TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 363, Number 8, August 2011, Pages 4225–4262
S 0002-9947(2011)05209-8
Article electronically published on March 15, 2011

HAUSDORFF MEASURES AND FUNCTIONS OF BOUNDED QUADRATIC VARIATION

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ABSTRACT. To each function f of bounded quadratic variation we associate a Hausdorff measure μ_f. We show that the map $f \to \mu_f$ is locally Lipschitz and onto the positive cone of $\mathcal{M}[0,1]$. We use the measures $\{\mu_f : f \in V_2\}$ to determine the structure of the subspaces of V_2^0 which either contain c_0 or the square stopping time space S^2.

1. INTRODUCTION

The functions of bounded quadratic variation, introduced by N. Wiener in [22], have been extensively studied in their own right as well as for their applications. For example, related results can be found in [4], [5], [9], [19] and also in the monograph [8] where several applications are included.

In the present work we study the subspace structure of V_2^0. In the sequel we shall denote by V_2 the space of all real-valued functions f with bounded quadratic variation, defined on the unit interval and satisfying $f(0) = 0$; V_2 endowed with the quadratic variation norm is a Banach space. The aforementioned space V_2^0 is a separable subspace of V_2 of significant importance; it is defined as the closed subspace of V_2 containing all the square absolutely continuous functions, a concept introduced by R. E. Love in the early 1950s (cf. [17]).

The space V_2^0 was introduced by S. V. Kisliakov in [14] as an isometric version of Lindenstrauss' space JF. His aim was to provide easier proofs of the fundamental properties of JF; V_2^0 is separable, not containing ℓ_1 and with non-separable dual. These properties were the most distinctive ones for JF, as such a space answers in the negative a problem posed by S. Banach. Earlier R.C. James [11] had presented the James Tree (JT) space which is the analogue of V_2^0 in the frame of the sequence spaces. It is notable that the class of the separable Banach spaces not containing ℓ_1 and with non-separable dual, which appears as an exotic subclass of Banach spaces, includes spaces, such as V_2^0, naturally arising from other branches of Analysis.

Kisliakov also proved an important relation between the spaces V_2^0 and V_2. Namely, V_2 naturally coincides with the second dual of V_2^0 and moreover the w*-topology on the bounded subsets of V_2 coincides with the topology of pointwise convergence. Among the consequences of the preceding remarkable property is that every $f \in V_2$ is the pointwise limit of a bounded sequence $(f_n)_n$ from V_2^0 (cf.

Received by the editors March 31, 2009 and, in revised form July 14, 2009.
2000 *Mathematics Subject Classification.* Primary 28A78, 46B20, 46B26.
Key words and phrases. Hausdorff measures, functions of bounded p-variation, Banach spaces with non-separable dual, James Function space.
This research was supported by PEBE 2007.

[18]). The variety of the classical Banach spaces which are isomorphic to a subspace of V_2^0 is large and rather unexpected. Indeed, beyond the space ℓ_2 which is the most relative to V_2^0, as was stated in [15] and proved later in [3] and [6], the space c_0 is also isomorphic to a subspace of V_2^0. Moreover for all $2 \leq p < \infty$, the space ℓ_p shares the same property (cf. [6]).

The study of the subspace structure of V_2^0 was initiated in [3] and [6] and continued in [1]. From our point of view a sufficient understanding of V_2^0 requires answers to the following problems.

Problem 1. Let X be a reflexive subspace of V_2^0. Does there exist some $2 \leq p < \infty$ such that ℓ_p is isomorphically embedded into X?

As we have mentioned earlier, all ℓ_p, $2 \leq p < \infty$, are embedded into V_2^0. Also, in [6] it was shown that no ℓ_p, $1 \leq p < 2$ is isomorphic to a subspace of V_2^0. It is worth pointing out that the proof of the embedding of ℓ_p, $2 \leq p < \infty$ is rather indirect and uses the quadratic time stopping space S^2, which is one of the central objects of the present paper. The space S^2 is the square convexification of the stopping time space S^1. The latter space was defined by H. P. Rosenthal as the unconditional analogue of the space $L^1(\lambda)$. Both spaces (i.e. S^1, S^2) belong to the wider class of the spaces S^p, $1 \leq p < \infty$ introduced by S. Buechler's Ph.D. Thesis ([6]). We recall the definition. We denote by $2^{<\mathbb{N}}$ the dyadic tree and by $c_{00}(2^{<\mathbb{N}})$ the vector space of all real-valued functions defined on $2^{<\mathbb{N}}$ with finite support. For $1 \leq p < \infty$ we define the $\|\cdot\|_{S^p}$ on $c_{00}(2^{<\mathbb{N}})$ as follows. For $x \in c_{00}(2^{<\mathbb{N}})$, we set

$$\|x\|_{S^p} = \sup \Big(\sum_{s \in A} |x(s)|^p \Big)^{1/p},$$

where the supremum is taken over all antichains A of $2^{<\mathbb{N}}$. The space S^p is the completion of $(c_{00}(2^{<\mathbb{N}}), \|\cdot\|_{S^p})$. The space S^1 has an unconditional basis and G. Schechtman, in an unpublished work, showed that it contains all ℓ_p, $1 \leq p < \infty$. This result was extended in [6] to all S^p spaces by showing that for every $p \leq q$, ℓ_q is embedded into S^p. An excellent and detailed study of the stopping time space S^1, in fact in a more general setting, is included in N. Dew's Ph.D. Thesis ([7]). The interested reader will also find there, among other things, a proof of Schechtman's unpublished result. Let us also point out that the analogous problem to Problem 1 for the spaces S^p, $1 \leq p < \infty$ also remains open. An important result in [6] shows that S^2 is isomorphic to a subspace of V_2^0, and this actually yields that V_2^0 contains isomorphs of all ℓ_p, $2 \leq p < \infty$. Before closing our discussion for Problem 1, let us also note that for every infinite chain C of $2^{<\mathbb{N}}$ the subspace of S^2 generated by $\{e_s : s \in C\}$ is isomorphic to c_0, while for every infinite antichain A the corresponding one is isomorphic to ℓ_2. Thus, if a subspace X of V_2^0 contains an isomorph of S^2, then it contains all possible classical spaces that are embedded into V_2^0.

Our next two problems concern non-reflexive subspaces of V_2^0. Let us begin with a result from [3] which asserts that every non-reflexive subspace X of V_2^0 contains an isomorph of ℓ_2 or c_0. To see this, we start with some $f \in X^{**} \setminus X$, where X^{**} is considered as a subspace of V_2. Since X does not contain ℓ_1, Odell-Rosenthal's theorem (cf. [18]) yields that there exists a bounded sequence $(f_n)_n$ in X pointwise converging to f. If f is discontinuous, then there exists a sequence $(g_k)_k$, $g_k = f_{n_k} - f_{m_k}$ equivalent to the ℓ_2 basis and hence ℓ_2 is embedded into X.

The case of a continuous f is more interesting. As is shown in [3], such an f is a difference of bounded semicontinuous functions (DBSC) when f is considered as a function with domain $(B_{(V_2^0)^*}, \mathrm{w}^*)$. A result of Haydon, Odell and Rosenthal (cf. [10]) yields that the sequence $(f_n)_n$ has a convex block subsequence $(g_n)_n$ equivalent to the summing basis of c_0. Let us note that the existence of a continuous function f in $X^{**} \setminus X$ is actually equivalent to the embedding of c_0 into X.

The second problem concerns subspaces of V_2^0 with non-separable dual and it is stated as follows.

Problem 2. Is it true that every subspace X of V_2^0 with X^* non-separable contains an isomorph of V_2^0 itself? Moreover, is every complemented subspace X of V_2^0 with non-separable dual isomorphic to V_2^0?

An affirmative answer to the second part of Problem 2 yields that V_2^0 is a primary space. In [1] it has been shown that the corresponding problem in James' space JT has an affirmative answer. This evidence supports the possibility for a positive solution to Problem 2. It is worth mentioning that, as is shown in [1], every subspace X of V_2^0 with non-separable dual contains the space TF. This is a sequence space with non-separable dual, introduced in [1]. It is isomorphic to any subspace of V_2^0 generated by a tree family $(f_s)_{s\in 2^{<\mathbb{N}}}$ of trapezoids. The latter spaces were introduced in [6] for showing that V_2^0 does not contain isomorphs of JT. In [1] it is also stated without proof that S^2 is embedded into TF which, as we have already mentioned, yields that every subspace of V_2^0 with non-separable dual contains isomorphs of all possible classical spaces that are embedded in V_2^0. In the present paper we give a proof of the embedding of S^2 into TF, granting from [6] that c_0 is embedded into TF.

The third problem concerns subspaces of V_2^0 with non-separable second dual.

Problem 3. Is it true that every subspace X of V_2^0 with X^{**} non-separable contains c_0?

The main goal of the present work is to provide a positive answer to this problem. Before we start explaining our proof, we point out that the preceding results on subspaces X with non-separable dual reduce the problem to those X with X^* separable and X^{**} non-separable. Also, as we noted above, the embedding of c_0 into X is equivalent to the existence of a function $f \in (X^{**} \setminus X) \cap C[0,1]$. In the early stages of our engagement to this problem we observed that when X^* is separable, the set $D_{X^{**}} = \{t \in [0,1] : \exists f \in X^{**}, oscf(t) > 0\}$ is at most countable, a fact supporting an affirmative solution to the problem. However we had no further progress until the moment where we discovered a new concept which plays a key role in our approach. This is a Hausdorff type measure μ_f associated to every $f \in V_2$. The measure μ_f is defined as follows. First we introduce some notation.

Given $f : [0,1] \to \mathbb{R}$ and $\mathcal{P} = \{t_0 < \ldots < t_p\} \subseteq [0,1]$, with $p \geq 1$, let $\|\mathcal{P}\|_{\max} = \max\{t_{i+1} - t_i : 0 \leq i \leq p-1\}$ and $v_2^2(f,\mathcal{P}) = \sum_{i=0}^{p-1}(f(t_{i+1}) - f(t_i))^2$. For every $f \in V_2$ and for every interval I of $[0,1]$ we set

$$\widetilde{\mu}_f(I) = \inf_{\delta>0} \widetilde{\mu}_{f,\delta}(I),$$

where for each $\delta > 0$, $\widetilde{\mu}_{f,\delta}(I) = \sup\{v_2^2(f,\mathcal{P}) : \mathcal{P} \subseteq I \text{ and } \|\mathcal{P}\|_{\max} < \delta\}$. The collection $\{\widetilde{\mu}_f(I) : I \text{ is an interval of } [0,1]\}$ defines an outer measure and μ_f is the regular measure induced by $\widetilde{\mu}_f$ on the Borel subsets of $[0,1]$. We should mention that N. Wiener himself had also considered the quantity $\sqrt{\widetilde{\mu}_f[0,1]}$, pointing out that it is a seminorm on V_2. The measure μ_f incorporates a sufficient amount of information concerning the function f. Thus $\mu_f = 0$ if and only if $f \in V_2^0$, μ_f is continuous (diffuse) if and only if f is continuous and also the discrete (atomic) part of μ_f is supported by the points of discontinuity of f. Furthermore the following hold.

Proposition 1. *Let $f \in V_2$. Then the set of the points of differentiability of f has μ_f-measure zero.*

As a consequence we obtain the following.

Corollary. *Let f be a continuous function in V_2. If the set of all non-differentiability points of f is at most countable, then f belongs to V_2^0. Moreover if $f \in (V_2 \setminus V_2^0) \cap C[0,1]$, then the set of all non-differentiability points of f contains a perfect set.*

The second result concerns the variety of the elements of V_2.

Proposition 2. *Let $\Phi : V_2 \to \mathcal{M}^+[0,1]$ be the function that maps f to μ_f. Then Φ is locally Lipschitz and onto. In particular for every continuous positive measure $\mu \in \mathcal{M}^+[0,1]$ there exists $f \in (V_2 \setminus V_2^0) \cap C[0,1]$ such that $\mu_f = \mu$.*

The solution of Problem 3 heavily relies on properties of the measure μ_f. In particular the following inequality is a key ingredient. For every $f \in V_2$ the following holds:

$$\sqrt{\|\mu_f\|} \leq dist(f, V_2^0) \leq \|\widetilde{osc}_{\mathcal{K}} f\|_\infty \leq \sqrt{\|\mu_f\|} + 2\sqrt{\|\mu_f^d\|}, \tag{1}$$

where $\mathcal{K}$ is a w*-closed subset of $B_{(V_2^0)^*}$, 1-norming V_2^0, $\widetilde{osc} f$ is as in [21] or [2] and was introduced in [13] and also μ_f^d is the discrete part of μ_f. Note that when f is continuous, then (1) becomes an equality and hence $dist(f, V_2^0) = \sqrt{\|\mu_f\|}$. Furthermore the measures $\{\mu_f : f \in V_2\}$ permit us to have a better and more precise understanding of the structure of X when X^{**} is non-separable. Thus we prove the following.

Theorem. *Let X be a closed subspace of V_2^0. Then the following hold.*

1. *The space X contains an isomorphic copy of c_0 if and only if X^{**} is non-separable.*
2. *The space X contains an isomorphic copy of S^2 if and only if $\mathcal{M}_{X^{**}} = \{\mu_f : f \in X^{**}\}$ is a non-separable subset of $\mathcal{M}[0,1]$.*

Note that when X^* is non-separable, then the stronger case (case (2)) of the above theorem occurs. When X is isomorphic to c_0, then both X^* and $\mathcal{M}_{X^{**}}$ are separable. On the other hand, any subspace X of V_2^0 isomorphic to S^2 is an example of a subspace X with separable dual and such that $\mathcal{M}_{X^{**}}$ is non-separable.

In the rest of the introduction we shall describe the basic steps towards a proof of the main theorem. Let us start by saying that a function $h \in V_2^0$ is $(C,\varepsilon)-$dominated by a measure $\mu \in \mathcal{M}^+[0,1]$, if for every finite family $\mathcal{I} = ([a_i, b_i])_{i=1}^n$ of non-overlapping intervals it follows that

$$\sum_{i=1}^n \Big(h(b_i) - h(a_i)\Big)^2 \leq C\mu(\cup\mathcal{I}) + \varepsilon.$$

This domination property allows us to relate measures with sequences $(h_n)_n$ which are equivalent to the usual basis of c_0 as follows.

Proposition 3. *Let $(h_n)_n$ be a seminormalized sequence of functions of V_2^0, $(\varepsilon_n)_n$ be a null sequence of positive real numbers and $\mu \in \mathcal{M}^+[0,1]$ such that for some $C > 0$ each h_n is $(C, \varepsilon_n) - \mu$ dominated and $\lim_n \|h_n\|_\infty = 0$. Then there is a subsequence of $(h_n)_n$ equivalent to the usual basis of c_0.*

The next result explains how we can extract information about X from the elements of $X^{**} \setminus X$.

