

NODAL SOLUTIONS FOR SOME SINGULARLY PERTURBED DIRICHLET PROBLEMS

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ABSTRACT. We consider the equation $-\varepsilon^2 \Delta u + u = f(u)$ in a bounded, smooth domain $\Omega \subset \mathbb{R}^N$ with homogeneous Dirichlet boundary conditions. We prove the existence of nodal solutions with multiple peaks concentrating at different points of Ω . The nonlinearity f grows superlinearly and subcritically. We do not require symmetry conditions on the geometry of the domain.

1. INTRODUCTION

Let us consider the following singularly perturbed elliptic problem:

$$(1.1) \quad \begin{cases} \varepsilon^2 \Delta u - u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain of \mathbb{R}^N , $N \geq 2$, $2 < p < \frac{2N}{N-2}$ if $N \geq 3$ and $p > 2$ if $N = 2$, and $\varepsilon > 0$ is a small parameter. Problem (1.1) arises in various mathematical models deriving from biological population theory, chemical reactor theory, etc. There have been many results which prove that solutions of (1.1) may exhibit sharp peaks near a certain number of points. In particular many papers have been devoted to establishing the existence of single and multiple peak solutions and to determining the location of the peaks as well as the profile of the spikes as $\varepsilon \rightarrow 0^+$. Ni and Wei in [23] showed that for $\varepsilon > 0$ sufficiently small problem (1.1) has a positive least energy solution v_ε which develops a spike layer at the *most centered* part of the domain, i.e. $d_{\partial\Omega}(P_\varepsilon) \rightarrow \max_{P \in \Omega} d_{\partial\Omega}(P)$, denoting by P_ε the unique maximum of v_ε . Hereafter $d_{\partial\Omega}(P)$ denotes the distance of P from $\partial\Omega$. Since then, there have been many papers looking for higher energy solutions. More specifically, several papers study the effect of the geometry of the domain on the existence of positive solutions with single and multiple peaks (see [3], [4], [7], [8], [9], [10], [11], [14], [15], [17], [21], [26] and the references therein). However, in a general domain the existence of k -peaked positive solutions when $k \geq 2$ is not guaranteed. Indeed, in [10] the authors showed that if Ω is a strictly convex domain, problem (1.1) does not admit k -peaked positive solutions for any $k \geq 2$ (see also [25] when $k = 2$).

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We point out that, for the corresponding Neumann problem, multiple interior and boundary positive spikes in a general domain have been constructed, for example, in [5], [18], [19], [20], and [22].

On the other hand, multiple peak nodal solutions always exist for problem (1.1) in a bounded and smooth domain Ω . The first result was due to Noussair and Wei in [24], where it is proved that for ε sufficiently small (1.1) has a least energy nodal solution with one positive boundary peak P_1^ε and one negative boundary peak P_2^ε whose location depends on the geometry of the Ω . More precisely, if $\bar{P}_1 := \lim_{\varepsilon \rightarrow 0} P_1^\varepsilon$ and $\bar{P}_2 := \lim_{\varepsilon \rightarrow 0} P_2^\varepsilon$, then $\varphi(\bar{P}_1, \bar{P}_2) = \max_{(P_1, P_2) \in \Omega \times \Omega} \varphi(P_1, P_2)$, where the function $\varphi : \Omega \times \Omega \rightarrow \mathbb{R}$ is defined by

$$\varphi(P_1, P_2) := \min \left\{ d_{\partial\Omega}(P_1), d_{\partial\Omega}(P_2), \frac{|P_1 - P_2|}{2} \right\}.$$

In [27] the authors showed that in the case of the unit ball such a solution is odd in one direction. We also mention the papers [1]–[2] where, by a different approach, a lower bound on the number of sign-changing solutions is provided; however, these papers are not concerned with the shape of such solutions. As far as we know the question of the existence of k -peaked nodal solutions for problem (1.1) with $k \geq 3$, without symmetry assumptions on the domain Ω , is largely open. It is the purpose of the present paper to establish this kind of result.

We are able to deal with more general nonlinearities. We consider the problem

$$(1.2) \quad \begin{cases} -\varepsilon^2 \Delta u + u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We will assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is of class $C^{1+\sigma}$ and satisfies the following conditions:

- (f1) $f(0) = f'(0) = 0$ and $f(u) = -f(-u)$ for any $u \in \mathbb{R}$.
- (f2) $f(u) = O(|u|^{p_1})$, $f'(u) = O(|u|^{p_2-1})$ as $|u| \rightarrow \infty$ for some $p_1, p_2 > 1$ and there exists $p_3 > 1$ such that

$$|f'(u + \phi) - f'(u)| \leq \begin{cases} c|\phi|^{p_3-1} & \text{if } p_3 > 2, \\ c(|\phi| + |\phi|^{p_3-1}) & \text{if } p_3 \leq 2. \end{cases}$$

- (f3) The problem

$$(1.3) \quad \begin{cases} \Delta w - w + f(w) = 0, & w > 0 \text{ in } \mathbb{R}^N, \\ w(0) = \max_{z \in \mathbb{R}^N} w(z), & \lim_{|z| \rightarrow +\infty} w(z) = 0 \end{cases}$$

has a unique solution w , which is nondegenerate; i.e., denoting by L the linearized operator

$$L : H^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N), \quad L[u] := \Delta u - u + f'(w)u,$$

then

$$(1.4) \quad \text{Kernel}(L) = \text{span} \left\{ \frac{\partial w}{\partial z_1}, \dots, \frac{\partial w}{\partial z_N} \right\}.$$

By the well-known result of Gidas, Ni and Nirenberg ([16]), w is radially symmetric and strictly decreasing in $r = |z|$. Moreover, by classical regularity results, the

following asymptotic result holds:

$$(1.5) \quad \lim_{|z| \rightarrow +\infty} |z|^{\frac{N-1}{2}} e^{|z|} w(|z|) = A > 0 \quad \text{and} \quad \lim_{|z| \rightarrow +\infty} \frac{w'(|z|)}{w(|z|)} = -1.$$

The main result of this paper states that, given two positive integers k_1, k_2 with $k_1 + k_2 \leq 6$ (with the exception of the couples $(1, 5)$ and $(5, 1)$ in the two-dimensional case), for ε sufficiently small (1.2) possesses a solution with k_1 positive peaks and k_2 negative peaks approaching different points of Ω . Furthermore, each peak has a profile similar to w suitably rescaled. More precisely we will prove the following theorem.

Theorem 1.1. *Assume that hypotheses (f1)–(f3) hold and that $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a smooth and bounded domain. Let $k_1, k_2 \in \mathbb{N}$ satisfy*

$$k_1, k_2 \geq 1, \quad k := k_1 + k_2 \leq 6, \quad (k_1, k_2) \neq (1, 5), (5, 1) \text{ if } N = 2.$$

Then, for $\varepsilon > 0$ sufficiently small, the problem (1.2) has a solution $v_\varepsilon \in H^2(\Omega)$.

Furthermore, there exist $P_1^\varepsilon, \dots, P_k^\varepsilon \in \Omega$ such that, as $\varepsilon \rightarrow 0^+$,

$$(a) \quad v_\varepsilon(x) = \sum_{i=1}^{k_1} w\left(\frac{x-P_i^\varepsilon}{\varepsilon}\right) - \sum_{i=k_1+1}^k w\left(\frac{x-P_i^\varepsilon}{\varepsilon}\right) + o(e^{-\frac{\delta}{\varepsilon}}) \text{ uniformly for } x \in \bar{\Omega},$$

$$(b) \quad |P_i^\varepsilon - P_h^\varepsilon| \geq \delta \text{ for } i \neq h, \quad d_{\partial\Omega}(P_i^\varepsilon) \geq \delta \text{ for all } i$$

for a suitable $\delta > 0$.

An easy computation shows that we obtain 9 pairs of nodal peak solutions for problem (1.2) if $N \geq 3$ and 8 pairs if $N = 2$, provided that ε is small enough.

We would like to remark that in the presence of symmetries we can localize the limit concentration points $\bar{P}_i := \lim_{\varepsilon \rightarrow 0} P_i^\varepsilon$, $i = 1, \dots, k$, as it is proved in Section 5. However, in general we can only conjecture that the location of the peaks is strictly related with the location of suitable critical points of the function $\varphi : \Omega^k \rightarrow \mathbb{R}$ defined by

$$(1.6) \quad \varphi(P_1, \dots, P_k) := \min_{\substack{i, h=1, \dots, k \\ i \neq h}} \left\{ d_{\partial\Omega}(P_i), \frac{|P_i - P_h|}{2} \right\}.$$

More precisely, we conjecture that $(\bar{P}_1, \dots, \bar{P}_k)$ is a critical point of φ such that

$$\frac{|\bar{P}_i - \bar{P}_h|}{2} > \varphi(\bar{P}_1, \dots, \bar{P}_k) \quad \text{if either } i, h = 1, \dots, k_1 \quad \text{or} \quad i, h = k_1 + 1, \dots, k.$$

Finally, we point out that a multiplicity result for nodal multi-peak solutions has been obtained in [12] for the Dirichlet problem in the presence of a positive potential V and in [13] for the corresponding Neumann problem. In both cases the peaks collapse to the same point, contrary to the present situation.

