible if and only if V_2 is as well.

Let S_1 and S_2 be two closed subgroups of G with continuous characters (not necessarily unitary)

$$\chi_{S_i} : S_i \to \mathbb{C}^\times, \quad i = 1, 2.$$

Let τ be a continuous anti-automorphism of G (not necessarily an anti-involution).

The following is a generalization of the usual Gelfand-Kazhdan criterion. See [SZ1] for a detailed proof. Recall that $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})^G$ is identified with the space of bi-invariant differential operators on G, as usual.

Proposition 10.1. *Assume that for every $f \in C^{-\xi}(G)$ which is an eigenvector of $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})^G$, the conditions*

$$f(sx) = \chi_{S_1}(s) f(x), \quad s \in S_1,$$

and

$$f(xs) = \chi_{S_2}(s)^{-1} f(x), \quad s \in S_2,$$

imply that

$$f(x^\tau) = f(x).$$

Then for any two irreducible representations V_1 *and* V_2 *of* G *in the class* $\mathcal{FH}$ *which are contragredient to each other, one has that*

$$\dim \mathrm{Hom}_{S_1}(V_1, \mathbb{C}_{\chi_{S_1}}) \dim \mathrm{Hom}_{S_2}(V_2, \mathbb{C}_{\chi_{S_2}}) \leq 1.$$

Now we finish the proof of Theorem 1.3. Assume that $V_1 = V$ is an irreducible representation of G in the class $\mathcal{FH}$. Define the irreducible representation V_2 of G in the class $\mathcal{FH}$ as follows. The representation V_2 equals V as a topological vector space, and the action ρ_2 of G on V_2 is given by

$$\rho_2(g)v = \rho_1(g^{-\tau})v, \quad g \in G, v \in V,$$

where ρ_1 is the action of G on V_1. Using character theory and the fact that g is always conjugate to g^τ, we conclude that V_1 and V_2 are contragredient to each other [AGS07, Theorem 2.4.2]. Now let

$$S_1 = S, \quad S_2 = S^\tau, \quad \chi_{S_1} = \chi_S,$$

and

$$\chi_{S_2}(g) = \chi_S(g^{-\tau}), \quad g \in S_2.$$

Theorem 1.4 says that the assumption of Proposition 10.1 is satisfied, and so

$$\dim \mathrm{Hom}_{S_1}(V_1, \mathbb{C}_{\chi_{S_1}}) \dim \mathrm{Hom}_{S_2}(V_2, \mathbb{C}_{\chi_{S_2}}) \leq 1.$$

Note that by the identification $V_1 = V_2 = V$ as well as the explicit actions, we have

$$\mathrm{Hom}_{S_1}(V_1, \mathbb{C}_{\chi_{S_1}}) = \mathrm{Hom}_{S_2}(V_2, \mathbb{C}_{\chi_{S_2}}) = \mathrm{Hom}_S(V, \mathbb{C}_{\chi_S}).$$

Hence

$$\dim \mathrm{Hom}_S(V, \mathbb{C}_{\chi_S}) \leq 1,$$

and the proof is complete.

11. Some consequences

11.1. **Uniqueness of trilinear forms.** The following theorem is proved in [L01] (in an exhaustive approach), and its p-adic analog was proved much earlier in [P90, Theorem 1.1].

Theorem 11.1. *Let* V *be an irreducible representation of* $\mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$ *in the class* $\mathcal{FH}$*. Then*

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{K})}(V, \mathbb{C}_{\chi_2}) \leq 1.$$

Here we view $\mathrm{GL}_2(\mathbb{K})$ *as the diagonal subgroup of* $\mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$*, and* $\chi_2 = \chi_{\mathbb{K}^\times} \circ \det$ *is a character of* $\mathrm{GL}_2(\mathbb{K})$*.*