Proposition 4. *Let X be a subspace of V_2^0, $f \in X^{**} \setminus X$ and $(f_n)_n$ be a bounded sequence in X pointwise convergent to f. Then for every $0 < \delta < dist(f, X)$ and every sequence $(\varepsilon_n)_n$ of positive real numbers there exists a convex block sequence $(h_n)_n$ of $(f_n)_n$ such that for all $n < m$ the following properties are satisfied.*

(i) *$\delta < \|h_m - h_n\|_{V_2} \leq 2M$, where $M = \sup_n \|f_n\|_{V_2}$.*
(ii) *$\|h_m - h_n\|_\infty \leq 2\|\widetilde{osc}_{[0,1]} f\|_\infty + \varepsilon_n \leq 4\|f\|_\infty + \varepsilon_n$.*
(iii) *$h_m - h_n$ is $(4, \widetilde{\varepsilon}_n)$-$\mu_f$ dominated, where $\widetilde{\varepsilon}_n = 32\|f\|_{V_2}\sqrt{\|\mu_f^d\|} + \varepsilon_n$.*

The proof of the proposition uses inequality (1) and also optimal sequences pointwise convergent to the function f (cf. [2]). Note that if we additionally assume that f is continuous, in which case $\mu_f^d = \widetilde{osc}_{[0,1]} f = 0$, Propositions 3 and 4 almost immediately yield that the space X contains c_0, a result initially proved with a different method in [3].

The proof of the main result is divided into two cases. In the first case we consider subspaces X of V_2^0 with X^* separable, X^{**} non-separable and $\mathcal{M}_{X^{**}}$ separable. Then using Proposition 4 and the separability of X^*, we may select a seminormalized sequence $(H_n)_n$ in X and a norm-converging sequence of measures $(\mu_n)_n$ in $\mathcal{M}_{X^{**}}$ such that each H_n is $(4, \varepsilon_n) - \mu_n$ dominated. Then an easy modification of Proposition 3 yields that there exists a subsequence of $(H_n)_n$ equivalent to the c_0-basis.

The second case, namely when $\mathcal{M}_{X^{**}}$ is non-separable, is more involved. Here, we first give sufficient conditions for the embedding of the space S^2 into a subspace X of V_2^0 . Moreover, again using Proposition 4, we construct a seminormalized tree family of functions $(H_s)_{s \in 2^{<\mathbb{N}}}$ in X and a bounded family of measures $(\mu_s)_{s \in 2^{<\mathbb{N}}}$. For each $s \in 2^n$ we define a finite subset $L_s \subseteq 2^{2n}$ with $card(L_s) = 2^n$ and we set $G_s = 2^{-n/2} \sum_{t \in L_s} H_t$ and $\nu_s = 2^{-n} \sum_{t \in L_s} \mu_t$. The proof ends by showing that these new tree families satisfy the requirements for containing a tree subfamily equivalent to the S^2-basis.

The present work can be considered as a step towards the understanding of the structure of V_2^0. Our approach has revealed a new component, the Hausdorff measure μ_f associated to a function f of bounded quadratic variation, which is of independent interest and could be useful to a further investigation of V_2 as well as in applications.

We close this introduction by pointing out that all the results contained here remain valid under obvious modifications for the space V_p^0, for all $1 < p < \infty$.

2. Preparatory work on V_2

This section is divided into three subsections. First we fix the notation that we shall use. In the second subsection we prove that the set of discontinuity points of the elements of X^{**} when X^* is separable is countable and also in this case $(X^*, \|\cdot\|_\infty)$ is separable. Finally, we introduce the biorthogonal families of functions of V_2^0. Such families share nice properties and as we will see they play a critical role in the proofs of almost all of our results.

2.1. **Preliminaries.** We start with some notation concerning intervals as well as families of intervals of $[0,1]$. The length of an interval I will be denoted by $|I|$. For a finite family $\mathcal{I}$ of intervals, $\|\mathcal{I}\|_{\max} = \max\{|I| : I \in \mathcal{I}\}$ and $\|\mathcal{I}\|_{\min} = \min\{|I| : I \in \mathcal{I}\}$.

By $\mathcal{A}$ we denote the set of all finite families of intervals of $[0,1]$ with pairwise disjoint interiors. A sequence $(\mathcal{I}_i)_i$ in $\mathcal{A}$ will be called *disjoint* if for every $i \neq j$, $I \in \mathcal{I}_i$ and $J \in \mathcal{I}_j$, the interiors of I and J are disjoint. Also by $\mathcal{F}$ we denote the set of all finite families of pairwise disjoint closed intervals of $[0,1]$. More generally, given a subset $S \subseteq [0,1]$, $\mathcal{F}(S)$ is the set of all $\mathcal{I} \in \mathcal{F}$ such that the endpoints of every $I \in \mathcal{I}$ belong to S.

Given $f : [0,1] \to \mathbb{R}$ and $\mathcal{P} = \{t_0 < \ldots < t_p\} \subseteq [0,1]$, with $p \geq 1$, the quadratic variation of f on $\mathcal{P}$ is the quantity

$$v_2(f,\mathcal{P}) = \Big(\sum_{i=0}^{p-1}(f(t_{i+1}) - f(t_i))^2\Big)^{1/2}.$$

Similarly for $\mathcal{I} = (I_i)_{i=1}^k$ in $\mathcal{A}$, we set $v_2(f,\mathcal{I}) = (\sum_{i=0}^k (f(b_i) - f(a_i))^2)^{1/2}$, where for each $1 \leq i \leq k$, a_i, b_i are the endpoints of I_i (if $\mathcal{I}$ is the empty sequence, then we define $v_2(f,\varnothing) = 0$). The quantity $v_2(f,\mathcal{I})$ has also been defined in [19] where the exponent $1/2$ is omitted.

Notice that every $\mathcal{P}$ as above determines the family $\mathcal{I}_\mathcal{P} = ((t_i, t_{i+1}))_{i=0}^{p-1}$ in $\mathcal{A}$ and $v_2(f,\mathcal{P}) = v_2(f,\mathcal{I}_\mathcal{P})$. It is easy to see that for every $f, g : [0,1] \to \mathbb{R}$ and every $\mathcal{I} \in \mathcal{A}$, we have that

$$(2) \qquad |v_2(f,\mathcal{I}) - v_2(g,\mathcal{I})| \leq v_2(f+g,\mathcal{I}) \leq v_2(f,\mathcal{I}) + v_2(g,\mathcal{I}).$$

Moreover for every disjoint partition $\mathcal{I} = \bigcup_i \mathcal{I}_i$ of $\mathcal{I} \in \mathcal{A}$,

$$(3) \qquad v_2(f,\mathcal{I}) \leq \sum_i v_2(f,\mathcal{I}_i) \quad \text{and} \quad v_2^2(f,\mathcal{I}) = \sum_i v_2^2(f,\mathcal{I}_i).$$

For $\widetilde{\mathcal{I}}, \mathcal{I}$ in $\mathcal{A}$, we write $\widetilde{\mathcal{I}} \preceq \mathcal{I}$ if for every $\widetilde{I} \in \widetilde{\mathcal{I}}$ there is $I \in \mathcal{I}$ such that $\widetilde{I} \subseteq I$.

For every $\varepsilon > 0$, $D \subseteq [0,1]$ and $H_1, \ldots, H_k$ in V_2^0 we will say that D *$\varepsilon-$determines the quadratic variation of the linear span* $\langle H_1, \ldots, H_k\rangle$ if for every $\mathcal{I} \in \mathcal{A}$ there is $\widetilde{\mathcal{I}} \preceq \mathcal{I}$ in $\mathcal{F}(D)$ such that

$$\Big| v_2^2\Big(\sum_{i=1}^k \lambda_i H_i, \mathcal{I}\Big) - v_2^2\Big(\sum_{i=1}^k \lambda_i H_i, \widetilde{\mathcal{I}}\Big)\Big| \leq \Big(\sum_{i=1}^k |\lambda_i|^2\Big)\varepsilon,$$

for every sequence of scalars $(\lambda_i)_{i=1}^k$. Using standard approximation arguments the following is easily proved.

Proposition 1. *Let $k \in \mathbb{N}$, $H_1, \ldots, H_k$ in V_2^0 and $\varepsilon > 0$. Then there exists $\delta > 0$ such that every $D \subseteq [0,1]$ which is $\delta-$dense in $[0,1]$ $\varepsilon-$determines the quadratic variation of $\langle H_1, \ldots, H_k\rangle$.*

Next we state some notation for the dyadic tree. For every $n \geq 0$, we set $2^n = \{0,1\}^n$ (where $2^0 = \{\emptyset\}$). Hence for $n \geq 1$, every $s \in 2^n$ is of the form

$s = (s(1), ..., s(n))$. For $0 \leq m < n$ and $s \in 2^n$, $s|m = (s(1), ..., s(m))$, where if $m = 0$, $s|0 = \emptyset$. Also, $2^{\leqslant n} = \bigcup_{i=0}^{n} 2^i$ and $2^{<\mathbb{N}} = \bigcup_{n=0}^{\infty} 2^n$. The *length* $|s|$ of an $s \in 2^{<\mathbb{N}}$ is the unique $n \geq 0$ such that $s \in 2^n$. The initial segment partial ordering on $2^{<\mathbb{N}}$ will be denoted by $\sqsubseteq$ (i.e. $s \sqsubseteq t$ if $m = |s| \leq |t|$ and $s = t|m$). For $s, t \in 2^{<\mathbb{N}}$, $s \perp t$ means that s, t are $\sqsubseteq$-incomparable (that is, neither $s \sqsubseteq t$ nor $t \sqsubseteq s$). For an $s \in 2^{<\mathbb{N}}$, $s^\frown 0$ and $s^\frown 1$ denote the two immediate successors of s which end with 0 and 1, respectively. More generally for $s, u \in 2^{<\mathbb{N}}$, $s^\frown u$ denotes the concatenation of s and u, namely the element $t \in 2^{<\mathbb{N}}$ with $|t| = |s| + |u|$, $t(i) = s(i)$ for all $1 \leq i \leq |s|$ and $t(|s| + i) = u(i)$ for all $1 \leq i \leq |u|$.

An *antichain* of $2^{<\mathbb{N}}$ is a subset of $2^{<\mathbb{N}}$ such that for every $s, t \in A$, $s \perp t$. A *branch* of $2^{<\mathbb{N}}$ is a maximal totally ordered subset of $2^{<\mathbb{N}}$. A *dyadic subtree* is a subset T of $2^{<\mathbb{N}}$ such that there is an order isomorphism $\phi : 2^{<\mathbb{N}} \to T$. In this case T is denoted by $T = (t_s)_{s \in 2^{<\mathbb{N}}}$, where $t_s = \phi(s)$.

In the sequel by the term *subspace* we always mean a closed infinite-dimensional subspace. We also use the standard notation for Banach spaces from [16].

2.2. **The discontinuities of X^{**} for subspaces X of V_2^0.** For every $f \in V_2$, by D_f we denote the set of all points of discontinuity of f. For all $t \in [0, 1]$ let $f(t^+) = \lim_{s \to t^+} f(s)$ and $f(t^-) = \lim_{s \to t^-} f(s)$ (where by convention we set $f(0^-) = f(0)$ and $f(1^+) = f(1)$). It is easily shown that for every $f \in V_2$, the set D_f is at most countable and so f is a Baire-1 function. Moreover for every $t \in D_f$, $f(t^-)$ and $f(t^+)$ always exist and $\sum_{t \in D_f} |f(t) - f(t^-)|^2 + |f(t) - f(t^+)|^2 \leq \|f\|_{V_2}^2$.

In this subsection we will study the set $D_{X^{**}} = \bigcup_{f \in X^{**}} D_f$, for subspaces X of V_2^0 with X^* separable and X^{**} non-separable. We will show that $D_{X^{**}}$ is a countable subset of $[0, 1]$ which as we will see implies that the space $(X^{**}, \|\cdot\|_\infty)$ is separable. We start with a characterization of the subspaces X of V_2^0 with separable dual through the discontinuity points of all $f \in X^{**}$.

Proposition 2. *Let X be a subspace of V_2^0. Then X^* is separable if and only if $D_{X^{**}}$ is countable.*

Proof. Suppose that $D_{X^{**}}$ is uncountable. Then, since for every $f \in X^{**}$, D_f is countable, we may choose uncountable sets $\mathcal{F} = \{f_\xi\}_{\xi < \omega_1} \subseteq B_{X^{**}}$ and $A = \{t_\xi\}_{\xi < \omega_1} \subseteq [0, 1]$ such that the following are satisfied.

(1) For every $\xi < \omega_1$, f_ξ is discontinuous at t_ξ.
(2) Exactly one of the following hold.
  (2a) For all $\xi < \omega_1$, $f_\xi(t_\xi^+) \neq f_\xi(t_\xi)$.
  (2b) For all $\xi < \omega_1$, $f_\xi(t_\xi^-) \neq f_\xi(t_\xi)$.

Suppose that (2a) holds (the other case is similar). Passing to an uncountable subset of $\mathcal{F}$ we may assume that there exists $\delta > 0$ such that $|f_\xi(t_\xi^+) - f_\xi(t_\xi)| > \delta$, for every $\xi < \omega_1$. Moreover, by passing to a further uncountable subset, we can suppose that there exist $0 < \varepsilon < \delta$ and an open interval I of $(0, 1)$ such that for every $\xi < \omega_1$ we have that (i) $t_\xi \in I$, (ii) for every $t \in I$ and $t < t_\xi$, $|f_\xi(t) - f_\xi(t_\xi^-)| < \varepsilon$ and (iii) for every $t \in I$ and $t > t_\xi$, $|f_\xi(t) - f_\xi(t_\xi^+)| < \varepsilon$.

Let $\xi < \xi'$. If $t_\xi < t_{\xi'}$, then we have that

$$|\delta_{t_\xi}(f_\xi) - \delta_{t_{\xi'}}(f_\xi)| \geq |f_\xi(t_\xi) - f_\xi(t_\xi^+)| - |f_\xi(t_\xi^+) - f_\xi(t_{\xi'})| > \delta - \varepsilon,$$

and if $t_{\xi'} < t_\xi$, then similarly $|\delta_{t_\xi}(f_{\xi'}) - \delta_{t_{\xi'}}(f_{\xi'})| > \delta - \varepsilon$.

This implies that $\|\delta_{t_\xi}|_X - \delta_{t_{\xi'}}|_X\| \geq \delta - \varepsilon$ for every $\xi \neq \xi'$ and therefore X^* is non-separable. Finally for the converse, suppose that X^* is non-separable. Then by Proposition 23 of [1] we have that X^{**} contains a non-separable family $\mathcal{H} \subseteq V_2^d = \overline{\langle\{\chi_t : t \in (0,1)\}\rangle}$ and therefore $D_{X^{**}}$ must be uncountable. □

Let us recall that by [1] (Theorem 15), we have that

$$V_2 = \overline{V_2^c + \langle\{\chi_{[t,1]} : 0 < t \leq 1\}\rangle + \langle\{\chi_t : 0 < t < 1\}\rangle}^{\|\cdot\|_{V_2}},$$

where $V_2^c = V_2 \cap C[0,1]$. The proof of this result is based on a series of lemmas (Lemmas 16-20). It is easy to see that the exact content of their proof is the following stronger result.