To introduce the main idea of the proof of Theorem 1.1, we first need to fix some notation. For any smooth bounded domain U , let $\mathcal{P}_U w$ be the projection of w into $H_0^1(U)$, i.e.

$$(1.7) \quad \begin{cases} -\Delta u + u = f(w) & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

For $P \in \Omega$ we set

$$\Omega_\varepsilon := \{y : \varepsilon y \in \Omega\} \quad \text{and} \quad \Omega_{\varepsilon, P} := \{y : \varepsilon y + P \in \Omega\}.$$

Let us scale problem (1.2), so that we get an equivalent problem

$$(1.8) \quad \begin{cases} -\Delta u + u = f(u) & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Associated with problem (1.8) is the rescaled energy functional

$$(1.9) \quad J_\varepsilon(u) := \frac{1}{2} \int_{\Omega_\varepsilon} (|\nabla u|^2 + u^2) - \int_{\Omega_\varepsilon} F(u), \quad u \in H_0^1(\Omega_\varepsilon),$$

where $F(u) = \int_0^u f(s)ds$. Fix $\mathbf{P} = (P_1, \dots, P_k) \in \Omega^k$ and set, for any $i = 1, \dots, k$,

$$w_i(y) = w_{P_i}(y) := w\left(y - \frac{P_i}{\varepsilon}\right) \quad \text{and} \quad \mathcal{P}w_i(y) := \mathcal{P}_{\Omega_\varepsilon} w_i(y), \quad y \in \Omega_\varepsilon.$$

We look for a solution to (1.8) as

$$(1.10) \quad u = \sum_{i=1}^k \lambda_i \mathcal{P}w_i + \Phi_{\varepsilon, \mathbf{P}},$$

where $\lambda_i \in \{-1, +1\}$ and the rest term $\Phi_{\varepsilon, \mathbf{P}} = \Phi_{\varepsilon, P_1, \dots, P_k}$ belongs to a suitable space. By using the classical Lyapunov-Schmidt reduction method, in order to find such a solution the location of the concentration points P_1, \dots, P_k should be critical for the reduced functional

$$\tilde{J}_\varepsilon(\mathbf{P}) = kI(w) + \frac{1}{2}(\gamma + o(1)) \sum_{i=1}^k e^{-2\frac{d_{\partial\Omega}(P_i)}{\varepsilon}} - \frac{1}{2}(\gamma + o(1)) \sum_{i \neq h} \lambda_i \lambda_h w\left(\frac{P_i - P_h}{\varepsilon}\right) + h.o.t.$$

on a suitable open set $\mathcal{D}_\varepsilon \subset \Omega^k$, where γ is a positive constant. The terms $e^{-2\frac{d_{\partial\Omega}(P_i)}{\varepsilon}}$ represent the boundary effect on each peak, created by the boundary condition, while the terms $-\lambda_i \lambda_h w\left(\frac{P_i - P_h}{\varepsilon}\right)$ are due to the interaction among the peaks which has an attractive or a repulsive effect according to its respective sign.

Let us observe that the sum $\sum_{i=1}^k e^{-2\frac{d_{\partial\Omega}(P_i)}{\varepsilon}}$ tends to cluster the points P_i at the *most centered* part of Ω , while, using (1.5), the sum $\sum_{\lambda_i = -\lambda_h} w\left(\frac{P_i - P_h}{\varepsilon}\right)$ tends to repel the points P_i from each other. Therefore, in the case $k_1 = k_2 = 1$ we have $\lambda_i \lambda_h = \lambda_1 \lambda_2 = -1$, and then we can easily conclude that the two sums above achieve an equilibrium for a suitable configuration of the points P_1, P_2 , which is a local minimum for the functional \tilde{J}_ε .

In the general case $k_1, k_2 \geq 1$ the presence of the term $-\sum_{\lambda_i = \lambda_h} w\left(\frac{P_i - P_h}{\varepsilon}\right)$ does not allow us to catch a critical point of \tilde{J}_ε as a local minimum or a local maximum. However the different interaction effects of the boundary and the peaks provide \tilde{J}_ε with a suitable local linking structure. Following the strategy of [12], we will use a max-min theorem to obtain the existence of a saddle point for \tilde{J}_ε . The choice of the set where we minimize \tilde{J}_ε will be crucial. Roughly speaking, such a set is made up of configurations P_1, \dots, P_k for which the terms $w\left(\frac{P_i - P_h}{\varepsilon}\right)$ with $\lambda_i = \lambda_h$ ($i \neq h$) are negligible (for example, in the case $k_1 = k_2$, we choose the set of the configurations which are aligned on a fixed direction with alternating sign), so that the remaining terms may balance.

Once we have obtained a local geometrical linking structure, in order to apply a max-min argument to conclude the existence of a critical point a kind of compactness property for \tilde{J}_ε is required. More precisely, we need to prove that the

tangential component of the gradient of \tilde{J}_ε on $\partial\mathcal{D}_\varepsilon$ is not zero at the max-min level. This is the most technical part of the paper, and the choice of \mathcal{D}_ε plays a key role in our computations. A restriction on the number of peaks $k_1 + k_2 \leq 6$, which occurs in Theorem 1.1, is required in this last step. Indeed, for larger numbers the method breaks down due to lack of compactness arising in some configurations, which are related to the solutions of certain sphere-packing type variational problem (see Proposition 4.1). Such crucial configurations may appear, for example, when 7 of the k points (P_1, \dots, P_k) are placed in a hexagonal arrangement (i.e. a single point surrounded by 6 other points at the vertices of a regular hexagon). The difficulty due to this loss of compactness for $k_1 + k_2 \geq 7$ seems the main obstacle in producing solutions to the equation (1.1) with a higher number of mixed positive and negative peaks when no symmetry on the domain Ω is assumed.

Since the Lyapunov-Schmidt procedure has been widely used, in this paper we shall only sketch the process, referring to [5] and [19] for the details, and concentrate on the max-min part.

The paper is organized as follows. Notation, preliminaries and some useful estimates are recalled in Section 2. In Section 3 we sketch the Lyapunov-Schmidt reduction method which reduces the problem to a finite-dimensional problem. Section 4 is devoted to applying a max-min argument to the reduced functional \tilde{J}_ε in order to catch a critical point. Finally, in Section 5, we deal with the symmetric case, where the use of natural constraints allows us to improve the above results for general domains.

2. PRELIMINARIES

We need the following results (see [23], Lemma 4.4). Set

$$\varphi_{\varepsilon,P}(x) := w\left(\frac{x-P}{\varepsilon}\right) - \mathcal{P}_{\Omega_{\varepsilon,P}} w\left(\frac{x-P}{\varepsilon}\right) \text{ and } \psi_{\varepsilon,P}(x) := -\varepsilon \log \varphi_{\varepsilon,P}(x), \quad x \in \Omega.$$

Lemma 2.1.

$$\psi_{\varepsilon,P}(x) \rightarrow \psi_0(x) := \inf_{z \in \partial\Omega} \{|z-P| + L(x,z)\} \text{ as } \varepsilon \rightarrow 0$$

uniformly in $\bar{\Omega}$, where $L(x,z)$ is the infimum of T such that there exists $\xi(s) \in \mathcal{C}^{0,1}([0,T], \bar{\Omega})$ with $\xi(0) = x$, $\xi(T) = z$, $|\frac{d\xi}{ds}| \leq 1$ a.e. in $[0,T]$. In particular $\psi_0(P) = 2d_{\partial\Omega}(P)$.

Let

$$V_{\varepsilon,P}(y) := \frac{\varphi_{\varepsilon,P}(\varepsilon y + P)}{\varphi_{\varepsilon,P}(P)}, \quad y \in \Omega_{\varepsilon,P}.$$

Lemma 2.2. *For every sequence $\varepsilon_n \rightarrow 0$, there is a subsequence $\varepsilon_{nh} \rightarrow 0$ such that $V_{\varepsilon_{nh},P} \rightarrow \tilde{V}$ uniformly on every compact set of \mathbb{R}^N , where \tilde{V} is a positive solution of*

$$\begin{cases} \Delta u - u = 0 & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(0) = 1. \end{cases}$$

Furthermore, for any $\sigma_1 > 0$,

$$\sup_{y \in \Omega_{\varepsilon_{nh},P}} e^{-(1+\sigma_1)|y|} |V_{\varepsilon_{nh},P}(y) - \tilde{V}(y)| \rightarrow 0 \text{ as } \varepsilon_{nh} \rightarrow 0.$$

Moreover, the following result holds (see [26], Lemma 2.4).

Lemma 2.3. *Assume that $P_\varepsilon \in \Omega$ is such that $P_\varepsilon \rightarrow P$ as $\varepsilon \rightarrow 0$ and $P \in \Omega$. Then there exists a bounded Borel measure $d\mu_P$ on $\partial\Omega$ with*

$$\int_{\partial\Omega} d\mu_P = 1 \quad \text{and} \quad \text{supp}(d\mu_P) \subset \Pi_{\partial\Omega}(P) := \{Q \in \partial\Omega : |Q - P| = d_{\partial\Omega}(P)\}$$

such that, up to a subsequence,

$$V_{\varepsilon, P_\varepsilon}(y) \rightarrow \tilde{V}(y) := \int_{\partial\Omega} e^{\langle \frac{Q-P}{|Q-P|}, y \rangle} d\mu_P \quad \text{as } \varepsilon \rightarrow 0.$$

It is well known that the distance function $d_{\partial\Omega} : \Omega \rightarrow \mathbb{R}$ is a Lipschitz continuous function, and by [6], Theorem 2.8.2, we deduce that for any $P \in \Omega$

$$(2.1) \quad \left. \begin{aligned} \partial d_{\partial\Omega}(P) &= \left\{ \int_{\partial\Omega} \nu(Q) d\mu_P(Q) : d\mu_P \text{ is a bounded Borel measure on } \partial\Omega, \right. \\ &\left. \int_{\partial\Omega} d\mu_P(Q) = 1 \quad \text{and} \quad \text{supp}(d\mu_P) \subset \Pi_{\partial\Omega}(P) \right\}, \end{aligned}$$

where $\Pi_{\partial\Omega}(P) := \{Q \in \partial\Omega : |Q - P| = d_{\partial\Omega}(P)\}$ and $\nu(Q)$ denotes the unit inward normal at the point $Q \in \partial\Omega$.

3. THE LYAPUNOV-SCHMIDT REDUCTION

Let $k \geq 1$ be a fixed integer. Let $\Lambda_\delta := \{\mathbf{P} := (P_1, \dots, P_k) : \varphi(\mathbf{P}) \geq \delta\}$ for some small $\delta > 0$, where the function φ is defined in (1.6).