Proof. By the Gelfand-Kazhdan criterion, one just needs to show the following: let $\mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$ act on

$$G_{2,2,2} = \mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$$

by

$$(g_1, g_2)(x, y, z) = (g_1 x g_2^\tau, g_1 y g_2^\tau, g_1 z g_2^\tau), \quad g_1, g_2 \in \mathrm{GL}_2(\mathbb{K}).$$

Denote by $\chi_{2,2}$ the character of $\mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$ given by

$$\chi_{2,2}(g_1, g_2) = \chi_{\mathbb{K}^\times}(\det(g_1))\chi_{\mathbb{K}^\times}(\det(g_2)), \quad g_1, g_2 \in \mathrm{GL}_2(\mathbb{K}).$$

Then for all $f \in C^{-\xi}_{\chi_{2,2}}(G_{2,2,2})$, we have

$$f(x^\tau, y^\tau, z^\tau) = f(x, y, z).$$

To show the above, we observe that $M_2 = \mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K}) \times \{I_2\}$ is a $\mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$ slice of $G_{2,2,2}$, which is stable under $H_2 = \{(x, x^{-\tau}) \mid x \in \mathrm{GL}_2(\mathbb{K})\} \subset \mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$ and τ. The result then follows from Proposition 7.1. □

As noted near the end of Section 9, if we replace $\mathrm{GL}_2(\mathbb{K})$ by $\mathbb{H}^\times$, the analog of Proposition 7.1 still holds (again by using Proposition 1.5). Thus the analog of Theorem 11.1 for $\mathbb{H}^\times$ holds. Of course this is well known and easier.

11.2. **Uniqueness of the Jacquet-Shalika model for** $\mathrm{GL}_3(\mathbb{K})$**.** Let L_3 and N_3 be the subgroups of $\mathrm{GL}_3(\mathbb{K})$, as in Section 8. Write $S_3 = L_3 N_3$ and

$$\chi_{S_3}\left(\begin{bmatrix} 1 & 0 & d \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} \right) = \chi_{\mathbb{K}^\times}(a)\psi_{\mathbb{K}}(d),$$

which defines a character of S_3.

Theorem 11.2. *Let V be an irreducible representation of* $\mathrm{GL}_3(\mathbb{K})$ *in the class $\mathcal{FH}$. Then*

$$\dim \mathrm{Hom}_{S_3}(V, \mathbb{C}_{\chi_{S_3}}) \le 1.$$

Proof. As a corollary of Proposition 8.2, we know that if $f \in C^{-\xi}(\mathrm{GL}_3(\mathbb{K}))$ satisfies

$$f(sx) = f(xs^\tau) = \chi_{S_3}(s) f(x), \text{ for all } s \in S_3,$$

then

$$f(x^\tau) = f(x).$$

The theorem then follows, as in Section 10. □

We remark that the p-adic analog of Theorem 11.2 holds true, as the same proof goes through.

Remark. By inducing the character χ_{S_3} to a Heisenberg group, one may obtain uniqueness of the Fourier-Jacobi model for $\mathrm{GL}_3(\mathbb{K})$.

11.3. **Uniqueness of a certain model for** $\mathrm{GL}_4(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$**.** Recall from the introduction that

$$S_{4,2} = (\mathrm{GL}_4(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})) \cap S = \left\{ \begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in G \right\}$$

and $\chi_{S_{4,2}} = \chi_S|_{S_{4,2}}$.

Theorem 11.3. *Let V be an irreducible representation of* $\mathrm{GL}_4(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$ *in the class $\mathcal{FH}$. Then*

$$\dim \mathrm{Hom}_{S_{4,2}}(V, \mathbb{C}_{\chi_{S_{4,2}}}) \le 1.$$

Proof. Denote by $\Delta_{4,2}$ the Casimir operator on $\mathrm{GL}_4(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$ associated to the real trace form. Arguing as in Section 10, we will just need to show that if $f \in C^{-\xi}(\mathrm{GL}_4(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K}))$ is an eigenvector of $\Delta_{4,2}$ and if

$$f(sx) = f(xs^\tau) = \chi_{S_{4,2}}(s) f(x), \text{ for all } s \in S_{4,2},$$

then

$$f(x^\tau) = f(x).$$

To conclude the above, we further assume that $f(x^\tau) = -f(x)$. We need to show that $f = 0$.

