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LINEAR σ-ADDITIVITY AND SOME APPLICATIONS

TAL ORENSHTEIN AND BOAZ TSABAN

Abstract. We show that countable increasing unions preserve a large family of well-studied covering properties, which are not necessarily σ-additive. Using this, together with infinite-combinatorial methods and simple forcing theoretic methods, we explain several phenomena, settle problems of Just, Miller, Scheepers and Szeptycki (1996), Gruenhage and Szeptycki (2005), Tsaban and Zdomskyy (2008), and Tsaban (2006), (2007), and construct topological groups with very strong combinatorial properties.

1. Introduction

The following natural definition unifies all results presented here.

Definition 1.1. Let $\mathcal{F}$ be a family of topological spaces. $\mathcal{F}$ is *linearly σ-additive* if it is preserved by countable increasing unions. That is: For each topological space $X=\bigcup_n X_n$ with $X_1\subseteq X_2\subseteq\dots$ and $X_n\in\mathcal{F}$ for all $n\in\mathbb{N}$, $X\in\mathcal{F}$.

Removing the restriction that $X_n\subseteq X_{n+1}$ for all n, we obtain the definition of a *σ-additive* family. We identify a topological property with the family of all topological spaces satisfying it. Thus, we may talk about linearly σ-additive *properties*.

We consider additivity in the context of topological selection principles, to which we now give a brief introduction.[1] This is a framework suggested by Scheepers in [24] to study in a uniform manner a variety of properties introduced in different mathematical disciplines, since the early 1920s, by Menger, Hurewicz, Rothberger, and Gerlits and Nagy, and many others.

Let X be a topological space. We say that $\mathcal{U}$ is a *cover* of X if $X=\bigcup\mathcal{U}$ but $X\notin\mathcal{U}$. Often, X is considered as a subspace of another space Y, and in this case we always consider covers of X by subsets of Y, and require instead that no member of the cover contains X. Let $\mathrm{O}(X)$ be the family of all open covers of X. Define the following subfamilies of $\mathrm{O}(X)$: $\mathcal{U}\in\Omega(X)$ if each finite subset of X is contained in some member of $\mathcal{U}$. $\mathcal{U}\in\Gamma(X)$ if $\mathcal{U}$ is infinite, and each element of X is contained in all but finitely many members of $\mathcal{U}$.

Some of the following statements may hold for families $\mathscr{A}$ and $\mathscr{B}$ of covers of X:

$\mathsf{S}_1(\mathscr{A},\mathscr{B})$**:** For all $\mathcal{U}_1,\mathcal{U}_2,\ldots\in\mathscr{A}$, there are $U_1\in\mathcal{U}_1,U_2\in\mathcal{U}_2,\ldots$ such that $\{U_n:n\in\mathbb{N}\}\in\mathscr{B}$.

$\mathsf{S}_{\mathrm{fin}}(\mathscr{A},\mathscr{B})$**:** For all $\mathcal{U}_1,\mathcal{U}_2,\ldots\in\mathscr{A}$, there are finite $\mathcal{F}_1\subseteq\mathcal{U}_1,\mathcal{F}_2\subseteq\mathcal{U}_2,\ldots$ such that $\bigcup_n\mathcal{F}_n\in\mathscr{B}$.

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[1] Extended introductions to this field are available in [19, 27, 31].

$\mathsf{U}_{\mathrm{fin}}(\mathscr{A},\mathscr{B})$**:** For all $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathscr{A}$, none containing a finite subcover, there are finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathscr{B}$.

We say, e.g., that X satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ if the statement $\mathsf{S}_1(\mathrm{O}(X),\mathrm{O}(X))$ holds. This way, $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ is a property of topological spaces, and similarly for all other statements and families of covers. Under some mild hypotheses on the considered topological spaces, each nontrivial property among these properties, where $\mathscr{A},\mathscr{B}$ range over O,Ω,Γ, is equivalent to one in Figure 1 [24, 17]. In this diagram, an arrow denotes implication.

$$\begin{tikzcd}
 & & \begin{matrix}\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)\\ \text{Hurewicz}\end{matrix} \arrow[r] & \mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega) \arrow[r] & \begin{matrix}\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})\\ \text{Menger}\end{matrix} \\
 & & \mathsf{S}_{\mathrm{fin}}(\Gamma,\Omega) \arrow[ur] & & \\
\mathsf{S}_1(\Gamma,\Gamma) \arrow[r] \arrow[uurr] & \mathsf{S}_1(\Gamma,\Omega) \arrow[ur] \arrow[rr] & & \mathsf{S}_1(\Gamma,\mathrm{O}) \arrow[uur] & \\
 & & \begin{matrix}\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)\\ \text{Menger}^{\uparrow}\end{matrix} \arrow[uu] & & \\
\begin{matrix}\mathsf{S}_1(\Omega,\Gamma)\\ \text{Gerlits-Nagy}\end{matrix} \arrow[r] \arrow[uu] & \begin{matrix}\mathsf{S}_1(\Omega,\Omega)\\ \text{Rothberger}^{\uparrow}\end{matrix} \arrow[ur] \arrow[rr] \arrow[uu] & & \begin{matrix}\mathsf{S}_1(\mathrm{O},\mathrm{O})\\ \text{Rothberger}\end{matrix} \arrow[uu] &
\end{tikzcd}$$

FIGURE 1. The Scheepers Diagram

In this diagram, the classical name of a property is indicated below it, as well as two names ending with a symbol $\uparrow$, by which we indicate that the properties $\mathsf{S}_1(\Omega,\Omega)$ and $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$ may also be viewed as classical ones [23, 17], in accordance with the following notation.

Definition 1.2. Let P be a property of topological spaces. X *satisfies* $P^{\uparrow}$ if all finite powers X^k of X satisfy P.

The Scheepers Diagram is at the heart of the field of topological selection principles, and many additional—classical and new—properties are studied in relation to it. The reader is encouraged to consider his favorite properties in light of the results presented here.

In Section 2, we prove that all properties in the Scheepers Diagram are linearly σ-additive and are thus hereditary for F_σ subsets. This solves a problem of Tsaban and Zdomskyy from [36].

A crucial part of the proof that the studied properties are linearly σ-additive is a recent theorem of F. Jordan [15]. Miller asked in [22] whether $\mathsf{S}_1(\Omega,\Gamma)$ is linearly σ-additive. A negative solution would have solved the notorious Gerlits-Nagy problem [13]. Using a brilliant argument, Jordan proved that this is not the case. We give a direct version of Jordan's solution and provide some applications.

In Section 3, we use Jordan's method in a proof that if there is an unbounded family of cardinality $\aleph_1$ in the Baire space, then there is an uncountable set of real numbers satisfying $\mathsf{S}_1(\Omega,\Gamma)$. This settles in the positive a problem from the seminal paper of Just, Miller, Scheepers, and Szeptycki [17]. Indeed, our result is more general and also solves a problem of Gruenhage and Szeptycki [14].

In Section 4, we apply linear σ-additivity to study heredity of properties, answer a question of Zdomskyy, and suggest a simple revision of a question of Bukovský, Recław, and Repický, which makes it possible to answer it in the positive. (The problem, as originally stated, was answered in the negative by Miller [20].) This section also explains the phenomenon observed in [3], that none of the considered properties is hereditary in the open case, whereas in the Borel case some are and some are not hereditary.

In Section 5, we apply our results to construct topological groups with strong combinatorial properties, and solve problems from [32] and [35].

1.1. **Generalizations.** The results presented here require little or no assumptions on the topology of the studied spaces. However, they are interesting even when restricting attention to, e.g., metric spaces or even subsets of $\mathbb{R}$.

For concreteness, we present the results only for the types of covers mentioned above, but the proofs show that they hold for many additional types. In particular, define B, B_Ω, B_Γ as O, Ω, Γ were defined, replacing *open cover* by *countable Borel cover*. The properties thus obtained have rich history of their own [28]. Mild assumptions on X imply that the considered open covers may be assumed to be countable, and this makes the Borel variants of the studied properties (strictly) stronger [28]. All of the results presented here also hold in the Borel case (after replacing *open* or *closed* by *Borel*). Moreover, unlike some of the results in the open case, none of the Borel variant requires any assumption on the topology of X.

