

SEMI-TERMINAL CONTINUA IN KELLEY SPACES

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ABSTRACT. A continuum K in a space X is said to be *semi-terminal* if at least one out of every two disjoint continua in X intersecting K is contained in K . Based on this concept, new structural results on Kelley continua are obtained. In particular, two decomposition theorems for Kelley continua are presented. One of these theorems is an improved version of the aposyndetic decomposition theorem for Kelley continua.

The concept of a *semi-terminal continuum*, which is introduced in this article, captures and generalizes commonly known examples. If a continuum has an *outlet point* in the sense of [13], then it is semi-terminal. There are an abundance of semi-terminal continua with outlet points in spaces such as arcs, trees, dendrites and dendroids. If every point of a continuum is its outlet point, the continuum is *terminal*. Thus *semi-terminality* generalizes *terminality* of continua. Terminal continua proved to be important in the study of hereditarily indecomposable continua and homogeneous continua. Indeed, hereditary indecomposability can be defined as having only terminal subcontinua, and there are hereditarily indecomposable continua in all dimensions [1]. Jones' aposyndetic decomposition theorem [11] asserts that nonaposyndetic homogeneous continua admit a decomposition into nondegenerate, terminal subcontinua with an aposyndetic quotient.

There exist semi-terminal continua, even in homogeneous spaces, having no outlet points. A class of such examples has been defined in a recent study [18] of the *mutually aposyndetic decomposition*, and reproduced below (Example 2.2) for the convenience of the reader. Semi-terminal continua in homogeneous spaces are a subject for another study. Here our approach is more general; we investigate semi-terminal continua in Kelley spaces.

The significance of the class of Kelley continua was noticed in the 1940s [12]. Since then, numerous results have been obtained showing that the division of all continua into those with regular properties, Kelley continua, and the others, non-Kelley continua, is fundamental. Though there exist intriguing, important non-Kelley continua, such as the Lelek fan [8] [14], most continua of interest tend to be Kelley. All locally connected continua are Kelley. On the other end of the spectrum, all hereditarily indecomposable continua are Kelley. So are all homogeneous continua. All absolute retracts for tree-like continua, and more generally, for hereditarily unicoherent continua, are also Kelley.

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To show that the theory developed here applies to wide collections of objects, we first introduce basic ideas and provide some classes of semi-terminal and semi-indecomposable continua (section 2). Then we collect a number of general results showing regular properties of semi-terminal continua in all continua (section 3) and in Kelley continua (section 4). The structural results are included in sections 4 and 5.

Besides having abundance of significant examples of semi-terminal and semi-indecomposable continua, two striking phenomena motivated the study presented in this article. First, our theory reveals a lot of new structural information in such general spaces as Kelley continua. Second, we find a deep analogy between the two following pairs of concepts for continua in Kelley spaces: (a) *terminality* and *indecomposability*; and (b) *semi-terminality* and *semi-indecomposability*.

1. PRELIMINARIES

All spaces are assumed to be metric. A *continuum* is a compact, connected, nonempty space. A one-dimensional continuum is called a *curve*. The symbol $\text{Lim}K_n$ stands for the limit of the sets K_n in the sense of the Hausdorff distance. If X is a space, the symbol $C(X)$ denotes the hyperspace of the subcontinua of X equipped with the Hausdorff distance. A continuum is *decomposable* if it can be represented as the union of two of its proper subcontinua. Otherwise it is called *indecomposable*. It is easy to see that a continuum X is indecomposable if and only if X is the only subcontinuum of X with nonempty interior. A *hereditarily indecomposable continuum* is a continuum whose every subcontinuum is indecomposable. A subcontinuum T of a space X is called *terminal* in X provided that for every subcontinuum Y of X that intersects T , either $Y \subset T$ or $T \subset Y$. Note that a continuum X is hereditarily indecomposable if and only if every subcontinuum of X is terminal. A point p in a subcontinuum Y of a space X is called an *outlet point* of Y provided that $p \in Z$ for every continuum Z in X such that $Z \cap Y \neq \emptyset \neq Z - Y$. We observe that a subcontinuum T of a space X is terminal if and only if every point of T is its outlet point.

All maps are assumed to be continuous. A map $f : X \rightarrow Y$ between spaces X and Y is called *monotone* if $f^{-1}(p)$ is connected for every $p \in Y$.

A space X is called a *Kelley space* if, whenever a sequence $\{x_n\} \subset X$ converges to a point $x \in X$ and x belongs to a subcontinuum K of X , there are continua K_n such that $x_n \in K_n$ and $\text{Lim}K_n = K$ [12]. A space X is said to be *homogeneous* provided that for every $x, y \in X$ there exists a homeomorphism $h : X \rightarrow X$ such that $h(x) = y$. Every homogeneous continuum is a Kelley space [30].

A space X is said to be *aposyndetic at x with respect to y* if there exists a continuum K in X such that $x \in \text{Int}K$ and $y \notin K$. If X is aposyndetic at x with respect to y for every two different points x and y in X , then X is called *aposyndetic*. If for every two different points $x, y \in X$ there are disjoint continua K_x and K_y such that $x \in \text{Int}K_x$ and $y \in \text{Int}K_y$, then X is called *mutually aposyndetic*.

The following concepts have been introduced in [21] and further developed in [18], [19], [20], [22], [23], and [24]. If X is a space, a subcontinuum K is called *filament* (in X) provided there exists a neighborhood N of K such that the component of N containing K has empty interior. If X is a continuum, $p \in X$, and every subcontinuum of $X - \{p\}$ has empty interior, then p is called a *cofilament point* of X (see [20] for a more systematic study of cofilament sets). Note that p is a

cofilament point of X if and only if every subcontinuum of $X - \{p\}$ is filament, and, X is indecomposable if and only if each $p \in X$ is a cofilament point of X . A subcontinuum A of X is called *ample* if for every neighborhood U of A there is a continuum B such that $A \subset \text{Int}B \subset B \subset U$. Proposition 1.1 below, which has been proved in [21, Proposition 2.3], is fundamental to the study of filament and ample continua in Kelley spaces. By this proposition, the set of cofilament points in a Kelley continuum is the intersection of all ample subcontinua of the space. More generally, in the study of concepts such as aposynthesis, mutual aposynthesis, or *semi-indecomposability* (see section 2) in Kelley spaces, ample continua can often replace subcontinua with interiors.

Proposition 1.1. *Every subcontinuum of a Kelley continuum X is either filament or ample. In particular, if a continuum Y in X contains a nonfilament subcontinuum of X , then Y is ample in X .*

2. TWO CRUCIAL CONCEPTS AND SOME EXAMPLES

The first of the two crucial concepts of this article is *semi-indecomposability*, which is an intrinsic topological property of a continuum. A continuum X is said to be *semi-indecomposable* provided each two subcontinua of X with interiors have nonempty intersection. Originally, this concept was introduced by Charles Hagopian in 1969 [9] under a different name of a *strictly non-mutually aposyndetic continuum*. Observe that a continuum X is semi-indecomposable if and only if out of every two disjoint subcontinua of X at least one is filament. Moreover, X is indecomposable if and only if each two disjoint subcontinua of X are filament. Thus semi-indecomposability is, in a sense, “half” or “semi” indecomposability.

If X is a Kelley continuum, then X is semi-indecomposable if and only if each two ample subcontinua of X have nonempty intersection (see Proposition 1.1). All examples discussed in this section are Kelley continua. Clearly, a continuum having a cofilament point is semi-indecomposable. The Cantor fan, which is defined as the cone $\text{Cone}(\mathcal{C})$ over the Cantor set \mathcal{C} , seems to be the simplest nondegenerate, semi-indecomposable continuum. Indeed, $\text{Cone}(\mathcal{C})$ is semi-indecomposable because the vertex of $\text{Cone}(\mathcal{C})$ is its cofilament point. It is also a hereditarily decomposable continuum.