Denote

$$C_{4,2} = \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & y \end{bmatrix} \in \mathrm{GL}_4(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K}) \mid y^{-1}a_{22} \text{ is nilpotent} \right\}.$$

This is the union of Z_4 (in Section 5) and Z_4', where

$$Z_4' = \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & y \end{bmatrix} \in \mathrm{GL}_4(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K}) \right\}.$$

By using Proposition 7.1 and Proposition 8.1, we first show that f is supported in $C_{4,2}$. Proposition 5.1 further implies that f can only be supported in Z_4'.

Now set

$$x_{4,\mathrm{left}} = \begin{bmatrix} 0 & I_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{gl}_4(\mathbb{K}) \times \mathfrak{gl}_2(\mathbb{K})$$

and denote by $X_{4,\mathrm{left}}$ the left invariant vector field on $\mathrm{GL}_4(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$ whose tangent vector at $x \in G$ is $xx_{4,\mathrm{left}}$. As in Section 4 one checks that $X_{4,\mathrm{left}}$ is transversal to Z_4'. We may then conclude that $f = 0$, as in Section 9. □

11.4. **Uniqueness of Whittaker models.** Let $\mathbf{G}$ be a quasi-split connected reductive algebraic group defined over $\mathbb{R}$. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$, with unipotent radical $\mathbf{N}$. Let

$$\chi_{\mathbf{N}} : \mathbf{N}(\mathbb{R}) \to \mathbb{C}^\times$$

be a generic unitary character. The meaning of "generic" will be explained later in the proof.

The following theorem is fundamental and well known. For $\mathbf{G} = \mathrm{GL}_n$, this is a celebrated result of Shalika [S74]. A proof in general may be found in [CHM00, Theorem 9.2]. We shall give a short proof based on the notion of unipotent χ-incompatibility.

Theorem 11.4. *Let V be an irreducible representation of $\mathbf{G}(\mathbb{R})$ in the class $\mathcal{FH}$. Then*

$$\dim \mathrm{Hom}_{\mathbf{N}(\mathbb{R})}(V, \mathbb{C}_{\chi_{\mathbf{N}}}) \leq 1.$$

Proof. We say that a representation is in the class $\mathcal{DH}$ if it is the strong dual of a representation in the class $\mathcal{FH}$. The current theorem can then be reformulated as follows: the space

$$U^{\chi_{\mathbf{N}}^{-1}} = \{u \in U \mid gu = \chi_{\mathbf{N}}^{-1}(g)u \text{ for all } g \in \mathbf{N}(\mathbb{R})\}$$

is at most one dimensional for every irreducible representation U of $\mathbf{G}(\mathbb{R})$ in the class $\mathcal{DH}$.

Let $\bar{\mathbf{B}}$ be a Borel subgroup opposite to $\mathbf{B}$, with unipotent radical $\bar{\mathbf{N}}$. Then $\mathbf{T} = \mathbf{B} \cap \bar{\mathbf{B}}$ is a maximal torus. Let

$$\chi_{\mathbf{T}} : \mathbf{T}(\mathbb{R}) \to \mathbb{C}^\times$$

be an arbitrary character. Then

$$U(\chi_{\mathbf{T}}) =\{f \in C^{-\infty}(\mathbf{G}(\mathbb{R})) \mid f(t\bar{n}x) = \chi_{\mathbf{T}}(t)f(x) \text{ for all } t \in \mathbf{T}(\mathbb{R}), \bar{n} \in \bar{\mathbf{N}}(\mathbb{R})\}$$

is the distributional version of nonunitary principal series representations. By Casselman's subrepresentation theorem (in the category of representations in the class $\mathcal{DH}$), it suffices to show that

$$\dim U(\chi_{\mathbf{T}})^{\chi_{\mathbf{N}}^{-1}} \leq 1, \quad \text{for any } \chi_{\mathbf{T}}. \tag{11.1}$$

