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THE BEREZIN TRANSFORM AND mTH-ORDER BERGMAN METRIC

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Abstract. We improve upon recent directional derivative estimates for Berezin's operator calculus, and consider the relation between the mth-order Bergman metric of Burbea and the classical Bergman metric in the analysis of higher order directional derivative estimates of the Berezin symbols of general bounded operators. A new metric, naturally arising in our analysis, is introduced and certain comparison theorems are established among this metric, the mth-order Bergman metric and the classical Bergman metric on the unit ball, the polydisk and $\mathbb{C}^n$.

1. Introduction

Recently, much attention has been concentrated on the study of the Berezin transform, introduced by F. A. Berezin in his quantization program [1], [2]. The Berezin transform provides a general symbol calculus for linear operators on any reproducing-kernel Hilbert space. More specifically, for H a Hilbert space of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ with reproducing kernel $K_w(z) = K(z,w)$ such that $K(z,z) > 0$, and $X \in Op(H)$ the algebra of bounded linear operators on H, the Berezin transform (or symbol) of X is given by

$$\widetilde{X}(z) = \langle Xk_z, k_z\rangle = tr(XA(z)),$$

where $k_z(\cdot) = K(\cdot, z)K(z,z)^{-\frac{1}{2}}$ is the normalized kernel function at z, and $A(z) = k_z \otimes k_z = \langle \cdot, k_z\rangle k_z$ is the projection onto the span of k_z.

The attraction and motivation for the recent focus on the study of the Berezin transform is Coburn's sharp Lipschitz estimate ([6], [7]):

$$\sup_{\substack{a\neq b\in\Omega \\ X\neq 0\in Op(A^2(\Omega))}} \frac{|\widetilde{X}(a) - \widetilde{X}(b)|}{\|X\| d_\Omega(a,b)} = 2.$$

Here $A^2(\Omega)$ is either the Bergman space $A^2(\Omega, dv)$ of square-integrable holomorphic functions defined on a bounded domain $\Omega \subset \mathbb{C}^n$ with the normalized volume

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measure $d\upsilon(z) = V(\Omega)^{-1}dm(z)$, where dm is the Lebesgue measure and $V(\Omega)$ is the volume of Ω with respect to dm, or the Segal-Bargmann space $H^2(\mathbb{C}^n, d\mu)$ of square-integrable entire functions on the n-dimensional complex space $\mathbb{C}^n$ with the Gaussian measure

$$d\mu(z) = (2\pi)^{-n}e^{-\frac{|z|^2}{2}}dm(z).$$

The function $d_\Omega(a,b)$ is the Bergman distance function induced by the (positive-definite) infinitesimal Bergman metric

$$B_1(z,v) = \Big\{ \sum_{j,k=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K(z,z) v_j \bar{v}_k \Big\}^{\frac{1}{2}} \tag{1}$$

for $v \in T_z(\Omega) \cong \mathbb{C}^n$, the tangent space at $z \in \Omega$.

Subsequently, Engliš and Zhang [9] further investigated the actions of invariant differential operators on the Berezin symbols in the context of bounded symmetric domains. Specifically, the action of the *Laplace-Beltrami* operator on the range of the Berezin transform was examined for the unit ball of $\mathbb{C}^n$ in [14], where the main techniques rely strongly on the homogeneity of the underlying domains similar to [9].

We observe that one of the main ingredients of the approach in [9] is the commutativity of differential operators and the trace operation; that is, for any differential operator L and $X \in Op(A^2(\Omega))$,

$$(L\widetilde{X})(z) = L[tr(XA(\cdot))](z) = tr[X(LA)(z)]. \tag{2}$$

In particular, for any differential operator L with $L1 = 0$,

$$tr[(LA)(z)] = 0 \tag{3}$$

since $\widetilde{I}(z) = tr(A(z)) = 1$ for any $z \in \Omega$ by taking $X = I$, the identity operator in (2). Moreover, the estimate

$$|(L\widetilde{X})(z)| \leq \|X\| \|(LA)(z)\|_{tr} \tag{4}$$

follows immediately from the standard fact ([10]) that

$$|tr(XT)| \leq \|X\| \|T\|_{tr}$$

for X bounded and T trace-class. Furthermore, if $\|(LA)(z)\|_{tr}$ is positive for any $z \in \Omega$, let $L^* = \|(LA)(z)\|_{tr}^{-1}L$ be the "weighted" differential operator associated to L. Then (4) is equivalent to

$$\|L^*\widetilde{X}\|_\infty \leq \|X\|, \tag{5}$$

which is analogous to the results in Theorem 1, Theorem 2 and the covariant differentiation conjecture of [9].

For the covariant differentiation conjecture of [9], the first step towards its solution was demonstrated in [8], where sharp directional derivative estimates were obtained with the aid of the Skwarczynski distance function ([15]); that is,

$$\max_{X \neq 0 \in Op(A^2(\Omega))} \frac{|(D_v\widetilde{X})(z)|}{B_1(z,v)\|X\|} = 2 \tag{6}$$

for $z \in \Omega, v \in \mathbb{C}^n \backslash \{0\}$, where $D_v = \partial_v + \bar{\partial}_v$ is the directional derivative operator. If we write $L_v = B_1(z,v)^{-1} D_v$, the weighted directional derivative operator, (6) yields

$$\|L_v \widetilde{X}\|_\infty \leq 2\|X\|,$$

which is of the form (5). In addition, in order to show the equality of (6), the following basic identities are established:

$$\|(D_v A)(z)\| = \frac{1}{2}\|(D_v A)(z)\|_{tr} = B_1(z,v), \quad \|[(D_v A)(z)]^2\|_{tr} = 2B_1(z,v)^2. \tag{7}$$

Therefore, in view of (6), it is natural to consider the actions of higher order directional derivatives operators D_v^m on the range of the Berezin transform and expect to extract the geometrical content of $\|(D_v^m A)(z)\|_{tr}$ like (7) if possible. We achieve this goal in part by connecting $\|(D_v^m A)(z)\|_{tr}$ with the mth-order Bergman metric $(B_m \partial_v)(z)$ of Burbea ([4]) in Section 3.2. On the other hand, thanks to the referee of [8] who raised the question of improving our "less sharp" multiplying constant in Corollary 2.2 of that paper, we elaborate upon (6) by considering the action of ∂_v and $\bar{\partial}_v$ separately, obtain the similar estimates and identities (Proposition 2.2 and Theorem 2.3), and answer the question affirmatively. Furthermore, since (6) and (7) can be recovered from these more detailed results, this suggests that it might be much more natural to investigate the actions of higher order holomorphic directional derivatives operators ∂_v^m on the range of the Berezin transform. Proposition 2.2 also motivates us to introduce a new biholomorphically invariant "metric"

$$(L_m \partial_v)(z) = \|(\partial_v^m A)(z)\|_{tr}. \tag{8}$$

It turns out that this new quantity (8) has an intimate relation with the mth-order Bergman metric $(B_m \partial_v)(z)$, which is one of the main issues discussed throughout the paper.

This paper is organized as follows. We will improve the directional derivative estimates (6) from different perspectives in Section 2. The relation between the mth-order Bergman metric $(B_m \partial_v)(z)$ and the classical Bergman metric $B_1(z,v)$ is investigated on certain domains in $\mathbb{C}^n$ in Section 3, where one of the major results is that $(B_m \partial_v)(z)$ is a constant multiple of $\{B_1(z,v)\}^m$ on the open unit ball $\mathbb{B}^n$. The new metric (8) is introduced in Section 4, and some equivalence relations are established among $(B_m \partial_v)(z)$, $\{B_1(z,v)\}^m$ and $(L_m \partial_v)(z)$ on certain domains. As a by-product, the Bloch type estimates of the Berezin symbols in terms of the higher order derivatives are obtained on the unit ball $\mathbb{B}^n$.

We conclude this section by setting up the standard notation for multi-indices in our presentation. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$ with Z_+ the set of nonnegative integers, its length is $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$ and $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}, \sum_{\beta=0}^{\alpha} = \sum_{\beta_1=0}^{\alpha_1} \cdots \sum_{\beta_n=0}^{\alpha_n}$ for $\beta = (\beta_1, \cdots, \beta_n)$ with $\beta_j \leq \alpha_j$ for each j. We write $\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $z, w \in \mathbb{C}^n$, and denote $\partial_j = \frac{\partial}{\partial z_j}$, $\bar{\partial}_j = \frac{\partial}{\partial \bar{z}_j}$ for $j = 1, 2, \cdots, n$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. For $v = (v_1, \cdots, v_n) \in \mathbb{C}^n \backslash \{0\}$, $\partial_v = \sum_{j=1}^n v_j \partial_j, \bar{\partial}_v = \sum_{j=1}^n \bar{v}_j \bar{\partial}_j$, $\partial_v^m = \partial_v(\partial_v^{m-1}), D_v^m = D_v(D_v^{m-1})$ for $m \geq 2$, and $\partial_{v,z}$ means $\partial_{v,z} f = (\partial_v f)(z)$ for a fixed point $z \in \mathbb{C}^n$ and a differentiable function f at z.

2. Improvement of directional derivative estimates

We begin with a fact about a class of selfadjoint operators of rank two.

Lemma 2.1. *For nonzero vectors $x, y \in H$ a Hilbert space and $\langle x, y\rangle = 0$, let T be the selfadjoint operator $x \otimes y + y \otimes x$. Then*

$$\|T^2\|_p = 2^{\frac{1}{p}}\|x\|^2\|y\|^2, \quad \|T\|_p = 2^{\frac{1}{p}}\|x\|\|y\|,$$

where $\|\cdot\|_p$ is the Schatten p-class norm for $p \geq 1$.

Proof. Note that

$$T^2 = \|y\|^2 x \otimes x + \|x\|^2 y \otimes y,$$

so the square root $|T|$ of the positive operator T^2 is

$$|T| = \|x\|\|y\|\frac{x}{\|x\|} \otimes \frac{x}{\|x\|} + \|x\|\|y\|\frac{y}{\|y\|} \otimes \frac{y}{\|y\|}.$$

Our claims follow readily. □

As an application of the preceding lemma, we can improve slightly Proposition 3.2 of [8] as follows:

Proposition 2.2.

$$\|(\partial_v A)(z)\|_p = \|(\bar{\partial}_v A)(z)\|_p = B_1(z, v), \|(D_v A)(z)\|_p = 2^{\frac{1}{p}} B_1(z, v),$$
$$\|[(D_v A)(z)]^2\|_p = 2^{\frac{1}{p}} B_1(z, v)^2.$$

Proof. Writing $(g \otimes h)(f) = \langle f, h\rangle g$ for a typical rank-one operator, we have

$$\begin{aligned}
(\partial_v A)(z) &= \partial_{v,z}\{K(w, w)^{-1}K(\cdot, w) \otimes K(\cdot, w)\} \\
&= \{\partial_{v,z}[K(w, w)^{-1}]\}K(\cdot, z) \otimes K(\cdot, z) + K(z, z)^{-1}\partial_{v,z}\{K(\cdot, w) \otimes K(\cdot, w)\} \\
&= K(\cdot, z) \otimes \{\bar{\partial}_{v,z}[K(w, w)^{-1}]\}K(\cdot, z) + K(\cdot, z) \otimes K(z, z)^{-1}(\bar{\partial}_v K)(\cdot, z) \\
&= K(\cdot, z) \otimes \{\bar{\partial}_{v,z}[K(w, w)^{-1}]K(\cdot, z) + K(z, z)^{-1}(\bar{\partial}_v K)(\cdot, z)\}.
\end{aligned}$$

As an operator of rank one,

$$\|(\partial_v A)(z)\|_p = \|K(\cdot, z)\| \times \|\bar{\partial}_{v,z}[K(w, w)^{-1}]K(\cdot, z) + K(z, z)^{-1}(\bar{\partial}_v K)(\cdot, z)\|,$$

since $\|g \otimes h\|_p = \|g\|\|h\|$.