Semi-indecomposability naturally occurs in the study of absolute retracts for hereditarily unicoherent continua (abbr. $AR(\mathcal{HU})$) and tree-like continua (abbr. $AR(\mathcal{TL})$) [3], [4], [5], [6], [7], though the name *semi-indecomposable* is not used in these papers. It can be shown that each such absolute retract is either a Kelley dendroid or a non-path connected, semi-indecomposable continuum. Some of these absolute retracts, for instance the Cantor fan, are both path connected and semi-indecomposable. It can be shown that if $X \in AR(\mathcal{HU})$ or $X \in AR(\mathcal{TL})$, then X is semi-indecomposable if and only if X has some cofilament point. If X is not path connected, the set of cofilament points coincides with the *kernel* of X introduced in [3]. Since every tree-like continuum can be such a kernel of an $AR(\mathcal{TL})$, [3], [5], the class $AR(\mathcal{TL})$ contains a large collection of decomposable, semi-indecomposable continua.

There exist semi-indecomposable continua X without cofilament points. Since being a cofilament point is a topological property, in homogeneous continua either all or none of the points are cofilament. In [9, Theorem 10, p. 621] Hagopian has shown that the product $P \times P$ of the pseudo-arc P with itself is semi-indecomposable.

In a recent paper [19] the author has shown that if S is a solenoid, then $S \times S$ is semi-indecomposable. Actually, a precise characterization of the pairs of solenoids whose products are semi-indecomposable is presented in this last article. The products $P \times P$ and $S \times S$ are homogenous, decomposable continua. Thus not every point of these spaces is cofilament. Hence $P \times P$ and $S \times S$ have no cofilament points.

Question 1. Does every decomposable, semi-indecomposable homogeneous continuum have dimension greater than 1?

The second crucial concept of this article is *semi-terminality*, which refers to the position of a continuum in a larger space.

Definition 2.1. A subcontinuum Y of a space X is called *semi-terminal*, provided that for each two disjoint subcontinua Z_1 and Z_2 of X such that $Z_1 \cap Y \neq \emptyset \neq Z_2 \cap Y$, either $Z_1 \subset Y$ or $Z_2 \subset Y$.

Note that a subcontinuum Y of a space X is terminal if and only if for each two disjoint subcontinua Z_1 and Z_2 of X such that $Z_1 \cap Y \neq \emptyset \neq Z_2 \cap Y$, both Z_1 and Z_2 are contained in Y . Thus semi-terminality indeed can be viewed as “half” or “semi” terminality.

In many cases semi-terminality of a subcontinuum is related to having an outlet point. Indeed, it is an immediate observation that a continuum having an outlet point is semi-terminal. As is mentioned in the introduction, an abundance of semi-terminal continua with outlet points can be found in various classes of tree-like continua. In the study of aposynthesis all L -sets (see the definition in section 5) are semi-terminal subcontinua with outlet points. In particular, all members of the generalized Jones’ aposyndetic decomposition of a Kelley continuum [23] are such continua.

Not all semi-terminal subcontinua have outlet points, even in homogeneous continua. The following examples have been introduced and studied in [18].

Example 2.2. Let X be a homogeneous one-dimensional continuum, and let \hat{X} be the continuous curve of pseudo-arcs with the quotient space X and the quotient map $q : \hat{X} \rightarrow X$, as defined in [15]. The fibers $q^{-1}(x)$ are terminal pseudo-arcs in \hat{X} . Define the diagonal $\Delta = \{(x, x) \mid x \in X\}$, and define $\hat{X}_\Delta^2 = (q \times q)^{-1}(\Delta)$ as a subspace of $\hat{X} \times \hat{X}$. It is shown [18, Proposition 5.4] that any subcontinuum of \hat{X}_Δ^2 that intersects a set $(q \times q)^{-1}(p)$ and its complement, for $p \in \Delta$, contains an ample subcontinuum of $(q \times q)^{-1}(p)$. On the other hand, $(q \times q)^{-1}(p)$ is the product $q^{-1}(p) \times q^{-1}(p)$ of two pseudo-arcs, and thus it is semi-indecomposable. Therefore, any two subcontinua intersecting $(q \times q)^{-1}(p)$ and its complement must intersect each other. Consequently, each continuum $(q \times q)^{-1}(p)$ is semi-terminal in \hat{X}_Δ^2 . By [18, Proposition 5.3], the group of self-homeomorphisms of \hat{X}_Δ^2 that respect the decomposition into the continua $(q \times q)^{-1}(p)$ acts transitively on \hat{X}_Δ^2 . In particular, \hat{X}_Δ^2 is homogeneous. Note that the continua $(q \times q)^{-1}(p)$ are not terminal in \hat{X}_Δ^2 . Indeed, the diagonal $\Delta_0 = \{(x, x) \mid x \in \hat{X}\}$ of $\hat{X} \times \hat{X}$ is a continuum in \hat{X}_Δ^2 intersecting each set $(q \times q)^{-1}(p)$ but containing none of them. Using the homeomorphisms that respect the decomposition of \hat{X}_Δ^2 into the sets $(q \times q)^{-1}(p)$, we show that every point of $(q \times q)^{-1}(p)$ belongs to the complement of some continuum intersecting $(q \times q)^{-1}(p)$ and $\hat{X}_\Delta^2 - (q \times q)^{-1}(p)$. Hence the sets $(q \times q)^{-1}(p)$ are semi-terminal subcontinua of \hat{X}_Δ^2 without outlet points.

Remark 2.3. If the continuum X in Example 2.2 is mutually aposyndetic, the decomposition \mathcal{Q} of \hat{X}_Δ^2 into the continua $(q \times q)^{-1}(p)$ is the *mutually aposyndetic decomposition* of \hat{X}_Δ^2 defined in [18]. The properties of \mathcal{Q} discussed in Example 2.2 may suggest, for homogeneous continua, a connection among the concepts of mutual aposyndesis, semi-indecomposability and semi-terminality, similar to the relation among aposyndesis, indecomposability and terminality known from Jones' aposyndetic decomposition theorem. This seems to be a promising idea for future research.

3. SEMI-TERMINAL CONTINUA: GENERAL PROPERTIES

Proposition 3.1. *If Y and Z are subcontinua of a space X such that Y is semi-terminal, then $Y \cap Z$ is connected.*

Proof. Suppose C_1 and C_2 are different components of $Y \cap Z$. Then there are continua Z_1 and Z_2 in Z , slightly larger than C_1 and C_2 , respectively, such that $Z_1 \cap Z_2 = \emptyset$. We have $Z_1 - Y \neq \emptyset \neq Z_2 - Y$, which contradicts the semi-terminality of Y . \square

Corollary 3.2. *Let X be a continuum and Y its semi-terminal subcontinuum. Then for every ample subcontinuum A of X the intersection $A \cap Y$ is an ample subcontinuum of Y .*

Proof. Since A is ample in X , it can be slightly enlarged to a continuum B such that $A \subset \text{Int}B$. By Proposition 3.1 the sets $A \cap Y$ and $B \cap Y$ are continua, and $A \cap Y$ is in the interior of $B \cap Y$ relative to Y . Clearly, if we take a sequence of such B 's converging to A , then the sequence of their intersections with Y converges to $A \cap Y$, and the needed conclusion follows. \square

Corollary 3.3. *If X is compact, $f : X \rightarrow Y$ is a monotone map, and Z is a semi-terminal subcontinuum of X , then the restriction $f|Z : Z \rightarrow Y$ is also monotone.*

An easy proof of the following proposition is left to the reader.