2. Linear σ-additivity in the Scheepers Diagram

One motivation for studying linear σ-additivity in the context of the Scheepers Diagram is an experimentally observed dichotomy concerning additivity of properties in this diagram: Each property is either provably σ-additive or not even provably *finitely* additive.[2] In this section, we prove the following.

Theorem 2.1. *All properties in the Scheepers Diagram (Figure 1) are linearly σ-additive.*

It remains to notice the following.

Lemma 2.2. *Each linearly σ-additive property is either σ-additive or not finitely additive.*

Proof. $\bigcup_n X_n = \bigcup_n (\bigcup_{m\leq n} X_m)$. □

Corollary 2.3. *Each property in the Scheepers Diagram is either σ-additive or not additive.* □

Before proving Theorem 2.1, we point out two additional consequences. In Problem 4.9 of [36] and Problem 6.2 of [35], Tsaban and Zdomskyy ask whether $\mathsf{S}_1(\Gamma,\Omega)$ and $\mathsf{S}_{\mathrm{fin}}(\Gamma,\Omega)$ are hereditary for F_σ subsets. We obtain a positive answer.

Corollary 2.4. *All properties in the Scheepers Diagram are hereditary for F_σ subsets.*

Proof. These properties are easily seen to be hereditary for closed subsets [17], and countable unions of closed subsets can be presented as countable increasing unions of closed subsets. Apply Theorem 2.1. □

[2] A survey of the involved results, with complete proofs, is available in [33].

The following was observed in the past for at least some properties in the Scheepers Diagram, each time using an ad-hoc argument.

Corollary 2.5. *Let P be a property in the Scheepers Diagram, and X, D be subspaces of some topological space. If X satisfies P and D is countable, then $X \cup D$ satisfies P.*

Proof. In light of Theorem 2.1, it remains to observe that for each singleton $\{a\}$, $X \cup \{a\}$ satisfies P. This is not hard to verify. $\square$

Theorem 2.1 is proved in parts. The properties $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, $\mathsf{S}_1(\Gamma,\mathrm{O})$, $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, $\mathsf{S}_1(\Gamma,\Gamma)$, and $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ are in fact σ-additive [17, 26] (see [33]).

Linear σ-additivity of $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega)$ was proved in [36] for sets of reals. It also follows from the following.

Theorem 2.6. *For all $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}, \mathsf{U}_{\mathrm{fin}}\}$ and $\mathscr{A} \in \{\Gamma, \Omega, \mathrm{O}\}$, $\Pi(\mathscr{A}, \Omega)$ is linearly σ-additive.*

Proof. We prove the theorem for $\Pi = \mathsf{S}_{\mathrm{fin}}$, the other proofs being similar.

Assume that $X = \bigcup_n X_n$ is an increasing union, with each X_n satisfying $\mathsf{S}_{\mathrm{fin}}(\mathscr{A},\Omega)$. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathscr{A}(X)$. We first exclude the trivial case: Assume that for infinitely many n, there are m_n and elements $U_{m_n} \in \mathcal{U}_{m_n}$ such that $X_n \subseteq U_{m_n}$. As X is not contained in any member of any $\mathcal{U}_n$ and the sets X_n increase to X, we may if necessary thin out the sequence m_n to make it increasing. Then $\{U_{m_n} : n \in \mathbb{N}\} \in \Omega(X)$, and this suffices.

Thus, we may assume that for all n, k, we have that $\mathcal{U}_k \in \mathscr{A}(X_n)$. Take a partition $\mathbb{N} = \bigcup_n I_n$ of $\mathbb{N}$ into infinite sets I_n. Fix n. As X_n satisfies $\mathsf{S}_{\mathrm{fin}}(\mathscr{A},\Omega)$, there are finite $\mathcal{F}_k \subseteq \mathcal{U}_k$, $k \in I_n$, such that $\bigcup_{k\in I_n} \mathcal{F}_k \in \Omega(X_n)$. Then $\bigcup_{k\in\mathbb{N}} \mathcal{F}_k \in \Omega(X)$. $\square$

The remaining property, $\mathsf{S}_1(\Omega,\Gamma)$, was treated by F. Jordan.

2.1. **Jordan's Theorem and some applications.** The following technical lemma will be useful in the proof of Jordan's Theorem below.

Lemma 2.7. *Let $Y \subseteq X$ be such that Y satisfies $\mathsf{S}_1(\Gamma,\Gamma)$. Assume that for each n,*

1. *$\mathcal{U}_n$ is an infinite family of open subsets of X; and*
2. *for each $y \in Y$, $y \in U$ for all but finitely many $U \in \mathcal{U}_n$.*

Then there are infinite $\mathcal{V}_1 \subseteq \mathcal{U}_1, \mathcal{V}_2 \subseteq \mathcal{U}_2, \ldots$, such that for each $y \in Y$, $y \in \bigcap \mathcal{V}_n$ for all but finitely many n.

Proof. It may be the case that no subset of $\mathcal{U}_n$ is in $\Gamma(Y)$.

Case 1: For all but finitely many n, $\mathcal{V}_n = \{U \in \mathcal{U}_n : Y \subseteq U\}$ is infinite. Then the sets $\mathcal{V}_n$ thus defined are as required.

Case 2: Let I be the set of all n such that $\mathcal{V}_n = \{U \in \mathcal{U}_n : Y \subseteq U\}$ is infinite, and $J = \mathbb{N} \setminus I$. For each $n \in J$, $\mathcal{W}_n = \{U \in \mathcal{U}_n : Y \not\subseteq U\}$ is infinite, and thus $\mathcal{W}_n \in \Gamma(Y)$. As J is infinite, there are by Theorem 15 of [34] infinite $\mathcal{V}_n \subseteq \mathcal{W}_n$, $n \in J$, such that each $y \in Y$ belongs to $\bigcap \mathcal{V}_n$ for all but finitely many n. (Briefly: By thinning out if needed, we may assume that each $\mathcal{W}_n$ is countable and that $\mathcal{W}_n \cap \mathcal{W}_m = \emptyset$ for $m \neq n$ [24]. Apply $\mathsf{S}_1(\Gamma,\Gamma)$ to the countable family of all cofinite subsets of all $\mathcal{W}_n$ to obtain $\mathcal{V} \in \Gamma(Y)$. Let $\mathcal{V}_n = \mathcal{V} \cap \mathcal{W}_n$.) The sets $\mathcal{V}_1, \mathcal{V}_2, \ldots$ are as required. $\square$

Theorem 2.8 (Jordan [15])**.** $\mathsf{S}_1(\Omega,\Gamma)$ *is linearly σ-additive.*

Proof. We give a direct proof, following what seems to be the essence of Jordan's arguments. The following statement can be deduced from Theorem 7 in [15].

Lemma 2.9. *Let $X=\bigcup_n X_n$ be an increasing union, where each X_n satisfies $\mathsf{S}_1(\Gamma,\Gamma)$. For all $\mathcal{U}_1\in\Gamma(X_1),\mathcal{U}_2\in\Gamma(X_2),\dots$, there are infinite $\mathcal{V}_1\subseteq\mathcal{U}_1,\mathcal{V}_2\subseteq\mathcal{U}_2,\dots$, such that for each $x\in X$, $x\in\bigcap\mathcal{V}_n$ for all but finitely many n.*

Proof. Step 1: By Lemma 2.7, we may thin out the families $\mathcal{U}_n$ so that they remain infinite, and each member of X_1 belongs to all but finitely many $\bigcap\mathcal{U}_n$. Let $\mathcal{V}_1=\mathcal{U}_1$.

Step 2: By the same lemma, we may further thin out the families $\mathcal{U}_n$, $n\geq 2$, so that they remain infinite, and each member of X_2 belongs to all but finitely many $\bigcap\mathcal{U}_n$, $n>1$. Let $\mathcal{V}_2=\mathcal{U}_2$.

Step k: By Lemma 2.7, we may further thin out the families $\mathcal{U}_n$, $n\geq k$, so that they remain infinite, and each member of X_k belongs to all but finitely many $\bigcap\mathcal{U}_n$, $n\geq k$. Let $\mathcal{V}_k=\mathcal{U}_k$.

