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HOLOMORPHIC SHADOWS IN THE EYES OF MODEL THEORY

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Abstract. We define a subset of an almost complex manifold (M, J) to be a *holomorphic shadow* if it is the image of a J-holomorphic map from a compact complex manifold. Notice that a J-holomorphic curve is a holomorphic shadow, and so is a complex subvariety of a compact complex manifold.

We show that under some conditions on an almost complex structure J on a manifold M, the holomorphic shadows in the Cartesian products of (M, J) form a Zariski-type structure. Checking this leads to non-trivial geometric questions and results. We then apply the work of Hrushovski and Zilber on Zariski-type structures.

We also restate results of Gromov and McDuff on J-holomorphic curves in symplectic geometry in the language of shadows structures.

1. Introduction

An almost complex manifold (M, J) is a manifold M with a complex structure J on the fibers of the tangent bundle TM. A smooth (C^∞) map is J-holomorphic if for every point in the domain the differential is a complex linear map between the tangent spaces. In this paper we study almost complex manifolds and J-holomorphic maps using the framework of Zariski-type structures from model theory.

Zariski-type structures were introduced and studied by Zilber and Hrushovski [Zi1, ZP], [HZ1], [HZ2] in their study of strongly minimal sets. A *Zariski-type structure* is a set M with a collection of compatible Noetherian topologies, one on each M^n for $n \in \mathbb{N}$, with an assignment of *dimension* to the closed sets, satisfying certain conditions that are reasonable to require if we think of the closed sets as subvarieties. A topology is called *Noetherian* if the Descending Chain Condition holds for closed sets; by *compatible* we mean that the coordinate projections are continuous and closed. Zilber showed that such a structure admits elimination of quantifiers, which is essential in applications of abstract model theory in concrete areas of mathematics. A motivating example is given by taking the complex subvarieties of Cartesian products of a compact complex manifold to be the closed sets and the dimension to be the complex dimension.

One motivation to our study is to give a good definition for an almost complex subvariety; namely, a holomorphic shadow. The interpretation of "good" is according to the axioms of a Zariski-type structure. Then we can apply results from model theory to characterize an almost complex manifold by properties of the structure of holomorphic shadows in its Cartesian products.

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In Section 2 we give the necessary background on almost complex manifolds and J-holomorphic maps. We denote by $\mathcal{J}_{sp}$ the set of almost complex structures J on M such that there are no J-holomorphic maps from complex manifolds to (M,J) except for curves and constant maps (see (1)). We define a holomorphic shadow and the holomorphic shadows structure. In Section 3 we present the axioms of a Zariski-type structure, as defined by [Zi1], [Zi2], and prove that for $J \in \mathcal{J}_{sp}$ the holomorphic shadows in Cartesian products of (M,J) form a Zariski-type structure. At the end of Section 3 we give an immediate application of the work of Hrushovski and Zilber to our case.

Another goal of our study is to restate results from Gromov's theory of J-holomorphic curves in symplectic manifolds in the language of shadows structures, as we do in Section 4.

2. Almost complex manifolds and holomorphic shadows

Almost complex manifolds and maps. An *almost complex structure* on a $2n$-dimensional manifold X is an automorphism of the tangent bundle, $J\colon TX \to TX$, such that $J^2 = -\operatorname{Id}$. The pair (X,J) is called an *almost complex manifold*. An almost complex structure is *integrable* if it is induced from a complex manifold structure. In dimension two any almost complex manifold is integrable (see, e.g., [MS1, Theorem 4.16]). In higher dimensions this is not true [Ca]. A submanifold Y of X is called an *almost complex submanifold* if $JTY = TY$. We denote by $\mathcal{J}(X)$ the space of all almost complex structures on X with the C^∞ topology.

A smooth (C^∞) map $f\colon X_1 \to X_2$ is called *J-holomorphic* if for all $p \in X_1$ the differential $df_p\colon T_p(X_1) \to T_{f(p)}(X_2)$ is a complex linear map, i.e.,

$$df_p \circ J_{1p} = J_{2f(p)} \circ df_p.$$

This coincides with the Cauchy Riemann equations if (X_1,J_1) and (X_2,J_2) are complex manifolds. The equation for holomorphic maps between two almost complex manifolds becomes overdetermined as soon as the complex dimension of the domain exceeds one, so for a generic almost complex structure J on a manifold X, there should not be any almost complex submanifolds of complex dimension strictly greater than one. We denote by

$$\mathcal{J}_{sp} \tag{1}$$

the subset of almost complex structures such that for every J-holomorphic map from a holomorphic disc $\mathbb{C}^k$ to (X,J) there is a neighbourhood of 0 in $\mathbb{C}^k$ such that the map factors through $\mathbb{C}^k \to \mathbb{C}$.

When the domain of a J-holomorphic map is a compact Riemann surface (i.e., a compact one-dimensional complex manifold), we call the map a *parameterized J-holomorphic curve* and its image a *J-holomorphic curve*. When the domain is $\mathbb{CP}^1$, with the standard complex structure, the map is a *parametrized J-holomorphic sphere*. A J-holomorphic map is called *simple* if it cannot be factored through a branched covering of the domain. In general, a J-holomorphic curve cannot be represented as the common zeroes of J-holomorphic functions into $\mathbb{C}$, not even locally. This makes the notion of an almost complex variety tricky.

2.1. Let (X,J) be an almost complex manifold. Let

$$\Sigma_k = (\Sigma_k, j)$$

be a compact complex manifold of complex dimension k. The J-holomorphic maps from (Σ_k, j) to (X, J) are the maps satisfying

$$\bar{\partial}_J(u) = 0,$$

where

$$\bar{\partial}_J(u) := \frac{1}{2}(du + J \circ du \circ j).$$

Let $A \in H_{2k}(X; \mathbb{Z})$ be a homology class. The $\bar{\partial}_J$ operator defines a section $S\colon \mathcal{B} \to \mathcal{E}$ by

$$S(u) := (u, \bar{\partial}_J(u)),$$

where $\mathcal{B} \subset C^\infty(\Sigma_k, X)$ denotes the space of all smooth maps $u\colon \Sigma_k \to X$ that represent the homology class A, and the bundle $\mathcal{E} \to \mathcal{B}$ is the infinite-dimensional vector bundle whose fiber at u is the space $\mathcal{E}_u = \Omega^{0,1}(\Sigma_k, u^*TX)$ of smooth J-antilinear 1-forms on Σ_k with values in u^*TX. The moduli space

$$\mathcal{M}(A, \Sigma_k, J) = \{u \mid u \text{ is a } (j, J)\text{-holomorphic map } \Sigma_k \to X \text{ in } A\}$$

is the zero set of this section. Denote by

$$D_u = DS(u)\colon \Omega^0(\Sigma_k, u^*TX) \to \Omega^{0,1}(\Sigma_k, u^*TX) \tag{2}$$

the composition of the differential $dS(u)\colon T_u\mathcal{B} \to T_{(u,0)}\mathcal{E}$ with the projection

$$\pi_u\colon T_{(u,0)} = T_u\mathcal{B} \oplus \mathcal{E}_u \to \mathcal{E}_u.$$

The operator D_u is the *vertical differential* of the section S at u.

If $k = 1$, then D_u is a real linear Cauchy Riemann operator. When $k > 1$, the image of the map (2) is of infinite codimension.

Consider the universal moduli space

$$\mathcal{M}(A, \Sigma_k, \mathcal{J}) = \{(u, J) \mid J \in \mathcal{J},\ u \text{ is a } (j, J)\text{-holomorphic map } \Sigma_k \to X \text{ in } A\}.$$

Here $\mathcal{J}$ is an open subset of $\mathcal{J}(X)$. When X has a symplectic form ω (see Section 4), we can take $\mathcal{J}$ to be $\mathcal{J}(X, \omega)$, the space of all ω-tame almost complex structures on X.

Consider the projection map

$$p_A\colon \mathcal{M}(A, \Sigma_k, \mathcal{J}) \to \mathcal{J}.$$

The differential dp_A at a point (u, J) is essentially the operator D_u and is surjective at (u, J) when D_u is onto.

We have the following consequence of the Sard-Smale theorem, the infinite-dimensional implicit function theorem, and the ellipticity of the Cauchy-Riemann equations. When $k = 1$, the set $\mathcal{J}_{reg}(A)$ of regular values for p_A is of the second category in $\mathcal{J}$. For any $J \in \mathcal{J}_{reg}(A)$, the space of simple J-holomorphic Σ_1-curves in A is a smooth manifold of dimension $2c_1(A) + n(2 - 2g)$, where c_1 is the first Chern class of the complex vector bundle (TX, J), $2n$ is the dimension of X, and g is the genus of Σ_1 [MS2, Theorem 3.1.5]. When $k > 1$, for a generic J the space $\mathcal{M}(A, \Sigma_k, J)$ is empty.

Holomorphic shadows.

2.2. **Definition.** A subset of an almost complex manifold (X, J) is a *holomorphic shadow*[1] if it is the image of a J-holomorphic map from a compact complex analytic manifold.