Fix $\mathbf{P} \in \Lambda_\delta$ and set

$$\begin{aligned} \mathcal{K}_{\varepsilon, \mathbf{P}} &= \text{span} \left\{ \frac{\partial \mathcal{P} w_i}{\partial P_i^j} : i = 1, \dots, k, j = 1, \dots, N \right\} \subset H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon), \\ \mathcal{C}_{\varepsilon, \mathbf{P}} &= \text{span} \left\{ \frac{\partial \mathcal{P} w_i}{\partial P_i^j} : i = 1, \dots, k, j = 1, \dots, N \right\} \subset L^2(\Omega_\varepsilon), \end{aligned}$$

denoting by P_i^j the j -th component of P_i for $j = 1, \dots, N$. We also need the following spaces:

$$\begin{aligned} \mathcal{K}_{\varepsilon, \mathbf{P}}^\perp &= \left\{ u \in H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} u \frac{\partial \mathcal{P} w_i}{\partial P_i^j} = 0, i = 1, \dots, k, j = 1, \dots, N \right\}, \\ \mathcal{C}_{\varepsilon, \mathbf{P}}^\perp &= \left\{ u \in L^2(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} u \frac{\partial \mathcal{P} w_i}{\partial P_i^j} = 0, i = 1, \dots, k, j = 1, \dots, N \right\}. \end{aligned}$$

First of all, we point out that solving problem (1.8) is equivalent to solving

$$(3.1) \quad S_\varepsilon(u) := \Delta u + u - f(u) = 0, \quad u \in H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon).$$

We have the following result.

Proposition 3.1. *For any $\delta > 0$ there exist $\varepsilon_0 > 0$ and $c > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and for any $\mathbf{P} \in \Lambda_\delta$ there exists a unique $\Phi_{\varepsilon, \mathbf{P}} \in \mathcal{K}_{\varepsilon, \mathbf{P}}^\perp$ such that*

$$S_\varepsilon \left(\sum_{i=1}^k \lambda_i \mathcal{P}w_i + \Phi_{\varepsilon, \mathbf{P}} \right) \in \mathcal{C}_{\varepsilon, \mathbf{P}}^\perp.$$

Moreover,

$$\|\Phi_{\varepsilon, \mathbf{P}}\|_{\mathbf{H}^2(\Omega_\varepsilon)} \leq c\varepsilon^{-(1+\frac{\sigma}{2})\frac{\varphi(\mathbf{P})}{\varepsilon}}.$$

Finally, the function $\mathbf{P} \rightarrow \Phi_{\varepsilon, \mathbf{P}} \in \mathbf{H}_0^1(\Omega_\varepsilon)$ is of class \mathcal{C}^1 .

Proof. We argue exactly as in [19], Section 3, and [5], Section 3. □

Let us introduce the reduced energy $\tilde{J}_\varepsilon : \Lambda_\delta \rightarrow \mathbb{R}$ as

$$\tilde{J}_\varepsilon(\mathbf{P}) := J_\varepsilon \left(\sum_{i=1}^k \lambda_i \mathcal{P}w_i + \Phi_{\varepsilon, \mathbf{P}} \right).$$

The following result holds.

Proposition 3.2. *If $\mathbf{P} \in \Lambda_\delta$ is a critical point of \tilde{J}_ε , then the function $u_\varepsilon = \sum_{i=1}^k \lambda_i \mathcal{P}w_i + \Phi_{\varepsilon, \mathbf{P}}$ is a critical point of J_ε , i.e. a solution to problem (1.8).*

Proof. We argue exactly as in [5], Proposition 3.6. □

We need the expansion of the reduced energy.

Lemma 3.3. *It holds that*

$$\tilde{J}_\varepsilon(\mathbf{P}) = kI(w) + \frac{1}{2}(\gamma + o(1)) \sum_{i=1}^k \varphi_{\varepsilon, P_i}(P_i) - \frac{1}{2}(\gamma + o(1)) \sum_{\substack{i, h=1 \\ i \neq h}}^k \lambda_i \lambda_h w \left(\frac{|P_i - P_h|}{\varepsilon} \right)$$

uniformly with respect to $\mathbf{P} \in \Lambda_\delta$. Here

$$(3.2) \quad I(w) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) dy - \int_{\mathbb{R}^N} F(w) dy,$$

$$(3.3) \quad \gamma := \frac{1}{2} \int_{\mathbb{R}^N} f(w) e^{-y_1} dy.$$

Proof. We argue exactly as in Lemma 2.6 and Proposition 4.1 of [19]. □

Lemma 3.4. *If $h = 1, \dots, k$ and $j = 1, \dots, N$, it holds that*

$$\begin{aligned} \frac{\partial \tilde{J}_\varepsilon}{\partial P_h^j}(\mathbf{P}) &= -\frac{\gamma}{\varepsilon} [\alpha(P_h)]^j \varphi_{\varepsilon, P_h}(P_h) + \frac{\gamma}{\varepsilon} \sum_{\substack{i=1 \\ i \neq h}}^k \lambda_i \lambda_h \frac{P_h^j - P_i^j}{|P_h - P_i|} w \left(\frac{|P_i - P_h|}{\varepsilon} \right) \\ &\quad + \frac{1}{\varepsilon} o \left(\sum_{i=1}^k \varphi_{\varepsilon, P_i}(P_i) + \sum_{\substack{i, l=1 \\ i \neq l}}^k w \left(\frac{|P_i - P_l|}{\varepsilon} \right) \right) \end{aligned}$$

uniformly with respect to $\mathbf{P} \in \Lambda_\delta$, where $\alpha(P_h) \in \partial d_{\partial\Omega}(P_h)$ (see (2.1)) and $[\alpha(P_h)]^j$ denotes the j -th component of $\alpha(P_h)$.

Proof. Arguing exactly as in Lemma 4.2 of [5], if $h = 1, \dots, k$ and $j = 1, \dots, N$, we have

$$\begin{aligned}
 \frac{\partial \tilde{J}_\varepsilon}{\partial P_h^j}(\mathbf{P}) &= J'_\varepsilon \left(\sum_{i=1}^k \lambda_i \mathcal{P}w_i + \Phi_{\varepsilon, \mathbf{P}} \right) \left[\lambda_h \frac{\partial \mathcal{P}w_h}{\partial P_h^j} + \frac{\partial \Phi_{\varepsilon, \mathbf{P}}}{\partial P_h^j} \right] \\
 &= J'_\varepsilon \left(\sum_{i=1}^k \lambda_i \mathcal{P}w_i \right) \left[\lambda_h \frac{\partial \mathcal{P}w_h}{\partial P_h^j} \right] \\
 &\quad + \left[J'_\varepsilon \left(\sum_{i=1}^k \lambda_i \mathcal{P}w_i + \Phi_{\varepsilon, \mathbf{P}} \right) - J'_\varepsilon \left(\sum_{i=1}^k \lambda_i \mathcal{P}w_i \right) \right] \left[\lambda_h \frac{\partial \mathcal{P}w_h}{\partial P_h^j} \right] \\
 &\quad + J'_\varepsilon \left(\sum_{i=1}^k \lambda_i \mathcal{P}w_i + \Phi_{\varepsilon, \mathbf{P}} \right) \left[\frac{\partial \Phi_{\varepsilon, \mathbf{P}}}{\partial P_h^j} \right] \\
 (3.4) \quad &= J'_\varepsilon \left(\sum_{i=1}^k \lambda_i \mathcal{P}w_i \right) \left[\lambda_h \frac{\partial \mathcal{P}w_h}{\partial P_h^j} \right] + h.o.t.,
 \end{aligned}$$

where

$$h.o.t. = \frac{1}{\varepsilon} o \left(\sum_{i=1}^k \varphi_{\varepsilon, P_i}(P_i) + \sum_{\substack{i, l=1 \\ i \neq l}}^k w \left(\frac{|P_i - P_l|}{\varepsilon} \right) \right).$$

It remains to estimate the leading term of (3.5):

$$\begin{aligned}
 (3.5) \quad & J'_\varepsilon \left(\sum_{i=1}^k \lambda_i \mathcal{P}w_i \right) \left[\lambda_h \frac{\partial \mathcal{P}w_h}{\partial P_h^j} \right] \\
 &= \sum_{i=1}^k \lambda_i \lambda_h \int_{\Omega_\varepsilon} \left(\nabla \mathcal{P}w_i \nabla \frac{\partial \mathcal{P}w_h}{\partial P_h^j} + \mathcal{P}w_i \frac{\partial \mathcal{P}w_h}{\partial P_h^j} \right) dy - \int_{\Omega_\varepsilon} f \left(\sum_{i=1}^k \lambda_i \mathcal{P}w_i \right) \lambda_h \frac{\partial \mathcal{P}w_h}{\partial P_h^j} \\
 &= \sum_{i=1}^k \lambda_i \lambda_h \int_{\Omega_\varepsilon} f(w_i) \frac{\partial \mathcal{P}w_h}{\partial P_h^j} - \int_{\Omega_\varepsilon} f \left(\sum_{i=1}^k \lambda_i \mathcal{P}w_i \right) \lambda_h \frac{\partial \mathcal{P}w_h}{\partial P_h^j} \\
 &= \sum_{\substack{i=1 \\ i \neq h}}^k \lambda_i \lambda_h \int_{\Omega_\varepsilon} f(w_i) \frac{\partial \mathcal{P}w_h}{\partial P_h^j} + \int_{\Omega_\varepsilon} [f(w_h) - f(\mathcal{P}w_h)] \frac{\partial \mathcal{P}w_h}{\partial P_h^j} \\
 &\quad - \int_{\Omega_\varepsilon} \left[f \left(\sum_{i=1}^k \lambda_i \mathcal{P}w_i \right) - f(\lambda_h \mathcal{P}w_h) \right] \lambda_h \frac{\partial \mathcal{P}w_h}{\partial P_h^j}.
 \end{aligned}$$

Arguing as in Lemma 4.1 of [17] (see estimate (4.6)) we get

$$(3.6) \quad \int_{\Omega_\varepsilon} [f(w_h) - f(\mathcal{P}w_h)] \frac{\partial \mathcal{P}w_h}{\partial P_h^j} = -\gamma [\alpha(P_h)]^j \frac{\varphi_{\varepsilon, P_h}(P_h)}{\varepsilon} + o \left(\frac{\varphi_{\varepsilon, P_h}(P_h)}{\varepsilon} \right),$$

where γ is defined in (3.3) and $\alpha(P_h) := \int_{\partial\Omega} \frac{Q-P}{|Q-P|} d\mu_P \in \partial d_{\partial\Omega}(P_h)$ (because of Lemma 2.3 and (2.1)).