Lemma 3. *For every $f \in V_2$, we have that*

$$f \in \overline{V_2^c + \langle\{\chi_{[t,1]} : t \in D_f\}\rangle + \langle\{\chi_t : t \in D_f\}\rangle}^{\|\cdot\|_{V_2}}.$$

Proposition 4. *Let $\mathcal{F}$ be a subset of V_2 such that $D_{\mathcal{F}} = \bigcup_{f\in\mathcal{F}} D_f$ is countable. Then the space $(\mathcal{F}, \|\cdot\|_\infty)$ is separable.*

Proof. Lemma 3 readily yields that

$$\mathcal{F} \subseteq \overline{V_2^c + Y}^{\|\cdot\|_{V_2}},$$

where $Y = \langle\{\chi_{[t,1]} : t \in D_{\mathcal{F}}\}\rangle + \langle\{\chi_t : t \in D_{\mathcal{F}}\}\rangle$. Using that $\|\cdot\|_\infty \leq \|\cdot\|_{V_2}$, we get

$$\mathcal{F} \subseteq \overline{V_2^c + Y}^{\|\cdot\|_{V_2}} \subseteq \overline{V_2^c + Y}^{\|\cdot\|_\infty} \subseteq \overline{C[0,1] + Y}^{\|\cdot\|_\infty}. \tag{4}$$

Since $D_{\mathcal{F}}$ is countable the space $(Y, \|\cdot\|_\infty)$ is separable. Hence $\overline{C[0,1] + Y}^{\|\cdot\|_\infty}$ is also separable which by (4) yields the desired result. □

Corollary 5. *Let X be a subspace of V_2^0 such that X^* is separable. Then the space $(X^{**}, \|\cdot\|_\infty)$ is separable.*

2.3. **Biorthogonal families in V_2^0.**

2.3.1. *Definition and existence.* In this subsection we introduce the concept of biorthogonality for families of functions in V_2^0.

Definition 6. Let $(H_i)_{i\in S}$ be a family of functions of V_2^0 and $(\varepsilon_i)_{i\in S}$ a family of positive real numbers, where S is a countable set. We will say that $(H_i)_{i\in S}$ is $(\varepsilon_i)_{i\in S}$-biorthogonal if for every $\mathcal{I} \in \mathcal{A}$ there is a disjoint partition $\mathcal{I} = \bigcup_{i\in S} \mathcal{I}^{(i)}$ such that for every $j \in S$,

$$\sum_{\{i\in S : i\neq j\}} v_2(H_i, \mathcal{I}^{(j)}) \leq \varepsilon_j. \tag{5}$$

Proposition 7. *Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of functions of V_2^0 with $\lim \|H_n\|_\infty = 0$. Then for every sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ of positive real numbers there exists a subsequence $(H_{n_i})_{i\in\mathbb{N}}$ such that $(H_{n_i})_{i\in\mathbb{N}}$ is $(\varepsilon_i)_{i\in\mathbb{N}}$-biorthogonal.*

For the proof of the above proposition we will need some specialized forms of biorthogonality. Let $k \geq 1$, $(\varepsilon_i)_{i=1}^k$ and $(\delta_i)_{i=0}^{k-1}$ be finite sequences of positive real numbers such that $0 < \delta_{k-1} < ... < \delta_1 < \delta_0 = 1$. We say that a sequence $(H_i)_{i=1}^k$ in V_2^0 is $((\varepsilon_i)_{i=1}^k, (\delta_i)_{i=0}^{k-1})$-biorthogonal if the inequality (5) of Definition 6 is satisfied for $S = \{1, ..., k\}$ and

$$\mathcal{I}^{(j)} = \{I \in \mathcal{I} : \delta_j < |I| \leq \delta_{j-1}\} \tag{6}$$

for all $1 \leq j \leq k$ (where for $j = k$, we set $\delta_k = 0$).

We will also use the following notation. For a sequence of positive real numbers $(\varepsilon_i)_{i\in\mathbb{N}}$ and every $1 \leq i \leq k$, we set $\varepsilon_i^k = (\sum_{r=1}^{k-i+1} 2^{-r})\varepsilon_i$. Clearly for all $1 \leq i \leq k$, $\varepsilon_i^k < \varepsilon_i^{k+1}$ and $\lim_k \varepsilon_i^k = \varepsilon_i$.

The proof of Proposition 7 is based on the next lemma.

Lemma 8. *Let $(H_i)_{i=1}^k$ be an $((\varepsilon_i^k)_{i=1}^k, (\delta_i)_{i=0}^{k-1})$-biorthogonal sequence of V_2^0. Let $0 < \delta_k < \delta_{k-1}$ be such that for every $\mathcal{I} \in \mathcal{A}$ with $\|\mathcal{I}\|_{\max} \leq \delta_k$, the following holds:*

$$\sum_{i=1}^{k} v_2(H_i, \mathcal{I}) \leq \frac{\varepsilon_{k+1}}{2}. \tag{7}$$

Then there is $\epsilon > 0$ such that for every $H_{k+1} \in V_2^0$ with $\|H_{k+1}\|_\infty < \epsilon$, the sequence $(H_i)_{i=1}^{k+1}$ is $((\varepsilon_i^{k+1})_{i=1}^{k+1}, (\delta_i)_{i=0}^{k})$-biorthogonal.

Proof. Notice that for every $\mathcal{J} \in \mathcal{A}$ with $\delta_k < \|\mathcal{J}\|_{\min}$, $card(\mathcal{J}) < \delta_k^{-1}$. Set $\epsilon = \sqrt{\delta_k} 2^{-(k+3)} \min\{\varepsilon_i\}_{i=1}^k$ and let $H_{k+1} \in V_2^0$ be such that $\|H_{k+1}\|_\infty < \epsilon$. Then for every $1 \leq j \leq k$ and $\mathcal{I} \in \mathcal{A}$, we have that $\delta_k < \delta_j < \|\mathcal{I}^{(j)}\|_{\min}$ and therefore

$$v_2(H_{k+1}, \mathcal{I}^{(j)}) \leq \Big(\delta_k^{-1}(2\|H_{k+1}\|_\infty)^2\Big)^{1/2} \leq \frac{\min\{\varepsilon_i\}_{i=1}^k}{2^{k+2}} \leq \frac{\varepsilon_j}{2^{k-j+2}}.$$

Hence for each $1 \leq j \leq k$ and for every $\mathcal{I} \in \mathcal{A}$,

$$\begin{aligned}\sum_{\{i:1\leq i\leq k+1, i\neq j\}} v_2(H_i, \mathcal{I}^{(j)}) &= \sum_{\{i:1\leq i\leq k, i\neq j\}} v_2(H_i, \mathcal{I}^{(j)}) + v_2(H_{k+1}, \mathcal{I}^{(j)}) \\ &\leq \varepsilon_j^k + \frac{\varepsilon_j}{2^{k-j+2}} = \Big(\sum_{r=1}^{k-j+2} 2^{-r}\Big)\varepsilon_j = \varepsilon_j^{k+1}.\end{aligned}$$

Finally $\|\mathcal{I}^{(k+1)}\|_{\max} \leq \delta_k$ and so by (7), we get that

$$\sum_{i=1}^{k} v_2(H_i, \mathcal{I}^{(k+1)}) \leq \frac{\varepsilon_{k+1}}{2} = \varepsilon_{k+1}^{k+1}.$$

□

Proof of Proposition 7. We inductively construct an increasing sequence $n_1 < n_2 < ...$ of natural numbers and a decreasing sequence of positive real numbers $0 < ... < \delta_2 < \delta_1 < 1 = \delta_0$, such that for every $k \geq 1$, the sequence $(H_{n_i})_{i=1}^k$ is $((\varepsilon_i^k)_{i=1}^k, (\delta_i)_{i=1}^{k-1})$-biorthogonal. We claim that $(H_{n_i})_i$ is $(\varepsilon_i)_i$-biorthogonal. Indeed, let $\mathcal{I} \in \mathcal{A}$ and let $\mathcal{I} = \bigcup_i \mathcal{I}^{(i)}$ be the partition of $\mathcal{I}$ induced by (6). Also let $k_0 \geq 1$ be such that $\delta_{k_0} < \|\mathcal{I}\|_{\min}$. Then for each $j \in \mathbb{N}$ with $j \geq k_0$, $\mathcal{I}^{(j)} = \emptyset$ and so (5) trivially holds. Otherwise for all $k \geq 1$, $\sum_{\{1\leq i\leq k:\, i\neq j\}} v_2(H_{n_i}, \mathcal{I}^{(j)}) < \varepsilon_j^k$ and so $\sum_{\{i\neq j\}} v_2(H_{n_i}, \mathcal{I}^{(j)}) \leq \varepsilon_j$. □

We will also need the analogue of the above in the case where $S = 2^{<\mathbb{N}}$. We omit the proof since it is an easy modification of the one of Proposition 7.

Proposition 9. *Let $(H_s)_{s\in 2^{<\mathbb{N}}}$ be a family of functions in V_2^0 such that for every $\sigma \in 2^{\mathbb{N}}$, $\lim_n \|H_{\sigma|n}\|_\infty = 0$. Then for every family $(\varepsilon_s)_{s\in 2^{<\mathbb{N}}}$ of positive real numbers, there exists a dyadic subtree $(t_s)_{s\in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ such that $(H_{t_s})_{s\in 2^{<\mathbb{N}}}$ is $(\varepsilon_s)_{s\in 2^{<\mathbb{N}}}$-biorthogonal.*

2.3.2. *Estimations on biorthogonal sequences.* In the next two lemmas and proposition, S stands for a countable set and $(H_i)_{i\in S}$ is an $(\varepsilon_i)_{i\in S}-$biorthogonal family in V_2^0 such that $\sum_{i\in S}\varepsilon_i = \varepsilon < \infty$.

Lemma 10. *Let $\mathcal{I} \in \mathcal{A}$, $F \subseteq S$ be finite and $(\lambda_i)_{i\in F}$ be a sequence of real numbers. Also let $(\mathcal{I}^{(i)})_{i\in S}$ be a disjoint partition of $\mathcal{I}$ such that for every $j \in S$, $\sum_{\{i\in S: i\neq j\}} v_2(H_i, \mathcal{I}^{(j)}) \le \varepsilon_j$ (see Definition 6). Then*

(i) *For every $i \in S$, $v_2(H_i, \mathcal{I}\setminus\mathcal{I}^{(i)}) < \varepsilon$.*
(ii) *For every $j \notin F$, $v_2(\sum_{i\in F}\lambda_i H_i, \mathcal{I}^{(j)}) \le \max_{i\in F}|\lambda_i|\varepsilon_j$.*
(iii) *For every $j \in F$, $v_2(\sum_{i\in F}\lambda_i H_i, \mathcal{I}^{(j)}) \le |\lambda_j| v_2(H_j, \mathcal{I}^{(j)}) + \max_{i\in F}|\lambda_i|\varepsilon_j$.*
(iv) *For every $j \in F$, $v_2(\sum_{i\in F}\lambda_i H_i, \mathcal{I}^{(j)}) \ge \Big| |\lambda_j| v_2(H_j, \mathcal{I}^{(j)}) - \max_{i\in F}|\lambda_i|\varepsilon_j \Big|$.*

Proof. (i) Let $i \in S$. Then

$$v_2(H_i, \mathcal{I}\setminus\mathcal{I}^{(i)}) = v_2(H_i, \bigcup_{j\neq i}\mathcal{I}^{(j)}) \le \sum_{j\neq i} v_2(H_i, \mathcal{I}^{(j)}) \le \sum_{j\neq i}\varepsilon_j < \varepsilon.$$

(ii) Let $j \notin F$. Then

$$v_2(\sum_{i\in F}\lambda_i H_i, \mathcal{I}^{(j)}) \le \sum_{i\in F} v_2(\lambda_i H_i, \mathcal{I}^{(j)}) = \sum_{i\in F}|\lambda_i| v_2(H_i, \mathcal{I}^{(j)}) \le \max_{i\in F}|\lambda_i|\varepsilon_j.$$

(iii) Let $j \in F$. Then using (ii) we get that

$$\begin{aligned} v_2(\sum_{i\in F}\lambda_i H_i, \mathcal{I}^{(j)}) &\le v_2(\lambda_j H_j, \mathcal{I}^{(j)}) + \sum_{i\neq j} v_2(\lambda_i H_i, \mathcal{I}^{(j)}) \\ &\le |\lambda_j| v_2(H_j, \mathcal{I}^{(j)}) + \max_{i\in F}|\lambda_i|\varepsilon_j. \end{aligned}$$

(iv) Since $v_2(\sum_{i\in F}\lambda_i H_i, \mathcal{I}^{(j)}) \ge |v_2(\lambda_j H_j, \mathcal{I}^{(j)}) - \sum_{i\neq j} v_2(\lambda_i H_i, \mathcal{I}^{(j)})|$, the proof is similar to that of (ii). □

Lemma 11. *Suppose that there is $M > 0$ such that $\|H_i\|_{V_2} \le M$, for all $i \in S$. Let $\mathcal{I} \in \mathcal{A}$, $F \subseteq G \subseteq S$ be finite and $(\lambda_i)_{i\in G}$ be a sequence of real numbers. Also let $(\mathcal{I}^{(i)})_{i\in S}$ be a disjoint partition of $\mathcal{I}$ such that for every $j \in S$, $\sum_{\{i\in S: i\neq j\}} v_2(H_i, \mathcal{I}^{(j)}) \le \varepsilon_j$. Then the following are satisfied.*

(i) $v_2^2\Big(\sum_{i\in F}\lambda_i H_i, \mathcal{I}\Big) \le \sum_{i\in F}|\lambda_i|^2 v_2^2(H_i, \mathcal{I}^{(i)}) + \max_{i\in F}|\lambda_i|^2(2M+\varepsilon)\varepsilon$.
(ii) $v_2^2\Big(\sum_{i\in G}\lambda_i H_i, \mathcal{I}\Big) > \sum_{i\in F}|\lambda_i|^2 v_2^2(H_i, \mathcal{I}^{(i)}) - \max_{i\in G}|\lambda_i|^2 2M\varepsilon$.
(iii) $v_2^2\Big(\sum_{i\in F}\lambda_i H_i, \mathcal{I}\Big) \le v_2^2\Big(\sum_{i\in G}\lambda_i H_i, \mathcal{I}\Big) + \max_{i\in G}|\lambda_i|^2(4M+\varepsilon)\varepsilon$.