2.3. *Remark.* As a compact set in a Hausdorff space, each holomorphic shadow is closed in the C^∞ topology on X.

2.4. For a complex analytic subvariety V of a complex analytic manifold (M, J_M), i.e., a subset given locally as the common zeros of a finite collection of holomorphic functions, we say that a map $f\colon V \to X$ is J-holomorphic if for one (hence every) resolution of V to a complex analytic manifold $\tilde{V}$, $\phi\colon \tilde{V} \to V$, the map $f \circ \phi$ is a proper J-holomorphic map from the complex analytic manifold $\tilde{V}$.

By [Hi], every complex analytic subvariety admits a resolution of singularities, i.e., a map $\phi\colon \tilde{V} \to V$, such that $\tilde{V}$ is a complex analytic manifold, the preimage of the non-singular points of V is a dense subset in $\tilde{V}$ on which ϕ is an isomorphism, and ϕ is a proper map (in particular, if V is compact so is $\tilde{V}$). On the other hand, by the proper mapping theorem, an image of a complex analytic subvariety by a proper holomorphic map is a complex analytic subvariety. As a result we get the following claim.

2.5. *Claim.* A subset of a compact complex analytic manifold is a complex analytic subvariety if and only if it is a holomorphic shadow.

For a compact complex analytic manifold M, taking the complex analytic subvarieties of M^n, $n \in \mathbb{N}$, to be the closed subsets and the dimension to be the complex dimension gives a Zariski-type structure. This follows from standard facts in complex geometry, as observed by B. Zilber [ZP]. We show a similar claim in the non-integrable case.

2.6. **Definition.** Given an almost complex $2r$-manifold (X, J) and a collection $\mathcal{H}$ of holomorphic shadows in the finite Cartesian products of (X, J), we consider the collection of:

- the holomorphic shadows in $\mathcal{H}$,
- the diagonals, i.e., sets of the form
$$\Delta^n{}_{(i_1,\ldots,i_k)} = \{\bar{x} \in X^n \mid x_{i_1} = \ldots = x_{i_k}\},$$
- subsets of X^n of the form $S \times D_1 \times \ldots \times D_k$, where $S \in \mathcal{H}$ is a shadow in X^l, each D_i is a diagonal in X^{d_i}, and $\sum_{i=1}^k d_i = n - l$,
- the images of sets as above under permutations of the coordinates,
- finite unions of the above sets.

We denote this collection $\mathcal{S}_{(X,J,\mathcal{H})}$.

Notation. When $\mathcal{H}$ is the collection of all holomorphic shadows in the finite Cartesian products of (X, J), we write $\mathcal{S}_{(X,J)}$ for $\mathcal{S}_{(X,J,\mathcal{H})}$. We call $\mathcal{S}_{(X,J)}$ the *holomorphic shadows structure*.

[1] We follow an attempt of Hardt [Ha] to give the name *semianalytic shadows* to subanalytic sets.

The holomorphic shadows structure admits a natural (partial) dimension function:

- the dimension of a point is 0;
- the dimension of a non-constant J-holomorphic curve is 1.

2.7. **Theorem.** *Let (X, J) be an almost complex manifold. Assume that $J \in \mathcal{J}_{sp}$. Then there exists a dimension function* dim *on $\mathcal{S}_{(X,J)}$ that is consistent with the natural partial dimension above, such that $(X, \mathcal{S}_{(X,J)}, \dim)$ is a Zariski-type structure that satisfies the essential uncountability (EU) property.*

The set $\mathcal{J}_{sp}$ is defined in (1). We will give explicit definitions and prove the theorem in the next section.

3. Zariski-type structures

A Zariski-type structure, as defined in [Zi1], is a set X with a collection $\mathcal{C}$ of subsets of its Cartesian products, X^n, to be called *Z-closed* sets, and a dimension assignment to $\mathcal{C}$, such that:

- (L1) The set X is Z-closed.
- (L2) Each point is Z-closed.
- (L3) Cartesian products of Z-closed sets are Z-closed.
- (L4) The diagonals are Z-closed.
- (L5) Finite unions and intersections of Z-closed sets are Z-closed.
- (P) Any of the coordinate projections

  $$\mathrm{pr}_{i_1,\dots,i_m} \colon (x_1, \dots, x_n) \mapsto (x_{1_1}, \dots, x_{i_m}),\ i_1, \dots, i_m \in \{1, \dots, n\}$$

  are closed and continuous, i.e., the images and inverse images of Z-closed sets under these projections are Z-closed;
- (DCC) Descending Chain Condition for Z-closed sets: For any Z-closed

  $$C_1 \supseteq C_2 \supseteq \ldots \supseteq C_i \supseteq \ldots$$

  there is k such that $C_i = C_k$ for all $i \geq k$.

This condition implies that for any Z-closed C there are Z-closed $C_1, \dots, C_m$ that are distinct and where none is a subset of the other, such that $C = C_1 \cup \ldots \cup C_m$, where m is maximal. These C_i are the *irreducible components* of C. They are defined up to permutation uniquely.

A Z-closed set S is called *irreducible* if there are no Z-closed subsets $S_1, S_2 \subsetneq S$ such that $S = S_1 \cup S_2$.

To any Z-closed subset C there is attached a natural number, called $\dim C$, such that:

- (DP) Dimension of a Point is 0.
- (DU) Dimension of a Union: $\dim(C_1 \cup C_2) = \max\{\dim C_1, \dim C_2\}$.
- (DI) $\dim C_1 < \dim C$ for C irreducible, $C_1 \subseteq C, C_1 \neq C$.
- (FC) For any $k \in \mathbb{N}$, any Z-closed $C \subset X^n$, and projection $pr \colon X^n \to X^m$, the set

  $$p(C, k) = \{a \mid \dim(C \cap pr^{-1}(a)) > k\}$$

  is constructible.

- (ADF) For any irreducible Z-closed $C \subset X^n$ and projection $pr \colon X^n \to X^m$,
$$\dim C = \dim pr(C) + \min_{a \in pr(C)} \dim(pr^{-1}(a) \cap C).$$

A *constructible* set is a finite Boolean combination of Z-closed sets.

We will call these axioms the *Z axioms*.

Other properties that will be relevant are the following:

- (EU) Essential Uncountability: If a Z-closed $C \subseteq X^n$ is a union of countably many Z-closed subsets, then there are finitely many among the subsets whose union is C. This implies that if X is not finite it must be uncountable.
- (PS) Pre-smoothness: For any irreducible Z-closed $S_1, S_2 \subseteq X^n$, the dimension of any irreducible component of $S_1 \cap S_2$ is no less than $\dim(S_1) + \dim(S_2) - \dim X^n$.

In the Zariski-type structure in which the Z-closed sets are the complex subvarieties of Cartesian products of a compact complex manifold and the dimension is the complex analytic dimension, the properties (EU) and (PS) are satisfied [Zi1], [Zi2].

Zilber [ZP], [Zi2] showed that any Zariski-type structure admits elimination of quantifiers: the projection of a constructible set is constructible.

Proof of Theorem 2.7. It follows from the definition of a J-holomorphic map that

3.1. *Claim.*
- J-holomorphic maps are closed under disjoint union, Cartesian product and composition (when defined).
- The canonical coordinate projections $\pi \colon X^{n+k} \to X^n$ are J-holomorphic.

As a corollary we get the following claim.

3.2. *Claim.*
(1) A finite union of holomorphic shadows is a holomorphic shadow.
(2) A finite Cartesian product of holomorphic shadows is a holomorphic shadow.
(3) The image of a holomorphic shadow under a J-holomorphic map is a holomorphic shadow.
(4) The image of a holomorphic shadow under the canonical coordinate projection $X^{n+k} \to X^n$ is a holomorphic shadow.

To continue, we show the following lemma.