Since $\|K(\cdot, z)\|^2 = K(z, z)$ and

$$\begin{aligned}
&\|\bar{\partial}_{v,z}[K(w, w)^{-1}]K(\cdot, z) + K(z, z)^{-1}(\bar{\partial}_v K)(\cdot, z)\|^2 \\
= \;& K(z, z)^{-3}\{K(z, z)(\partial_v\bar{\partial}_v K)(z, z) - (\partial_v K)(z, z)(\bar{\partial}_v K)(z, z)\},
\end{aligned}$$

we have

$$\begin{aligned}
\|(\partial_v A)(z)\|_p &= \left\{K(z, z)^{-2}[K(z, z)(\partial_v\bar{\partial}_v K)(z, z) - (\partial_v K)(z, z)(\bar{\partial}_v K)(z, z)]\right\}^{\frac{1}{2}} \\
&= \{(\partial_v\bar{\partial}_v \log K)(z, z)\}^{\frac{1}{2}} \\
&= B_1(z, v).
\end{aligned}$$

Observing (3) for $L = \partial_v$ and $tr(x \otimes y) = \langle x, y\rangle$, it follows that

$$tr[(\partial_v A)(z)] = \langle K(\cdot, z), \bar{\partial}_{v,z}[K(w, w)^{-1}]K(\cdot, z) + K(z, z)^{-1}(\bar{\partial}_v K)(\cdot, z)\rangle = 0.$$

The remaining identities just follow from Lemma 2.1 by noting that $(\partial_v A)(z)$ is the adjoint of $(\bar{\partial}_v A)(z)$, and

$$(D_v A)(z) = (\partial_v A)(z) + (\bar{\partial}_v A)(z).$$ □

Similar to Theorem 3.3 of [8], we can also obtain the sharpness of the estimates for ∂_v and $\bar{\partial}_v$ separately.

Theorem 2.3.

$$\max_{X\neq 0\in Op(A^2(\Omega))} \frac{|(\partial_v \widetilde{X})(z)|}{B_1(z,v)\|X\|} = 1, \qquad \max_{X\neq 0\in Op(A^2(\Omega))} \frac{|(\bar{\partial}_v \widetilde{X})(z)|}{B_1(z,v)\|X\|} = 1.$$

Proof. In view of Proposition 2.2 and (4) for L equal to ∂_v or $\bar{\partial}_v$, it suffices to show that the equality can be realized at each point in each estimate.

For the first one, let $X_{z,v} = (\bar{\partial}_v A)(z)$ for any fixed $z \in \Omega$ and $v \in \mathbb{C}^n \backslash \{0\}$. Then we know that $X_{z,v} \in Op(A^2(\Omega))$ and

$$(\partial_v \widetilde{X}_{z,v})(w) = tr[X_{z,v}(\partial_v A)(w)].$$

Evaluating at $w = z$,

$$(\partial_v \widetilde{X}_{z,v})(z) = tr[X_{z,v}(\partial_v A)(z)] = tr[(\bar{\partial}_v A)(z)(\partial_v A)(z)],$$

and by Proposition 2.2,

$$\begin{aligned} & tr[(\bar{\partial}_v A)(z)(\partial_v A)(z)] \\ = \quad & \|K(\cdot,z)\|^2 \times \|\bar{\partial}_{v,z}[K(w,w)^{-1}]K(\cdot,z) + K(z,z)^{-1}(\bar{\partial}_v K)(\cdot,z)\|^2 \\ = \quad & \|(\partial_v A)(z)\|_{tr}\|(\bar{\partial}_v A)(z)\|, \end{aligned}$$

so

$$\frac{(\partial_v \widetilde{X}_{z,v})(z)}{\|(\partial_v A)(z)\|_{tr}} = \frac{(\partial_v \widetilde{X}_{z,v})(z)}{B_1(z,v)} = \|(\bar{\partial}_v A)(z)\| = \|X_{z,v}\|.$$

Similarly, for $Y_{z,v} = (\partial_v A)(z)$,

$$\frac{(\bar{\partial}_v \widetilde{Y}_{z,v})(z)}{\|(\bar{\partial}_v A)(z)\|_{tr}} = \frac{(\bar{\partial}_v \widetilde{Y}_{z,v})(z)}{B_1(z,v)} = \|Y_{z,v}\|.$$

□

Remark 2.4. The estimate (6) can be recovered by Theorem 2.3 for $|(D_v \widetilde{X})(z)| \leq |(\partial_v \widetilde{X})(z)| + |(\bar{\partial}_v \widetilde{X})(z)|$. Moreover, as an application of the ∂_v part of Theorem 2.3, Corollary 2.2 of [8] can be improved to recover Hahn's Theorem in [11] again with the exact constant as his, which answers the question of the referee of [8] as mentioned in the introduction.

3. Burbea's mth-order Bergman metric

In this section, we formulate at first the definition of Burbea's mth-order Bergman metric in a more general setting, namely, in the space of holomorphic vector fields. It will help us clarify the difference between Burbea's metric when $m > 1$ and the classical Bergman metric geometrically, even though the former is a natural generalization of the latter in the sense of Hilbert space theory.

For Ω any bounded domain in $\mathbb{C}^n$ and a fixed point $a \in \Omega$, let $V = \sum_{j=1}^n v_j(z)\partial_j$ be a nonzero holomorphic vector field on Ω and for $m \geq 1$, define

$$S_m(V,a) = \{f \in A^2(\Omega) : \|f\| \leq 1, (V^j f)(a) = 0, j = 0, 1, \cdots, m-1\}, \tag{9}$$

where $V^0 = I$, the identity operator. We denote by $S_0(V,a) = \{f \in A^2(\Omega) : \|f\| \leq 1\}$ the closed unit ball of $A^2(\Omega)$. It is clear that $S_m(V,a)$ is nonempty since it

contains the functions of the form $c(z-a)^\alpha = c\prod_{j=1}^n (z_j - a_j)^{\alpha_j}$ with $|\alpha| \geq m$, where $c = \|(z-a)^\alpha\|^{-1}$. Let

$$(R_m V)(a) = \sup_{f \in S_m(V,a)} |(V^m f)(a)|^2, \tag{10}$$

and write

$$\{(B_m V)(a)\}^2 = K(a,a)^{-1}(R_m V)(a). \tag{11}$$

So we can view B_m as a mapping defined on the space of holomorphic vector fields. In particular, $R_0(a) = (R_0 V)(a) = K(a,a)$ and $(B_0 V)(a) = 1$ ([12]).

Note that for $m \geq 1$, the iteration V^m of a holomorphic vector field V can be viewed as a holomorphic differential operator of the form $V^m = \sum_{|\alpha|=0}^m f_\alpha(z)\partial^\alpha$ for some holomorphic functions f_α. For fixed $a \in \Omega$, the linear functionals $V_a^m : f \to (V^m f)(a)$ defined on $A^2(\Omega)$ are bounded by Cauchy Estimates and the fact that the L^2 norm dominates the supremum norm over any compact set for $f \in A^2(\Omega)$. So there exists a unique element $K_{V^m,a} \in A^2(\Omega)$ such that

$$(V^m f)(a) = \langle f, K_{V^m,a}\rangle$$

and

$$K_{V^m,a}(z) = \langle K_{V^m,a}, K_z\rangle = \overline{\langle K_z, K_{V^m,a}\rangle} = \overline{(V^m K_z)(a)}.$$

Let $e_{a,j}(z) = K_{V^j,a}(z)$ for $j = 0, 1, \cdots, m$. In turn, (9) can be rewritten as

$$S_m(V,a) = \{f \in S_0(V,a) : \langle f, e_{a,j}\rangle = 0, j = 0, 1, \cdots, m-1\}$$

and

$$(R_m V)(a) = \sup_{f \in S_m(V,a)} |\langle f, e_{a,m}\rangle|^2. \tag{12}$$

Let $\mathcal{H}_{a,m}$ be the closed subspace spanned by $\{e_{a,j}\}_{j=0}^{m-1}$ and $P_{\mathcal{H}_{a,m}^\perp}$ be the orthogonal projection onto $\mathcal{H}_{a,m}^\perp$, the orthogonal complement space of $\mathcal{H}_{a,m}$ in $A^2(\Omega)$. Then $(R_m V)(a)$ is exactly the square of the operator norm of the linear functional V_a^m restricted to $\mathscr{H}_{a,m}^\perp$ by (10), and

$$(R_m V)(a) = \|P_{\mathcal{H}_{a,m}^\perp} e_{a,m}\|^2 \tag{13}$$

by (12) in the point of view of Hilbert space geometry.

Proposition 3.1. *Let $\psi : \Omega_1 \to \Omega_2$ be a biholomorphic mapping, and $\psi_*(V)$ the push-forward of the vector field V under ψ. Then $(B_m V)(a)$ is biholomorphically invariant on the space of holomorphic vector fields on Ω_1 in the sense:*

$$(B_m V)^{(\Omega_1)}(a) = (B_m \psi_*(V))^{(\Omega_2)}(\psi(a)). \tag{14}$$

Proof. For $w = \psi(z)$, consider the associated unitary operator $U_\psi : A^2(\Omega_2) \longrightarrow A^2(\Omega_1)$ defined by

$$(U_\psi f)(z) = (f \circ \psi)(z) J_\psi(z) \tag{15}$$

for $f \in A^2(\Omega_2)$, where $J_\psi(z) = \det[\psi'(z)]$ is the complex Jacobian of ψ.

We claim that

$$U_\psi\{S_m^{(\Omega_2)}(\psi_*(V), \psi(a))\} = S_m^{(\Omega_1)}(V, a).$$

For $f \in S_m^{(\Omega_2)}(\psi_*(V), \psi(a))$, we have that for $j = 0, \cdots, m-1$,

$$0 = [(\psi_*(V))^j f](\psi(a)) = [V^j(f \circ \psi)](a) \tag{16}$$

by the general chain rule for vector fields. It follows by (16) and the Leibnitz rule that for $j = 0, \cdots, m-1$,

(17)
$$[V^j(U_\psi f)](a) = [V^j(f \circ \psi(\cdot) J_\psi(\cdot))](a) = \sum_{i=0}^{j} \binom{j}{i} [V^i(f \circ \psi)](a)[V^{j-i} J_\psi](a) = 0.$$

So we have $U_\psi f \in S_m^{(\Omega_1)}(V, a)$ since $\|U_\psi f\| = \|f\|$. A similar argument can be used to prove the opposite inclusion, which justifies our claim.

