

LAW OF LARGE NUMBERS FOR THE MAXIMAL FLOW THROUGH A DOMAIN OF \mathbb{R}^d IN FIRST PASSAGE PERCOLATION

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ABSTRACT. We consider the standard first passage percolation model in the rescaled graph \mathbb{Z}^d/n for $d \geq 2$, and a domain Ω of boundary Γ in \mathbb{R}^d . Let Γ^1 and Γ^2 be two disjoint open subsets of Γ , representing the parts of Γ through which some water can enter and escape from Ω . We investigate the asymptotic behaviour of the flow ϕ_n through a discrete version Ω_n of Ω between the corresponding discrete sets Γ_n^1 and Γ_n^2 . We prove that under some conditions on the regularity of the domain and on the law of the capacity of the edges, ϕ_n converges almost surely towards a constant ϕ_Ω , which is the solution of a continuous non-random min-cut problem. Moreover, we give a necessary and sufficient condition on the law of the capacity of the edges to ensure that $\phi_\Omega > 0$.

1. FIRST DEFINITIONS AND MAIN RESULT

We use many notation introduced in [18] and [19]. Let $d \geq 2$. We consider the graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$ having for vertices $\mathbb{Z}_n^d = \mathbb{Z}^d/n$ and for edges \mathbb{E}_n^d the set of pairs of nearest neighbours for the standard L^1 norm. With each edge e in \mathbb{E}_n^d we associate a random variable $t(e)$ with values in \mathbb{R}^+ . We suppose that the family $(t(e), e \in \mathbb{E}_n^d)$ is independent and identically distributed, with a common law Λ : this is the standard model of first passage percolation on the graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$. We interpret $t(e)$ as the capacity of the edge e ; it means that $t(e)$ is the maximal amount of fluid that can go through the edge e per unit of time.

We consider an open bounded connected subset Ω of \mathbb{R}^d such that the boundary $\Gamma = \partial\Omega$ of Ω is piecewise of class \mathcal{C}^1 (in particular Γ has finite area: $\mathcal{H}^{d-1}(\Gamma) < \infty$). It means that Γ is included in the union of a finite number of hypersurfaces of class \mathcal{C}^1 , i.e., in the union of a finite number of C^1 submanifolds of \mathbb{R}^d of codimension 1. Let Γ^1, Γ^2 be two disjoint subsets of Γ that are open in Γ . We want to define the maximal flow from Γ^1 to Γ^2 through Ω for the capacities $(t(e), e \in \mathbb{E}_n^d)$. We consider a discrete version $(\Omega_n, \Gamma_n, \Gamma_n^1, \Gamma_n^2)$ of $(\Omega, \Gamma, \Gamma^1, \Gamma^2)$ defined by

$$\begin{cases} \Omega_n = \{x \in \mathbb{Z}_n^d \mid d_\infty(x, \Omega) < 1/n\}, \\ \Gamma_n = \{x \in \Omega_n \mid \exists y \notin \Omega_n, \langle x, y \rangle \in \mathbb{E}_n^d\}, \\ \Gamma_n^i = \{x \in \Gamma_n \mid d_\infty(x, \Gamma^i) < 1/n, d_\infty(x, \Gamma^{3-i}) \geq 1/n\} \text{ for } i = 1, 2, \end{cases}$$

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where d_∞ is the L^∞ -distance and the notation $\langle x, y \rangle$ corresponds to the edge of endpoints x and y (see Figure 1).

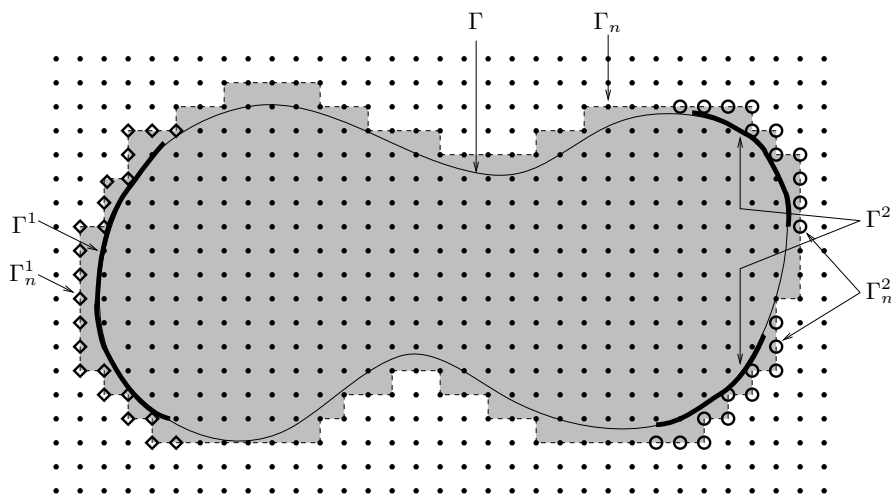


FIGURE 1. Domain Ω .

We shall study the maximal flow from Γ_n^1 to Γ_n^2 in Ω_n . Let us properly define the maximal flow $\phi(F_1 \rightarrow F_2 \text{ in } C)$ from F_1 to F_2 in C , for $C \subset \mathbb{R}^d$ (or by commodity the corresponding graph $C \cap \mathbb{Z}^d/n$). We will say that an edge $e = \langle x, y \rangle$ belongs to a subset A of \mathbb{R}^d , which we denote by $e \in A$, if the interior of the segment joining x to y is included in A . We define $\tilde{\mathbb{E}}_n^d$ as the set of all the oriented edges; i.e., an element \tilde{e} in $\tilde{\mathbb{E}}_n^d$ is an ordered pair of vertices which are nearest neighbours. We denote an element $\tilde{e} \in \tilde{\mathbb{E}}_n^d$ by $\langle \langle x, y \rangle \rangle$, where $x, y \in \mathbb{Z}_n^d$ are the endpoints of \tilde{e} and the edge is oriented from x towards y . We consider the set \mathcal{S} of all pairs of functions (g, o) , with $g : \mathbb{E}_n^d \rightarrow \mathbb{R}^+$ and $o : \mathbb{E}_n^d \rightarrow \tilde{\mathbb{E}}_n^d$ such that $o(\langle x, y \rangle) \in \{\langle \langle x, y \rangle \rangle, \langle \langle y, x \rangle \rangle\}$, satisfying:

- for each edge e in C we have

$$0 \leq g(e) \leq t(e),$$

- for each vertex v in $C \setminus (F_1 \cup F_2)$ we have

$$\sum_{e \in C : o(e) = \langle \langle v, \cdot \rangle \rangle} g(e) = \sum_{e \in C : o(e) = \langle \langle \cdot, v \rangle \rangle} g(e),$$

where the notation $o(e) = \langle \langle v, \cdot \rangle \rangle$ (respectively $o(e) = \langle \langle \cdot, v \rangle \rangle$) means that there exists $y \in \mathbb{Z}_n^d$ such that $e = \langle v, y \rangle$ and $o(e) = \langle \langle v, y \rangle \rangle$ (respectively $o(e) = \langle \langle y, v \rangle \rangle$). A couple $(g, o) \in \mathcal{S}$ is a possible stream in C from F_1 to F_2 : $g(e)$ is the amount of fluid that goes through the edge e , and $o(e)$ gives the direction in which the fluid goes through e . The two conditions on (g, o) express only the fact that the amount of fluid that can go through an edge is bounded by its capacity and that there is no loss of fluid in the graph. With each possible stream we associate the corresponding flow

$$\text{flow}(g, o) = \sum_{u \in F_2, v \notin C : \langle u, v \rangle \in \mathbb{E}_n^d} g(\langle \langle u, v \rangle \rangle) \mathbb{1}_{o(\langle \langle u, v \rangle \rangle) = \langle \langle u, v \rangle \rangle} - g(\langle \langle u, v \rangle \rangle) \mathbb{1}_{o(\langle \langle u, v \rangle \rangle) = \langle \langle v, u \rangle \rangle}.$$

This is the amount of fluid that crosses C from F_1 to F_2 if the fluid respects the stream (g, o) . The maximal flow through C from F_1 to F_2 is the supremum of this quantity over all possible choices of streams

$$\phi(F_1 \rightarrow F_2 \text{ in } C) = \sup\{\text{flow}(g, o) \mid (g, o) \in \mathcal{S}\}.$$

We denote by

$$\phi_n = \phi(\Gamma_n^1 \rightarrow \Gamma_n^2 \text{ in } \Omega_n)$$

the maximal flow from Γ_n^1 to Γ_n^2 in Ω_n . We will investigate the asymptotic behaviour of ϕ_n/n^{d-1} when n goes to infinity. More precisely, we will show that $(\phi_n/n^{d-1})_{n \geq 1}$ converges towards a constant ϕ_Ω (depending on Ω , Γ^1 , Γ^2 , Λ and d) when n goes to infinity and that this constant is strictly positive if and only if $\Lambda(0) < 1 - p_c(d)$, where $p_c(d)$ is the critical parameter for the bond percolation on \mathbb{Z}^d . The description of ϕ_Ω will be given in section 2. Here we state the precise theorem:

Theorem 1. *We suppose that Ω is a Lipschitz domain and that Γ is included in the union of a finite number of oriented hypersurfaces $\mathcal{S}_1, \dots, \mathcal{S}_r$ of class \mathcal{C}^1 which are transverse to each other. We also suppose that Γ^1 and Γ^2 are open in Γ , that their relative boundaries $\partial_\Gamma \Gamma^1$ and $\partial_\Gamma \Gamma^2$ in Γ have null \mathcal{H}^{d-1} measure, and that $d(\Gamma^1, \Gamma^2) > 0$. We suppose that the law Λ of the capacity of an edge admits an exponential moment:*

$$\exists \theta > 0 \quad \int_{\mathbb{R}^+} e^{\theta x} d\Lambda(x) < +\infty.$$

Then there exists a finite constant $\phi_\Omega \geq 0$ such that

$$\lim_{n \rightarrow \infty} \frac{\phi_n}{n^{d-1}} = \phi_\Omega \quad a.s.$$

Moreover, this equivalence holds:

$$\phi_\Omega > 0 \iff \Lambda(0) < 1 - p_c(d).$$

Remark 1. In the two companion papers [7] and [8], we prove in fact that the lower large deviations of ϕ_n/n^{d-1} below ϕ_Ω are of surface order and that the upper large deviations of ϕ_n/n^{d-1} above ϕ_Ω are of volume order (see section 3.2 where these results are presented).

2. COMPUTATION OF ϕ_Ω

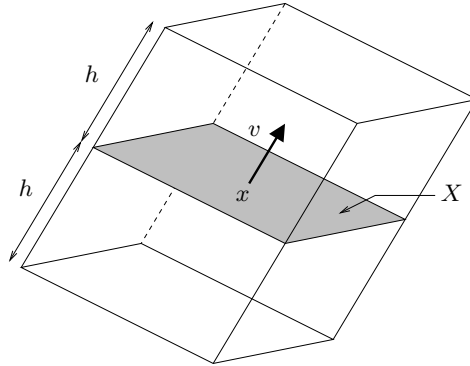
2.1. Geometric notation. We start with some geometric definitions. For a subset X of \mathbb{R}^d , we denote by $\mathcal{H}^s(X)$ the s -dimensional Hausdorff measure of X (we will use $s = d - 1$ and $s = d - 2$). The r -neighbourhood $\mathcal{V}_i(X, r)$ of X for the distance d_i , that can be the Euclidean distance if $i = 2$ or the L^∞ -distance if $i = \infty$, is defined by

$$\mathcal{V}_i(X, r) = \{y \in \mathbb{R}^d \mid d_i(y, X) < r\}.$$

If X is a subset of \mathbb{R}^d included in a hyperplane of \mathbb{R}^d and of codimension 1 (for example a non-degenerate hyperrectangle), we denote by $\text{hyp}(X)$ the hyperplane spanned by X , and we denote by $\text{cyl}(X, h)$ the cylinder of basis X and of height $2h$ defined by

$$\text{cyl}(X, h) = \{x + tv \mid x \in X, t \in [-h, h]\},$$

where v is one of the two unit vectors orthogonal to $\text{hyp}(X)$ (see Figure 2). For $x \in \mathbb{R}^d$, $r \geq 0$ and a unit vector v , we denote by $B(x, r)$ the closed ball centered at x of radius r , by $\text{disc}(x, r, v)$ the closed disc centered at x of radius r and normal

FIGURE 2. Cylinder $\text{cyl}(X, h)$.

vector v , and by $\text{hyp}(x, v)$ the hyperplane containing x and orthogonal to v . We denote by α_d the volume of a unit ball in \mathbb{R}^d and by α_{d-1} the \mathcal{H}^{d-1} measure of a unit disc.

2.2. Flow in a cylinder. Here are some particular definitions of flows through a box. Let A be a non-degenerate hyperrectangle, i.e., a box of dimension $d-1$ in \mathbb{R}^d . All hyperrectangles will be assumed to be closed in \mathbb{R}^d . We denote by v one of the two unit vectors orthogonal to $\text{hyp}(A)$. For h a positive real number, we consider the cylinder $\text{cyl}(A, h)$. The set $\text{cyl}(A, h) \setminus \text{hyp}(A)$ has two connected components, which we denote by $\mathcal{C}_1(A, h)$ and $\mathcal{C}_2(A, h)$. For $i = 1, 2$, let A_i^h be the set of the points in $\mathcal{C}_i(A, h) \cap \mathbb{Z}_n^d$ which have a nearest neighbour in $\mathbb{Z}_n^d \setminus \text{cyl}(A, h)$:

$$A_i^h = \{x \in \mathcal{C}_i(A, h) \cap \mathbb{Z}_n^d \mid \exists y \in \mathbb{Z}_n^d \setminus \text{cyl}(A, h), \langle x, y \rangle \in \mathbb{E}_n^d\}.$$

Let $T(A, h)$ (respectively $B(A, h)$) be the top (respectively the bottom) of $\text{cyl}(A, h)$; i.e.,

$$T(A, h) = \{x \in \text{cyl}(A, h) \mid \exists y \notin \text{cyl}(A, h), \langle x, y \rangle \in \mathbb{E}_n^d \text{ and } \langle x, y \rangle \text{ intersects } A + hv\}$$

and

$$B(A, h) = \{x \in \text{cyl}(A, h) \mid \exists y \notin \text{cyl}(A, h), \langle x, y \rangle \in \mathbb{E}_n^d \text{ and } \langle x, y \rangle \text{ intersects } A - hv\}.$$

For a given realisation $(t(e), e \in \mathbb{E}_n^d)$ we define the variable $\tau(A, h) = \tau(\text{cyl}(A, h), v)$ by

$$\tau(A, h) = \tau(\text{cyl}(A, h), v) = \phi(A_1^h \rightarrow A_2^h \text{ in } \text{cyl}(A, h))$$

and the variable $\phi(A, h) = \phi(\text{cyl}(A, h), v)$ by

$$\phi(A, h) = \phi(\text{cyl}(A, h), v) = \phi(B(A, h) \rightarrow T(A, h) \text{ in } \text{cyl}(A, h)),$$

where $\phi(F_1 \rightarrow F_2 \text{ in } C)$ is the maximal flow from F_1 to F_2 in C , for $C \subset \mathbb{R}^d$ (or by commodity the corresponding graph $C \cap \mathbb{Z}^d/n$) defined previously. The dependence in n is implicit here. In fact we can also write $\tau_n(A, h)$ and $\phi_n(A, h)$ if we want to emphasize this dependence on the mesh of the graph.

2.3. Max-flow min-cut theorem. The maximal flow $\phi(F_1 \rightarrow F_2 \text{ in } C)$ can be expressed differently thanks to the max-flow min-cut theorem (see [5]). We need some definitions to state this result. A path on the graph \mathbb{Z}_n^d from v_0 to v_m is a sequence $(v_0, e_1, v_1, \dots, e_m, v_m)$ of vertices v_0, \dots, v_m alternating with edges e_1, \dots, e_m such that v_{i-1} and v_i are neighbours in the graph, joined by the edge e_i , for i in $\{1, \dots, m\}$. A set E of edges in C is said to cut F_1 from F_2 in C if there is no path from F_1 to F_2 in $C \setminus E$. We call E an (F_1, F_2) -cut if E cuts F_1 from F_2 in C and if no proper subset of E does. With each set E of edges we associate its capacity, which is the variable

$$V(E) = \sum_{e \in E} t(e).$$

The max-flow min-cut theorem states that

$$\phi(F_1 \rightarrow F_2 \text{ in } C) = \min\{V(E) \mid E \text{ is a } (F_1, F_2)\text{-cut}\}.$$

In fact, as we will see in section 2.5, ϕ_Ω is a continuous equivalent of the discrete min-cut.

2.4. Definition of ν . The asymptotic behaviour of the rescaled expectation of $\tau_n(A, h)$ for large n is well known, thanks to the almost subadditivity of this variable. We recall the following result:

Theorem 2. *We suppose that*

$$\int_{[0, +\infty[} x d\Lambda(x) < \infty.$$

Then for each unit vector v there exists a constant $\nu(d, \Lambda, v) = \nu(v)$ (the dependence on d and Λ is implicit) such that for every non-degenerate hyperrectangle A orthogonal to v and for every strictly positive constant h , we have

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[\tau_n(A, h)]}{n^{d-1} \mathcal{H}^{d-1}(A)} = \nu(v).$$

For a proof of this proposition, see [25]. We emphasize the fact that the limit depends on the direction of v , but not on h or on the hyperrectangle A itself.