Proof. (i) By (ii) and (iii) of Lemma 10, we have that

$$v_2^2\Big(\sum_{i\in F}\lambda_i H_i,\mathcal{I}\Big)=\sum_{j\in F}v_2^2\Big(\sum_{i\in F}\lambda_i H_i,\mathcal{I}^{(j)}\Big)+\sum_{j\in S\setminus F}v_2^2\Big(\sum_{i\in F}\lambda_i H_i,\mathcal{I}^{(j)}\Big)$$
$$\leq\sum_{j\in F}\Big(|\lambda_j|v_2(H_j,\mathcal{I}^{(j)})+\max_{i\in F}|\lambda_i|\varepsilon_j\Big)^2+\sum_{j\in S\setminus F}\max_{i\in F}|\lambda_i|^2\varepsilon_j^2$$
$$\leq\sum_{j\in F}|\lambda_j|^2v_2^2(H_j,\mathcal{I}^{(j)})+\max_{j\in F}|\lambda_j|^2(2M+\varepsilon)\varepsilon.$$

(ii) Using (iv) of Lemma 10, we obtain that

$$v_2^2\Big(\sum_{i\in G}\lambda_i H_i,\mathcal{I}\Big)\geq v_2^2(\sum_{i\in G}\lambda_i H_i,\bigcup_{j\in F}\mathcal{I}^{(j)})=\sum_{j\in F}v_2^2(\sum_{i\in G}\lambda_i H_i,\mathcal{I}^{(j)})$$
$$\geq\sum_{j\in F}\Big||\lambda_j|v_2(H_j,\mathcal{I}^{(j)})-\max_{i\in G}|\lambda_i|\varepsilon_j\Big|^2\geq\sum_{j\in F}|\lambda_j|^2v_2^2(H_j,\mathcal{I}^{(j)})-\max_{i\in G}|\lambda_i|^2 2M\varepsilon.$$

Finally (iii) follows easily from (i) and (ii). □

Proposition 12. *Let $M>\theta>2\varepsilon>0$ and suppose that $\theta<\|H_i\|_{V_2}\leq M$, for all $i\in S$. Then $(H_i)_{i\in S}$ is an unconditional family.*

Proof. Let $F\subseteq G$ be finite subsets of S and let $|\lambda_{i_0}|=\max_{i\in G}|\lambda_i|$. By (iii) of Lemma 11, we easily get that

$$\Big\|\sum_{i\in F}\lambda_i H_i\Big\|_{V_2}^2\leq\Big\|\sum_{i\in G}\lambda_i H_i\Big\|_{V_2}^2+|\lambda_{i_0}|^2(4M+\varepsilon)\varepsilon. \tag{8}$$

Let $\mathcal{I}_0\in\mathcal{A}$ such that $v_2(H_{i_0},\mathcal{I}_0)>\theta$. Since $(H_i)_{i\in S}$ is $(\varepsilon_i)_{i\in S}-$biorthogonal, we get that there is a disjoint partition $(\mathcal{I}^{(i)})_{i\in S}$ of $\mathcal{I}$ such that for every $j\in S$, $\sum_{\{i\in S:i\neq j\}}v_2(H_i,\mathcal{I}^{(j)})\leq\varepsilon_j$. Then

$$v_2(H_{i_0},\mathcal{I}_0^{(i_0)})\geq v_2(H_{i_0},\mathcal{I}_0)-v_2(H_{i_0},\mathcal{I}\setminus\mathcal{I}_0^{(i_0)})>\theta-\varepsilon. \tag{9}$$

Moreover by (ii) of Lemma 10, we have that

$$\Big|v_2(\lambda_{i_0}H_{i_0},\mathcal{I}_0^{(i_0)})-v_2\Big(\sum_{i\in G}\lambda_i H_i,\mathcal{I}_0^{(i_0)}\Big)\Big|\leq v_2\Big(\sum_{i\in G,i\neq i_0}\lambda_i H_i,\mathcal{I}_0^{(i_0)}\Big)\leq|\lambda_{i_0}|\varepsilon,$$

and so

$$|\lambda_{i_0}|v_2(H_{i_0},\mathcal{I}_0^{(i_0)})\leq v_2\Big(\sum_{i\in G}\lambda_i H_i,\mathcal{I}_0^{(i_0)}\Big)+|\lambda_{i_0}|\varepsilon\leq\Big\|\sum_{i\in G}\lambda_i H_i\Big\|_{V_2}+|\lambda_{i_0}|\varepsilon. \tag{10}$$

By (9) and (10), we get that

$$|\lambda_{i_0}|\leq\frac{1}{\theta-2\varepsilon}\Big\|\sum_{i\in G}\lambda_i H_i\Big\|_{V_2}.$$

Hence by (8), we have that

$$\Big\|\sum_{i\in F}\lambda_i H_i\Big\|_{V_2}\leq\Big(1+\frac{(4M+\varepsilon)\varepsilon}{(\theta-2\varepsilon)^2}\Big)^{1/2}\Big\|\sum_{i\in G}\lambda_i H_i\Big\|_{V_2}$$

and the proof of the proposition is complete. □

3. Hausdorff measures associated to functions of bounded quadratic variation

The aim of this section is to introduce and study the fundamental properties of the measure μ_f corresponding to a function $f \in V_2$. It is divided into three subsections. The first includes the definition and initial properties of the measure μ_f. The second is mainly devoted to the proof of Theorem 20, and the last contains a study of the points of non-differentiability of a function $f \in V_2$.

3.1. **Definition and elementary properties.** For every $f \in V_2$ and for every interval I of $[0,1]$ we set

$$\widetilde{\mu}_f(I) = \inf_{\delta>0} \widetilde{\mu}_{f,\delta}(I),$$

where for each $\delta > 0$, $\widetilde{\mu}_{f,\delta}(I) = \sup\{v_2^2(f,\mathcal{P}) : \mathcal{P} \subseteq I \text{ and } \|\mathcal{P}\|_{\max} < \delta\}$.

We also define the function $\widetilde{F}_f : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{F}_f(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \widetilde{\mu}_f[0,x], & \text{if } 0 < x < 1, \\ \widetilde{\mu}_f[0,1], & \text{if } x \geq 1. \end{cases}$$

Notice that $\widetilde{F}_f$ is a non-negative increasing function on $\mathbb{R}$ and so taking the upper envelope $F_f(x) = \widetilde{F}_f(x^+)$ of $\widetilde{F}_f$, we have that F_f is in addition a right continuous function. Moreover since $\lim_{x\to-\infty} F_f(x) = 0$ and $\lim_{x\to+\infty} F_f(x) = \widetilde{\mu}_f[0,1]$, F_f is the distribution function of a finite positive Borel measure on $\mathbb{R}$ which we will denote by μ_f. Notice that $\mu_f = 0$ if and only if $f \in V_2^0$ and also that $\mu_f \leq \|f\|_{V_2}$. Actually, it is easy to see that defining for any function $f : [0,1] \to \mathbb{R}$, the measure μ_f as above, then μ_f is finite if and only if $f \in V_2$. Since $\mu_f(-\infty,0) = \mu_f(1,+\infty) = 0$, in the sequel we will identify μ_f with its restriction on $[0,1]$.

Remark 1. The definition of the measure μ_f is generalized as follows. Let (X,ρ) be a metric space and $f : X \to \mathbb{R}$ be a real-valued function. Following C.A. Rogers in [20], for a function $h : [0,+\infty] \to [0,+\infty]$ satisfying the conditions of p.50 of [20] and every open subset G of X, we define the premeasure $h_f(G) = h(diam f[G])$. Next following Method II (see [20]), we induce the measure μ_f^h defined on the Borel subsets of X. It is easy to see that in the case of $f \in V_2$ and for $h(x) = x^2$, the measure μ_f^h coincides with the measure μ_f defined above. Although the measures μ_f^h are not mentioned as Hausdorff measures in the literature, their definition and geometrical properties motivate us to include them in the latter class. It seems interesting to examine the regularity conditions that a function $f : X \to \mathbb{R}$ must satisfy so that the corresponding measure μ_f^h is a finite Borel measure. For example, this easily yields that f has at most countably many discontinuities. Hence if X is a Polish space, then f is a Baire-1 function.

The following two lemmas are easily proved. The second one is essentially contained in [1] (Lemma 18).

Lemma 13. *Let $0 \leq a < x < b \leq 1$ such that f is continuous at x. Then for every $I = I_1 \cup I_2$, where I is an interval with endpoints a, b and I_1, I_2 are intervals with $\sup I_1 = x = \inf I_2$, we have that*

$$\widetilde{\mu}_f(I) = \widetilde{\mu}_f(I_1) + \widetilde{\mu}_f(I_2).$$

Lemma 14. *(a) For every $x \in (0,1]$ and every $\varepsilon > 0$ there exists $0 < \delta < x$ such that $\sup\{v_2^2(f,\mathcal{P}) : \mathcal{P} \subseteq [x-\delta, x)\} \leq \varepsilon$. In particular $\widetilde{\mu}_f[x-\delta,x) \leq \varepsilon$.*

(b) Similarly for every $x \in [0,1)$ and every $\varepsilon > 0$ there exists $0 < \delta < 1-x$ such that $\sup\{v_2^2(f,\mathcal{P}) : \mathcal{P} \subseteq (x, x+\delta]\} \leq \varepsilon$. In particular $\widetilde{\mu}_f(x,x+\delta] \leq \varepsilon$.

Proposition 15. (a) *$D_f = D_{F_f} = D_{\widetilde{F}_f}$ and so f is continuous if and only if μ_f is continuous. Also for $x \in [0,1] \setminus D_f$, $\widetilde{F}_f(x) = F_f(x)$.*

(b) *If f is continuous at x, then $\mu_f[0,x] = \widetilde{\mu}_f[0,x]$ and $\mu_f[x,1] = \widetilde{\mu}_f[x,1]$.*

(c) *For all continuity points $x < y$ of f, $\mu_f[x,y] = \widetilde{\mu}_f[x,y]$.*

(d) *For every open interval (α,β) of $[0,1]$, $\mu_f(\alpha,\beta) = \widetilde{\mu}_f(\alpha,\beta)$.*

Proof. (a) By the monotonicity of $\widetilde{F}_f$ we have that for all $x_0 \in \mathbb{R}$, $\widetilde{F}_f(x_0^+) = F_f(x_0^+)$ and $\widetilde{F}_f(x_0^-) = F_f(x_0^-)$. Therefore $D_{F_f} = D_{\widetilde{F}_f}$ and for every $x_0 \in [0,1] \setminus D_{\widetilde{F}_f}$, $\widetilde{F}_f(x_0) = F_f(x_0)$. It remains to show that $D_f = D_{\widetilde{F}_f}$. Let $x_0 \in [0,1]$ be a continuity point of f. If $x_0 < 1$, by Lemma 13 we have that for every $x_0 < y \leq 1$, $\widetilde{F}_f(y) - \widetilde{F}_f(x_0) = \widetilde{\mu}_f[0,y] - \widetilde{\mu}_f[0,x_0] = \widetilde{\mu}_f(x_0,y]$ and so by Lemma 14(b), $\widetilde{F}_f(x_0^+) = \widetilde{F}_f(x_0)$. If $0 < x_0$, again by Lemma 13, for every $0 \leq y < x_0$ such that y is a continuity point of f we have that $\widetilde{F}_f(x_0) - \widetilde{F}_f(y) = \widetilde{\mu}_f(y,x_0]$. Since $[0,1] \setminus D_f$ is dense in $[0,1]$, by Lemma 14(b) we get that $\widetilde{F}_f$ is continuous at x_0. Conversely suppose that $x_0 \in D_f$. Then either $f(x_0^+) \neq f(x_0)$ or $f(x_0^-) \neq f(x_0)$. Suppose that $f(x_0^+) \neq f(x_0)$ (the other case is similarly treated). Then it is easy to see that for every $0 < \delta < 1 - x_0$, $\widetilde{\mu}_f[x_0, x_0+\delta] \geq |f(x_0^+) - f(x_0)|^2$ and using Lemma 13, we obtain that

$$\begin{aligned}\widetilde{F}_f(x_0^+) = \lim_{\delta\to 0} \widetilde{F}_f(x_0+\delta) &\geq \lim_{\delta\to 0}(\widetilde{\mu}_f[0,x_0] + \widetilde{\mu}_f[x_0,x_0+\delta]) \\ &\geq \widetilde{F}_f(x_0) + |f(x_0^+) - f(x_0)|^2 > \widetilde{F}_f(x_0).\end{aligned}$$

Hence $x_0 \in D_{\widetilde{F}_f}$.

(b) If f is continuous at x, then by (a) we have that $\widetilde{F}_f(x) = F_f(x)$ or $\mu_f[0,x] = \widetilde{\mu}_f[0,x]$. Again by (a) we have that F_f is continuous at x and so $\mu_f(\{x\}) = 0$. Hence $\mu_f[x,1] = \mu_f[0,1] - \mu_f[0,x) = \mu_f[0,1] - \mu_f[0,x] = \widetilde{\mu}_f[0,1] - \widetilde{\mu}_f[0,x] = \widetilde{\mu}_f[x,1]$, where the last equality follows from Lemma 13.

(c) Indeed, using (b) and Lemma 13, $\mu_f[x,y] = \mu_f[0,y] - \mu_f[0,x) = \mu_f[0,y] - \mu_f[0,x] = \widetilde{\mu}_f[0,y] - \widetilde{\mu}_f[0,x] = \widetilde{\mu}_f[x,y]$.

(d) Let $a < a_n < b_n < b$ such that a_n, b_n are continuity points of f, $\lim_n a_n = a$ and $\lim_n b_n = b$. By Lemma 14, $\lim_n \widetilde{\mu}_f(a,a_n] = \lim_n \widetilde{\mu}_f[b_n,b) = 0$ and by Lemma 13, $\widetilde{\mu}_f(a,b) = \widetilde{\mu}_f(a,a_n] + \widetilde{\mu}_f[a_n,b_n] + \widetilde{\mu}_f[b_n,b)$. Hence $\widetilde{\mu}_f(a,b) = \lim_n \widetilde{\mu}_f[a_n,b_n] = \lim_n \mu_f[a_n,b_n] = \mu_f(a,b)$. □

In order to proceed, we need the following notation. For every $f \in V_2$ and $x_0 \in [0,1]$, let

$$\tau_f(x_0) = \max\{|f(x_0^+) - f(x_0^-)|^2, |f(x_0^+) - f(x_0)|^2 + |f(x_0^-) - f(x_0)|^2\},$$

where $f(0^-) = f(0)$ and $f(1^+) = f(1)$. Moreover, for every $\delta > 0$, let

$$\tau_{f,\delta}(x_0) = \sup\{|f(y) - f(x_0)|^2 + |f(x_0) - f(z)|^2, |f(y) - f(z)|^2\},$$

where the max is taken for all $0 \leq y \leq x_0 \leq z \leq 1$ with $|y - z| \leq \delta$. Clearly $\lim_{\delta\to 0} \tau_{f,\delta}(x_0) = \tau_f(x_0)$ and f is continuous at x_0 if and only if $\tau_f(x_0) = 0$.

The quantity $\tau_f(x_0)$ is defined (with different notation) in [19], p.1464. An equivalent definition was introduced earlier by L.C. Young [23] in order to characterise the class $\mathcal{W}_2^*$. The proof of the next proposition uses similar arguments to those in [19].

Proposition 16. *For every $f \in V_2$ and $x_0 \in [0,1]$, $\mu_f(\{x_0\}) = \tau_f(x_0)$. Therefore $\mu_f^d[0,1] = \sum_{x\in D_f} \tau_f(x)$, where μ_f^d denotes the discrete part of μ_f.*

Remark 2. Under the current terminology, it follows that $\mathcal{W}_2^* = \{f \in V_2 : \mu_f = \mu_f^d\}$.