Proposition 3.4. *If X is a compact space, $f : X \rightarrow Y$ is a monotone surjective map, and Z is a semi-terminal subcontinuum of X , then $f(Z)$ is a semi-terminal subcontinuum of Y .*

Corollary 3.5. *Let X be a compact space, $f : X \rightarrow Y$ a monotone surjective map, and Z a subcontinuum of X such that every subcontinuum of Z is semi-terminal in X . Then every subcontinuum of $f(Z)$ is semi-terminal in Y .*

Proof. Let K be a subcontinuum of $f(Z)$. The restriction $f_Z = f|Z : Z \rightarrow f(Z)$ is monotone by Corollary 3.3, and thus $L = f_Z^{-1}(K)$ is a subcontinuum of Z such that $f(L) = K$. By assumption, L is semi-terminal in X . Thus $K = f(L)$ is semi-terminal in Y by Proposition 3.4. \square

Proposition 3.6. *If Y_1 and Y_2 are semi-terminal subcontinua of a space X and $Y_1 \cap Y_2 \neq \emptyset$, then $Y_1 \cup Y_2$ is a semi-terminal subcontinuum of X .*

Proof. If either $Y_1 \subset Y_2$ or $Y_2 \subset Y_1$, the conclusion is trivial. Assume $Y_1 - Y_2 \neq \emptyset \neq Y_2 - Y_1$. Suppose Z_1 and Z_2 are disjoint subcontinua of X such that $Z_1 \cap (Y_1 \cup Y_2) \neq \emptyset \neq Z_2 \cap (Y_1 \cup Y_2)$ and $Z_1 - (Y_1 \cup Y_2) \neq \emptyset \neq Z_2 - (Y_1 \cup Y_2)$. Then either $Z_1 \cap Y_1 \neq \emptyset$ or $Z_1 \cap Y_2 \neq \emptyset$. In the latter case, since Y_2 is semi-terminal, it follows that $Z_1 \cap Y_1 \neq \emptyset$.

Thus $Z_1 \cap Y_1 \neq \emptyset$ in both cases. Similarly we argue that $Z_2 \cap Y_1 \neq \emptyset$. Since Y_1 is semi-terminal, either $Z_1 \subset Y_1 \subset Y_1 \cup Y_2$ or $Z_2 \subset Y_1 \subset Y_1 \cup Y_2$, a contradiction. \square

Definition 3.7. We say that a continuum X is *hereditarily semi-indecomposable* whenever every subcontinuum of X is semi-indecomposable.

Remark 3.8. Hereditarily semi-indecomposable continua are related to hereditarily indecomposable continua, which have been known and extensively studied since the 1920s. It is obvious that each hereditarily indecomposable continuum is hereditarily semi-indecomposable. The converse does not always hold. For instance, the one point union of two hereditarily indecomposable, nondegenerate continua (for instance, pseudo-arcs) is a decomposable, hereditarily semi-indecomposable continuum. In the next section it is shown (Corollary 4.10), however, that among Kelley continua the properties of being hereditarily semi-indecomposable and hereditarily indecomposable are equivalent.

It is known that for every continuum X the collections of indecomposable and hereditarily indecomposable subcontinua of X are G_δ subsets of $C(X)$. Here we observe the following.

Proposition 3.9. *For every continuum X the collections of semi-indecomposable and hereditarily semi-indecomposable subcontinua of X are G_δ subsets of $C(X)$.*

Proof. Let d be a metric on X , and let \mathcal{K}_n be the collection of subcontinua K of X such that K contains subcontinua L and M satisfying: (i) $d(x, y) \geq 1/n$ for all $x \in L$ and $y \in M$; and (ii) both L and M contain open balls relative to K each of radius $1/n$. We observe that \mathcal{K}_n is a closed collection in $C(X)$, and the set of semi-indecomposable subcontinua of X equals $C(X) - \bigcup_n \mathcal{K}_n$. The proof that the collections of hereditarily semi-indecomposable subcontinua of X are a G_δ subset of $C(X)$ is left to the reader. \square

It is clear that a continuum is hereditarily indecomposable if and only if each of its subcontinua is terminal. Here we show a similar property for the hereditary semi-indecomposability and semi-terminality of continua.

Proposition 3.10. *A continuum X is hereditarily semi-indecomposable if and only if every subcontinuum of X is semi-terminal in X .*

Proof. Assume X contains a non-semi-terminal subcontinuum Y . By definition there are disjoint continua Z_1 and Z_2 in X such that $Z_1 \cap Y \neq \emptyset \neq Z_2 \cap Y$ and $Z_1 - Y \neq \emptyset \neq Z_2 - Y$. Thus Z_1 and Z_2 are disjoint subcontinua of the continuum $C = Z_1 \cup Y \cup Z_2 \subset X$ with interiors relative to C . Hence a subcontinuum C of X is not semi-indecomposable.

Assume X contains a non-semi-indecomposable continuum Y . Throughout the remaining part of the proof all interiors are relative to Y . By definition there are disjoint subcontinua Y_0 and Y_1 of Y each having nonempty interior. Let p_0 and p_1 be interior points of Y_0 and Y_1 , respectively. Define a map $f : Y \rightarrow [0, 1]$, letting

$$f(y) = \frac{d(p_0, y)}{d(p_0, y) + d(y, p_1)},$$

and note that f is continuous, $f^{-1}(0) = \{p_0\}$ and $f^{-1}(1) = \{p_1\}$. Consequently, $f^{-1}(t_0) \subset \text{Int}Y_0$ and $f^{-1}(t_1) \subset \text{Int}Y_1$, for $t_0, t_1 \in (0, 1)$, whenever t_0 and t_1 are sufficiently near to 0 and 1, respectively. Let t_0 and t_1 be such numbers. Since

every map from a continuum onto $[0, 1]$ is weakly confluent [16, Theorems 12.29 and 13.52], there exists a continuum $K \subset Y$ such that $f(K) = [t_0, t_1]$. Thus $K \cap Y_0 \neq \emptyset \neq K \cap Y_1$, and neither Y_0 nor Y_1 is contained in K . Hence a subcontinuum K of X is not semi-terminal. The proof is complete. \square

4. SEMI-TERMINAL CONTINUA IN KELLEY SPACES

Locally connected continua form an important and distinctive category within Kelley spaces. We begin this section with the following characterization of semi-terminal subcontinua of locally connected continua.

Proposition 4.1. *If X is a locally connected continuum, then a proper subcontinuum Y of X is semi-terminal in X if and only if $\text{card}(\text{bd} Y) = 1$. In particular, each semi-terminal subcontinuum of X is locally connected and has an outlet point.*

Proof. Suppose $a, b \in \text{bd} Y$ with $a \neq b$. Since X is locally connected, there are disjoint continua A and B such that $a \in \text{Int} A$ and $b \in \text{Int} B$. We have $A \cap Y \neq \emptyset \neq A - Y$ and $B \cap Y \neq \emptyset \neq B - Y$. Thus Y is not semi-terminal, a contradiction. Therefore, the boundary of Y is either a singleton or the empty set. It is easy to observe that if $p \in X$, then for every component C of $X - \{p\}$ the closure $\text{cl} C$ is a locally connected, semi-terminal subcontinuum of X . The proof is complete. \square

Proposition 4.2. *Each semi-terminal subcontinuum of a Kelley continuum is a Kelley continuum.*

Proof. Let X be a Kelley continuum, and Y be a semi-terminal subcontinuum of X . To prove that Y is a Kelley continuum, take a continuum $K \subset Y$ and a sequence $\{p_n\} \subset Y$ converging to some point $p \in K$. Since X is a Kelley continuum, there are continua $K'_n \subset X$ such that $p_n \in K'_n$ for each n , and $\text{Lim} K'_n = K$. Without loss of generality assume $K'_n - Y \neq \emptyset$ for each n . By Proposition 3.1, the sets $L_n = K'_n \cap Y$ are continua. Since Y is semi-terminal, $K'_m \cap K'_n \neq \emptyset$ for each m and n . Again by Proposition 3.1, the sets $L_m \cup L_n = (K'_m \cup K'_n) \cap Y$ are connected. Thus $L_m \cup L_n$ is a continuum for each m and n . For fixed n , the sequence $P_m = L_m \cap L_n$ has some limit point in K , and thus $L_n \cap K \neq \emptyset$. Therefore $K_n = L_n \cup K$ is a subcontinuum of Y . Observe that $p_n \in K_n \subset Y$ for each n , and $\text{Lim} K_n = K$, which completes the proof. \square