Arguing as in Lemma 4.1 of [5] (see the proof of estimate (4.2)) we get for $i \neq h$

$$(3.7) \quad \int_{\Omega_\varepsilon} f(w_i) \frac{\partial \mathcal{P}w_h}{\partial P_h^j} = \frac{\gamma}{\varepsilon} w \left(\frac{|P_i - P_h|}{\varepsilon} \right) \frac{P_i - P_h}{|P_i - P_h|} + h.o.t.,$$

and arguing as in Lemma 4.2 of [5] (see the estimate of I_3) we get

$$(3.8) \quad \begin{aligned} & \int_{\Omega_\varepsilon} \left[f \left(\sum_{i=1}^k \lambda_i \mathcal{P}w_i \right) - f(\lambda_h \mathcal{P}w_h) \right] \lambda_h \frac{\partial \mathcal{P}w_h}{\partial P_h^j} \\ &= 2 \frac{\gamma}{\varepsilon} \sum_{\substack{i=1 \\ i \neq h}}^k \lambda_i \lambda_h w \left(\frac{|P_i - P_h|}{\varepsilon} \right) \frac{P_i - P_h}{|P_i - P_h|} + h.o.t. \end{aligned}$$

Collecting all the previous estimates, the claim follows. □

By Lemma 3.3 and Lemma 3.4 we deduce the following remark.

Remark 3.5. For any $i = 1, \dots, k$ and $j = 1, \dots, N$ it holds that

$$J_\varepsilon(\mathcal{P}w_i) = I(w) + \frac{1}{2}(\gamma + o(1))\varphi_{\varepsilon, P_i}(P_i)$$

and

$$\frac{\partial}{\partial P_i^j} J_\varepsilon(\mathcal{P}w_i) = -\frac{\gamma}{\varepsilon} [\alpha(P_i)]^j \varphi_{\varepsilon, P_i}(P_i)(1 + o(1))$$

uniformly with respect to P_i in compact sets of Ω , where $\alpha(P_i) \in \partial d_{\partial\Omega}(P_i)$ (see (2.1)).

4. THE MAX-MIN ARGUMENT

According to Proposition 3.2 we just need to prove that the reduced functional \tilde{J}_ε has a critical point to find a solution of (1.2). Let $k_1, k_2 \geq 1$ be such that

$$(4.1) \quad k := k_1 + k_2 \leq 6 \quad \text{and} \quad (k_1, k_2) \neq (1, 5), (5, 1) \text{ if } N = 2.$$

We need the following auxiliary lemma.

Lemma 4.1. *There exists a continuous function*

$$\begin{aligned} \mathbf{S} : \mathbb{R}^N \times (0, \infty)^{k-1} &\longrightarrow \mathbb{R}^{kN}, \\ (P, \mathbf{r}) &\longmapsto \mathbf{S}_{P, \mathbf{r}} = (S_1(P, \mathbf{r}), \dots, S_k(P, \mathbf{r})) \end{aligned}$$

such that, setting $\mathbf{r} = (r_2, \dots, r_k)$,

$$(4.2) \quad r_h = \min_{i < h} |S_h(P, \mathbf{r}) - S_i(P, \mathbf{r})|, \quad h = 2, \dots, k,$$

$$(4.3) \quad |S_i(P, \mathbf{r}) - S_h(P, \mathbf{r})| \geq \sqrt{2} \min_{i=2, \dots, k} r_i \quad \text{if } \lambda_i \lambda_h = 1 \text{ and } i \neq h,$$

$$(4.4) \quad |S_i(P, \mathbf{r}) - P| \leq k \max_{2 \leq i \leq k} r_i.$$

Proof. For the sake of definiteness, we will assume $k_1 \leq k_2$ and set

$$(4.5) \quad \lambda_i = (-1)^{i+1} \text{ if } i = 1, \dots, 2k_1, \quad \lambda_i = -1 \text{ if } i = 2k_1 + 1, \dots, k.$$

If $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ denote the standard basis in \mathbb{R}^N , let us choose $k_2 - k_1 + 1$ vectors (note that $k_2 - k_1 + 1 \leq 2N$ by (4.1)) $Z_1, Z_{2k_1+1}, \dots, Z_k$ such that

$$\{Z_1, Z_{2k_1+1}, \dots, Z_k\} \subset \{\pm \mathbf{e}_1, \pm \mathbf{e}_2, \dots, \pm \mathbf{e}_N\}, \quad Z_i \neq Z_h \text{ for } i \neq h.$$

We define \mathbf{S} in the following way: for any $P \in \mathbb{R}^N$ and $\mathbf{r} = (r_2, \dots, r_k) \in (0, \infty)^{k-1}$ set

$$\begin{cases} S_1(P, \mathbf{r}) = P, \\ S_i(P, \mathbf{r}) = P + \sum_{s=2}^i r_s Z_s & \text{if } 2 \leq i \leq 2k_1, \\ S_i(P, \mathbf{r}) = P + r_i Z_i & \text{if } 2k_1 + 1 \leq i \leq k. \end{cases}$$

We remark that it holds that

$$(4.6) \quad |S_h(P, \mathbf{r}) - S_i(P, \mathbf{r})|^2 \begin{cases} = (r_{i+1} + \dots + r_h)^2 & \text{if } 1 \leq i < h \leq 2k_1, \\ \geq (r_2 + \dots + r_i)^2 + r_h^2 & \text{if } 2 \leq i \leq 2k_1 < h \leq k, \\ \geq r_i^2 + r_h^2 & \text{if } 2k_1 + 1 \leq i < h \leq k \end{cases}$$

and

$$(4.7) \quad |S_i(P, \mathbf{r}) - P| = \begin{cases} 0 & \text{if } i = 1, \\ r_2 + \dots + r_i & \text{if } 2 \leq i \leq 2k_1, \\ r_i & \text{if } 2k_1 + 1 \leq i \leq k. \end{cases}$$

By (4.6) and (4.7), recalling (4.5), we deduce (4.2)–(4.4). □

We want to apply a max-min argument to characterize a topologically nontrivial critical value of J_ε . More precisely, we are going to prove the existence of sets $\mathcal{D}_\varepsilon, K, K_0 \subset \mathbb{R}^{kN}$ satisfying the following properties:

- (P1) \mathcal{D}_ε is an open set with smooth boundary $\partial\mathcal{D}_\varepsilon$, K_0 and K are compact sets, K is connected and

$$(4.8) \quad K_0 \subset K \subset \mathcal{D}_\varepsilon \subset \overline{\mathcal{D}_\varepsilon} \subset \Omega^k.$$

- (P2) If we define the complete metric space \mathcal{F} by

$$\mathcal{F} = \{\eta : K \rightarrow \mathcal{D}_\varepsilon \mid \eta \text{ continuous, } \eta(\mathbf{P}) = \mathbf{P} \forall \mathbf{P} \in K_0\},$$

then

$$(4.9) \quad c_\varepsilon := \sup_{\eta \in \mathcal{F}} \min_{\mathbf{P} \in K} \tilde{J}_\varepsilon[\eta(\mathbf{P})] < \min_{\mathbf{P} \in K_0} \tilde{J}_\varepsilon[\mathbf{P}].$$

- (P3) For every $\mathbf{P} \in \partial\mathcal{D}_\varepsilon$ such that $\tilde{J}_\varepsilon[\mathbf{P}] = c_\varepsilon$, there exists a vector $\tau_{\mathbf{P}}$ tangent to $\partial\mathcal{D}_\varepsilon$ at \mathbf{P} so that $\partial_{\tau_{\mathbf{P}}} \tilde{J}_\varepsilon[\mathbf{P}] \neq 0$.

Under these assumptions a critical point $\mathbf{P}_\varepsilon \in \mathcal{D}_\varepsilon$ of \tilde{J}_ε with $\tilde{J}_\varepsilon[\mathbf{P}_\varepsilon] = c_\varepsilon$ exists, as a standard deformation argument involving the gradient flow of \tilde{J}_ε shows.

4.1. **Definition of the sets $\mathcal{D}_\varepsilon, K, K_0$.** Let $\delta > 0$ be a sufficiently small number and define¹

$$\begin{aligned} \mathcal{D}_\varepsilon &= \left\{ \mathbf{P} \in \Omega^k \mid F_\varepsilon(\mathbf{P}) := \sum_{i=1}^k J_\varepsilon(\mathcal{P}w_i) + \frac{\gamma}{2} \sum_{\substack{i,h=1 \\ i \neq h}}^k w \left(\frac{|P_i - P_h|}{\varepsilon} \right) < kI(w) + e^{-\frac{3\delta}{\varepsilon}} \right\} \\ (4.10) \quad &= \left\{ \mathbf{P} \in \Omega^k \mid \frac{1}{2}(\gamma + o(1)) \sum_{i=1}^k e^{-\frac{\psi_\varepsilon(P_i)}{\varepsilon}} + \frac{\gamma}{2} \sum_{\substack{i,h=1 \\ i \neq h}}^k w \left(\frac{|P_i - P_h|}{\varepsilon} \right) < e^{-\frac{3\delta}{\varepsilon}} \right\}, \end{aligned}$$

because of Remark 3.5.

Observe that by Lemma 2.1 and (1.5)

$$\mathcal{D}_\varepsilon \subset \left\{ \mathbf{P} \in \Omega^k \mid |P_i - P_h| \geq 2\delta \ \forall i \neq h, \ d_{\partial\Omega}(P_i) \geq \delta \right\}.$$

Moreover, we easily deduce that

$$(4.11) \quad -\varepsilon \log \left(\sum_{i=1}^k e^{-\frac{\psi_\varepsilon(P_i)}{\varepsilon}} + \sum_{\substack{i,h=1 \\ i \neq h}}^k w \left(\frac{|P_i - P_h|}{\varepsilon} \right) \right) \rightarrow 2\varphi(\mathbf{P}) \text{ as } \varepsilon \rightarrow 0,$$

where φ is defined in (1.6).