Let

$$H_{\mathbf{G}} = \bar{\mathbf{B}}(\mathbb{R}) \times \mathbf{N}(\mathbb{R}),$$

which acts on $\mathbf{G}(\mathbb{R})$ by

$$(\bar{b}, n)x = \bar{b}xn^{-1}.$$

Write

$$\chi_{\mathbf{G}}(t\bar{n}, n) = \chi_{\mathbf{T}}(t)\chi_{\mathbf{N}}(n),$$

which defines a character of $H_{\mathbf{G}}$. Then (11.1) is equivalent to

$$\dim C^{-\infty}_{\chi_{\mathbf{G}}}(\mathbf{G}(\mathbb{R})) \leq 1. \tag{11.2}$$

Let W be the Weyl group of $\mathbf{G}(\mathbb{R})$ with respect to $\mathbf{T}$. We have the Bruhat decomposition

$$\mathbf{G}(\mathbb{R}) = \bigsqcup_{w \in W} \mathbf{G}_w, \quad \text{with} \quad \mathbf{G}_w = \bar{\mathbf{B}}(\mathbb{R})w\mathbf{N}(\mathbb{R}).$$

From this we form an $H_{\mathbf{G}}$ stable filtration

$$\emptyset = \mathbf{G}^0 \subset \mathbf{G}^1 \subset \mathbf{G}^2 \subset \cdots \subset \mathbf{G}^r = \mathbf{G}(\mathbb{R})$$

of $\mathbf{G}(\mathbb{R})$ by open subsets, with $\mathbf{G}^1 = \bar{\mathbf{B}}(\mathbb{R})\mathbf{N}(\mathbb{R})$ and every difference $\mathbf{G}^i \setminus \mathbf{G}^{i-1}$ a Bruhat cell $\mathbf{G}_w$, for $i \geq 2$.

Clearly (11.2) is implied by the following two assertions:

$$\dim C^{-\infty}_{\chi_{\mathbf{G}}}(\mathbf{G}^1) = 1 \tag{11.3}$$

and

$$\text{if } f \in C^{-\infty}_{\chi_{\mathbf{G}}}(\mathbf{G}^i) \text{ vanishes on } \mathbf{G}^{i-1}, \text{ then } f = 0, \tag{11.4}$$

for $i \geq 2$. The equality (11.3) is clear as $\mathbf{G}^1 = \bar{\mathbf{B}}(\mathbb{R})\mathbf{N}(\mathbb{R})$. For (11.4), we write

$$\mathbf{G}^i \setminus \mathbf{G}^{i-1} = \mathbf{G}_w, \quad \text{with } w \text{ a nonidentity element of } W.$$

The genericity means that $\chi_{\mathbf{N}}$ has nontrivial restriction to $\mathbf{N}(\mathbb{R}) \cap w^{-1}(\bar{\mathbf{N}}(\mathbb{R}))w$. Pick

$$n = w^{-1}\bar{n}w \in \mathbf{N}(\mathbb{R}) \cap w^{-1}(\bar{\mathbf{N}}(\mathbb{R}))w$$

so that $\chi_{\mathbf{N}}(n) \neq 1$. Then $(\bar{n}, n) \in H_{\mathbf{G}}$ satisfies

$$(\bar{n}, n)w = w \quad \text{and } \chi_{\mathbf{G}}(\bar{n}, n) = \chi_{\mathbf{N}}(n) \neq 1.$$

Consequently, $\mathbf{G}_w$ is unipotently $\chi_{\mathbf{G}}$-incompatible. Now (11.4) follows from Lemma 3.4. $\square$

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