Proposition 4.3. *If Y is a semi-terminal subcontinuum of a Kelley continuum X , and Z is a subcontinuum of X such that $Z \cap Y \neq \emptyset \neq Z - Y$, then $Z \cap Y$ is an ample subcontinuum of Y .*

Proof. Suppose $K = Z \cap Y$ is a filament subcontinuum of Y and let $p \in K$. Since K is filament, there is a neighborhood U of K in Y such that the component C of U containing K has empty interior relative to Y . Therefore, there is a sequence $\{p_n\} \subset U - C$ such that $\lim p_n = p$. Since X is a Kelley continuum, there are continua Z_n in X converging to Z , in the sense of the Hausdorff distance, such that $p_n \in Z_n$. We have $Z - Y \neq \emptyset$, and thus $Z_n - Y \neq \emptyset$ for almost all n . Since Y is semi-terminal, it follows that $Z_n \cap Z \neq \emptyset$ for almost all n . For sufficiently large n we also have $Z_n \cap Y \subset U$. Thus the continuum $L = Z \cup Z_n$, for such n , satisfies $L \cap Y \subset U$ and $L \cap C \neq \emptyset \neq U - C$. Hence $L \cap K$ is not connected, which is impossible by Proposition 3.1.

We have shown that K is not filament in Y . Since Y is a Kelley continuum by Proposition 4.2, the continuum K is ample in Y by Proposition 1.1. \square

Corollary 4.4. *Let X be a Kelley continuum and Y its semi-terminal subcontinuum. Then every cofilament point of Y is an outlet point of Y . In particular, every indecomposable semi-terminal subcontinuum of a Kelley continuum X is terminal in X .*

Proof. If p is not an outlet point of Y , then some continuum $Z \subset X$ satisfies $Z \cap Y \neq \emptyset \neq Z - Y$ and $p \notin Z$. The set $Z \cap Y$ is an ample subcontinuum of Y by Proposition 4.3, and thus p is not cofilament in Y .

If Y is indecomposable, p is cofilament for each $p \in Y$, and thus each $p \in Y$ is an outlet point of Y . Hence Y is terminal in X . \square

It is known that the terminal subcontinua of a Kelley continuum form a closed collection in $C(X)$. Here we show a similar result for semi-terminal continua.

Proposition 4.5. *If X is a compact Kelley space, then the collection $\mathcal{ST}(X)$ of semi-terminal subcontinua of X is closed in $C(X)$.*

Proof. Let K be a non-semi-terminal subcontinuum of X , and K_n a sequence of subcontinua of X converging to K in the sense of the Hausdoff distance. Since K is non-semi-terminal, there are disjoint continua P and Q such that $P \cap K \neq \emptyset \neq Q \cap K$ and $P - K \neq \emptyset \neq Q - K$. Let $p \in P \cap Y$ and $q \in Q \cap Y$. We have $\lim K_n = K$, and thus there exist $\{p_n\} \subset X$ and $\{q_n\} \subset X$ such that $p_n, q_n \in K_n$, $\lim p_n = p$ and $\lim q_n = q$. Since X is a Kelley space, there are continua P_n and Q_n such that $p_n \in P_n$, $q_n \in Q_n$, $\lim P_n = P$, and $\lim Q_n = Q$. Since $P \cap Q = \emptyset$, it follows that $P_n \cap Q_n = \emptyset$ for almost all n . Similarly, since $P - K \neq \emptyset \neq Q - K$, we have $P_n - K_n \neq \emptyset \neq Q_n - K_n$ for almost all n . Thus almost all K_n are non-semi-terminal. Consequently, the collection of non-semi-terminal subcontinua of X is open in $C(X)$. Hence $\mathcal{ST}(X)$ is closed. \square

If P is a subset of X and $x \in X$, let $\mathcal{ST}(X, P, x)$ be the collection of semi-terminal subcontinua S of X such that $x \in S \subset P$. Note that $\mathcal{ST}(X, P, x) \neq \emptyset$ if and only if $x \in P$.

Theorem 4.6. *If X is a compact Kelley space, P is a closed subset of X and $x \in P$, then the union $\Lambda(X, P, x) = \bigcup \mathcal{ST}(X, P, x)$ is the largest member of $\mathcal{ST}(X, P, x)$. Moreover, the collection $\Lambda(X, P) = \{\Lambda(X, P, x) \mid x \in P\}$ is an upper semi-decomposition of P into the maximal semi-terminal subcontinua of X contained in P .*

Proof. Since $\mathcal{ST}(X)$ is closed in $C(X)$ by Proposition 4.5, so is $\mathcal{ST}(X, P, x)$. By Proposition 3.6 we have $Y_1 \cup \dots \cup Y_n \in \mathcal{ST}(X, P, x)$ whenever $Y_1, \dots, Y_n \in \mathcal{ST}(X, P, x)$. Finite unions of members of $\mathcal{ST}(X, P, x)$ approximate $\Lambda(X, P, x)$, and thus $\Lambda(X, P, x)$ is the largest member of $\mathcal{ST}(X, P, x)$. The continua $\Lambda(X, P, x)$, for varying x , are maximal semi-terminal subcontinua of X in P which are mutually disjoint by Proposition 3.6. Since the collection of semi-terminal subcontinua of X contained in P is closed in $C(X)$, the decomposition $\Lambda(X, P)$ is upper semi-continuous. \square

If X is a space, let $\mathcal{HST}(X)$ be the collection of *hereditarily semi-terminal subcontinua* of X , that is, the collection of the subcontinua K of X such that every subcontinuum of K is semi-terminal in X .

Proposition 4.7. *If X is a Kelley continuum, then the collection $\mathcal{HST}(X)$ of hereditarily semi-terminal subcontinua of X is closed in $C(X)$.*

Proof. Suppose a sequence $\{K_n\} \subset \mathcal{HST}(X)$ converges in $C(X)$ to a continuum $K \in C(X) - \mathcal{HST}(X)$. Thus there exists a subcontinuum L of K that is not semi-terminal in X . Consequently, there exist disjoint continua A and B in X such that $A \cap L \neq \emptyset \neq B \cap L$ and $A - L \neq \emptyset \neq B - L$. Let $a \in A \cap L$, $b \in B \cap L$, and $\{a_n\} \subset X$, $\{b_n\} \subset X$ be sequences converging to a and b , respectively, such that $a_n, b_n \in K_n$ for each n . Since X is a Kelley continuum, there exist sequences $\{A_n\} \subset C(X)$ and $\{B_n\} \subset C(X)$ converging to A and B , respectively, such that $a_n \in A_n$ and $b_n \in B_n$ for each n . Note that $A_n \cap B_n = \emptyset$ for almost all n . Similarly, there exist sequences $\{L_{a_n}\} \subset C(X)$ and $\{L_{b_n}\} \subset C(X)$, each converging to L , such that $a_n \in L_{a_n}$ and $b_n \in L_{b_n}$ for each n .

Case 1. Either infinitely many L_{a_n} 's or infinitely many L_{b_n} 's are contained in the corresponding K_n 's.

Since $K_n \in \mathcal{HST}(X)$, every subcontinuum of K_n is semi-terminal. In this case, L is the limit semi-terminal continua, and thus it is semi-terminal by Proposition 4.5, a contradiction.

Case 2. $L_{a_n} - K_n \neq \emptyset \neq L_{b_n} - K_n$ for almost all n .