Moreover, for $f \in S_m^{(\Omega_2)}(\psi_*(V), \psi(a))$, (16) and (17) imply that

$$[V^m(U_\psi f)](a) = [V^m(f \circ \psi)](a) J_\psi(a) = [(\psi_*(V))^m f](\psi(a)) J_\psi(a).$$

Therefore, we have

$$(R_m V)^{(\Omega_1)}(a) = (R_m \psi_*(V))^{(\Omega_2)}(\psi(a)) |J_\psi(a)|^2.$$

Then (14) follows easily from the definition (11) and the transformation laws for reproducing kernels (Proposition 1.4.12 of [13]). □

3.1. **mth-order Bergman metric.** Now we specialize the vector field V to be the constant vector field $v \cong \partial_v = \sum_{j=1}^n v_j \partial_j$ with $v \in \mathbb{C}^n \backslash \{0\}$. In this case,

$$(B_1 \partial_v)(a) = B_1(a, v), \tag{18}$$

the classical Bergman metric defined in (1) (see [3] or [12]). Moreover,

$$(B_m \partial_v)(a) = B_m(a, v), \tag{19}$$

the mth-order Bergman metric introduced by Burbea in [4]. We would like to point out that $(B_m \partial_v)(a)$ is homogeneous in v of degree m, and is NOT, strictly speaking, an infinitesimal metric for $m > 1$, since the length of a C^1 curve with respect to $(B_m \partial_v)(a)$ is not invariant after reparametrization for $m > 1$.

It is well known that for the classical Bergman metric,

$$B_1^{(\Omega_1)}(a, v) = B_1^{(\Omega_2)}(\psi(a), \psi_*(v)(a)), \tag{20}$$

where $w = \psi(z)$ is a biholomorphic mapping from Ω_1 to Ω_2, and $\psi_*(v)(a) = \psi'(a)v$ is the tangent vector at point $\psi(a)$. In view of (18),

$$B_1^{(\Omega_2)}(\psi(a), \psi_*(v)(a)) = (B_1 \psi_*(v)(a))^{(\Omega_2)}(\psi(a)) = (B_1 \psi_*(v))^{(\Omega_2)}(\psi(a)). \tag{21}$$

Here $\psi_*(v)(a)$ is identified as the constant vector field $\partial_{\psi'(a)v} = \sum_{j=1}^n [\psi'(a)(v)]_j \partial_{w_j}$, $\psi_*(v) = \psi'(z)v$ is the push-forward of the constant vector field $\partial_v = \sum_{j=1}^n v_j \partial_j$ under ψ and they are equal at $\psi(a)$. The second equality in (21) follows from the simple fact that for any vector field X and any function f differentiable at point a in Ω, the directional derivative $(Xf)(a)$ depends simply on $X(a)$, the value of X at a, while $(X^m f)(a)$ is determined by the local behavior of X near a for $m > 1$.

Since Burbea's metric $B_m(a, v)$ is NOT an infinitesimal metric but is denoted in the form of an infinitesimal metric, care must be taken in considering the "transformation laws" for $B_m(a, v)$ with $m > 1$ ([4]):

$$B_m^{(\Omega_1)}(a, v) = B_m^{(\Omega_2)}(\psi(a), \psi_*(v)). \tag{22}$$

It looks very similar to the usual biholomorphically invariant formula for intrinsic metrics such as (20). However, in order to avoid any confusion, we would like to

emphasize that both sides of (22) should be understood in the sense of vector fields such as (19). So

$$B_m^{(\Omega_2)}(\psi(a), \psi_*(v)) = (B_m \psi_*(v))^{(\Omega_2)}(\psi(a)) \tag{23}$$

while

$$B_m^{(\Omega_2)}(\psi(a), \psi_*(v)(a)) = (B_m \psi_*(v)(a))^{(\Omega_2)}(\psi(a)). \tag{24}$$

Generally, (23) and (24) are not identical, even though they coincide for some particular vector fields on the unit ball (see Proposition 3.13). We note that another detailed proof of (22) is included in the proof of Theorem 4 of [5], which confirms our understanding of (22) in the language of vector fields.

For $V = \partial_v$ with $v \in \mathbb{C}^n \backslash \{0\}$, we write $e_j = e_{a,j} = K_{\partial_v^m, a}(\cdot) = (\bar{\partial}_v^j K)(\cdot, a)$ if no confusion arises. Then the set $\{e_j\}_{j=0}^m$ is linearly independent since the set of holomorphic polynomials is contained in $A^2(\Omega)$. It follows immediately from (13) that $(B_m \partial_v)(a)$ is strictly positive at each point, and the nonzero supremum is attained in (10). Furthermore, an application of the Gram-Schmidt orthogonalization process gives an orthonormal basis $\{\varphi_i\}_{i=0}^m$ for $\mathcal{H}_{a,m+1}$ ([3]):

$$\begin{aligned} \varphi_0 &= J_0^{-\frac{1}{2}} e_0, \\ \varphi_i &= \{J_{i-1} J_i\}^{-\frac{1}{2}} \begin{vmatrix} \langle e_0, e_0 \rangle & \cdots & \langle e_0, e_{i-1} \rangle & e_0 \\ \vdots & \ddots & \vdots & \vdots \\ \langle e_i, e_0 \rangle & \cdots & \langle e_i, e_{i-1} \rangle & e_i \end{vmatrix} \end{aligned} \tag{25}$$

for $i = 1, \cdots, m$, where

$$J_i = J_i(\partial_v, a) = \begin{vmatrix} \langle e_0, e_0 \rangle & \cdots & \langle e_0, e_i \rangle \\ \vdots & \ddots & \vdots \\ \langle e_i, e_0 \rangle & \cdots & \langle e_i, e_i \rangle \end{vmatrix}$$

is the determinant of the Gram-matrix of the system $\{e_j\}_{j=0}^i$ for $i = 0, 1, \cdots, m$.

Note that $J_i > 0$ due to the linear independence of $\{e_j\}_{j=0}^i$ (see [3]) for $i = 0, 1, \cdots, m$, and $(B_m \partial_v)(a)$ can be expressed in a closed form in terms of the kernel function, i.e. ([3])

$$(R_m \partial_v)(a) = J_{m-1}^{-1} J_m, \quad \{(B_m \partial_v)(a)\}^2 = \{J_0 J_{m-1}\}^{-1} J_m, \quad m \geq 1. \tag{26}$$

3.2. **Characterizations of mth-order Bergman metric.** We are going to provide two characterizations of the mth-order Bergman metric in terms of the operators $(D_v^m A)(a)$ and $(\partial_v^i \bar{\partial}_v^j A)(a)$ in Berezin's operator calculus. First, we note that by the Leibnitz rule,

$$(D_v^m A)(a) = \sum_{i+j+l=m} \frac{m!}{i!j!l!} (D_v^i K^{-1})(a, a) e_j \otimes e_l = \sum_{j=0}^m \sum_{i=0}^{m-j} a_{ij} e_j \otimes e_i, \tag{27}$$

where $a_{ij} = \frac{m!}{i!j!(m-i-j)!} (D_v^{m-i-j} K^{-1})(a, a)$. Since a_{ij} is real and $a_{ij} = a_{ji}$, it is clear that $(D_v^m A)(a)$ is selfadjoint and of rank at most $m+1$ with range contained in the space $\mathcal{H}_{a,m+1}$.

Proposition 3.2. *For $m, i, j \in \mathbb{Z}_+$,*

$$\begin{aligned}(B_m\partial_v)(a) &= \max_{\substack{f\in S_m(\partial_v,a)\\ g\in S_0(\partial_v,a)}} |tr((D_v^m A)(a) f\otimes g)|,\\ (B_i\partial_v)(a)(B_j\partial_v)(a) &= \max_{\substack{f\in S_i(\partial_v,a)\\ g\in S_j(\partial_v,a)}} |tr((\partial_v^i\bar{\partial}_v^j A)(a) f\otimes g)|.\end{aligned}$$

Proof. Based on (27),

$$\begin{aligned}tr[(D_v^m A)(a) f\otimes g] &= \sum_{j=0}^{m}\sum_{i=0}^{m-j} a_{i,j}\langle f, e_i\rangle\langle e_j, g\rangle\\ &= \sum_{j=0}^{m}\sum_{i=0}^{m-j} a_{i,j}(\partial_v^i f)(a)\overline{(\partial_v^j g)(a)}.\end{aligned}$$

For $f\in S_m(\partial_v, a)$ and $g\in S_0(\partial_v, a)$, the preceding identity reduces to

$$tr[(D_v^m A)(a) f\otimes g] = K(a,a)^{-1}(\partial_v^m f)(a)\overline{g(a)}.$$

Consequently,

$$\begin{aligned}\max_{\substack{f\in S_m(\partial_v,a)\\ g\in S_0(\partial_v,a)}} |tr((D_v^m A)(a) f\otimes g)| &= K(a,a)^{-1}\max_{f\in S_m(\partial_v,a)}|(\partial_v^m f)(a)|\max_{g\in S_0(\partial_v,a)}|g(a)|\\ &= K(a,a)^{-1}K(a,a)^{\frac{1}{2}}(B_m\partial_v)(a)K(a,a)^{\frac{1}{2}}\\ &= (B_m\partial_v)(a).\end{aligned}$$

Essentially the same argument leads to the second characterization. □

Corollary 3.3. *For $m, i, j \in \mathbb{Z}_+$,*

$$\|(D_v^m A)(a)\|_{tr} \geq (B_m\partial_v)(a), \quad \|(\partial_v^i\bar{\partial}_v^j A)(a)\|_{tr} \geq (B_i\partial_v)(a)(B_j\partial_v)(a).$$

Proof. Let $\mathscr{S}_1$ be the space of trace-class operators on $A^2(\Omega)$, and $\mathscr{S}_0$ be the space of compact operators on $A^2(\Omega)$. By the standard duality relation $(\mathscr{S}_0)^* = \mathscr{S}_1$ ([18]) under the pairing $\langle T, S\rangle = tr(TS) = tr(ST)$ for $T\in\mathscr{S}_0$ and $S\in\mathscr{S}_1$,

$$\begin{aligned}\|(D_v^m A)(a)\|_{tr} &= \sup\{|tr((D_v^m A)(a)T)| : \|T\|\leq 1\}\\ &\geq \sup\{|tr((D_v^m A)(a) f\otimes g)| : f\in S_m(\partial_v,a), g\in S_0(\partial_v,a)\}\\ &= (B_m\partial_v)(a).\end{aligned}$$

The second inequality follows similarly. □

Next, we are able to improve the lower bound in the first inequality in Corollary 3.3 by some simple facts from operator theory.

Proposition 3.4. *Let T be a rank l normal operator on a Hilbert space H with $\{\lambda_j\}_{j=1}^l$ the l eigenvalues of T repeated according to multiplicity. Then*

$$\|T^2\|_{tr} \leq \|T\|_{tr}^2 \leq l\|T^2\|_{tr}.$$

Proof. Note that $\|T\|_{tr} = \sum_{j=1}^l |\lambda_j|$. Since T^2 is also of rank l and normal with the eigenvalues $\{\lambda_j^2\}_{j=1}^l$ by Functional Calculus,

$$\|T^2\|_{tr} = \sum_{j=1}^{l}|\lambda_j|^2.$$

So the first inequality is trivial, while the second follows from the Cauchy-Schwarz inequality. □

Since $\{\varphi_j\}_{j=0}^m$ is an orthonormal basis for $\mathcal{H}_{a,m+1}$ defined in (25), it is easy to see that

$$\langle (D_v^m A)(a)\varphi_i, \varphi_j\rangle = \begin{cases} 0, & i+j>m; \\ \binom{m}{i}(B_i\partial_v)(a)(B_{m-i}\partial_v)(a), & i+j=m. \end{cases}$$

Now applying Proposition 3.4 to $T = (D_v^m A)(a)$ and by Theorem 1.4.2 in [18], we obtain

$$\begin{aligned} \|(D_v^m A)(a)\|_{tr}^2 \geq \|[(D_v^m A)(a)]^2\|_{tr} &= \sum_{0\leq i,j\leq m} |\langle (D_v^m A)(a)\varphi_i, \varphi_j\rangle|^2 \\ &\geq \sum_{i+j=m} |\langle (D_v^m A)(a)\varphi_i, \varphi_j\rangle|^2 \\ &= \sum_{i=0}^m \binom{m}{i}^2 \{(B_i\partial_v)(a)\}^2\{(B_{m-i}\partial_v)(a)\}^2. \end{aligned}$$

3.3. $(B_m\partial_v)(a)$ **on the unit ball.** It seems difficult to compute $(B_m\partial_v)(a)$ directly for any bounded domain based on its definition or the formula stated in (26). Nevertheless, we can still make a little progress by computing it for a particular point in nice domains.