We recall some geometric properties of the map $\nu : v \in S^{d-1} \mapsto \nu(v)$, under the only condition on Λ that $\mathbb{E}(t(e)) < \infty$. They have been stated in section 4.4 of [25]. There exists a unit vector v_0 such that $\nu(v_0) = 0$ if and only if for all unit vectors v , $\nu(v) = 0$, and this happens if and only if $\Lambda(\{0\}) \geq 1 - p_c(d)$. This property has been proved by Zhang in [27]. Moreover, ν satisfies the weak triangle inequality; i.e., if (ABC) is a non-degenerate triangle in \mathbb{R}^d and v_A, v_B and v_C are the exterior normal unit vectors to the sides $[BC]$, $[AC]$, $[AB]$ in the plane spanned by A, B, C , then

$$\mathcal{H}^1([AB])\nu(v_C) \leq \mathcal{H}^1([AC])\nu(v_B) + \mathcal{H}^1([BC])\nu(v_A).$$

This implies that the homogeneous extension ν_0 of ν to \mathbb{R}^d , defined by $\nu_0(0) = 0$ and for all w in \mathbb{R}^d ,

$$\nu_0(w) = |w|_2 \nu(w/|w|_2),$$

is a convex function; in particular, since ν_0 is finite, it is continuous on \mathbb{R}^d . We denote by ν_{\min} (respectively ν_{\max}) the infimum (respectively supremum) of ν on S^{d-1} .

2.5. Continuous min-cut. Here we give a definition of ϕ_Ω and of another constant $\widetilde{\phi}_\Omega$ in terms of the map ν . For a subset F of \mathbb{R}^d , we define the perimeter of F in Ω by

$$\mathcal{P}(F, \Omega) = \sup \left\{ \int_F \operatorname{div} f(x) d\mathcal{L}^d(x), f \in \mathcal{C}_c^\infty(\Omega, B(0, 1)) \right\},$$

where $\mathcal{C}_c^\infty(\Omega, B(0, 1))$ is the set of the functions of class \mathcal{C}^∞ from \mathbb{R}^d to $B(0, 1)$, the ball centered at 0 and of radius 1 in \mathbb{R}^d , having a compact support included in Ω , and div is the usual divergence operator. The perimeter $\mathcal{P}(F)$ of F is defined as $\mathcal{P}(F, \mathbb{R}^d)$. We denote by ∂F the boundary of F and by $\partial^* F$ the reduced boundary of F . At any point x of $\partial^* F$, the set F admits a unit exterior normal vector $v_F(x)$ at x in a measure theoretic sense (for definitions see for example [9], section 13). For all $F \subset \mathbb{R}^d$ of finite perimeter in Ω , we define

$$\begin{aligned} \mathcal{I}_\Omega(F) = & \int_{\partial^* F \cap \Omega} \nu(v_F(x)) d\mathcal{H}^{d-1}(x) + \int_{\Gamma^2 \cap \partial^*(F \cap \Omega)} \nu(v_{(F \cap \Omega)}(x)) d\mathcal{H}^{d-1}(x) \\ & + \int_{\Gamma^1 \cap \partial^*(\Omega \setminus F)} \nu(v_\Omega(x)) d\mathcal{H}^{d-1}(x). \end{aligned}$$

If $\mathcal{P}(F, \Omega) = +\infty$, we define $\mathcal{I}_\Omega(F) = +\infty$. Finally, we define

$$\phi_\Omega = \inf \{ \mathcal{I}_\Omega(F) \mid F \subset \mathbb{R}^d \} = \inf \{ \mathcal{I}_\Omega(F) \mid F \subset \Omega \}.$$

In the case where ∂F is \mathcal{C}^1 , $\mathcal{I}_\Omega(F)$ has the following simpler expression:

$$\begin{aligned} \mathcal{I}_\Omega(F) = & \int_{\partial F \cap \Omega} \nu(v_F(x)) d\mathcal{H}^{d-1}(x) + \int_{\Gamma^2 \cap \partial(F \cap \Omega)} \nu(v_{(F \cap \Omega)}(x)) d\mathcal{H}^{d-1}(x) \\ & + \int_{\Gamma^1 \cap \partial(\Omega \setminus F)} \nu(v_\Omega(x)) d\mathcal{H}^{d-1}(x). \end{aligned}$$

The localization of the set along which the previous integrals are done is illustrated in Figure 3.

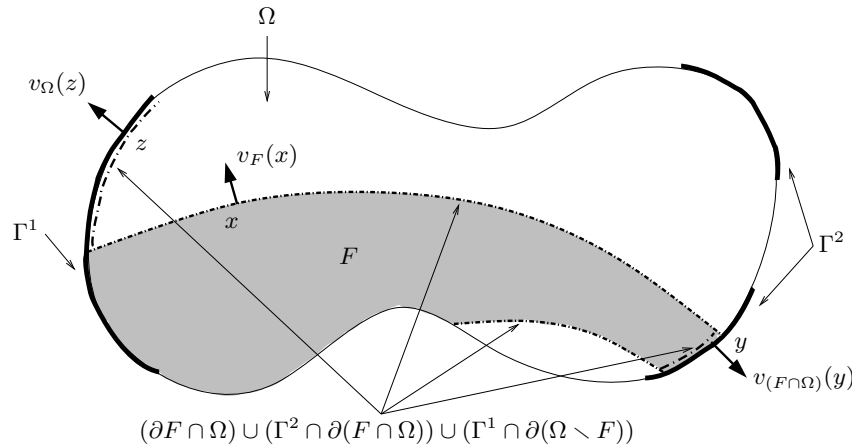


FIGURE 3. The set $(\partial F \cap \Omega) \cup (\Gamma^2 \cap \partial(F \cap \Omega)) \cup (\Gamma^1 \cap \partial(\Omega \setminus F))$.

When a hypersurface \mathcal{S} is piecewise of class \mathcal{C}^1 , we say that \mathcal{S} is transverse to Γ if for all $x \in \mathcal{S} \cap \Gamma$, the normal unit vectors to \mathcal{S} and Γ at x are not collinear.

If the normal vector to \mathcal{S} (respectively to Γ) at x is not well defined, this property must be satisfied by all the vectors which are limits of normal unit vectors to \mathcal{S} (respectively Γ) at $y \in \mathcal{S}$ (respectively $y \in \Gamma$) when we send y to x - there is at most a finite number of such limits. We say that a subset P of \mathbb{R}^d is polyhedral if its boundary ∂P is included in the union of a finite number of hyperplanes. For each point x of such a set P which is on the interior of one face of ∂P , we denote by $v_P(x)$ the exterior unit vector orthogonal to P at x . For $A \subset \mathbb{R}^d$, we denote by $\overset{\circ}{A}$ the interior of A . We define $\widetilde{\phi}_\Omega$ by

$$\widetilde{\phi}_\Omega = \inf \left\{ \mathcal{I}_\Omega(P) \left| \begin{array}{l} P \subset \mathbb{R}^d, \overline{\Gamma^1} \subset \overset{\circ}{P}, \overline{\Gamma^2} \subset \widehat{\mathbb{R}^d \setminus P} \\ P \text{ is polyhedral, } \partial P \text{ is transverse to } \Gamma \end{array} \right. \right\}.$$

Notice that if P is a set such that

$$\overline{\Gamma^1} \subset \overset{\circ}{P} \quad \text{and} \quad \overline{\Gamma^2} \subset \widehat{\mathbb{R}^d \setminus P},$$

then

$$\mathcal{I}_\Omega(P) = \int_{\partial P \cap \Omega} \nu(v_P(x)) d\mathcal{H}^{d-1}(x).$$

See Figure 4 for an example of such a polyhedral set P .

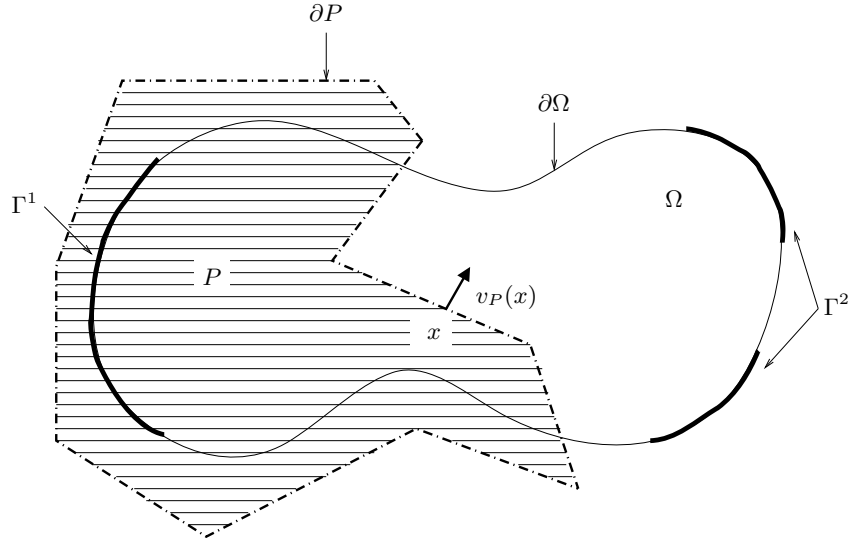


FIGURE 4. A polyhedral set P as in the definition of $\widetilde{\phi}_\Omega$.

The definitions of the constants ϕ_Ω and $\widetilde{\phi}_\Omega$ are not very intuitive. We propose to define the notion of a continuous cutset to have a better understanding of these constants. We say that $\mathcal{S} \subset \mathbb{R}^d$ cuts Γ^1 from Γ^2 in $\overline{\Omega}$ if every continuous path from Γ^1 to Γ^2 in $\overline{\Omega}$ intersects \mathcal{S} . In fact, if P is a polyhedral set of \mathbb{R}^d such that

$$\overline{\Gamma^1} \subset \overset{\circ}{P} \quad \text{and} \quad \overline{\Gamma^2} \subset \widehat{\mathbb{R}^d \setminus P},$$

then $\partial P \cap \overline{\Omega}$ is a continuous cutset from Γ^1 to Γ^2 in $\overline{\Omega}$. Since $\nu(v)$ is the average amount of fluid that can cross a hypersurface of area one in the direction v per

unit of time, it can be interpreted as the capacity of a unitary hypersurface. Thus $\mathcal{I}_\Omega(P)$ can be interpreted as the capacity of the continuous cutset $\partial P \cap \overline{\Omega}$. The constant $\widetilde{\phi}_\Omega$ is the solution of a min-cut problem, because it is equal to the infimum of the capacity of a continuous cutset that satisfies some specific properties. We can define two other constants that are solutions of possibly more intuitive min-cut problems. If \mathcal{S} is a hypersurface which is piecewise of class \mathcal{C}^1 , we denote by $v_{\mathcal{S}}(x)$ one of the two normal unit vectors to \mathcal{S} at x for every point x at which \mathcal{S} is regular. The \mathcal{H}^{d-1} measure of the points at which \mathcal{S} is not regular is null. We define

$$\widehat{\phi}_\Omega = \inf \left\{ \int_{\mathcal{S} \cap \overline{\Omega}} \nu(v_{\mathcal{S}}(x)) d\mathcal{H}^{d-1}(x) \mid \begin{array}{l} \mathcal{S} \text{ hypersurface piecewise of class } \mathcal{C}^1 \\ \mathcal{S} \text{ cuts } \Gamma^1 \text{ from } \Gamma^2 \text{ in } \overline{\Omega} \end{array} \right\}$$

and

$$\widehat{\phi}_\Omega = \inf \left\{ \int_{\mathcal{S} \cap \overline{\Omega}} \nu(v_{\mathcal{S}}(x)) d\mathcal{H}^{d-1}(x) \mid \begin{array}{l} \mathcal{S} \text{ polyhedral hypersurface} \\ \mathcal{S} \text{ cuts } \Gamma^1 \text{ from } \Gamma^2 \text{ in } \overline{\Omega} \end{array} \right\}.$$

We remark that by definition,

$$\widehat{\phi}_\Omega \leq \widehat{\phi}_\Omega \leq \widetilde{\phi}_\Omega.$$

We claim that $\phi_\Omega \leq \widehat{\phi}_\Omega$. Let \mathcal{S} be a hypersurface which is piecewise of class \mathcal{C}^1 , which cuts Γ^1 from Γ^2 in $\overline{\Omega}$, and such that

$$\int_{\mathcal{S} \cap \overline{\Omega}} \nu(v_{\mathcal{S}}(x)) d\mathcal{H}^{d-1}(x) \leq \widehat{\phi}_\Omega + \eta$$

for some positive η . Let F be the set of the points of $\overline{\Omega} \setminus \mathcal{S}$ that can be joined to a point of Γ^1 by a continuous path. Then

$$(\partial F \cap \Omega) \cup (\Gamma^1 \cap \partial(\Omega \setminus F)) \cup (\Gamma^2 \cap \partial(F \cap \Omega)) \subset \mathcal{S} \cap \overline{\Omega}.$$

Thus F is of finite perimeter in Ω and $\mathcal{I}_\Omega(F)$ satisfies

$$\mathcal{I}_\Omega(F) \leq \int_{\mathcal{S} \cap \overline{\Omega}} \nu(v_{\mathcal{S}}(x)) d\mathcal{H}^{d-1}(x) \leq \widehat{\phi}_\Omega + \eta.$$

Thus we have proved that

$$\phi_\Omega \leq \widehat{\phi}_\Omega \leq \widehat{\phi}_\Omega \leq \widetilde{\phi}_\Omega.$$

3. STATE OF THE ART

3.1. Existing laws of large numbers. Only in this section do we consider the standard first passage percolation model on the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ instead of the rescaled graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$. Here we present some laws of large numbers that have been proved about maximal flows.

Using a subadditive argument and concentration inequalities, Rossignol and Théret have proved in [25] that $\tau(nA, h(n))$ satisfies a law of large numbers:

Theorem 3 (Rossignol and Théret). *We suppose that*

$$\int_{[0, \infty[} x d\Lambda(x) < \infty.$$

For every unit vector v , for every non-degenerate hyperrectangle A orthogonal to v , and for every height function $h : \mathbb{N} \rightarrow \mathbb{R}^+$ satisfying $\lim_{n \rightarrow \infty} h(n) = +\infty$ we have

$$\lim_{n \rightarrow \infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \nu(v) \quad \text{in } L^1.$$

Moreover, if the origin of the graph belongs to A or if

$$\int_{[0, \infty[} x^{1+\frac{1}{d-1}} d\Lambda(x) < \infty,$$

then

$$\lim_{n \rightarrow \infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \nu(v) \quad \text{a.s.}$$

Kesten, Zhang, Rossignol and Th  ret have studied the maximal flow between the top and the bottom of straight cylinders. Let us denote by $D(\mathbf{k}, m)$ the cylinder

$$D(\mathbf{k}, m) = \prod_{i=1}^{d-1} [0, k_i] \times [0, m],$$

where $\mathbf{k} = (k_1, \dots, k_{d-1}) \in \mathbb{R}^{d-1}$. We denote by $\phi(\mathbf{k}, m)$ the maximal flow in $D(\mathbf{k}, m)$ from its top $\prod_{i=1}^{d-1} [0, k_i] \times \{m\}$ to its bottom $\prod_{i=1}^{d-1} [0, k_i] \times \{0\}$. Kesten proved in [19] the following result:

Theorem 4 (Kesten). *Let $d = 3$. We suppose that $\Lambda(0) < p_0$ for some fixed $p_0 \geq 1/27$ and that*

$$\exists \gamma > 0 \quad \int_{[0, +\infty[} e^{\gamma x} d\Lambda(x) < \infty.$$

If $m = m(\mathbf{k})$ goes to infinity with $k_1 \geq k_2$ in such a way that

$$\exists \delta > 0 \quad \lim_{k_1 \geq k_2 \rightarrow \infty} k^{-1+\delta} \log m(\mathbf{k}) = 0,$$

then

$$\lim_{k_1 \geq k_2 \rightarrow \infty} \frac{\phi(\mathbf{k}, m)}{k_1 k_2} = \nu((0, 0, 1)) \quad \text{a.s. and in } L^1.$$

Moreover, if $\Lambda(0) > 1 - p_c(d)$, where $p_c(d)$ is the critical parameter for the standard bond percolation model on \mathbb{Z}^d , and if

$$\int_{[0, +\infty[} x^6 d\Lambda(x) < \infty,$$

there exists a constant $C = C(F) < \infty$ such that for all $m = m(\mathbf{k})$ that goes to infinity with $k_1 \geq k_2$ and satisfies

$$\liminf_{k_1 \geq k_2 \rightarrow \infty} \frac{m(\mathbf{k})}{k_1 k_2} > C,$$

for all $k_1 \geq k_2$ sufficiently large, we have

$$\phi(\mathbf{k}, m) = 0 \quad \text{a.s.}$$

Zhang improved this result in [28] where he proved the following theorem.

Theorem 5 (Zhang). *Let $d \geq 2$. We suppose that*

$$\exists \gamma > 0 \quad \int_{[0, +\infty[} e^{\gamma x} d\Lambda(x) < \infty.$$

Then for all $m = m(\mathbf{k})$ that goes to infinity when all the k_i , $i = 1, \dots, d-1$, go to infinity in such a way that

$$\exists \delta \in]0, 1] \quad \log m(\mathbf{k}) \leq \max_{i=1, \dots, d-1} k_i^{1-\delta},$$

we have

$$\lim_{k_1, \dots, k_{d-1} \rightarrow \infty} \frac{\phi(\mathbf{k}, m)}{\prod_{i=1}^{d-1} k_i} = \nu((0, \dots, 0, 1)) \quad \text{a.s. and in } L^1.$$

Moreover, this limit is positive if and only if $\Lambda(0) < 1 - p_c(d)$.