Without loss of generality, assume $L_{a_n} - K_n \neq \emptyset \neq L_{b_n} - K_n$ for each n . Since each K_n is semi-terminal, it follows that $L_{a_n} \cap L_{b_n} \neq \emptyset$ for each n , and thus the sets $M_n = L_{a_n} \cup L_{b_n}$ are continua. By Proposition 3.1 the intersections $P_n = M_n \cap K_n$ are continua, which satisfy $a_n, b_n \in P_n$ for each n . Since the continua M_n converge to L , we also have $A_n - P_n \neq \emptyset \neq B_n - P_n$ for almost all n . Hence almost all K_n 's contain subcontinua, the corresponding P_n 's, that are not semi-terminal. A contradiction. \square

Proposition 4.8. *Each hereditarily semi-terminal subcontinuum of a Kelley continuum is indecomposable.*

Proof. Suppose a decomposable subcontinuum Y of a Kelley continuum X is hereditarily semi-terminal. Then Y is the union $K \cup L$ of two of its proper subcontinua K and L . Since Y is also hereditarily semi-terminal in itself, by Proposition 3.10 each subcontinuum of Y is semi-indecomposable. By Proposition 4.2, the continuum K is Kelley. By Proposition 4.3 the set $A = K \cap L$ is an ample subcontinuum of K . The set $K - A$ contains no ample subcontinuum relative to K because K is semi-indecomposable. The continuum K is Kelley, and thus every subcontinuum of $K - A$ is filament relative to K . Let A_0 be a subcontinuum of K containing A in its interior such that $K - A_0 \neq \emptyset$. There are two points $p_1, p_2 \in K - A_0$ such that every continuum in K containing p_1 and p_2 intersects A_0 ; otherwise, K would contain an ample subcontinuum, $\text{cl}(K - A_0)$, disjoint with A , which is impossible.

Since A_0 is ample in K , there is a continuum B_0 in K containing A_0 in its interior relative to K , such that $p_1, p_2 \notin B_0$. Let C_1 and C_2 be the components of $K - B_0$ containing p_1 and p_2 , respectively. Then the closures $\text{cl}C_1$ and $\text{cl}C_2$ are disjoint continua, both intersecting B_0 , such that $\text{cl}C_1 - B_0 \neq \emptyset \neq \text{cl}C_2 - B_0$. Hence Y contains a non-semi-terminal subcontinuum, B_0 , a contradiction. \square

If K is a subcontinuum of a space X such that every subcontinuum of K is terminal in X , we say that K is a *hereditarily terminal subcontinuum* of X . The next result follows from Corollary 4.4 and Proposition 4.8.

Corollary 4.9. *Each hereditarily semi-terminal subcontinuum of a Kelley continuum is hereditarily indecomposable and hereditarily terminal.*

The next corollary follows from Proposition 3.10 and Corollary 4.9.

Corollary 4.10. *For each Kelley continuum X the following conditions are equivalent:*

- (a) X is hereditarily indecomposable,
- (b) X is hereditarily semi-indecomposable,
- (c) each subcontinuum of X is terminal in X , and
- (d) each subcontinuum of X is semi-terminal in X .

We end this section with the following “hereditarily (semi-)terminal decomposition theorem” for Kelley continua. The circle of pseudo-arcs [2] and subsequent generalizations [15] are primary examples of continua in which this decomposition is nontrivial and has nondegenerate members. Nevertheless, each hereditarily indecomposable continuum can be a member of a nontrivial decomposition described in this theorem.

Theorem 4.11. *Each Kelley continuum X admits a finest monotone, upper semi-continuous decomposition \mathcal{H} such that the quotient space X/\mathcal{H} has no nondegenerate hereditarily semi-terminal subcontinua. Moreover, the decomposition \mathcal{H} can be defined as the collection of the maximal hereditarily terminal (or equivalently, hereditarily semi-terminal) subcontinua of X , and the members of \mathcal{H} are hereditarily indecomposable.*

Proof. Since the collection $\mathcal{HST}(X)$ is closed by Proposition 4.7, it follows by the Kuratowski-Zorn lemma that every point x in X belongs to a maximal member H_x of $\mathcal{HST}(X)$. By Corollary 4.9, the members of $\mathcal{HST}(X)$ are terminal. Therefore, there is only one such set H_x for each x , and the collection $\mathcal{H} = \{H_x \mid x \in X\}$ partitions X . Moreover, by Proposition 4.7, the decomposition \mathcal{H} is upper semi-continuous. Let $q : X \rightarrow X/\mathcal{H}$ be the quotient map of this decomposition. First, we show that X/\mathcal{H} has no proper, nondegenerate hereditarily semi-terminal subcontinuum.

Indeed, suppose the space X/\mathcal{H} has a proper, nondegenerate, hereditarily semi-terminal subcontinuum Y . Let $Z = q^{-1}(Y)$, and let T be a subcontinuum of Z . We want to show that T is semi-terminal in X . If T is contained in a single member H of \mathcal{H} , then it is semi-terminal in X by the definition of \mathcal{H} . Otherwise, since each $H \in \mathcal{H}$ is terminal, T is the union of some members of \mathcal{H} . Thus $q^{-1}(q(T)) = T$.

Let K and L be disjoint continua in X with $K \cap T \neq \emptyset \neq L \cap T$. If either K or L is contained in a single member of \mathcal{H} , then either $K \subset T$ or $L \subset T$, because T is saturated with respect to q .

Assume neither K nor L is contained in a single member of \mathcal{H} . Then $q^{-1}(q(K)) = K$ and $q^{-1}(q(L)) = L$ by the terminality of the members of \mathcal{H} . Consequently, $q(K)$ and $q(L)$ are disjoint continua both intersecting $q(T)$. The continuum $q(T)$ is a subcontinuum of Y , and thus it is semi-terminal in X/\mathcal{H} . Therefore, either $q(K) \subset q(T)$ or $q(L) \subset q(T)$. Since T is saturated with respect to q , either $K \subset T$ or $L \subset T$.

Thus T is semi-terminal in X . Consequently $Z = q^{-1}(Y)$ is hereditarily semi-terminal in X and $Y = q(Z)$ is a singleton by the definition of q , a contradiction. We have shown that X/\mathcal{H} has no proper, nondegenerate hereditarily semi-terminal subcontinua.

Suppose \mathcal{F} is a monotone, upper semi-continuous decomposition of X such that \mathcal{H} is not finer than \mathcal{F} , which means that a member H of \mathcal{H} is not contained in a single member of \mathcal{F} . Let $f : X \rightarrow X/\mathcal{F}$ be the quotient map of \mathcal{F} . Then $f(H)$ is a nondegenerate subcontinuum of X/\mathcal{F} , which is hereditarily semi-terminal in X/\mathcal{F} by Proposition 3.4. The proof is complete. \square

5. APOSYNDETTIC DECOMPOSITION OF KELLEY CONTINUA

In this section we add some improvements to the aposyndetic decomposition theorem for Kelley continua presented in [23] and show other facts about the structure of Kelley continua related to that decomposition. This theorem generalizes Jones' aposyndetic theorem [11], later improved by Rogers [25], [26], [28], [29].

Given a proper subcontinuum Y of a continuum X , the set of all outlet points of Y can be expressed as $\bigcap\{Z \in C(X) \mid Z \cap Y \neq \emptyset \neq Z - Y\}$. Thus the set of outlet points of Y is compact. If $Y = X$, then each point of Y is its outlet point. As is noticed in section 2, every continuum having an outlet point is semi-terminal. Example 2.2 shows that, in general, the converse is not true. An easy proof of the following proposition is left to the reader.

Proposition 5.1. *If X is a Kelley continuum, then the collections $\{(K, x) \mid x \text{ is an outlet point of } K \in C(X)\}$ and $\{K \in C(X) \mid K \text{ has an outlet point}\}$ are closed in $C(X) \times X$ and $C(X)$, respectively.*

The following proposition is a consequence of Proposition 4.2.

Proposition 5.2. *If a subcontinuum Y of a Kelley space X has an outlet point, then Y is a semi-terminal, Kelley subcontinuum of X .*

If X is a continuum and $x \in X$, define

$$L(x) = \{y \in X \mid X \text{ is not aposyndetic at } y \text{ with respect to } x\}.$$

Directly from this definition we have the following.