Proposition 3.5. *For Ω a bounded circular domain in $\mathbb{C}^n$ containing 0,*

$$\{(B_m\partial_v)(0)\}^2 = (\partial_v^m\bar{\partial}_v^m K)(0,0).$$

Proof. For any U in the circle group on $\mathbb{C}^n$ of the form

$$U(z_1,\cdots,z_n) = (e^{i\theta}z_1,\cdots,e^{i\theta}z_n) = e^{i\theta}(z_1,\cdots,z_n)$$

for some $\theta\in[0,2\pi]$, the transformation laws of the Bergman kernel function imply that

$$K(Uz,Uz) = K(z,z).$$

Moreover, the chain rule yields that

$$\partial_v\{K(Uz,Uz)\} = (\partial_{Uv}K)(Uz,Uz), \quad \bar{\partial}_v\{K(Uz,Uz)\} = (\bar{\partial}_{Uv}K)(Uz,Uz).$$

Consequently, for any $(j,l)\in\mathbb{Z}_+\times Z_+\backslash\{(0,0)\}$,

$$(\partial_{Uv}^j\bar{\partial}_{Uv}^l K)(Uz,Uz) = \partial_v^j\bar{\partial}_v^l\{K(Uz,Uz)\} = \partial_v^j\bar{\partial}_v^l\{K(z,z)\}.$$

Evaluating at $z=0$, we have

$$\{e^{i(j-l)\theta}-1\}(\partial_v^j\bar{\partial}_v^l K)(0,0) = 0$$

for any $\theta\in[0,2\pi]$. It follows that

$$(\partial_v^j\bar{\partial}_v^l K)(0,0) = 0 \quad \text{for} \quad j\neq l,$$

which means that all off-diagonal entries of $J_m(\partial_v,0)$ vanish. So

$$J_m(\partial_v,0) = \prod_{j=0}^m (\partial_v^j\bar{\partial}_v^j K)(0,0).$$

Thus

$$\{(B_m\partial_v)(0)\}^2 = K(0,0)^{-1}J_{m-1}(\partial_v,0)^{-1}J_m(\partial_v,0) = (\partial_v^m\bar{\partial}_v^m K)(0,0)$$

since $K(0,0)=1$ due to the normalized volume measure dv. □

Corollary 3.6. *For Ω a bounded Reinhardt domain in $\mathbb{C}^n$ containing* 0,

$$\{(B_m\partial_v)(0)\}^2 = (m!)^2 \sum_{|\alpha|=m} \frac{v^\alpha \overline{v}^\alpha}{\|z^\alpha\|^2}.$$

Proof. It is known that the set $\{\frac{z^\alpha}{\|z^\alpha\|} : \alpha \geq 0\}$ forms an orthonormal basis for $A^2(\Omega)$ in this case ([3]), so that

$$K(z,z) = \sum_{\alpha\geq 0} \frac{z^\alpha \bar{z}^\alpha}{\|z^\alpha\|^2},$$

which converges absolutely and uniformly on compacta. A straightforward calculation shows that

$$(\partial_v^m\bar{\partial}_v^m K)(0,0) = (m!)^2 \sum_{|\alpha|=m} \frac{v^\alpha \bar{v}^\alpha}{\|z^\alpha\|^2},$$

which finishes the proof with an application of Proposition 3.5. □

Now, we restrict our attention to the unit ball $\mathbb{B}^n$, and show that $(B_m\partial_v)(a)$ is a constant multiple of $\{(B_1\partial_v)(a)\}^m$ as the main result of this section. It is standard that for each a in $\mathbb{B}^n$, there is an involution $\psi_a \in \mathrm{Aut}(\mathbb{B}^n)$, the group of all biholomorphic self-maps of $\mathbb{B}^n$, interchanging a and 0. More precisely,

$$w = \psi_a(z) = \frac{a - P_a z - \sqrt{1-|a|^2}Q_a z}{1-\langle z,a\rangle}, \tag{28}$$

where $P_0 = 0$, $P_a = \frac{1}{|a|^2} a\otimes a$ for $a \neq 0$, and $Q_a = I - P_a$([17]).

For simplicity, we define $g(w) = (\psi_a)_*(v) = \psi_a'(z)v = (g_1(w),\cdots,g_n(w))$. Then $g(0) = \psi_a'(a)v$. Moreover, let ∂_g^m stand for the operator $\left\{\sum_{j=1}^n g_j(w)\partial_{w_j}\right\}^m$ and $\partial_{g(0)}^m = \partial_{\psi_a'(a)v}^m$ for the operator $\left\{\sum_{j=1}^n g_j(0)\partial_{w_j}\right\}^m$. This notation will also be used in Section 4.

Lemma 3.7. *For $\Omega = \mathbb{B}^n$,*

$$\{(B_1\partial_v)(a)\}^2 = (n+1)|g(0)|^2.$$

Proof. For the unit ball $\mathbb{B}^n$, the classical Bergman metric is given by

$$\{(B_1\partial_v)(a)\}^2 = (n+1)\frac{(1-|a|^2)|v|^2 + |\langle a,v\rangle|^2}{(1-|a|^2)^2}$$

(Proposition 1.4.22 in [13]). Note that ψ_a is an isometry of $\mathbb{B}^n$ in the Bergman metric, so by (20) and (21),

$$\{(B_1\partial_v)(a)\}^2 = \{(B_1\partial_{g(0)})(0)\}^2 = (n+1)|g(0)|^2,$$

as desired. □

Remark 3.8. The above lemma could be derived by using the explicit expression of

$$\psi_a'(a) = -(1-|a|^2)^{-1}P_a - (1-|a|^2)^{-\frac{1}{2}}Q_a \tag{29}$$

(see [17]) along with a direct calculation.

Proposition 3.9. *For* $\Omega = \mathbb{B}^n$,

$$(B_m \partial_{g(0)})(0) = \left\{ \frac{m!(m+n)!}{n!(n+1)^m} \right\}^{\frac{1}{2}} \{(B_1 \partial_v)(a)\}^m.$$

Proof. We know that $\|z^\alpha\|^2 = \frac{n!\alpha!}{(n+|\alpha|)!}$ for $\alpha \geq 0$ ([17]). It follows by Corollary 3.6 that

$$(\partial_v^j \bar{\partial}_v^i K)(0,0) = \begin{cases} 0, & i \neq j; \\ \frac{i!(i+n)!}{n!}|v|^{2i}, & i = j. \end{cases} \tag{30}$$

Thus

$$\{(B_m \partial_{g(0)})(0)\}^2 = (\partial_{g(0)}^m \bar{\partial}_{g(0)}^m K)(0,0) = \frac{m!(m+n)!}{n!}|g(0)|^{2m}.$$

Then an application of Lemma 3.7 finishes our proof. □

Next we would like to establish a relation similar to Proposition 3.9 between $(B_m \partial_g)(0)$ and $(B_m \partial_{g(0)})(0)$.

Lemma 3.10.

$$g(w) = (1 - \langle w, a \rangle)\{g(0) + h(a,v)w\},$$

where $h(a,v) = -\langle g(0), a \rangle = (1 - |a|^2)^{-1}\langle v, a \rangle$.

Proof. For $w = \psi_a(z)$, its complex Jacobian matrix $\psi_a'(z)$ has the following decomposition (Lemma 2.5 [16]):

$$\psi_a'(z) = (1 - \langle \psi_a(z), a \rangle)(I - \psi_a(z) \otimes a)\psi_a'(a).$$

Thus

$$\begin{aligned} g(w) &= \psi_a'(\psi_a(w))v \\ &= (1 - \langle w, a \rangle)(I - w \otimes a)\psi_a'(a)v \\ &= (1 - \langle w, a \rangle)(I - w \otimes a)g(0) \\ &= (1 - \langle w, a \rangle)\{g(0) - \langle g(0), a \rangle w\} \\ &= (1 - \langle w, a \rangle)\{g(0) + h(a,v)w\}, \end{aligned}$$

where $h(a,v) = -\langle g(0), a \rangle = -\langle v, \psi_a'(a)a \rangle = (1 - |a|^2)^{-1}\langle v, a \rangle$, since $\psi_a'(a)$ is selfadjoint and $\psi_a'(a)a = -(1 - |a|^2)^{-1}a$ by (29). □

Lemma 3.11. *For* $j = 1, \cdots, n$,

$$(\partial_{g(0)} g_j)(w)|_{w=0} = 2h(a,v)g_j(0), \quad (\partial_{g(0)}^2 g_j)(w)|_{w=0} = 2h(a,v)^2 g_j(0).$$

Proof. By Lemma 3.10,

$$\begin{aligned} (\partial_{g(0)} g_j)(w)|_{w=0} &= \sum_{p=1}^n g_p(0)\partial_p\{g_j(0)(1 - \langle w, a \rangle) + h(a,v)w_j(1 - \langle w, a \rangle)\}|_{w=0} \\ &= \sum_{p=1}^n g_p(0)\{g_j(0)(-\bar{a}_p) + h(a,v)\delta_{pj}\} \\ &= g_j(0)\{-\langle g(0), a \rangle + h(a,v)\} \\ &= 2h(a,v)g_j(0). \end{aligned}$$

Similarly,

$$\begin{aligned}(\partial^2_{g(0)}g_j)(w)|_{w=0} &= \sum_{|\alpha|=2}\frac{2}{\alpha!}g(0)^\alpha\partial^\alpha\{g_j(0)(1-\langle w,a\rangle)+h(a,v)w_j(1-\langle w,a\rangle)\}|_{w=0}\\ &= \sum_{|\alpha|=2}\frac{2}{\alpha!}g(0)^\alpha h(a,v)\partial^\alpha\{-w_j\langle w,a\rangle\}|_{w=0}\\ &= g_j(0)^2h(a,v)(-2\bar{a}_j)+\sum_{p\neq j}2h(a,v)g_j(0)g_p(0)(-\bar{a}_p)\\ &= 2h(a,v)g_j(0)\sum_{p=1}^n g_p(0)(-\bar{a}_p)\\ &= 2h(a,v)^2g_j(0).\end{aligned}$$

□

Let $\mathscr{X}$ be a Banach space and $C^\infty(\mathbb{B}^n,\mathscr{X})$ be the set of smooth mappings from $\mathbb{B}^n$ into $\mathscr{X}$ possessing *strong derivatives* of all orders (see e.g. [9]). Our main result depends on an induction formula for the action of ∂_g^m on $C^\infty(\mathbb{B}^n,\mathscr{X})$ at $w=0$.