Lemma 17. *Let $f_1, f_2 \in V_2$ and τ be a positive Borel measure on $[0,1]$ such that $\mu_{f_1} \perp \tau$ and $\tau \leq \mu_{f_2}$. Then $\tau \leq \mu_{f_2 - f_1}$.*

Proof. Let $F = f_1 - f_2$. It suffices to show that $\mu_F(V) \geq \tau(V)$, for every open subset V of $[0,1]$. So fix an open subset V of $[0,1]$ and let $\varepsilon > 0$. Since $\mu_{f_1} \perp \tau$, there is $V_\varepsilon \subseteq V$ such that $V_\varepsilon = \bigcup_{i=1}^k I_i$, where $(I_i)_{i=1}^k$ is a finite family of pairwise disjoint open intervals of $[0,1]$, $\mu_{f_1}(V_\varepsilon) < \varepsilon$ and $\tau(V_\varepsilon) > \tau(V) - \varepsilon$. Let $\delta > 0$ be such that $|\widetilde{\mu}_{f_1,\delta}(V_\varepsilon) - \mu_{f_1}(V_\varepsilon)| < \varepsilon$, $|\widetilde{\mu}_{f_2,\delta}(V_\varepsilon) - \mu_{f_2}(V_\varepsilon)| < \varepsilon$ and $|\widetilde{\mu}_{F,\delta}(V_\varepsilon) - \mu_F(V_\varepsilon)| < \varepsilon$. Also for all $0 \leq i \leq k$, let $\mathcal{P}_i \subseteq I_i$ with $|\mathcal{P}_i| < \delta$ and $\Big|\sum_{i=1}^k \upsilon_2^2(f_2, \mathcal{P}_i) - \mu_{f_2}(V_\varepsilon)\Big| < \varepsilon$. Hence $\sum_{i=1}^k \upsilon_2^2(f_1, \mathcal{P}_i) < 2\varepsilon$ and

$$\begin{aligned}
\mu_F(V) \geq \mu_F(V_\varepsilon) > \widetilde{\mu}_{F,\delta}(V_\varepsilon) - \varepsilon &\geq \sum_{i=1}^k \upsilon_2^2(F, \mathcal{P}_i) - \varepsilon \\
&\geq \sum_{i=1}^k [\upsilon_2(f_2, \mathcal{P}_i) - \upsilon_2(f_1, \mathcal{P}_i)]^2 - \varepsilon \\
&\geq \mu_{f_2}(V_\varepsilon) - 2\Big(\sum_{i=1}^k \upsilon_2^2(f_2, \mathcal{P}_i)\Big)^{1/2}\Big(\sum_{i=1}^k \upsilon_2^2(f_1, \mathcal{P}_i)\Big)^{1/2} - 2\varepsilon \\
&\geq \mu_{f_2}(V_\varepsilon) - 2\|f_2\|_{V_2}\sqrt{2\varepsilon} - 2\varepsilon \geq \tau(V_\varepsilon) - 2\|f_2\|_{V_2}\sqrt{2\varepsilon} - 2\varepsilon \\
&\geq \tau(V) - 2\|f_2\|_{V_2}\sqrt{2\varepsilon} - 3\varepsilon.
\end{aligned}$$

Hence, letting $\varepsilon \to 0$, we get that $\mu_F(V) \geq \tau(V)$ and the proof is complete. □

3.2. **The correspondence between the functions of V_2 and the measures on the unit interval.** By $\mathcal{M}[0,1]$ we denote the space of all Borel measures on $[0,1]$ endowed by the norm $\|\mu\| = \sup\{|\mu(B)| : B$ is a Borel subset of $[0,1]\}$. The positive cone of $\mathcal{M}[0,1]$ will be denoted by $\mathcal{M}^+[0,1]$. Recall that for every $\mu \in \mathcal{M}^+[0,1]$, $\|\mu\| = \mu[0,1]$. In this subsection we study the properties of the function $\Phi : V_2 \to \mathcal{M}[0,1]$, defined by $\Phi(f) = \mu_f$, for all $f \in V_2$. We start with the following easily established proposition.

Proposition 18. *The following hold.*

(i) *For every $f_1, f_2 \in V_2$, $\mu_{f_1+f_2} \leq 2\mu_{f_1} + 2\mu_{f_2}$.*
(ii) *For every $f \in V_2$ and $\lambda \in \mathbb{R}$, $\mu_{(\lambda f)} = \lambda^2 \mu_f$.*
(iii) *For every $f \in V_2$ and every $g \in V_2^0$, $\mu_{f+g} = \mu_f$.*
(iv) *The map $\Phi : V_2 \to \mathcal{M}[0,1]$, defined by $\Phi(f) = \mu_f$, is locally Lipschitz. More precisely,*

$$\|\mu_{f_1} - \mu_{f_2}\| \leq (\|f_1\|_{V_2} + \|f_2\|_{V_2})\|f_1 - f_2\|_{V_2}.$$

Remark 3. One could not expect that Φ is a linear map as its range is a subset of the positive cone $\mathcal{M}^+[0,1]$ of $\mathcal{M}[0,1]$ (next we shall show that Φ is actually onto $\mathcal{M}^+[0,1]$). However there are special cases where the additivity of the function Φ is established. For example it can be shown that for every pair $f_1, f_2 \in V_2$ with $\mu_{f_1} \perp \mu_{f_2}$ we have that $\mu_{f_1+f_2} = \mu_{f_1} + \mu_{f_2}$. Finally the map Φ is not w*-w* continuous. For example let $f \in V_2$ such that $f = \sum_n g_n$, where $(g_n)_n$ is a sequence in V_2^0. Then setting $f_n = \sum_{k\geq n} g_k$, we have that $(f_n)_n$ pointwise converges to 0; however, by (iii) of Proposition 18, $\mu_{f_n} = \mu_f$, for all $n \in \mathbb{N}$.

Lemma 19. *Let μ be a finite positive discrete measure on $[0,1]$. Then there is $h \in V_2^d$ such that $\mu_h = \mu$.*

Proof. Let $S = \{t_n\}_n$ be an enumeration of the support of μ. Then $\mu^d = \sum_n \lambda_n \delta_{t_n}$, where $\lambda_n = \mu^d(\{t_n\})$. We define $h = \sum_n \sqrt{\lambda_n}\chi_{t_n}$ and let $h_n = \sum_{k=1}^n \sqrt{\lambda_k}\chi_{t_k}$. Then $h \in V_2^d$, $(h_n)_n$ $\|\cdot\|_{V_2}$-converges to h and so the $(\mu_{h_n})_n$-norm converges to μ_h in $\mathcal{M}[0,1]$. Since $\mu_{h_n} = \sum_{k=1}^n \lambda_k\delta_{t_k}$, $\mu_h = \sum_n \lambda_n\delta_{t_n} = \mu$. $\square$

Theorem 20. *For every finite positive Borel measure μ on $[0,1]$ there is $f \in V_2$ such that $\mu = \mu_f$.*

Proof. Since $\mu = \mu^c + \mu^d$, where μ^c is the continuous and μ^d is the discrete part of μ, by Lemma 19, it suffices to find $f \in V_2 \cap C[0,1]$ such that $\mu^c = \mu_f$ (it is then easy to see that $\mu_{f+h} = \mu$, where $h \in V_2^d$ satisfies that $\mu_h = \mu^d$). Hence we suppose for the sequel that μ is continuous.

For an interval $I = [a,b]$ in $[0,1]$, let $F_I : I \to \mathbb{R}$ be defined by $F_I(x) = \mu[a,x]$, for all $x \in I$. Then F_I is continuous, $F_I(a) = 0$ and $F_I(b) = \mu(I)$. Hence we may choose $\xi_I \in (a,b)$ such that $F_I(\xi_I) = \mu[a,\xi_I] = \mu(I)/2$. Now consider the function $G_I : I \to \mathbb{R}$ defined by $G_I(x) = F_I(x)$ if $a \leq x \leq \xi_I$ and by $G_I(x) = \mu(I) - F_I(x)$ if $\xi_I \leq x \leq b$. Clearly $\|G_I\|_\infty = \mu(I)/2$.

Claim 1. For every interval I of $[0,1]$, let $H_I = \sqrt{G_I}$.

(i) For every $x < y \in I$, $|H_I(y) - H_I(x)|^2 \leq \mu(x,y]$.

(ii) For all intervals I_1, I_2 in $[0,1]$ such that $\max I_1 \leq \min I_2$ and for all $x_1 \in I_1$, $x_2 \in I_2$, $|H_{I_2}(x_2) - H_{I_1}(x_1)|^2 \leq \mu(x_1,x_2]$.

Proof. (i) Notice that for $\alpha, \beta > 0$, $|\alpha - \beta|^2 \leq |\alpha^2 - \beta^2|$. Hence $|H_I(y) - H_I(x)|^2 \leq |G_I(y) - G_I(x)|$ and so it suffices to show that $|G_I(y) - G_I(x)| \leq \mu(x,y]$. By the definition of G_I, we immediately get that for $x < y \leq \xi_I$ or for $\xi_I \leq x < y$, $|G_I(y) - G_I(x)| = \mu(x,y]$. In the case $x < \xi_I < y$, we may assume that $G_I(x) < G_I(y)$ (the other case is similarly treated). Then there is $x < z < \xi_I$ with $G_I(z) = G_I(y)$ and so $|G_I(y) - G_I(x)| = |G_I(z) - G_I(x)| = \mu(x,z] < \mu(x,y]$.
(ii) As above it suffices to show that $|G_{I_2}(x_2) - G_{I_1}(x_1)| \leq \mu(x_1,x_2]$. Let b_1 be the right end of I_1 and a_2 be the left end of I_2. Then $G_{I_1}(b_1) = G_{I_2}(a_2) = 0$ and by the proof of part (i), $|G_{I_2}(x_2) - G_{I_1}(x_1)| = |G_{I_2}(x_2) - G_{I_2}(a_2) - G_{I_1}(x_1) + G_{I_1}(b_1)| \leq |G_{I_2}(x_2) - G_{I_2}(a_2)| + |G_{I_1}(b_1) - G_{I_1}(x_1)| \leq \mu(a_2,x_2] + \mu(x_1,b_1] \leq \mu(x_1,x_2]$. $\square$

Claim 2. Let $H_n = \sum_{i=1}^{2^{n+1}} H_{I_i^n}$, where $I_i^n = [\frac{i-1}{2^n}, \frac{i+1}{2^n}]$, $n \geq 0$, $1 \leq i \leq 2^n$.

(i) $H_n \in V_2^0$ and $\|H_n\|_\infty = \sqrt{2^{-(n+1)}\mu[0,1]}$.

(ii) For all $\mathcal{I} \in \mathcal{A}$, $v_2^2(H_n, \mathcal{I}) \leq \mu(\bigcup \mathcal{I})$. Therefore $\|H_n\|_{V_2^0} \leq \sqrt{\mu[0,1]}$.

(iii) Let $\mathcal{P}_n = \{i/2^n\}_{i=0}^{2^n} \cup \{\xi_{I_i^n}\}_{i=0}^{2^n}$. Then $v_2^2(H_n, \mathcal{P}_n \cap [0,t]) = \mu[0,t]$, for all $t \in \mathcal{P}_n$.

Proof. This is straightforward by Claim 1. □

Claim 3. There exist a subsequence $(H_{n_i})_i$ of $(H_n)_n$ and a strictly decreasing null sequence $(\delta_i)_{i=0}^{\infty}$, with $\delta_0 = 0$, satisfying the following.

(i) $\sum_i \|H_{n_i}\|_\infty < \infty$.
(ii) $(H_{n_i})_i$ is $((2^{-i})_{i=1}^{\infty}, (\delta_i)_{i=0}^{\infty})$-biorthogonal.
(iii) For every $i \in \mathbb{N}$, $\delta_i < \|\mathcal{P}_{n_i}\|_{\min} \leq \|\mathcal{P}_{n_i}\|_{\max} \leq \delta_{i-1}$.

Proof. We inductively define a strictly increasing sequence $n_1 < n_2 < \dots$ in $\mathbb{N}$ and a strictly decreasing sequence $1 = \delta_0 > \delta_1 > \delta_2 > \dots > 0$ such that for every $k \geq 1$, the following hold.

(1) For all $1 \leq i \leq k$, $\|H_{n_i}\|_\infty < 2^{-i}$.
(2) The finite sequence $(H_{n_i})_{i=1}^{k}$ is $((2^{-i})_{i=1}^{k}, (\delta_i)_{i=0}^{k-1})$-biorthogonal.
(3) For all $1 \leq i \leq k$, $\delta_i < \|\mathcal{P}_{n_i}\|_{\min} \leq \|\mathcal{P}_{n_i}\|_{\max} \leq \delta_{i-1}$.
(4) For every $\mathcal{I} \in \mathcal{A}$ with $\|\mathcal{I}\|_{\max} \leq \delta_k$, $\sum_{i=1}^{k} v_2(H_i, \mathcal{I}) < 2^{-(k+1)}$.

The general inductive step of the construction goes as follows. Suppose that for some $k \geq 1$, we have chosen $(n_i)_{i \leq k}$ and $(\delta_i)_{i \leq k}$ satisfying the above. Applying Lemma 8 for $\varepsilon_{k+1} = 2^{-(k+1)}$, we have that there exists $\varepsilon > 0$ such that for every $H \in V_2^0$ with $\|H\|_{V_2^0} < \varepsilon$, the sequence $(H_{n_1}, \dots, H_{n_k}, H)$ is $((2^{-i})_{i=1}^{k+1}, (\delta_i)_{i=0}^{k})$-biorthogonal. By Claim 2, we have that $\lim \|H_n\|_\infty = 0$ and $\lim \|\mathcal{P}_n\|_{\max} = 0$. Hence we may choose $n_{k+1} > n_k$ such that $\|H_{n_{k+1}}\|_\infty < \min\{2^{-k+1}, \varepsilon\}$ and $\|\mathcal{P}_{n_{k+1}}\|_{\max} \leq \delta_k$. Finally we choose $0 < \delta_{k+1} < \|\mathcal{P}_{n_{k+1}}\|_{\min}$ such that for every $\mathcal{I} \in \mathcal{A}$ with $\|\mathcal{I}\|_{\max} \leq \delta_{k+1}$, $\sum_{i=1}^{k+1} v_2(H_{n_i}, \mathcal{I}) < 2^{-(k+2)}$ and the proof of the inductive step of the construction is complete. □

Claim 4. Let $f = \sum_i H_{n_i}$. Then $f \in V_2 \cap C[0,1]$ and $\mu_f = \mu$.

Proof. Since $\sum_i \|H_{n_i}\|_\infty < \infty$ and H_{n_i} are continuous we have that $f \in C[0,1]$. By (ii) of Claim 2, $(H_n)_n$ is a bounded (by $M = \sqrt{\mu[0,1]}$) sequence in V_2^0. Moreover by Claim 3, $(H_{n_i})_i$ is biorthogonal and so by Lemma 11 and (ii) of Claim 2, we have that for every $\mathcal{I} \in \mathcal{A}$,

$$v_2^2(f, \mathcal{I}) \leq \sum_i \mu\big(\bigcup \mathcal{I}^{(i)}\big) + 2\sqrt{\mu[0,1]} + 1 \leq (\sqrt{\mu[0,1]} + 1)^2$$

and therefore $f \in V_2$. To prove that $\mu_f = \mu$, let $D = \bigcup_i D_{n_i}$ where for all $i \in \mathbb{N}$, $D_{n_i} = \{m2^{-n_i} :\ 0 \leq m \leq 2^{n_i}\}$. Since D is dense in $[0,1]$, it suffices to show that $\mu_f[0,t] = \mu[0,t]$, for all $t \in D$.

Fix i_0 and $0 \leq m_0 \leq 2^{n_{i_0}}$ and let $t = m_0/2^{n_{i_0}}$. By the definition of $(\mathcal{P}_n)_n$, we have that for all $j \geq i_0$, $t \in \mathcal{P}_{n_i}$, and so by (iii) of Claim 2,

$$\mu[0,t] = v_2^2(H_{n_j}, \mathcal{P}_{n_j} \cap [0,t]).$$

Hence by (iii) of Claim 3, for all $j \geq i_0$,

$$v_2(f, \mathcal{P}_{n_j} \cap [0,t]) = v_2(H_{n_j} + \sum_{i \neq j} H_{n_i}, \mathcal{P}_{n_j} \cap [0,t]) \geq \mu[0,t] - 2^{-j};$$

hence since $\lim_j \|\mathcal{P}_{n_j}\|_{\max} = 0$, $\mu_f[0,t] \geq \lim_j v_2^2(f, \mathcal{P}_{n_j} \cap [0,t]) \geq \mu[0,t]$.

