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ON CERTAIN VANISHING IDENTITIES FOR GROMOV-WITTEN INVARIANTS

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ABSTRACT. In this paper we study certain vanishing identities for Gromov-Witten invariants conjectured by K. Liu and H. Xu. We will prove their conjectures when the summation range is big compared to the genus. In such cases, we actually obtained vanishing identities which are stronger than their conjectures. We also prove these conjectures in low genus cases.

Let V be a compact symplectic manifold and $\{\gamma_\alpha \mid \alpha = 1, \dots, N\}$ be a basis for $H^*(V;\mathbb{C})$. We choose γ_1 to be the identity of the cohomology ring of V. Let $\gamma^\alpha = \eta^{\alpha\beta}\gamma_\beta$ with $(\eta^{\alpha\beta})$ representing the inverse matrix of the Poincaré intersection pairing. As a convention, repeated Greek letter indices are summed over their entire ranges from 1 to N. Recall that the *big phase space* for Gromov-Witten invariants is $\prod_{n=0}^{\infty} H^*(V;\mathbb{C})$ with standard basis $\{\tau_n(\gamma_\alpha) \mid \alpha = 1, \dots, N, \quad n \geq 0\}$. Let t_n^α be the coordinates on the big phase space with respect to the standard basis. The genus-g generating function F_g is a formal power series of $t = (t_n^\alpha)$ with coefficients given by genus-g Gromov-Witten invariants. Derivatives of F_g with respect to $t_{n_1}^{\alpha_1}, \dots, t_{n_k}^{\alpha_k}$ are denoted by $\langle\!\langle \tau_{n_1}(\alpha_1) \cdots \tau_{n_k}(\alpha_k) \rangle\!\rangle_g$.

The following conjecture was proposed by K. Liu and H. Xu in [7]:

Conjecture A. *For $m > 2g$, genus-g Gromov-Witten invariants of V satisfy the following identity:*

$$\sum_{j=0}^{m} (-1)^j \langle\!\langle \tau_j(\gamma_\alpha)\ \tau_{m-j}(\gamma^\alpha) \rangle\!\rangle_g = 0.$$

Note that when m is odd, this conjecture is trivial due to the symmetry of the indices. One can see this by simply replacing j by $m - j$ on the left hand side. So we only need to consider the case when m is even.

In this paper, we prove the following

Theorem 0.1. *Conjecture A is true if $m \geq 3g + \delta_{g,0}$ for all genera.*

In particular, this theorem implies that Conjecture A is true for $g \leq 2$.

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Another conjecture proposed in [7] is the following:

Conjecture B. *For $k \geq g$,*

$$\begin{aligned} & -\sum_{n,\alpha} \tilde{t}_n^\alpha \langle\!\langle \tau_{n+2k-1}(\gamma_\alpha) \rangle\!\rangle_g + \frac{1}{2}\sum_{h=0}^{g}\sum_{j=0}^{2k-2}(-1)^j \langle\!\langle \tau_j(\gamma_\alpha) \rangle\!\rangle_h \langle\!\langle \tau_{2k-2-j}(\gamma^\alpha) \rangle\!\rangle_{g-h} \\ = \quad & \frac{1}{2}\delta_{g,0}\delta_{k,0}\eta_{\alpha\beta}t_0^\alpha t_0^\beta, \end{aligned}$$

where $\tilde{t}_n^\alpha = t_n^\alpha - \delta_{\alpha,1}\delta_{n,1}$.

In this paper, we also prove the following

Theorem 0.2. *(a) For $m = 2k > 2g$, Conjecture A for genus $g-1$ is equivalent to conjecture B for genus g.*

(b) Conjecture B is true if $2k \geq 3g-1$ for all genera.

(c) Conjecture B is true for genus $g \leq 3$.

Part (a) of this theorem follows from a result of Faber and Pandharipande [3], which will be discussed in more detail in Section 1.

The following conjecture, which is more general than Conjecture B, was also proposed in [7]:

Conjecture C. *For all $x_i, y_i \in H^*(V;\mathbb{C})$ and $m \geq 2g-3+a+b$,*

$$\sum_{j\in\mathbb{Z}}\sum_{h=0}^{g}(-1)^j \left\langle\!\!\left\langle \tau_j(\gamma_\alpha)\prod_{i=1}^{a}\tau_{p_i}(x_i) \right\rangle\!\!\right\rangle_h \left\langle\!\!\left\langle \tau_{m-j}(\gamma^\alpha)\prod_{i=1}^{b}\tau_{q_i}(y_i) \right\rangle\!\!\right\rangle_{g-h} = 0.$$

Here j is allowed to be an arbitrary integer. To interpret this conjecture correctly, one has to use the convention that $\langle \tau_{-2}(\gamma_1) \rangle_{0,0} = 1$ and

$$\langle \tau_m(\gamma_\alpha)\tau_{-1-m}(\gamma_\beta) \rangle_{0,0} = (-1)^{\max(m,-1-m)}\eta_{\alpha\beta}, \qquad m \in \mathbb{Z}.$$

Conjecture B is the special case of Conjecture C for $a = b = 0$. Note that we will not use this convention in the rest of this paper. We will reformulate these conjectures in Theorem 3.3 in order to see their relations with topological recursion relations. We will also prove the following vanishing identity for all genera:

Theorem 0.3. *Assume $g \geq h \geq 0$, $a, b \geq 0$, $m \geq 3g-3+a+b$. Moreover we also assume that $a \geq 2$ if $h = 0$ and $b \geq 2$ if $h = g$. Then for all $p_i, q_j \geq 0$ and $x_i, y_j \in H^*(V;\mathbb{C})$, we have*

$$\sum_{j=0}^{m}(-1)^j \left\langle\!\!\left\langle \tau_j(\gamma_\alpha)\prod_{i=1}^{a}\tau_{p_i}(x_i) \right\rangle\!\!\right\rangle_h \left\langle\!\!\left\langle \tau_{m-j}(\gamma^\alpha)\prod_{i=1}^{b}\tau_{q_i}(y_i) \right\rangle\!\!\right\rangle_{g-h} = 0. \tag{0.1}$$

In the case that $h = 0$, $g \geq 1$ and $a = 1$, we also have the following identity:

$$\begin{aligned} & \sum_{j=0}^{m}(-1)^j \langle\!\langle \tau_j(\gamma_\alpha)\tau_p(x) \rangle\!\rangle_0 \left\langle\!\!\left\langle \tau_{m-j}(\gamma^\alpha)\prod_{i=1}^{b}\tau_{q_i}(y_i) \right\rangle\!\!\right\rangle_g \\ = \quad & \left\langle\!\!\left\langle \tau_{p+m+1}(x)\prod_{i=1}^{b}\tau_{q_i}(y_i) \right\rangle\!\!\right\rangle_g \end{aligned} \tag{0.2}$$

for $m \geq 3g-2+b$.

Note that the vanishing identity in this theorem is stronger than the corresponding cases for Conjecture C since there is no summation over the genus. The remaining cases of Conjecture C will be proved in a later paper with R. Pandharipande [11].

This paper is organized as follows: In Section 1*, we will discuss a result in* [3] *which implies part (a) of Theorem* 0.2*. In Section* 2*, we will discuss some consequences and generalizations of low genus topological recursion relations. The proofs of the above theorems will be presented in Section* 3*. Our proofs only use topological recursion relations. This indicates that all these conjectures should be some kind of combination of topological recursion relations for all genera.*

The author would like to thank H. Xu for presenting more detailed formulation on Conjecture C.