The sets $\mathcal{V}_1,\mathcal{V}_2,\dots$ are as required. □

Now, let $X=\bigcup_n X_n$ be an increasing union, where each X_n satisfies $\mathsf{S}_1(\Omega,\Gamma)$. Let $\mathcal{U}_1,\mathcal{U}_2,\dots\in\Omega(X)$. As in the argument of the proof of Lemma 2.7, we may assume that $\mathcal{U}_n\in\Omega(X_n)$ for all n. As X_n satisfies $\mathsf{S}_1(\Omega,\Gamma)$, we may thin out $\mathcal{U}_n$ so that $\mathcal{U}_n\in\Gamma(X_n)$.[3] Apply Lemma 2.9, and pick for each n some $U_n\in\mathcal{V}_n\setminus\{U_1,\dots,U_{n-1}\}$. $\{U_n:n\in\mathbb{N}\}\in\Gamma(X)$. □

Let $C(X)$ be the family of continuous functions $f:X\to\mathbb{R}$. $C(X)$ has the *Arhangel'skiĭ property* α_2 if the following holds for all $f^n_m\in C(X)$, $n,m\in\mathbb{N}$: If for each n, $\lim_{m\to\infty}f^n_m(x)=0$ for all $x\in X$, then there are m_n such that $\lim_n f^n_{m_n}(x)=0$ for each $x\in X$.

For the same reason briefly mentioned in the proof of Lemma 2.7, one may require in the definition of α_2 that there are infinite $I_1,I_2,\dots\subseteq\mathbb{N}$ such that some (equivalently, any) enumeration of the countable set $\bigcup_n\{f^n_m(x):m\in I_n\}$ converges to 0, for each $x\in X$. Indeed, this was Arhangel'skiĭ's original definition of α_2.

Definition 2.10. Let $X=\bigcup_n X_n$ be an increasing union. $C(X)$ is $\{X_n\}_{n\in\mathbb{N}}$-α_2 if the following holds whenever $f^n_m\in C(X_n)$ for all m,n: If for each n, $\lim_m f^n_m(x)=0$ for all $x\in X_n$, then there are infinite $I_1,I_2,\dots\subseteq\mathbb{N}$, such that for each $x\in X$, $\lim_n\sup\{|f^n_m(x)|:m\in I_n\}=0$.[4]

Clearly, $\{X_n\}_{n\in\mathbb{N}}$-α_2 implies α_2. The proof of Lemma 2.9, with minor modifications, yields the following new result about function spaces.

Theorem 2.11. *Let $X=\bigcup_n X_n$ be an increasing union such that $C(X_n)$ is α_2 for all n. Then $C(X)$ is $\{X_n\}_{n\in\mathbb{N}}$-α_2.*

Proof. Assume that $f^n_m\in C(X_n)$ for all m,n, and for each n, $\lim_m f^n_m(x)=0$ for all $x\in X_n$.

[3] Clearly, if $\mathcal{U}\in\Omega(X)$ and X satisfies $\mathsf{S}_1(\Omega,\Gamma)$, then $\mathcal{U}$ contains a subcover $\mathcal{V}$ such that $\mathcal{V}\in\Gamma(X)$ [13].

[4] For each $x\in X$, $x\in X_n$ for all large enough n. Thus, for all large enough n, $r_n(x)=\sup\{|f^n_m(x)|:m\in I_n\}$ is defined, and therefore the question as to whether $\lim_n r_n(x)$ converges to 0 or not makes sense.

Step 1: As $C(X_1)$ is α_2, there are infinite $J_1, J_2, \ldots \subseteq \mathbb{N}$ such that any enumeration of the countable set $\bigcup_n \{f^n_m(x) : m \in J_n\}$ converges to 0, for each $x \in X_1$. In particular, $\lim_n \sup\{|f^n_m(x)| : m \in J_n\} = 0$ for all $x \in X_1$. Let $I_1 = J_1$.

Step k: For each $n \geq k$, $\lim_{m\in J_n} f^n_m(x) = 0$ for all $x \in X_k$. As $C(X_k)$ is α_2, the index sets $J_k, J_{k+1}, \ldots$ may be thinned out so that they remain infinite, and $\lim_n \sup\{|f^n_m(x)| : m \in J_n\} = 0$ for all $x \in X_k$. Let $I_k = J_k$.

The index sets $I_1, I_2, \ldots$ are as required. □

Corollary 2.12. *Assume that for each subset Y of X, $C(Y)$ is α_2. For all $X_1 \subseteq X_2 \subseteq \ldots \subseteq X$, $C(\bigcup_n X_n)$ is $\{X_n\}_{n\in\mathbb{N}}$-α_2.* □

A topological space is *perfectly normal* if for each pair of disjoint closed sets $C_0, C_1 \subseteq X$, there is $f \in C(X)$ such that $f^{-1}[\{0\}] = C_0$ and $f^{-1}[\{1\}] = C_1$.

Corollary 2.13. *Assume that X is a perfectly normal space and $C(X)$ is α_2. For all closed sets $X_1 \subseteq X_2 \subseteq \ldots \subseteq X$, $C(\bigcup_n X_n)$ is $\{X_n\}_{n\in\mathbb{N}}$-α_2.*

Proof. If $C(X)$ is α_2 and Y is a closed subset of X, then $C(Y)$ is α_2.[5] Apply Theorem 2.11. □

In addition to these applications, Jordan's Lemma 2.9 is an ingredient in a solution of a thus far open problem, to which we now turn.

3. γ-sets of reals from a weak hypothesis

In this section, we construct sets of reals satisfying $\mathsf{S}_1(\Omega, \Gamma)$. Traditionally, general topological spaces satisfying $\mathsf{S}_1(\Omega, \Gamma)$ are called *γ-spaces*, and if they happen to be (homeomorphic to) sets of real numbers, they are called *γ-sets*.

The problem settled by our construction has some history, which we now survey briefly. This involves combinatorial cardinal characteristics of the continuum [4]. We give the necessary definitions as we proceed. Readers who are new to this field may skip this section in their first reading.

γ-spaces were introduced by Gerlits and Nagy in [13], their most influential paper, as the third property in a list numbered α through ϵ. This turned out to be the most important property in the list and obtained its alphabetic number as its name. One of the main results in [13] is that for Tychonoff spaces X, $C(X)$ with the topology of pointwise convergence is Fréchet-Urysohn if, and only if, X is a γ-space.

While uncountable γ-spaces exist in ZFC[6] [30], Borel's Conjecture (which is consistent with, but not provable within, ZFC) implies that all *metrizable* γ-spaces are countable.

Since we are dealing with constructions rather than general results, we restrict our attention in this section to subsets of $\mathbb{R}$ (or, since the property is preserved by continuous images, subsets of any topological space which can be embedded in $\mathbb{R}$).

[5] The argument is as in Theorem 4.1 of [7]: As X is perfectly normal, the open set $X \setminus Y$ is a countable increasing union of closed sets, $X \setminus Y = \bigcup_n C_n$. For each n, extend each f^n_m to an element of $C(X)$ which is constantly 0 on C_n. Applying α_2 on the new sequences, we obtain m_n such that $\lim_n f^n_{m_n}(x) = 0$ for all $x \in X$, and thus this is also the case for the original functions, for all $x \in Y$.

[6] The axioms of Zermelo and Fraenkel, together with the axiom of choice, the ordinary axioms of mathematics.

Thus, we may restrict our attention to *countable* open covers. As mentioned above, by a γ-*set* we mean a γ-space which is (homeomorphic to) a set of real numbers.