To show this theorem, Zhang first obtains an important control on the number of edges in a minimal cutset. Finally, Rossignol and Théret improved Zhang's result in [25] in the particular case where the dimensions of the basis of the straight cylinder go to infinity all at the same speed. They obtain the following result:

Theorem 6 (Rossignol and Théret). *We suppose that*

$$\int_{[0, \infty[} x d\Lambda(x) < \infty.$$

For every straight hyperrectangle

$$A = \prod_{i=1}^{d-1} [0, a_i] \times \{0\}$$

with $a_i > 0$ for all i and for every height function $h : \mathbb{N} \rightarrow \mathbb{R}^+$ satisfying $\lim_{n \rightarrow \infty} h(n) = +\infty$ and $\lim_{n \rightarrow \infty} \log h(n)/n^{d-1} = 0$, we have

$$\lim_{n \rightarrow \infty} \frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \nu((0, \dots, 0, 1)) \quad \text{a.s. and in } L^1.$$

In dimension two, more results are known. Here we present two of them. Rossignol and Théret have studied in [24] the maximal flow from the top to the bottom of a tilted cylinder in dimension two, and they have proved the following theorem (Corollary 2.10 in [24]):

Theorem 7 (Rossignol and Théret). *Let v be a unit vector, A a non-degenerate line-segment orthogonal to v , and $h : \mathbb{N} \rightarrow \mathbb{R}^+$ a height function satisfying $\lim_{n \rightarrow \infty} h(n) = +\infty$ and $\lim_{n \rightarrow \infty} \log h(n)/n = 0$. We suppose that there exists $\alpha \in [0, \pi/2]$ such that*

$$\lim_{n \rightarrow \infty} \frac{2h(n)}{\mathcal{H}^1(nA)} = \tan \alpha.$$

Then, if

$$\int_{[0, \infty[} x d\Lambda(x) < \infty,$$

we have

$$\lim_{n \rightarrow \infty} \frac{\phi(nA, h(n))}{\mathcal{H}^1(nA)} = \inf \left\{ \frac{\nu(v')}{v \cdot v'} \mid v' \text{ satisfies } v \cdot v' \geq \cos \alpha \right\} \quad \text{in } L^1.$$

Moreover, if the origin of the graph is the middle of A or if

$$\int_{[0,\infty[} x^2 d\Lambda(x) < \infty,$$

then we have

$$\lim_{n \rightarrow \infty} \frac{\phi(nA, h(n))}{\mathcal{H}^1(nA)} = \inf \left\{ \frac{\nu(v')}{v \cdot v'} \mid v' \text{ satisfies } v \cdot v' \geq \cos \alpha \right\} \quad a.s.$$

In [15] Garet studied the maximal flow $\sigma(A)$ between a convex bounded set A and infinity in the case $d = 2$. By an extension of the max-flow min-cut theorem to non-finite graphs, in [15] Garet proves that this maximal flow is equal to the minimal capacity of a set of edges that cuts all paths from A to infinity. Let ∂A be the boundary of A , and let $\partial^* A$ be the set of the points $x \in \partial A$ at which A admits a unique exterior normal unit vector $v_A(x)$ in a measure theoretic sense (see [9], section 13, for a precise definition). If A is a convex set, the set $\partial^* A$ is also equal to the set of the points $x \in \partial A$ at which A admits a unique exterior normal vector in the classical sense, and this vector is $v_A(x)$. Garet proved the following theorem:

Theorem 8 (Garet). *Let $d = 2$. We suppose that $\Lambda(0) < 1 - p_c(2) = 1/2$ and that*

$$\exists \gamma > 0 \quad \int_{[0,+\infty[} e^{\gamma x} d\Lambda(x) < \infty.$$

Then for each convex bounded set A containing 0 in its interior, we have

$$\lim_{n \rightarrow \infty} \frac{\sigma(nA)}{n} = \int_{\partial^* A} \nu(v_A(x)) d\mathcal{H}^1(x) = \mathcal{I}(A) > 0 \quad a.s.$$

Moreover, for all $\varepsilon > 0$, there exist constants $C_1, C_2 > 0$ depending on ε and Λ such that

$$\forall n \geq 0 \quad \mathbb{P} \left[\frac{\sigma(nA)}{n\mathcal{I}(A)} \notin [1 - \varepsilon, 1 + \varepsilon] \right] \leq C_1 \exp(-C_2 n).$$

Nevertheless, a law of large numbers for the maximal flow from the top to the bottom of a tilted cylinder for $d \geq 3$ was not yet proved. In fact, the lack of symmetry of the graph induced by the slope of the box is a major issue to extend the existing results concerning straight cylinders to tilted cylinders. The theorem of Garet was not extended to dimension $d \geq 3$ either. Theorem 1 applies to the maximal flow from the top to the bottom of a tilted cylinder. Thus it is a generalisation of the laws of large numbers of Kesten, Zhang, Rossignol and Th  ret for the variable ϕ in straight cylinders, in the particular case where all the dimensions of the cylinder go to infinity at the same speed (or, equivalently, the cylinder is fixed and the mesh of the graph go to zero isotropically). Moreover, it gives a hint of what could be a generalisation of the result of Garet in higher dimension, all the more since the expression of the constant ϕ_Ω is reminiscent of the value of the limit in Garet's Theorem: the capacity \mathcal{I}_Ω of a continuous cutset is exactly the same as the one defined by Garet in [15] in dimension two, except that we consider a maximal flow through a bounded domain, so our capacity is adapted to deal with specific boundary conditions.

From now on, we work in the rescaled graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$.

3.2. Large deviations for ϕ_n . Here we present the two existing results concerning ϕ_n . We consider an open bounded connected subset Ω of \mathbb{R}^d , whose boundary Γ is piecewise of class \mathcal{C}^1 , and two disjoint open subsets Γ^1 and Γ^2 of Γ . The first result states that the lower large deviations below ϕ_Ω are of surface order and is proved by the authors in [7]:

Theorem 9. *If the law Λ of the capacity of an edge admits an exponential moment,*

$$\exists \theta > 0 \quad \int_{\mathbb{R}^+} e^{\theta x} d\Lambda(x) < +\infty,$$

and if $\Lambda(0) < 1 - p_c(d)$, then for all $\lambda < \phi_\Omega$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \log \mathbb{P}[\phi_n \leq \lambda n^{d-1}] < 0.$$

The second result states that the upper large deviations of ϕ_n above $\widetilde{\phi}_\Omega$ are of volume order and is proved by the authors in [8]:

Theorem 10. *We suppose that $d(\Gamma^1, \Gamma^2) > 0$. If the law Λ of the capacity of an edge admits an exponential moment,*

$$\exists \theta > 0 \quad \int_{\mathbb{R}^+} e^{\theta x} d\Lambda(x) < +\infty,$$

then for all $\lambda > \widetilde{\phi}_\Omega$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n^d} \log \mathbb{P}[\phi_n \geq \lambda n^d] < 0.$$

By a simple Borel-Cantelli lemma, these results imply that if Λ admits an exponential moment and if $d(\Gamma^1, \Gamma^2) > 0$, then

$$\phi_\Omega \leq \liminf_{n \rightarrow \infty} \frac{\phi_n}{n} \leq \limsup_{n \rightarrow \infty} \frac{\phi_n}{n} \leq \widetilde{\phi}_\Omega.$$

Notice here that Theorem 9 allows us to obtain the first inequality only under the additional hypothesis that $\Lambda(0) < 1 - p_c(d)$. However, if $\Lambda(0) \geq 1 - p_c(d)$ we know that $\nu(v) = 0$ for all v , so $\phi_\Omega = 0$ and the first inequality remains valid.

Thus, to prove Theorem 1, it remains to prove that $\phi_\Omega = \widetilde{\phi}_\Omega$ and to study the positivity of ϕ_Ω . The equality $\phi_\Omega = \widetilde{\phi}_\Omega$ is a consequence of a polyhedral approximation of sets having finite perimeter that will be done in section 4. The positivity of ϕ_Ω is proved in section 5, using tools of differential geometry such as the tubular neighbourhood of paths. These two results are proved by purely geometrical studies. Since the probabilistic part of the proof of Theorem 1 is contained in Theorems 9 and 10, we propose a sketch of the proofs of these two theorems in sections 3.2.1 and 3.2.2 to help in the understanding of the law of large numbers proved in this paper.

Before these two sketches of proofs, we would like to make two remarks. The first one is that the large deviations that are obtained in Theorems 9 and 10 are of relevant order. Indeed, if all the edges in Ω_n have a capacity which is abnormally large, then the maximal flow ϕ_n will be abnormally large, too. The probability for these edges to have an abnormally large capacity is of order $\exp -Cn^d$ for a constant C , because the number of edges in Ω_n is $C'n^d$ for a constant C' . On the other hand, if all the edges in a flat layer that separates Γ_n^1 from Γ_n^2 in Ω_n have abnormally small capacity, then ϕ_n will be abnormally small. Since the cardinality

of such a set of edges is $D'n^{d-1}$ for a constant D' , the probability of this event is of order $\exp -Dn^{d-1}$ for a constant D .

The second remark we would like to make is that the condition $d(\Gamma^1, \Gamma^2) > 0$ is relevant in Theorem 10. First, without this condition, we cannot be sure that there exists a polyhedral set P as in the definition of $\widetilde{\phi}_\Omega$, and thus the polyhedral approximation (see section 4) cannot be performed. Moreover, if $d(\Gamma^1, \Gamma^2) = 0$, there exists a set of edges of constant cardinality (not depending on n) that contains paths from Γ_n^1 to Γ_n^2 through Ω_n for all n along the common boundary of Γ^1 and Γ^2 , and so it may be sufficient for these edges to have a huge capacity to obtain the fact that ϕ_n is abnormally large, too. Thus, we cannot hope to obtain upper large deviations of volume order (see [26] for a counterexample). However, we do not know if this condition is essential for Theorem 1 to hold.

3.2.1. Lower large deviations. To prove Theorem 9, we have to study the probability

$$(1) \quad \mathbb{P}[\phi_n \leq (\phi_\Omega - \varepsilon)n^{d-1}]$$

for a positive ε . The proof is divided into three steps.

First step: We consider a set of edges \mathcal{E}_n that cuts Γ_n^1 from Γ_n^2 in Ω_n , of minimal capacity (so $\phi_n = V(\mathcal{E}_n)$) and having the minimal number of edges among those cutsets. We see it as the (edge) boundary of a set E_n which is included in Ω . Zhang's estimate of the number of edges in a minimal cutset (Theorem 1 in [28]) states that with high probability, the perimeter $\mathcal{P}(E_n, \Omega)$ of E_n in Ω is smaller than a constant β . Thus, E_n belongs to the set

$$\mathcal{C}_\beta = \{F \subset \Omega \mid F \subset \Omega, \mathcal{P}(F, \Omega) \leq \beta\}.$$

We endow \mathcal{C}_β with the topology L^1 associated to the following distance d :

$$d(F_1, F_2) = \mathcal{L}^d(F_1 \triangle F_2),$$

where \mathcal{L}^d is the d -dimensional Lebesgue measure. For this topology, the set \mathcal{C}_β is compact. Thus, if we associate to each set F in \mathcal{C}_β a positive constant ε_F , and if we denote by $\mathcal{V}(F, \varepsilon_F)$ the neighbourhood of F of radius ε_F for the distance d defined above, the collection of these neighbourhoods is an open covering of \mathcal{C}_β , and thus by compactness of \mathcal{C}_β we can extract a finite covering:

$$\exists F_1, \dots, F_N \quad \mathcal{C}_\beta \subset \bigcup_{i=1}^N \mathcal{V}(F_i, \varepsilon_{F_i}).$$

If we find an upper bound on the probability

$$(2) \quad \mathbb{P}[\phi_n \leq (\phi_\Omega - \varepsilon)n^{d-1} \text{ and } d(E_n, F) \leq \varepsilon_F]$$

for each F in \mathcal{C}_β and a corresponding ε_F , then we will obtain an upper bound on the probability (1).

Second step: We consider a fixed set F in \mathcal{C}_β , and we want to evaluate the probability (2). So we suppose that E_n is close to F for the distance d , and we denote it by $E_n \approx F$ to simplify the notation. Here we skip all the problems of boundary conditions that arise in the proof of Theorem 9: we suppose that $\mathcal{I}_\Omega(F)$ is equal to the integral of ν along $\partial^* F \cap \Omega$.

We make a zoom along ∂F . Using the Vitali covering theorem (Theorem 12 in section 4), we know that there exists a finite number of disjoint balls $B_j = B(x_j, r_j)$

for $j = 1, \dots, \mathcal{N}$ with $x_j \in \partial F$ such that ∂F is “almost flat” in each ball, and the part of ∂F that is missing in the covering has a very small area. We denote by v_j the exterior normal unit vector of F at x_j (we suppose that it exists). Here “almost flat” means that

- (i) the capacity of ∂F inside B_j is very close to the capacity of the flat disc $\text{hyp}(x_j, v_j) \cap B_j$, i.e., very close to $\alpha_{d-1} r_j^{d-1} \nu(v_j)$;
- (ii) $F \cap B_j \approx B_j^-$, where B_j^- is the lower half part of the ball B_j in the direction given by v_j :

$$B_j^- = \{y \in B_j \mid (y - x_j) \cdot v_j < 0\}.$$

Thanks to property (i) and the fact that only a very small area of ∂F is missing in the covering, we know that

$$(3) \quad \mathcal{I}_\Omega(F) \text{ is close to } \sum_{j=1}^{\mathcal{N}} \alpha_{d-1} r_j^{d-1} \nu(v_j).$$

On the other hand, thanks to property (ii), we obtain that

$$\mathcal{E}_n \cap B_j \approx F \cap B_j \approx B_j^-$$

for the distance d . It means that in volume, E_n is very similar to B_j^- inside B_j . However, there might exist some thin but long strands in B_j that belong to $E_n \cap (B_j^-)^c$ or to $E_n^c \cap B_j^-$. We want to compare $V(\mathcal{E}_n \cap B_j)$ with the maximal flow $\tau_n(D_j, \gamma)$ in a cylinder of basis $D_j = \text{disc}(x_j, r'_j, v_j)$, where r'_j is a little bit smaller than r_j and γ is a very small height, so that the cylinder is included in B_j and is almost flat. To make this comparison, we have to cut the above-mentioned strands by adding edges to \mathcal{E}_n . We do it very carefully, in order to control the number of edges we add, together with their capacity, and we obtain that

$$(4) \quad V(\mathcal{E}_n \cap B_j) \leq \tau_n(D_j, \gamma) + \text{error},$$

where *error* is a corrective term that is very small. Combining (3) and (4), since $\mathcal{I}_\Omega(F) \geq \phi_\Omega$, we conclude that if $\phi_n \leq (\phi_\Omega - \varepsilon)n^{d-1}$ and $E_n \approx F$, then there exists $j \in \{1, \dots, \mathcal{N}\}$ such that

$$\tau_n(D_j, \gamma) \leq (\nu(v_j) - \varepsilon/2) \alpha_{d-1} r_j'^{d-1} n^{d-1}.$$

Third step: There remains to study the probability

$$\mathbb{P}[\tau_n(D_j, \gamma) \leq (\nu(v_j) - \varepsilon/2) \alpha_{d-1} r_j'^{d-1} n^{d-1}].$$

In fact, it has already been done by Rossignol and Théret in [25]. It is easy to compare $\tau_n(D_j, \gamma)$ with a sum of maximal flows through cylinders whose bases are hyperrectangles. Then, we can directly use Theorem 3.9 in [25] that states that the lower large deviations of these maximal flows below their limits are of surface order.

3.2.2. Upper large deviations. To prove Theorem 10, we have to study the probability

$$(5) \quad \mathbb{P}[\phi_n \geq (\widetilde{\phi}_\Omega + \varepsilon)n^{d-1}]$$

for a positive ε . First of all, we can check that $\widetilde{\phi}_\Omega$ is finite. In fact, we have to construct a polyhedral set P that satisfies all the conditions in the definition of $\widetilde{\phi}_\Omega$. This is done with the help of techniques very similar to some of those we will use

in section 4 to complete our polyhedral approximation, so we will not explain these techniques here. The proof of Theorem 10 is divided into three steps.

First step: We consider a polyhedral set P as in the definition of $\widetilde{\phi}_\Omega$ such that $\mathcal{I}_\Omega(P)$ is very close to this constant. We want to construct sets of edges near $\partial P \cap \Omega$ that cut Γ_n^1 from Γ_n^2 in Ω_n . Because we took a discrete approximation of Ω from the outside, we need to enlarge Ω a little, because some flow might go from Γ_n^1 to Γ_n^2 using paths that lie partly in $\Omega_n \setminus \Omega$. Thus we construct a set Ω' which contains a small neighbourhood of Ω (hence also Ω_n for all n large enough), which is transverse to ∂P , and which is small enough to ensure that $\mathcal{I}_{\Omega'}(P)$ is still very close to ϕ_Ω . To construct this set, we cover $\partial\Omega$ with small cubes, by compactness we extract a finite subcover of $\partial\Omega$, and finally we add the cubes of the subcover to Ω to obtain Ω' . We construct these cubes so that their boundaries are transverse to ∂P , and their diameters are uniformly smaller than a small constant, so that Ω' is included in a neighbourhood of Ω as small as we need. Since ∂P is transverse to Γ , if we take this constant small enough, we can control $\mathcal{H}^{d-1}(\partial P \cap (\Omega' \setminus \Omega))$, and thus the difference between $\mathcal{I}_{\Omega'}(P)$ and $\mathcal{I}_\Omega(P)$.

Then we construct a family of Cn (where $C > 0$) disjoint sets of edges that cut Γ_n^1 from Γ_n^2 in Ω_n and that lie near ∂P . We consider the neighbourhood P' of P inside Ω' at a distance smaller than a tiny constant h , and we partition $P' \setminus P$ into slabs $\mathcal{M}'(k)$ of width of order $1/n$, so we have Cn such slabs which look like translates of $\partial P \cap \Omega'$ that are slightly deformed and thickened. We prove that each path from Γ_n^1 to Γ_n^2 in Ω_n must contain at least one edge that lies in the set $\mathcal{M}'(k)$ for each k , i.e., each set $\mathcal{M}'(k)$ contains a cutset. Thus we have found a family of Cn disjoint cutsets.

Second step: We almost cover $\partial P \cap \Omega'$ by a finite family of disjoint cylinders B_j , $j \in J$, whose bases are hyperrectangles of sidelength l , that are orthogonal to ∂P , of height larger than h , and such that the part of ∂P which is missing in this covering is very small. Thus, we obtain that

$$(6) \quad \mathcal{I}_{\Omega'}(P) \text{ is close to } \sum_{j \in J} \nu(v_j) l^{d-1},$$

where v_j gives the direction towards which the cylinder B_j is tilted (it is the unit vector which is orthogonal to the face of ∂P that cuts B_j).