1. Equivalence of Conjectures A and B

In [3], Faber and Pandharipande considered following Hodge integrals over moduli stacks of maps to V: For $\gamma_{\alpha_i} \in H^*(V;\mathbb{C})$, non-negative integers k_i and b_j,

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_{\alpha_i}) \prod_{j=1}^{m} ch_{b_j}(\mathbb{E}) \right\rangle_{g,A}, := \int_{[\overline{\mathcal{M}}_{g,n}(V,A)]^{vir}} \prod_{i=1}^{n} \psi_i^{k_i} \cup ev_i^*(\gamma_{\alpha_i}) \cup \prod_{j=1}^{m} ch_{b_j}(\mathbb{E}),$$

where $[\overline{\mathcal{M}}_{g,n}(V,A)]^{vir}$ is the virtual fundamental cycle in the moduli space of stable maps from prestable curves of genus-g with n-marked points into V with degree $A \in H^2(V;\mathbb{Z})$ (cf. [6] and [1]), ψ_i is the first Chern class of the tautological line bundle over $\overline{\mathcal{M}}_{g,n}(V,A)$ defined by cotangent lines at the i-th marked points on the domain curves, ev_i is the evaluation map from the moduli space to V defined by the image of the i-th marked point under stable maps, $\mathbb{E}$ is the Hodge bundle over the moduli spaces, and $ch(\mathbb{E})$ is the Chern character of $\mathbb{E}$. If classes $ch_{b_j}(\mathbb{E})$ are omitted in the above expression, this is precisely the descendant Gromov-Witten invariants. Let $F_{g,\mathbb{E}}(t,s)$ be the generating function for the above Hodge integrals, where $t = (t_n^\alpha)$ is the variable for the usual Gromov-Witten invariants and $s = (s_n)$ is the variable for $ch_n(\mathbb{E})$. More precisely

$$F_{g,\mathbb{E}} := \sum_{A\in H^2(V;\mathbb{Z})} q^A \sum_{m,n\geq 0} \frac{1}{m!\,n!} \sum_{\substack{\alpha_1,\cdots,\alpha_n \\ k_1,\cdots,k_n \\ b_1,\cdots,b_m}} \prod_{i=1}^{n} t_{k_i}^{\alpha_i} \prod_{j=1}^{m} s_{b_j} \cdot \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_{\alpha_i}) \prod_{j=1}^{m} ch_{b_j}(\mathbb{E}) \right\rangle_{g,A}.$$

In particular $F_g := F_{g,\mathbb{E}}\,|_{s=0}$ is the usual generating function of genus-g Gromov-Witten invariants. Moreover, define

$$Z_{\mathbb{E}} := \exp\{\sum_{g\geq 0} \hbar^{g-1} F_{g,\mathbb{E}}\},$$

where $\hbar$ is a parameter used to separate information from different genera.

For convenience, we identify $\tau_n(\gamma_\alpha)$ with $\frac{\partial}{\partial t_n^\alpha}$ and $ch_n(\mathbb{E})$ with $\frac{\partial}{\partial s_n}$. Moreover we use the convention that $\tau_0(\gamma_\alpha)=\gamma_\alpha$ and $\tau_n(\gamma_\alpha)=0$ if $n<0$. Define

$$(1.1)\quad D_{2l-1} := -ch_{2l-1}(\mathbb{E}) - \frac{B_{2l}}{(2l)!}\left\{\sum_{n,\alpha}\tilde{t}_n^\alpha\ \tau_{n+2l-1}(\gamma_\alpha) - \frac{\hbar}{2}\sum_{i=0}^{2l-2}(-1)^i\,\tau_i(\gamma_\alpha)\,\tau_{2l-2-i}(\gamma^\alpha)\right\},$$

where B_{2l} are the Bernoulli numbers defined by

$$\frac{x}{e^x-1}=\sum_{r=0}^{\infty}\frac{B_r}{r!}x^r.$$

The following formula was proved in [3]:

$$(1.2)\qquad D_{2l-1}Z_{\mathbb{E}}=0$$

for $l\geq 1$. Taking the coefficients of $\hbar^{g-1}$ in $Z_{\mathbb{E}}^{-1}D_{2l-1}Z_{\mathbb{E}}$ we have

$$\begin{aligned}0 &= -\frac{(2l)!}{B_{2l}}\langle\!\langle\, ch_{2l-1}(\mathbb{E})\,\rangle\!\rangle_{g,\mathbb{E}} - \sum_{n,\alpha}\tilde{t}_n^\alpha\,\langle\!\langle\,\tau_{n+2l-1}(\gamma_\alpha)\,\rangle\!\rangle_{g,\mathbb{E}}\\ &\quad+\frac{1}{2}\sum_{i=0}^{2l-2}(-1)^i\left\{\langle\!\langle\,\tau_i(\gamma_\alpha)\tau_{2l-2-i}(\gamma^\alpha)\,\rangle\!\rangle_{g-1,\mathbb{E}}\right.\\ &\qquad\left.+\sum_{h=0}^{g}\langle\!\langle\,\tau_i(\gamma_\alpha)\,\rangle\!\rangle_{h,\mathbb{E}}\,\langle\!\langle\,\tau_{2l-2-i}(\gamma^\alpha)\,\rangle\!\rangle_{g-h,\mathbb{E}}\right\}\end{aligned}$$

for $g\geq 0$ and $l\geq 1$. Here we use $\langle\!\langle\cdots\rangle\!\rangle_{g,\mathbb{E}}$ to represent derivatives of $F_{g,\mathbb{E}}$. Moreover we also adopt the convention that $\langle\!\langle\cdots\rangle\!\rangle_{g,\mathbb{E}}=0$ if $g<0$. This is needed when we consider the genus-0 case of the above formula. Define

$$P_{l,\mathbb{E}} := -\sum_{n,\alpha}\tilde{t}_n^\alpha\tau_{n+2l-1}(\gamma_\alpha)+\sum_{i=0}^{2l-2}(-1)^i\,\langle\!\langle\,\tau_i(\gamma_\alpha)\,\rangle\!\rangle_{0,\mathbb{E}}\,\tau_{2l-2-i}(\gamma^\alpha).$$

Then the above equation can be rewritten as

$$\begin{aligned}(1.3)\qquad &\frac{(2l)!}{B_{2l}}\langle\!\langle\, ch_{2l-1}(\mathbb{E})\,\rangle\!\rangle_{g,\mathbb{E}} - \langle\!\langle\, P_{l,\mathbb{E}}\,\rangle\!\rangle_{g,\mathbb{E}}\\ &= \frac{1}{2}\sum_{i=0}^{2l-2}(-1)^i\left\{\langle\!\langle\,\tau_i(\gamma_\alpha)\tau_{2l-2-i}(\gamma^\alpha)\,\rangle\!\rangle_{g-1,\mathbb{E}}\right.\\ &\qquad+\sum_{h=1}^{g-1}\langle\!\langle\,\tau_i(\gamma_\alpha)\,\rangle\!\rangle_{h,\mathbb{E}}\,\langle\!\langle\,\tau_{2l-2-i}(\gamma^\alpha)\,\rangle\!\rangle_{g-h,\mathbb{E}}\\ &\qquad\left.-\delta_{g,0}\,\langle\!\langle\,\tau_i(\gamma_\alpha)\,\rangle\!\rangle_{0,\mathbb{E}}\,\langle\!\langle\,\tau_{2l-2-i}(\gamma^\alpha)\,\rangle\!\rangle_{0,\mathbb{E}}\right\}\end{aligned}$$

for $g\geq 0$ and $l\geq 1$.

It is also observed in [3] that

$$\langle\!\langle\, ch_{2l-1}(\mathbb{E})\,\rangle\!\rangle_{g,\mathbb{E}} = 0 \tag{1.4}$$

if $l > g$. So in this case, if we set $s = 0$, equation (1.3) implies the following relation for pure Gromov-Witten invariants:

$$\begin{aligned} 0 \;=\; & \langle\!\langle\, P_l\,\rangle\!\rangle_g - \frac{1}{2}\,\delta_{g,0}\sum_{i=0}^{2l-2}(-1)^i\,\langle\!\langle\,\tau_i(\gamma_\alpha)\,\rangle\!\rangle_0\,\langle\!\langle\,\tau_{2l-2-i}(\gamma^\alpha)\,\rangle\!\rangle_0 \\ & +\frac{1}{2}\sum_{i=0}^{2l-2}(-1)^i\Big\{\langle\!\langle\,\tau_i(\gamma_\alpha)\tau_{2l-2-i}(\gamma^\alpha)\,\rangle\!\rangle_{g-1} \\ & \qquad\qquad +\sum_{h=1}^{g-1}\langle\!\langle\,\tau_i(\gamma_\alpha)\,\rangle\!\rangle_h\,\langle\!\langle\,\tau_{2l-2-i}(\gamma^\alpha)\,\rangle\!\rangle_{g-h}\Big\} \end{aligned} \tag{1.5}$$

for $l > g$, where

$$P_l := P_{l,\mathbb{E}}\,|_{s=0} = -\sum_{n,\alpha}\tilde{t}^\alpha_n\tau_{n+2l-1}(\gamma_\alpha) + \sum_{i=0}^{2l-2}(-1)^i\,\langle\!\langle\,\tau_i(\gamma_\alpha)\,\rangle\!\rangle_0\,\tau_{2l-2-i}(\gamma^\alpha). \tag{1.6}$$

For convenience, we extend the notation $\langle\!\langle\cdots\rangle\!\rangle_g$ in such a way that $\langle\!\langle\,\mathcal{W}_1\cdots\mathcal{W}_k\,\rangle\!\rangle_g$ means the covariant derivative of F_g with respect to any vector fields $\mathcal{W}_1,\ldots,\mathcal{W}_k$ on the big phase space. This notation has been used in the above equations. Note that equation (1.5) is just a combination of Conjecture A and Conjecture B. It implies that Conjecture A holds for genus $g \geq 0$ if and only if Conjecture B holds for genus-$(g+1)$. This proves part (a) of Theorem 0.2.