Lemma 3.12. *For $f\in C^\infty(\mathbb{B}^n,\mathscr{X})$ and $m\in\mathbb{N}$,*

$$(\partial_g^m f)(w)|_{w=0}=\sum_{i=1}^m C_i^{(m)}h(a,v)^{m-i}(\partial^i_{g(0)}f)(w)|_{w=0}, \tag{31}$$

where the $C_i^{(m)}$'s are constants depending on m and i, and $C_m^{(m)}=1$.

Proof. We proceed by induction on m. It holds for $m=1$ clearly. Assume (31) is true for m and any $f\in C^\infty(\mathbb{B}^n,\mathscr{X})$. Then by our assumption,

$$\begin{aligned}(\partial_g^{m+1}f)(w)|_{w=0} &= \{\partial_g^m(\partial_g f)\}(w)|_{w=0}\\ &= \sum_{i=1}^m C_i^{(m)}h(a,v)^{m-i}\{\partial^i_{g(0)}(\partial_g f)\}(w)|_{w=0}\\ &= \sum_{i=1}^m C_i^{(m)}h(a,v)^{m-i}\sum_{j=1}^n\Big\{\partial^i_{g(0)}[g_j(w)(\partial_j f)(w)]\Big\}\Big|_{w=0}\\ &= \sum_{i=1}^m C_i^{(m)}h(a,v)^{m-i}\sum_{j=1}^n\Big\{\sum_{l=0}^i\binom{i}{l}[(\partial^l_{g(0)}g_j)(w)][(\partial^{i-l}_{g(0)}\partial_j f)(w)]\Big\}\Big|_{w=0}.\end{aligned}$$

Note that by Lemma 3.10, $g_j(w)$ is a polynomial in w of degree 2 for $j=1,\cdots,n$, so for $l>2$,

$$(\partial^l_{g(0)}g_j)(w)=0.$$

It follows that

$$\begin{aligned}
&(\partial_g^{m+1} f)(w)|_{w=0} \\
= \quad & C_1^{(m)} h(a,v)^{m-1} \sum_{j=1}^{n} \Big\{ \sum_{l=0}^{1} \binom{1}{l} [(\partial_{g(0)}^{l} g_j)(w)][(\partial_{g(0)}^{1-l} \partial_j f)(w)] \Big\}\Big|_{w=0} \\
& + \sum_{i=2}^{m} C_i^{(m)} h(a,v)^{m-i} \sum_{j=1}^{n} \Big\{ \sum_{l=0}^{2} \binom{i}{l} [(\partial_{g(0)}^{l} g_j)(w)][(\partial_{g(0)}^{i-l} \partial_j f)(w)] \Big\}\Big|_{w=0} \\
= \quad & I_1 + I_2.
\end{aligned}$$

By Lemma 3.11,

$$\begin{aligned}
I_1 &= C_1^{(m)} h(a,v)^{m-1} \sum_{j=1}^{n} \Big\{ \sum_{l=0}^{1} \binom{1}{l} [(\partial_{g(0)}^{l} g_j)(w)][(\partial_{g(0)}^{1-l} \partial_j f)(w)] \Big\}\Big|_{w=0} \\
&= C_1^{(m)} h(a,v)^{m-1} (\partial_{g(0)}^{2} f)(w)|_{w=0} + 2C_1^{(m)} h(a,v)^{m} (\partial_{g(0)} f)(w)|_{w=0}
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \sum_{i=2}^{m} C_i^{(m)} h(a,v)^{m-i} \sum_{j=1}^{n} \Big\{ \sum_{l=0}^{2} \binom{i}{l} [(\partial_{g(0)}^{l} g_j)(w)][(\partial_{g(0)}^{i-l} \partial_j f)(w)] \Big\}\Big|_{w=0} \\
&= \sum_{i=2}^{m} C_i^{(m)} h(a,v)^{m-i} \Big\{ (\partial_{g(0)}^{i+1} f)(w)|_{w=0} + 2ih(a,v)(\partial_{g(0)}^{i} f)(w)|_{w=0} \\
&\quad + i(i-1)h(a,v)^2 (\partial_{g(0)}^{i-1} f)(w)|_{w=0} \Big\}.
\end{aligned}$$

Now adding up I_1 and I_2 and reindexing the sum will finish the proof, while $C_{m+1}^{(m+1)} = C_m^{(m)} = \cdots = C_1^{(1)} = 1$. □

Based on Lemma 3.12, we can demonstrate a "commutativity" of evaluation and differentiation on the unit ball, which is of independent interest.

Proposition 3.13. *For the unit ball* $\mathbb{B}^n$,

$$(B_m \partial_g)(0) = (B_m \partial_{g(0)})(0).$$

Proof. We first claim that

$$S_m(\partial_g, 0) = S_m(\partial_{g(0)}, 0).$$

Let $f \in S_m(\partial_{g(0)}, 0)$. Then $\|f\| \le 1$ and

$$(\partial_{g(0)}^{j} f)(0) = 0 \qquad \text{for} \quad j = 0, \cdots, m-1.$$

By Lemma 3.12,

$$(\partial_g^j f)(0) = \sum_{i=1}^{j} C_i^{(j)} h(a,v)^{j-i} (\partial_{g(0)}^{i} f)(0),$$

which implies that

$$S_m(\partial_{g(0)}, 0) \subset S_m(\partial_g, 0).$$

For the opposite inclusion, let $f \in S_m(\partial_g, 0)$. Then $\|f\| \le 1$ and

$$(\partial_g^j f)(0) = 0, \ j = 0, 1, \cdots, m-1.$$

By induction, applying Lemma 3.12 repeatedly and noting that $C_j^{(j)} = 1$, we have

$$(\partial_{g(0)}^j f)(0) = 0,\ j = 0, 1, \cdots, m-1.$$

So

$$S_m(\partial_g, 0) \subset S_m(\partial_{g(0)}, 0).$$

Furthermore, for $f \in S_m(\partial_g, 0)$,

$$(\partial_g^m f)(0) = (\partial_{g(0)}^m f)(0)$$

by Lemma 3.12 again. It follows that

$$\begin{aligned}(R_m\partial_g)(0) &= \max_{f\in S_m(\partial_g,0)} |(\partial_g^m f)(0)|^2 = \max_{f\in S_m(\partial_g,0)} |(\partial_{g(0)}^m f)(0)|^2 \\ &= \max_{f\in S_m(\partial_{g(0)},0)} |(\partial_{g(0)}^m f)(0)|^2 = (R_m\partial_{g(0)})(0).\end{aligned}$$

Therefore,

$$\{(B_m\partial_g)(0)\}^2 = K(0,0)^{-1}(R_m\partial_g)(0) = K(0,0)^{-1}(R_m\partial_{g(0)})(0) = \{(B_m\partial_{g(0)})(0)\}^2.$$

□

Now our main result can be stated as

Theorem 3.14. *For the unit ball $\mathbb{B}^n$,*

$$(B_m\partial_v)(a) = \left\{\frac{m!(m+n)!}{n!(n+1)^m}\right\}^{\frac{1}{2}} \{(B_1\partial_v)(a)\}^m. \tag{32}$$

Proof. The proof is immediate by combining (14) with $V = \partial_v$, Proposition 3.13 and Proposition 3.9. □

Recall [4] that the mth-order Carathéodory-Reiffen metric C_m is defined as

$$(C_m\partial_v)(a) = \sup_{f\in T_m(\partial_v,a)} |(\partial_v^m f)(a)|,$$

where $T_m(\partial_v, a) = \{f \in H^\infty(\Omega) : \|f\|_\infty \le 1, (\partial_v^j f)(z) = 0, j = 0, 1, \cdots, m-1\}$ and $H^\infty(\Omega)$ is the Banach algebra of bounded holomorphic functions on Ω. Here we use the vector field notation rather than the "infinitesimal" notation in [4]. $(C_1\partial_v)(a)$ is just the classical Carathéodory-Reiffen metric $C_1(a, v)$ in the infinitesimal form ([13]). The following chain of inequalities was established for any bounded domain by Burbea in [4]:

$$(m!)^2\{(C_1\partial_v)(a)\}^{2m} \le \{(C_m\partial_v)(a)\}^2 < \{(B_m\partial_v)(a)\}^2. \tag{33}$$

Consequently, he conjectured that $(C_m\partial_v)(a)$ and $(B_m\partial_v)(a)$ are equivalent on any bounded domain. As an instant application of Theorem 3.14, we can answer affirmatively this conjecture on the unit ball.

Corollary 3.15. *For the unit ball $\mathbb{B}^n$,*

$$\{(C_m\partial_v)(a)\}^2 < \{(B_m\partial_v)(a)\}^2 \le \frac{(m+n)!}{m!n!}\{(C_m\partial_v)(a)\}^2.$$

Proof. Since $\{(B_1\partial_v)(a)\}^2 = (n+1)\{(C_1\partial_v)(a)\}^2$ on $\mathbb{B}^n$ ([5]), Theorem 3.14 implies that

$$\begin{aligned}\{(B_m\partial_v)(a)\}^2 &= \frac{m!(m+n)!}{n!(n+1)^m}\{(B_1\partial_v)(a)\}^{2m}\\ &= \frac{m!(m+n)!}{n!}\{(C_1\partial_v)(a)\}^{2m} \le \frac{(m+n)!}{m!n!}\{(C_m\partial_v)(a)\}^2,\end{aligned}$$

as desired. □

It seems hard to find a relation such as (32) between $(B_m\partial_v)(a)$ and $(B_1\partial_v)(a)$ for other bounded domains in $\mathbb{C}^n$. As another application of Theorem 3.14, however, we can give some estimates for the boundary behavior of $(B_m\partial_v)(a)$.

Proposition 3.16. *For Ω a bounded domain in $\mathbb{C}^n$,*

$$C(m,n)\frac{d_\Omega^{2n}(a)|v|^{2m}}{\rho_\Omega^{2(m+n)}} \le \{(B_m\partial_v)(a)\}^2 \le C(m,n)\frac{\rho_\Omega^{2n}|v|^{2m}}{d_\Omega^{2(m+n)}(a)},$$

where $C(m,n) = \frac{m!(m+n)!}{n!}$, $d_\Omega(a) = \operatorname{dist}(a,\partial\Omega)$ is the distance of a to the boundary of Ω, and ρ_Ω is the radius of the smallest ball Ω_1 containing Ω.