3.3. **Lemma.** *Let (X, J) be an almost complex manifold. Assume that $J \in \mathcal{J}_{sp}$. Then any J-holomorphic map $f \colon M_f \to X^{n(f)}$ from a compact complex manifold to a Cartesian product of (X, J) satisfies the "pulling back diagonals property": the preimage of any diagonal $\Delta^{n(f)}_{i_1,\dots,i_k}$ is a complex subvariety of M_f.*

Proof. We first show that the pulling back diagonals property holds for J-holomorphic maps of the form

$$\prod_{j=1}^{n} g_j \colon \prod_{j=1}^{n} \Sigma^{(j)} \to X^n, \tag{3}$$

where the $\Sigma^{(j)}$-s are compact connected Riemann surfaces and the maps $g_j \colon \Sigma^{(j)} \to X$ are J-holomorphic and satisfy the following assumptions:

$$g_j \text{ is simple } \forall j \tag{4}$$

and

$$g_{i_1}(\Sigma^{i_1}) = g_{i_2}(\Sigma^{i_2}) \Rightarrow \Sigma^{(i_1)} = \Sigma^{(i_2)} \text{ and } g_{i_1} = g_{i_2}. \tag{5}$$

Indeed, for an almost complex manifold (X, J), for two J-holomorphic maps $g_i\colon \Sigma^{(i)} \to X$, $i = 1, 2$, from compact connected Riemann surfaces $\Sigma^{(i)}$, either the set

$$(g_1, g_2)^{-1}(\Delta^2_{1,2}(X)) = \{(z_1, z_2) \in \Sigma^{(1)} \times \Sigma^{(2)} \mid g_1(z_1) = g_2(z_2)\}$$

consists of finitely many points or

$$g_1(\Sigma^{(1)}) = g_2(\Sigma^{(2)}), \text{ hence by (5), } \Sigma^{(i_1)} = \Sigma^{(i_2)} = \Sigma \text{ and } g_{i_1} = g_{i_2}.$$

For a simple J-holomorphic $g\colon \Sigma \to X$ from a compact Riemann surface,

$$(g, g)^{-1}(\Delta^2_{1,2}(X)) = \Delta^2_{1,2}(\Sigma) \cup S,$$

where S consists of finitely many points. See [MS2, Theorem E.1.2, Exercise E.1.4] and [MW]. By induction, this implies that the pulling back diagonals property holds for maps of the form (3).

Now, for $J \in \mathcal{J}_{sp}$, a holomorphic shadow in (X, J) is either a J-holomorphic curve or a point. Hence for a holomorphic shadow $S \subset (X^n, J^n)$, the projection of S on any of the coordinates is either a point or a J-holomorphic curve. So every J-holomorphic map $f\colon M \to X^n$ from a compact complex manifold M decomposes as

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "h"] & f(M) \arrow[r, "\subset"] \arrow[d, "\prod_{j=1}^n \mathrm{pr}_j|_{f(M)}"] & X^n \\
\prod_{j=1}^n \Sigma^{(j)} \arrow[r, "\prod_{j=1}^n g_j"'] & \prod_{j=1}^n C_j \arrow[r, "\subset"'] & X^n
\end{tikzcd}$$

where $h\colon M \to \prod_{j=1}^n \Sigma^j$ is a holomorphic map, and for all j, $g_j\colon \Sigma^{(j)} \to X$ is a J-holomorphic map from a compact Riemann surface $\Sigma^{(j)}$. Some of these maps might be constant; in that case replace $\Sigma^{(j)}$ with a point. We can also assume that (4) and (5) hold. This reduces this case to the special case (3) discussed above. □

We show that the pulling back diagonals property implies being closed under finite intersections, the Descending Chain Condition, the fact that the image of an irreducible set under a coordinate projection is irreducible, and the Essential Uncountability in $\mathcal{S}_{(X,J)}$.

Notation. In this subsection, $\mathcal{F}$ is the collection of all J-holomorphic maps from compact complex manifolds to Cartesian products of (X, J) for $J \in \mathcal{J}_{sp}$, and $\mathcal{H}$ denotes the collection of all holomorphic shadows in the finite Cartesian products of (X, J) for $J \in \mathcal{J}_{sp}$.

3.4. *Claim.* Consider

$$f_1\colon M_1 \to X^n,$$
$$f_2\colon M_2 \to X^n$$

maps in $\mathcal{F}$. Then ${f_1}^{-1}[f_1(M_1) \cap f_2(M_2)]$ is a complex subvariety of M_1.

Proof. By the "pulling back diagonals" property, $Z = (f_1 \times f_2)^{-1}[\Delta_{X^n}]$ is a complex subvariety of the complex manifold $M_1 \times M_2$.

The preimage ${f_1}^{-1}[f_1(M_1) \cap f_2(M_2)]$ is the image of the complex subvariety Z by the canonical projection $\pi_1 \colon M_1 \times M_2 \to M_1$. Hence, by the proper mapping theorem, it is a complex subvariety of M_1. □

Now, the intersection $f_1(M_1) \cap f_2(M_2)$ is the image of $f_1 \circ \pi_1 \circ \phi$, where $\phi \colon \tilde{Z} \to Z$ is a resolution of $Z = (f_1 \times f_2)^{-1}[\Delta_{X^n}]$ to a compact complex analytic manifold $\tilde{Z}$.

As a result,

3.5. **Corollary.** *The holomorphic shadows in $\mathcal{H}$ are closed under finite intersections.*

Similarly, consider

$$f_1 \colon M_1 \to X^{n+k},$$
$$f_2 \colon M_2 \to X^n$$

in $\mathcal{F}$, and the coordinate projection map

$$\mathrm{pr}_{1,\dots,n} \colon X^{n+k} \to X^n.$$

The preimage $Z = (\mathrm{pr}_{1,\dots,n} \circ f_1, f_2)^{-1}[\Delta_{X^n}]$ is a complex subvariety of the complex manifold $M_1 \times M_2$. Hence its image by f_1 composed on the canonical projection $M_1 \times M_2 \to M_1$ (composed on a resolution of Z) is a holomorphic shadow $\in \mathcal{H}$ in X^{n+k}.

3.6. **Corollary.** *For $S_1 \in \mathcal{H}$ a holomorphic shadow in X^{n+k} and $S_2 \in \mathcal{H}$ a holomorphic shadow in X^n, the intersection $S_1 \cap S_2 \times X^k$ is a holomorphic shadow $\in \mathcal{H}$ in X^{n+k}.*

3.7. **Corollary.** *Let $A \in \mathcal{H}$ be a holomorphic shadow in X^n. Let D be a diagonal in X^n. Then the intersection $A \cap D$ is a holomorphic shadow in X^n.*

Proof. A is the image under a J-holomorphic map from a compact complex manifold M into X^n. Then $A \cap D$ is the image under f of $f^{-1}[D]$. By Lemma 3.3, $f^{-1}[D]$ is a complex subvariety in M. Hence, this image is a holomorphic shadow. □

3.8. *Claim.* Let $S \in \mathcal{H}$ be a holomorphic shadow in X^{n+m}. Then the inverse image of a holomorphic shadow $C \subseteq X^n$ under the projection $\mathrm{pr}_{1,\dots,n}|_S \colon S \to X^n$ is a holomorphic shadow in $\mathcal{H}$.

Proof. Given

$$M_S \xrightarrow{f_S} S \xrightarrow{\mathrm{pr}_{1,\dots,n}|_S} X^n,$$

set

$$A = \mathrm{pr}_{1,\dots,n}(S) \cap C \subseteq X^n.$$

By Claim 3.4, A is the image under f_S of a complex subvariety $M_A \subseteq M_S$. It remains to notice that $f_S(M_A) = \mathrm{pr}_{1,\dots,n}|_S{}^{-1}(C)$. □

3.9. *Claim.* The descending chain condition holds for holomorphic shadows in $\mathcal{H}$.

Proof. Consider a descending chain

$$S_1 \supseteq S_2 \supseteq \dots \supseteq S_i \supseteq \dots$$

of holomorphic shadows $S_i = f_i(M_i)$ in $\mathcal{H}$. By Claim 3.4,

$$M_1 = {f_1}^{-1}[S_1] \supseteq {f_1}^{-1}[S_1 \cap S_2] \supseteq \ldots \supseteq {f_1}^{-1}[S_1 \cap S_i] \supseteq \ldots$$

is a descending chain of complex subvarieties of M_1. Since $S_1 \cap S_i = S_i$, we get

$${f_1}^{-1}[S_1] \supseteq {f_1}^{-1}[S_2] \supseteq \ldots \supseteq {f_1}^{-1}[S_i] \supseteq \ldots .$$

By the descending chain condition for complex subvarieties of a compact complex manifold (see, e.g., [ZP]), there is k such that for all $i \geq k$, ${f_1}^{-1}[S_i] = {f_1}^{-1}[S_k]$; hence so are their images under f_1, i.e., $S_i = S_k$. □

This implies that for any holomorphic shadow S in $\mathcal{H}$ there are distinct holomorphic shadows $S_1, \ldots, S_m$ in $\mathcal{H}$ such that $S = S_1 \cup \ldots \cup S_m$, where m is maximal. These S_i are the *irreducible components* of S.

We make the following observation.

3.10. *Claim.* The image of an irreducible holomorphic shadow $C \in \mathcal{H}$ under a projection $pr\colon X^{n+m} \to X^n$ is an irreducible holomorphic shadow.

Proof. Otherwise $pr(C) = S_1 \cup S_2$, where S_1 and S_2 are distinct holomorphic shadows that $\neq pr(C)$. By Claim 3.8, ${pr|_C}^{-1}(S_1)$ and ${pr|_C}^{-1}(S_2)$ are holomorphic shadows in $\mathcal{H}$, they are distinct, $\neq C$ and their union equals C, to obtain a contradiction. □

3.11. *Claim* (EU). If a holomorphic shadow $S \in \mathcal{H}$ is a union of countably many holomorphic shadows in $\mathcal{H}$, then there are finitely many among the subsets whose union is S.