To get more information from equation (1.5), we need a better understanding of the vector field P_l. For this purpose, we will use the following operator, which was studied in [8]:

$$T(\mathcal{W}) := \tau_+(\mathcal{W}) - \langle\!\langle\,\mathcal{W}\,\gamma^\alpha\,\rangle\!\rangle_0\,\gamma_\alpha \tag{1.7}$$

for any vector field $\mathcal{W}$ on the big phase space, where $\tau_+(\mathcal{W})$ is a linear operator defined by $\tau_+(\tau_n(\gamma_\alpha)) := \tau_{n+1}(\gamma_\alpha)$. We will also use $\tau_k(\mathcal{W}) := \tau_+^k(\mathcal{W})$. The operator T is very useful in the study of topological recursion relations. For example, a topological recursion relation discovered by Eguchi-Xiong [2] can be reformulated as

$$\langle\!\langle\, T^{3g-1}(\mathcal{W})\,\rangle\!\rangle_g = 0 \tag{1.8}$$

for any $g > 0$ and any vector field $\mathcal{W}$ (cf. [8]). This equation follows from the simple fact that $\psi_1^{3k-1} = 0$ on the moduli space $\overline{\mathcal{M}}_{g,1}$ due to a dimension count.

Let

$$\mathcal{S} := -\sum_{n,\alpha}\tilde{t}^\alpha_n\tau_{n-1}(\gamma_\alpha)$$

be the *string vector field*. The following lemma follows from [10, Equation (33)]. It also follows from equation (3.3) below and the dilaton equation.

Lemma 1.1. *For $l \geq 1$,*

$$P_l = T^{2l}(\mathcal{S}).$$

Therefore combining equations (1.5) and (1.8), we have

$$\sum_{i=0}^{2l-2}(-1)^i\left\{\langle\!\langle\, \tau_i(\gamma_\alpha)\tau_{2l-2-i}(\gamma^\alpha)\,\rangle\!\rangle_{g-1}+\sum_{h=1}^{g-1}\langle\!\langle\, \tau_i(\gamma_\alpha)\,\rangle\!\rangle_h\,\langle\!\langle\, \tau_{2l-2-i}(\gamma^\alpha)\,\rangle\!\rangle_{g-h}\right\}=0$$

if $g>0$ and $l\geq\max\{g+1,\frac{3g-1}{2}\}$. Replacing l by $k+1$, we have

Lemma 1.2. *For $g>0$ and $k\geq\max\{g,\frac{3g-3}{2}\}$,*

$$\sum_{i=0}^{2k}(-1)^i\left\{\langle\!\langle\, \tau_i(\gamma_\alpha)\tau_{2k-i}(\gamma^\alpha)\,\rangle\!\rangle_{g-1}+\sum_{h=1}^{g-1}\langle\!\langle\, \tau_i(\gamma_\alpha)\,\rangle\!\rangle_h\,\langle\!\langle\, \tau_{2k-i}(\gamma^\alpha)\,\rangle\!\rangle_{g-h}\right\}=0.$$

Note that this lemma is a special case of Theorem 0.1 and Theorem 0.3. Note that when $g=1$, Lemma 1.2 is equivalent to a genus-0 equation

$$\sum_{i=0}^{2k}(-1)^i\,\langle\!\langle\, \tau_i(\gamma_\alpha)\;\tau_{2k-i}(\gamma^\alpha)\,\rangle\!\rangle_0=0 \tag{1.9}$$

for $k>0$. This is precisely the genus-0 case of Conjecture A.

2. Low genus TRR

For simplicity, "topological recursion relations" will be abbreviated as TRR. A discussion of relations among TRR of genus ≤ 2 can be found in [9]. Recall the *genus-0 TRR*:

$$\langle\!\langle\, \tau_1(\mathcal{W})\,\mathcal{U}\,\mathcal{V}\,\rangle\!\rangle_0=\langle\!\langle\, \mathcal{W}\,\gamma^\alpha\,\rangle\!\rangle_0\,\langle\!\langle\, \gamma_\alpha\,\mathcal{U}\,\mathcal{V}\,\rangle\!\rangle_0$$

for any vector fields $\mathcal{W},\mathcal{U}$ and $\mathcal{V}$. As observed by Witten [13], this formula implies the *generalized WDVV equation*:

$$\langle\!\langle\, \mathcal{W}_1\mathcal{W}_2\gamma^\alpha\,\rangle\!\rangle_0\,\langle\!\langle\, \gamma_\alpha\,\mathcal{W}_3\mathcal{W}_4\,\rangle\!\rangle_0=\langle\!\langle\, \mathcal{W}_1\mathcal{W}_3\gamma^\alpha\,\rangle\!\rangle_0\,\langle\!\langle\, \gamma_\alpha\,\mathcal{W}_2\mathcal{W}_4\,\rangle\!\rangle_0\,.$$

We now prove some other useful consequences of the genus-0 TRR. First, we prove the following formula.

Lemma 2.1. *For $m\geq 0$,*

$$\begin{aligned}&\sum_{i=0}^{m}(-1)^i\,\langle\!\langle\, \mathcal{W}\,\tau_i(\gamma^\alpha)\,\rangle\!\rangle_0\,\langle\!\langle\, \tau_{m-i}(\gamma_\alpha)\,\mathcal{V}\,\rangle\!\rangle_0\\ =\;&\langle\!\langle\, \tau_{m+1}(\mathcal{W})\;\mathcal{V}\,\rangle\!\rangle_0+(-1)^m\,\langle\!\langle\, \mathcal{W}\,\tau_{m+1}(\mathcal{V})\,\rangle\!\rangle_0\end{aligned} \tag{2.1}$$

for any vector fields $\mathcal{W}$ and $\mathcal{V}$.

Proof. We first note that when $m=0$ this formula has the following form:

$$\langle\!\langle\, \mathcal{W}\,\gamma^\alpha\,\rangle\!\rangle_0\,\langle\!\langle\, \gamma_\alpha\,\mathcal{V}\,\rangle\!\rangle_0=\langle\!\langle\, \tau_1(\mathcal{W})\;\mathcal{V}\,\rangle\!\rangle_0+\langle\!\langle\, \mathcal{W}\,\tau_1(\mathcal{V})\,\rangle\!\rangle_0\,. \tag{2.2}$$

This is exactly [12, Equation (10)], which was proved using the string equation and the genus-0 TRR. When $m=1$, this formula is exactly [12, Lemma 4.3 (iii)]. We now prove the lemma by induction on m. Assume the lemma is true for $m=k$.

For $m=k+1$, we apply equation (2.2) to obtain the following formula for the left hand side of equation (2.1):

$$\begin{aligned}&\text{LHS}\\ =\;&\sum_{i=0}^{k}(-1)^i\,\langle\!\langle\, \mathcal{W}\,\tau_i(\gamma^\alpha)\,\rangle\!\rangle_0\left\{-\langle\!\langle\, \tau_{k-i}(\gamma_\alpha)\,\tau_1(\mathcal{V})\,\rangle\!\rangle_0+\langle\!\langle\, \tau_{k-i}(\gamma_\alpha)\,\gamma^\beta\,\rangle\!\rangle_0\,\langle\!\langle\, \gamma_\beta\,\mathcal{V}\,\rangle\!\rangle_0\right\}\\ &+(-1)^{k+1}\,\langle\!\langle\, \mathcal{W}\,\tau_{k+1}(\gamma_\alpha)\,\rangle\!\rangle_0\,\langle\!\langle\, \gamma_\alpha\,\mathcal{V}\,\rangle\!\rangle_0\,.\end{aligned}$$

Applying the induction hypothesis to the first two terms, we obtain

$$\begin{aligned}\text{LHS} &= -\left\{\langle\!\langle\, \tau_{k+1}(\mathcal{W})\, \tau_1(V)\, \rangle\!\rangle_0 + (-1)^k \langle\!\langle\, \mathcal{W}\, \tau_{k+2}(\mathcal{V})\, \rangle\!\rangle_0\right\} \\ &\qquad + \left\langle\!\!\left\langle\, \tau_{k+1}(\mathcal{W})\, \gamma^\beta\, \right\rangle\!\!\right\rangle_0 \langle\!\langle\, \gamma_\beta\, \mathcal{V}\, \rangle\!\rangle_0 .\end{aligned}$$

The lemma follows by applying equation (2.2) to the last term. □

An immediate consequence of Lemma 2.1 is the following

Lemma 2.2 (Generalized genus-0 TRR)**.** *For $m \geq 0$,*

$$\sum_{i=0}^{m} (-1)^i \langle\!\langle\, \mathcal{W}\, \tau_i(\gamma^\alpha)\, \rangle\!\rangle_0 \langle\!\langle\, \tau_{m-i}(\gamma_\alpha)\, \mathcal{U}\, \mathcal{V}\, \rangle\!\rangle_0 = \langle\!\langle\, \tau_{m+1}(\mathcal{W})\, \mathcal{U}\, \mathcal{V}\, \rangle\!\rangle_0$$

for any vector fields $\mathcal{W}$, $\mathcal{U}$ and $\mathcal{V}$.