Proof. For fixed $a \in \Omega$, let

$$\Omega_2 = \{w \in \Omega : |w-a| < d_\Omega(a)\}.$$

Then $\Omega_2 \subset \Omega \subset \Omega_1$. So we have

$$(R_j\partial_v)^{(\Omega_1)}(a) \le (R_j\partial_v)^{(\Omega)}(a) \le (R_j\partial_v)^{(\Omega_2)}(a)$$

for $j = 0, 1, \ldots$. Hence by the definition of the mth-order Bergman metric,

(34)
$$\{(B_m\partial_v)^{(\Omega_1)}(a)\}^2\frac{R_0^{(\Omega_1)}(a)}{R_0^{(\Omega_2)}(a)} \le \{(B_m\partial_v)^{(\Omega)}(a)\}^2 \le \{(B_m\partial_v)^{(\Omega_2)}(a)\}^2\frac{R_0^{(\Omega_2)}(a)}{R_0^{(\Omega_1)}(a)}.$$

For the ball $\mathbb{B}(z_0,r) = \{w \in \mathbb{C}^n : |w-z_0| < r\}$, the mapping $\psi(w) = \frac{w-z_0}{r}$ maps $\mathbb{B}(z_0,r)$ biholomorphically onto the unit ball $\mathbb{B}^n$, and $\psi_*(v) = \frac{v}{r}$. It follows from the biholomorphic invariance (14) of $(B_m\partial_v)(z)$ and Theorem 3.14 that

$$\begin{aligned}&\{(B_m\partial_v)^{\mathbb{B}(z_0,r)}(w)\}^2 = \{(B_m\psi_*(v))^{\mathbb{B}^n}(\psi(w))\}^2\\ &= \frac{C(m,n)}{(n+1)^m}\{(B_1\psi_*(v))^{\mathbb{B}^n}(\psi(w))\}^{2m} \ge \frac{C(m,n)|v|^{2m}}{r^{2m}}.\end{aligned}$$

Consequently,

$$\{(B_m\partial_v)^{(\Omega_1)}(a)\}^2 \ge C(m,n)\frac{|v|^{2m}}{\rho_\Omega^{2m}}, \quad \{(B_m\partial_v)^{(\Omega_2)}(a)\}^2 = C(m,n)\frac{|v|^{2m}}{d_\Omega^{2m}(a)}. \tag{35}$$

Moreover, the transformation laws of the Bergman kernel function $R_0(a) = K(a,a)$ give us

$$\frac{R_0^{(\Omega_2)}(a)}{R_0^{(\Omega_1)}(a)} \le \frac{\rho_\Omega^{2n}}{d_\Omega^{2n}(a)}. \tag{36}$$

Our estimates now follow from (34), (35) and (36). □

3.4. $(B_m\partial_v)(a)$ **on the polydisk.** We shall show that Theorem 3.14 fails to hold in the case of the polydisk with dimension $n \geq 2$ and order $m \geq 2$. Nevertheless, an equivalent relation between $(B_m\partial_v)(a)$ and $\{(B_1\partial_v)(a)\}^m$ will be derived on a more general setting including the polydisk in Section 4.4.

For $\Omega = \mathbb{D}^n$ the polydisk, the kernel function is given by

$$K(z,w) = \prod_{j=1}^{n}(1 - z_j\bar{w}_j)^{-2}.$$

Note that the Bergman metric on $\mathbb{D}^n$ equals

$$\{(B_1\partial_v)(a)\}^2 = 2\sum_{j=1}^{n}\frac{|v_j|^2}{(1-|a_j|^2)^2}.$$

For a contradiction, suppose that $\{(B_m\partial_v)(a)\}^2 = C(n,m)\{(B_1\partial_v)(a)\}^{2m}$ for some constant $C(n,m)$ with $n \geq 2$ and $m \geq 2$. In particular, for $a = 0$,

$$\{(B_m\partial_v)(0)\}^2 = C(n,m)\{(B_1\partial_v)(0)\}^{2m} = 2^m C(n,m)|v|^{2m}.$$

On the other hand, by Corollary 3.6, we have

$$\{(B_m\partial_v)(0)\}^2 = (m!)^2\sum_{|\alpha|=m}\frac{v^\alpha\bar{v}^\alpha}{\|z^\alpha\|^2} = (m!)^2\sum_{|\alpha|=m}\Big\{\prod_{j=1}^{n}(\alpha_j+1)\Big\}v^\alpha\bar{v}^\alpha,$$

since $\|z^\alpha\|^2 = \{\prod_{j=1}^{n}(\alpha_j+1)\}^{-1}$. Therefore, we can conclude that

$$C(n,m) = \frac{(m!)^2}{2^m|v|^{2m}}\sum_{|\alpha|=m}\Big\{\prod_{j=1}^{n}(\alpha_j+1)\Big\}v^\alpha\bar{v}^\alpha. \tag{37}$$

This is impossible for $n \geq 2$ and $m \geq 2$, since it is not hard to check that the right side of (37), as a function of v and $\bar{v}$, is not constant.

3.5. $(B_m\partial_v)(a)$ **on** $\mathbb{C}^n$**.** For the Segal-Bargmann space $H^2(\mathbb{C}^n, d\mu)$, the kernel function is given by $K(z,w) = e^{\frac{\langle z,w\rangle}{2}}$. The induced Bergman metric coincides with the usual Euclidean metric on $\mathbb{C}^n$ up to a constant $\frac{\sqrt{2}}{2}$.

The concept of the mth-order Bergman metric (11) can also be defined identically for $\mathbb{C}^n$ and the expressions (26) remain true. Moreover,

Theorem 3.17.

$$(B_m\partial_v)(z) = (B_m\psi_*(v))(\psi(z)) \tag{38}$$

for ψ in the group of all orientation-preserving rigid motions (generated by translations and unitary transformations) of $\mathbb{C}^n$.

Proof. It suffices to show that (38) is true for unitary transformations and translations. Let $U \in U(n)$ be the unitary group on $\mathbb{C}^n$. Similar to the proof of Proposition 3.5, we have for any $j, l \in \mathbb{Z}_+$,

$$(\partial_{Uv}^j\bar{\partial}_{Uv}^l K)(Uz,Uz) = \partial_v^j\bar{\partial}_v^l\{K(Uz,Uz)\} = \partial_v^j\bar{\partial}_v^l\{K(z,z)\} = (\partial_v^j\bar{\partial}_v^l K)(z,z),$$

since $K(Uz, Uz) = K(z, z)$. It follows that

$$J_m(\partial_v, z) = J_m(\partial_{Uv}, Uz),$$

so (38) holds for $\psi = U$.

For $a \in \mathbb{C}^n$ and the associated translation $\psi_a(z) = z - a$, we know that

$$K(z, z) = K(\psi_a(z), \psi_a(z))|k_a(z)|^2,$$

where $k_a(z) = e^{\frac{\langle z,a\rangle}{2} - \frac{|a|^2}{4}}$. We consider the Weyl operator

$$W_a f(z) = f(\psi_a(z))k_a(z) = f(z - a)k_a(z) \tag{39}$$

on $H^2(\mathbb{C}^n, d\mu)$. It is known that W_a is unitary, so

$$W_a\{S_m((\psi_a)_*(v), \psi_a(z))\} = S_m(\partial_v, z)$$

as we showed for U_ψ in Proposition 3.1, and the rest of the proof is also the same as the argument in Proposition 3.1. □

As a consequence of Theorem 3.17, we have

Theorem 3.18. *On $\mathbb{C}^n$,*

$$\{(B_m\partial_v)(a)\}^2 = 2^{-m}m!|v|^{2m}.$$

Consequently,

$$(B_m\partial_v)(a) = \sqrt{m!}\{(B_1\partial_v)(a)\}^m.$$

Proof. For $a \in \mathbb{C}^n$, let $\psi_a(z) = z - a$. Then $(\psi_a)_*(v) = v$. It follows that

$$\begin{aligned}(B_m\partial_v)(a) &= \{(B_m(\psi_a)_*(v))(\psi_a(a))\}^2 = \{(B_m\partial_v)(0)\}^2 \\ &= (\partial_v^m\bar{\partial}_v^m K)(0, 0) = 2^{-m}m!|v|^{2m}.\end{aligned}$$

□

4. A new metric $(L_m\partial_v)(z)$

Motivated by Proposition 2.2 and the definition of the mth-order Bergman metric, we might replace ∂_v in $\|(\partial_v A)(z)\|_{tr}$ by any nonzero (holomorphic) vector field. That is, we can introduce another mapping from the space of vector fields to $C^\infty(\Omega)$ defined by

$$(L_m V)(z) = \|(V^m A)(z)\|_{tr} \tag{40}$$

for any smooth vector field V on Ω. It turns out that L_m enjoys the same transformation laws as B_m.

Proposition 4.1. *Let $\psi : \Omega_1 \to \Omega_2$ be a biholomorphic mapping, and $\psi_*(V)$ the push-forward of the vector field V under ψ. Then $(L_m V)(z)$ is biholomorphically invariant on the space of holomorphic vector fields on Ω_1 in the sense:*

$$(L_m V)^{(\Omega_1)}(z) = (L_m\psi_*(V))^{(\Omega_2)}(\psi(z)).$$

Proof. Let $w = \psi(z) : \Omega_1 \longrightarrow \Omega_2$ be a biholomorphic mapping. Then the adjoint U_ψ^* of the operator U_ψ in (15) is given by

$$(U_\psi^* f)(w) = (f \circ \psi^{-1})(w)J_{\psi^{-1}}(w)$$

for $f \in A^2(\Omega_1)$.

For $k_z^{(\Omega_1)}(\cdot) = \frac{K^{(\Omega_1)}(\cdot,z)}{\sqrt{K^{(\Omega_1)}(z,z)}}$ the normalized kernel function at z in $A^2(\Omega_1)$, by Proposition 1.4.12 of [13] we have

$$(U_\psi^* k_z^{(\Omega_1)})(\cdot) \quad = \quad \overline{J_\psi(z)}|J_\psi(z)|^{-1} k_{\psi(z)}^{(\Omega_2)}(\cdot).$$

So for the rank-one operators $A^{(\Omega_1)}(z) = k_z^{(\Omega_1)} \otimes k_z^{(\Omega_1)}$ and $A^{(\Omega_2)}(w) = k_w^{(\Omega_2)} \otimes k_w^{(\Omega_2)}$,

$$U_\psi^*[A^{(\Omega_1)}(z)]U_\psi = \left\{U_\psi^* k_z^{(\Omega_1)}\right\} \otimes \left\{U_\psi^* k_z^{(\Omega_1)}\right\} = k_{\psi(z)}^{(\Omega_2)} \otimes k_{\psi(z)}^{(\Omega_2)} = A^{(\Omega_2)}(\psi(z)).$$

For $X \in Op(A^2(\Omega_2))$ and V a holomorphic vector field on Ω_1,

$$\begin{aligned}
\left\{V^m(\widetilde{X} \circ \psi)\right\}(z) &= \left\{V^m[tr(X(A^{(\Omega_2)} \circ \psi(\cdot)))]\right\}(z) \\
&= \left\{tr[XV^m(A^{(\Omega_2)} \circ \psi)(\cdot)]\right\}(z) \\
&= \left\{tr[XV^m(U_\psi^* A^{(\Omega_1)}(\cdot)U_\psi)]\right\}(z) \\
&= \left\{tr[XU_\psi^*(V^m A^{(\Omega_1)})(\cdot)U_\psi]\right\}(z). \qquad (41)
\end{aligned}$$

On the other hand, we have

(42)
$$\left\{V^m(\widetilde{X} \circ \psi)\right\}(z) = \left\{[\psi_*(V)]^m \widetilde{X}\right\}(\psi(z)) = \left\{tr[X((\psi_*(V))^m A^{(\Omega_2)})(\cdot)]\right\}(\psi(z)).$$

Therefore, (41) and (42) together imply that

$$tr\left\{X[U_\psi^*(V^m A^{(\Omega_1)})(z)U_\psi - ((\psi_*(V))^m A^{(\Omega_2)})(\psi(z))]\right\} = 0$$

for any $X \in Op(A^2(\Omega_2))$. It follows from the standard duality relation between $Op(A^2(\Omega_2))$ and the trace-class operators on $A^2(\Omega_2)$ that

$$U_\psi^*(V^m A^{(\Omega_1)})(z)U_\psi = ((\psi_*(V))^m A^{(\Omega_2)})(\psi(z)).$$