Gerlits and Nagy proved in [13] that Martin's Axiom implies that all sets of reals of cardinality less than $\mathfrak{c}$ are γ-sets. There is a simple reason for this: The *critical cardinality* of a property P, denoted $\mathrm{non}(P)$, is the minimal cardinality of a set not satisfying P. Let $\binom{\Omega}{\Gamma}$ be the property: Each $\mathcal{U} \in \Omega(X)$ contains a set $\mathcal{V} \in \Gamma(X)$. Gerlits and Nagy proved that $\mathsf{S}_1(\Omega, \Gamma) = \binom{\Omega}{\Gamma}$ [13]. Let $A \subseteq^* B$ mean that $A \setminus B$ is finite. A is a *pseudointersection* of $\mathcal{F}$ if $A \subseteq^* B$ for all $B \in \mathcal{F}$. Let $\mathfrak{p}$ be the minimal cardinality of a family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ which is closed under finite intersections and has no pseudointersection. Then $\mathrm{non}\binom{\Omega}{\Gamma} = \mathfrak{p}$ [12], and Martin's Axiom implies $\mathfrak{p} = \mathfrak{c}$ [12].

By definition, for each property P of sets of reals, every set of reals whose cardinality is smaller than $\mathrm{non}(P)$ satisfies P. Thus, the real question is whether there is a set of reals X of cardinality at least $\mathrm{non}(P)$ which satisfies P. Galvin and Miller [12] proved a result of this type: $\mathfrak{p} = \mathfrak{c}$ implies that there is a γ-set of cardinality $\mathfrak{p}$. Just, Miller, Scheepers and Szeptycki [17] have improved the construction of [12]. We introduce their construction in a slightly more general form that will be useful later.

Cantor's space $\{0,1\}^{\mathbb{N}}$ is equipped with the Tychonoff product topology, and $P(\mathbb{N})$ is identified with $\{0,1\}^{\mathbb{N}}$ using characteristic functions. This defines the topology of $P(\mathbb{N})$. The partition $P(\mathbb{N}) = [\mathbb{N}]^{\infty} \cup [\mathbb{N}]^{<\infty}$, into the infinite and the finite sets, respectively, is useful here.

For $f, g \in \mathbb{N}^{\mathbb{N}}$, let $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n. $\mathfrak{b}$ is the minimal cardinality of a $\leq^*$-unbounded subset of $\mathbb{N}^{\mathbb{N}}$. A set $B \subseteq [\mathbb{N}]^{\infty}$ is *unbounded* if the set of all increasing enumerations of elements of B is unbounded in $\mathbb{N}^{\mathbb{N}}$, with respect to $\leq^*$. It follows that $|B| \geq \mathfrak{b}$. For $m, n \in \mathbb{N}$, let $(m, n) = \{k : m < k < n\}$.

Lemma 3.1 (Folklore). *If $B \subseteq [\mathbb{N}]^{\infty}$ is unbounded, then for each increasing $f \in \mathbb{N}^{\mathbb{N}}$, there is $x \in B$ such that $x \cap (f(n), f(n+1)) = \emptyset$ for infinitely many n.*

Proof. Assume that f is a counterexample. Let g dominate all functions $f_m(n) = f(n+m)$, $m \in \mathbb{N}$. Then for each $x \in B$, $x \leq^* g$. Indeed, let m be such that for all $n \geq m$, $x \cap (f(n), f(n+1)) \neq \emptyset$. Then for each n, the n-th element of x is smaller than $f_{m+1}(n)$. □

Definition 3.2. A *tower* of cardinality κ is a set $T \subseteq [\mathbb{N}]^{\infty}$ which can be enumerated bijectively as $\{x_\alpha : \alpha < \kappa\}$, such that for all $\alpha < \beta < \kappa$, $x_\beta \subseteq^* x_\alpha$.

An *unbounded tower* of cardinality κ is an unbounded set $T \subseteq [\mathbb{N}]^{\infty}$ which is a tower of cardinality κ. (Necessarily, $\kappa \geq \mathfrak{b}$.)

Let $\mathfrak{t}$ be the minimal cardinality of a tower which has no pseudointersection. Rothberger proved that $\mathfrak{t} \leq \mathfrak{b}$ [4].

Lemma 3.3 (Folklore). *$\mathfrak{t} = \mathfrak{b}$ if, and only if, there is an unbounded tower of cardinality $\mathfrak{t}$.*

Proof. ($\Rightarrow$) Construct x_α by induction on α. Let $\{b_\alpha : \alpha < \mathfrak{b}\} \subseteq \mathbb{N}^{\mathbb{N}}$ be unbounded. At step α, let a be a pseudointersection of $\{x_\beta : \beta < \alpha\}$, and take $x_\alpha \subseteq a$ such that the increasing enumeration of x_α dominates b_α.

($\Leftarrow$) $\mathfrak{t} \leq \mathfrak{b} \leq |T| = \mathfrak{t}$. □

Just, Miller, Scheepers and Szeptycki [17] proved that if T is an unbounded tower of cardinality $\aleph_1$, then $T \cup [\mathbb{N}]^{<\infty}$ satisfies $\mathsf{S}_1(\Omega, \Omega)$, as well as a property, which was later proved by Scheepers [26] to be equivalent to $\mathsf{S}_1(\Gamma,\Gamma)$. In Problem 7 of [17], we are asked the following.

Problem 3.4 (Just-Miller-Scheepers-Szeptycki [17])**.** Assume that $T \subseteq [\mathbb{N}]^\infty$ is an unbounded tower of cardinality $\aleph_1$ (so that $\aleph_1 = \mathfrak{b}$). Is $T \cup [\mathbb{N}]^{<\infty}$ a γ-set, i.e., does it satisfy $\mathsf{S}_1(\Omega,\Gamma)$?

Scheepers proves in [25] that for each unbounded tower T of cardinality $\mathfrak{t} = \mathfrak{b}$, $T \subseteq [\mathbb{N}]^\infty$ satisfies $\mathsf{S}_1(\Gamma,\Gamma)$.

Miller [21] proves that in the Hechler model there are no uncountable γ-sets. In this model, $\aleph_1 = \mathfrak{p} = \mathfrak{t} < \mathfrak{b}$, and thus $\aleph_1 = \mathfrak{t}$ does not suffice to have an uncountable γ-set. At the end of [21] and in its appendix, Miller proves that $\diamondsuit(\mathfrak{b})$, a property strictly stronger than $\aleph_1 = \mathfrak{b}$, implies that there is an uncountable γ-set.[7] He concludes that it is still an open question as to whether $\mathfrak{b} = \aleph_1$ is enough to construct an uncountable γ-set.

We show that the answer is positive, and indeed also answer a question of Gruenhage and Szeptycki [14]: A classical problem of Malykhin asks whether there is a countable Fréchet-Urysohn topological group which is not metrizable. Gruenhage and Szeptycki prove that $F \subseteq \mathbb{N}^\mathbb{N}$ is a γ-set if, and only if, a certain construction associated to F provides a positive answer to Malykhin's Problem [14]. They define a generalization of the γ-set, called a *weak γ-set*, and combine their results with results of Nyikos to prove the following.

Theorem 3.5 (Gruenhage-Szeptycki [14])**.** *If $\mathfrak{p} = \mathfrak{b}$, then there is a weak γ-set in $\mathbb{N}^\mathbb{N}$.*

They write: "The relationship between γ-sets and weak γ-sets is not known. Perhaps $\mathfrak{b} = \mathfrak{p}$ implies the existence of a γ-set" [14]. Our solution confirms their conjecture.

$\mathfrak{p} \le \mathfrak{t} \le \mathfrak{b}$, and in all known models of set theory, $\mathfrak{p} = \mathfrak{t}$. When $\mathfrak{p} = \mathfrak{c}$, our theorem shows that in Galvin and Miller's construction from [12], even if the possible open covers are not considered at all, the resulting set is still a γ-set.

Theorem 3.6. *For each unbounded tower T of cardinality $\mathfrak{p}$ in $[\mathbb{N}]^\infty$, $T \cup [\mathbb{N}]^{<\infty}$ satisfies $\mathsf{S}_1(\Omega,\Gamma)$.*

Proof. By Lemma 3.3, there is an unbounded tower T of cardinality $\mathfrak{p}$ if, and only if, $\mathfrak{p} = \mathfrak{b}$. Let $T = \{x_\alpha : \alpha < \mathfrak{b}\}$ be an unbounded tower of cardinality $\mathfrak{b}$. For each α, let $X_\alpha = \{x_\beta : \beta < \alpha\} \cup [\mathbb{N}]^{<\infty}$. We will show that $T \cup [\mathbb{N}]^{<\infty}$ satisfies $\binom{\Omega}{\Gamma}$. Let $\mathcal{U} \in \Omega(T \cup [\mathbb{N}]^{<\infty})$. We use the following modification of Lemma 1.2 of [12].