We want to compare ϕ_n with the sum of the maximal flows $\phi(B_j, v_j)$. For each j , let E_j be a set of edges that cuts the top from the bottom of B_j . The set $\bigcup_{j \in J} E_j$ does not cut Γ_n^1 from Γ_n^2 in Ω_n in general. To create such a cutset we must add two sets of edges:

- (i) a set of edges that covers the part of $\partial P \cap \Omega'$ that is missing in the covering by the cylinders B_j ,
- (ii) a set of edges that glues together all the previous sets of edges (the sets E_j and the set described in (i)).

In fact, we have already constructed Cn possible sets of edges as in (i): the edges that lie in $\mathcal{M}'(k) \setminus (\bigcup_{j \in J} B_j)$ for $k = 1, \dots, Cn$. We denote these sets by $M(k)$. We can also find $C'n$ ($C' > 0$) disjoint sets of edges that can be the glue described in (ii); we denote these sets by $W(l)$ for $l = 1, \dots, C'n$. We do not provide a precise description of these sets. In fact, we can choose different sets because we provide

the glue more or less in the interior of the cylinders B_j . Thus we obtain that

$$\forall k \in \{1, \dots, Cn\} \quad \forall l \in \{1, \dots, C'n\} \quad \bigcup_{j \in J} E_j \cup M(k) \cup W(l) \quad \text{cuts } \Gamma_n^1 \text{ from } \Gamma_n^2 \text{ in } \Omega_n.$$

We obtain that

$$(7) \quad \phi_n \leq \sum_{j \in J} \phi(B_j, v_j) + \min_{k=1, \dots, Cn} V(M(k)) + \min_{l=1, \dots, C'n} V(W(l)).$$

Combining (6) and (7), we see that if $\phi_n \geq (\widetilde{\phi}_\Omega + \varepsilon)n^{d-1}$, one of the following events must happen:

- (a) $\exists j \in J \quad \phi(B_j, v_j) \geq (\nu(v_j) + \varepsilon/2)l^{d-1}n^{d-1},$
- (b) $\forall k \in \{1, \dots, Cn\} \quad V(M(k)) \geq \eta n^{d-1},$
- (c) $\forall l \in \{1, \dots, C'n\} \quad V(W(l)) \geq \eta n^{d-1},$

where η is a very small constant (depending on ε and ϕ_Ω).

Third step: It consists in taking care of the probability that the events (a), (b) or (c) happen. The probability of (a) has already been studied in [26]: the upper large deviations of the variable ϕ in a cylinder above ν are of volume order. The events (b) and (c) are of the same type, and their probability is of the form

$$(8) \quad \mathbb{P} \left[\sum_{m=1}^{\alpha n^{d-1}} t_m \geq \eta n^{d-1} \right]^{Dn},$$

where $(t_m)_{m \in \mathbb{N}}$ is a family of i.i.d. variables of distribution function Λ , D is a constant, η is a very small constant and αn^{d-1} is the cardinality of the family of variables we consider. If $\alpha < \eta \mathbb{E}[t_1]^{-1}$ and if the law Λ admits one exponential moment, the Cramér Theorem in \mathbb{R} states that the probability (8) decays exponentially fast with n^d . Note the role of the optimization over Dn different probabilities to obtain the correct speed of decay. To complete the proof, it is enough to control the cardinality of the sets $M(k)$ and $W(l)$ for each k, l . This can be done by using the geometrical properties of ∂P (it is polyhedral and transverse to $\partial \Omega'$).

4. POLYHEDRAL APPROXIMATION: $\phi_\Omega = \widetilde{\phi}_\Omega$

We consider an open bounded domain Ω in \mathbb{R}^d . We denote its topological boundary by $\Gamma = \partial \Omega$. Also let Γ^1, Γ^2 be two disjoint subsets of Γ .

Hypothesis on Ω . We suppose that Ω is a Lipschitz domain, i.e., its boundary Γ can be locally represented as the graph of a Lipschitz function defined on some open ball of \mathbb{R}^{d-1} . Moreover, there exists a finite number of oriented hypersurfaces S_1, \dots, S_p of class C^1 which are transverse to each other and such that Γ is included in their union $S_1 \cup \dots \cup S_p$.

This hypothesis is automatically satisfied when Ω is a bounded open set with a C^1 boundary or when Ω is a polyhedral domain. The Lipschitz condition can be expressed as follows: each point x of $\Gamma = \partial \Omega$ has a neighbourhood U such that $U \cap \Omega$ is represented by the inequality $x_n < f(x_1, \dots, x_{n-1})$ in some cartesian coordinate system where f is a function satisfying a Lipschitz condition. Such domains are usually called Lipschitz domains in the literature. The boundary Γ

of a Lipschitz domain is $d - 1$ rectifiable (in the terminology of Federer's book [14]) so that its Minkowski content is equal to $\mathcal{H}^{d-1}(\Gamma)$. In addition, a Lipschitz domain Ω is admissible (in the terminology of Ziemer's book [29]), and in particular $\mathcal{H}^{d-1}(\Gamma \setminus \partial^*\Omega) = 0$. Moreover, each point of Γ is accessible from Ω through a rectifiable arc.

Hypothesis on Γ^1, Γ^2 . *The sets Γ^1, Γ^2 are open subsets of Γ . The relative boundaries $\partial_\Gamma \Gamma^1, \partial_\Gamma \Gamma^2$ of Γ^1, Γ^2 in Γ have null \mathcal{H}^{d-1} measure. The distance between Γ^1 and Γ^2 is positive.*

We recall that the relative topology of Γ is the topology induced on Γ by the topology of \mathbb{R}^d . Hence each of the sets Γ^1, Γ^2 is the intersection of Γ with an open set of \mathbb{R}^d . For F a subset of Ω having finite perimeter in Ω , the capacity of F is

$$\begin{aligned} \mathcal{I}_\Omega(F) &= \int_{\Omega \cap \partial^* F} \nu(v_F(y)) d\mathcal{H}^{d-1}(y) \\ &\quad + \int_{\Gamma^2 \cap \partial^* F} \nu(v_F(y)) d\mathcal{H}^{d-1}(y) + \int_{\Gamma^1 \cap \partial^*(\Omega \setminus F)} \nu(v_{\Omega \setminus F}(y)) d\mathcal{H}^{d-1}(y). \end{aligned}$$

For all $A \subset \mathbb{R}^d$, \bar{A} is the closure of A , $\overset{\circ}{A}$ its interior and $A^c = \mathbb{R}^d \setminus A$. We will prove the following theorem:

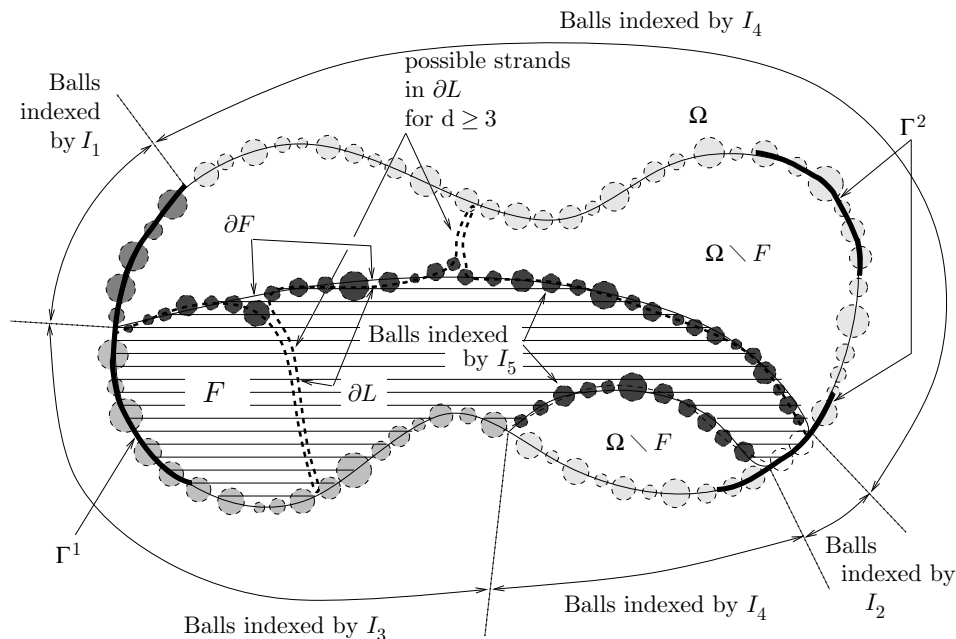
Theorem 11. *Let F be a subset of Ω having finite perimeter. For any $\varepsilon > 0$, there exists a polyhedral set P whose boundary ∂P is transverse to Γ and such that*

$$\begin{aligned} \bar{\Gamma}^1 &\subset \overset{\circ}{P}, \quad \bar{\Gamma}^2 \subset \widehat{\mathbb{R}^d \setminus P}, \quad \mathcal{L}^d(F \Delta (P \cap \Omega)) < \varepsilon, \\ \int_{\partial^* P \cap \Omega} \nu(v_P(x)) d\mathcal{H}^{d-1}(x) &= \mathcal{I}_\Omega(P) \leq \mathcal{I}_\Omega(F) + \varepsilon. \end{aligned}$$

First we notice that Theorem 11 implies that $\phi_\Omega = \widetilde{\phi_\Omega}$, and thus the convergence of ϕ_n (see section 3.2). It is obvious since $\phi_\Omega \leq \widetilde{\phi_\Omega}$ (see section 2.5), and Theorem 11 implies that $\phi_\Omega \geq \widetilde{\phi_\Omega}$.

The main difficulty of the proof of Theorem 11 is in properly handling the approximation close to Γ in order to push all the interfaces back inside Ω . The essential tools of the proof are the Besicovitch differentiation theorem, the Vitali covering theorem and an approximation technique due to De Giorgi. Let us summarise the global strategy.

Sketch of the proof. We fix $\gamma > 0$. We cover $\partial^*\Omega$ up to a set of \mathcal{H}^{d-1} measure less than γ by a finite collection of disjoint balls $B(x_i, r_i)$, $i \in I_1 \cup I_2 \cup I_3 \cup I_4$, centered on Γ , whose radii are sufficiently small to ensure that the surface and volume estimates within the balls are controlled by the factor γ . The indices of I_1 correspond to balls centered on $\Gamma^1 \cap \partial^*(\Omega \setminus F)$, the indices of I_2 to balls centered on $\Gamma^2 \cap \partial^* F$, the indices of I_3 to balls centered on $(\Gamma \setminus \Gamma^2) \cap \partial^* F$, and the indices of I_4 to balls centered on $(\Gamma \setminus \Gamma^1) \cap \partial^*(\Omega \setminus F)$ (see Figure 5). The remaining part of Γ is covered by a finite collection of balls $B(y_j, s_j)$, $j \in J_0 \cup J_1 \cup J_2$. The indices of J_1 correspond to balls covering the remaining part of $\bar{\Gamma}^1$, and the indices of J_2 correspond to balls covering the remaining part of $\bar{\Gamma}^2$. We choose $\varepsilon > 0$ sufficiently small, depending on γ and on the previous families of balls, and we approximate the set F by a smooth set L inside Ω , whose capacity and volume are at a distance

FIGURE 5. The balls indexed by I_i for $i = 1, \dots, 5$.

less than ε from those of F . We then build two further family of balls:

- $B(x_i, r_i)$, $i \in I_5$, cover $\Omega \cap \partial L$, up to a set of \mathcal{H}^{d-1} measure ε .
- $B(y_j, s_j)$, $j \in J_3$, cover the remaining set $\Omega \cap \partial L \setminus \bigcup_{i \in I_5} B(x_i, r_i)$.

Inside each ball $B(x_i, r_i)$, $i \in I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5$, up to a small fraction, the interfaces are located on hypersurfaces and the radii of the balls are so small that these hypersurfaces are almost flat. Hence we can enclose the interfaces into small flat polyhedral cylinders D_i , $i \in I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5$, and by aggregating adequately the cylinders to the set F or to its complement $\Omega \setminus F$, we move these interfaces on the boundaries of these cylinders. The remaining interfaces are enclosed in the balls $B(y_j, s_j)$, $j \in J_0 \cup J_1 \cup J_2 \cup J_3$, and we approximate these balls from the outside by polyhedra.

We have to delicately define the whole process, in order not to lose too much capacity and to control the possible interaction between interfaces close to Γ and interfaces in Ω . The presence of boundary conditions creates a substantial additional difficulty compared to the polyhedral approximation performed in [9]. Indeed, the most difficult interfaces to handle are those corresponding to D_i , $i \in I_3 \cup I_4$. We first choose the balls $B(x_i, r_i)$, $i \in I_1 \cup I_2 \cup I_3 \cup I_4$, corresponding to γ . We cover the remaining portion of Γ with the balls $B(y_j, s_j)$, $j \in J_0 \cup J_1 \cup J_2$. At this point we can already in principle define the cylinders D_i , $i \in I_1 \cup I_2$. Then we choose ε small enough, depending on γ and the balls $B(x_i, r_i)$, $i \in I_1 \cup I_2 \cup I_3 \cup I_4$, to ensure that the perturbation of volume ε caused when smoothing the set F inside Ω will not significantly alter the situation inside the balls $B(x_i, r_i)$, $i \in I_3 \cup I_4$. Then we move inside Ω and we build the cylinders D_i , $i \in I_5$. Then we come back to the boundary and we build the cylinders D_i , $i \in I_3 \cup I_4$. We cover the remaining

interfaces in Ω by the balls $B(y_j, s_j)$, $j \in J_3$. Finally we aggregate successively each flat polyhedral cylinder D_i to the set L or to its complement. \square

Preparation of the proof. Let us consider a subset F of Ω having finite perimeter. Let γ belong to $]0, 1/16[$. We start by handling the boundary Γ , for which we make locally flat approximations controlled by the factor γ . By hypothesis, there exists a finite number of oriented hypersurfaces S_1, \dots, S_p of class C^1 such that Γ is included in their union $S_1 \cup \dots \cup S_p$. In particular, we have

$$\Gamma \setminus \partial^* \Omega \subset S = \bigcup_{1 \leq k < l \leq p} S_k \cap S_l.$$

Since the hypersurfaces S_1, \dots, S_r are transverse to each other, this implies that $\mathcal{H}^{d-1}(S) = 0$.

• **Continuity of the normal vectors.** The hypersurfaces S_1, \dots, S_p being C^1 and the set Γ compact, the maps $x \in \Gamma \mapsto v_{S_k}(x)$, $1 \leq k \leq p$ (where $v_{S_k}(x)$ is the unit normal vector to S_k at x) are uniformly continuous:

$$\forall \delta > 0 \quad \exists \eta > 0 \quad \forall k \in \{1, \dots, p\} \quad \forall x, y \in S_k \cap \Gamma \quad |x - y|_2 \leq \eta \Rightarrow |v_{S_k}(x) - v_{S_k}(y)|_2 < \delta.$$

Let η^* be associated to $\delta = 1$ by this property. We will also use a more refined property.

• **Localisation of the interfaces.** We first prove a geometric lemma:

Lemma 1. *Let Γ be a hypersurface (that is, a C^1 submanifold of \mathbb{R}^d of codimension 1) and let K be a compact subset of Γ . There exists a positive $M = M(\Gamma, K)$ such that*

$$\forall \varepsilon > 0 \quad \exists r > 0 \quad \forall x, y \in K \quad |x - y|_2 \leq r \quad \Rightarrow \quad d_2(y, \tan(\Gamma, x)) \leq M \varepsilon |x - y|_2$$

($\tan(\Gamma, x)$ is the tangent hyperplane of Γ at x).

Proof. By a standard compactness argument, it is enough to prove the following local property:

$$\forall x \in \Gamma \quad \exists M(x) > 0 \quad \forall \varepsilon > 0 \quad \exists r(x, \varepsilon) > 0 \quad \forall y, z \in \Gamma \cap B(x, r(x, \varepsilon)),$$

$$d_2(y, \tan(\Gamma, z)) \leq M(x) \varepsilon |y - z|_2.$$

Indeed, if this property holds, we cover K by the open balls $\overset{\circ}{B}(x, r(x, \varepsilon)/2)$, $x \in K$, we extract a finite subcover $\overset{\circ}{B}(x_i, r(x_i, \varepsilon)/2)$, $1 \leq i \leq k$, and we set

$$M = \max\{M(x_i) : 1 \leq i \leq k\}, \quad r = \min\{r(x_i, \varepsilon)/2 : 1 \leq i \leq k\}.$$

Now let y, z belong to K with $|y - z|_2 \leq r$. Let i be such that y belongs to $\overset{\circ}{B}(x_i, r(x_i, \varepsilon)/2)$. Since $r \leq r(x_i, \varepsilon)/2$, both y, z belong to the ball $B(x_i, r(x_i, \varepsilon))$ and it follows that

$$d_2(y, \tan(\Gamma, z)) \leq M(x_i) \varepsilon |y - z|_2 \leq M \varepsilon |y - z|_2.$$

We now turn to the proof of the above local property. Since Γ is a hypersurface, for any x in Γ there exists a neighbourhood V of x in \mathbb{R}^d , a diffeomorphism $f : V \mapsto \mathbb{R}^d$ of class C^1 and a $(d-1)$ -dimensional vector space Z of \mathbb{R}^d such that $Z \cap f(V) = f(\Gamma \cap V)$ (see for instance [14], 3.1.19). Let A be a compact neighbourhood of x included in V . Since f is a diffeomorphism, the maps $y \in A \mapsto df(y) \in \text{End}(\mathbb{R}^d)$ and $u \in f(A) \mapsto df^{-1}(u) \in \text{End}(\mathbb{R}^d)$ are continuous. Therefore they are bounded:

$$\exists M > 0 \quad \forall y \in A \quad \|df(y)\| \leq M, \quad \forall u \in f(A) \quad \|df^{-1}(u)\| \leq M$$

(here $\|df(x)\| = \sup\{|df(x)(y)|_2 : |y|_2 \leq 1\}$ is the standard operator norm in $\text{End}(\mathbb{R}^d)$). Since $f(A)$ is compact, the differential map df^{-1} is uniformly continuous on $f(A)$:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall u, v \in f(A) \quad |u - v|_2 \leq \delta \quad \Rightarrow \quad \|df^{-1}(u) - df^{-1}(v)\| \leq \varepsilon.$$