Recalling the definition (40) and the unitary invariance of the trace-class norm, we have

$$\begin{aligned}
(L_m V)^{(\Omega_1)}(z) = \left\|(V^m A^{(\Omega_1)})(z)\right\|_{tr} &= \left\|((\psi_*(V))^m A^{(\Omega_2)})(\psi(z))\right\|_{tr} \\
&= (L_m \psi_*(V))^{(\Omega_2)}(\psi(z)),
\end{aligned}$$

which justifies our claim. □

In particular, for $V = \partial_v$ on Ω_1 with $v \in \mathbb{C}^n \backslash \{0\}$,

$$(L_m \partial_v)^{(\Omega_1)}(z) = (L_m \psi_*(v))^{(\Omega_2)}(\psi(z)).$$

Obviously, $(L_1\partial_v)(z)$ coincides with the classical Bergman metric $(B_1\partial_v)(z) = B_1(z,v)$ in view of Proposition 2.2, and Corollary 3.3 implies that on any bounded domain,

$$(B_m\partial_v)(z) \leq (L_m\partial_v)(z). \qquad (43)$$

Furthermore, we can give an explicit formula for $(L_m\partial_v)(z)$ by observing that $(\partial_v^m A)(z)$ is an operator of rank one. Actually, we can even prove more:

Proposition 4.2. *For $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{Z}_+^k$ with $|\alpha| = m$, $v = (v_1, \cdots, v_k) \in \mathbb{C}^{nk}$, where $v_i \in \mathbb{C}^n \backslash \{0\}$ for $i = 1, \cdots, k$ and $\partial_v^\alpha = \partial_{v_k}^{\alpha_k} \cdots \partial_{v_1}^{\alpha_1}$,*

(44)

$$\begin{aligned}
&\|(\partial_v^\alpha A)(z)\|_{tr}^2 \\
&= K(z,z) \sum_{\beta,\gamma=0}^{\alpha} \binom{\alpha}{\beta}\binom{\alpha}{\gamma} \{(\bar\partial_v^{\alpha-\beta} K^{-1})(z,z)\}\{(\partial_v^{\alpha-\gamma} K^{-1})(z,z)\}\{(\partial_v^\gamma \bar\partial_v^\beta K)(z,z)\}.
\end{aligned}$$

In particular for $k = 1$,

$$\begin{aligned}
&\{(L_m\partial_v)(z)\}^2 \\
&= K(z,z) \sum_{i,j=0}^{m} \binom{m}{i}\binom{m}{j} \{(\bar\partial_v^{m-i} K^{-1})(z,z)\}\{(\partial_v^{m-j} K^{-1})(z,z)\}\{(\partial_v^j \bar\partial_v^i K)(z,z)\}.
\end{aligned}$$

Proof. The proof is essentially the same as the proof of Proposition 2.2 by noticing the facts that

$$(\partial_v^\alpha A)(z) = K(\cdot, z) \otimes \Big\{ \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} (\bar\partial_v^{\alpha-\beta} K^{-1})(z,z)(\bar\partial_v^\beta K)(\cdot, z) \Big\}$$

and $\langle (\bar\partial_v^\beta K)(\cdot, z), (\bar\partial_v^\gamma K)(\cdot, z)\rangle = (\partial_v^\gamma \bar\partial_v^\beta K)(z,z)$. □

4.1. $(L_m\partial_v)(z)$ **on the unit ball.** For $w = \psi_a(z)$ as in (28),

$$(L_m\partial_v)(z) = (L_m(\psi_a)_*(v))(\psi_a(z)) = \|(\partial_g^m A)(w)\|_{tr}.$$

Since $(L_m\partial_v)(z) = \|(\partial_v^m A)(z)\|_{tr}$ is smooth as a function of z with respect to the trace norm $\|\cdot\|_{tr}$, then by Lemma 3.12 with $\mathscr{X}$ = the space of trace-class operators on $A^2(\mathbb{B}^n)$,

$$(L_m\partial_v)(a) = \|(\partial_g^m A)(0)\|_{tr} = \|\sum_{i=1}^{m} C_i^{(m)} h(a,v)^{m-i} (\partial_{g(0)}^i A)(w)|_{w=0}\|_{tr}.$$

Recalling that $A(w) = K(w,w)^{-1} K(\cdot, w) \otimes K(\cdot, w)$ and

$$\{\partial_{g(0)}^{i-j} K(w,w)^{-1}\}|_{w=0} = 0, \quad i > j,$$

we have

$$\begin{aligned}
(\partial_{g(0)}^i A)(0) &= \sum_{j=0}^{i} \binom{i}{j} \{\partial_{g(0)}^{i-j} K(w,w)^{-1}\}|_{w=0} \{\partial_{g(0)}^j [K(\cdot, w) \otimes K(\cdot, w)]\}|_{w=0} \\
&= \sum_{j=0}^{i} \binom{i}{j} \{\partial_{g(0)}^{i-j} K(w,w)^{-1}\}|_{w=0} \{K(\cdot, w) \otimes \bar\partial_{g(0)}^j K(\cdot, w)\}|_{w=0} \\
&= K(\cdot, 0) \otimes \{\bar\partial_{g(0)}^i K(\cdot, w)\}|_{w=0}.
\end{aligned}$$

Therefore, $(\partial_g^m A)(0)$ is an operator of rank one and by (30),

$$\begin{aligned}\{(L_m\partial_v)(a)\}^2 &= \|K(\cdot,0)\|^2 \times \|\{\sum_{i=1}^m \bar{C}_i^{(m)}\bar{h}(a,v)^{m-i}\bar{\partial}^i_{g(0)}K(\cdot,w)\}|_{w=0}\|^2 \\ &= \sum_{i,j=1}^m \bar{C}_i^{(m)}C_j^{(m)}\bar{h}(a,v)^{m-i}h(a,v)^{m-j}(\partial^j_{g(0)}\bar{\partial}^i_{g(0)}K)(w,w)|_{w=0} \\ &= \sum_{i=1}^m |C_i^{(m)}|^2|h(a,v)|^{2(m-i)}(\partial^i_{g(0)}\bar{\partial}^i_{g(0)}K)(0,0).\end{aligned}$$

Then, by (30) again, Lemma 3.7 and Theorem 3.14, we have

$$\begin{aligned}\{(L_m\partial_v)(a)\}^2 &\geq |C_m^{(m)}|^2\frac{m!(m+n)!}{n!}|g(0)|^{2m} = \frac{m!(m+n)!}{n!}\left\{\frac{[(B_1\partial_v)(a)]^2}{n+1}\right\}^m \\ &= \frac{m!(m+n)!}{n!(n+1)^m}\{(B_1\partial_v)(a)\}^{2m} = \{(B_m\partial_v)(a)\}^2.\end{aligned}$$

So (43) is recovered here on the unit ball by an alternative argument.

On the other hand, by Lemma 3.7 again, we have

$$|h(a,v)|^2 = \frac{|\langle v,a\rangle|^2}{(1-|a|^2)^2} \leq \frac{(1-|a|^2)|v|^2+|\langle a,v\rangle|^2}{(1-|a|^2)^2} = |g(0)|^2. \tag{45}$$

Then by Theorem 3.14,

$$\begin{aligned}&\{(L_m\partial_v)(a)\}^2 \leq |g(0)|^{2m}\sum_{i=1}^m |C_i^{(m)}|^2\frac{i!(i+n)!}{n!} \\ = \ &\frac{\{(B_1\partial_v)(a)\}^{2m}}{(n+1)^m}\sum_{i=1}^m |C_i^{(m)}|^2\frac{i!(i+n)!}{n!} \leq \{(B_m\partial_v)(a)\}^2\sum_{i=1}^m |C_i^{(m)}|^2.\end{aligned}$$

Thus the preceding argument can be summarized as

Theorem 4.3. *On the unit ball $\mathbb{B}^n$, there is a constant $C'(n,m)$ such that*

$$(B_m\partial_v)(a) \leq (L_m\partial_v)(a) \leq C'(n,m)(B_m\partial_v)(a).$$

4.2. **Bloch type estimates on the unit ball.** For $k=n$ and for $v_i=(0,\cdots,1,\cdots,0)$ the standard i-th basis vector for $i=1,\cdots,n$ in (44), with direct calculations we can have

$$\|(\partial^\alpha A)(z)\|_{tr}^2 \leq C(\alpha,n)(1-|z|^2)^{-2|\alpha|}$$

for some constant $C(\alpha,n)$. Then by (4), for $X \in Op(A^2(\mathbb{B}^n))$ we obtain

$$(1-|z|^2)^{|\alpha|}|(\partial^\alpha\widetilde{X})(z)| \leq C(\alpha,n)\|X\|.$$

These estimates are consistent with the higher order derivatives characterization of the Bloch space on the unit ball $\mathbb{B}^n$ (Theorem 3.5 in [19]) since $H^\infty(\mathbb{B}^n)$ is the intersection of the Bloch space and the range of the Berezin transform.

4.3. $(L_m\partial_v)(z)$ **on** $\mathbb{C}^n$**.** For the Weyl operators W_a defined in (39), it is easy to check that

$$W_a A(z) W_a^* = A(z+a)$$

(or see [9]). Similar to the proof of Proposition 4.1, we have

$$(L_m\partial_v)(a) = (L_m\partial_v)(0).$$

By the rotation-invariance of the kernel function $K(z,w) = e^{\frac{\langle z,w\rangle}{2}}$, the formula in Proposition 4.2 yields that

$$\{(L_m\partial_v)(a)\}^2 = \{(L_m\partial_v)(0)\}^2 = (\partial_v^m \bar{\partial}_v^m K)(0,0) = 2^{-m} m! |v|^{2m}.$$

So in this case,

$$(B_m\partial_v)(a) = (L_m\partial_v)(a).$$

4.4. **Equivalence relations on the polyball.** As promised in Section 3.4, we shall prove in this section that $(B_m\partial_v)(a)$ and $\{(B_1\partial_v)(a)\}^m$ are equivalent on the polydisk. It turns out that our argument even works on the polyball $\Omega = \prod_{j=1}^l \mathbb{B}^{n_j}$ for $l \geq 2$, including the case of the polydisk. Moreover, the new metric $(L_m\partial_v)(a)$ plays an important role in our proof and can be shown to also be equivalent to the preceding two quantities, inspired by Theorem 3.14 and Theorem 4.3. However, just for the notational simplicity, we limit our argument to the case $l = 2$ even though the results and proofs remain unchanged for the general case $l \geq 2$.

We believe that the next lemma is well known in the study of intrinsic metrics and the proof remains valid for any two biholomorphically invariant (positive) metrics of order 1. For our purpose, we only state it for the Bergman metric and the Carathéodory metric.

Lemma 4.4. *For Ω any bounded homogeneous domain, $(B_1\partial_v)(a)$ and $(C_1\partial_v)(a)$ are equivalent.*

Proof. Without loss of generality we assume $0 \in \Omega$. For any point $a \in \Omega$, there exists $\psi_a \in \mathrm{Aut}(\Omega)$ such that $\psi_a(a) = 0$ by the homogeneity of Ω.