Lemma 3.7. *Assume that $[\mathbb{N}]^{<\infty} \subseteq X \subseteq P(\mathbb{N})$ and X satisfies $\binom{\Omega}{\Gamma}$. For each $\mathcal{U} \in \Omega(X)$, there are $m_1 < m_2 < \dots$ and distinct $U_1, U_2, \dots \in \mathcal{U}$ such that $\{U_n : n \in \mathbb{N}\} \in \Gamma(X)$, and for each $x \subseteq \mathbb{N}$, $x \in U_n$ whenever $x \cap (m_n, m_{n+1}) = \emptyset$.*

Proof. As X satisfies $\binom{\Omega}{\Gamma}$, we may thin out $\mathcal{U}$ so that $\mathcal{U} \in \Gamma(X)$.

We proceed as in the proof of Lemma 1.2 of [12]. Let $m_1 = 1$. For each $n \ge 1$: As $\mathcal{U} \in \Omega(X)$, each finite subset of X is contained in infinitely many elements of $\mathcal{U}$. Take $U_n \in \mathcal{U} \setminus \{U_1, \dots, U_{n-1}\}$, such that $P(\{1, \dots, m_n\}) \subseteq U_n$. As U_n is open, for each $s \subseteq \{1, \dots, m_n\}$ there is k_s such that for each $x \in P(\mathbb{N})$ with $x \cap \{1, \dots, k_s - 1\} = s$, $x \in U_n$. Let $m_{n+1} = \max\{k_s : s \subseteq \{1, \dots, m_n\}\}$.

[7] $\diamondsuit(\mathfrak{b})$ is defined in Dzamonja-Hrusak-Moore [10].

As $\{U_n : n \in \mathbb{N}\}$ is an infinite subset of $\mathcal{U} \in \Gamma(X)$, $\{U_n : n \in \mathbb{N}\} \in \Gamma(X)$, too. $\square$

We may assume that $\mathcal{U}$ is countable. As $\mathfrak{b}$ is regular, there is $\alpha_1 < \mathfrak{b}$ such that X_{α_1} is not contained in any member of $\mathcal{U}$. This guarantees that $\mathcal{U} \in \Omega(X_\alpha)$ for all $\alpha \geq \alpha_1$.

As $|X_{\alpha_1}| < \mathfrak{p}$, X_{α_1} satisfies $\binom{\Omega}{\Gamma}$. As $\mathcal{U} \in \Omega(X_{\alpha_1})$, by Lemma 3.7 there are $m^1_1 < m^1_2 < \dots$ and distinct $U^1_1, U^1_2, \dots \in \mathcal{U}$ such that $\{U^1_n : n \in \mathbb{N}\} \in \Gamma(X_{\alpha_1})$, and for each $x \in P(\mathbb{N})$, $x \in U^1_n$ whenever $x \cap (m^1_n, m^1_{n+1}) = \emptyset$. Let $D_1 = \mathbb{N}$.

As $\alpha_1 < \mathfrak{b}$, $\{x_\alpha : \alpha_1 < \alpha < \mathfrak{b}\}$ is unbounded. By Lemma 3.1, there is $\alpha_2 > \alpha_1$ such that $D_2 = \{n : x_{\alpha_2} \cap (m^1_n, m^1_{n+1}) = \emptyset\}$ is infinite. As $|X_{\alpha_2}| < \mathfrak{p}$, X_{α_2} satisfies $\binom{\Omega}{\Gamma}$. As $\mathcal{U} \in \Omega(X_{\alpha_2})$, by Lemma 3.7 there are $m^2_1 < m^2_2 < \dots$ and distinct $U^2_1, U^2_2, \dots \in \mathcal{U}$ such that $\{U^2_n : n \in \mathbb{N}\} \in \Gamma(X_{\alpha_2})$, and for each $x \in P(\mathbb{N})$, $x \in U^2_n$ whenever $x \cap (m^2_n, m^2_{n+1}) = \emptyset$. As D_2 is infinite, $\{U^2_n : n \in D_2\} \in \Gamma(X_{\alpha_2})$.

Continue in the same manner to define for each $k > 1$ elements with the following properties:

(1) $\alpha_k > \alpha_{k-1}$;
(2) $D_k = \{n : x_{\alpha_k} \cap (m^{k-1}_n, m^{k-1}_{n+1}) = \emptyset\}$ is infinite;
(3) $m^k_1 < m^k_2 < \dots$;
(4) $U^k_1, U^k_2, \dots \in \mathcal{U}$ are distinct;
(5) $\{U^k_n : n \in D_k\} \in \Gamma(X_{\alpha_k})$; and
(6) for each $x \in P(\mathbb{N})$, $x \in U^k_n$ whenever $x \cap (m^k_n, m^k_{n+1}) = \emptyset$.

Let $\alpha = \sup_k \alpha_k$. As $\mathfrak{b}$ is regular, $\alpha < \mathfrak{b}$. $X_\alpha = \bigcup_k X_{\alpha_k}$ is a countable increasing union. For each k, $|X_{\alpha_k}| < \mathfrak{b}$, and thus X_{α_k} satisfies $\mathsf{S}_1(\Gamma, \Gamma)$. By Lemma 2.9, there are infinite $I_1 \subseteq D_1, I_2 \subseteq D_2, \dots$ such that each $x \in X_\alpha$ belongs to $\bigcap_{n \in I_k} U^k_n$ for all but finitely many $k \in \mathbb{N}$.

Take $n_1 \in I_2$. For $k > 1$, take $n_k \in I_{k+1}$ such that $m^k_{n_k} > m^{k-1}_{n_{k-1}+1}$, $x_\alpha \cap (m^k_{n_k}, m^k_{n_k+1}) \subseteq x_{\alpha_{k+1}} \cap (m^k_{n_k}, m^k_{n_k+1})$, and $U^k_{n_k} \notin \{U^1_{n_1}, \dots, U^{k-1}_{n_{k-1}}\}$. We claim that $\{U^k_{n_k} : k \in \mathbb{N}\} \in \Gamma(T \cup [\mathbb{N}]^{<\infty})$. As $\{U^k_{n_k} : k \in \mathbb{N}\} \in \Gamma(X_\alpha)$, it remains to show that for each $x \subseteq^* x_\alpha$, $x \in U^k_{n_k}$ for all but finitely many k. For each large enough k, $m^k_{n_k}$ is large enough, so that

$$x \cap (m^k_{n_k}, m^k_{n_k+1}) \subseteq x_\alpha \cap (m^k_{n_k}, m^k_{n_k+1}) \subseteq x_{\alpha_{k+1}} \cap (m^k_{n_k}, m^k_{n_k+1}) = \emptyset,$$

since $n_k \in D_{k+1}$. Thus, $x \in U^k_{n_k}$. $\square$

Remark 3.8. Zdomskyy points out that our proof actually shows that a wider family of sets are γ-sets. For example, if we start with T an unbounded tower of cardinality $\mathfrak{p}$, and thin out its elements arbitrarily, $T \cup [\mathbb{N}]^{<\infty}$ remains a γ-set. This may be useful for constructions of examples with additional properties, since this way each element of T may be chosen arbitrarily from a certain perfect set.

In particular, we have that in each model of ZFC where $\mathfrak{p} = \mathfrak{b}$, there are γ-sets of cardinality $\mathfrak{p}$. In the following corollary, by an "X model" we mean the model obtained by generically extending a model of the Continuum Hypothesis by adding $\aleph_2$ (or more, in the case $X \in \{\text{Cohen, Random}\}$) X reals in the standard way; see [4] for details.

Corollary 3.9. *In each of the Cohen, Random, Sacks, and Miller models of ZFC, there are γ-sets of reals with cardinality $\mathfrak{p}$.* $\square$

Earlier, Corollary 3.9 was shown for the Sacks model by Ciesielski, Millán, and Pawlikowski in [9], and for the Cohen and Miller models by Miller [21], using specialized arguments. It seems that the result, that there are uncountable γ-sets in the random reals model (constructed by extending a model of the Continuum Hypothesis), is new. We point out that the Random poset alone is not the reason for having uncountable γ-sets in the generic extension: Judah, Shelah, and Woodin prove in [16] that there are no uncountable strong measure zero sets (and in particular, no uncountable γ-sets) in an extension of Laver's model by random reals.