Proof. Given $f \in \mathcal{F}$ such that

$$f(M_f) = S = \bigcup_{i \in \mathbb{N}} S_i,$$

where S_i are holomorphic shadows in $\mathcal{H}$, then

$$S \cap S_i = S_i,$$

so

$$M_f = f^{-1}[S] = f^{-1}[\bigcup_{i \in \mathbb{N}} S_i] = \bigcup_{i \in \mathbb{N}} f^{-1}[S_i] = \bigcup_{i \in \mathbb{N}} f^{-1}[S \cap S_i].$$

By Claim 3.4, for all i, the set $f^{-1}[S \cap S_i]$ is a complex subvariety of M.

By the (EU) claim for complex subvarieties in a compact complex manifold (see [ZP]), there are finitely many among the subsets $f^{-1}[S \cap S_i]$ whose union is M. Hence there are finitely many among the subsets S_i whose union is S. □

To complete the proof of Theorem 2.7, we need to define a dimension of a holomorphic shadow and show that it satisfies the dimension axioms. For that we need the following lemma.

3.12. **Lemma.** *Let A be a holomorphic shadow in $\mathcal{H}$. Then there is a subset U_A in A that satisfies the following:*

(1) *U_A is isomorphic to a complex manifold in the almost complex sense. In particular, it is an integrable submanifold.*
(2) *U_A is dense in A, in the usual (C^∞) topology.*
(3) *The set $A \setminus U_A$ is contained in a holomorphic shadow P_A in $\mathcal{H}$ that is a proper subset of A.*

Proof. By definition, A is the image of a compact complex manifold M under a J-holomorphic map

$$f\colon M \to X^n.$$

By the "pulling back diagonals property", the inverse image under $(f,f)\colon M\times M \to X^n \times X^n$ of the diagonal of $X^n \times X^n$ is a complex subvariety in $M \times M$, i.e., the relation $\sim_f$, where

$$m_1 \sim_f m_2 \Leftrightarrow f(m_1) = f(m_2)$$

is a complex equivalence relation in M. Moosa [Mo2] showed that in this case there exist a degenerate complex subvariety P of M (i.e., the intersection of P with each of the irreducible components of M is a proper subvariety of the component, hence of a lower dimension), a compact complex analytic space N with a degenerate complex subvariety Q of N, and a holomorphic map

$$g\colon U = M \smallsetminus P \to N$$

such that $N - g(U) \subset Q$ and for all $a, b \in U$, $g(a) = g(b)$ if and only if $a \sim_f b$. The set $V = N \smallsetminus Q$ is a Zariski-open set that is dense in N and contained in $g(U)$. By replacing U with $g^{-1}V = M \smallsetminus P \smallsetminus g^{-1}Q$ we can assume that $g(U) = V$. By reducing U and V, we can assume that g is a submersion, i.e., for any $u \in U$, dg_u is onto $T_{g(u)}V$. By the local submersion theorem, for any $u \in U$ there are holomorphic local coordinates around u and $g(u)$ such that $g(u_1, \ldots, u_k) = (u_1, \ldots, u_l)$.

We define

$$h\colon V \to X^n$$

by $h(c) = x$ if there exists $m \in g^{-1}(c)$ such that $f(m) = x$. In the holomorphic local coordinates of U and $V = g(U)$ chosen above, $h(u_1, \ldots, u_l) = f(u_1, \ldots, u_k)$, i.e., locally f is the composition of h with the canonical submersion. Then, h is a well defined one-to-one map that is smooth. Since

$$J_{X^n} dh \circ dg = J_{X^n} d(h\circ g) = J_{X^n} df = df J_M = d(h\circ g) J_{X^n} = dh \circ dg J_M = dh J_N \circ dg$$

and dg is a onto, h is J-holomorphic.

The zero set Z of the holomorphic function $\det h\colon V \to \mathbb{C}$ is a proper complex subvariety of V (of lower dimension). Replacing V by $V \smallsetminus Z$ we get that the map h^{-1} is also J-holomorphic; see Lemma 3.13. The set

$$U_A = h(g(U))$$

is dense in $A = f(M)$ since

$$A = f(M) = f(\mathrm{cl}_{\mathbb{C}}(U)) \subseteq \mathrm{cl}(f(U)) = \mathrm{cl}(h(g(U))),$$

where $\mathrm{cl}_{\mathbb{C}}(\cdot)$ is the Zariski-closure and $\mathrm{cl}(\cdot)$ is closure in the C^∞-topology. $A \smallsetminus U_A$ is contained in the image P_A of f restricted to $P = M \smallsetminus U$. □

Notation. We will call such U_A an *umbra* of the holomorphic shadow A, and call *penumbra* a holomorphic shadow as in part (3) of the lemma. The compact complex variety $N_A = N$ in which $h^{-1}[U_A] = V_A$ is dense is called a *shadow caster* of A. We call $g\colon U \to N$ the *map induced by the complex equivalence relation* $\sim_f$.

3.13. **Lemma** (The inverse function theorem in the almost complex category)**.** *Suppose that $f\colon X \to Y$ is a J-holomorphic map whose derivative df_p at the point p is an isomorphism. Then f is a local J-holomorphic isomorphism at p.*

Proof. Suppose that $f\colon X \to Y$ is a J-holomorphic map whose derivative df_p at the point p is an isomorphism. By the inverse function theorem in the smooth category, locally there exists an inverse map f^{-1} to f. It is enough to notice that

$$df_p \circ J_{1p} = J_{2f(p)} \circ df_p$$

implies

$$J_{1p} \circ {df_p}^{-1} = {df_p}^{-1} \circ J_{2f(p)},$$

i.e., f^{-1} is J-holomorphic. $\square$

3.14. *Remark.* Let M, N be compact complex manifolds, $U \subset M$, $V \subset N$ open dense subsets, and $g\colon U \to V$ a holomorphic map whose graph $G \subset M \times N$ is constructible, i.e., it is a Boolean combination of complex analytic subvarieties of $M \times N$. Resolve Γ to a compact complex manifold $\tilde{\Gamma}$ by Hironaka's [Hi] resolution of singularities $\phi\colon \tilde{\Gamma} \to \Gamma$. The set U is naturally embedded in $\Gamma \subset M \times N$. The proper transform $\phi^{-1}U$ of U is a Zariski-open dense subset of $\tilde{\Gamma}$. By restricting U we may assume that $\phi|_{\phi^{-1}U}$ is an isomorphism onto U. Composing $\phi|_{\phi^{-1}U}$ with the projection $\pi_N\colon M \times N \to N$, we get the map g. Thus the holomorphic map $\tilde{g} = \pi_N \circ \phi$ can be considered an expansion of g.

Now, consider a holomorphic shadow $A \subset X^n$ (in $\mathcal{H}$) that is the image of a J-holomorphic $f\colon M \to X^n$ (in $\mathcal{F}$). The map $g\colon U \to N$ induced by the complex equivalence relation $\sim_f$ is constructible. (See [Mo1, Section 2.2].) Hence (by expanding g to $\tilde{g}$ and replacing M by $\tilde{\Gamma}$) we may assume that the map induced by $\sim_f$ is $g\colon M \to N$.

3.15. *Remark.* Given a J-holomorphic $f\colon M \to X^{n+k}$ and a projection $\pi\colon X^{n+k} \to X^n$, let U_S and $U_{\pi(S)}$ be umbras constructed as in Lemma 3.12. We can assume that the restriction of π to U_S is a holomorphic and proper projection onto $U_{\pi(S)}$. To see this, first apply Lemma 3.12 and Remark 3.14 to get $g_1\colon M \to N_1$ and $g_2\colon M \to N_2$, such that for $i = 1, 2$, the map g_i restricted to a Zariski-open dense subset U of M is a submersion onto a Zariski-open dense subset V_i of N_i and that ${g_i|_U}^{-1}(V_i) = U$. Locally (up to a holomorphic isomorphism), there are systems of holomorphic coordinates in which $g_i\colon U \to V_i$ is given by the projection $(u_1, \ldots, u_l) \to (u_1, \ldots, u_{k_i})$, where $l > k_1 > k_2$. This gives a map $\psi\colon V_1 \to V_2$, mapping $(u_1, \ldots, u_{k_1})$ to $(u_1, \ldots, u_{k_2})$. Applying Remark 3.14, we expand ψ to $\tilde{\psi}\colon \tilde{N}_1 \to N_2$ between compact complex manifolds (hence proper) using a resolution of singularities $\phi\colon \tilde{N}_1 \to N_1$. By restricting U, V_1, V_2, we assume that $\phi|_{\phi^{-1}V_1}$ is an isomorphism onto V_1 and $\tilde{\psi}^{-1}V_2 = \phi^{-1}V_1$ (we denote $\phi^{-1}V_1$ again by V_1). So the following diagram commutes:

$$\begin{tikzcd}
X^{n+k} & M \arrow[l, "f"'] & U \arrow[l, "\supset"'] \arrow[r, "\mathrm{Id}"] \arrow[d, "\phi^{-1}|_{V_1}\circ g_1"'] & U \arrow[r, "\subset"] \arrow[d, "g_2"] & M \arrow[r, "\pi\circ f"] & X^n \\
& \tilde{N}_1 & V_1 \arrow[l, "\supset"'] \arrow[r, "\tilde{\psi}"] \arrow[ull, "h_1"] & V_2 \arrow[r, "\subset"] \arrow[urr, "h_2"'] & N_2.
\end{tikzcd}$$

The map ψ as the restriction $\tilde{\psi}_{V_1}\colon V_1 \to V_2$ is proper. (A compact set A in the open set V_2 is compact in N_2, so $A' = \tilde{\psi}^{-1}A = \psi^{-1}A$ is compact in N_1 and contained in V_1; hence is compact in V_1.)