For the nonzero vector field ∂_v, we know that $(B_1\partial_v)(a)$ and $(C_1\partial_v)(a)$ are strictly positive since Ω is bounded. Then we define the positive function

$$F(a,v) = \frac{(B_1\partial_v)(a)}{(C_1\partial_v)(a)}.$$

It is easy to see that $F(a,\lambda v) = F(a,v)$ for any $\lambda \in \mathbb{C}\backslash\{0\}$. It follows from the biholomorphic invariance of $(B_1\partial_v)(a)$ and $(C_1\partial_v)(a)$ that

$$F(a,v) = F(0,\psi_a'(a)v) = F\left(0, \frac{\psi_a'(a)v}{\|\psi_a'(a)v\|}\right),$$

where the last equality uses the fact that the matrix $\psi_a'(a)$ is nonsingular. So we only need to consider the function $F(0,u)$ with $u \in S$ the unit sphere of $\mathbb{C}^n$. As a positive continuous function defined on the compact set S , we know that $F(0,u)$ attains its minimum and maximum, which implies that $(B_1\partial_v)(a)$ and $(C_1\partial_v)(a)$ are equivalent on Ω. □

Let us also recall some facts about the Bergman geometry on product domains. For example, the kernel function $K(z,w)$ on the product domain is the product of the kernel function of each factor domain, and the square of the Bergman metric is

the sum of the squares of the Bergman metrics on each factor domain ([12]). Now we are ready to prove the main result.

Theorem 4.5. *For the polyball* $\Omega = \mathbb{B}^{n_1} \times \mathbb{B}^{n_2} (n_1 \leq n_2)$, $a \in \Omega$ *and* $v \in \mathbb{C}^{n_1+n_2} \backslash \{0\}$,

(46)
$$C_1(\Omega, m)\{(B_1\partial_v)(a)\}^m \leq (B_m\partial_v)(a) \leq (L_m\partial_v)(a) \leq C_2(\Omega, m)\{(B_1\partial_v)(a)\}^m$$

for some constants $C_1(\Omega, m)$ *and* $C_2(\Omega, m)$.

Proof. In view of (43), it suffices to prove the leftmost and the rightmost inequalities. Actually, Lemma 4.4 and Burbea's inequalities (33) imply that the leftmost inequality holds on any bounded homogeneous domain. Thus we only need to prove the rightmost inequality on the polyball.

For a mapping f defined on Ω, let $(e_0 f)(w_1, w_2) = f(0, 0)$, $(e_1 f)(w_1, w_2) = f(0, w_2)$ and $(e_2 f)(w_1, w_2) = f(w_1, 0)$. Formally, we can write $e_0 = e_1 e_2 = e_2 e_1$. For $w = (w_1, w_2) = \psi_a(z) = (\psi_{a_1}(z_1), \psi_{a_2}(z_2))$ with $w_j = \psi_{a_j}(z_j) \in \mathrm{Aut}(\mathbb{B}^{n_j})$ of the form (28) for $j = 1, 2$, we know that $\psi_a \in \mathrm{Aut}(\Omega)$ and $\psi_a(a) = 0$. For $v = (v_1, v_2) \in \mathbb{C}^{n_1+n_2} \backslash \{0\}$, write $g(w) = (\psi_a)_* v = \psi_a'(z)v = (g_1(w_1), g_2(w_2)) = (\psi_{a_1}'(z_1)v_1, \psi_{a_2}'(z_2)v_2)$. Note that $\partial_{g_1}\partial_{g_2} = \partial_{g_2}\partial_{g_1}$ and $e_i\partial_{g_j} = \partial_{g_j} e_i$ for $i \neq j$,

$$(\partial_g^m A)(0) = e_0[(\partial_{g_1} + \partial_{g_2})^m A](w) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left(\prod_{j=1}^{2} e_j \partial_{g_j}^{\alpha_j} \right) A(w).$$

Applying Lemma 3.12 repeatedly to $A(w) = k_w \otimes k_w$, a trace-class-operator-valued mapping with the codomain $\mathscr{X}$ = the space of trace-class operators on $A^2(\Omega)$, but with the domain varying from $\mathbb{B}^{n_1}$ to $\mathbb{B}^{n_2}$ correspondingly, we have

$$(\partial_g^m A)(0) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \sum_{\beta_1=0}^{\alpha_1} \sum_{\beta_2=0}^{\alpha_2} \left\{ \prod_{j=1}^{2} C_{\beta_j}^{(\alpha_j)} h(a_j, v_j)^{\alpha_j - \beta_j} \right\} \left\{ \partial_{g_1(0)}^{\beta_1} \partial_{g_2(0)}^{\beta_2} A \right\} (0).$$

Since the kernel function satisfies $K(z, w) = K_1(z_1, w_1) K_2(z_2, w_2)$, it is easy to see that

$$\left(\partial_{g_1(0)}^{\beta_1} \partial_{g_2(0)}^{\beta_2} K^{-1} \right)(0, 0) = \left(\partial_{g_1(0)}^{\beta_1} K_1^{-1} \right)(0, 0) \left(\partial_{g_2(0)}^{\beta_2} K_2^{-1} \right)(0, 0) = 0$$

for $(\beta_1, \beta_2) \neq (0, 0)$. Then by the Leibnitz rule,

$$\begin{aligned}
&(\partial_g^m A)(0) \\
&= \sum_{|\alpha|=m} \frac{m!}{\alpha!} \sum_{\beta=0}^{\alpha} \left\{ \prod_{j=1}^{2} C_{\beta_j}^{(\alpha_j)} h(a_j, v_j)^{\alpha_j - \beta_j} \right\} \left\{ K(\cdot, w) \otimes \bar{\partial}_{g_1(0)}^{\beta_1} \bar{\partial}_{g_2(0)}^{\beta_2} K(\cdot, w) \right\} \Big|_{w=0} \\
&= K(\cdot, 0) \otimes \sum_{|\alpha|=m} \frac{m!}{\alpha!} \sum_{\beta=0}^{\alpha} \left\{ \prod_{j=1}^{2} \bar{C}_{\beta_j}^{(\alpha_j)} \bar{h}(a_j, v_j)^{\alpha_j - \beta_j} \right\} \left\{ \bar{\partial}_{g_1(0)}^{\beta_1} \bar{\partial}_{g_2(0)}^{\beta_2} K(\cdot, w) \right\} \Big|_{w=0}.
\end{aligned}$$

Since $\|K(\cdot,0)\|=1$, we have

$$\begin{aligned}
&\|(\partial_g^m A)(0)\|_{tr}\\
&=\left\|\sum_{|\alpha|=m}\frac{m!}{\alpha!}\sum_{\beta=0}^{\alpha}\left\{\prod_{j=1}^{2}\bar{C}_{\beta_j}^{(\alpha_j)}\bar{h}(a_j,v_j)^{\alpha_j-\beta_j}\right\}\left\{\bar{\partial}_{g_1(0)}^{\beta_1}\bar{\partial}_{g_2(0)}^{\beta_2}K(\cdot,w)\right\}\big|_{w=0}\right\|\\
&\le\sum_{|\alpha|=m}\frac{m!}{\alpha!}\sum_{\beta=0}^{\alpha}\left\{\prod_{j=1}^{2}\left|\bar{C}_{\beta_j}^{(\alpha_j)}\bar{h}(a_j,v_j)^{\alpha_j-\beta_j}\right|\right\}\left\|\left\{\bar{\partial}_{g_1(0)}^{\beta_1}\bar{\partial}_{g_2(0)}^{\beta_2}K(\cdot,w)\right\}\big|_{w=0}\right\|\\
&=\sum_{|\alpha|=m}\frac{m!}{\alpha!}\sum_{\beta=0}^{\alpha}\left\{\prod_{j=1}^{2}\left|\bar{C}_{\beta_j}^{(\alpha_j)}\bar{h}(a_j,v_j)^{\alpha_j-\beta_j}\right|\right\}\left\{\prod_{j=1}^{2}(\partial_{g_j(0)}^{\beta_j}\bar{\partial}_{g_j(0)}^{\beta_j}K_j)(0,0)\right\}^{\frac{1}{2}}.
\end{aligned}$$

Therefore, by (30) and (45), we have

$$\begin{aligned}
\|(\partial_g^m A)(0)\|_{tr}&\le\sum_{|\alpha|=m}\frac{m!}{\alpha!}\sum_{\beta=0}^{\alpha}\left\{\prod_{j=1}^{2}\left|C_{\beta_j}^{(\alpha_j)}\right||g_j(0)|^{\alpha_j-\beta_j}\right\}\\
&\quad\times\left\{\prod_{j=1}^{2}\frac{\beta_j!(\beta_j+n_j)!}{n_j!}|g_j(0)|^{2\beta_j}\right\}^{\frac{1}{2}}\\
&=\sum_{|\alpha|=m}\frac{m!}{\alpha!}\left\{\sum_{\beta=0}^{\alpha}\prod_{j=1}^{2}\left|C_{\beta_j}^{(\alpha_j)}\right|\left[\frac{\beta_j!(\beta_j+n_j)!}{n_j!}\right]^{\frac{1}{2}}\right\}\left\{\prod_{j=1}^{2}|g_j(0)|^{\alpha_j}\right\}.
\end{aligned}$$

Let

$$C(\Omega,m)=\max\left\{\sum_{\beta=0}^{\alpha}\prod_{j=1}^{2}\left|C_{\beta_j}^{(\alpha_j)}\right|\left[\frac{\beta_j!(\beta_j+n_j)!}{n_j!}\right]^{\frac{1}{2}}:|\alpha|=m\right\}.$$

Then

$$\begin{aligned}
\|(\partial_g^m A)(0)\|_{tr}&\le C(\Omega,m)\sum_{|\alpha|=m}\frac{m!}{\alpha!}\prod_{j=1}^{2}|g_j(0)|^{\alpha_j}\\
&=C(\Omega,m)\left\{\sum_{j=1}^{2}|g_j(0)|\right\}^{m}\\
&\le C(\Omega,m)\left\{2\sum_{j=1}^{2}|g_j(0)|^2\right\}^{\frac{m}{2}}\\
&\le C(\Omega,m)2^{\frac{m}{2}}\{(B_1\partial_v)(a)\}^m,
\end{aligned}$$

since $|g_j(0)|^2\le\{(B_1\partial_{v_j})(a_j)\}^2$ by Lemma 3.7. Thus,

$$(L_m\partial_v)(a)=\|(\partial_v^m A)(a)\|_{tr}=\left\|(\partial_g^m A)(0)\right\|_{tr}\le C_2(\Omega,m)\{(B_1\partial_v)(a)\}^m,$$

where $C_2(\Omega,m)=C(\Omega,m)2^{\frac{m}{2}}$. So we complete the proof of the theorem. □

Similar to the unit ball case, we can answer Burbea's conjecture on the polyball.

Corollary 4.6. *For the polyball* $\Omega = \mathbb{B}^{n_1} \times \mathbb{B}^{n_2} (n_1 \leq n_2)$,

$$(C_m \partial_v)(a) < (B_m \partial_v)(a) \leq C_3(\Omega, m)(C_m \partial_v)(a)$$

for some constant $C_3(\Omega, m)$.

Proof. It follows from Theorem 4.5, Lemma 4.4 and (33). □

As another application of Theorem 4.5 to Berezin's operator calculus, we also have the Bloch type estimates on the polyball based on (4):

Corollary 4.7. *For* $X \in Op(A^2(\Omega))$, $a \in \Omega = \mathbb{B}^{n_1} \times \mathbb{B}^{n_2} (n_1 \leq n_2)$ *and* $v \in \mathbb{C}^{n_1+n_2} \backslash \{0\}$,

$$\{(B_1 \partial_v)(a)\}^{-m} |(\partial_v^m \widetilde{X})(a)| \leq C_4(\Omega, m) \|X\|,$$

for some constant $C_4(\Omega, m)$.

We conclude by posing the conjecture that Theorem 4.5 holds on any bounded homogeneous domain in $\mathbb{C}^n$, which implies Burbea's conjecture clearly. In fact, based on the proof of Theorem 4.5, only the rightmost inequality in (46) needs to be verified in this general case.

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