As discussed above, there are no uncountable γ-sets in the Hechler model [21]. Since the Laver and Mathias models satisfy Borel's Conjecture, there are no uncountable γ-sets in these models either.

4. Heredity

A topological space X satisfies a property P *hereditarily* if each subspace of X satisfies P. In our context, heredity was observed to be tightly connected to the following property. X is a *σ-space* if each Borel subset of X is an F_σ subset of X.

A combination of results of Fremlin-Miller [11], Bukovský-Recław-Repický [8], and Bukovský-Haleš [6] implies that a Tychonoff space X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ hereditarily if, and only if, X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ and is a σ-space (see also [37]).[8] A similar result was proved for $\mathsf{S}_1(\Gamma,\Gamma)$ in [6]. Problem 7.9 in [8] asks whether every (nice enough) σ-space X satisfying $\mathsf{S}_1(\Omega,\Gamma)$ satisfies $\mathsf{S}_1(\Omega,\Gamma)$ hereditarily. A negative answer was given by Miller [20].

We have, in light of the previous section, a simple reason for the difference between $\mathsf{S}_1(\Omega,\Gamma)$ on one hand, and $\mathsf{S}_1(\Gamma,\Gamma)$ and $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ on the other hand.

Definition 4.1. Let $\mathscr{A}\in\{\Gamma,\Omega,\mathrm{O},\dots\}$. $\mathscr{A}$ is *Borel superset covering* for X if, for each subspace $Y\subseteq X$ and each $\mathcal{U}\in\mathscr{A}(Y)$, there are $\mathcal{V}\subseteq\mathcal{U}$ and a Borel $B\subseteq X$, such that $Y\subseteq B$ and $\mathcal{V}\in\mathscr{A}(B)$.

Many classical types of covers are Borel superset covering. In particular, we have the following.

Lemma 4.2. *Γ and O are Borel superset covering (for all X).*

Proof. If $\mathcal{U}\in\Gamma(X)$, take a countable infinite subset $\mathcal{V}=\{U_n : n\in\mathbb{N}\}\subseteq\mathcal{U}$, and $B=\bigcup_m\bigcap_{n\ge m}U_n$. If $\mathcal{U}\in\mathrm{O}(X)$, take $B=\bigcup\mathcal{U}$. $\square$

By Miller's mentioned result and the following theorem, Ω need not be Borel superset covering.

Theorem 4.3. *Let $\mathscr{A},\mathscr{B}\in\{\Gamma,\Omega,\mathrm{O}\}$ and $\Pi\in\{\mathsf{S}_1,\mathsf{S}_{\mathrm{fin}},\mathsf{U}_{\mathrm{fin}}\}$. Assume that X is a σ-space and satisfies $\Pi(\mathscr{A},\mathscr{B})$, and $\mathscr{A}$ is Borel superset covering for X. Then X satisfies $\Pi(\mathscr{A},\mathscr{B})$ hereditarily.*

Proof. Let $Y\subseteq X$ and assume that $\mathcal{U}_1,\mathcal{U}_2,\dots\in\mathscr{A}(Y)$. For each n, pick intermediate Borel sets $Y\subseteq B_n\subseteq X$ such that $\mathcal{U}_n\in\mathscr{A}(B_n)$, and let $B=\bigcap_n B_n$. Then

[8]The assumptions on X in the cited references are stronger, but: A Tychonoff hereditarily-$\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ space cannot have the unit interval $[0,1]$ as a continuous image, and thus is zero-dimensional. As it is hereditarily Lindelöf, each open set is a countable union of clopen sets, and this suffices for the arguments in the cited references.

$\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathscr{A}(B)$. As B is Borel and X is a σ-space, B is F_σ in X. By Corollary 2.4, B satisfies $\Pi(\mathscr{A}, \mathscr{B})$, and we can thus obtain from the covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$ a cover $\mathcal{V} \in \mathscr{B}(B)$. Then $\mathcal{V} \in \mathscr{B}(Y)$. □

As for the other direction, we have, by the discussion at the beginning of this section, the following.

Lemma 4.4. *If a property P of Tychonoff spaces implies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ and X satisfies P hereditarily, then X is a σ-space.* □

During his work with the second author on [37], L. Zdomskyy asked whether every subset of $\mathbb{R}$ satisfying $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Omega)$ hereditarily is a σ-space. We show that the answer is "no", in a very strong sense. We will see that, in the context of Scheepers Diagram, Lemma 4.4 becomes a *criterion*; that is, satisfying P hereditarily implies being a σ-space if, and only if, P implies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.

We use, in our proof, the method of "forcing", following an elegant approach of Brendle [5]. A proof avoiding this method (e.g., using the methods of [32]) would be more lengthy and technically involved.

Theorem 4.5. *Assume the Continuum Hypothesis. There is $L \subseteq \mathbb{R}$ such that all finite powers of L satisfy $\mathsf{S}_1(\mathrm{B}_\Omega, \mathrm{B}_\Omega)$ hereditarily, but L is not a σ-space.*

Proof. Let M_α, $\alpha < \aleph_1$, be an increasing sequence of countable submodels of $H(\lambda)$ (λ large enough) such that:

1. $\mathbb{R} \subseteq \bigcup_{\alpha<\aleph_1} M_\alpha$;
2. for each $\alpha < \aleph_1$, $M_{\alpha+1} \models M_\alpha$ is countable; and
3. $(M_\beta : \beta \leq \alpha) \in M_{\alpha+1}$.

For each α choose a real c_α Cohen generic over M_α. (As M_α is countable, there is such c_α.) Let $L = \{c_\alpha : \alpha < \aleph_1\}$. It is well known that a set L constructed this way is a *Luzin set*, i.e., L is uncountable, but for each meager $M \subseteq \mathbb{R}$, $L \cap M$ is countable.[9]

Lemma 4.6 (Folklore). *No Luzin set L is a σ-space.*

Proof. Assume otherwise, and take a countable dense $D \subseteq L$. As $L \setminus D$ is Borel, it is F_σ. Then $D = \bigcap_n U_n$, an intersection of open sets. Then for each n, $L \setminus U_n$ is meager, and thus countable. It follows that $L \setminus D$ is countable, a contradiction. □

Recall that according to Definition 1.2, X satisfies $P^\uparrow$ if all finite powers of X satisfy P. By [3], or alternatively Lemma 4.2 and Remark 4.9, $\mathsf{S}_1(\mathrm{B}, \mathrm{B})$ is hereditary. By [28], $\mathsf{S}_1(\mathrm{B}_\Omega, \mathrm{B}_\Omega)^\uparrow = \mathsf{S}_1(\mathrm{B}, \mathrm{B})^\uparrow$.

Lemma 4.7. *If P is hereditary, then so is $P^\uparrow$.*

Proof. If $Y \subseteq X$ and X satisfies $P^\uparrow$, then for each k, $Y^k \subseteq X^k$ and X^k satisfies P. Thus, Y^k satisfies P for all k. □

Thus, it suffices to prove that all finite powers of L satisfy $\mathsf{S}_1(\mathrm{B}, \mathrm{B})$. Consider for example L^2. We have to show (essentially, [28]) that for each Borel $\Psi : \mathbb{R}^2 \to \mathbb{N}^\mathbb{N}$, there is $g \in \mathbb{N}^\mathbb{N}$ such that for each $(x, y) \in L^2$, $\Psi(x, y)(n) = g(n)$ for some n.