Let ε be positive and let δ be associated to ε as above. Let ρ be positive and small enough so that $\rho < \delta/2$ and $B(f(x), \rho) \subset f(A)$ (since f is a C^1 diffeomorphism, $f(A)$ is a neighbourhood of $f(x)$). Let r be such that $0 < r < \rho/M$ and $B(x, r) \subset A$. We claim that M associated to x and r associated to ε, x answer the problem. Let y, z belong to $\Gamma \cap B(x, r)$. Since $[y, z] \subset B(x, r) \subset A$, and $\|df(\zeta)\| \leq M$ on A ,

$$\begin{aligned} |f(y) - f(x)|_2 &\leq M|y - x|_2 \leq Mr < \rho, & |f(z) - f(x)|_2 &< \rho, \\ |f(y) - f(z)|_2 &< \delta, & |f(y) - f(z)|_2 &< M|y - z|_2. \end{aligned}$$

We next apply a classical lemma of differential calculus (see [20], I, 4, Corollary 2) to the map f^{-1} , the interval $[f(z), f(y)]$ (which is included in $B(f(x), \rho) \subset f(A)$) and the point $f(z)$:

$$\begin{aligned} |y - z - df^{-1}(f(z))(f(y) - f(z))|_2 \\ \leq |f(y) - f(z)|_2 \sup\{\|df^{-1}(\zeta) - df^{-1}(f(z))\| : \zeta \in [f(z), f(y)]\}. \end{aligned}$$

The right-hand member is less than $M|y - z|_2 \varepsilon$. Since $z + df^{-1}(f(z))(f(y) - f(z))$ belongs to $\tan(\Gamma, z)$, we are done. \square

We come back to our case. Let $k \in \{1, \dots, p\}$. The set $S_k \cap \Gamma$ is a compact subset of the hypersurface S_k . Applying Lemma 1, we get

$$\exists M_k \quad \forall \delta_0 > 0 \quad \exists \eta_k > 0 \quad \forall x, y \in S_k \cap \Gamma \quad |x - y|_2 \leq \eta_k \Rightarrow d_2(y, \tan(S_k, x)) \leq M_k \delta_0 |x - y|_2.$$

Let $M_0 = \max_{1 \leq k \leq p} M_k$ and let δ_0 in $]0, 1/2[$ be such that $M_0 \delta_0 < \gamma$. For each k in $\{1, \dots, p\}$, let η_k be associated to δ_0 as in the above property and let

$$\eta_0 = \min \left(\min_{1 \leq k \leq p} \eta_k, \eta^*, \frac{1}{8d} \text{dist}(\Gamma^1, \Gamma^2) \right).$$

• **Covering of Γ by transverse cubes.** We build a family of cubes $Q(x, r)$, indexed by $x \in \Gamma$ and $r \in]0, r_\Gamma[$ such that $Q(x, r)$ is a cube centered at x of side length r which is transverse to Γ . For $x \in \mathbb{R}^d$ and $k \in \{1, \dots, p\}$, let $p_k(x)$ be a point of $S_k \cap \Gamma$ such that

$$|x - p_k(x)|_2 = \inf \{ |x - y|_2 : y \in S_k \cap \Gamma \}.$$

Such a point exists since $S_k \cap \Gamma$ is compact. We then define for $k \in \{1, \dots, p\}$

$$\forall x \in \mathbb{R}^d \quad v_k(x) = v_{S_k}(p_k(x)).$$

We also define

$$d_r = \inf_{v_1, \dots, v_p \in S^{d-1}} \max_{b \in \mathcal{B}_d} \min_{\substack{1 \leq k \leq p \\ e \in b}} (|e - v_i|_2, |-e - v_i|_2),$$

where \mathcal{B}_d is the collection of the orthonormal basis of \mathbb{R}^d and S^{d-1} is the unit sphere of \mathbb{R}^d . Let η be associated to $d_r/4$ as in the above continuity property. We set

$$r_\Gamma = \frac{\eta}{2d}.$$

Let $x \in \Gamma$. By the definition of d_r , there exists an orthonormal basis b_x of \mathbb{R}^d such that

$$\forall e \in b_x \quad \forall k \in \{1, \dots, p\} \quad \min(|e - v_k(x)|_2, |-e - v_k(x)|_2) > \frac{d_r}{2}.$$

Let $Q(x, r)$ be the cube centered at x of sidelength r whose sides are parallel to the vectors of b_x . We claim that $Q(x, r)$ is transverse to Γ for $r < r_\Gamma$. Indeed, let $y \in Q(x, r) \cap \Gamma$. Suppose that $y \in S_k$ for some $k \in \{1, \dots, p\}$, so that $v_k(y) = v_{S_k}(y)$ and $|x - p_k(x)|_2 < dr_\Gamma$. In particular, we have $|y - p_k(x)|_2 < 2dr_\Gamma < \eta$ and $|v_{S_k}(y) - v_k(x)|_2 < d_r/4$. For $e \in b_x$,

$$\frac{d_r}{2} \leq |e - v_k(x)|_2 \leq |e - v_{S_k}(y)|_2 + |v_{S_k}(y) - v_k(x)|_2,$$

whence

$$|e - v_{S_k}(y)|_2 \geq \frac{d_r}{2} - \frac{d_r}{4} = \frac{d_r}{4}.$$

This is also true for $-e$, therefore the faces of the cube $Q(x, r)$ are transverse to S_k .

• **Vitali covering theorem for \mathcal{H}^{d-1} .** A collection of sets \mathcal{U} is called a Vitali class for a Borel set E of \mathbb{R}^d if for each $x \in E$ and $\delta > 0$ there exists a set $U \in \mathcal{U}$ containing x such that $0 < \text{diam } U < \delta$, where $\text{diam } U$ is the diameter of the set U . We now recall the Vitali covering theorem for \mathcal{H}^{d-1} (see for instance [13], Theorem 1.10), since it will be useful during the proof:

Theorem 12. *Let E be an \mathcal{H}^{d-1} measurable subset of \mathbb{R}^d and \mathcal{U} be a Vitali class of closed sets for E . Then we may select a (countable) disjoint sequence $(U_i)_{i \in I}$ from \mathcal{U} such that*

$$\text{either } \sum_{i \in I} (\text{diam } U_i)^{d-1} = +\infty \text{ or } \mathcal{H}^{d-1}(E \setminus \bigcup_{i \in I} U_i) = 0.$$

If $\mathcal{H}^{d-1}(E) < \infty$, then given $\varepsilon > 0$, we may also require that

$$\mathcal{H}^{d-1}(E) \leq \frac{\alpha_{d-1}}{2^{d-1}} \sum_{i \in I} (\text{diam } U_i)^{d-1}.$$

Start of the main argument. We first handle the interfaces along Γ . Let $\mathcal{R}(\Gamma)$ be the set of the points x of $\Gamma \setminus S$ such that

$$\begin{aligned} \lim_{r \rightarrow 0} (\alpha_d r^d)^{-1} \mathcal{L}^d(B(x, r) \setminus \Omega) &= 1/2, \\ \lim_{r \rightarrow 0} (\alpha_{d-1} r^{d-1})^{-1} \mathcal{H}^{d-1}(B(x, r) \cap \Gamma) &= 1. \end{aligned}$$

Let $\mathcal{R}(\Omega \setminus F)$ be the set of the points x belonging to $\partial^*(\Omega \setminus F) \cap \mathcal{R}(\Gamma)$ such that

$$\begin{aligned} \lim_{r \rightarrow 0} (\alpha_{d-1} r^{d-1})^{-1} \mathcal{H}^{d-1}(B(x, r) \cap \partial^*(\Omega \setminus F)) &= 1, \\ \lim_{r \rightarrow 0} (\alpha_d r^d)^{-1} \mathcal{L}^d(B(x, r) \cap (\Omega \setminus F)) &= 1/2, \\ \lim_{r \rightarrow 0} (\alpha_{d-1} r^{d-1})^{-1} \int_{B(x, r) \cap \partial^*(\Omega \setminus F)} \nu(v_{\Omega \setminus F}(y)) d\mathcal{H}^{d-1}(y) &= \nu(v_\Omega(x)). \end{aligned}$$

Let $\mathcal{R}(F)$ be the set of the points x belonging to $\partial^* F \cap \mathcal{R}(\Gamma)$ such that

$$\begin{aligned} \lim_{r \rightarrow 0} (\alpha_{d-1} r^{d-1})^{-1} \mathcal{H}^{d-1}(B(x, r) \cap \partial^* F) &= 1, \\ \lim_{r \rightarrow 0} (\alpha_d r^d)^{-1} \mathcal{L}^d(B(x, r) \cap F) &= 1/2, \\ \lim_{r \rightarrow 0} (\alpha_{d-1} r^{d-1})^{-1} \int_{B(x, r) \cap \partial^* F} \nu(v_F(y)) d\mathcal{H}^{d-1}(y) &= \nu(v_\Omega(x)). \end{aligned}$$

Thanks to the hypothesis on Γ and the structure of the sets of finite perimeter (see either Lemma 1, section 5.8 of [12], Lemma 5.9.5 in [29] or Theorem 3.61 of [1]), we have

$$\mathcal{H}^{d-1}(\Gamma \setminus (\mathcal{R}(F) \cup \mathcal{R}(\Omega \setminus F))) = 0.$$

For x in $\mathcal{R}(\Gamma)$, there exists a positive $r_0(x, \gamma)$ such that, for any $r < r_0(x, \gamma)$,

$$\begin{aligned} |\mathcal{L}^d(B(x, r) \setminus \Omega) - \alpha_d r^d / 2| &\leq \gamma \alpha_d r^d, \\ |\mathcal{H}^{d-1}(B(x, r) \cap \Gamma) - \alpha_{d-1} r^{d-1}| &\leq \gamma \alpha_{d-1} r^{d-1}. \end{aligned}$$

For x in $\mathcal{R}(\Omega \setminus F)$, there exists a positive $r(x, \gamma) < r_0(x, \gamma)$ such that, for any $r < r(x, \gamma)$,

$$\begin{aligned} |\mathcal{H}^{d-1}(B(x, r) \cap \partial^*(\Omega \setminus F)) - \alpha_{d-1} r^{d-1}| &\leq \gamma \alpha_{d-1} r^{d-1}, \\ |\mathcal{L}^d(B(x, r) \cap (\Omega \setminus F)) - \alpha_d r^d / 2| &\leq \gamma \alpha_d r^d, \\ \left| (\alpha_{d-1} r^{d-1})^{-1} \int_{B(x, r) \cap \partial^*(\Omega \setminus F)} \nu(v_{\Omega \setminus F}(y)) d\mathcal{H}^{d-1}(y) - \nu(v_\Omega(x)) \right| &\leq \gamma. \end{aligned}$$

For x in $\mathcal{R}(F)$, there exists a positive $r(x, \gamma) < r_0(x, \gamma)$ such that, for any $r < r(x, \gamma)$,

$$\begin{aligned} |\mathcal{H}^{d-1}(B(x, r) \cap \partial^* F) - \alpha_{d-1} r^{d-1}| &\leq \gamma \alpha_{d-1} r^{d-1}, \\ |\mathcal{L}^d(B(x, r) \cap F) - \alpha_d r^d / 2| &\leq \gamma \alpha_d r^d, \\ \left| (\alpha_{d-1} r^{d-1})^{-1} \int_{B(x, r) \cap \partial^* F} \nu(v_F(y)) d\mathcal{H}^{d-1}(y) - \nu(v_\Omega(x)) \right| &\leq \gamma. \end{aligned}$$

Let us define the sets

$$\begin{aligned} \Gamma^{1*} &= \Gamma^1 \cap \mathcal{R}(\Omega \setminus F), \quad \Gamma^{2*} = \Gamma^2 \cap \mathcal{R}(F), \\ \Gamma^{3*} &= (\Gamma \setminus \bar{\Gamma}_2) \cap \mathcal{R}(F), \quad \Gamma^{4*} = (\Gamma \setminus \bar{\Gamma}_1) \cap \mathcal{R}(\Omega \setminus F). \end{aligned}$$

The family of balls

$$\begin{aligned} B(x, r), \quad x \in \Gamma^{1*} \cup \Gamma^{2*}, \quad r < \min(r(x, \gamma), \gamma, \eta_0, \frac{1}{2} \text{dist}(x, S)), \\ B(x, r), \quad x \in \Gamma^{3*}, \quad r < \min(r(x, \gamma), \gamma, \eta_0, \frac{1}{2} \text{dist}(x, S), \frac{1}{2} \text{dist}(x, \bar{\Gamma}_2)), \\ B(x, r), \quad x \in \Gamma^{4*}, \quad r < \min(r(x, \gamma), \gamma, \eta_0, \frac{1}{2} \text{dist}(x, S), \frac{1}{2} \text{dist}(x, \bar{\Gamma}_1)) \end{aligned}$$

is a Vitali relation for $\Gamma^{1*} \cup \Gamma^{2*} \cup \Gamma^{3*} \cup \Gamma^{4*}$. Recall that S is the set of points belonging to two or more of the hypersurfaces S_1, \dots, S_p , and since S is disjoint from $\Gamma^{1*}, \Gamma^{2*}, \Gamma^{3*}, \Gamma^{4*}$, then $\text{dist}(x, S) > 0$ for $x \in \Gamma^{1*} \cup \Gamma^{2*} \cup \Gamma^{3*} \cup \Gamma^{4*}$. By the standard Vitali covering theorem (see Theorem 12), we may select a finite

or countable collection of disjoint balls $B(x_i, r_i)$, $i \in I$, such that: for $i \in I$, $x_i \in \Gamma^{1*} \cup \Gamma^{2*} \cup \Gamma^{3*} \cup \Gamma^{4*}$, $r_i < \min(r(x_i, \gamma), \gamma, \eta_0, \frac{1}{2}\text{dist}(x_i, S))$ and

$$\text{either } \mathcal{H}^{d-1}\left(\Gamma \setminus \bigcup_{i \in I} B(x_i, r_i)\right) = 0 \quad \text{or} \quad \sum_{i \in I} r_i^{d-1} = \infty.$$

Because for each i in I , r_i is smaller than $r(x_i, \gamma)$,

$$\alpha_{d-1}(1-\gamma) \sum_{i \in I} r_i^{d-1} \leq \mathcal{H}^{d-1}(\Gamma) < \infty,$$

and therefore the first case occurs, so that we may select four finite subsets I_1, I_2, I_3, I_4 of I such that

$$\begin{aligned} \forall k \in \{1, \dots, 4\} \quad \forall i \in I_k \quad x_i \in \Gamma^{k*}, \\ \mathcal{H}^{d-1}\left(\Gamma \setminus \bigcup_{1 \leq k \leq 4} \bigcup_{i \in I_k} B(x_i, r_i)\right) < \gamma. \end{aligned}$$

Let i belong to $I_1 \cup I_2 \cup I_3 \cup I_4$. We have

$$\begin{aligned} \mathcal{H}^{d-1}(\Gamma \cap B(x_i, r_i) \setminus B(x_i, r_i(1-2\sqrt{\gamma}))) \\ = \mathcal{H}^{d-1}(\Gamma \cap B(x_i, r_i)) - \mathcal{H}^{d-1}(\Gamma \cap B(x_i, r_i(1-2\sqrt{\gamma}))) \\ \leq (1+\gamma)\alpha_{d-1}r_i^{d-1} - (1-\gamma)\alpha_{d-1}r_i^{d-1}(1-2\sqrt{\gamma})^{d-1} \\ = \alpha_{d-1}r_i^{d-1}(1+\gamma - (1-\gamma)(1-2\sqrt{\gamma})^{d-1}) \\ \leq \alpha_{d-1}r_i^{d-1}2d\sqrt{\gamma}. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i \in I_1 \cup I_2 \cup I_3 \cup I_4} \mathcal{H}^{d-1}(\Gamma \cap B(x_i, r_i) \setminus B(x_i, r_i(1-2\sqrt{\gamma}))) \\ \leq 2d\sqrt{\gamma} \sum_{i \in I_1 \cup I_2 \cup I_3 \cup I_4} \alpha_{d-1}r_i^{d-1} \leq 4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma) \end{aligned}$$

and

$$\mathcal{H}^{d-1}\left(\Gamma \setminus \bigcup_{i \in I_1 \cup I_2 \cup I_3 \cup I_4} B(x_i, r_i(1-2\sqrt{\gamma}))\right) < \gamma + 4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma).$$

We have a finite number of disjoint closed balls $B(x_i, r_i(1-2\sqrt{\gamma}))$, $i \in I_1 \cup I_2 \cup I_3 \cup I_4$. By slightly increasing all the radii r_i , we can keep the balls disjoint, ensure that each radius r_i satisfies the same strict inequalities for i in $I_1 \cup I_2 \cup I_3 \cup I_4$, and get the inequality

$$\mathcal{H}^{d-1}\left(\Gamma \setminus \bigcup_{i \in I_1 \cup I_2 \cup I_3 \cup I_4} \overset{\circ}{B}(x_i, r_i(1-2\sqrt{\gamma}))\right) < 2\gamma + 4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma).$$

The above set is a compact subset of Γ . For $k = 1, 2$, we define

$$R_k = \bar{\Gamma}_k \setminus \bigcup_{i \in I_1 \cup I_2 \cup I_3 \cup I_4} \overset{\circ}{B}(x_i, r_i(1-2\sqrt{\gamma})).$$

The sets R_1 and R_2 are compact, and their \mathcal{H}^{d-1} measure is less than $2\gamma + 4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma)$ (recall that $\partial_\Gamma \Gamma^1$ and $\partial_\Gamma \Gamma^2$ have a null \mathcal{H}^{d-1} measure). For $k = 1, 2$,

by the definition of the Hausdorff measure \mathcal{H}^{d-1} , there exists a collection of balls $B(y_j, s_j)$, $j \in J_k$, such that

$$\begin{aligned} \forall j \in J_k \quad 0 < s_j < \min\left(\eta_0, \frac{r_\Gamma}{2}\right), \quad B(y_j, s_j) \cap R_k \neq \emptyset, \\ \sum_{j \in J_k} \alpha_{d-1} s_j^{d-1} < 3\gamma + 4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma), \\ R_k \subset \bigcup_{j \in J_k} \overset{\circ}{B}(y_j, s_j). \end{aligned}$$