Proof. When $m = 0$, this formula is precisely the genus-0 TRR. Now assume $m > 0$. Applying the genus-0 TRR to the left hand side of this formula, we obtain

$$\begin{aligned}\text{LHS} &= \sum_{i=0}^{m-1} (-1)^i \langle\!\langle\, \mathcal{W}\, \tau_i(\gamma^\alpha)\, \rangle\!\rangle_0 \left\langle\!\!\left\langle\, \tau_{m-i-1}(\gamma_\alpha)\, \gamma^\beta\, \right\rangle\!\!\right\rangle_0 \langle\!\langle\, \gamma_\beta\, \mathcal{U}\, \mathcal{V}\, \rangle\!\rangle_0 \\ &\quad + (-1)^m \langle\!\langle\, \mathcal{W}\, \tau_m(\gamma^\alpha)\, \rangle\!\rangle_0 \langle\!\langle\, \gamma_\alpha\, \mathcal{U}\, \mathcal{V}\, \rangle\!\rangle_0 .\end{aligned}$$

Applying Lemma 2.1 to the first term, we have

$$\text{LHS} = \left\langle\!\!\left\langle\, \tau_m(\mathcal{W})\, \gamma^\beta\, \right\rangle\!\!\right\rangle_0 \langle\!\langle\, \gamma_\beta\, \mathcal{U}\, \mathcal{V}\, \rangle\!\rangle_0 = \langle\!\langle\, \tau_{m+1}(\mathcal{W})\, \mathcal{U}\, \mathcal{V}\, \rangle\!\rangle_0$$

by the genus-0 TRR. The lemma is proved. □

Note that a special case of Theorem 0.3 is the following: If $k, n \geq 2$ and $m \geq k + n - 3$,

$$\sum_{i=0}^{m} (-1)^i \langle\!\langle\, \mathcal{W}_1 \cdots \mathcal{W}_n\, \tau_i(\gamma^\alpha)\, \rangle\!\rangle_0 \langle\!\langle\, \tau_{m-i}(\gamma_\alpha)\, \mathcal{V}_1 \cdots \mathcal{V}_k\, \rangle\!\rangle_0 = 0 \tag{2.3}$$

for any vector fields $\mathcal{W}_1, \ldots, \mathcal{W}_n$ and $\mathcal{V}_1, \ldots, \mathcal{V}_k$. Therefore if we take derivatives of the formula in Lemma 2.2, we obtain

$$\sum_{i=0}^{m} (-1)^i \langle\!\langle\, \mathcal{W}\, \tau_i(\gamma^\alpha)\, \rangle\!\rangle_0 \langle\!\langle\, \tau_{m-i}(\gamma_\alpha)\, \mathcal{V}_1 \cdots \mathcal{V}_k\, \rangle\!\rangle_0 = \langle\!\langle\, \tau_{m+1}(\mathcal{W})\, \mathcal{V}_1 \cdots \mathcal{V}_k\, \rangle\!\rangle_0 \tag{2.4}$$

for $k \geq 2$, $m \geq k - 2$, and any vector fields $\mathcal{W}$, $\mathcal{V}_1, \ldots, \mathcal{V}_k$.

Note that together with the genus-0 part of Conjecture B and Lemma 2.1, equations (2.3) and (2.4) cover most cases of the genus-0 part of Conjecture C. The remaining cases follow from the derivatives of Conjecture B.

We now study analogous results for genus-1 Gromov-Witten invariants. The *genus-1 TRR* is the following:

$$\langle\!\langle\, \tau_1(\mathcal{W})\, \rangle\!\rangle_1 = \langle\!\langle\, \mathcal{W}\, \gamma^\alpha\, \rangle\!\rangle_0 \langle\!\langle\, \gamma_\alpha\, \rangle\!\rangle_1 + \frac{1}{24} \langle\!\langle\, \mathcal{W}\, \gamma^\alpha\, \gamma_\alpha\, \rangle\!\rangle_0$$

for any vector field $\mathcal{W}$. Similar to Lemma 2.2, we also have the following

Lemma 2.3 (Generalized genus-1 TRR)**.** *For $m \geq 0$,*

$$\sum_{i=0}^{m} (-1)^i \langle\!\langle\, \mathcal{W}\, \tau_i(\gamma^\alpha)\, \rangle\!\rangle_0 \langle\!\langle\, \tau_{m-i}(\gamma_\alpha)\, \rangle\!\rangle_1 = \langle\!\langle\, \tau_{m+1}(\mathcal{W})\, \rangle\!\rangle_1 - \delta_{m,0}\, \frac{1}{24} \langle\!\langle\, \mathcal{W}\, \gamma^\alpha\, \gamma_\alpha\, \rangle\!\rangle_0$$

for any vector field $\mathcal{W}$.

Proof. When $m = 0$, this formula is precisely the genus-1 TRR. Now assume $m > 0$. Applying the genus-1 TRR to the left hand side of this formula, we obtain

$$\begin{aligned}\text{LHS} \quad = \quad & \sum_{i=0}^{m-1}(-1)^i \langle\!\langle \mathcal{W}\, \tau_i(\gamma^\alpha) \rangle\!\rangle_0 \big\{ \langle\!\langle \tau_{m-1-i}(\gamma_\alpha)\, \gamma^\beta \rangle\!\rangle_0 \langle\!\langle \gamma_\beta \rangle\!\rangle_1 \\ & \qquad\qquad + \frac{1}{24} \langle\!\langle \tau_{m-1-i}(\gamma_\alpha)\ \gamma^\beta\, \gamma_\beta \rangle\!\rangle_0 \big\} \\ & + (-1)^m \langle\!\langle \mathcal{W}\, \tau_m(\gamma^\alpha) \rangle\!\rangle_0 \langle\!\langle \gamma_\alpha \rangle\!\rangle_1 .\end{aligned}$$

Applying Lemma 2.1 to the first term and Lemma 2.2 to the second term, we have

$$\text{LHS} = \langle\!\langle \tau_m(\mathcal{W})\, \gamma^\beta \rangle\!\rangle_0 \langle\!\langle \gamma_\beta \rangle\!\rangle_1 + \frac{1}{24} \langle\!\langle \tau_m(\mathcal{W})\ \gamma^\beta\, \gamma_\beta \rangle\!\rangle_0 = \langle\!\langle \tau_{m+1}(\mathcal{W}) \rangle\!\rangle_1$$

by the genus-1 TRR. This proves the lemma. □

Note that a special case of Theorem 0.3 is the following: For $n \geq 2$, $k \geq 0$ and $m \geq k + n$,

$$\sum_{i=0}^{m}(-1)^i \langle\!\langle \mathcal{W}_1 \cdots \mathcal{W}_n\ \tau_i(\gamma^\alpha) \rangle\!\rangle_0 \langle\!\langle \tau_{m-i}(\gamma_\alpha)\, \mathcal{V}_1 \cdots \mathcal{V}_k \rangle\!\rangle_1 = 0 \tag{2.5}$$

for any vector fields $\mathcal{W}_1, \ldots, \mathcal{W}_n$ and $\mathcal{V}_1, \ldots, \mathcal{V}_k$. Therefore if we take derivatives of the formula in Lemma 2.3, we obtain

$$\sum_{i=0}^{m}(-1)^i \langle\!\langle \mathcal{W}\, \tau_i(\gamma^\alpha) \rangle\!\rangle_0 \langle\!\langle \tau_{m-i}(\gamma_\alpha)\, \mathcal{V}_1 \cdots \mathcal{V}_k \rangle\!\rangle_1 = \langle\!\langle \tau_{m+1}(\mathcal{W})\ \mathcal{V}_1 \cdots \mathcal{V}_k \rangle\!\rangle_1 \tag{2.6}$$

for $k \geq 0$, $m \geq k + 1$, and any vector fields $\mathcal{W}$, $\mathcal{V}_1, \ldots, \mathcal{V}_k$.