Then let W_ε be the following open set of \mathbb{R}^{N+k-1} :

$$W_\varepsilon = \left\{ (P, \mathbf{r}) \mid \mathbf{S}_{P,\mathbf{r}} \in \Omega^k, F_\varepsilon(\mathbf{S}_{P,\mathbf{r}}) < kI(w) + \frac{1}{2}e^{-\frac{3\delta}{\varepsilon}} \right\},$$

where $\mathbf{S}_{P,\mathbf{r}}$ is defined in Lemma 4.1.

Let $P_0 \in \Omega$ be such that $d_{\partial\Omega}(P_0) = \max_{P \in \Omega} d_{\partial\Omega}(P)$. Let us point that

$$(4.12) \quad (P_0, \mathbf{r}_0) \in W_\varepsilon, \text{ where } \mathbf{r}_0 := (r_0, \dots, r_0) \in \mathbb{R}^{k-1} \text{ and } r_0 = \frac{d_{\partial\Omega}(P_0)}{k+1}.$$

Indeed, according to (4.2), $|S_i(P_0, \mathbf{r}_0) - S_h(P_0, \mathbf{r}_0)| \geq \frac{d_{\partial\Omega}(P_0)}{k+1}$ for $i \neq h$, and by (1.5) we immediately check $w \left(\frac{|S_i(P_0, \mathbf{r}_0) - S_h(P_0, \mathbf{r}_0)|}{\varepsilon} \right) = o \left(e^{-\frac{d_{\partial\Omega}(P_0)}{\varepsilon(k+1)}} \right)$ for $i \neq h$. Moreover, according to (4.4) $|S_i(P_0, \mathbf{r}_0) - P_0| \leq \frac{k d_{\partial\Omega}(P_0)}{k+1}$, which implies $d_{\partial\Omega}(S_i(P_0, \mathbf{r}_0)) \geq \frac{d_{\partial\Omega}(P_0)}{k+1}$, and consequently, by Lemma 2.1, we deduce $\varphi_{\varepsilon, S_i(P_0, \mathbf{r}_0)}(S_i(P_0, \mathbf{r}_0)) = (1 + o(1))e^{-2\frac{d_{\partial\Omega}(S_i(P_0, \mathbf{r}_0))}{\varepsilon}} = O \left(e^{-\frac{2d_{\partial\Omega}(P_0)}{\varepsilon(k+1)}} \right)$. Hence Remark 3.5 gives $J_\varepsilon(\mathcal{P}w_{S_i(P,\mathbf{r})}) = I(w) + O \left(e^{-\frac{d_{\partial\Omega}(P_0)}{\varepsilon(k+1)}} \right)$. If we choose $3\delta < \frac{d_{\partial\Omega}(P_0)}{k+1}$, we obtain $(P_0, \mathbf{r}_0) \in W_\varepsilon$.

Let U_ε be the connected component of W_ε containing (P_0, \mathbf{r}_0) and let us define

$$K = \{ \mathbf{S}_{P,\mathbf{r}} \in \mathbb{R}^{kN} \mid (P, \mathbf{r}) \in \overline{U}_\varepsilon \} \text{ and } K_0 = \{ \mathbf{S}_{P,\mathbf{r}} \in \mathbb{R}^{kN} \mid (P, \mathbf{r}) \in \partial U_\varepsilon \}.$$

K is connected and closed by construction since U_ε is connected and closed. Furthermore, it is obvious that $K_0 \subset K \subset \mathcal{D}_\varepsilon$. K_0 can be rewritten as

$$(4.13) \quad K_0 = \left\{ \mathbf{S}_{P,\mathbf{r}} \mid (P, \mathbf{r}) \in \overline{U}_\varepsilon, F_\varepsilon(\mathbf{S}_{P,\mathbf{r}}) = kI(w) + \frac{1}{2}e^{-\frac{3\delta}{\varepsilon}} \right\}.$$

¹Hereafter, for the sake of simplicity, we will write $\psi_\varepsilon(P)$ in place of $\psi_{\varepsilon,P}(P)$.

It is useful to point out that
 (4.14)

$$\tilde{J}_\varepsilon[\mathbf{S}_{P,\mathbf{r}}] \geq kI(w) + \gamma(1 + o(1)) \sum_{\substack{i,h=1 \\ i \neq h}}^k w\left(\frac{|S_i(P, \mathbf{r}) - S_h(P, \mathbf{r})|}{\varepsilon}\right) \quad \text{unif. for } \mathbf{S}_{P,\mathbf{r}} \in K$$

and

$$(4.15) \quad \tilde{J}_\varepsilon[\mathbf{S}_{P,\mathbf{r}}] = F_\varepsilon[\mathbf{S}_{P,\mathbf{r}}] + o(e^{-\frac{3\delta}{\varepsilon}}) \quad \text{unif. for } \mathbf{S}_{P,\mathbf{r}} \in K.$$

In fact, by (4.2) and (4.3) we get that there exist \bar{i}, \bar{h} such that $\lambda_{\bar{i}} = -\lambda_{\bar{h}}, |S_{\bar{i}}(P, \mathbf{r}) - S_{\bar{h}}(P, \mathbf{r})| = \min_{i \neq h} |S_i(P, \mathbf{r}) - S_h(P, \mathbf{r})|$ and $\min_{i \neq h, \lambda_i = \lambda_h} |S_i(P, \mathbf{r}) - S_h(P, \mathbf{r})| > \sqrt{2}|S_{\bar{i}}(P, \mathbf{r}) - S_{\bar{h}}(P, \mathbf{r})|$. Hence, by (1.5)

$$\sum_{\substack{i,h=1 \\ i \neq h, \lambda_i = \lambda_h}}^k w\left(\frac{S_i(P, \mathbf{r}) - S_h(P, \mathbf{r})}{\varepsilon}\right) = o\left(w\left(\frac{S_{\bar{i}}(P, \mathbf{r}) - S_{\bar{h}}(P, \mathbf{r})}{\varepsilon}\right)\right),$$

and, using Lemma 3.3 and Remark 3.5, (4.14)-(4.15) follow.

4.2. Proof of (4.9). Let $\eta \in \mathcal{F}$; namely $\eta : K \rightarrow \mathcal{D}_\varepsilon$ is a continuous function such that $\eta(\mathbf{P}) = \mathbf{P}$ for any $\mathbf{P} \in K_0$. Setting $\eta = (\eta_1, \dots, \eta_k)$ where $\eta_i : K \rightarrow \mathbb{R}^N$, let $\tilde{\eta} : \bar{U}_\varepsilon \rightarrow \mathbb{R}^N \times \mathbb{R}^{k-1}$ be defined by

$$\tilde{\eta}_1(P, \mathbf{r}) = \eta_1(\mathbf{S}_{P,\mathbf{r}}) \quad \text{and} \quad \tilde{\eta}_i(P, \mathbf{r}) = \min_{h < i} |\eta_i(\mathbf{S}_{P,\mathbf{r}}) - \eta_h(\mathbf{S}_{P,\mathbf{r}})| \quad \text{for } i = 2, \dots, k.$$

First of all $\tilde{\eta}$ is a continuous function because of the continuity of η . Secondly, we claim that $\tilde{\eta}(P, \mathbf{r}) = (P, \mathbf{r})$ for any $(P, \mathbf{r}) \in \partial U_\varepsilon$. In fact, if $(P, \mathbf{r}) \in \partial U_\varepsilon$, then by definition $\mathbf{S}_{P,\mathbf{r}} \in K_0$; consequently $\eta(\mathbf{S}_{P,\mathbf{r}}) = \mathbf{S}_{P,\mathbf{r}}$, by which

$$\tilde{\eta}_1(P, \mathbf{r}) = \eta_1(\mathbf{S}_{P,\mathbf{r}}) = S_1(P, \mathbf{r}) = P,$$

while, using (4.2), for $i \geq 2$

$$\tilde{\eta}_i(P, \mathbf{r}) = \min_{h < i} |\eta_i(\mathbf{S}_{P,\mathbf{r}}) - \eta_h(\mathbf{S}_{P,\mathbf{r}})| = \min_{h < i} |S_i(P, \mathbf{r}) - S_h(P, \mathbf{r})| = r_i.$$

Hence the theory of the topological degree ensures that there exists $(\bar{P}, \bar{\mathbf{r}}) \in U_\varepsilon$ such that $\tilde{\eta}(\bar{P}, \bar{\mathbf{r}}) = (P_0, \mathbf{r}_0)$; that is (see (4.12)),

$$\eta_1(\mathbf{S}_{\bar{P},\bar{\mathbf{r}}}) = P_0 \quad \text{and} \quad \min_{h < i} |\eta_i(\mathbf{S}_{\bar{P},\bar{\mathbf{r}}}) - \eta_h(\mathbf{S}_{\bar{P},\bar{\mathbf{r}}})| = \frac{d_{\partial\Omega}(P_0)}{k+1}, \quad i = 2, \dots, k.$$

In particular

$$|\eta_i(\mathbf{S}_{\bar{P},\bar{\mathbf{r}}}) - \eta_h(\mathbf{S}_{\bar{P},\bar{\mathbf{r}}})| \geq \frac{d_{\partial\Omega}(P_0)}{k+1} \quad \text{if } i \neq h,$$

which implies (because of (1.5))

$$(4.16) \quad w\left(\frac{\eta_i(\mathbf{S}_{\bar{P},\bar{\mathbf{r}}}) - \eta_h(\mathbf{S}_{\bar{P},\bar{\mathbf{r}}})}{\varepsilon}\right) = o\left(e^{-\frac{d_{\partial\Omega}(P_0)}{\varepsilon(k+1)}}\right) \quad \text{if } i \neq h.$$