Let $\Psi : \mathbb{R}^2 \to \mathbb{N}^\mathbb{N}$ be Borel. Ψ is coded by a real. Let α be such that this code belongs to M_α. $\mathbb{N}^\mathbb{N} \cap M_\alpha$ is countable. Take a canonical partition $\mathbb{N} = \bigcup_n I_n$ with

[9] Indeed, each meager set is contained in a meager F_σ set M, which in turn is coded by a single real r. Let α be such that $r \in M_\alpha$. Then for each $\beta > \alpha$, $c_\beta \notin M$.

each I_n infinite, so that $(I_n : n \in \mathbb{N}) \in M_\alpha$. Enumerate $\mathbb{N}^\mathbb{N} \cap M_\alpha = \{f_n : n \in \mathbb{N}\}$, and take $g \in \mathbb{N}^\mathbb{N}$ such that for each n, $g\restriction_{I_n} = f_n\restriction_{I_n}$. For $x, y \in L$, $\Psi(x,y) \in M_\alpha[x,y]$, an extension of M_α by finitely many Cohen reals, which is in fact either M_α or an extension of M_α by a single Cohen real. As the Cohen forcing does not add eventually different reals (e.g., [2]), there is n such that $f_n\restriction_{I_n}$ coincides with $\Psi(x,y)\restriction_{I_n}$ on infinitely many values. But $f_n\restriction_{I_n} = g\restriction_{I_n}$. $\square$

We conclude the section with the following characterization, which may be viewed as a necessary revision of Problem 7.9 of Bukovský-Recław-Repický [8], so that it has a (provably) positive answer.[10]

Theorem 4.8. *A Tychonoff space X satisfies $\mathsf{S}_1(\Omega,\Gamma)$ hereditarily if, and only if:*

1. *X satisfies $\mathsf{S}_1(\Omega,\Gamma)$;*
2. *X is a σ-space; and*
3. *Ω is Borel superset covering for X.*

Proof. Theorem 4.3 provides the "if" part. For the "only if" part, by Lemma 4.4 it remains to prove (3). Let $Y \subseteq X$, $\mathcal{U} \in \Omega(Y)$. As Y satisfies $\mathsf{S}_1(\Omega,\Gamma)$, there are $U_n \in \mathcal{U}$ such that $\{U_n : n \in \mathbb{N}\} \in \Gamma(Y)$. Let $B = \bigcup_m \bigcap_{n \geq m} U_n$. Then $Y \subseteq B$ and $\{U_n : n \in \mathbb{N}\} \in \Omega(B)$. $\square$

Remark 4.9 (The case of countable Borel covers). In the Borel case, the properties are hereditary for Borel subsets [28], and thus in Theorem 4.3, there is no need to assume that X is a σ-space. This explains why some of the Borel properties are hereditary [3] whereas others are not [20], and why none of the open properties is hereditary [3].

Two major open problems concerning $\mathsf{S}_1(\mathrm{B}_\Omega,\mathrm{B}_\Omega)$ and $\mathsf{S}_{\mathrm{fin}}(\mathrm{B}_\Omega,\mathrm{B}_\Omega)$ are whether these properties are hereditary, and whether they are—like their open variant—preserved by finite powers [3, 35, 20]. By Lemma 4.7, the problems are related: If $\mathsf{S}_1(\mathrm{B}_\Omega,\mathrm{B}_\Omega)$ is preserved by finite powers, then it is equal to $\mathsf{S}_1(\mathrm{B},\mathrm{B})^\uparrow$, which is hereditary. Similarly for $\mathsf{S}_{\mathrm{fin}}(\mathrm{B}_\Omega,\mathrm{B}_\Omega)$.

Item (2) may be removed from Theorem 4.8 in the Borel case. On the other hand, $\mathsf{U}_{\mathrm{fin}}(\mathrm{B},\mathrm{B}_\Gamma)$ is hereditary and (for Lindelöf spaces) implies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$, and thus implies being a σ-set. Theorem 4.5 completes the picture.

5. Topological groups with strong combinatorial properties

Problem 10.7 in [35], which is also implicit in Theorem 20 of [32] and in the discussion around it, asks whether the Continuum Hypothesis implies the existence of an uncountable subgroup G of $\mathbb{Z}^\mathbb{N}$ satisfying $\mathsf{S}_1(\Omega,\Gamma)$. We will show that the answer is "yes", and indeed the weaker hypothesis $\mathfrak{p} = \mathfrak{b}$ suffices to obtain such groups (see the discussion in Section 3). Indeed, Theorem 3.6 gives an uncountable subset of $\mathbb{Z}^\mathbb{N}$ satisfying $\mathsf{S}_1(\Omega,\Gamma)$. We will show that for a wide class of properties P, including $\mathsf{S}_1(\Omega,\Gamma)$, if X satisfies P, then so is the group generated by X in any Tychonoff topological group G.

There is some restriction on P: Let $F(X)$ be the free topological group generated by X. That is, the group such that each continuous function f from X into a topological group H can be extended uniquely to a continuous homomorphism from $F(X)$ to H.[11]

[10] See the discussion at the beginning of the present section.

[11] A thorough introduction to free topological groups is available in [29].

Lemma 5.1. *Assume that P is a property of Tychonoff topological spaces, which is hereditary for closed subsets and preserved by homeomorphic images. If for each X satisfying P, $F(X)$ satisfies P, then $P = P^{\uparrow}$.*

Proof. Assume that X satisfies P. Then $F(X)$ satisfies P. For each k, X^k is homeomorphic to $\{x_1 \cdots x_k : x_1, \dots, x_k \in X\}$, a closed subspace of $F(X)$, and thus satisfies P. $\square$

Thus, we must consider properties P such that $P = P^{\uparrow}$. For each Q in the Scheepers Diagram, $P = Q^{\uparrow}$ is as required. By the discussion in the introduction [17]:

(1) $\mathsf{S}_1(\mathrm{O},\mathrm{O})^{\uparrow} = \mathsf{S}_1(\Omega,\Omega)^{\uparrow} = \mathsf{S}_1(\Omega,\Omega)$;
(2) $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})^{\uparrow} = \mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega)^{\uparrow} = \mathsf{S}_{\mathrm{fin}}(\Gamma,\Omega)^{\uparrow} = \mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)^{\uparrow} = \mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$.

Moreover, $\mathsf{S}_1(\Omega,\Gamma)^{\uparrow} = \mathsf{S}_1(\Omega,\Gamma)$ [12], and $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)^{\uparrow}$ also has a simple characterization [18]. The properties $\mathsf{S}_1(\Gamma,\Gamma)^{\uparrow}$, $\mathsf{S}_1(\Gamma,\Omega)^{\uparrow}$, and $\mathsf{S}_1(\Gamma,\mathrm{O})^{\uparrow}$ seem, however, to be unexplored. By the results of [17], each of these properties is strictly stronger than its non-$\uparrow$-ed version.

All P in the Scheepers Diagram have the properties required in the following theorem.

Theorem 5.2. *Assume that P is a property of Tychonoff topological spaces, which is hereditary for closed subsets and preserved by continuous images. If P is linearly σ-additive, then for each X satisfying $P^{\uparrow}$ and each Tychonoff topological group G containing X, the group $\langle X\rangle \le G$ generated by X satisfies $P^{\uparrow}$.*

Proof. It is not difficult to prove the following.

Lemma 5.3. *Assume that P is a property of topological spaces, which is hereditary for closed subsets and preserved by continuous images, and P is linearly σ-additive. Then $P^{\uparrow}$ also has these three properties.* $\square$

Fix distinct $a, b \in X \setminus \{1\}$, 1 being the identity element of G. As $\{1,a\}, \{a,b\}$ are discrete subspaces of G, $X \times \{1,a\}$ is homeomorphic to $X \times \{a,b\}$. As X^2 satisfies $P^{\uparrow}$, so does its closed subset $X \times \{a,b\}$, and thus so does $X \times \{1,a\}$. Thus, so does the image Y of $X \times \{1,a\}$ under the continuous map $(x,y) \mapsto xy^{-1}$. $Y = X \cup Xa^{-1}$, and as $a \in X$, $1 \in Y$. As $a \in X$, $\langle X\rangle = \langle Y\rangle$. Thus, we may assume that $1 \in X$.

$(\{1\} \times X) \cup (X \times \{1\})$ is a closed subset of X^2 which satisfies $P^{\uparrow}$. Thus, $(\{1\} \times X) \cup (X \times \{1\})$ satisfies $P^{\uparrow}$, and therefore so does its image under the same map, $Y = X \cup X^{-1}$. As $\langle Y\rangle = \langle X\rangle$, we may assume that $X = X^{-1}$ and $1 \in X$.