Note that equations (2.5) and (2.6) are stronger than the corresponding cases of the genus-1 part of Conjecture C in the sense that we do not sum over the genus. This implies most cases of the genus-1 part of conjecture C. The remaining cases follow from the genus-1 part of Conjecture B and its derivatives.

Using the operator T, we can reformulate the genus-0 TRR as

$$\langle\!\langle T(\mathcal{W}_1)\, \mathcal{W}_2\, \mathcal{W}_3 \rangle\!\rangle_0 = 0$$

and the genus-1 TRR as

$$\langle\!\langle T(\mathcal{W}) \rangle\!\rangle_1 = \frac{1}{24} \langle\!\langle \mathcal{W}\, \gamma^\alpha\, \gamma_\alpha \rangle\!\rangle_0$$

for any vector fields $\mathcal{W}$ and $\mathcal{W}_i$. Replacing $\mathcal{W}$ by $T^k(\mathcal{W})$ in the genus-1 TRR, we obtain

$$\langle\!\langle T^k(\mathcal{W}) \rangle\!\rangle_1 = 0$$

for all vector fields $\mathcal{W}$ if $k \geq 2$.

For later use, we also recall the genus-2 Mumford relation (cf. [4]) as formulated in [8]:

$$\begin{aligned}\langle\!\langle T^2(\mathcal{W}) \rangle\!\rangle_2 \quad = \quad & \frac{7}{10} \langle\!\langle \gamma_\alpha \rangle\!\rangle_1 \langle\!\langle \{\gamma^\alpha \circ \mathcal{W}\} \rangle\!\rangle_1 + \frac{1}{10} \langle\!\langle \gamma_\alpha\, \{\gamma^\alpha \circ \mathcal{W}\} \rangle\!\rangle_1 \\ & - \frac{1}{240} \langle\!\langle \mathcal{W}\, \{\gamma_\alpha \circ \gamma^\alpha\} \rangle\!\rangle_1 + \frac{13}{240} \langle\!\langle \mathcal{W}\, \gamma_\alpha\, \gamma^\alpha\, \gamma^\beta \rangle\!\rangle_0 \langle\!\langle \gamma_\beta \rangle\!\rangle_1 \\ & + \frac{1}{960} \langle\!\langle \mathcal{W}\, \gamma^\alpha\, \gamma_\alpha\, \gamma^\beta\, \gamma_\beta \rangle\!\rangle_0\end{aligned} \tag{2.7}$$

for any vector field $\mathcal{W}$. Here we have used the quantum product for vector fields on the big phase space defined by

$$\mathcal{W}_1 \circ \mathcal{W}_2 := \langle\!\langle \mathcal{W}_1 \, \mathcal{W}_2 \, \gamma^\alpha \rangle\!\rangle_0 \; \gamma_\alpha.$$

Basic properties of this product can be found in [8]. Replacing $\mathcal{W}$ by $T^i(\mathcal{W})$ in equation (2.7) and using the genus-0 and genus-1 TRR, we obtain

$$\begin{aligned}
(2.8)\quad \langle\!\langle T^3(\mathcal{W}) \rangle\!\rangle_2 &= \frac{1}{20} \langle\!\langle \{\mathcal{W} \circ \gamma^\alpha \circ \gamma_\alpha\} \rangle\!\rangle_1 + \frac{1}{1152} \langle\!\langle \mathcal{W} \; \gamma^\alpha \; \gamma_\alpha \; \{\gamma^\beta \circ \gamma_\beta\} \rangle\!\rangle_0 \\
&\quad + \frac{1}{480} \langle\!\langle \{\mathcal{W} \circ \gamma^\alpha\} \; \gamma_\alpha \; \gamma^\beta \; \gamma_\beta \rangle\!\rangle_0 , \\
(2.9)\quad \langle\!\langle T^4(\mathcal{W}) \rangle\!\rangle_2 &= \frac{1}{1152} \langle\!\langle \{\mathcal{W} \circ \gamma^\alpha \circ \gamma_\alpha\} \; \gamma^\beta \; \gamma_\beta \rangle\!\rangle_0 ,
\end{aligned}$$

and

$$\langle\!\langle T^k(\mathcal{W}) \rangle\!\rangle_2 = 0$$

for any vector field $\mathcal{W}$ if $k \geq 5$.

To prove Theorem 0.2, we also need to use the following genus-3 equation (cf. [5]):

$$\begin{aligned}
(2.10)\quad & \langle\!\langle T^3(\mathcal{W}) \rangle\!\rangle_3 \\
= \; & -\frac{1}{252} \langle\!\langle \mathcal{W} \, T(\gamma_\alpha \circ \gamma^\alpha) \rangle\!\rangle_2 + \frac{5}{42} \langle\!\langle T(\gamma_\alpha) \, \{\mathcal{W} \circ \gamma^\alpha\} \rangle\!\rangle_2 \\
& + \frac{13}{168} \langle\!\langle T(\gamma^\alpha) \rangle\!\rangle_2 \langle\!\langle \gamma_\alpha \, \mathcal{W} \, \gamma^\beta \, \gamma_\beta \rangle\!\rangle_0 + \frac{41}{21} \langle\!\langle T(\gamma^\alpha) \rangle\!\rangle_2 \langle\!\langle \{\gamma_\alpha \circ \mathcal{W}\} \rangle\!\rangle_1 \\
& - \frac{13}{168} \langle\!\langle \{\mathcal{W} \circ \gamma_\alpha \circ \gamma^\alpha\} \rangle\!\rangle_2 + \frac{1}{280} \langle\!\langle \mathcal{W} \gamma^\alpha \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \; \{\gamma^\beta \circ \gamma_\beta\} \rangle\!\rangle_1 \\
& - \frac{23}{5040} \langle\!\langle \gamma^\alpha \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \mathcal{W} \, \{\gamma^\beta \circ \gamma_\beta\} \rangle\!\rangle_1 \\
& - \frac{47}{5040} \langle\!\langle \gamma^\alpha \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \gamma^\beta \rangle\!\rangle_1 \langle\!\langle \gamma_\beta \, \mathcal{W} \, \gamma^\mu \, \gamma_\mu \rangle\!\rangle_0 \\
& - \frac{5}{1008} \langle\!\langle \mathcal{W} \, \gamma^\alpha \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \gamma^\beta \gamma_\beta \gamma^\mu \rangle\!\rangle_0 \langle\!\langle \gamma_\mu \rangle\!\rangle_1 \\
& + \frac{23}{504} \langle\!\langle \gamma^\alpha \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \mathcal{W} \gamma^\beta \gamma_\beta \gamma^\mu \rangle\!\rangle_0 \langle\!\langle \gamma_\mu \rangle\!\rangle_1 \\
& + \frac{11}{140} \langle\!\langle \gamma^\alpha \gamma^\beta \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \; \{\gamma_\beta \circ \mathcal{W}\} \rangle\!\rangle_1 - \frac{4}{35} \langle\!\langle \gamma^\alpha \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \gamma^\beta \rangle\!\rangle_1 \langle\!\langle \{\gamma_\beta \circ \mathcal{W}\} \rangle\!\rangle_1 \\
& + \frac{2}{105} \langle\!\langle \mathcal{W} \, \gamma^\alpha \rangle\!\rangle_1 \langle\!\langle \{\gamma_\alpha \circ \gamma_\beta\} \rangle\!\rangle_1 \langle\!\langle \gamma^\beta \rangle\!\rangle_1 \\
& + \frac{89}{210} \langle\!\langle \gamma^\alpha \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \, \mathcal{W} \, \gamma^\beta \, \gamma^\mu \rangle\!\rangle_0 \langle\!\langle \gamma_\beta \rangle\!\rangle_1 \langle\!\langle \gamma_\mu \rangle\!\rangle_1 \\
& - \frac{1}{210} \langle\!\langle \gamma^\alpha \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \gamma^\beta \, \{\gamma_\beta \circ \mathcal{W}\} \rangle\!\rangle_1 + \frac{1}{140} \langle\!\langle \mathcal{W} \, \gamma^\alpha \gamma^\beta \rangle\!\rangle_1 \langle\!\langle \{\gamma_\alpha \circ \gamma_\beta\} \rangle\!\rangle_1 \\
& + \frac{23}{140} \langle\!\langle \gamma^\alpha \gamma^\beta \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \gamma_\beta \mathcal{W} \gamma^\mu \rangle\!\rangle_0 \langle\!\langle \gamma_\mu \rangle\!\rangle_1 - \frac{3}{140} \langle\!\langle \gamma^\alpha \gamma^\beta \rangle\!\rangle_1 \langle\!\langle \{\gamma_\alpha \circ \gamma_\beta\} \, \mathcal{W} \rangle\!\rangle_1 \\
& - \frac{1}{4480} \langle\!\langle \mathcal{W} \gamma^\alpha \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \gamma_\beta \gamma^\beta \gamma_\mu \gamma^\mu \rangle\!\rangle_0 + \frac{13}{8064} \langle\!\langle \gamma^\alpha \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \mathcal{W} \gamma^\beta \gamma_\beta \gamma^\mu \gamma_\mu \rangle\!\rangle_0 \\
& - \frac{1}{2240} \langle\!\langle \mathcal{W} \gamma^\alpha \gamma^\beta \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \gamma_\beta \gamma^\mu \gamma_\mu \rangle\!\rangle_0 + \frac{41}{6720} \langle\!\langle \gamma^\alpha \gamma^\beta \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \gamma_\beta \mathcal{W} \gamma^\mu \gamma_\mu \rangle\!\rangle_0 \\
& + \frac{1}{53760} \langle\!\langle \mathcal{W} \gamma^\alpha \gamma_\alpha \gamma^\beta \gamma_\beta \gamma^\mu \gamma_\mu \rangle\!\rangle_0 - \frac{1}{210} \langle\!\langle \{\mathcal{W} \circ \gamma^\alpha\} \rangle\!\rangle_1 \langle\!\langle \gamma_\alpha \gamma^\beta \gamma_\beta \rangle\!\rangle_1
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{5760}\left\langle\!\!\left\langle \mathcal{W}\gamma^\alpha\gamma_\alpha\left\{\gamma^\beta\circ\gamma_\beta\right\}\right\rangle\!\!\right\rangle_1-\frac{1}{2688}\left\langle\!\!\left\langle \gamma^\alpha\gamma_\alpha\gamma^\beta\right\rangle\!\!\right\rangle_1\left\langle\!\!\left\langle \gamma_\beta\mathcal{W}\gamma^\mu\gamma_\mu\right\rangle\!\!\right\rangle_0\\
&-\frac{1}{5040}\left\langle\!\!\left\langle \gamma^\alpha\gamma_\alpha\gamma^\beta\left\{\gamma_\beta\circ\mathcal{W}\right\}\right\rangle\!\!\right\rangle_1+\frac{1}{3780}\left\langle\!\!\left\langle \mathcal{W}\gamma_\alpha\gamma_\beta\gamma_\mu\right\rangle\!\!\right\rangle_1\left\langle\!\!\left\langle \gamma^\alpha\gamma^\beta\gamma^\mu\right\rangle\!\!\right\rangle_0\\
&+\frac{1}{252}\left\langle\!\!\left\langle \gamma_\alpha\gamma_\beta\gamma_\mu\right\rangle\!\!\right\rangle_1\left\langle\!\!\left\langle \mathcal{W}\gamma^\alpha\gamma^\beta\gamma^\mu\right\rangle\!\!\right\rangle_0.
\end{aligned}
$$