Moreover, it is not difficult to check that

$$|\eta_i(\mathbf{S}_{\bar{P},\bar{\mathbf{r}}}) - P_0| = |\eta_i(\mathbf{S}_{\bar{P},\bar{\mathbf{r}}}) - \eta_1(\mathbf{S}_{\bar{P},\bar{\mathbf{r}}})| \leq \frac{k}{k+1} d_{\partial\Omega}(P_0), \quad i = 2, \dots, k,$$

by which $d_{\partial\Omega}(\eta_i(\mathbf{S}_{\bar{P},\bar{\mathbf{r}}})) \geq \frac{d_{\partial\Omega}(P_0)}{k+1}$ and consequently, reasoning as above,

$$(4.17) \quad J_\varepsilon(\mathcal{P}w_{\eta_i(\mathbf{S}_{\bar{P},\bar{\mathbf{r}}})}) = I(w) + O\left(e^{-\frac{d_{\partial\Omega}(P_0)}{\varepsilon(k+1)}}\right).$$

Then by Lemma 3.3, (4.16) and (4.17), we deduce that

$$\min_{\mathbf{S}_{P,r} \in K} \tilde{J}_\varepsilon[\eta(\mathbf{S}_{P,r})] \leq kI(w) + O\left(e^{-\frac{d_{\partial\Omega}(P_0)}{\varepsilon(k+1)}}\right).$$

Hence

$$(4.18) \quad c_\varepsilon = \sup_{\eta \in \mathcal{F}} \min_{\mathbf{S}_{P,r} \in K} \tilde{J}_\varepsilon[\eta(\mathbf{S}_{P,r})] \leq kI(w) + O\left(e^{-\frac{d_{\partial\Omega}(P_0)}{\varepsilon(k+1)}}\right).$$

On the other hand, by taking $\eta(\mathbf{S}_{P,r}) = \mathbf{S}_{P,r}$ and using (4.14),

$$(4.19) \quad c_\varepsilon \geq \min_{\mathbf{S}_{P,r} \in K} \tilde{J}_\varepsilon[\mathbf{S}_{P,r}] \geq kI(w).$$

Combining (4.18)-(4.19) we get

$$(4.20) \quad kI(w) \leq c_\varepsilon \leq kI(w) + o\left(e^{-\frac{3\delta}{\varepsilon}}\right).$$

On the other hand, combining (4.13) and (4.15)

$$\min_{\mathbf{S}_{P,r} \in K_0} \tilde{J}_\varepsilon[\mathbf{S}_{P,r}] = kI(w) + \frac{1}{2}(1 + o(1))e^{-\frac{3\delta}{\varepsilon}},$$

and (4.9) follows.

4.3. Proof of (P3). Assume by contradiction that there exists $\mathbf{P} \in \partial\mathcal{D}_\varepsilon$ such that $\tilde{J}_\varepsilon(\mathbf{P}) = c_\varepsilon$ and $\tau_{\mathbf{P}} \cdot \nabla \tilde{J}_\varepsilon(\mathbf{P}) = 0$ for any vector $\tau_{\mathbf{P}}$ tangent to $\partial\mathcal{D}_\varepsilon$ at \mathbf{P} . Therefore, there exists $\mu \in \mathbb{R}$ such that \mathbf{P} is a critical point of the function

$$\mathbf{P} \rightarrow \tilde{J}_\varepsilon(\mathbf{P}) + \mu \sum_{i=1}^k J_\varepsilon(\mathcal{P}w_i) + \mu \frac{\gamma}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^k w \left(\frac{|P_i - P_j|}{\varepsilon} \right).$$

By Lemma 3.4, Remark 3.5 and estimate (1.5) we deduce that

$$\begin{aligned} & -\gamma \frac{\alpha(P_h) + \mu\beta(P_h)}{\varepsilon} e^{-\frac{\psi_\varepsilon(P_h)}{\varepsilon}} + \sum_{\substack{i=1 \\ i \neq h}}^k \gamma \frac{\lambda_i \lambda_h - \mu}{\varepsilon} \frac{P_h - P_i}{|P_h - P_i|} w \left(\frac{|P_i - P_h|}{\varepsilon} \right) \\ & + o\left(\frac{e^{-\frac{3\delta}{\varepsilon}}}{\varepsilon}\right) = 0, \quad h = 1, \dots, k, \end{aligned}$$

where $\alpha(P_h), \beta(P_h) \in \partial d_{\partial\Omega}(P_h)$ (see (2.1)), which can be rewritten as

$$(4.21) \quad \begin{aligned} & -(\alpha(P_h) + \mu\beta(P_h)) e^{-\frac{\psi_\varepsilon(P_h)}{\varepsilon}} - (1 + \mu) \sum_{\substack{i=1 \\ \lambda_i \lambda_h = -1}}^k \frac{P_h - P_i}{|P_h - P_i|} w \left(\frac{|P_i - P_h|}{\varepsilon} \right) \\ & + (1 - \mu) \sum_{\substack{i=1 \\ i \neq h, \lambda_i \lambda_h = +1}}^k \frac{P_h - P_i}{|P_h - P_i|} w \left(\frac{|P_i - P_h|}{\varepsilon} \right) + o\left(e^{-\frac{3\delta}{\varepsilon}}\right) = 0, \quad h = 1, \dots, k, \end{aligned}$$

or equivalently

$$\begin{aligned}
 (4.22) \quad & -\alpha(P_h)e^{-\frac{\psi_\varepsilon(P_h)}{\varepsilon}} - \sum_{\substack{i=1 \\ \lambda_i \lambda_h = -1}}^k \frac{P_h - P_i}{|P_h - P_i|} w\left(\frac{|P_i - P_h|}{\varepsilon}\right) + \sum_{\substack{i=1 \\ i \neq h, \lambda_i \lambda_h = 1}}^k \frac{P_h - P_i}{|P_h - P_i|} w\left(\frac{|P_i - P_h|}{\varepsilon}\right) \\
 & = \mu \left[\beta(P_h)e^{-\frac{\psi_\varepsilon(P_h)}{\varepsilon}} + \sum_{\substack{i=1 \\ i \neq h}}^k \frac{P_h - P_i}{|P_h - P_i|} w\left(\frac{|P_i - P_h|}{\varepsilon}\right) \right] + o\left(e^{-\frac{3\delta}{\varepsilon}}\right), \quad h = 1, \dots, k.
 \end{aligned}$$

Up to a subsequence, we can assume that $P_i \rightarrow \bar{P}_i$ as $\varepsilon \rightarrow 0$. Therefore

$$(4.23) \quad |\bar{P}_i - \bar{P}_j| \geq 3\delta \quad \text{for } i \neq j \quad \text{and } d_{\partial\Omega}(\bar{P}_i) \geq \frac{3}{2}\delta.$$

Let us point out that μ is bounded, and so (up to a subsequence) $\mu \rightarrow \bar{\mu}$ as $\varepsilon \rightarrow 0$. In fact, let $\mathbf{X} := (X_1, \dots, X_k)$, where

$$\begin{aligned}
 X_h := & \beta(P_h) \frac{e^{-\frac{\psi_\varepsilon(P_h)}{\varepsilon}}}{\sum_{i=1}^k e^{-\frac{\psi_\varepsilon(P_i)}{\varepsilon}} + \sum_{\substack{i,j=1 \\ i \neq j}}^k w\left(\frac{|P_i - P_j|}{\varepsilon}\right)} \\
 & + \sum_{\substack{i=1 \\ i \neq h}}^k \frac{P_h - P_i}{|P_h - P_i|} \frac{w\left(\frac{|P_i - P_h|}{\varepsilon}\right)}{\sum_{i=1}^k e^{-\frac{\psi_\varepsilon(P_i)}{\varepsilon}} + \sum_{\substack{i,j=1 \\ i \neq j}}^k w\left(\frac{|P_i - P_j|}{\varepsilon}\right)}, \quad h = 1, \dots, k.
 \end{aligned}$$

By (4.22), taking into account (4.10), we obtain

$$(4.24) \quad |\mu| |X_h| + o(1) \leq c,$$

recalling $|\alpha(P_h)| \leq 1$ and $\mathbf{P} \in \partial\mathcal{D}_\varepsilon$. Moreover, since $\mathbf{P} \in \partial\mathcal{D}_\varepsilon$, by (4.11) we deduce that $\varphi(\bar{\mathbf{P}}) = 3\delta/2$ where $\bar{\mathbf{P}} := (\bar{P}_1, \dots, \bar{P}_k)$, and by Lemma 2.6 in [18] we deduce that $\mathbf{X} \rightarrow \bar{\mathbf{X}} \in \partial\varphi(\bar{\mathbf{P}})$. Finally, by Proposition 2.7 in [18] we get that if $\delta > 0$ is small enough, then

$$(4.25) \quad \inf \{|\mathbf{X}| : \mathbf{X} \in \partial\varphi(\mathbf{P}), \varphi(\mathbf{P}) \leq 3\delta/2\} > 0.$$

(4.24) and (4.25) imply that μ is bounded.

Let

$$(4.26) \quad a_{ih} := \lim_{\varepsilon \rightarrow 0} w\left(\frac{|P_i - P_h|}{\varepsilon}\right) e^{\frac{3\delta}{\varepsilon}} \quad \text{and} \quad c_i := \lim_{\varepsilon \rightarrow 0} e^{-\frac{\psi_\varepsilon(P_i)}{\varepsilon}} e^{\frac{3\delta}{\varepsilon}}.$$

We point out that

$$a_{ih} > 0 \quad \Rightarrow \quad |\bar{P}_i - \bar{P}_h| = 3\delta$$

and

$$c_i > 0 \quad \Rightarrow \quad 2d_{\partial\Omega}(\bar{P}_i) = 3\delta \quad \text{and} \quad \partial d_{\partial\Omega}(\bar{P}_i) = \{\nu_i\},$$

where ν_i denotes the unit inward normal at the unique point $Q_i \in \partial\Omega$ such that $|Q_i - \bar{P}_i| = d_{\partial\Omega}(\bar{P}_i)$.