3.16. *Claim.* Let A be a holomorphic shadow in $\mathcal{H}$. If A_1 and A_2 are shadow casters (or shadow umbras) of A, then there is a holomorphic isomorphism between open dense subsets of them.

3.17. *Claim.* Let C be a holomorphic shadow in $\mathcal{H}$. Then C is irreducible as a holomorphic shadow if and only if a shadow caster of C is irreducible as a complex variety.

3.18. **Proposition.** *A holomorphic shadow $S \subset X^n$ in $\mathcal{H}$ can be decomposed as*

$$S = S^{(r)} \cup S^{(r-1)} \cup \ldots \cup S^{(0)},$$

where for all i, $S^{(i)}$ is an i-dimensional integrable almost complex submanifold of X^n and $\mathrm{cl}(S^{(i)}) \subseteq S^{(i)} \cup S^{(i-1)} \cup \ldots \cup S^{(0)}$. (Here cl *can be interpreted in the Z-topology or in the C^∞ topology.)*

3.19. **Definition.** If $S^{(r)} \neq \emptyset$, we say that r is the *dimension* of S and denote it by $\dim S$.

The dimension of a holomorphic shadow $C \in \mathcal{H}$ equals the dimension as a complex manifold of a shadow umbra U_C of C, which equals the dimension of a related shadow caster N_C. By Claim 3.16, the dimension of a holomorphic shadow is well defined.

The following claim is clear from our definition of dimension.

3.20. *Claim.* Let C_1 and C_2 be holomorphic shadows in $\mathcal{H}$. Then:

(DP): The dimension of a point is 0;

(DU): $\dim(C_1 \cup C_2) = \max\{\dim C_1, \dim C_2\}$.

Claim 3.17 and axiom (DI) for subvarieties of a compact complex manifold imply the following claim.

3.21. *Claim* (DI). If C_2 is irreducible and $C_1 \subseteq C_2, C_1 \neq C_2$, then $\dim C_1 < \dim C_2$.

3.22. *Claim* (FC). (FC) Let $S \in \mathcal{H}$ be a holomorphic shadow in X^{n+m}. Let pr stand for the projection $X^{n+m} \to X^n$. Then

$$p(S,k) = \{a \in X^n \mid \dim(S \cap pr^{-1}(a)) > k\} \tag{6}$$

is h-constructible. If $k \geq \min \dim(pr^{-1}a \cap S)$, then $p(S,k)$ is contained in a holomorphic shadow that is a proper subset of $pr(S)$.

Notice that by Claim 3.8, there is a meaning to (6).

Notation. We say that a set is *h-constructible* if it is constructible from holomorphic shadows, i.e., of the form $\bigcup_{i \leq k} A_i \setminus B_i$, where k is a natural number, A_i, B_i are holomorphic shadows, and $B_i \subseteq A_i$.

Proof. First, we notice that for the decomposition $S = \bigcup_{i=1}^n S_i$ to be irreducible components, $\dim(pr^{-1}(a) \cap S) = \max \dim(pr^{-1}(a) \cap S_i)$. Hence

$$p(S,k) = \bigcup_{i=1}^{n} p(S_i, pr|_{S_i}, k).$$

So we may assume that S is irreducible. Hence, by Claim 3.10 and Claim 3.17, so are $pr(S)$ and the shadow casters N_S and $N_{pr(S)}$.

By Remark 3.15, we may assume that the restriction of pr to the umbra U_S is a holomorphic and proper projection onto the umbra $U_{pr(S)}$. We identify U_S and

$U_{pr(S)}$ with the corresponding isomorphic Zariski-open sets in the shadow casters N_S and $N_{pr(S)}$. For every $a \in pr(U_S)(= U_{pr(S)})$ we have that $pr^{-1}(a) \cap U_S$ is an umbra in $p^{-1}(a) \cap S$; hence $\dim(pr^{-1}(a) \cap S) = \dim(pr^{-1}(a) \cap U_S)$. Assume that $pr|_{U_S}$ is given by holomorphic functions $p_1, ..., p_l$. For a point in the fiber $U_S \cap pr^{-1}a$, the fact that $\dim(pr^{-1}a \cap U_S) > k$ implies that

(∗) the complex rank of the Jacobian of $(p_1, ..., p_l)$ at b is $\leq \dim U_S - k$.

This condition is equivalent to the vanishing of all $\dim U_S - k$ minors of the Jacobian. Let Z be defined by (*) and let W be the Zariski closure in N_S of Z. Let S' be a penumbra in S such that $S \smallsetminus U_S \subset S'$, and $(pr(S))'$ be a penumbra in $pr(S)$ such that $pr(S) \smallsetminus U_{pr(S)} \subset (pr(S))'$. Then $p(S,k) = p(S',k) \cup ((pr(S) \smallsetminus (pr(S))') \cap f(W))$ (where $f\colon M_S \to X$ is the map giving the holomorphic shadow $S = f(M_S)$). By induction $p(S',k)$ is h-constructible; hence so is $p(S,k)$. □

3.23. *Claim* (ADF). Let $S \in \mathcal{H}$ be an irreducible holomorphic shadow in X^{n+m} and let pr denote the canonical projection $X^{n+m} \to X^n$. Then

$$\dim S = \dim pr(S) + \min_{a \in pr(S)} \{\dim(pr^{-1}(a) \cap S)\}. \tag{7}$$

Proof. By Remark 3.15, we may assume that the restriction of pr to U_S is a proper holomorphic projection onto $U_{pr(S)}$. By the corresponding claim for complex analytic subvarieties in a complex manifold, this is true for shadow umbras U_S and $U_{pr(S)}$. So,

$$\dim U_S = \dim U_{pr(S)} + \min_{a \in U_{pr(S)}} \{\dim(pr^{-1}(a) \cap U_S)\}.$$

For every $a \in pr(U_S)(= U_{pr(S)})$ we have that $pr^{-1}(a) \cap U_S$ is an umbra in $p^{-1}(a) \cap S$; hence $\dim(pr^{-1}(a) \cap S) = \dim(pr^{-1}(a) \cap U_S)$. By Claim 3.22, the minimal dimension of a fiber $pr^{-1}(a) \cap S$ cannot be attained on a holomorphic shadow that is a proper subset of the irreducible shadow $pr(S)$. Therefore,

$$\min_{a \in U_{pr(S)}} \{\dim(pr^{-1}(a) \cap U_S)\} = \min_{a \in pr(S)} \{\dim(pr^{-1}(a) \cap S)\}.$$

Since $\dim S = \dim U_S$ and $\dim pr(S) = \dim U_{pr(S)}$, we get (7). □

3.24. *Remark.* It follows from the axioms that for any holomorphic shadow $S \subseteq X^{n+m}$,

$$\dim S \geq \dim pr(S) + \min_{a \in pr(S)} \{\dim(pr^{-1}(a) \cap S)\}.$$

See Fact 2.2 in [ZP].

Notation. If X is a holomorphic shadow, its dimension is already defined. Otherwise, we assign $\dim X$ to be half the dimension of X as a real manifold. The dimension of ${\Delta^n}_{(i_1,\dots,i_k)}$ is $(n-k+1)\dim X$.

For a shadow $S \in \mathcal{H}$ in X^l and a diagonal D in X^{n-l}, the *dimension* of $S \times D$ is the sum of $\dim S$ (as a holomorphic shadow) and $\dim D$ (as above). The *dimension* of the image of $S \times D$ under permutations of the coordinates is the dimension of $S \times D$. The dimension of a finite union of such sets is the maximum of the dimensions of the sets in the union.

It is easy to check that the dimension axioms still hold in $\mathcal{S}_{(X,J,\mathcal{H})}$. This completes the proof of Theorem 2.7.

3.25. *Remark.* By the arguments in the proof of Theorem 2.7, we get that $\mathcal{S}_{(X,J,\mathcal{H})}$ with dimension as in Definition 3.19 is a Zariski-type structure, with the (EU) property, whenever $\mathcal{H}$ consists of the images of a collection of J-holomorphic maps $\mathcal{F}$ such that $\mathcal{F}$ satisfies the pulling back diagonals property, $\mathcal{F}$ is closed under compositions and inverses, and $\mathcal{F}$ contains all the coordinate projections.