By compactness of R_1 and R_2 , the sets J_1 and J_2 can be chosen to be finite. It remains to cover

$$R_0 = \Gamma \setminus \bigcup_{i \in I_1 \cup I_2 \cup I_3 \cup I_4} \overset{\circ}{B}(x_i, r_i(1 - 2\sqrt{\gamma})) \setminus \bigcup_{j \in J_1 \cup J_2} \overset{\circ}{B}(y_j, s_j).$$

The set R_0 is a closed subset of Γ which is at a positive distance from Γ^1 and Γ^2 . There exists a collection of balls $B(y_j, s_j)$, $j \in J_0$, such that

$$\begin{aligned} \forall j \in J_0 \quad 0 < s_j < \min\left(\eta_0, \frac{r_\Gamma}{2}, \frac{1}{8d} \text{dist}(R_0, \Gamma^1 \cup \Gamma^2)\right), \quad B(y_j, s_j) \cap R_0 \neq \emptyset, \\ \sum_{j \in J_0} \alpha_{d-1} s_j^{d-1} < 3\gamma + 4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma), \\ R_0 \subset \bigcup_{j \in J_0} \overset{\circ}{B}(y_j, s_j). \end{aligned}$$

Now the collection of balls

$$\overset{\circ}{B}(x_i, r_i(1 - 2\sqrt{\gamma})), i \in I_1 \cup I_2 \cup I_3 \cup I_4, \quad B(y_j, s_j), j \in J_0 \cup J_1 \cup J_2$$

completely covers Γ . We will next replace these balls by polyhedra. For $j \in J_0 \cup J_1 \cup J_2$, let x_j belong to $B(y_j, s_j) \cap \Gamma$ and let Q_j be the cube $Q(x_j, 4s_j)$. For i in $I_1 \cup I_2 \cup I_3 \cup I_4$, the point x_i belongs to exactly one hypersurface among S_1, \dots, S_p , which we denote by $S_{s(i)}$. In particular Γ admits a normal vector $v_\Omega(x_i)$ at x_i in the classical sense. For each i in $I_1 \cup I_2 \cup I_3 \cup I_4$, let P_i be a convex open polygon inside the hyperplane $\text{hyp}(x_i, v_\Omega(x_i))$ such that

$$\begin{aligned} \text{disc}(x_i, r_i(1 - 2\sqrt{\gamma}), v_\Omega(x_i)) \subset P_i \subset \text{disc}(x_i, r_i(1 - \sqrt{\gamma}), v_\Omega(x_i)), \\ |\mathcal{H}^{d-2}(\partial P_i) - \alpha_{d-2} r_i^{d-2} (1 - \sqrt{\gamma})^{d-2}| \leq \delta_0 \alpha_{d-2} r_i^{d-2} (1 - \sqrt{\gamma})^{d-2}, \\ |\mathcal{H}^{d-1}(P_i) - \alpha_{d-1} r_i^{d-1} (1 - \sqrt{\gamma})^{d-1}| \leq \delta_0 \alpha_{d-1} r_i^{d-1} (1 - \sqrt{\gamma})^{d-1}. \end{aligned}$$

Thanks to the choices of the radius r_i and the constants M_0, η_0 , we then have

$$\begin{aligned} \Gamma \cap B(x_i, r_i(1 - 2\sqrt{\gamma})) \subset S_{s(i)} \cap B(x_i, r_i(1 - 2\sqrt{\gamma})) \subset \overset{\circ}{\text{cyl}}(P_i, 2\gamma r_i), \\ \Gamma \cap B(x_i, r_i) \subset S_{s(i)} \cap B(x_i, r_i) \subset \text{cyl}(\text{disc}(x_i, r_i, v_\Omega(x_i)), M_0 \delta_0 r_i), \\ \forall x \in B(x_i, r_i) \cap \Gamma \quad |v_\Omega(x) - v_\Omega(x_i)|_2 < 1. \end{aligned}$$

The choice of δ_0 guarantees that $M_0 \delta_0 (1 + \delta_0) r_i < 2\gamma r_i$. Let t be such that

$$M_0 \delta_0 (1 + \delta_0) r_i \leq t < \sqrt{\gamma} r_i.$$

We have

$$-tv_\Omega(x_i) + P_i \subset \Omega \cap B(x_i, r_i), \quad \Gamma \cap (-tv_\Omega(x_i) + P_i) = \emptyset.$$

In particular, the set Γ can intersect the cylinder $\text{cyl}(P_i, t)$ only along its lateral sides, which are parallel to $v_\Omega(x_i)$. Let x belong to $\Gamma \cap \partial \text{cyl}(P_i, t)$. Then

$$|v_{\text{cyl}(P_i, t)}(x) - v_\Omega(x)|_2 \geq |v_{\text{cyl}(P_i, t)}(x) - v_\Omega(x_i)|_2 - |v_\Omega(x_i) - v_\Omega(x)|_2 \geq \sqrt{2} - 1.$$

Therefore the cylinder $\text{cyl}(P_i, t)$ is transverse to Γ . We will replace the ball $B(x_i, r_i(1 - 2\sqrt{\gamma}))$ by the cylinder $\text{cyl}(P_i, t_i)$, for a carefully chosen value of t_i in the interval $[M_0\delta_0(1 + \delta_0)r_i, \sqrt{\gamma}r_i]$. However, we must delay the choices of the values t_i , $i \in I_3 \cup I_4$, until we have modified the set F inside Ω . We next deal with the interfaces inside Ω , and we make an approximation of F controlled by a factor ε . We choose ε sufficiently small compared to γ so that, when we perturb the set F by a volume ε , the resulting effect close to Γ is still of order γ . Let ε be such that $0 < \varepsilon < \gamma$ and

$$\varepsilon < \gamma \alpha_d \min_{i \in I_1 \cup I_2 \cup I_3 \cup I_4} r_i^d.$$

We next use a classical approximation result: there exists a relatively closed subset L of Ω having finite perimeter such that $\Omega \cap \partial L$ is a hypersurface of class C^∞ and

$$\mathcal{L}^d(F \Delta L) < \varepsilon, \quad \left| \int_{\Omega \cap \partial^* F} \nu(v_F(y)) d\mathcal{H}^{d-1}(y) - \int_{\Omega \cap \partial L} \nu(v_L(y)) d\mathcal{H}^{d-1}(y) \right| < \varepsilon.$$

In the case where ν is constant, this result is stated in Lemma 4.4 of [23]. In the non-constant case, the argument should be slightly modified, as explained in the proof of proposition 14.8 of [9], where the approximation is performed in \mathbb{R}^d instead of in Ω . When working inside Ω , the extra difficulty is in dealing with regions close to the boundary (see the proof of Proposition 4.3 of [23]). For $r > 0$, we define

$$\partial L_r = \{x \in \partial L : d(x, \Gamma) \geq r\}.$$

By continuity of the measure $\mathcal{H}^{d-1}|_{\partial L}$, there exists $r^* > 0$ such that

$$\mathcal{H}^{d-1}(\Omega \cap \partial L \setminus \partial L_{2r^*}) \leq \varepsilon.$$

We apply Lemma 1 to the set ∂L_{r^*} and the hypersurface $\Omega \cap \partial L$:

$$\exists M > 0 \quad \forall \delta > 0 \quad \exists \eta > 0 \quad \forall x, y \in \partial L_{r^*},$$

$$|x - y|_2 \leq \eta \Rightarrow d_2(y, \tan(\partial L, x)) \leq M\delta|x - y|_2.$$

For a point x belonging to ∂L_{r^*} , the tangent hyperplane of $\Omega \cap \partial L$ at x is precisely $\text{hyp}(x, v_L(x))$. Let M be as above. We can assume that $M > 1$. Let δ in $]0, \delta_0[$ be such that $2\delta M < \varepsilon$. Let η be associated to δ as in the above property. For $x \in \partial L_{2r^*}$,

$$\lim_{r \rightarrow 0} (\alpha_{d-1}r^{d-1})^{-1} \mathcal{H}^{d-1}(B(x, r) \cap \partial L) = 1,$$

$$\lim_{r \rightarrow 0} (\alpha_{d-1}r^{d-1})^{-1} \int_{B(x, r) \cap \partial L} \nu(v_L(y)) d\mathcal{H}^{d-1}(y) = \nu(v_L(x)).$$

For any x in ∂L_{2r^*} , there exists a positive $r(x, \varepsilon)$ such that, for any $r < r(x, \varepsilon)$,

$$|\mathcal{H}^{d-1}(B(x, r) \cap \partial L) - \alpha_{d-1}r^{d-1}| \leq \varepsilon \alpha_{d-1}r^{d-1},$$

$$\left| (\alpha_{d-1}r^{d-1})^{-1} \int_{B(x, r) \cap \partial L} \nu(v_L(y)) d\mathcal{H}^{d-1}(y) - \nu(v_L(x)) \right| \leq \varepsilon.$$

The family of balls $B(x, r)$, $x \in \partial L_{2r^*}$, $r < \min(r^*, \eta_0, r(x, \varepsilon), \varepsilon, \eta)$, is a Vitali relation for ∂L_{2r^*} . By the standard Vitali covering theorem, we may select a finite

or countable collection of disjoint balls $B(x_i, r_i)$, $i \in I'$, such that: for any i in I' , $x_i \in \partial L_{2r^*}$,

$$r_i < \min(r^*, \eta_0, r(x_i, \varepsilon), \varepsilon, \eta)$$

and

$$\text{either } \mathcal{H}^{d-1}\left(\partial L_{2r^*} \setminus \bigcup_{i \in I'} B(x_i, r_i)\right) = 0 \quad \text{or} \quad \sum_{i \in I'} r_i^{d-1} = \infty.$$

Because for each i in I' , r_i is smaller than $r(x_i, \varepsilon)$,

$$\alpha_{d-1}(1 - \varepsilon) \sum_{i \in I'} r_i^{d-1} \leq \mathcal{H}^{d-1}(\Omega \cap \partial L) < \infty,$$

and therefore the first case occurs so that we may select a finite subset I_5 of I' such that

$$\mathcal{H}^{d-1}\left(\partial L_{2r^*} \setminus \bigcup_{i \in I_5} B(x_i, r_i)\right) < \varepsilon.$$

We have a finite number of disjoint closed balls $B(x_i, r_i)$, $i \in I_5$. By slightly increasing all the radii r_i , we can keep the balls disjoint, with each r_i strictly smaller than $\min(r^*, \eta_0, r(x_i, \varepsilon), \varepsilon, \eta)$ for i in I_5 , and get the stronger inequality

$$\mathcal{H}^{d-1}\left(\partial L_{2r^*} \setminus \bigcup_{i \in I_5} \overset{\circ}{B}(x_i, r_i)\right) < \varepsilon.$$

For each i in I_5 , let P_i be a convex open polygon inside the hyperplane $\text{hyp}(x_i, v_L(x_i))$ such that

$$\begin{aligned} \text{disc}(x_i, r_i, v_L(x_i)) &\subset P_i \subset \text{disc}(x_i, r_i(1 + \delta), v_L(x_i)), \\ |\mathcal{H}^{d-2}(\partial P_i) - \alpha_{d-2} r_i^{d-2}| &\leq \delta \alpha_{d-2} r_i^{d-2}, \\ |\mathcal{H}^{d-1}(P_i) - \alpha_{d-1} r_i^{d-1}| &\leq \delta \alpha_{d-1} r_i^{d-1}. \end{aligned}$$

We set $\psi = M\delta(1 + \delta)$ (hence $\psi < \varepsilon < 1$). Let i belong to I_5 . Let D_i be the cylinder

$$D_i = \text{cyl}(P_i, M\delta(1 + \delta)r_i)$$

of basis P_i and height $2\psi r_i$. The point x_i belongs to ∂L_{2r^*} and the radius r_i is smaller than η and r^* , so that

$$\forall x \in \partial L \cap B(x_i, r_i) \quad d_2(x, \text{hyp}(x_i, v_L(x_i))) \leq M\delta|x - x_i|_2,$$

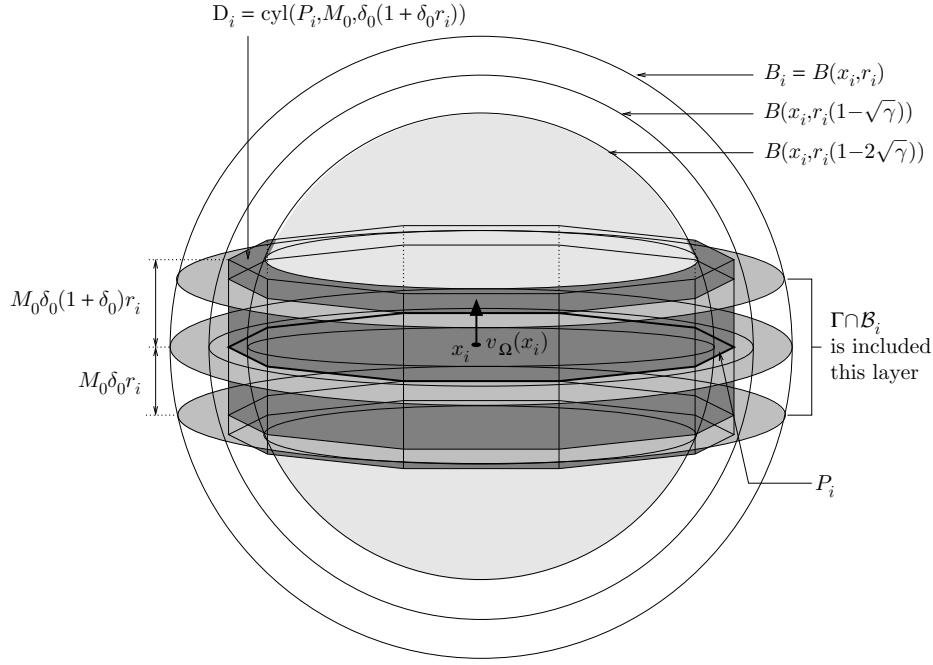
whence

$$\partial L \cap B(x_i, r_i) \subset \text{cyl}(\text{disc}(x_i, r_i, v_L(x_i)), M\delta r_i) \subset \overset{\circ}{D}_i.$$

We will approximate F by L inside Ω , and we will push the interfaces $\Gamma^1 \cap \partial^*(\Omega \setminus F)$ and $\Gamma^2 \cap \partial^* F$ into Ω . We next handle the regions close to Γ inside the family of balls $B(x_i, r_i)$, $i \in I_1 \cup I_2 \cup I_3 \cup I_4$. We will adequately modify the set F to ensure that no significant interface is created within these balls. Our technique consists in building a small flat cylinder centered on Γ which we add (for indices in $I_1 \cup I_3$) or remove (for indices in $I_2 \cup I_4$) to the set F . We have to design this operation carefully in order not to create any significant additional interface. This is the place where we tie together the covering of the boundary and the inner approximation. Recall that we already chose a family of polygons P_i , $i \in I_1 \cup I_2 \cup I_3 \cup I_4$. For $i \in I_1 \cup I_2$, we simply define D_i to be the cylinder

$$D_i = \text{cyl}(P_i, M_0\delta_0(1 + \delta_0)r_i);$$

see Figure 6. The construction of the cylinders associated to the indices $i \in I_3 \cup I_4$ is

FIGURE 6. The cylinder D_i for $i \in I_1 \cup I_2$.

more complicated. Our technique consists in carefully choosing the height t_i of the cylinders $\text{cyl}(P_i, t_i)$ for $i \in I_3 \cup I_4$. We examine the indices in I_3 and I_4 separately.

• **Balls indexed by I_3 .** Let i belong to I_3 . Because of the condition imposed on ε , we have

$$|\mathcal{L}^d(B(x_i, r_i) \cap L) - \alpha_d r_i^d / 2| \leq \gamma \alpha_d r_i^d + \varepsilon \leq 2\gamma \alpha_d r_i^d.$$

Since in addition

$$|\mathcal{L}^d(B(x_i, r_i) \setminus \Omega) - \alpha_d r_i^d / 2| \leq \gamma \alpha_d r_i^d,$$

it follows that

$$\mathcal{L}^d(B(x_i, r_i) \cap (\Omega \setminus \overset{\circ}{L})) \leq 3\gamma \alpha_d r_i^d.$$

Thanks to the choice of the polygon P_i , we then have

$$\int_{2\gamma r_i < t < \sqrt{\gamma} r_i} \mathcal{H}^{d-1}((-tv_\Omega(x_i) + P_i) \setminus \overset{\circ}{L}) dt \leq \mathcal{L}^d(B(x_i, r_i) \cap (\Omega \setminus \overset{\circ}{L})) \leq 3\gamma \alpha_d r_i^d.$$

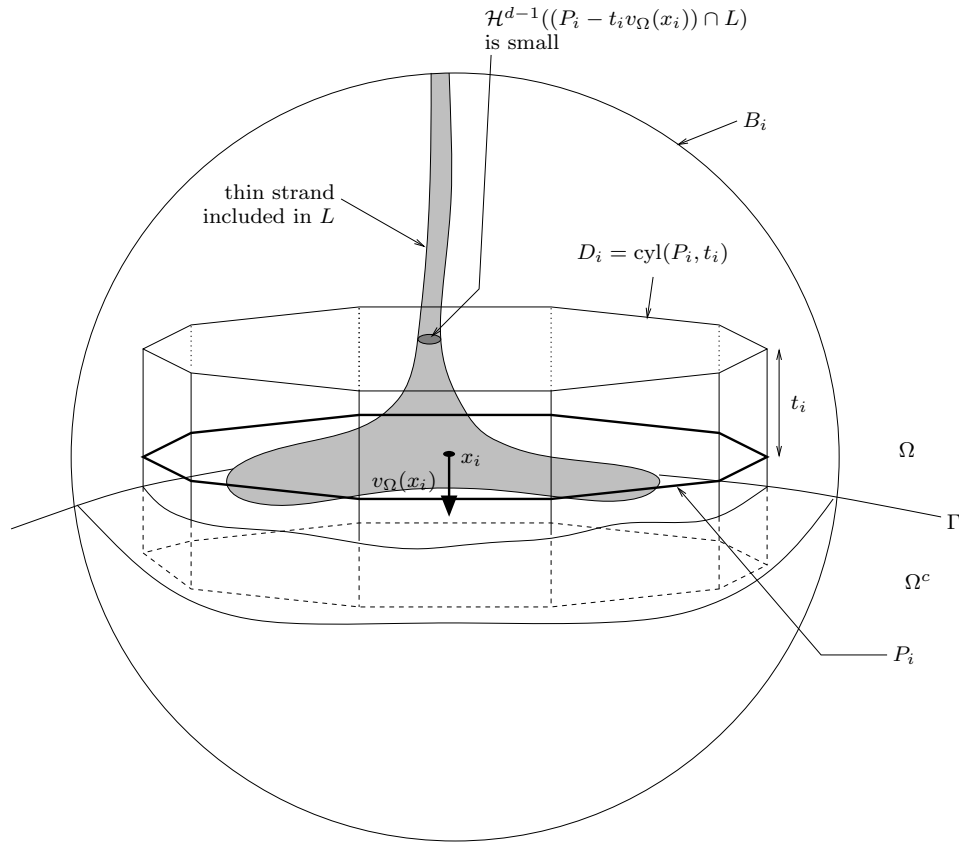
The condition on γ yields in particular $\sqrt{\gamma} - 2\gamma \geq \sqrt{\gamma}/2$. Hence there exists $t_i \in]2\gamma r_i, \sqrt{\gamma} r_i[$ such that

$$\mathcal{H}^{d-1}((-t_i v_\Omega(x_i) + P_i) \setminus \overset{\circ}{L}) \leq 6\sqrt{\gamma} \alpha_d r_i^{d-1}.$$

Let D_i be the cylinder $D_i = \text{cyl}(P_i, t_i)$.