We multiply (4.21) by $e^{\frac{3\delta}{\varepsilon}}$. Then we pass to the limit as $\varepsilon \rightarrow 0$ and get the system

$$(4.27) \quad \begin{aligned} & -(\alpha(\bar{P}_h) + \bar{\mu}\beta(\bar{P}_h))c_h - (1 + \bar{\mu}) \sum_{\substack{i=1 \\ \lambda_i \lambda_h = -1}}^k a_{ih} \frac{\bar{P}_h - \bar{P}_i}{|\bar{P}_h - \bar{P}_i|} \\ & + (1 - \bar{\mu}) \sum_{\substack{i=1 \\ i \neq h, \lambda_i \lambda_h = +1}}^k a_{ih} \frac{\bar{P}_h - \bar{P}_i}{|\bar{P}_h - \bar{P}_i|} = 0, \quad h = 1, \dots, k. \end{aligned}$$

Here $\alpha(\bar{P}_h), \beta(\bar{P}_h) \in \partial d_{\partial\Omega}(\bar{P}_h)$ (see (2.1)).

Let us point out that

$$(4.28) \quad \sum_{\substack{i,j=1 \\ i \neq j, \lambda_i \lambda_j = +1}}^k w\left(\frac{|P_i - P_j|}{\varepsilon}\right) \geq C e^{-\frac{3\delta}{\varepsilon}}$$

for some constant $C > 0$. In fact, by (4.20) we deduce that

$$(4.29) \quad \sum_{i=1}^k e^{-\frac{\psi_\varepsilon(P_i)}{\varepsilon}} + \sum_{\substack{i,j=1 \\ \lambda_i \lambda_j = -1}}^k w\left(\frac{|P_i - P_j|}{\varepsilon}\right) - \sum_{\substack{i,j=1 \\ i \neq j, \lambda_i \lambda_j = +1}}^k w\left(\frac{|P_i - P_j|}{\varepsilon}\right) = o\left(e^{-\frac{3\delta}{\varepsilon}}\right).$$

On the other hand, if $\mathbf{P} \in \partial\mathcal{D}_\varepsilon$, then

$$(4.30) \quad (1 + o(1)) \sum_{i=1}^k e^{-\frac{\psi_\varepsilon(P_i)}{\varepsilon}} + \sum_{\substack{i,j=1 \\ \lambda_i \lambda_j = -1}}^k w\left(\frac{|P_i - P_j|}{\varepsilon}\right) + \sum_{\substack{i,j=1 \\ i \neq j, \lambda_i \lambda_j = 1}}^k w\left(\frac{|P_i - P_j|}{\varepsilon}\right) = \frac{2}{\gamma} e^{-\frac{3\delta}{\varepsilon}}.$$

If $\sum_{\substack{i,j=1 \\ i \neq j, \lambda_i \lambda_j = +1}}^k w\left(\frac{|P_i - P_j|}{\varepsilon}\right) = o\left(e^{-\frac{3\delta}{\varepsilon}}\right)$, by (4.29) and (4.30) we will get a contradiction. By (4.28) we deduce that

$$(4.31) \quad \sum_{\substack{i,j=1 \\ i \neq j, \lambda_i \lambda_j = +1}}^k a_{ij} > 0.$$

Moreover, it holds that

$$(4.32) \quad \sum_{i=1}^k e^{-\frac{\psi_\varepsilon(P_i)}{\varepsilon}} + \sum_{\substack{i,j=1 \\ \lambda_i \lambda_j = -1}}^k w\left(\frac{|P_i - P_j|}{\varepsilon}\right) \geq C e^{-\frac{3\delta}{\varepsilon}}$$

for some $C > 0$. In fact, by (4.20) we deduce that

$$(4.33) \quad \sum_{i=1}^k e^{-\frac{\psi_\varepsilon(P_i)}{\varepsilon}} + \sum_{\substack{i,j=1 \\ \lambda_i \lambda_j = -1}}^k w\left(\frac{|P_i - P_j|}{\varepsilon}\right) - \sum_{\substack{i,j=1 \\ i \neq j, \lambda_i \lambda_j = 1}}^k w\left(\frac{|P_i - P_j|}{\varepsilon}\right) + o\left(e^{-\frac{3\delta}{\varepsilon}}\right) \geq 0.$$

If $\sum_{i=1}^k e^{-\frac{\psi_\varepsilon(P_i)}{\varepsilon}} + \sum_{\substack{i,j=1 \\ \lambda_i \lambda_j = -1}}^k w\left(\frac{|P_i - P_j|}{\varepsilon}\right) = o\left(e^{-\frac{3\delta}{\varepsilon}}\right)$, by (4.28) and (4.33) we will get a contradiction. By (4.32) we achieve

$$(4.34) \quad \sum_{i=1}^k c_i + \sum_{\substack{i,j=1 \\ i \neq j, \lambda_i \lambda_j = -1}}^k a_{ij} > 0.$$

If $c_i = 0$ for any $i = 1, \dots, k$, system (4.27) reduces to

$$(4.35) \quad \sum_{\substack{i=1 \\ i \neq h}}^k A_{ih} \frac{\bar{P}_h - \bar{P}_i}{|\bar{P}_i - \bar{P}_h|} = 0, \quad h = 1, \dots, k,$$

where

$$A_{ih} := \begin{cases} -(1 + \bar{\mu})a_{ih} & \text{if } \lambda_i \lambda_h = -1, \\ (1 - \bar{\mu})a_{ih} & \text{if } \lambda_i \lambda_h = +1. \end{cases}$$

By (4.31) and (4.34) we deduce that $A_{ih} \neq 0$ for at least one pair of indexes i and h . A contradiction arises because of Proposition 4.1.

Assume now that $c_i > 0$ for some i . Therefore system (4.27) reduces to the system

$$(4.36) \quad \left\{ \begin{array}{l} -(1 + \bar{\mu})c_h \nu_h - (1 + \bar{\mu}) \sum_{\substack{i=1 \\ \lambda_i \lambda_h = -1}}^k a_{ih} \frac{\bar{P}_h - \bar{P}_i}{|\bar{P}_i - \bar{P}_h|} \\ \quad + (1 - \bar{\mu}) \sum_{\substack{i=1 \\ i \neq h, \lambda_i \lambda_h = +1}}^k a_{ih} \frac{\bar{P}_h - \bar{P}_i}{|\bar{P}_i - \bar{P}_h|} = 0, \text{ if } c_h > 0, \\ -(1 + \bar{\mu}) \sum_{\substack{i=1 \\ \lambda_i \lambda_h = -1}}^k a_{ih} \frac{\bar{P}_h - \bar{P}_i}{|\bar{P}_i - \bar{P}_h|} \\ \quad + (1 - \bar{\mu}) \sum_{\substack{i=1 \\ i \neq h, \lambda_i \lambda_h = +1}}^k a_{ih} \frac{\bar{P}_h - \bar{P}_i}{|\bar{P}_i - \bar{P}_h|} = 0, \text{ otherwise.} \end{array} \right.$$

If $\bar{\mu} = -1$ we get the system $\sum_{\substack{i=1 \\ i \neq h, \lambda_i \lambda_h = +1}}^k a_{ih} \frac{\bar{P}_h - \bar{P}_i}{|\bar{P}_i - \bar{P}_h|} = 0, \quad h = 1, \dots, k$. By (4.31)

we deduce that $a_{ih} \neq 0$ for at least a pair of indexes i and h with $\lambda_i = \lambda_h$, and a contradiction arises again because of Proposition 4.1.

Assume $\bar{\mu} \neq -1$. Without loss of generality we can assume $c_1 > 0$. Let

$$\mathcal{H} := \{1\} \cup \{h \mid \exists i_1, \dots, i_l \text{ s.t. } a_{1i_1} > 0, a_{1i_2} > 0, \dots, a_{1i_l} > 0\}$$

and $\mathcal{I} := \{1, \dots, k\} \setminus \mathcal{H}$. \mathcal{H} could reduce to $\{1\}$. It is easy to check that if $i \in \mathcal{I}$ and $h \in \mathcal{H}$, then $a_{ih} = 0$. We point out that if $h \in \mathcal{H}$, then $|\bar{P}_1 - \bar{P}_h| \leq 3k\delta$ and

$d_{\partial\Omega}(\bar{P}_h) \leq (6k + 3)\delta/2$. Therefore, if δ is small enough,
 (4.37) for any $h \in \mathcal{H}$ $\partial d_{\partial\Omega}(\bar{P}_h) = \{\nu_h\}$ and $\nu_1 \cdot \nu_h > 0$.

Finally, by adding all the equations in system (4.36) with index $h \in \mathcal{H}$, we get $\sum_{h \in \mathcal{H}} c_h \nu_h = 0$, and a contradiction arises because of (4.37). That concludes the proof.

Proposition 4.1. *Let $k \geq 2$, $N \geq 2$ and consider the function*

$$(4.38) \quad (X_1, \dots, X_k) \in \mathbb{R}^{Nk} \mapsto \sum_{\substack{i,h=1 \\ i \neq h}}^k A_{ih} |X_i - X_h|,$$

where $A_{ih} = A_{hi}$. If (4.38) is not identically zero and there exists $(\bar{X}_1, \dots, \bar{X}_k)$ a critical point of (4.38) satisfying

$$|\bar{X}_i - \bar{X}_h| \geq 1 \text{ for } i \neq h \quad \text{and} \quad |\bar{X}_i - \bar{X}_h| = 1 \text{ if } A_{ih} \neq 0,$$

then $k \geq 7$.