Proposition 5.3. *If X is a continuum and $x \in X$, then $x \in A$ for every ample subcontinuum A of X intersecting $L(x)$.*

The sets $L(x)$ are called *L-sets*. It is known that the *L-sets* are continua [10]. If X is a Kelley continuum, we have $L(x) = X - \bigcup\{A \in C(X) \mid A \text{ is an ample and } x \notin A\}$, and also $x \in L(y)$ if and only if $L(x) \subset L(y)$ [23, Corollary 2.8, p. 5994]. Thus the relation $x \preceq^* y$, defined by $x \in L(y)$, is a quasi-order, and it is a closed subset of $X \times X$ [23, Proposition 3.1, p. 5996]. Consequently, in Kelley continua there are maximal and minimal *L-sets*. The following has been proved in [23, Proposition 4.2, p. 5997].

Proposition 5.4. *If X is a Kelley continuum, then x is an outlet point of $L(x)$ for every $x \in X$.*

The next proposition follows from the definition.

Proposition 5.5. *If X is a continuum, then $L(x) = X$ if and only if x is a cofilament point of X .*

Corollary 5.6. *If X is a Kelley continuum, then each L -set is a semi-terminal, Kelley continuum, and each minimal L -set is terminal.*

Proof. The first conclusion is a consequence of Propositions 5.2 and 5.4. Note that if $L(x)$ is a minimal L -set and $y \in L(x)$, then $L(y) = L(x)$. Therefore, each point of $L(x)$ is its outlet point by Proposition 5.3. Hence $L(x)$ is terminal in X . \square

Proposition 5.7. *Let X be a Kelley continuum, Y a semi-terminal subcontinuum of X , and $x \in Y$. If $L_Y(x)$ is the L -set in Y determined by x , then x is an outlet point of $L_Y(x)$ in X . In particular, every L -set in Y is a semi-terminal subcontinuum of X .*

Proof. Let K be a subcontinuum of X such that $K \cap L_Y(x) \neq \emptyset \neq K - L_Y(x)$ and $A = K \cap Y$. If $K \subset Y$, then $x \in K$, because x is an outlet point of $L_Y(x)$ in Y (Proposition 5.4). If $K - Y \neq \emptyset$, then, by Proposition 4.3, A is an ample subcontinuum of Y intersecting $L_Y(x)$. Thus $x \in A \subset K$ by Proposition 5.3. \square

Let $\mathcal{G} = \mathcal{G}(X)$ be the collection of all maximal L -sets in a continuum X , i.e.

$$\mathcal{G} = \{L(x) \mid x \text{ is maximal in } X \text{ with respect to } \preceq^*\}.$$

In [23] a generalization of Jones' aposyndetic decomposition theorem for Kelley continua was shown. In that result, the collection $\mathcal{G}(X)$ is the generalized aposyndetic decomposition of a Kelley continuum X . Here we present an improved version of this theorem. The new addition is that the fibers of the decomposition are semi-terminal, Kelley subcontinua of the space.

Theorem 5.8 (Aposyndetic Decomposition Theorem for Kelley continua). *If X is a Kelley continuum, then $\mathcal{G}(X)$ is an upper semi-continuous decomposition of X into semi-terminal, Kelley continua, and the quotient space $X/\mathcal{G}(X)$ is an aposyndetic Kelley continuum.*

Moreover, $\mathcal{G}(X)$ is the finest monotone, upper semi-continuous decomposition of X such that the quotient space is aposyndetic. That is, for every monotone, upper semi-continuous decomposition \mathcal{D} of X with an aposyndetic quotient, each member of $\mathcal{G}(X)$ is a subset of some member of \mathcal{D} .

The next corollary follows from Proposition 5.5 and Theorem 5.8.

Corollary 5.9. *If X is a Kelley continuum, then $\mathcal{G}(X)$ is trivial (that is, it has only one member, X) if and only if X has a cofilament point.*

Corollary 5.10. *If X is a Kelley continuum and $\mathcal{G}(X)$ is trivial, then X is semi-indecomposable.*

Remark 5.11. The converse to Corollary 5.10 in general is not true even among homogeneous continua. Indeed, if X is the product of two pseudo-arcs, then X is aposyndetic. Therefore, its aposyndetic decomposition is composed only of singletons. Nevertheless, this space is known to be semi-indecomposable [9, Theorem 10, p. 621].

Proposition 5.12. *Let Y be a semi-terminal subcontinuum of a Kelley continuum X . If X is aposyndetic at x with respect to y and $x, y \in Y$, then Y is aposyndetic at x with respect to y . Consequently, if X is aposyndetic, then so is Y .*

Proof. Let P be a continuum in X such that $x \in \text{Int}P$ and $y \notin P$. Then $P \cap Y$ is a continuum by Proposition 3.1, it contains x in its interior relative to Y , and $y \notin P \cap Y$. \square

Note that if Y is a semi-terminal subcontinuum of a Kelley continuum X , then, by Corollary 3.3, the collection $\{D \cap Y \mid D \in \mathcal{G}(X)\}$ is a monotone, upper semi-continuous decomposition of Y .

Proposition 5.13. *Let Y be a semi-terminal subcontinuum of a Kelley continuum X . Then the aposyndetic decomposition $\mathcal{G}(Y)$ of Y is finer than the decomposition $\mathcal{F} = \{D \cap Y \mid D \in \mathcal{G}(X)\}$.*

Proof. Let $q : X \rightarrow X/\mathcal{G}(X)$ be the quotient map of $\mathcal{G}(X)$. By Corollary 3.3 the restriction $q|_Y : Y \rightarrow q(Y)$ is also monotone. The continuum $q(X) = X/\mathcal{G}(X)$ is aposyndetic and Kelley by Theorem 5.8, and $q(Y)$ is its semi-terminal subcontinuum by Proposition 3.4. Therefore, $q(Y)$ is aposyndetic by Proposition 5.12. The continuum Y/\mathcal{F} is homeomorphic to $q(Y)$, and thus it is aposyndetic. Hence $\mathcal{G}(Y)$ is finer than \mathcal{F} by Theorem 5.8. \square

If X is a continuum, let $\text{Cfm}(X)$ be the set of cofilament points of X . For any topological space X , let $\widehat{\mathcal{A}}(X)$ be the collection of all semi-terminal continua Y in X such that $\text{Cfm}(Y) \neq \emptyset$. Recall that for a Kelley continuum X and $Y \in \widehat{\mathcal{A}}(X)$, the defining condition $\text{Cfm}(Y) \neq \emptyset$ is essentially stronger than the semi-indecomposability of Y , and it is equivalent to having the intersection of ample subcontinua of Y nonempty. Moreover, in such X each member of $\widehat{\mathcal{A}}(X)$ has an outlet point by Corollary 4.4.

Proposition 5.14. *Let X be a space and Y a member of $\widehat{\mathcal{A}}(X)$. For every Kelley continuum Z in X such that $Y \subset Z$, the continuum Y is contained in a single member of the aposyndetic decomposition $\mathcal{G}(Z)$ of Z .*

Proof. Observe that $Y \in \widehat{\mathcal{A}}(Z)$. By Corollary 5.9, the decomposition $\mathcal{G}(Y)$ is trivial, and it is finer than $\{Y \cap D \mid D \in \mathcal{G}(Z)\}$ by Proposition 5.13. The conclusion follows. \square

In the proofs of the next two propositions we use the following observation. Let $\mathcal{A}(X)$ be the collection of ample continua in a Kelley continuum X , and assume $\text{Cfm}(X) = \emptyset$. Since $\text{Cfm}(X) = \bigcap \mathcal{A}(X)$, we have $\bigcap \mathcal{A}(X) = \emptyset$, and thus $\bigcup \{X - A \mid A \in \mathcal{A}(X)\} = X$. Since X is compact, $X = (X - A_1) \cup \dots \cup (X - A_n)$, or equivalently $A_1 \cap \dots \cap A_n = \emptyset$, for some finite collection $\{A_1, \dots, A_n\}$ of ample subcontinua of X .