Almost complex manifolds that are ample as Zariski-type structures are of real dimension two. We say that a Zariski-type structure on X is *very ample* if there exists a family $Y \subset X^2 \times X^n$ of irreducible one-dimensional Z-closed subsets of X^2, parametrized by a Z-closed irreducible set in X^n, such that

- through any two points in X^2 there is a curve in the family passing through both and
- for any two points in X^2 there is a curve in the family passing through exactly one of the points.

If only the first condition is satisfied, the structure is called *ample.*

3.26. *Claim.* Let X be a manifold of real dimension $2n$, and $J \in \mathcal{J}(X)_{sp}$. If $(X, \mathcal{S}_{(X,J)})$ is ample, then $\dim_{\mathbb{R}} X \leq 2$; in particular J is integrable.

Proof. Assume $\dim_{\mathbb{R}} X > 2$. Since $J \in \mathcal{J}(X)_{sp}$, the projection of a holomorphic shadow $Y \subset X^k$ on every coordinate is either a point or a J-holomorphic curve. Hence, since $\dim_{\mathbb{R}} X > 2$, a holomorphic shadow $Y \subset X^2 \times X^n$ does not project onto X^2. Similarly, if Y is a diagonal, a Cartesian product of holomorphic shadows and diagonals, or the image of such a set under coordinate-permutation, it does not project onto X^2. Thus there is no $Y \subset X^2 \times X^n$ that can serve as a family demonstrating ampleness. $\square$

To generalize this claim we use results on Zariski geometries.

Zariski geometry is defined by [HZ1], [HZ2]. A set X with a collection of compatible Noetherian topologies, one on each X^n, $n \in \mathbb{N}$, and the Noetherian dimension as dimension, such that X is irreducible and of Noetherian dimension one, and the Pre-smoothness (PS) property is satisfied, is a (one-dimensional) Zariski geometry. (A topological space has *Noetherian dimension* n if n is the maximal length of a chain of closed irreducible sets $C_n \supset C_{n-1} \supset \ldots \supset C_0$.)

Any smooth algebraic curve $X = C$ can be viewed as a Zariski geometry. The Z-closed subsets are taken to be the Zariski closed subsets of C^n for each n. This Zariski geometry is very ample.

In [HZ2, Theorem 1], Hrushovski and Zilber show that if X is a very ample Zariski geometry, then there exists a smooth curve C over an algebraically closed field F such that X and C are isomorphic as Zariski geometries. In [HZ2, Theorem 2], they show that if X is an ample Zariski geometry, then there exist an algebraically closed field F and a surjective map $f\colon X \to \mathbb{CP}^1(F)$, such that off a finite set f induces a closed continuous map on each Cartesian power.

3.27**.** Let X be a $2n$-manifold and J an almost complex structure on X. Assume that the pulling back diagonals property is satisfied for J-holomorphic maps from compact complex manifolds to Cartesian products of X. Thus the Z-axioms (L1)–(L5), (P), and (DCC), as well as the (EU) property, are satisfied in the holomorphic shadows structures. Assume also that X is of Noetherian dimension one in this structure, i.e., any proper Z-closed subset of X is a finite set of points.

If X itself is a holomorphic shadow, then outside the penumbra, a proper Z-closed subset, hence a finite set, it is a complex manifold (the umbra).

If X is not a holomorphic shadow, then it does not contain holomorphic shadows except for finite sets of points. In this case, for any $n \in \mathbb{N}$, the space X^n contains no holomorphic shadows (that are not finite) either, since for an infinite holomorphic shadow in X^n its image under one of the coordinate projections $X^n \to X$ is an infinite holomorphic shadow in X. Thus the holomorphic shadows structure is trivial, i.e., consists only of diagonals and points.

In both cases we have the Pre-smoothness (PS) property, with Noetherian dimension.

3.28. **Corollary.** *Let X be a manifold and J an almost complex structure on X. Assume that the pulling back diagonals property is satisfied for J and that X is irreducible and of Noetherian dimension one in the holomorphic shadows structure. Then this structure, with Noetherian dimension, is a Zariski geometry that also satisfies the (EU) property.*

3.29. **Corollary.** *Let X be a manifold and J an almost complex structure on X. Assume that the pulling back diagonals property is satisfied for J and that X is irreducible and of Noetherian dimension one in the holomorphic shadows structure. If this structure is ample, then (X, J) is of real dimension two; in particular J is integrable.*

Proof. By Theorem 2 in [HZ2], there exists an algebraically closed field $\mathcal{F}$ and a map $\pi\colon X \to P^1(\mathcal{F})$. The map π maps constructible sets to algebraically constructible sets. Off a certain finite set, π is surjective and induces a closed continuous map on each Cartesian power. $\mathcal{F}$ is interpretable on X, i.e., there is an equivalence relation $\sim_\pi$ on X such that for some finite subset A' the quotient by $\sim_\pi$ of $\bar{X} \times \bar{X}$, where $\bar{X} = X - A'$, is a closed subset of $\bar{X} \times \bar{X}$. There are definable subsets $A, M \subset \bar{X} \times \bar{X} \times \bar{X}$ such that their quotient by $\sim_\pi$, restricted to products of a coordinate neighbourhood, give the graphs of the field operations (addition and multiplication) in $\mathcal{F}$.

By removing finite sets from $\bar{X}$ and $\mathcal{F} \subseteq P^1(\mathcal{F})$, we have $\pi\colon \tilde{X} \to \tilde{\mathcal{F}}$ that is surjective, continuous, maps constructible sets to algebraically constructible sets, and finite to one.

"Finite to one" follows from the fact that the Zariski geometries X and $P^1(\mathcal{F})$ are both of Noetherian dimension 1, and in a generic point y in $P^1(\mathcal{F})$, $\dim_X(X) = \dim_{P^1(\mathcal{F})}(\pi(X)) + \dim_X(\pi^{-1}y)$.

For $k \in \mathbb{N}$, let $E_k = \{f \in \tilde{\mathcal{F}} | \pi^{-1}(f) \cap \tilde{X}| = k\}$. E_k is a definable set in $P^1(\mathcal{F})$ as interpreted in X. Since $P^1(\mathcal{F})$ is a strongly minimal set, either E_k is finite or $\mathcal{F} - E_k$ is finite. If for every $k \in \mathbb{N}$, E_k is finite, we get that $\mathcal{F}$ is countable (recall that π is finite to one), contradicting axiom (EU). ($\mathcal{F}$ is infinite since it is algebraically closed.) Thus there exists $n \in \mathbb{N}$ such that for $f \in \tilde{\mathcal{F}}$ minus a finite set (also to be denoted $\tilde{\mathcal{F}}$), $|(\pi^{-1}(f) \cap \tilde{\mathcal{F}})| = n$.

For $p \in \tilde{\mathcal{F}}$, take a coordinate neighbourhood $U \subset \tilde{X}$ around a point in the fiber $\pi^{-1}p$ and define its image in $\tilde{F}$ by π to be a coordinate neighbourhood. This gives $\tilde{\mathcal{F}}$ a manifold structure. The map π is continuous with respect to this topology on $\tilde{\mathcal{F}}$ and the given topology on $\tilde{X}$ (induced from the topology on X).

Moreover, covering $\mathcal{F}$ by translates of $\tilde{\mathcal{F}}$ (by the translations $f \to f+b$, $f \to f*b$, induced from the addition and multiplication in $\mathcal{F}$), we obtain $\mathcal{F}$ as a manifold.

The field operations $+\colon \mathcal{F}\times\mathcal{F}\to\mathcal{F}$ and $*\colon \mathcal{F}\times\mathcal{F}\to\mathcal{F}$ are continuous with respect to this topology, as the graph of the operations restricted to products of coordinate neighbourhoods are given by the quotient by $\sim_\pi$ of constructible subsets $A, M \subset \tilde{X}\times\tilde{X}\times\tilde{X}$, as above. (By elimination of quantifiers, a set is constructible iff it is definable.) So the manifold structure on $\mathcal{F}$ is consistent with the field operations.

In particular $\mathcal{F}$ is locally compact. Since it is also algebraically closed, $\mathcal{F}=\mathbb{C}$. Since $\pi\colon \tilde{X}\to\tilde{F}$ is finite to one and $\pi^{-1}(\mathcal{F}-$ finite set) is open in X, the almost complex manifold X must be of real dimension 2. □

4. Holomorphic shadows in symplectic geometry

The theory of J-holomorphic curves has been an active study area and a powerful tool in symplectic geometry since the pioneering paper of Gromov [Gr].