It remains to show that $\mu_f[0,t] \leq \mu[0,t]$. Since $\lim \delta_k = 0$, we have that

$$\mu_f[0,t] = \lim_k \sup\{v_2^2(f,\mathcal{P}) : \mathcal{P} \subseteq [0,t], \|\mathcal{P}\|_{\max} < \delta_k\}. \tag{11}$$

Fix $k \geq 1$ and $\mathcal{P} \subseteq [0,t]$ such that $\|\mathcal{P}\|_{\max} \leq \delta_{k-1}$. Let $\mathcal{I} = \mathcal{I}_{\mathcal{P}}$ be the corresponding family of intervals with endpoints successive points of $\mathcal{P}$. Then $\mathcal{I} = \bigcup_{j \geq k} \mathcal{I}^{(j)}$, where $\mathcal{I}^{(j)} = \{I \in \mathcal{I} : \delta_j < |I| \leq \delta_{j-1}\}$, and

$$v_2^2(f,\mathcal{P}) = v_2^2(f,\mathcal{I}) = v_2^2(f, \bigcup_{j \geq k} \mathcal{I}^{(j)}) = \sum_{j \geq k} v_2^2(f,\mathcal{I}^{(j)}). \tag{12}$$

Moreover by (ii) of Claim 2,

$$v_2(f,\mathcal{I}^{(j)}) \leq v_2(H_{n_j} + \sum_{i \neq j} H_{n_i}, \mathcal{I}^{(j)}) \leq v_2(H_{n_j}, \mathcal{I}^{(j)}) + 2^{-j} \leq \sqrt{\mu(\bigcup \mathcal{I}^{(j)})} + 2^{-j}.$$

Therefore $v_2^2(f,\mathcal{I}^{(j)}) \leq \mu(\bigcup \mathcal{I}^{(j)}) + (2M+1)2^{-j}$, where $M = \sqrt{\mu[0,1]}$, and so by (12) we obtain that

$$\begin{aligned} v_2^2(f,\mathcal{P}) &\leq \sum_{j \geq k} \mu(\bigcup \mathcal{I}^{(j)}) + (2M+1) \sum_{j \geq k} 2^{-j} \\ &= \mu(\bigcup \mathcal{I}) + (2M+1)2^{-(k-1)} = \mu[0,t] + (2M+1)2^{-(k-1)}. \end{aligned}$$

Hence by (11) we get that $\mu_f[0,t] \leq \mu[0,t]$. □

By Claim 4 the proof of the theorem is complete. □

Remark 4. Let us present here a concrete example which illustrates the method of the proof of Theorem 20. Let λ be the Lebesgue measure on $[0,1]$ and let $(R_n)_n$ be defined by

$$R_n(t) = 2^{n/2} \int_0^t r_n(x)dx, \tag{13}$$

where $(r_n)_n$ is the sequence of Rademacher functions. As in Claims 1 and 2 of Theorem 20, it can be shown that

(i) $v_2^2(R_n,\mathcal{I}) \leq \lambda(\bigcup \mathcal{I})$,
(ii) for every $m \geq n$ and $1 \leq i \leq 2^n$, $v_2^2(R_n, \mathcal{P}_m \cap [0, \frac{i}{2^n}]) = \lambda[0, \frac{i}{2^n}]$,

where here $\mathcal{P}_n = \{\frac{i}{2^n} : 0 \leq i \leq 2^n\}$. Then as in Claim 3 we may show that there is a subsequence $(R_{n_i})_i$ of $(R_n)_n$ such that the sum $f = \sum_i R_{n_i}$ satisfies that $\mu_f = \lambda$. We note that, as has been stated in [15], the above-defined sequence $(R_n)_n$ contains subsequences equivalent to the c_0-basis. In the sequel (Corollary 32) we shall provide a proof of this statement.

3.3. On the points of non-differentiability of functions in V_2.

Lemma 21. *Let $f \in V_2$ and let $(\mathcal{P}_n)_n$ be a sequence of finite subsets of $[0,1]$ such that $\lim \|\mathcal{P}_n\|_{\max} = 0$ and $\lim v_2^2(f,\mathcal{P}_n) = \mu_f[0,1]$. Then for every sequence $(\mathcal{I}_n)_n$ in $\mathcal{A}$ such that $\mathcal{I}_n \subseteq \mathcal{I}_{\mathcal{P}_n}$ and $(v_2^2(f,\mathcal{I}_n))_n$ converges, we have that $(\mu_f(\bigcup \mathcal{I}_n))_n$ also converges and $\lim v_2^2(f,\mathcal{I}_n) = \lim \mu_f(\bigcup \mathcal{I}_n)$.*

Proof. Let $\alpha = \lim v_2^2(f,\mathcal{I}_n)$ and assume that $(\mu_f(\bigcup \mathcal{I}_n))_n$ does not converge to α. Then by passing to a subsequence, we may suppose that $\lim \mu_f(\bigcup \mathcal{I}_n) = \beta \neq \alpha$.

Let $\mathcal{J}_n = \mathcal{I}_{\mathcal{P}_n} \setminus \mathcal{I}_n$. Then

$$\lim v_2^2(f, \mathcal{J}_n) = \mu_f[0,1] - \alpha \text{ and } \lim \mu_f(\bigcup \mathcal{J}_n) = \mu_f[0,1] - \beta.$$

Since $\mathcal{I}_n$ and $\mathcal{J}_n$ consist of open intervals of $[0,1]$, we can choose $\mathcal{I}'_n \preceq \mathcal{I}_n$ and $\mathcal{J}'_n \preceq \mathcal{J}_n$ such that $|\mu_f(\bigcup \mathcal{I}_n) - v_2^2(f, \mathcal{I}'_n)| < 1/n$ and $|\mu_f(\bigcup \mathcal{J}_n) - v_2^2(f, \mathcal{J}'_n)| < 1/n$. Therefore we get that

$$\lim v_2^2(f, \mathcal{I}'_n) = \beta \text{ and } \lim v_2^2(f, \mathcal{J}'_n) = \mu_f[0,1] - \beta.$$

Since $\mathcal{I}_n, \mathcal{J}'_n$ are disjoint and $\lim \|\mathcal{I}_n\|_{\max} = \lim \|\mathcal{J}'_n\|_{\max} = 0$ we obtain that

$$\mu_f[0,1] \geq \lim v_2^2(f, \mathcal{I}_n \cup \mathcal{J}'_n) = \alpha + (\mu_f[0,1] - \beta),$$

which implies that $\beta \geq \alpha$. Similarly,

$$\mu_f[0,1] \geq \lim v_2^2(f, \mathcal{I}'_n \cup \mathcal{J}_n) = \beta + (\mu_f[0,1] - \alpha),$$

which gives that $\alpha \geq \beta$. Hence $\alpha = \beta$, which is a contradiction. □

Theorem 22. *Let $f \in V_2$. Then the set of all points $x \in [0,1]$ such that f is differentiable at x has μ_f-measure zero.*

Proof. Let $\mathcal{P}_n = \{0 = t_0^n < ... < t_{k_n}^n = 1\} \subseteq [0,1]$ such that $\lim \|\mathcal{P}_n\|_{\max} = 0$ and $\lim v_2^2(f, \mathcal{P}_n) = \mu_f([0,1])$. It suffices to show that for every $C > 0$, $\mu_f(A_C) = 0$, where

$$A_C = \{x \in [0,1] : \exists f'(x) \text{ and } |f'(x)| < C\}.$$

Fix $C > 0$ and for every $k \in \mathbb{N}$ let A_C^k be the set of all $x \in (0,1) \setminus \bigcup_n \mathcal{P}_n$ such that for every $y, z \in (x - 1/k, x + 1/k)$ with $0 \leq y < x < z \leq 1$, $|\frac{f(z)-f(y)}{z-y}| < C$.

Notice that

$$A_C \subseteq \Big(\bigcup_n \mathcal{P}_n \setminus D_f\Big) \cup \bigcup_{k=1}^{\infty} A_C^k.$$

By Proposition 16, we get that $\mu_f(\bigcup_n \mathcal{P}_n \setminus D_f) = 0$, and therefore it remains to show that for every $k \in \mathbb{N}$, $\mu_f(A_C^k) = 0$. To this end, fix $k \in \mathbb{N}$. Then for every $x \in A_C^k$ and $n \in \mathbb{N}$ there exists $0 \leq i \leq k_n - 1$ such that $x \in (t_i^n, t_{i+1}^n)$. Since $\lim \|\mathcal{P}_n\|_{\max} = 0$ there is n_0 such that for all $n \geq n_0$, $t_{i+1}^n - t_i^n < 1/k$. For $n \geq n_0$ let F_n be the set of all $0 \leq i \leq k_n - 1$ such that $A_C^k \cap (t_i^n, t_{i+1}^n) \neq \emptyset$ and let $\mathcal{I}_n = ((t_i^n, t_{i+1}^n))_{i \in F_n}$. Then $A_C^k \subseteq \bigcup \mathcal{I}_n$ and therefore

$$v_2^2(f, \mathcal{I}_n) = \sum_{i \in F_n} |f(t_{i+1}^n) - f(t_i^n)|^2 \leq C^2 \sum_{i \in F_n} |t_{i+1}^n - t_i^n|^2 \leq C^2 \max_{i \in F_n} |t_{i+1}^n - t_i^n|.$$

Hence $\lim v_2^2(f, \mathcal{I}_n) = 0$. By Lemma 21, $\lim \mu_f(\bigcup \mathcal{I}_n) = 0$ and so $\mu_f(A_C^k) = 0$. □

Corollary 23. *Let $f \in V_2 \cap C[0,1]$. If the set of all points $x \in [0,1]$ such that f is not differentiable at x is countable, then $f \in V_2^0$. Moreover if $f \in (V_2 \setminus V_2^0) \cap C[0,1]$, then the set of all non-differentiability points of f contains a perfect set.*

Proof. Let B be the set of all $x \in [0,1]$ such that f is not differentiable at x. By Theorem 22, we have that $\mu_f(B) = \mu_f[0,1]$. Also since $f \in C[0,1]$, μ_f is continuous. Therefore if B is countable, then $\mu_f[0,1] = 0$ and so $f \in V_2^0$. In the case $f \in (V_2 \setminus V_2^0) \cap C[0,1]$, $\mu_f(B) > 0$ and so B contains a perfect set. □

4. Geometric properties of the measure

In this section we are mainly concerned with connecting the norm of the measure μ_f with the distance of f from V_2^0. This first requires some results from [2], included in the first subsection, related to the oscillation function $\widetilde{osc} f$ defined by A. Kechris and A. Louveau [13] and further studied by H. P. Rosenthal in [21]. The second subsection contains the statement and the proof of the basic inequality and in the third subsection we use these geometric properties of the measure to obtain optimal approximations for the functions of $V_2 \setminus V_2^0$.

4.1. **The oscillation function.** Recall that for a function $f : K \to \mathbb{R}$, where K is a compact metric space, $\widetilde{osc}_K f$ is defined as follows. For every $t \in V \subseteq K$ let $s(V,t) = \sup\{|f(x) - f(t)| : x \in V\}$. Then for each $t \in K$,

$$\widetilde{osc}_K f(t) = \inf\{s(V,t) : V \text{open neighborhood of } t\}.$$

It can be easily shown that for every sequence $(f_n)_n$ of continuous real-valued functions on K pointwise converging to a function f and every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $\|\widetilde{osc}_K f\|_\infty - \varepsilon < \|f_n - f\|_\infty$.

The next lemma is included in the more general Lemma 1.2 in [2] and shows that, passing to convex blocks, the above inequality can be reversed.

Lemma 24. *Let $(f_n)_n$ be a uniformly bounded sequence of continuous real-valued functions on a compact metric space K pointwise converging to a function f. Then for every null sequence of positive reals $(\delta_n)_n$ there exists a convex block sequence $(g_n)_n$ of $(f_n)_n$ such that $\|g_n - f\|_\infty < \|\widetilde{osc}_K f\|_\infty + \delta_n$.*

Corollary 25. *Let E be a separable Banach space, X be a subspace of E and K be a w^*-compact subset of B_{E^*} which is 1-norming for E^{**} (that is, for all $x^{**} \in E^{**}$, $\|x^{**}\| = \sup_{x^* \in K} |x^{**}(x^*)|$). Then for all $x^{**} \in X^{**}$ which are w^*-limits of sequences in X, we have that*

$$dist(x^{**}, X) \leq \|\widetilde{osc}_K x^{**}\|_\infty.$$

In particular this holds for all $x^{**} \in X^{**}$ if ℓ_1 is not embedded into X.

Proof. Let K be a w* subset of B_{E^*} which is 1-norming for X^{**}. Let $x^{**} \in X^{**}$ be the w*- limit of a sequence $(x_n)_n$ in X. Denoting again by x_n and x^{**} the restrictions of x_n and x^{**} on K, we have that $(x_n)_n$ is a uniformly bounded sequence of continuous functions on the compact metric space K, pointwise convergent to x^{**}. Therefore, by Lemma 24 there is a convex block sequence $(y_n)_n$ of $(x_n)_n$ such that for all $n \in \mathbb{N}$,

$$\sup_{x^* \in K} |y_n(x^*) - x^{**}(x^*)| \leq \|\widetilde{osc}_K x^{**}\|_\infty + 1/n.$$

Since K is 1-norming for E^{**}, we have that the left side of the above inequality is the norm of $y_n - x^{**}$. Hence

$$dist(x^{**}, X) \leq \|y_n - x^{**}\| \leq \|\widetilde{osc}_K x^{**}\|_\infty + 1/n,$$

for all $n \in \mathbb{N}$ and the result follows. Finally if ℓ_1 does not embed into X, then by Odell-Rosenthal's theorem [18], all $x^{**} \in X^{**}$ are w*-limits of sequences of X. □

Remark 5. Let E, X and K be as in Corollary 25. Then notice that for every $x^{**} \in X^{**}$ and every $x \in X$, $\|\widetilde{osc}_K x^{**}\|_\infty = \|\widetilde{osc}_K(x^{**} - x)\|_\infty \leq 2\|x^{**} - x\|$ and so $\|\widetilde{osc}_K x^{**}\|_\infty \leq 2dist(x^{**}, X)$. Hence by Corollary 25 we have that if X does not contain ℓ_1, $\|\widetilde{osc}_K x^{**}\|_\infty$ is an equivalent norm on the quotient space X^{**}/X.

4.2. **Connection of the measure μ_f with the distance of f from V_2^0.** Returning to V_2, we consider the following subset of $(V_2^0)^*$ introduced in [3]:

$$\mathcal{K} = \Big\{ \sum_{i=1}^{\infty} \alpha_i(\delta_{s_i} - \delta_{t_i}) : \big((s_i, t_i)\big)_i \text{ are disjoint intervals and } \sum_i \alpha_i^2 \leq 1 \Big\}. \tag{14}$$

We recall ([3]) that $\mathcal{K}$ is a w*-compact subset of $B_{(V_2^0)^*}$, 1-norming V_2.