• **Balls indexed by I_4 .** Let i belong to I_4 . Because of the condition imposed on ε , we have

$$|\mathcal{L}^d(B(x_i, r_i) \cap (\Omega \setminus L)) - \alpha_d r_i^d / 2| \leq \gamma \alpha_d r_i^d + \varepsilon \leq 2\gamma \alpha_d r_i^d.$$

FIGURE 7. The cylinder D_i for $i \in I_4$.

Since in addition

$$|\mathcal{L}^d(B(x_i, r_i) \setminus \Omega) - \alpha_d r_i^d / 2| \leq \gamma \alpha_d r_i^d,$$

it follows that

$$\mathcal{L}^d(B(x_i, r_i) \cap L) \leq 3\gamma \alpha_d r_i^d.$$

Thanks to the choice of the polygon P_i , we then have

$$\int_{2\gamma r_i < t < \sqrt{\gamma} r_i} \mathcal{H}^{d-1}((-tv_\Omega(x_i) + P_i) \cap L) dt \leq \mathcal{L}^d(B(x_i, r_i) \cap L) \leq 3\gamma \alpha_d r_i^d.$$

The condition on γ yields in particular $\sqrt{\gamma} - 2\gamma \geq \sqrt{\gamma}/2$. Hence there exists $t_i \in]2\gamma r_i, \sqrt{\gamma} r_i[$ such that

$$\mathcal{H}^{d-1}((-t_i v_\Omega(x_i) + P_i) \cap L) \leq 6\sqrt{\gamma} \alpha_d r_i^{d-1}.$$

Let D_i be the cylinder $D_i = \text{cyl}(P_i, t_i)$ (see Figure 7). We have now built the whole family of cylinders D_i , $i \in I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5$. Moreover, the sets

$$\overset{\circ}{D}_i, \quad i \in I_1 \cup I_2 \cup I_3 \cup I_4, \quad \overset{\circ}{B}(y_j, s_j), \quad j \in J_0 \cup J_1 \cup J_2,$$

completely cover Γ . It now remains to cover the region

$$R_3 = \Omega \cap \partial L \setminus \bigcup_{i \in I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5} \overset{\circ}{D}_i \setminus \bigcup_{j \in J_0 \cup J_1 \cup J_2} \overset{\circ}{B}(y_j, s_j).$$

Since R_3 does not intersect Γ , the distance

$$\rho = \frac{1}{8d} \text{dist}(\Gamma, R_3)$$

is positive and also R_3 is compact. From the preceding inequalities, we deduce that

$$\begin{aligned} \mathcal{H}^{d-1}(R_3) &\leq \mathcal{H}^{d-1}(\Omega \cap \partial L \setminus \partial L_{2r^*}) + \mathcal{H}^{d-1}\left(\partial L_{2r^*} \setminus \bigcup_{i \in I_5} \overset{\circ}{D}_i\right) \\ &\leq \varepsilon + \mathcal{H}^{d-1}\left(\partial L_{2r^*} \setminus \bigcup_{i \in I_5} \overset{\circ}{B}(x_i, r_i)\right) \leq 2\varepsilon. \end{aligned}$$

By the definition of the Hausdorff measure \mathcal{H}^{d-1} , there exists a collection of balls $B(y_j, s_j)$, $j \in J_3$, such that

$$\begin{aligned} \forall j \in J_3 \quad 0 < s_j < \rho, \quad B(y_j, s_j) \cap R_3 &\neq \emptyset, \\ R_3 &\subset \bigcup_{j \in J_3} \overset{\circ}{B}(y_j, s_j), \\ \sum_{j \in J_3} \alpha_{d-1} s_j^{d-1} &\leq 3\varepsilon. \end{aligned}$$

By compactness, we might assume in addition that J_3 is finite. For $j \in J_3$, let x_j belong to $B(y_j, s_j) \cap R_3$ and let Q_j be the cube $Q(x_j, 4s_j)$. We set

$$P = \left((\Omega \cap L) \cup \bigcup_{i \in I_1 \cup I_3 \cup I_5} D_i \cup \bigcup_{j \in J_1} \overset{\circ}{Q}_j \right) \setminus \bigcup_{i \in I_2 \cup I_4} D_i \setminus \bigcup_{j \in J_0 \cup J_2 \cup J_3} Q_j.$$

The sets $\overset{\circ}{Q}_j$, $j \in J_0 \cup J_1 \cup J_2 \cup J_3$, and $\overset{\circ}{D}_i$, $i \in I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5$, cover $\partial L \cup \Gamma$; therefore

$$\partial P \subset \bigcup_{i \in I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5} \partial D_i \cup \bigcup_{j \in J_0 \cup J_1 \cup J_2 \cup J_3} \partial Q_j.$$

Thus P is polyhedral and ∂P is transverse to Γ . Since the sets

$$\overset{\circ}{D}_i, \quad i \in I_1 \cup I_3, \quad \overset{\circ}{Q}_j, \quad j \in J_1,$$

completely cover $\bar{\Gamma}^1$, while the sets

$$D_i, \quad i \in I_2 \cup I_4 \cup I_5, \quad Q_j, \quad j \in J_0 \cup J_2 \cup J_3,$$

do not intersect $\bar{\Gamma}^1$, then $\bar{\Gamma}^1$ is included in the interior of P . Similarly, the sets

$$\overset{\circ}{D}_i, \quad i \in I_2 \cup I_4, \quad \overset{\circ}{Q}_j, \quad j \in J_2,$$

completely cover $\bar{\Gamma}^2$, while the sets

$$D_i, \quad i \in I_1 \cup I_3 \cup I_5, \quad Q_j, \quad j \in J_0 \cup J_1 \cup J_3,$$

do not intersect $\bar{\Gamma}^2$. Thus $\bar{\Gamma}^2$ is included in the interior of the complement of P . We next check that the set $P \cap \Omega$ approximates the initial set F with respect to the volume. We have

$$(P \cap \Omega) \Delta F \subset (L \Delta F) \cup \bigcup_{i \in I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5} D_i \cup \bigcup_{j \in J_0 \cup J_1 \cup J_2 \cup J_3} Q_j,$$

whence

$$\mathcal{L}^d((P \cap \Omega) \Delta F) \leq \varepsilon + \sum_{i \in I_1 \cup I_2 \cup I_3 \cup I_4} 2\alpha_{d-1}r_i^{d-1}(1+\delta_0)\sqrt{\gamma}r_i + \sum_{i \in I_5} 2\alpha_{d-1}r_i^{d-1}(1+\delta)\psi r_i + \sum_{j \in J_0 \cup J_1 \cup J_2 \cup J_3} \alpha_d(2s_j)^d.$$

Yet each r_i is smaller than γ ,

$$\begin{aligned} \sum_{i \in I_1 \cup I_2 \cup I_3 \cup I_4} \alpha_{d-1}r_i^{d-1} &\leq 2\mathcal{H}^{d-1}(\Gamma), \\ \sum_{i \in I_5} \alpha_{d-1}r_i^{d-1} &\leq 2\mathcal{H}^{d-1}(\Omega \cap \partial L) \leq \frac{2}{\nu_{\min}}(\nu_{\max}\mathcal{H}^{d-1}(\partial^* F \cap \Omega) + \varepsilon), \\ \sum_{j \in J_0 \cup J_1 \cup J_2 \cup J_3} \alpha_{d-1}s_j^{d-1} &\leq 3(3\gamma + 4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma)) + 3\varepsilon, \end{aligned}$$

so that

$$\begin{aligned} \mathcal{L}^d((P \cap \Omega) \Delta F) &\leq \varepsilon + 6\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma) + \frac{6\varepsilon}{\nu_{\min}}(\nu_{\max}\mathcal{H}^{d-1}(\partial^* F \cap \Omega) + \varepsilon) \\ &\quad + 3 \cdot 2^d \frac{\alpha_d}{\alpha_{d-1}}(3\gamma + 4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma) + \varepsilon). \end{aligned}$$

We next estimate the capacity of P . To do this, we examine the intersection of $\partial P \cap \Omega$ with each polyhedral cylinder. For $i \in I_1 \cup I_2$, we use the obvious inclusion

$$P \cap \Omega \cap \partial D_i \subset \Omega \cap \partial D_i.$$

For $i \in I_3 \cup I_4 \cup I_5$, the sets $\partial P \cap \Omega \cap \partial D_i$ require more attention. We consider separately the indices of I_3 , I_4 and I_5 .

- **Cylinders indexed by I_3 .** Let i be in I_3 . We have

$$\Omega \cap \partial P \cap \partial D_i \subset \Omega \cap (\partial D_i \setminus \overset{\circ}{L}) \cup \bigcup_{j \in J_0 \cup J_1 \cup J_2 \cup J_3} \partial Q_j.$$

Yet, thanks to the construction of the cylinder D_i ,

$$\begin{aligned} \mathcal{H}^{d-1}(\Omega \cap \partial D_i \setminus \overset{\circ}{L}) &\leq \mathcal{H}^{d-1}((-t_i v_\Omega(x_i) + P_i) \setminus \overset{\circ}{L}) + \mathcal{H}^{d-2}(\partial P_i)2\sqrt{\gamma}r_i \\ &\leq 6\sqrt{\gamma}\alpha_d r_i^{d-1} + 2\alpha_{d-2}r_i^{d-2}2\sqrt{\gamma}r_i \leq 6\sqrt{\gamma}(\alpha_d + \alpha_{d-2})r_i^{d-1}. \end{aligned}$$

- **Cylinders indexed by I_4 .** Let i be in I_4 . We have

$$\Omega \cap \partial P \cap \partial D_i \subset \Omega \cap (\partial D_i \cap L) \cup \bigcup_{j \in J_0 \cup J_1 \cup J_2 \cup J_3} \partial Q_j.$$

Yet, thanks to the construction of the cylinder D_i ,

$$\begin{aligned} \mathcal{H}^{d-1}(\Omega \cap \partial D_i \cap L) &\leq \mathcal{H}^{d-1}((-t_i v_\Omega(x_i) + P_i) \cap L) + \mathcal{H}^{d-2}(\partial P_i)2\sqrt{\gamma}r_i \\ &\leq 6\sqrt{\gamma}\alpha_d r_i^{d-1} + 2\alpha_{d-2}r_i^{d-2}2\sqrt{\gamma}r_i \leq 6\sqrt{\gamma}(\alpha_d + \alpha_{d-2})r_i^{d-1}. \end{aligned}$$

- **Cylinders indexed by I_5 .** Let i be in I_5 . We set

$$G_i = \text{disc}(x_i - \psi r_i v_L(x_i), \sqrt{1 - \psi^2} r_i, v_L(x_i)).$$

We claim that the set G_i is included in the interior of L . Indeed, $G_i \subset B(x_i, r_i) \cap \partial D_i$, yet $\partial L \cap B(x_i, r_i) \subset D_i$; therefore G_i does not intersect ∂L . Since $v_L(x_i)$ is

the exterior normal vector to L at x_i , then G_i is included in $\overset{\circ}{L}$. The definition of the set P implies that

$$\partial P \cap G_i \subset \bigcup_{j \in J_0 \cup J_1 \cup J_2 \cup J_3} \partial Q_j,$$

whence

$$\Omega \cap \partial P \cap \partial D_i \subset (\partial D_i \setminus G_i) \cup \bigcup_{j \in J_0 \cup J_1 \cup J_2 \cup J_3} \partial Q_j.$$

Yet

$$\begin{aligned} \mathcal{H}^{d-1}(\partial D_i \setminus (P_i + \psi r_i v_L(x_i)) \setminus G_i) &\leq 2\alpha_{d-2} r_i^{d-2} 2\psi r_i + \alpha_{d-1} r_i^{d-1} (1 + \delta - (1 - \psi^2)^{(d-1)/2}) \\ &\leq \alpha_{d-1} r_i^{d-1} \left(4 \frac{\alpha_{d-2}}{\alpha_{d-1}} \psi + 1 + \delta - (1 - \psi^2)^{(d-1)/2} \right). \end{aligned}$$

Finally, we conclude that

$$\begin{aligned} \Omega \cap \partial P \subset & \bigcup_{i \in I_1 \cup I_2} (\Omega \cap \partial D_i) \cup \bigcup_{i \in I_3} (\Omega \cap D_i \setminus \overset{\circ}{L}) \cup \bigcup_{i \in I_4} (\Omega \cap \partial D_i \cap L) \\ & \cup \bigcup_{i \in I_5} (\partial D_i \setminus G_i) \cup \bigcup_{j \in J_0 \cup J_1 \cup J_2 \cup J_3} \partial Q_j. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{I}_\Omega(P) &\leq \sum_{i \in I_1 \cup I_2} \int_{\Omega \cap \partial D_i} \nu(v_P(x)) d\mathcal{H}^{d-1}(x) + \nu_{\max} \sum_{i \in I_3} \mathcal{H}^{d-1}(\Omega \cap \partial D_i \setminus \overset{\circ}{L}) \\ &\quad + \nu_{\max} \sum_{i \in I_4} \mathcal{H}^{d-1}(\Omega \cap \partial D_i \cap L) \\ &\quad + \sum_{i \in I_5} \left(\nu(v_L(x_i)) \mathcal{H}^{d-1}(P_i) + \nu_{\max} \mathcal{H}^{d-1}(\partial D_i \setminus (P_i + \psi r_i v_L(x_i)) \setminus G_i) \right) \\ &\quad + \nu_{\max} \sum_{j \in J_0 \cup J_1 \cup J_2 \cup J_3} \mathcal{H}^{d-1}(\partial Q_j). \end{aligned}$$

We now use the various estimates obtained in the course of the approximation. We get

$$\begin{aligned} \mathcal{I}_\Omega(P) &\leq \sum_{i \in I_1 \cup I_2} \left(\alpha_{d-1} r_i^{d-1} (1 + \delta_0) \nu(v_\Omega(x_i)) + \nu_{\max} \alpha_{d-2} r_i^{d-1} 2M_0 \delta_0 (1 + \delta_0)^2 \right) \\ &\quad + \sum_{i \in I_3 \cup I_4} \nu_{\max} \left(6\sqrt{\gamma} (\alpha_d + \alpha_{d-2}) r_i^{d-1} \right) \\ &\quad + \sum_{i \in I_5} \left(\alpha_{d-1} r_i^{d-1} (1 + \delta) \nu(v_L(x_i)) \right. \\ &\quad \left. + \nu_{\max} \alpha_{d-1} r_i^{d-1} \left(4 \frac{\alpha_{d-2}}{\alpha_{d-1}} \psi + 1 + \delta - (1 - \psi^2)^{(d-1)/2} \right) \right) \\ &\quad + \sum_{j \in J_0 \cup J_1 \cup J_2 \cup J_3} \nu_{\max} \alpha_{d-1} 2^{d-1} s_j^{d-1} \\ &\leq \frac{1 + \delta_0}{1 - \gamma} \sum_{i \in I_1} \int_{B(x_i, r_i) \cap \partial^*(\Omega \setminus F)} \nu(v_\Omega(y)) d\mathcal{H}^{d-1}(y) \end{aligned}$$

$$\begin{aligned}
& + \frac{1+\delta_0}{1-\gamma} \sum_{i \in I_2} \int_{B(x_i, r_i) \cap \partial^* F} \nu(v_\Omega(y)) d\mathcal{H}^{d-1}(y) \\
& + \frac{1+\delta}{1-\varepsilon} \sum_{i \in I_5} \int_{B(x_i, r_i) \cap \partial L} \nu(v_L(y)) d\mathcal{H}^{d-1}(y) \\
& + \sum_{i \in I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5} \nu_{\max} \alpha_{d-1} r_i^{d-1} \left(\frac{\alpha_{d-2}}{d \alpha_{d-1}} 5\gamma + 6\sqrt{\gamma} \frac{\alpha_d + \alpha_{d-2}}{\alpha_{d-1}} + 4 \frac{\alpha_{d-2}}{\alpha_{d-1}} \psi \right. \\
& \quad \left. + 1 + \delta - (1 - \psi^2)^{(d-1)/2} \right) + \nu_{\max} 2^{d-1} 3(3\gamma + 4d\sqrt{\gamma} \mathcal{H}^{d-1}(\Gamma) + \varepsilon) \\
& \leq \frac{1+\delta_0}{1-\gamma} \left(\int_{\Gamma^1 \cap \partial^*(\Omega \setminus F)} \nu(v_\Omega(y)) d\mathcal{H}^{d-1}(y) + \int_{\Gamma^2 \cap \partial^* F} \nu(v_\Omega(y)) d\mathcal{H}^{d-1}(y) \right. \\
& \quad \left. + \int_{\Omega \cap \partial L} \nu(v_L(y)) d\mathcal{H}^{d-1}(y) \right) \\
& \quad + 2(\mathcal{H}^{d-1}(\Gamma) + \mathcal{H}^{d-1}(\Omega \cap \partial L)) \nu_{\max} \left(\frac{\alpha_{d-2}}{\alpha_{d-1}} 5\gamma + 6\sqrt{\gamma} \frac{\alpha_d + \alpha_{d-2}}{\alpha_{d-1}} + 4 \frac{\alpha_{d-2}}{\alpha_{d-1}} \psi \right. \\
& \quad \left. + 1 + \delta - (1 - \psi^2)^{(d-1)/2} \right) + \nu_{\max} \left(2^{d-1} 3(3\gamma + 4d\sqrt{\gamma} \mathcal{H}^{d-1}(\Gamma)) + 3\varepsilon \right) \\
& \leq \frac{1+\delta_0}{1-\gamma} (\mathcal{I}_\Omega(F) + \varepsilon) \\
& \quad + 2 \left(\mathcal{H}^{d-1}(\Gamma) + \frac{\nu_{\max} \mathcal{I}_\Omega(F) + \varepsilon}{\nu_{\min}} \right) \nu_{\max} \left(\frac{\alpha_{d-2}}{\alpha_{d-1}} 5\gamma + 6\sqrt{\gamma} \frac{\alpha_d + \alpha_{d-2}}{\alpha_{d-1}} + \delta\varepsilon + 4 \frac{\alpha_{d-2}}{\alpha_{d-1}} \varepsilon \right) \\
& \quad + \nu_{\max} \left(2^{d-1} 3(3\gamma + 4d\sqrt{\gamma} \mathcal{H}^{d-1}(\Gamma)) + 3\varepsilon \right),
\end{aligned}$$

where we have used the inequality $\psi < \varepsilon$ in the last step. We have also used the inclusions

$$\begin{aligned}
\forall i \in I_1 \quad B(x_i, r_i) \cap \partial^*(\Omega \setminus F) & \subset \Gamma^1 \cap \partial^*(\Omega \setminus F), \\
\forall i \in I_2 \quad B(x_i, r_i) \cap \partial^* F & \subset \Gamma^2 \cap \partial^* F.
\end{aligned}$$