Proof. See [12]. □

5. THE SYMMETRIC CASE

Theorem 5.1. *Assume that hypotheses (f1)–(f3) hold and that Ω is symmetric with respect to the reflection at $\mathbb{R} \times \{0\}$ and $\Omega \cap \mathbb{R} \times \{0\} \neq \emptyset$. Let $k \geq 1$ be a fixed integer. Then, for $\varepsilon > 0$ sufficiently small, the problem (1.2) as a solution $v_\varepsilon \in H^2(\Omega)$.*

Furthermore there exist $P_i^\varepsilon \in \Sigma := \Omega \cap \mathbb{R} \times \{0\}$, $P_i^\varepsilon := (t_i^\varepsilon, 0)$ with $t_1^\varepsilon < \dots < t_k^\varepsilon$ such that, as $\varepsilon \rightarrow 0^+$,

- (i) $v_\varepsilon(x) = \sum_{i=1}^k (-1)^i w\left(\frac{x - P_i^\varepsilon}{\varepsilon}\right) + o(e^{-\frac{\delta}{\varepsilon}})$ uniformly for $x \in \bar{\Omega}$ for some $\delta > 0$;
- (ii) $\mathbf{t}^\varepsilon := (t_1^\varepsilon, \dots, t_k^\varepsilon) \rightarrow \mathbf{t}^*$ as ε goes to zero, where

$$\varphi(\mathbf{t}^*) := \max_{\substack{\mathbf{t} \in \Sigma^k \\ t_1 < \dots < t_k}} \min_i \left\{ d_{\partial\Sigma}(t_i), \frac{t_{i+1} - t_i}{2} \right\}.$$

Proof. For any integer k we look for a symmetric solution to (1.2) as in (1.10) with $\lambda_i = (-1)^i$ and

$$P_i := (t_i, 0), \quad t_i \in \Sigma := \Omega \cap \mathbb{R} \times \{0\} \quad \text{and} \quad t_1 < \dots < t_k.$$

The rest term $\Phi_{\varepsilon, \mathbf{P}}$ will be chosen even with respect to the variables x_2, \dots, x_N .

Arguing in a standard way, we are led to find a critical point of the reduced energy

$$(5.1) \quad \tilde{J}_\varepsilon(\mathbf{t}) = kI(w) + \frac{1}{2}(\gamma + o(1)) \sum_{i=1}^k e^{-\frac{\psi_\varepsilon(t_i, 0)}{\varepsilon}} + (\gamma + o(1)) \sum_{i=2}^k w\left(\frac{t_i - t_{i-1}}{\varepsilon}\right),$$

where the previous expansion holds uniformly on compact sets of $\{\mathbf{t} \in \Sigma^k : t_1 < \dots < t_k\}$. Moreover, it is easy to check that \tilde{J}_ε has a minimum point \mathbf{t}_ε on the set $\{\mathbf{t} \in \Sigma^k : t_1 < \dots < t_k\}$ and that (ii) holds. □

Theorem 5.2. *Assume that hypotheses (f1)–(f3) hold and that Ω is the unit ball. Let $5 \geq k \geq 1$ be a fixed integer. Then, for $\varepsilon > 0$ sufficiently small, the problem (1.2) has a solution $v_\varepsilon \in H^2(\Omega)$.*

Furthermore, there exist $P_j^\varepsilon := (\rho_\varepsilon e^{\frac{2\pi i}{k}j}, 0)$ (\mathbf{i} denotes the imaginary unit), $j = 1, \dots, k$, with $\rho_\varepsilon \in (0, 1)$ such that, as $\varepsilon \rightarrow 0^+$,

- (i) $v_\varepsilon(x) = \sum_{j=1}^k w\left(\frac{x - P_j^\varepsilon}{\varepsilon}\right) - w\left(\frac{x}{\varepsilon}\right) + o(e^{-\frac{\delta}{\varepsilon}})$ uniformly for $x \in \overline{\Omega}$ for some $\delta > 0$;
- (ii) $\rho_\varepsilon \rightarrow \frac{2}{3}$ as ε goes to zero.

Proof. For any integer k we look for a symmetric solution to (1.2) as in (1.10) with $\lambda_1 = \dots = \lambda_k = +1$ and $\lambda_{k+1} = -1$, $P_{k+1} = 0$ and $P_j = (\rho e^{\frac{2\pi i}{k}j}, 0)$, for $j = 1, \dots, k$ with $\rho \in (0, 1)$. The rest term $\Phi_{\varepsilon, \mathbf{P}}$ is chosen even with respect to each x_h , $h = 3, \dots, N$, and, using the complex notation $z = x_1 + ix_2$,

$$\Phi_{\varepsilon, \mathbf{P}}(z, x_3, \dots, x_N) = \Phi_{\varepsilon, \mathbf{P}}(ze^{\frac{2\pi i}{k}}, x_3, \dots, x_N) \quad \forall (z, x_2, \dots, x_N) \in \mathbb{R}^N.$$

Arguing in a standard way, we are led to find a critical point of the reduced energy

$$\begin{aligned} \tilde{J}_\varepsilon(\rho) &= kI(w) + \frac{1}{2}(\gamma + o(1))e^{-\frac{\psi_\varepsilon(0)}{\varepsilon}} + \frac{k}{2}(\gamma + o(1))e^{-\frac{\psi_\varepsilon(\rho e^{2\pi i}, 0)}{\varepsilon}} \\ &\quad + k(\gamma + o(1))w\left(\frac{\rho}{\varepsilon}\right) - k(\gamma + o(1))\sum_{j=1}^{k-1} w\left(\rho \frac{|e^{2\pi i} - e^{\frac{2\pi i}{k}j}|}{\varepsilon}\right) \\ &= kI(w) + \frac{1}{2}(\gamma + o(1))e^{-\frac{\psi_\varepsilon(0)}{\varepsilon}} + \frac{k}{2}(\gamma + o(1))e^{-\frac{\psi_\varepsilon(\rho e^{2\pi i}, 0)}{\varepsilon}} \\ (5.2) \quad &\quad + k(\gamma + o(1))w\left(\frac{\rho}{\varepsilon}\right) - 2k(\gamma + o(1))w\left(\rho \frac{(2 - 2\cos \frac{2\pi}{k})^{1/2}}{\varepsilon}\right), \end{aligned}$$

where the previous expansion holds uniformly on compact sets of $(0, 1)$. If $2 - 2\cos \frac{2\pi}{k} > 1$, i.e. $k < 6$, by (5.2) we deduce that

$$\tilde{J}_\varepsilon(\rho) = kI(w) + \frac{1}{2}(\gamma + o(1))e^{-\frac{\psi_\varepsilon(0)}{\varepsilon}} + \frac{k}{2}(\gamma + o(1))e^{-\frac{\psi_\varepsilon(\rho e^{2\pi i}, 0)}{\varepsilon}} + k(\gamma + o(1))w\left(\frac{\rho}{\varepsilon}\right),$$

and so by (1.5) and Lemma 2.1 there exists a minimum point $\rho_\varepsilon \in (0, 1)$. In order to get such a minimum the last two terms of $\tilde{J}_\varepsilon(\rho)$ have to balance, hence they must have the same order, by which, using again (1.5) and Lemma 2.1, as ε goes to zero, $\rho_\varepsilon \rightarrow \rho^*$ such that

$$2(1 - \rho^*) = \rho^*,$$

and (ii) follows. □

Theorem 5.3. *Assume that hypotheses (f1)–(f3) hold and that Ω is the unit ball. Let $k \geq 1$ be a fixed even integer. Then, for $\varepsilon > 0$ sufficiently small, the problem (1.2) has a solution $v_\varepsilon \in H^2(\Omega)$.*

Furthermore there exist $P_j^\varepsilon := (\rho_\varepsilon e^{\frac{2\pi i}{k}j}, 0)$, $j = 1, \dots, k$, with $\rho_\varepsilon \in (0, 1)$ such that, as $\varepsilon \rightarrow 0^+$,

- (i) $v_\varepsilon(x) = \sum_{j=1}^k (-1)^j w\left(\frac{x - P_j^\varepsilon}{\varepsilon}\right) + o(e^{-\frac{\delta}{\varepsilon}})$ uniformly for $x \in \overline{\Omega}$ for some $\delta > 0$;
- (ii) $\rho_\varepsilon \rightarrow \frac{2}{2 + (2 - 2\cos(2\pi/k))^{1/2}}$ as ε goes to zero.

Proof. For any even integer k we look for a symmetric solution to (1.2) as in (1.10) with $P_j = (\rho e^{\frac{2\pi i}{k}j}, 0)$, $\rho \in (0, 1)$ and $\lambda_j = (-1)^j$ for $j = 1, \dots, k$. The rest term

$\Phi_{\varepsilon, \mathbf{P}}$ is chosen even with respect to each x_h , $h = 3, \dots, N$, and, using the complex notation $z = x_1 + ix_2$,

$$\Phi_{\varepsilon, \mathbf{P}}(z, x_3, \dots, x_N) = -\Phi_{\varepsilon, \mathbf{P}}(ze^{\frac{2\pi i}{k}}, x_3, \dots, x_N) \quad \forall (z, x_2, \dots, x_N) \in \mathbb{R}^N.$$

Arguing in a standard way, we are led to find a critical point of the reduced energy

$$\begin{aligned} \tilde{J}_\varepsilon(\rho) &= kI(w) + \frac{k}{2}(\gamma + o(1))e^{-\frac{\psi_\varepsilon(\rho e^{\frac{2\pi i}{k}}, 0)}{\varepsilon}} - k(\gamma + o(1)) \sum_{j=1}^{k-1} \lambda_j w \left(\rho \frac{|e^{2\pi i} - e^{\frac{2\pi i}{k}j}|}{\varepsilon} \right) \\ &= kI(w) + \frac{k}{2}(\gamma + o(1))e^{-\frac{\psi_\varepsilon(\rho e^{\frac{2\pi i}{k}}, 0)}{\varepsilon}} + 2k(\gamma + o(1))w \left(\rho \frac{(2 - 2\cos \frac{2\pi}{k})^{1/2}}{\varepsilon} \right), \end{aligned}$$

where the previous expansion holds uniformly on compact sets of $(0, 1)$. Therefore by (1.5) and Lemma 2.1 there exists a minimum point ρ_ε . In order to get such a minimum the last two terms of $\tilde{J}_\varepsilon(\rho)$ have to balance, consequently they must have the same order, by which, again using (1.5) and Lemma 2.1, as ε goes to zero, $\rho_\varepsilon \rightarrow \rho^*$ such that

$$2(1 - \rho^*) = \rho^* (2 - 2\cos(2\pi/k))^{1/2},$$

and (ii) follows. □

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