Theorem 26. *Let X be a subspace of V_2^0. Then for every $f \in X^{**}$,*

$$\sqrt{\|\mu_f\|} \leq dist(f, V_2^0) \leq dist(f, X) \leq \|\widetilde{osc}_{\mathcal{K}} f\|_\infty \leq \sqrt{\|\mu_f\|} + 2\sqrt{\|\mu_f^d\|}.$$

Proof. By Proposition 18, we have that for every $g \in V_2^0$, $\mu_{f+g} = \mu_f$. Hence for every $g \in V_2^0$, $\|\mu_f\| = \|\mu_{f+g}\| \leq \|f+g\|_{V_2}^2$, which gives that $\sqrt{\|\mu_f\|} \leq dist(f, V_2^0) \leq dist(f, X)$. Since $\mathcal{K}$ is a w*-compact subset of $(V_2^0)^*$ which is 1-norming for V_2 and ℓ_1 is not embedded into V_2^0, by Corollary 25, we have that $dist(f, X) \leq \|\widetilde{osc}_{\mathcal{K}} f\|_\infty$ and so it remains to prove that $\|\widetilde{osc}_{\mathcal{K}} f\|_\infty \leq \sqrt{\|\mu_f\|} + 2\sqrt{\|\mu_f^d\|}$. To show this, let $x^* \in \mathcal{K}$ and let $\{x_n^*\}_n$ be a sequence in $\mathcal{K}$, w^*-converging to x^* such that $\widetilde{osc}_{\mathcal{K}} f(x^*) = \lim_n |f(x_n^*) - f(x^*)|$ (clearly there exists such a sequence). Notice that for every $y^* = \sum_i \alpha_i(\delta_{s_i} - \delta_{t_i}) \in \mathcal{K}$ and every $f \in V_2$, $f(y^*) = \sum_i \alpha_i(f(s_i) - f(t_i))$, where the series $\sum_i \alpha_i(f(s_i) - f(t_i))$ is absolutely convergent. So we may reorder each x_n^* and in this way we may assume that $x_n^* = \sum_{i=1}^\infty \alpha_i^n(\delta_{s_i^n} - \delta_{t_i^n})$, where for every i, $t_i^n - s_i^n \geq t_{i+1}^n - s_{i+1}^n$. Moreover by passing to a subsequence we may also suppose that for each $i \in \mathbb{N}$, the sequences $\{s_i^n\}_n$, $\{t_i^n\}_n$ are monotone and that $\alpha_i^n \to \alpha_i$, $s_i^n \to s_i$, and $t_i^n \to t_i$.

Therefore $x^* = \sum_{i=1}^\infty \alpha_i(\delta_{s_i} - \delta_{t_i})$ with $t_i - s_i \geq t_{i+1} - s_{i+1}$ and so since $((s_i, t_i))_i$ consists of pairwise disjoint open intervals in $[0,1]$ of decreasing length, $t_i - s_i \leq 1/i$. Also by the monotonicity of $(s_i^n)_n$, $(t_i^n)_n$, there are $\varepsilon_{t_i}, \varepsilon_{s_i} \in \{0, +, -\}$, such that $\lim_n f(t_i^n) = f(t_i^{\varepsilon_{t_i}})$ and $\lim_n f(s_i^n) = f(s_i^{\varepsilon_{s_i}})$, where $t_i^0 = t_i$ and $s_i^0 = s_i$.

Let $\varepsilon > 0$. We choose $i_1 \in \mathbb{N}$ such that $\mu_f[0,1] \leq \widetilde{\mu}_{f,1/i_1}[0,1] < \mu_f[0,1] + \varepsilon^2$ and $\sum_{i=i_1+1}^\infty |\alpha_i|^2 < \varepsilon^2/\|f\|_{V_2}^2$. We set

$$x_1^* = \sum_{i=1}^{i_1} \alpha_i(\delta_{s_i} - \delta_{t_i}) \quad \text{and} \quad x_2^* = \sum_{i=i_1+1}^{\infty} \alpha_i(\delta_{s_i} - \delta_{t_i})$$

and for each $n \in \mathbb{N}$, let

$$x_{n,1}^* = \sum_{i=1}^{i_1} \alpha_i^n(\delta_{s_i^n} - \delta_{t_i^n}) \quad \text{and} \quad x_{n,2}^* = \sum_{i=i_1+1}^{\infty} \alpha_i^n(\delta_{s_i^n} - \delta_{t_i^n}).$$

Then by the Cauchy-Schwarz inequality we get that $|f(x_2^*)|^2 \leq \varepsilon^2$ and $|f(x_{n,2}^*)|^2 \leq \mu_f[0,1] + \varepsilon^2$, for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned}
\widetilde{osc}_{\mathcal{K}} f(x^*) &= \lim_n |f(x_n^*) - f(x^*)| \leq |\lim_n f(x_{n,1}^*) - f(x_1^*)| + \overline{\lim_n} |f(x_{n,2}^*) - f(x_2^*)| \\
&\leq \Big| \sum_{i=1}^{i_1} \alpha_i (f(s_i^{\varepsilon_{s_i}}) - f(s_i)) \Big| + \Big| \sum_{i=1}^{i_1} \alpha_i (f(t_i^{\varepsilon_{t_i}}) - f(t_i)) \Big| + \overline{\lim_n} |f(x_{n,2}^*)| + |f(x_2^*)| \\
&\leq \Big(\sum_{i=1}^{i_1} |f(s_i^{\varepsilon_{s_i}}) - f(s_i)|^2 \Big)^{1/2} + \Big(\sum_{i=1}^{i_1} |f(t_i^{\varepsilon_{t_i}}) - f(t_i)|^2 \Big)^{1/2} + (\sqrt{\mu_f[0,1]} + \varepsilon) + \varepsilon \\
&\leq \Big(\sum_{i=1}^{i_1} \mu_f(\{s_i\}) \Big)^{1/2} + \Big(\sum_{i=1}^{i_1} \mu_f(\{t_i\}) \Big)^{1/2} + \sqrt{\mu_f[0,1]} + 2\varepsilon.
\end{aligned}$$

Hence for every $\varepsilon > 0$, $\|\widetilde{osc}_{\mathcal{K}} f\|_\infty \leq \sqrt{\|\mu_f\|} + 2\sqrt{\|\mu_f^d\|} + 2\varepsilon$ and the conclusion follows. □

Since $\mu_f^d = 0$ if f is continuous, we easily get the following.

Corollary 27. *For every subspace X of V_2^0 and every $f \in X^{**} \cap C[0,1]$,*

$$dist(f, V_2^0) = dist(f, X) = \|\widetilde{osc}_{\mathcal{K}} f\|_\infty = \sqrt{\|\mu_f\|}.$$

Remark 6. Let us note that there exist non-continuous functions $f \in V_2$ satisfying the proper inequalities $\sqrt{\|\mu_f\|} < dist(f, V_2^0) < \|\widetilde{osc}_{\mathcal{K}} f\|_\infty$. For example, it can be easily shown that for $0 < s < t < 1$ and $f = \chi_{[s,t]} + 2\chi_{(t,1]}$, we have that $\|\widetilde{osc}_{\mathcal{K}} f\|_\infty = 2$, μ_f is the sum of the Dirac measures on s and t (so $\sqrt{\mu_f} = \sqrt{\mu_f^d} = \sqrt{2}$) and $\sqrt{2} < dist(f, V_2^0) < 2$.

4.3. Optimal approximation of functions of $V_2 \setminus V_2^0$.

Lemma 28. *Let $f \in V_2 \setminus V_2^0$ and let $(f_n)_n$ be a bounded sequence in V_2^0 pointwise convergent to f. Then for every sequence $(\varepsilon_n)_n$ of positive real numbers there exists a convex block sequence $(h_n)_n$ of $(f_n)_n$ satisfying the following properties:*

(i) $\|h_n - f\|_\infty \leq \|\widetilde{osc}_{[0,1]} f\|_\infty + \varepsilon_n$;

(ii) *for every $\mathcal{I} \in \mathcal{F}([0,1] \setminus D_f)$, $v_2^2(h_n - f, \mathcal{I}) \leq \mu_f(\bigcup \mathcal{I}) + 8\|f\|_{V_2}\sqrt{\|\mu_f^d\|} + \varepsilon_n$.*

Proof. We may assume that $\varepsilon_n < 1$. Also let $(\delta_n)_n$ be a decreasing sequence of positive real numbers such that $\delta_n \leq (1 + 6\|f\|_{V_2})^{-1}\varepsilon_n$, for all n. By our assumptions we have that $(f_n)_n$ is a uniformly bounded sequence in $C[0,1]$ pointwise convergent to f. Hence by Lemma 24 (for $K = [0,1]$), there exists a convex block sequence $(g_n)_n$ of $(f_n)_n$ such that

$$\|g_n - f\|_\infty \leq \|\widetilde{osc}_{[0,1]} f\|_\infty + \delta_n. \tag{15}$$

Next we view each g_n as well as f as functions acting on $\mathcal{K}$. Since $\mathcal{K}$ is a subset of the unit ball of $B_{(V_2^0)^*}$, 1-norming V_2, we conclude that the sequence $(g_n)_n$ is uniformly bounded in $C(\mathcal{K}, \mathrm{w}^*)$ pointwise convergent to f and also $\|g_n - f\|_{\mathcal{K}} = \|g_n - f\|_{V_2}$. Hence again by Lemma 24 (for $K = \mathcal{K}$) and Theorem 26, there exists a convex block sequence $(h_n)_n$ of $(g_n)_n$ such that

$$\|h_n - f\|_{V_2} = \|h_n - f\|_{\mathcal{K}} \leq \|\widetilde{osc}_{\mathcal{K}} f\|_\infty + \delta_n \leq \sqrt{\|\mu_f\|} + 2\sqrt{\|\mu_f^d\|} + \delta_n. \tag{16}$$

Let $(F_n)_n$ be a block sequence of finite subsets of $\mathbb{N}$ and $(\lambda_i)_i$ be a sequence of positive real numbers of $[0,1]$ such that for each $n \in \mathbb{N}$, $\sum_{i\in F_n} \lambda_i = 1$ and $h_n = \sum_{i\in F_n} \lambda_i g_i$. Then considering h_n and f as functions on $[0,1]$, by (15) we obtain that

$$\begin{aligned} \|h_n - f\|_\infty &\le \sum_{i\in F_n} \lambda_i \|g_i - f\|_\infty \le \sum_{i\in F_n} \lambda_i \|\widetilde{osc}_{[0,1]} f\|_\infty + \sum_{i\in F_n} \lambda_i \delta_i \\ &\le \|\widetilde{osc}_{[0,1]} f\|_\infty + \delta_n, \end{aligned} \tag{17}$$

as $n \le \min F_n$ and $(\delta_n)_n$ is decreasing. Moreover, using that $\|\mu_f^d\| = \sqrt{\|\mu_f^d\|}\sqrt{\|\mu_f^d\|}$ $\le \|f\|_{V_2}\sqrt{\|\mu_f^d\|}$ and taking squares in (16) we easily obtain that

$$\|h_n - f\|_{V_2}^2 \le \|\mu_f\| + 8\|f\|_{V_2}\sqrt{\|\mu_f^d\|} + \varepsilon_n. \tag{18}$$

Let $\mathcal{I} \in \mathcal{F}([0,1]\setminus D_f)$. Then $[0,1]\setminus \bigcup \mathcal{I} = \bigcup_{i=1}^m I'_i$, where each I'_i is a non-trivial open interval in $[0,1]$. By Proposition 15, $\mu_f(I'_i) = \widetilde{\mu}_f(I'_i)$ and so for each $1 \le i \le m$, we can choose a sequence $(\mathcal{P}_k^i)_k$ of finite subsets of I'_i, such that $\lim_k \|\mathcal{P}_k^i\|_{\max} = 0$ and $\lim_k v_2^2(f, \mathcal{P}_k^i) = \mu_f(I'_i)$. Since $h_n \in V_2^0$, for all $1 \le i \le m$,

$$\lim_k v_2^2(h_n - f, \mathcal{P}_k^i) = \lim_k v_2^2(f, \mathcal{P}_k^i) = \mu_f(I'_i). \tag{19}$$

Also setting $\mathcal{I}' = (I'_i)_{i=1}^m$, $\|\mu_f\| = \mu_f[0,1] = \mu_f(\bigcup \mathcal{I}) + \mu_f(\bigcup \mathcal{I}')$. Hence (18) gives that

$$v_2^2(h_n - f, \mathcal{I}) + \sum_{i=1}^m v_2^2(h_n - f, \mathcal{P}_k^i) \le \mu_f(\bigcup \mathcal{I}) + \mu_f(\bigcup \mathcal{I}') + 8\|f\|_{V_2}\sqrt{\|\mu_f^d\|} + \varepsilon_n.$$

Letting $k \to \infty$ and using (19), part (ii) of the lemma follows. □

Proposition 29. *Let X be a subspace of V_2^0, $f \in X^{**} \setminus X$ and $(f_n)_n$ be a bounded sequence in X pointwise convergent to f. Then for every $0 < \delta < dist(f, X)$ and for every sequence $(\varepsilon_n)_n$ of positive real numbers there exists a convex block sequence $(h_n)_n$ of $(f_n)_n$ such that for all $n < m$ the following properties are satisfied:*

(i) $\delta < \|h_m - h_n\|_{V_2} \le 2M$, *where* $M = \sup_n \|f_n\|_{V_2}$;
(ii) $\|h_m - h_n\|_\infty \le 2\|\widetilde{osc}_{[0,1]} f\|_\infty + \varepsilon_n \le 4\|f\|_\infty + \varepsilon_n$;
(iii) *for every* $\mathcal{I} \in \mathcal{A}$, $v_2^2(h_m - h_n, \mathcal{I}) \le 4\mu_f(\bigcup \mathcal{I}) + 32\|f\|_{V_2}\sqrt{\|\mu_f^d\|} + \varepsilon_n$.

Moreover given $k \in \mathbb{N}$ and an open subset V of $[0,1]$ with $\mu_f(V) > \theta > 0$ there exist $\mathcal{J} \in \mathcal{A}$ with $\bigcup \mathcal{J} \subseteq V$ and $l > k$ such that

(iv) $v_2^2(h_l - h_k, \mathcal{J}) > \theta$.

Proof. Let (ε'_n) be a decreasing sequence of positive real numbers with $\varepsilon'_n < 4\varepsilon_n$. Let $(h_n)_n$ be the convex block sequence of $(f_n)_n$ resulting from Lemma 28. Since $(h_n)_n$ is w*- convergent to f, for every $n \in \mathbb{N}$ there are finitely many $m > n$ such that $\|h_m - h_n\|_{V_2} \le \delta$ (otherwise, $\|f - h_n\|_{V_2} \le \delta < dist(f, X)$, which is impossible). Therefore by passing to a subsequence we may assume that for all $n < m$, $\delta < \|h_m - h_n\|_{V_2}$. Also since $(h_n)_n$ is a convex block sequence of $(f_n)_n$, $\|h_n\|_{V_2} \le M$ and so $\|h_m - h_n\|_{V_2} \le 2M$.