Since $\delta_0, \delta, \gamma, \varepsilon$ can be chosen arbitrarily small, we have obtained the desired approximation. \square

5. POSITIVITY OF $\widetilde{\phi}_\Omega$

We suppose that

$$(9) \quad \int_{[0, +\infty[} x d\Lambda(x) < \infty.$$

We will prove that $\widetilde{\phi}_\Omega > 0$ if and only if $\Lambda(0) < 1 - p_c(d)$. In fact we know that if the condition (9) is satisfied,

$$\Lambda(0) < 1 - p_c(d) \iff \exists v, \nu(v) > 0 \iff \forall v, \nu(v) > 0.$$

Thus, the implication

$$\Lambda(0) \geq 1 - p_c(d) \implies \widetilde{\phi}_\Omega = 0$$

is trivial. We suppose that $\Lambda(0) < 1 - p_c(d)$. Since ν satisfies the weak triangle inequality, the function $v \mapsto \nu(v)$ is continuous, and so as soon as $\Lambda(0) < 1 - p_c(d)$

and (9) is satisfied, we have

$$\nu_{\min} = \min_{\mathbb{S}^1} \nu > 0.$$

If P is a polyhedral set, then $\mathcal{H}^{d-1}((\partial P \cap \Omega) \setminus (\partial^* P \cap \Omega)) = 0$. We then obtain that

$$\widetilde{\phi}_{\Omega} \geq \nu_{\min} \times \inf \left\{ \mathcal{H}^{d-1}(\mathcal{S} \cap \Omega) \left| \begin{array}{l} \mathcal{S} \text{ hypersurface that cuts } \Gamma^1 \text{ from } \Gamma^2 \text{ in } \overline{\Omega}, \\ d(\mathcal{S}, \Gamma^1 \cup \Gamma^2) > 0 \end{array} \right. \right\}.$$

We recall that the hypersurface \mathcal{S} cuts Γ^1 from Γ^2 in $\overline{\Omega}$ if \mathcal{S} intersects any continuous path from a point in Γ^1 to a point in Γ^2 that is included in $\overline{\Omega}$. We consider such a hypersurface $\mathcal{S} \subset \mathbb{R}^d$, and we want to bound from below the quantity $\mathcal{H}^{d-1}(\mathcal{S} \cap \Omega)$ independently on \mathcal{S} .

The idea of the proof is the following. We consider a path from Γ^1 to Γ^2 in Ω . We construct a tubular neighbourhood of this path of diameter depending only on the domain and not on the path itself that lies in Ω except at its endpoints. Then we prove that it is not very deformed compared to a straight tube. Since \mathcal{S} has to cut this tube, we obtain the desired lower bound $\mathcal{H}^{d-1}(\mathcal{S} \cap \Omega)$.

For $i = 1, 2$, we can find x_i in Γ^i and $r_i > 0$ such that $\Gamma \cap B(x_i, r_i) \subset \Gamma^i$ and $\Gamma \cap B(x_i, r_i)$ is a \mathcal{C}^1 hypersurface. We denote by $v_{\Omega}(x_i)$ the exterior normal unit vector to Ω at x_i and by $T_{\Omega}(x_i)$ the hyperplane tangent to Γ at x_i . Since Γ is of class \mathcal{C}^1 in a neighbourhood of x_i and Ω is a Lipschitz domain, by applying Lemma 1 we know that for all $\theta > 0$ there exists $\varepsilon > 0$ depending on $(\Omega, \Gamma, \Gamma^1, \Gamma^2, x_1, x_2)$ such that for $i = 1, 2$ we have

$$\begin{cases} \Omega \cap B(x_i, 2\varepsilon) \text{ is connected,} \\ \Gamma \cap B(x_i, 2\varepsilon) \subset \mathcal{V}_2(T_{\Omega}(x_i), 2\varepsilon \sin \theta) \cap B(x_i, 2\varepsilon), \\ \Gamma \cap B(x_i, 2\varepsilon) \subset \Gamma^i. \end{cases}$$

We fix θ small enough to have $2\varepsilon \sin \theta < \varepsilon/2$. We define

$$A_i = T_{\Omega}(x_i) \cap B(x_i, \varepsilon) \quad \text{and} \quad D_i = \text{cyl}(A_i, \varepsilon),$$

and then

$$\widehat{\Omega} = \Omega \cup \mathring{D}_1 \cup \mathring{D}_2,$$

where \mathring{D}_i is the interior of D_i for $i = 1, 2$. We define

$$X_i = \{z \in \mathring{D}_i \mid x_i z \cdot v_{\Omega}(x_i) > \varepsilon/2\} \subset \widehat{\Omega}.$$

Then $X_i \subset \widehat{\Omega} \setminus \Omega$. Each path r from a point $y_1 \in X_1$ to a point $y_2 \in X_2$ contains a path r' from a point $y'_1 \in \Gamma^1$ to a point $y'_2 \in \Gamma^2$ such that $r' \subset \overline{\Omega}$; thus \mathcal{S} intersects r . We consider the set

$$V_i = \{z \in X_i \mid d_2(z, \partial X_i) > \varepsilon/8\}.$$

Let $\hat{y}_1 \in V_1$, $\hat{y}_2 \in V_2$ such that $d_2(\hat{y}_i, \partial X_i) > \varepsilon/4$ for $i = 1, 2$. Since $\widehat{\Omega}$ is obviously path-connected, there exists a path \hat{r} from \hat{y}_1 to \hat{y}_2 in $\widehat{\Omega}$. The path \hat{r} is compact and $\widehat{\Omega}$ is open, so $\delta = d_2(\hat{r}, \partial \widehat{\Omega}) > 0$. We thus can find a path r included in $\mathcal{V}_2(\hat{r}, \min(\delta/2, \varepsilon/8))$ which is a \mathcal{C}^{∞} submanifold of \mathbb{R}^d of dimension 1 and which has one endpoint, denoted by y_1 , in V_1 , and the other one, denoted by y_2 , in V_2 .

As we previously explained, $d_2(r, \partial \widehat{\Omega}) > 0$, so there exists a positive η_1 such that $\mathcal{V}_2(r, \eta_1) \subset \widehat{\Omega}$. We can suppose that $\eta_1 < \varepsilon/16$ in order to obtain that $B(y_i, \eta_1) \subset X_i$ for $i = 1, 2$. For all z in r we denote by $N_r(z)$ the hyperplane orthogonal to r at z , and by $N_r^{\eta}(z)$ the subset of $N_r(z)$ composed of the points of $N_r(z)$ that are at a


$$\text{tub}(r, \eta) = \bigcup_{z \in r} N_r^\eta(z)$$
$$Nr^\eta = \{(z, v), z \in r, v \in N_r^\eta(z)\}$$

to $\text{tub}(r, \eta)$. We choose a positive η smaller than $\min(\eta_1, \eta_2)$. We stress the fact that this η depends on $(\Omega, \Gamma, \Gamma^1, \Gamma^2)$ but not on \mathcal{S} .

Let (I, h) be a parametrisation of class \mathcal{C}^∞ of r ; i.e., $I = [a, b]$ is a closed interval of \mathbb{R} and $h : I \rightarrow r$ is a \mathcal{C}^∞ -diffeomorphism which is an immersion. Let z be in r , and $u_z = h^{-1}(z) \in I$. The vector $h'(u_z)$ is tangent to r at z , and there exists some vectors $(e_2(z), \dots, e_d(z))$ such that $(h'(u_z), e_2(z), \dots, e_d(z))$ is a direct basis of \mathbb{R}^d . There exists a neighbourhood U_z of u_z in I such that for all $u \in U_z$, $(h'(u), e_2(z), \dots, e_d(z))$ is still a basis of \mathbb{R}^d , since h' is continuous. Indeed the condition for a family of vectors $(\alpha_1, \dots, \alpha_d)$ to be a basis of \mathbb{R}^d is an open condition, because it corresponds to $\det((\alpha_1, \dots, \alpha_d)) > 0$, where \det is the determinant of the matrix. We apply the Gram-Schmidt process to the basis $(h'(u), e_2(z), \dots, e_d(z))$ to obtain a direct orthonormal basis $(h'(u)/\|h'(u)\|, v_2(u, z), \dots, v_d(u, z))$ of \mathbb{R}^d for all $u \in U_z$, such that the dependence of $(h'(u)/\|h'(u)\|, v_2(u, z), \dots, v_d(u, z))$ on $u \in U_z$ is of class \mathcal{C}^∞ . We remark that the family $(v_2(u, z), \dots, v_d(u, z))$ is a direct orthonormal basis of $N_r(h(u))$ for all $u \in U_z$. We have associated with each $z \in r$ a neighbourhood U_z of $u_z = h^{-1}(z)$ in I ; we can obviously suppose that U_z is an interval which is open in I . Since $(U_z, z \in r)$ is a covering of the compact I , we can extract from it a finite covering $(U_j, j = 1, \dots, n)$. We can choose this family to be minimal, i.e., such that $(U_j, j \in \{1, \dots, n\} \setminus j_0)$ is not a covering of I for any $j_0 \in \{1, \dots, n\}$. We then reorder the $(U_j, j = 1, \dots, n)$ (keeping the same notation) by the increasing order of their left end point in $I \subset \mathbb{R}$. Since the family (U_j) is minimal, each point of I belongs either to a unique set U_j , $j \in \{1, \dots, n\}$, or to exactly two sets U_j and U_{j+1} for $j \in \{1, \dots, n-1\}$. We denote by a_j the middle of the non-empty open interval $U_j \cap U_{j+1}$ for $j \in \{1, \dots, n-1\}$, and by $(h'(u)/\|h'(u)\|, v_2(u, j), \dots, v_d(u, j))$ the direct orthonormal basis defined previously on U_j for $j \in \{1, \dots, n\}$. We want to construct a family of direct orthonormal basis $(h'(u)/\|h'(u)\|, f_2(u), \dots, f_d(u))$ of \mathbb{R}^d such that the function

$$\psi : u \in I \mapsto (h'(u)/\|h'(u)\|, f_2(u), \dots, f_d(u))$$

is of class \mathcal{C}^∞ . We have to define a concatenation of the $(h'(u)/\|h'(u)\|, v_2(u, j), \dots, v_d(u, j))$ over the different sets U_j . For $u \in [a, a_1]$, we define

$$\psi(u) = (h'(u)/\|h'(u)\|, v_2(u, 1), \dots, v_d(u, 1)).$$

Thus the function ψ defined on $[a, a_1]$ is of class \mathcal{C}^∞ . On $U_1 \cap U_2$ we have defined two different direct orthonormal bases $(h'(u)/\|h'(u)\|, v_2(u, j), \dots, v_d(u, j))$ for $j = 1$ and $j = 2$ that have the same first vector. Let $\phi_1 : U_1 \cap U_2 \rightarrow SO_{d-1}(\mathbb{R})$ be the function of class \mathcal{C}^∞ that associates to each $u \in U_1 \cap U_2$ the matrix of change of basis from $(v_2(u, 2), \dots, v_d(u, 2))$ to $(v_2(u, 1), \dots, v_d(u, 1))$.

If b_1 is the right end point of $U_1 \cap U_2$, then ϕ_1 is in particular defined on $[a_1, b_1[$. Let g_1 be a \mathcal{C}^∞ -diffeomorphism from $[a_1, b_1[$ to $[a_1, \infty[$ which is strictly increasing (so $g_1(a_1) = a_1$) and such that all the derivatives of g_1 at a_1 are null. Then $\phi_1 \circ g_1^{-1}$ is defined on $[a_1, +\infty[$, and all its derivatives at a_1 are equal to those of ϕ_1 . We then transform all the orthonormal bases $(v_2(u, j), \dots, v_d(u, j))$ of \mathbb{R}^{d-1} for $j \geq 2$ and $u \geq a_1$ by the change of basis $\phi_1 \circ g_1^{-1}$, and we denote the new direct orthonormal basis of \mathbb{R}^{d-1} obtained this way by $(\tilde{v}_2(u, j), \dots, \tilde{v}_d(u, j))$. We then define ψ on $[a_1, a_2]$ by

$$\psi(u) = (h'(u)/\|h'(u)\|, \tilde{v}_2(u, 2), \dots, \tilde{v}_d(u, 2)),$$

and we remark that $\psi(u)$ still defines a direct orthonormal basis of \mathbb{R}^d . The function ψ is of class \mathcal{C}^∞ on $[a, a_2]$, including at a_1 . We iterate this process with the family

of basis

$$(h'(u)/\|h'(u)\|, \tilde{v}_2(u, j), \dots, \tilde{v}_d(u, j)), j = 2, \dots, n,$$

at a_2 , etc., finitely many times since we work with a finite covering of I . In the end we obtain a function

$$\psi \circ h^{-1} : r \rightarrow SO_{d-1}(\mathbb{R})$$

which is of class \mathcal{C}^∞ , and for all $z \in r$, the set of the points of \mathbb{R}^d that have for the first coordinate 0 in the basis $\psi \circ h^{-1}(z)$ is exactly the hyperplane $N_r(z)$.

For each $t = (t_2, \dots, t_{d-1}) \in \{z \in \mathbb{R}^{d-1} \mid d(z, 0) \leq \eta\}$, the set

$$r_t = \{y \in \mathbb{R}^d \mid \exists z \in r, y \text{ has coordinates } (0, t_2, \dots, t_{d-1}) \text{ in the basis } \psi \circ h^{-1}(z)\}$$

is a continuous path (even of class \mathcal{C}^∞) from a point in X_1 to a point in X_2 ; therefore

$$r_t \cap \mathcal{S} \cap \overline{\Omega} \neq \emptyset.$$

Moreover, since $d(\mathcal{S}, \Gamma^1 \cup \Gamma^2) > 0$, we obtain that

$$(10) \quad r_t \cap \mathcal{S} \cap \Omega \neq \emptyset.$$

For each $y \in \text{tub}(r, \eta)$, there exists a unique $z_y \in r$ such that $y \in N_r(z_y)$, so we can associate to y its coordinates $(0, t_2(y), \dots, t_d(y))$ in the basis $\psi \circ h^{-1}(z_y)$. We define the projection p of $\text{tub}(r, \eta)$ on $N_r^\eta(y_1)$ that associates to each y in $\text{tub}(r, \eta)$ the point of coordinate $(0, t_2(y), \dots, t_d(y))$ in the basis $\psi \circ h^{-1}(y_1)$. Then p is of class \mathcal{C}^∞ , as is $\psi \circ h^{-1}$. If z belongs to $N_r^\eta(y_1)$ and if $t(z) = (t_2(z), \dots, t_d(z))$, then we know by equation (10) that there exists a point on $r_{t(z)}$ that intersects \mathcal{S} in Ω . Moreover, $r_{t(z)}$ is exactly the set of the points y of $\text{tub}(r, \eta)$ whose image $p(y)$ by this projection is the point z . Thus

$$p(\mathcal{S} \cap \text{tub}(r, \eta) \cap \Omega) = N_r^\eta(y_1).$$

Since $\text{tub}(r, \eta)$ is compact, p is a Lipschitz function on $\text{tub}(r, \eta)$, and so there exists a constant K , depending on p , hence on Ω , r and η but not on \mathcal{S} , such that

$$\mathcal{H}^{d-1}(\mathcal{S} \cap \Omega) \geq \mathcal{H}^{d-1}(\mathcal{S} \cap \text{tub}(r, \eta) \cap \Omega) \geq K \mathcal{H}^{d-1}(p(\mathcal{S} \cap \text{tub}(r, \eta))) \geq K \alpha_{d-1} \eta^{d-1}.$$

This ends the proof of the positivity of $\widetilde{\phi}_\Omega$ when $\Lambda(0) < 1 - p_c(d)$.

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