Thus,

$$\langle X\rangle = \bigcup_n \{x_1 \cdots x_n : x_1, \dots, x_n \in X\}$$

is an increasing union. For each n, $\{x_1 \cdots x_n : x_1, \dots, x_n \in X\}$ is a continuous image of X^n, which satisfies $P^{\uparrow}$. Thus, $\langle X\rangle$ satisfies $P^{\uparrow}$. $\square$

Theorem 5.2 can also be stated in the language of free topological groups. We obtain results analogous (but incomparable) to ones of Banakh, Repovš, and Zdomskyy [1].

Theorem 5.4. *Let P be as in Theorem 5.2. For each topological space X satisfying $P^{\uparrow}$, the free topological group $F(X)$ satisfies $P^{\uparrow}$.* $\square$

It follows that for each P in the Scheepers Diagram, and each X satisfying $P^{\uparrow}$, $\langle X\rangle$ satisfies $P^{\uparrow}$. Previous constructions of topological groups satisfying $P^{\uparrow}$ for these properties were much more involved than constructions of topological *spaces* satisfying $P^{\uparrow}$.

By Theorem 3.6 and Theorem 5.2, we have the following.

Corollary 5.5. *Assume that $\mathfrak{p} = \mathfrak{b}$. There is a subgroup of $\mathbb{R}$ of cardinality $\mathfrak{b}$, satisfying $\mathsf{S}_1(\Omega, \Gamma)$.* $\square$

We conclude with an application of the results of this paper. To put it in context, we draw in Figure 2 the Scheepers Diagram, extended to also contain the Borel properties [28]. In the Borel case, some additional equivalences hold, and thus there are fewer distinct properties. In particular, $\mathsf{U}_{\mathrm{fin}}(\mathrm{B}, \mathrm{B}_\Gamma)$, the Borel counterpart of $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$, is equivalent to $\mathsf{S}_1(\mathrm{B}_\Gamma, \mathrm{B}_\Gamma)$ [28].

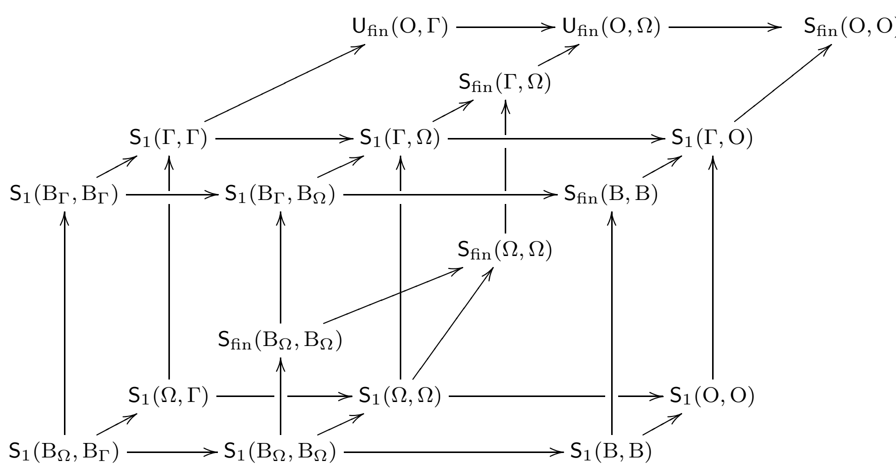

FIGURE 2. The extended Scheepers Diagram

Theorem 5.6. *Assume the Continuum Hypothesis.*

(1) *There is a subgroup G of $\mathbb{R}$ of cardinality continuum such that each finite power of G satisfies $\mathsf{S}_1(\mathrm{B}_\Omega, \mathrm{B}_\Omega)$ hereditarily, but G is not a σ-space and does not satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.*
(2) *There is a subgroup H of $\mathbb{R}$ of cardinality continuum such that all finite powers of H satisfy $\mathsf{S}_1(\mathrm{B}_\Gamma, \mathrm{B}_\Gamma)$ and $\mathsf{S}_{\mathrm{fin}}(\mathrm{B}_\Omega, \mathrm{B}_\Omega)$ hereditarily, but H does not satisfy $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$.*

Proof. By [3], or alternatively Lemma 4.2 and Remark 4.9, $\mathsf{S}_1(\mathrm{B}, \mathrm{B})$, $\mathsf{S}_1(\mathrm{B}_\Gamma, \mathrm{B}_\Gamma)$, and $\mathsf{S}_{\mathrm{fin}}(\mathrm{B}, \mathrm{B})$ are all hereditary. By [28], $\mathsf{S}_1(\mathrm{B}_\Omega, \mathrm{B}_\Omega)^{\uparrow} = \mathsf{S}_1(\mathrm{B}, \mathrm{B})^{\uparrow}$, and similarly for $\mathsf{S}_{\mathrm{fin}}$. Thus, by Lemma 4.7, it suffices to prove the statements with the words "hereditarily" removed.

(1) In the proof of Theorem 4.5, we have constructed a Luzin set $L \subseteq \mathbb{R}$ satisfying $\mathsf{S}_1(\mathrm{B}_\Omega, \mathrm{B}_\Omega)^\uparrow$. L is not a σ-space (Lemma 4.6). Let $G = \langle L \rangle$. By Theorem 5.2, G satisfies $\mathsf{S}_1(\mathrm{B}_\Omega, \mathrm{B}_\Omega)^\uparrow$.

As $L \subseteq G$ and being a σ-space is hereditary, G is not a σ-space. As $\mathsf{S}_1(\mathrm{B}, \mathrm{B})$ is hereditary and is not satisfied by perfect subsets of $\mathbb{R}$,[12] G does not contain any perfect subset of $\mathbb{R}$. Assume that G satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$. Theorem 5.5 of [17] tells us that sets satisfying $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ and not containing perfect sets are perfectly meager. In particular, G is meager, and thus so is L, a contradiction.

(2) By Theorem 23 and Lemma 24 of [32], there is a Sierpiński set[13] $S \subseteq \mathbb{R}$ satisfying $\mathsf{S}_1(\mathrm{B}_\Gamma, \mathrm{B}_\Gamma)^\uparrow$. (This result can alternatively and more easily be proved like Theorem 4.5 by using *random reals* instead of Cohen reals.) Let $H = \langle S \rangle$. By Theorem 5.2, H satisfies $\mathsf{S}_1(\mathrm{B}_\Gamma, \mathrm{B}_\Gamma)^\uparrow$. As $\mathsf{S}_1(\mathrm{B}_\Gamma, \mathrm{B}_\Gamma)$ implies $\mathsf{S}_{\mathrm{fin}}(\mathrm{B}, \mathrm{B})$, $\mathsf{U}_{\mathrm{fin}}(\mathrm{B}, \mathrm{B}_\Gamma)^\uparrow$ implies $\mathsf{S}_{\mathrm{fin}}(\mathrm{B}, \mathrm{B})^\uparrow$, which is $\mathsf{S}_{\mathrm{fin}}(\mathrm{B}_\Omega, \mathrm{B}_\Omega)^\uparrow$.

It remains to prove that H does not satisfy $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$. Assume otherwise. By Remark 4.9, as H satisfies $\mathsf{S}_1(\mathrm{B}_\Gamma, \mathrm{B}_\Gamma)$, it is a σ-space [28]. By Theorem 4.3, its subset S also satisfies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$, and thus has Lebesgue measure zero, a contradiction. □

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[12] Assume that a perfect set satisfies $\mathsf{S}_1(\mathrm{B}, \mathrm{B})$. Then so does the unit interval $[0, 1]$, its continuous image. But even $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ implies Lebesgue measure zero.

[13] $S \subseteq \mathbb{R}$ is a *Sierpiński set* if it is uncountable, but has countable intersection with each Lebesgue measure zero set.

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Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel
E-mail address: `talo@weizmann.ac.il`
URL: `http://www.orenshtein.com`

Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel
E-mail address: `tsaban@math.biu.ac.il`
URL: `http://www.cs.biu.ac.il/~tsaban`