A *symplectic* structure on a smooth $2n$-dimensional manifold X is a closed 2-form ω which is non-degenerate (i.e., ω^n does not vanish anywhere). Two symplectic manifolds (X_1,ω_1) and (X_2,ω_2) are called *symplectomorphic* if there is a diffeomorphism $\phi\colon X_1\to X_2$ such that $\phi^*\omega_2=\omega_1$. A symplectic form ω is said to *tame* an almost complex structure J if ω is J-positive, i.e.,

$$\omega(v,Jv)>0$$

for all non-zero $v\in TX$. This implies that for every embedded submanifold $C\subset M$, if $J(TC)=TC$, then $\omega|_{TC}$ is non-degenerate. Given ω, we denote (in this section) by

$$\mathcal{J}=\mathcal{J}(X,\omega)$$

the space of almost complex structures J on X that are tamed by ω. The space $\mathcal{J}$ is non-empty and contractible [MS1, Proposition 2.50(iii)]; in particular, it is path connected. As a result, the first Chern class of the complex vector bundle (TX,J) is independent of the choice of $J\in\mathcal{J}$.

We say that $A\in H_2(M;\mathbb{Z})$ is *J-indecomposable* if it does not split as a sum $A_1+\ldots+A_k$ of classes, all of which can be represented by non-constant J-holomorphic curves. The class A is called *indecomposable* if it is J-indecomposable for all ω-tame J. Notice that if A cannot be written as a sum $A=A_1+A_2$ where $A_i\in H_2(M;\mathbb{Z})$ and $\int_{A_i}\omega>0$, then it is indecomposable.

Restating claims in the language of shadows structures. We now restate claims of Gromov [Gr] and McDuff [McD] in the language of Zariski-type structures; we will sketch some of the ideas of the (geometric) proofs.

Notation. An *isomorphism* of two Zariski-type structures $(X,\mathcal{C}_1,\dim_X)$ and $(Y,\mathcal{C}_2,\dim_Y)$ is a map $\mathcal{C}_1\to\mathcal{C}_2$ which is an isomorphism of topologies between $\mathcal{C}_1\cap\mathcal{P}(X^n)$ and $\mathcal{C}_2\cap\mathcal{P}(Y^n)$ for all $n\in\mathbb{N}$ that commutes with coordinate projections, Cartesian products and dimension assigning. An *embedding* of one Zariski-type structure into the other is a one-to-one map that is an isomorphism with its image.

Let $\mathcal{Z}=(X,\mathcal{C}_Z,\dim_Z)$ be a Zariski-type structure, and let $\mathcal{S}=(X,\mathcal{C}_S,\dim_S)$ (not necessarily a Zariski-type structure) consist of a collection $\mathcal{C}_S$ of subsets of the Cartesian products of X and a partial dimension function $\dim_S$ on $\mathcal{C}_S$. We say that the Zariski-type structure $\mathcal{Z}$ is *embedded* into $\mathcal{S}$ if there is a one-to-one map $e\colon\mathcal{C}_Z\to\mathcal{C}_S$ that preserves the inclusion relation and commutes with coordinate projections and Cartesian products, such that $I_{\mathcal{Z}} = (X, e(\mathcal{C}_Z), \dim_S|_{e(\mathcal{C}_Z)})$ is a Zariski-type structure and the map e induces an isomorphism $\mathcal{Z} \to I_{\mathcal{Z}}$ of Zariski-type structures.

4.1. **Example.** Consider $(S^2 \times S^2, J_0 \oplus J_0)$, where J_0 is the standard complex structure on the sphere $S^2 = \mathbb{CP}^1$. Denote by ω_0 an area form on the sphere S^2 whose orientation agrees with the orientation induced by J_0. The form $\omega_0 \oplus \omega_0$, defined as the sum of the pullbacks of ω_0 to $S^2 \times S^2$ via the coordinate projections, is a symplectic form on $S^2 \times S^2$ that tames $J_0 \oplus J_0$.

Denote by $\mathrm{Striv}(S^2 \times S^2)$ the structure generated by finite unions and Cartesian products from

(1) points $(s, r) \in S^2 \times S^2$,
(2) the set $S^2 \times S^2$,
(3) the sets $\{s\} \times S^2$, $S^2 \times \{s\}$, for any $s \in S^2$,
(4) the diagonals

$$\Delta^n_{i_1,\dots,i_k} = \{(x_1, \dots, x_n) | x_{i_1} = \dots = x_{i_k}, x_i \in S^2 \times S^2\},$$

with the natural dimension assigned as:

- the dimension of a set in (1) is 0,
- the dimension of a set in (2) is 2,
- the dimension of a set in (3) is 1,
- $\dim \Delta^n_{i_1,\dots,i_k} = 2(n - k + 1)$,
- $\dim(S_1 \cup S_2) = \max\{\dim S_2, \dim S_2\}$,
- $\dim S \times T = \dim S + \dim T$.

Then $\mathrm{Striv}(S^2 \times S^2)$ is a Zariski-type structure.

For any $s \in S^2$, the sphere $\{s\} \times S^2$ is embedded as a symplectic sphere in $(S^2 \times S^2, \omega_0 \oplus \omega_0)$ and as a $J_0 \oplus J_0$-sphere. This implies that

4.2. *Claim.* $\mathrm{Striv}(S^2 \times S^2)$ is embedded as a Zariski-type structure in the holomorphic shadows structure $\mathcal{S}_{(S^2 \times S^2, J_0 \oplus J_0)}$.

We show a similar claim for $J \in \mathcal{J}(S^2 \times S^2, \omega_0 \oplus \omega_0)$ that does not necessarily split as a product of almost complex structures on S^2. This follows from known results of Gromov [Gr, 2.4 A1] and [McD, Lemma 4.1, Lemma 4.2]. We prove Claim 4.3 here to give the reader the flavor of the arguments.

4.3. *Claim.* For every almost complex structure J that is tamed by $\omega_0 \oplus \omega_0$, for each point in $S^2 \times S^2$ there exists a unique embedded J-holomorphic sphere in

$$A = [S^2 \times \mathrm{pt}],$$
$$(B = [\mathrm{pt} \times S^2]).$$

In addition, any J-holomorphic sphere in A and any J-holomorphic sphere in B intersect once and transversally.

Proof. Through each point in $S^2 \times S^2$ there is an embedded J_0-holomorphic sphere $f_0 \colon S^2 \to S^2 \times S^2$ in $A = [S^2 \times \mathrm{pt}]$ $(B = [\mathrm{pt} \times S^2])$. In particular, A is the homology class of an embedded $\omega_0 \oplus \omega_0$-sphere of minimal symplectic area. Hence A is indecomposable and every J-holomorphic sphere $f \colon S^2 \to S^2 \times S^2$ in A is simple.

By the adjunction inequality (in a four-dimensional manifold), if A is represented by a simple J-holomorphic curve f, then

$$A \cdot A - c_1(A) + 2 \geq 0,$$

with equality if and only if f is an embedding; see [MS2, Cor. E.1.7]. Applying this to (f_0, J_0), we get that the homology class A satisfies $A \cdot A - c_1(A) + 2 = 0$. Applying the adjunction inequality to any $(f, J) \in \mathcal{M}(A, S^2, \mathcal{J})$, we get that f is an embedding. Hence, by the Hofer-Lizan-Sikorav regularity criterion, (f, J) is regular for p_A. (The Hofer-Lizan-Sikorav regularity criterion asserts that if f is an immersed J-holomorphic curve in a four-dimensional manifold and $c_1([f]) \geq 1$, then (f, J) is a regular point for the projection p_A [HLS].) By the implicit function theorem, every p_A-regular sphere (f, J) persists when J is perturbed; see, e.g., [MS2, Remark 3.2.8]. On the other hand, since A is indecomposable, Gromov's compactness theorem [Gr, 1.5.B] implies that if J_n converges in $\mathcal{J}$, then every sequence (f_n, J_n) in $\mathcal{M}(A, S^2, \mathcal{J})$ has a (C^∞)-convergent subsequence. We conclude that for each point pt $\in S^2 \times S^2$, the set of $J \in \mathcal{J}(= \mathcal{J}(S^2 \times S^2, \omega_0))$ for which there is an embedded J-holomorphic sphere through pt in $A = [S^2 \times \mathrm{pt}]$ $(B = [\mathrm{pt} \times S^2])$ is non-empty open and closed in the connected space $\mathcal{J}$; hence it equals $\mathcal{J}(= \mathcal{J}(S^2 \times S^2, \omega_0 \oplus \omega_0))$.

Let $J \in \mathcal{J}$. If there are two different simple J-holomorphic spheres f_1, f_2 through a point in $S^2 \times S^2$, then, by positivity of intersections in almost complex four-manifolds [MS2, Theorem E.1.5], the intersection number $[f_1(S^2)] \cdot [f_2(S^2)]$ is positive. Thus, since, $A \cdot A = 0 = B \cdot B$, there cannot be two different J-holomorphic spheres in A (B) through a point in $S^2 \times S^2$. Also, again by positivity of intersections, for simple J-holomorphic spheres f_1, f_2, the intersection number $[f_1(S^2)] \cdot [f_2(S^2)]$ equals 1 if and only if the spheres meet exactly once and transversally. Hence a J-holomorphic sphere in A and a J-holomorphic sphere in B intersect once and transversally. □

In the language of shadows structures, this claim has the following translation.