Proposition 5.15. *If X is a Kelley continuum, $Y, Z \in \widehat{\mathcal{A}}(X)$, and $Y \cap Z \neq \emptyset$, then $Y \cup Z \in \widehat{\mathcal{A}}(X)$.*

Proof. If either $Y \subset Z$ or $Z \subset Y$, the conclusion is trivial. Assume $Y - Z \neq \emptyset \neq Z - Y$. By Proposition 3.6, $Y \cup Z$ is semi-terminal in X .

Suppose $\text{Cfm}(Y \cup Z) = \emptyset$. Then $A_1 \cap \dots \cap A_n = \emptyset$ for some ample subcontinua A_1, \dots, A_n of $Y \cup Z$. Note that $A_j \cap Y \neq \emptyset$ for each $j \in \{1, \dots, n\}$. Indeed, otherwise

$A_j \subset Z - Y$, and A_j is also ample in Z . The set $Y \cap Z$ is an ample subcontinuum of Z by Proposition 4.3, which is disjoint with A_j . Thus $\text{Cfm}(Z) = \emptyset$, contrary to the assumption.

Since A_1, \dots, A_n are ample continua in $Y \cup Z$ and Y is semi-terminal in $Y \cup Z$, by Corollary 3.2 it follows that $A_1 \cap Y, \dots, A_n \cap Y$ are ample subcontinua of Y . Consequently, $\emptyset \neq \text{Cfm}(Y) \subset (A_1 \cap Y) \cap \dots \cap (A_n \cap Y) \subset A_1 \cap \dots \cap A_n$, a contradiction. \square

Proposition 5.16. *If X is a Kelley continuum and $p \in X$, there exists a largest member of $\widehat{\mathcal{A}}(X)$ containing p .*

Proof. Let $\widehat{\mathcal{A}}(X, p) = \{K \in \widehat{\mathcal{A}}(X) \mid p \in K\}$, and note that $\{p\} \in \widehat{\mathcal{A}}(X, p)$. Thus $\widehat{\mathcal{A}}(X, p) \neq \emptyset$. By Proposition 5.15, it suffices to show that every member of $\widehat{\mathcal{A}}(X, p)$ is contained in a maximal member of $\widehat{\mathcal{A}}(X, p)$. We apply the Kuratowski-Zorn lemma to show this.

Let M_ξ be a collection of members of $\widehat{\mathcal{A}}(X, p)$, where ξ is in a set Ξ strictly ordered by a relation \preceq , and $M_{\xi_1} \subset M_{\xi_2}$ whenever $\xi_1 \preceq \xi_2$. Since the sets M_ξ are closed subsets of a compact space, and they are strictly ordered by inclusion, it follows that $\bigcup\{M_\xi \mid \xi \in \Xi\} = \bigcup\{M_n \mid n \in \{1, 2, \dots\}\}$ for some sets M_n selected from the collection $\{M_\xi \mid \xi \in \Xi\}$ and satisfying $M_n \subset M_{n+1}$ for each n . The continuum $M_0 = \text{Lim}M_n$ is an upper bound of the collection $\{M_\xi \mid \xi \in \Xi\}$. To use the Kuratowski-Zorn lemma as desired, it suffices to show that $M_0 \in \widehat{\mathcal{A}}(X, p)$.

Clearly, $p \in M_0$. Being the limit of the semi-terminal continua M_n , the continuum M_0 is semi-terminal by Proposition 4.5. To complete the proof it suffices to show that $\text{Cfm}(M_0) \neq \emptyset$.

Suppose $\text{Cfm}(M_0) = \emptyset$. Then $A_1 \cap \dots \cap A_m = \emptyset$ for some ample subcontinua A_1, \dots, A_m of M_0 . These continua can be slightly enlarged to the corresponding continua B_1, \dots, B_m such that for each j the set A_j is in the interior of B_j relative to M_0 and $B_1 \cap \dots \cap B_m = \emptyset$. Since the relative interior of B_j in M_0 is nonempty, $B_j \cap M_n \neq \emptyset$ for almost all n . Thus $B_j \cap M_n \neq \emptyset$ for all $j \in \{1, \dots, m\}$, and n is greater than some number C . Fix an $n > C$. By Corollary 3.2, the set $B_j \cap M_n$ is an ample subcontinuum of M_n for each j . Since $M_n \in \widehat{\mathcal{A}}(X, p) \subset \widehat{\mathcal{A}}(X)$, it follows that $\emptyset \neq (B_1 \cap M_n) \cap \dots \cap (B_m \cap M_n) \subset B_1 \cap \dots \cap B_m$, a contradiction. The proof is complete. \square

By Propositions 5.15 and 5.16, the maximal members of $\widehat{\mathcal{A}}(X)$ form a unique partition of a Kelley continuum X . Let $\Gamma(X)$ be that partition, and let $\Gamma(X, x)$ be the member of $\Gamma(X)$ containing a point $x \in X$. Similarly, let $\mathcal{G}(X, x)$ be the member of the aposyndetic decomposition $\mathcal{G}(X)$ of X containing $x \in X$. By Proposition 5.14 we have the following.

Corollary 5.17. *If X is a Kelley continuum, then the partition $\Gamma(X)$ is finer than the aposyndetic decomposition $\mathcal{G}(X)$; that is, $\Gamma(X, x) \subset \mathcal{G}(X, x)$ for each $x \in X$.*

By the Aposyndetic Decomposition Theorem, Theorem 5.8, the sets $\mathcal{G}(X, x)$ are Kelley continua if X is Kelley. Thus we can apply this theorem to each $\mathcal{G}(X, x)$ and obtain the aposyndetic decomposition $\mathcal{G}(\mathcal{G}(X, x))$. A semi-terminal subcontinuum of a semi-terminal continuum does not have to be semi-terminal in the whole space. There are counterexamples, even in $X = [0, 1]$. Nevertheless, by Proposition 5.7 the members of $\mathcal{G}(\mathcal{G}(X, x))$ are semi-terminal, not only in $\mathcal{G}(X, x)$ but also in X .

Given a Kelley continuum X and $x \in X$, let $\mathcal{G}_0(X, x) = X$. Suppose for some ordinal α a partition of X is defined into semi-terminal, Kelley subcontinua $\mathcal{G}_\alpha(X, x)$, where $\mathcal{G}_\alpha(X, x)$ is the member of that partition containing x . Then we let $\mathcal{G}_{\alpha+1}(X, x) = \mathcal{G}(\mathcal{G}_\alpha(X, x), x)$. By Proposition 5.7 it follows that $\mathcal{G}_{\alpha+1}(X, x)$ is semi-terminal. Therefore, $\mathcal{G}_{\alpha+1}(X, x)$ is a Kelley continuum by Proposition 4.2. For a limit ordinal $\beta \neq 0$ define $\mathcal{G}_\beta(X, x) = \bigcap \{\mathcal{G}_\alpha(X, x) \mid \alpha < \beta\}$. Note that such $\mathcal{G}_\beta(X, x)$ is the limit of the sets $\mathcal{G}_\alpha(X, x)$ for $\alpha < \beta$. If all $\mathcal{G}_\alpha(X, x)$'s are semi-terminal in X , then so is $\mathcal{G}_\beta(X, x)$ by Proposition 4.5. Consequently, $\mathcal{G}_\beta(X, x)$ is a Kelley continuum by Proposition 4.2. By induction, a nested, well-ordered sequence of Kelley, semi-terminal continua $\mathcal{G}_\alpha(X, x)$ is associated with each $x \in X$. For each α the collection $\mathcal{G}_\alpha(X) = \{\mathcal{G}_\alpha(X, x) \mid x \in X\}$ is a partition of X . Using Propositions 5.1 and 5.7, we also observe that for every α and $x \in X$ the set $\mathcal{G}_\alpha(X, x)$ is a continuum having an outlet point. Since X is a metric compactum, for every $x \in X$ there is the smallest countable ordinal γ such that $\mathcal{G}_\gamma(X, x) = \mathcal{G}_{\gamma+1}(X, x)$. Let $\gamma(x) = \gamma(X, x)$ be that ordinal. We note that $\{\mathcal{G}_{\gamma(x)}(X, x) \mid x \in X\}$ is a partition of X into semi-terminal continua.