Replacing $\mathcal{W}$ by $T^3(\mathcal{W})$ in this equation, we get

$$
\begin{aligned}
(2.11)\qquad &\left\langle\!\!\left\langle T^6(\mathcal{W})\right\rangle\!\!\right\rangle_3\\
=\ &\frac{7}{5760}\left\langle\!\!\left\langle \{\mathcal{W}\circ\Delta\circ\Delta\}\right\rangle\!\!\right\rangle_1+\frac{11}{2903040}\left\langle\!\!\left\langle \mathcal{W}\,\Delta\,\Delta\,\Delta\right\rangle\!\!\right\rangle_0\\
&+\frac{19}{967680}\left\langle\!\!\left\langle \{\mathcal{W}\circ\Delta\}\ \Delta\,\gamma^\alpha\,\gamma_\alpha\right\rangle\!\!\right\rangle_0+\frac{1}{120960}\left\langle\!\!\left\langle \mathcal{W}\ \{\Delta\circ\Delta\}\ \gamma^\alpha\,\gamma_\alpha\right\rangle\!\!\right\rangle_0\\
&+\frac{1}{60480}\left\langle\!\!\left\langle \{\mathcal{W}\circ\gamma^\alpha\}\ \gamma_\alpha\,\Delta\,\Delta\right\rangle\!\!\right\rangle_0\\
&+\frac{1}{11520}\left\langle\!\!\left\langle \{\mathcal{W}\circ\Delta\circ\gamma^\alpha\}\ \gamma_\alpha\,\gamma^\beta\,\gamma_\beta\right\rangle\!\!\right\rangle_0
\end{aligned}
$$

for any vector field $\mathcal{W}$, where $\Delta := \gamma^\alpha\circ\gamma_\alpha$.

3. Proof of the main theorems

In order to see the connection between TRR and the conjectures in the introduction, we give a new formulation to these conjectures and prove that these conjectures are correct for all genera if m is sufficiently large. The key point here is that we will use the operator T instead of the descendant operator τ. The following lemma gives the relation between $T^k(\mathcal{W})$ and $\tau_k(\mathcal{W})$.

Lemma 3.1. *For any $k\geq 0$ and any vector field $\mathcal{W}$ on the big phase space,*

$$
\tau_k(\mathcal{W})=T^k(\mathcal{W})+\sum_{i=0}^{k-1}\left\langle\!\!\left\langle \tau_{k-1-i}(\mathcal{W})\,\gamma^\alpha\right\rangle\!\!\right\rangle_0 T^i(\gamma_\alpha).
$$

Proof. We prove the lemma by induction on k. If $k=0$, it is trivial. If $k=1$, it is just the definition of T. Assuming it is true for $k=r$, then for $k=r+1$, we have

$$
\tau_k(\mathcal{W})=\tau_+(\tau_{k-1}(\mathcal{W}))=T(\tau_{k-1}(\mathcal{W}))+\left\langle\!\!\left\langle \tau_{k-1}(\mathcal{W})\,\gamma^\alpha\right\rangle\!\!\right\rangle_0\gamma_\alpha
$$

by the definition of T. By the induction hypothesis,

$$
\begin{aligned}
\tau_k(\mathcal{W})\ &=\ T\left(T^{k-1}(\mathcal{W})+\sum_{i=0}^{k-2}\left\langle\!\!\left\langle \tau_{k-2-i}(\mathcal{W})\,\gamma^\alpha\right\rangle\!\!\right\rangle_0 T^i(\gamma_\alpha)\right)+\left\langle\!\!\left\langle \tau_{k-1}(\mathcal{W})\,\gamma^\alpha\right\rangle\!\!\right\rangle_0\gamma_\alpha\\
&=\ T^k(\mathcal{W})+\sum_{i=0}^{k-1}\left\langle\!\!\left\langle \tau_{k-1-i}(\mathcal{W})\,\gamma^\alpha\right\rangle\!\!\right\rangle_0 T^i(\gamma_\alpha).
\end{aligned}
$$

The lemma is proved. □

The following proposition shows that in certain combinations, τ_k can be replaced by T^k.

Proposition 3.2. *For any contravariant tensors P and Q on the big phase space and $m \geq 0$,*

$$\sum_{j=0}^{m}(-1)^j P(\tau_j(\gamma^\alpha))\ Q(\tau_{m-j}(\gamma_\alpha)) = \sum_{j=0}^{m}(-1)^j P(T^j(\gamma^\alpha))\ Q(T^{m-j}(\gamma_\alpha)).$$

Proof. By Lemma 3.1, we have

$$P(\tau_j(\gamma^\alpha)) = P(T^j(\gamma^\alpha)) + \sum_{r=0}^{j-1} \left\langle\!\!\left\langle \tau_{j-1-r}(\gamma^\alpha)\,\gamma^\beta \right\rangle\!\!\right\rangle_0 P(T^r(\gamma_\beta)).$$