4.4. *Claim.* The Zariski-type structure $\mathrm{Striv}(S^2 \times S^2)$ can be embedded into the shadows structure $\mathcal{S}_{(S^2 \times S^2, J)}$ for every $J \in \mathcal{J}(S^2 \times S^2, \omega_0 \oplus \omega_0)$.

Proof. Fix $J \in \mathcal{J}(S^2 \times S^2, \omega_0 \oplus \omega_0)$. Choose $s_0 \in S^2$. By Claim 4.3, there is a unique J-holomorphic curve C_1 in A (C_2 in B) through (s_0, s_0). Choose a $J|_{C_1}$-holomorphic diffeomorphism a_1 of C_1 onto S^2 such that $a_1(s_0, s_0) = s_0$, and a $J|_{C_2}$-holomorphic diffeomorphism a_2 of C_2 onto S^2 such that $a_2(s_0, s_0) = s_0$. Now, send $S^2 \times \{s\}$ to the unique curve in A that intersects C_2 in v such that $a_2(v) = s$, and send $\{s\} \times S^2$ to the unique curve in B that intersects C_1 in v such that $a_1(v) = s$. This is a well defined and one-to-one map. It maps the two families $\{s\} \times S^2$, $S^2 \times \{s\}$, $s \in S^2$, to two families of J-holomorphic spheres such that each member of one family intersects each member of the other exactly once and transversally. This induces an embedding of $\mathrm{Striv}(S^2 \times S^2)$ into the shadows structure $\mathcal{S}_{(S^2 \times S^2, J)}$ (a point (r, t) is sent to the intersection point of the J-holomorphic sphere in A that is the image of $S^2 \times \{t\}$, and the J-holomorphic sphere in B that is the image of $\{r\} \times S^2$, $S^2 \times S^2$ is sent to $S^2 \times S^2$; each diagonal is sent to itself). □

4.5. *Remark.* The above embedding is not onto: for almost complex structures in $\mathcal{J}(S^2 \times S^2, \omega_0 \oplus \omega_0)$ there are J-holomorphic curves in $S^2 \times S^2$ in homology classes other than $[S^2 \times \{\mathrm{pt}\}]$ or $[\{\mathrm{pt}\} \times S^2]$; e.g., in $[\{s, s\}_{s \in S^2}]$.

Similar arguments to the ones in Claim 4.4 give the following claim. See [McD, Lemma 4.2, Lemma 4.6].

4.6. *Claim.* Let ω be a symplectic form on $S^2 \times S^2$ such that there exist symplectically embedded spheres in $A = [S^2 \times \mathrm{pt}]$ and $B = [\mathrm{pt} \times S^2]$ that intersect exactly once and transversally. Then the Zariski-type structure $\mathrm{Striv}(S^2 \times S^2)$ can be embedded into the shadows structure $\mathcal{S}_{(S^2 \times S^2, J)}$ for every $J \in \mathcal{J}(S^2 \times S^2, \omega)$. $\square$

4.7. **Example.** We denote by ω_{FS} the Fubini-Study form on $\mathbb{CP}^2$.

Denote by $\mathrm{Striv}(\mathbb{CP}^2)$ the structure generated by finite unions and Cartesian products from

(1) the points of $\mathbb{CP}^2$, with dimension assigned 0,
(2) the set $\mathbb{CP}^2$, with dimension assigned 2,
(3) a family $\mathcal{F} = \{C(p_0, p)\}_{p \in \mathbb{CP}^1}$, where p_0 is a fixed point in $\mathbb{CP}^2$ and p is a point on a $\mathbb{CP}^1$-line L in $\mathbb{CP}^2$ such that p_0 is not on L, of (spheres) $C(p_0, p)$ in $\mathbb{CP}^2$ such that for $p \neq q$, the intersection $C(p_0, p) \cap C(p_0, q)$ is the point $p_0 \in \mathbb{CP}^2$; each $C(p_0, p)$ is assigned dimension 1,
(4) diagonals $\Delta^n_{i_1, \ldots, i_k}$ in $(\mathbb{CP}^2)^n$, with dimension assigned $2(n - k + 1)$.

Then $\mathrm{Striv}(\mathbb{CP}^2)$ is a Zariski-type structure, embedded in the holomorphic shadows structure $\mathcal{S}_{(\mathbb{CP}^2, J_0 \oplus J_0)}$, where J_0 is the standard complex structure on $\mathbb{CP}^2$.

4.8. *Claim.* Let an almost complex structure J on $\mathbb{CP}^2$ be tamed by the standard symplectic form ω_{FS} on $\mathbb{CP}^2$. Then there is a J-holomorphic sphere $C \subset \mathbb{CP}^2$ through two given points v and v' in $\mathbb{CP}^2$ which is homologous to the projective line $\mathbb{CP}^1 \subset \mathbb{CP}^2$. Since the algebraic intersection number between two such spheres equals one, any two of them, say C and C' in $\mathbb{CP}^2$, necessarily meet at a single point, say at $v \in C \cap C'$, unless $C = C'$. Furthermore, the spheres C and C' are regular at v and meet transversally. Hence, C is regular at all points $v \in C$ and is uniquely determined by v and $v' \neq v$. Moreover, $C = C(v, v')$ smoothly depends on (v, v').

This is [Gr, 2.4 A]. In the language of shadows structures this claim has the following translation.

4.9. *Claim.* For every $J \in \mathcal{J}(\mathbb{CP}^2, \omega_{\mathrm{FS}})$, the Zariski-type structure $\mathrm{Striv}(\mathbb{CP}^2)$ is embedded into $\mathcal{S}_{(\mathbb{CP}^2, J)}$. $\square$

5. General observations and future research

The arguments in the proof of Theorem 2.7 prove the following proposition. Given an almost complex manifold (X, J), let $\mathcal{H}_c$ denote the collection of holomorphic shadows in the Cartesian products of X^n that are each obtained either as the image of a J-holomorphic map from a compact Riemann surface (i.e., is a J-holomorphic curve) or as the product of J-holomorphic curves.

5.1. **Proposition.** *Let (X, J) be an almost complex manifold. Then there exists a dimension function* $\dim$ *on $\mathcal{S}_{(X, J, \mathcal{H}_c)}$ that is consistent with the natural partial dimension assigning* 0 *to a point and* 1 *to a non-constant J-holomorphic curve, such that $(X, \mathcal{S}_{(X, J, \mathcal{H}_c)}, \dim)$ is a Zariski-type structure that satisfies the essential uncountability (EU) property.*

This observation raises the following question: how does the Zariski-type structure $\mathcal{S}_{(X, J, \mathcal{H}_c)}$ change as J is perturbed to another almost complex structure in $\mathcal{J}(X)$ or in $\mathcal{J}(X, \omega)$? The answer should apply the infinite-dimensional implicit function theorem and the ellipticity of the Cauchy-Riemann equations (see Section

2), as well as Gromov's compactness theorem (see Section 4). We will also have to develop ways to talk about families of Zariski-type structures in model theory.

In Theorem 2.7 we showed that for J in $\mathcal{J}_{sp}(X)$, taking $\mathcal{H}$ to be the collection of all holomorphic shadows in the finite Cartesian products of (X, J), we get that $\mathcal{S}_{(X,J,\mathcal{H})}$ is a Zariski-type structure. It is natural to ask whether this is true in general for J in $\mathcal{J}(X)$ or in $\mathcal{J}(X, \omega)$. Moreover, does Theorem 2.7 hold if the definition of a holomorphic shadow is extended so that it traces more than images of compact complex manifolds? A natural extension is to define an *almost complex shadow* to be the image of a J-holomorphic map from a compact almost complex manifold. Checking if $\mathcal{S}_{(X,J,\mathcal{H})}$ resulting when we take $\mathcal{H}$ to be the collection of almost complex shadows is a Zariski-type structure points us to geometric questions. In particular, to apply arguments we used to deduce that holomorphic shadows are closed under intersection to this extended definition, we will need a resolution of singularities in the almost complex category.

Notice that a further relaxation of the definition by dropping the compactness condition will not work. For example, let X, Y be smooth (non-singular C^∞) surfaces in $\mathbb{R}^3$, with intersection $X \cap Y$ of dimension 1. Take X to be $\mathbb{R}^2$ in $\mathbb{R}^3$ and Y to be a graph of a C^∞-function over X which is positive outside of the x_1-axis in $X = \mathbb{R}^2$ and equal to zero on the x_1-axis. Now, TX and TY are almost complex submanifolds of $T\mathbb{R}^3$ with the natural almost complex structure $J\colon T(T\mathbb{R}^3) \to T(T\mathbb{R}^3)$ ($J(\frac{\partial}{\partial p_i}) = \frac{\partial}{\partial q_i}$; $J(\frac{\partial}{\partial q_i}) = -\frac{\partial}{\partial p_i}$, where $p_i = \frac{\partial}{\partial x_i}$ and $q_i = \frac{\partial}{\partial p_i}$ for $i = 1, 2, 3$). The intersection $TX \cap TY$ is of real dimension 3 and thus cannot be the image of a J-holomorphic map.

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