Proposition 5.18. *The partition $\{\mathcal{G}_{\gamma(x)}(X, x) \mid x \in X\}$ is identical with the partition $\Gamma(X)$ of X into the maximal members of $\widehat{\mathcal{A}}(X)$.*

Proof. Let $x \in X$. Since $\mathcal{G}_{\gamma(x)}(X, x) = \mathcal{G}_{\gamma(x)+1}(X, x)$, the aposyndetic decomposition of $\mathcal{G}_{\gamma(x)}(X, x)$ is trivial, and thus $\mathcal{G}_{\gamma(x)}(X, x)$ has a cofilament point by Corollary 5.9. The continuum $\mathcal{G}_{\gamma(x)}(X, x)$ is semi-terminal in X . Consequently $\mathcal{G}_{\gamma(x)}(X, x) \in \widehat{\mathcal{A}}(X)$. Since $\Gamma(X, x)$ is the largest member of $\widehat{\mathcal{A}}(X)$ containing x , it follows that $\mathcal{G}_{\gamma(x)}(X, x) \subset \Gamma(X, x)$.

The continuum $\Gamma(X, x)$ has an outlet point with respect to the space X . Therefore it has an outlet point with respect to each continuum $Y \subset X$ containing $\Gamma(X, x)$. Thus, for any such Y , the continuum $\Gamma(X, x)$ is semi-terminal in Y , and $\Gamma(X, x) \in \widehat{\mathcal{A}}(Y)$. By Proposition 5.14 we have $\Gamma(X, x) \subset \mathcal{G}(X, x) = \mathcal{G}_1(X, x)$. More generally, using Proposition 5.14 we conclude that $\Gamma(X, x) \subset \mathcal{G}_{\alpha+1}(X, x)$ whenever $\Gamma(X, x) \subset \mathcal{G}_\alpha(X, x)$. By induction, $\Gamma(X, x) \subset \mathcal{G}_\alpha(X, x)$ for every α . In particular $\Gamma(X, x) \subset \mathcal{G}_{\gamma(x)}(X, x)$. The proof is complete. \square

6. FOUR DECOMPOSITIONS OF KELLEY CONTINUA

If \mathcal{P}_1 and \mathcal{P}_2 are partitions of a set S , we write $\mathcal{P}_1 \sqsubseteq \mathcal{P}_2$ whenever \mathcal{P}_1 is finer than \mathcal{P}_2 ; that is, each member of \mathcal{P}_1 is contained in some member of \mathcal{P}_2 . The following three partitions of a Kelley continuum X into semi-terminal subcontinua are discussed in sections 4 and 5:

- (i) the partition $\mathcal{H}(X)$ into the maximal hereditarily terminal subcontinua of X ,
- (ii) the partition $\Gamma(X)$ into the maximal semi-terminal subcontinua of X having a cofilament point, and
- (iii) the partition $\mathcal{G}(X)$ into the maximal L -sets in X .

Each member of the partitions $\mathcal{H}(X)$, $\Gamma(X)$ and $\mathcal{G}(X)$ has an outlet point, and both $\mathcal{H}(X)$ and $\mathcal{G}(X)$ are upper semi-continuous decompositions. The partition $\Gamma(X)$, in general, is neither upper nor lower semi-continuous. These partitions satisfy the relations $\mathcal{H}(X) \sqsubseteq \Gamma(X) \sqsubseteq \mathcal{G}(X)$.

Remark 6.1. There exists a finest upper semi-continuous decomposition $\Gamma_0(X)$ of a Kelley continuum X into closed sets such that $\Gamma(X)$ is finer than $\Gamma_0(X)$. The members of $\Gamma_0(X)$ are also semi-terminal subcontinua of X with outlet points. Clearly $\Gamma(X) \subseteq \Gamma_0(X) \subseteq \mathcal{G}(X)$. The decomposition $\Gamma_0(X)$ is not in the focus of this paper, though it may be an interesting subject for future study.

The fourth partition from the title of this section is the *mutually aposyndetic decomposition* $\mathcal{Q}(X)$ of a Kelley continuum X defined in [18]. The decomposition $\mathcal{Q}(X)$ is upper semi-continuous, and its members are closed. It is defined by the following equivalence relation: If $x, y \in X$, then x is equivalent to y if and only if every ample continua A_x and A_y in X containing x and y , respectively, have nonempty intersection. Observe that each L -set in X is contained in a single member of the decomposition $\mathcal{Q}(X)$ by the definition of $\mathcal{Q}(X)$. Thus $\mathcal{G}(X) \subseteq \mathcal{Q}(X)$. As it is noted in Remark 2.3, the decomposition of the continuum \hat{X}_Δ^2 defined in Example 2.2, into the continua $(q \times q)^{-1}(p)$, equals $\mathcal{Q}(\hat{X}_\Delta^2)$ whenever X is a mutually aposyndetic, homogeneous curve. The members of this decomposition are semi-terminal continua without outlet points. Note that these members are products of two pseudo-arcs, and thus they are semi-indecomposable continua without cofilament points.

For a Kelley continuum X and an ordinal $\alpha > 0$, let $\mathcal{G}_\alpha(X)$ be the partition of X defined in the previous section. The partitions $\mathcal{G}_\alpha(X)$, for $\alpha > 1$, and $\Gamma(X)$ are neither upper nor lower semi-continuous. If $\alpha_2 \geq \alpha_1 > 0$, we have the following:

$$\mathcal{H}(X) \subseteq \Gamma(X) \subseteq \mathcal{G}_{\alpha_2}(X) \subseteq \mathcal{G}_{\alpha_1}(X) \subseteq \mathcal{G}_1(X) = \mathcal{G}(X) \subseteq \mathcal{Q}(X).$$

If X is a homogeneous continuum, it follows from Jones' aposyndetic theorem [11] that $\gamma(x) = 1$ for every $x \in X$, and $\mathcal{G}(X) = \mathcal{G}_1(X) = \Gamma(X)$ is the aposyndetic decomposition of X . For each of the decompositions $\mathcal{H}(X)$, $\mathcal{G}(X)$ and $\mathcal{Q}(X)$ we have numerous examples of homogeneous continua X such that the decomposition is nontrivial and has nondegenerate members. We do not know whether $\mathcal{H}(X) = \mathcal{G}(X)$ for every decomposable homogeneous continuum X . By the main results of [13] and [27] this last problem is equivalent to [28, Question 1, p. 377].

Question 2. Does there exist a homogeneous continuum X such that at least two of the decompositions $\mathcal{H}(X)$, $\mathcal{G}(X)$ and $\mathcal{Q}(X)$: (i) are different; (ii) are nontrivial; and (iii) have nondegenerate members?

The following question, which was originally asked in [18, Question 2], is related to Question 1 from section 2.

Question 3. Is every aposyndetic, homogeneous curve mutually aposyndetic?

In the general case, the following two fundamental questions concerning the mutually aposyndetic decomposition $\mathcal{Q}(X)$ remain open (compare [18, Question 1]).

Question 4. If X is a Kelley continuum, is the quotient $X/\mathcal{Q}(X)$ a Kelley continuum?

Question 5. If X is a Kelley continuum, is each member of $\mathcal{Q}(X)$ connected? If yes, is it a semi-indecomposable continuum?

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