We also have a similar formula for $Q(\tau_{m-j}(\gamma_\alpha))$. Therefore the difference of the two sides of the equation in the proposition is given by

$$\begin{aligned}
&\text{LHS} - \text{RHS}\\
= \ & \sum_{j=0}^{m}(-1)^j \sum_{s=0}^{m-j-1} \left\langle\!\!\left\langle \tau_{m-j-1-s}(\gamma_\alpha)\gamma^\mu \right\rangle\!\!\right\rangle_0 P(T^j(\gamma^\alpha))Q(T^s(\gamma_\mu))\\
& + \sum_{j=0}^{m}(-1)^j \sum_{r=0}^{j-1} \left\langle\!\!\left\langle \tau_{j-1-r}(\gamma^\alpha)\gamma^\beta \right\rangle\!\!\right\rangle_0 P(T^r(\gamma_\beta))Q(T^{m-j}(\gamma_\alpha))\\
& + \sum_{j=0}^{m}(-1)^j \sum_{r=0}^{j-1}\sum_{s=0}^{m-j-1} \left\langle\!\!\left\langle \tau_{j-1-r}(\gamma^\alpha)\gamma^\beta \right\rangle\!\!\right\rangle_0 \left\langle\!\!\left\langle \tau_{m-j-1-s}(\gamma_\alpha)\gamma^\mu \right\rangle\!\!\right\rangle_0\\
& \qquad\qquad \cdot P(T^r(\gamma_\beta))Q(T^s(\gamma_\mu)).
\end{aligned}$$

The third term can be written as

$$\begin{aligned}
& \sum_{r=0}^{m-1}\sum_{s=0}^{m-1} P(T^r(\gamma_\beta))Q(T^s(\gamma_\mu))\\
& \qquad \cdot \sum_{j=r+1}^{m-1-s}(-1)^j \left\langle\!\!\left\langle \tau_{j-1-r}(\gamma^\alpha)\gamma^\beta \right\rangle\!\!\right\rangle_0 \left\langle\!\!\left\langle \tau_{m-j-1-s}(\gamma_\alpha)\gamma^\mu \right\rangle\!\!\right\rangle_0\\
= \ & \sum_{r=0}^{m-1}\sum_{s=0}^{m-1}(-1)^{r+1} P(T^r(\gamma_\beta))Q(T^s(\gamma_\mu))\\
& \qquad \cdot \left\{ \left\langle\!\!\left\langle \tau_{m-r-s-1}(\gamma^\beta)\,\gamma^\mu \right\rangle\!\!\right\rangle_0 + (-1)^{m-r-s} \left\langle\!\!\left\langle \gamma^\beta\,\tau_{m-r-s-1}(\gamma^\mu) \right\rangle\!\!\right\rangle_0 \right\}
\end{aligned}$$

by Lemma 2.1. Hence it can be canceled with the other two terms in the above expression. The proposition is thus proved. □

As a consequence of this proposition, we have the following

Theorem 3.3. *In Conjectures A, B, C in the introduction, $\tau_j(\gamma^\alpha)$ and $\tau_{m-j}(\gamma_\alpha)$ can be replaced by $T^j(\gamma^\alpha)$ and $T^{m-j}(\gamma_\alpha)$, respectively.*

We are now ready to prove the main theorems of this paper.

Proof of Theorem 0.3. By Proposition 3.2,

$$\begin{aligned}(3.1)\qquad &\sum_{j=0}^{m}(-1)^j \langle\!\langle \tau_j(\gamma_\alpha)\ \mathcal{W}_1 \cdots \mathcal{W}_a \rangle\!\rangle_h \langle\!\langle \tau_{m-j}(\gamma^\alpha)\ \mathcal{V}_1 \cdots \mathcal{V}_b \rangle\!\rangle_{g-h} \\ = &\sum_{j=0}^{m}(-1)^j \left\langle\!\!\left\langle T^j(\gamma_\alpha)\ \mathcal{W}_1 \cdots \mathcal{W}_a \right\rangle\!\!\right\rangle_h \left\langle\!\!\left\langle T^{m-j}(\gamma^\alpha)\ \mathcal{V}_1 \cdots \mathcal{V}_b \right\rangle\!\!\right\rangle_{g-h}\end{aligned}$$

for any vector fields $\mathcal{W}_i$ and $\mathcal{V}_j$. Since the dimension of the moduli space $\overline{\mathcal{M}}_{g,k}$ is $3g-3+k$, $\psi_1^j = 0$ on $\overline{\mathcal{M}}_{g,k}$ if $j > 3g-3+k$. Translating this fact to a universal equation for Gromov-Witten invariants, we get

$$\left\langle\!\!\left\langle T^j(\mathcal{V})\, \mathcal{W}_1 \cdots \mathcal{W}_k \right\rangle\!\!\right\rangle_g = 0$$

for any vector fields $\mathcal{V}$ and $\mathcal{W}_i$ if $j > 3g-2+k$. In the genus-0 case, we also require $k \geq 2$ since $\overline{\mathcal{M}}_{g,n}$ does not exist if $n < 3$. So any term in equation (3.1) which is non-zero requires that $j \leq 3h-2+a$ and $m-j \leq 3(g-h)-2+b$. But this requires that $m \leq 3g+a+b-4$. Equation (0.1) in Theorem 0.3 is thus proved.

To prove equation (0.2), we first prove the special case that $b = 0$. Note that equation (2.2) can be written in the following form:

$$(3.2)\qquad \langle\!\langle T(\mathcal{W})\ \mathcal{V} \rangle\!\rangle_0 = -\langle\!\langle \mathcal{W}\ \tau_+(\mathcal{V}) \rangle\!\rangle_0$$

for all vector fields $\mathcal{W}$ and $\mathcal{V}$. So by Proposition 3.2, Lemma 3.1 can be rewritten as

$$(3.3)\qquad T^k(\mathcal{W}) = \tau_k(\mathcal{W}) - \sum_{i=0}^{k-1}(-1)^i \langle\!\langle \mathcal{W}\, \tau_i(\gamma^\alpha) \rangle\!\rangle_0\, \tau_{k-1-i}(\gamma_\alpha).$$

Note that this is exactly [10, Equation (32)]. We also notice that if we replace $\mathcal{W}$ in this equation by the dilaton vector field, then we obtain Lemma 1.1 by the dilaton equation.

By equation (1.8), $\left\langle\!\!\left\langle T^k(\mathcal{W}) \right\rangle\!\!\right\rangle_g = 0$ if $k \geq 3g-1$. So by equation (3.3) with $m = k-1$, we have

$$(3.4)\qquad \sum_{i=0}^{m}(-1)^i \langle\!\langle \mathcal{W}\, \tau_i(\gamma^\alpha) \rangle\!\rangle_0 \langle\!\langle \tau_{m-i}(\gamma_\alpha) \rangle\!\rangle_g = \langle\!\langle \tau_{m+1}(\mathcal{W}) \rangle\!\rangle_g$$

for any vector field $\mathcal{W}$ if $g \geq 1$ and $m \geq 3g-2$. This is a special case of equation (0.2) with $b = 0$. We can view this equation as a generalization of Lemma 2.3. Taking derivatives of this equation and using equation (0.1), we obtain equation (0.2). Theorem 0.3 is thus proved. □

Proof of Theorem 0.1. The genus-0 part of Theorem 0.1 is precisely equation (1.9). To illustrate the idea that Conjecture A should follow from topological recursion relations, we provide another proof here. Since Conjecture A is trivial when m is odd, we assume that m is an even positive integer. By an argument similar to the

proof of Proposition 3.2 and the genus-0 TRR, we have

$$\begin{aligned}(3.5)\qquad & \sum_{j=0}^{m}(-1)^j \langle\!\langle \tau_j(\gamma_\alpha)\ \mathcal{W}\ \tau_{m-j}(\gamma^\alpha)\rangle\!\rangle_0 \\ = & \sum_{j=0}^{m}(-1)^j \left\langle\!\!\left\langle T^j(\gamma_\alpha)\ \mathcal{W}\ T^{m-j}(\gamma^\alpha)\right\rangle\!\!\right\rangle_0 \ = \ 0\end{aligned}$$

for any vector field $\mathcal{W}$.

Now we consider the special case that $\mathcal{W}$ is the *Euler vector field*

$$\mathcal{X} := -\sum_{n,\alpha}\left(n + b_\alpha - \frac{3-d}{2}\right)\tilde{t}^\alpha_n \tau_n(\gamma_\alpha) - \sum_{n,\alpha,\beta}\mathcal{C}^\beta_\alpha \tilde{t}^\alpha_n \tau_{n-1}(\gamma_\beta),$$

where $d = \frac{1}{2}\dim_{\mathbb{R}}(V)$, $b_\alpha = \frac{1}{2}\{\deg(\gamma_\alpha) - d + 1\}$, and $\mathcal{C}^\beta_\alpha$ represents the multiplication by the first Chern class of V, i.e. $c_1(V)\cup\gamma_\alpha = \mathcal{C}^\beta_\alpha\gamma_\beta$. By [12, Lemma 1.4 (3)], which follows from the quasi-homogeneity equation, we have

$$\begin{aligned}& \sum_{j=0}^{m}(-1)^j \langle\!\langle \tau_j(\gamma_\alpha)\ \mathcal{X}\ \tau_{m-j}(\gamma^\alpha)\rangle\!\rangle_0 \\ = & \sum_{j=0}^{m}(-1)^j \Big\{(m+1)\langle\!\langle \tau_j(\gamma_\alpha)\ \tau_{m-j}(\gamma^\alpha)\rangle\!\rangle_0 + \mathcal{C}^\mu_\alpha\eta_{\mu\nu}\langle\!\langle \tau_{j-1}(\gamma^\nu)\,\tau_{m-j}(\gamma^\alpha)\rangle\!\rangle_0 \\ & \qquad + \eta^{\alpha\beta}\mathcal{C}^\mu_\beta\langle\!\langle \tau_j(\gamma_\alpha)\,\tau_{m-j-1}(\gamma_\mu)\rangle\!\rangle_0\Big\}.\end{aligned}$$

Since both $\mathcal{C}^\mu_\alpha\eta_{\mu\nu}$ and $\eta^{\alpha\beta}\mathcal{C}^\mu_\beta$ are (super)symmetric with respect to the free indices, the last two terms are equal to 0. Therefore we have

$$\sum_{j=0}^{m}(-1)^j \langle\!\langle \tau_j(\gamma_\alpha)\ \mathcal{X}\ \tau_{m-j}(\gamma^\alpha)\rangle\!\rangle_0 = (m+1)\sum_{j=0}^{m}(-1)^j \langle\!\langle \tau_j(\gamma_\alpha)\ \tau_{m-j}(\gamma^\alpha)\rangle\!\rangle_0 .$$

So the genus-0 part of Theorem 0.1 follows from equation (3.5) with $\mathcal{W}$ replaced by $\mathcal{X}$.

For $g > 0$, we have

$$\left\langle\!\!\left\langle T^j(\mathcal{W})\,T^{m-j}(\mathcal{V})\right\rangle\!\!\right\rangle_g = 0$$

for any vector fields $\mathcal{W}$ and $\mathcal{V}$ if $m \geq 3g$. This follows from the fact that $\psi_1^j\psi_2^{m-j} = 0$ on $\overline{\mathcal{M}}_{g,2}$ since the dimension of $\overline{\mathcal{M}}_{g,2}$ is $3g-1$. Therefore Theorem 0.1 follows from Proposition 3.2. □

Proof of Theorem 0.2. Part (a) of Theorem 0.2 was proved in Section 1. When $g = k = 0$, Conjecture B is just the genus-0 string equation. Moreover, when $g = 0$ and $l > 0$, equation (1.5) is exactly the genus-0 part of Conjecture B with $k = l$. By Lemma 1.1, for genus bigger than 0, Conjecture B can be reformulated as

$$(3.6)\qquad \left\langle\!\!\left\langle T^{2k}(\mathcal{S})\right\rangle\!\!\right\rangle_g + \frac{1}{2}\Phi_{g,k-1} = 0$$

for $k \geq g$, where $\mathcal{S}$ is the string vector field and

$$\Phi_{g,k} := \sum_{j=0}^{2k}(-1)^j\sum_{h=1}^{g-1}\langle\!\langle \tau_j(\gamma^\alpha)\rangle\!\rangle_h \langle\!\langle \tau_{2k-j}(\gamma_\alpha)\rangle\!\rangle_{g-h} .$$

Since $\langle\!\langle T^2(\mathcal{W})\rangle\!\rangle_1 = 0$ for any vector field $\mathcal{W}$ by the genus-1 TRR, the genus-1 case of Conjecture B follows from equation (3.6) rather trivially. So we only need to consider the case of $g > 1$. Part (b) of Theorem 0.2 follows from equation (1.8) and Theorem 0.3. Part (c) of Theorem 0.2 follows from part (b) except for the cases $k = g = 2$ and $k = g = 3$.

We first look at the genus-2 case. By Proposition 3.2 and the genus-1 TRR,

$$\begin{aligned}\Phi_{2,1} &= \sum_{j=0}^{2}(-1)^j \langle\!\langle T^j(\gamma^\alpha)\rangle\!\rangle_1 \langle\!\langle T^{2-j}(\gamma_\alpha)\rangle\!\rangle_1 = -\langle\!\langle T(\gamma^\alpha)\rangle\!\rangle_1 \langle\!\langle T(\gamma_\alpha)\rangle\!\rangle_1 \\ &= -\frac{1}{576}\langle\!\langle \gamma^\alpha\gamma^\beta\gamma_\beta\rangle\!\rangle_0 \langle\!\langle \gamma_\alpha\gamma^\mu\gamma_\mu\rangle\!\rangle_0 = -\frac{1}{576}\langle\!\langle \Delta\gamma^\beta\gamma_\beta\rangle\!\rangle_0,\end{aligned}$$

where

$$\Delta := \gamma^\alpha \circ \gamma_\alpha.$$

On the other hand, since $\mathcal{S}\circ\Delta = \Delta$, by equation (2.9),

$$\langle\!\langle T^4(\mathcal{S})\rangle\!\rangle_2 = \frac{1}{1152}\langle\!\langle \Delta\gamma^\alpha\gamma_\alpha\rangle\!\rangle_0.$$

By equation (3.6), this proves the $g = k = 2$ case of Conjecture B.

Now we consider the $g = k = 3$ case. By Proposition 3.2 and the genus-1 TRR, we have

$$\begin{aligned}\frac{1}{2}\Phi_{3,2} &= \sum_{j=0}^{4}(-1)^j \langle\!\langle T^j(\gamma^\alpha)\rangle\!\rangle_1 \langle\!\langle T^{4-j}(\gamma_\alpha)\rangle\!\rangle_2 \\ &= \langle\!\langle \gamma^\alpha\rangle\!\rangle_1 \langle\!\langle T^4(\gamma_\alpha)\rangle\!\rangle_2 - \langle\!\langle T(\gamma^\alpha)\rangle\!\rangle_1 \langle\!\langle T^3(\gamma_\alpha)\rangle\!\rangle_2.\end{aligned}$$

By the genus-1 TRR and equations (2.8) and (2.9),

$$\begin{aligned}\frac{1}{2}\Phi_{3,2} &= \frac{1}{1152}\langle\!\langle \gamma^\alpha\rangle\!\rangle_1 \langle\!\langle (\gamma_\alpha\circ\Delta)\gamma^\beta\gamma_\beta\rangle\!\rangle_0 \\ &\quad -\frac{1}{24}\langle\!\langle \gamma^\alpha\gamma^\mu\gamma_\mu\rangle\!\rangle_0 \left\{\frac{1}{20}\langle\!\langle (\gamma_\alpha\circ\Delta)\rangle\!\rangle_1 + \frac{1}{1152}\langle\!\langle \gamma_\alpha\gamma^\beta\gamma_\beta\Delta\rangle\!\rangle_0 \right. \\ &\qquad \left. + \frac{1}{480}\langle\!\langle (\gamma_\alpha\circ\gamma^\beta)\gamma_\beta\gamma^\nu\gamma_\nu\rangle\!\rangle_0\right\} \\ &= -\frac{7}{5760}\langle\!\langle (\Delta\circ\Delta)\rangle\!\rangle_1 - \frac{1}{27648}\langle\!\langle \Delta\Delta\gamma^\alpha\gamma_\alpha\rangle\!\rangle_0 \\ &\quad -\frac{1}{11520}\langle\!\langle (\Delta\circ\gamma^\alpha)\gamma_\alpha\gamma^\beta\gamma_\beta\rangle\!\rangle_0.\end{aligned}$$

On the other hand, since $\mathcal{S}\circ\gamma_\alpha = \gamma_\alpha$ and $\langle\!\langle \mathcal{S}\gamma_\alpha\gamma_\beta\gamma_\mu\rangle\!\rangle_0 = 0$ for all α, β and μ, by equation (2.11), we have

$$\begin{aligned}\langle\!\langle T^6(\mathcal{S})\rangle\!\rangle_3 &= \frac{7}{5760}\langle\!\langle (\Delta\circ\Delta)\rangle\!\rangle_1 + \frac{1}{27648}\langle\!\langle \Delta\Delta\gamma^\alpha\gamma_\alpha\rangle\!\rangle_0 \\ &\quad + \frac{1}{11520}\langle\!\langle (\Delta\circ\gamma^\alpha)\gamma_\alpha\gamma^\beta\gamma_\beta\rangle\!\rangle_0.\end{aligned}$$

So the $g = k = 3$ case of Conjecture B follows from equation (3.6). This finishes the proof of Theorem 0.2. □

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