To show (ii), notice that by (1) above,

$$\|h_m - h_n\|_\infty \le \|h_n - f\|_\infty + \|h_m - f\|_\infty \le 2\|\widetilde{osc}_{[0,1]} f\|_\infty + \varepsilon_n \le 4\|f\|_\infty + \varepsilon_n.$$

For property (iii) observe that since $h_m - h_n$ is a continuous function on $[0,1]$, and $[0,1]\setminus D_f$ is dense in $[0,1]$, it suffices to check it for $\mathcal{I}\in\mathcal{F}([0,1]\setminus D_f)$. In this case, by Lemma 28 (ii), we have that

$$\begin{aligned} v_2^2(h_m-h_n,\mathcal{I}) &\le 2v_2^2(h_m-f,\mathcal{I})+2v_2^2(h_n-f,\mathcal{I})\\ &\le 4\mu_f(\bigcup\mathcal{I})+32\|f\|_{V_2}\sqrt{\|\mu_f^d\|}+\varepsilon_n. \end{aligned}$$

Finally fix $k\in\mathbb{N}$ and let V be an open subset of $[0,1]$ with $\mu_f(V)>\theta>0$. Choose a family $(I_i)_{i=1}^m$ of disjoint open intervals of $[0,1]$ such that $I_i\subseteq V$ and $\mu_f(\bigcup_{i=1}^m I_i)>\theta$ and let

$$0<\varepsilon<\frac{\mu_f(\bigcup_{i=1}^m I_i)-\theta}{2(1+M)m}.$$

Since $h_k\in V_2^0$, there is some $\delta>0$ such that

$$\sup\{v_2(h_k,\mathcal{Q}):\mathcal{Q}\subseteq[0,1],\ \|\mathcal{Q}\|_{\max}<\delta\}<\varepsilon. \tag{20}$$

For every $1\le i\le m$, there exists $\mathcal{P}_i\subseteq I_i$ with $\|\mathcal{P}_i\|_{\max}\le\delta$ and

$$\mu_f(I_i)-\varepsilon<v_2^2(f,\mathcal{P}_i). \tag{21}$$

Moreover since $(h_n)_n$ converges pointwise to f, there is $l>k$ such that for all $1\le i\le m$,

$$|v_2^2(f,\mathcal{P}_i)-v_2^2(h_l,\mathcal{P}_i)|<\varepsilon. \tag{22}$$

Then for every $1\le i\le m$ we have that

$$\begin{aligned} v_2^2(h_l-h_k,\mathcal{P}_i) &\ge (v_2(h_l,\mathcal{P}_i)-v_2(h_k,\mathcal{P}_i))^2\ge v_2^2(h_l,\mathcal{P}_i)-2v_2(h_k,\mathcal{P}_i)v_2(h_l,\mathcal{P}_i)\\ &\ge v_2^2(f,\mathcal{P}_i)-\varepsilon-2\varepsilon\|h_l\|_{V_2}\ge\mu_f(I_i)-2(1+M)\varepsilon. \end{aligned}$$

Therefore, setting $\mathcal{J}=\bigcup_{i=1}^k\mathcal{I}_{\mathcal{P}_i}$, we obtain that

$$v_2^2(h_l-h_k,\mathcal{J})=\sum_{i=1}^m v_2^2(h_l-h_k,\mathcal{P}_i)\ge\sum_{i=1}^m\mu_f(I_i)-2(1+M)m\varepsilon>\theta. \qquad\square$$

Remark 7. Notice that since ℓ_1 is not embedded into V_2^0, from [18] and Goldstine's theorem, there is a sequence $(f_n)_n$ in X pointwise converging to f with $\|f_n\|_{V_2}\le\|f\|_{V_2}$. Hence, in Proposition 29 we can assume that $(f_n)_n$ (and thus also $(h_n)_n$) is a sequence in X with $\|f_n\|_{V_2}\le\|f\|_{V_2}=M$.

5. On the embedding of c_0 into subspaces of V_2^0

In this section we show that every subspace X of V_2^0 with X^* separable, X^{**} non-separable and $\mathcal{M}_{X^{**}}=\{\mu_f:f\in X^{**}\}$ separable contains an isomorphic copy of c_0. This is the first step towards the proof of the main theorem integrated in the next section.

5.1. **Sequences of V_2^0 dominated by measures.**

Definition 30. Let μ be a positive finite Borel measure on $[0,1]$ and C,ε be positive constants. We will say that a function G of V_2^0 is $(C,\varepsilon)-$dominated by μ if for every $\mathcal{I}\in\mathcal{A}$,

$$v_2^2(G,\mathcal{I})\le C\mu(\bigcup\mathcal{I})+\varepsilon.$$

More generally for a sequence $(G_n)_n$ in V_2^0 and a sequence $(\varepsilon_n)_n$ of positive real numbers, we say that $(G_n)_n$ is $(C,(\varepsilon_n)_n)$-dominated by μ if for every $n\in\mathbb{N}$ and every $\mathcal{I}\in\mathcal{A}$, $v_2^2(G_n,\mathcal{I})\leq C\mu(\bigcup\mathcal{I})+\varepsilon_n$.

Remark 8. Suppose that the sequence $(G_n)_n$ is $(C,(\varepsilon_n)_n)$-dominated by μ and $\sum_n\varepsilon_n=\varepsilon<\infty$. Then by the countable additivity and the monotonicity of μ, it is easy to see that for every disjoint sequence $(\mathcal{I}_n)_n$ in $\mathcal{A}$, we have

$$\sum_n v_2^2(G_n,\mathcal{I}_n)\leq C\|\mu\|+\varepsilon. \tag{23}$$

Proposition 31. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of functions of V_2^0 which is $(C,(\varepsilon_n)_n)$-dominated by a positive measure $\mu\in\mathcal{M}[0,1]$, for some null sequence $(\varepsilon_n)_n$ of positive real numbers. Assume also that $(G_n)_n$ is a seminormalized sequence in V_2^0 and $\lim_n\|G_n\|_\infty=0$. Then there is a subsequence of $(G_n)_{n\in\mathbb{N}}$ equivalent to the usual basis of c_0.*

Proof. Let $0<\delta<\|G_n\|_{V_2}\leq M$. Since $(\varepsilon_n)_n$ is a null sequence, by passing to a subsequence we may assume that $(\varepsilon_n)_n$ is decreasing and $\sum_n\varepsilon_n=\varepsilon<\infty$. Since $(G_n)_n$ is seminormalized and pointwise convergent to zero, it is weakly null and so we may also suppose, by passing again to a subsequence, that it is a basic sequence. Also since $\lim_n\|G_n\|_\infty=0$, by Proposition 7 and passing to a further subsequence, we may suppose that $(G_n)_n$ is $(\varepsilon_n)_n$-biorthogonal.

As $(G_n)_n$ is a basic sequence, trivially $(G_n)_n$ has a lower c_0-estimate. To show that $(G_n)_n$ is dominated by the c_0-basis, let $(\lambda_k)_{k=1}^n$ be a sequence of scalars and let $|\lambda_{k_0}|=\max_{1\leq k\leq n}|\lambda_k|$. Then by Lemma 11, we have that

$$v_2^2\Big(\sum_{k=1}^n\lambda_kG_k,\mathcal{I}\Big)\leq|\lambda_{k_0}|^2\Big(\sum_{k=1}^n v_2^2(G_k,\mathcal{I}^{(k)})+\varepsilon(2M+\varepsilon)\Big), \tag{24}$$

for every $\mathcal{I}\in\mathcal{A}$, which by (23) gives that

$$v_2^2\Big(\sum_{k=1}^n\lambda_kG_k,\mathcal{I}\Big)\leq\Big(C\|\mu\|+\varepsilon(2M+1+\varepsilon)\Big)|\lambda_{k_0}|^2. \tag{25}$$

Therefore setting $K=\sqrt{C\|\mu\|+\varepsilon(2M+1+\varepsilon)}$, we conclude that

$$\Big\|\sum_{k=1}^n\lambda_kG_k\Big\|_{V_2}\leq K\max_{1\leq k\leq n}|\lambda_k|$$

and the proof of the proposition is complete. □

Remark 4 and the above proposition yield the following.

Corollary 32. *The sequence $(R_n)_n$ defined in* (13) *contains a subsequence equivalent to the c_0-basis.*

We also state the following generalization of Proposition 31 for later use.

Proposition 33. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of V_2^0, $(\mu_n)_n$ be a sequence in $\mathcal{M}^+[0,1]$ and $(\varepsilon_n)_n$ be a null sequence of positive real numbers with the following properties.*

(1) *$(G_n)_n$ is a seminormalized sequence in V_2^0.*
(2) $\lim_n\|G_n\|_\infty=0$.

(3) *There is a constant* $C > 0$ *such that* G_n *is* $(C, \varepsilon_n)-$*dominated by* μ_n, *for all* $n \in \mathbb{N}$.

(4) *There is a measure* $\mu \in \mathcal{M}^+[0,1]$ *such that* $(\mu_n)_n$ *is norm convergent to* μ.

Then there is a subsequence of $(G_n)_{n\in\mathbb{N}}$ *equivalent to the usual basis of* c_0.

Proof. By passing to a subsequence of $(G_n)_n$ we may suppose that for all $n \in \mathbb{N}$,

$$\|\mu_n - \mu\| < 2^{-n}. \tag{26}$$

Let $0 < \delta < \|G_n\|_{V_2} \leq M$. As in the proof of Proposition 31, by passing to a further subsequence we may also assume that $\sum_n \varepsilon_n = \varepsilon < \infty$ and that $(G_n)_n$ is again a basic sequence which in addition is $(\varepsilon_n)_n$-biorthogonal. To show that $(G_n)_n$ is dominated by the c_0-basis, equations (24) and (26) give that for every $\mathcal{I} \in \mathcal{A}$,

$$\begin{aligned} v_2^2\Big(\sum_{k=1}^n \lambda_k G_k, \mathcal{I}\Big) &\leq |\lambda_{k_0}|^2\Big(C\sum_{k=1}^n \mu_k(\bigcup \mathcal{I}^{(k)}) + \varepsilon(2M+1+\varepsilon)\Big) \\ &\leq |\lambda_{k_0}|^2(C\|\mu\| + C + \varepsilon(2M+1+\varepsilon)), \end{aligned}$$

which, setting $K = \sqrt{C\|\mu\| + C + \varepsilon(2M+1+\varepsilon)}$, yields that

$$\Big\|\sum_{k=1}^n \lambda_k G_k\Big\|_{V_2} \leq K \max\{|\lambda_k| : 1 \leq k \leq n\}.$$ ☐

Proposition 34. *Let* X *be a subspace of* V_2^0 *and let* $(f_n)_n$ *be a bounded sequence in* X *pointwise convergent to a function* $f \in (X^{**} \setminus X) \cap C[0,1]$. *Then for every* $0 < \delta < dist(f, X)$ *and for every sequence* $(\varepsilon_n)_n$ *of positive real numbers there exists a convex block sequence* $(h_n)_n$ *of* $(f_n)_n$ *such that for all* $n < m$ *the following are satisfied:*

(i) $\delta < \|h_m - h_n\|_{V_2} \leq 2M$, *where* $M = \sup_n \|f_n\|_{V_2}$;

(ii) $\|h_m - h_n\|_\infty \leq \varepsilon_n$;

(iii) *the function* $h_m - h_n$ *is* $(4, \varepsilon_n)$*-dominated by the measure* μ_f.

Proof. Since f is continuous, we have that $\widetilde{osc}_{[0,1]} f = 0$ and $\mu_f^d = 0$. The result now follows by Proposition 29. ☐

Corollary 35. *Let* X *be a subspace of* V_2^0 *and* $f \in (X^{**} \setminus X) \cap C[0,1]$. *Then* c_0 *is embedded into* X.

Proof. As we have already mentioned (see Remark 7), there is a sequence $(f_n)_n$ in X pointwise convergent to f with $\|f_n\|_{V_2} \leq \|f\|_{V_2}$. Let $(\varepsilon_n)_n$ be a null sequence of positive real numbers and let $(h_n)_n$ be a convex block sequence of $(f_n)_n$ satisfying the properties of Proposition 34. For each $n \in \mathbb{N}$, let $G_n = h_{2n} - h_{2n-1}$. By (i)-(iii) of Proposition 34, we have that $(G_n)_n$ is a seminormalized sequence of functions in X, $\lim_n \|G_n\|_\infty = 0$ and $(G_n)_n$ is $(4, (\varepsilon_{2n-1})_n)$-dominated by the measure μ_f. By Proposition 31 the result follows. ☐

Remark 9. Exploiting Lemma 34 more carefully, we may pass to an appropriate subsequence of $(h_n)_n$ which is equivalent to the summing basis. This gives an alternative proof of the known result that every $f \in V_2 \cap C[0,1]$ is a difference of bounded semicontinuous functions on the compact metric space $(B_{(V_2^0)^*}, \mathrm{w}^*)$. Moreover the converse of Corollary 35 also holds; that is, $(X^{**} \setminus X) \cap C[0,1] \neq \emptyset$ if and only if c_0 is embedded into X (cf. [3]).

5.2. **The embedding of c_0 into X when $\mathcal{M}_{X^{**}}$ is separable.**

Lemma 36. *Let X be a subspace of V_2^0 and $\mathcal{F}$ be an uncountable subset of X^{**}. If $D_{\mathcal{F}} = \bigcup_{f\in\mathcal{F}} D_f$ is countable, then for every $\varepsilon > 0$ there is an uncountable subset $\mathcal{F}' \subseteq \mathcal{F}$ such that for every $f_1, f_2 \in \mathcal{F}'$, $\|\mu^d_{f_1-f_2}\| < \varepsilon$.*

Proof. Let $D_{\mathcal{F}} = \{t_n\}_n$. By Proposition 16, we have that $\|\mu^d_f\| = \sum_n \tau_f(t_n) \leq \|f\|^2_{V_2}$, for every $f \in \mathcal{F}$. By the definition of $\tau_f(t)$, we easily get the following inequalities.

(a) For every $f \in V_2$, $\tau_f(t) \leq 4(|f(t^+)|^2 + |f(t^+)|^2 + |f(t)|^2)$.
(b) For every f_1, f_2 in V_2, $\tau_{f_1-f_2}(t) \leq 2\tau_{f_1}(t) + 2\tau_{f_2}(t)$.

By passing to an uncountable subset $\mathcal{F}'$ of $\mathcal{F}$, we may suppose that the following hold.

(i) There is $n_0 \in \mathbb{N}$ such that $\sum_{n>n_0} \tau_f(t_n) < \varepsilon/8$, for all $f \in \mathcal{F}'$.
(ii) For all $1 \leq n \leq n_0$, $j \in \{-,0,+\}$ and $f_1, f_2 \in \mathcal{F}'$, $|(f_1 - f_2)(t_n^j)| < \sqrt{\frac{\varepsilon}{24n_0}}$.

Then by (ii) and (a), we get that for every $f_1, f_2 \in \mathcal{F}'$,

$$\sum_{n=1}^{n_0} \tau_{f_1-f_2}(t_n) \leq 4 \sum_{j\in\{-,0,+\}} \sum_{n=1}^{n_0} |(f_1 - f_2)(t_n^j)|^2 < \varepsilon/2. \tag{27}$$

Moreover by (i) and (b),

$$\sum_{n>n_0} \tau_{f_1-f_2}(t_n) \leq 2 \sum_{n>n_0} \tau_{f_1}(t_n) + 2 \sum_{n>n_0} \tau_{f_2}(t_n) < \varepsilon/2. \tag{28}$$

Hence, by (27) and (28), $\|\mu^d_{f_1-f_2}\| = \sum_{n=1}^{n_0} \tau_{f_1-f_2}(t_n) + \sum_{n>n_0} \tau_{f_1-f_2}(t_n) < \varepsilon$. □

Proposition 37. *Let X be a subspace of V_2^0 such that X^* is separable, X^{**} non-separable. If $\mathcal{M}_{X^{**}}$ is a separable subset of $\mathcal{M}[0,1]$, then c_0 is embedded into X.*

Proof. Let $\mathcal{F}$ be an uncountable subset of the unit sphere $S_{X^{**}}$ of X^{**} such that for all $f_1 \neq f_2$ in $\mathcal{F}$, $\|f_1 - f_2\|_{V_2} > 3\delta > 0$. Since X is separable, it is easy to see that by passing to a further uncountable subset, we may assume that for all $f_1 \neq f_2$ in $\mathcal{F}$, $dist(f_1 - f_2, X) > \delta$. Moreover since X^* is separable, by Proposition 2 the set $D_{\mathcal{F}}$ is countable.

Let $(\varepsilon_n)_n$ be a sequence of positive real numbers with $\varepsilon = \sum_n \varepsilon_n < \infty$. Using Lemma 36, Proposition 4 and our assumption that $\mathcal{M}_{X^{**}}$ is separable, we easily construct a decreasing sequence $(\mathcal{F}_n)_n$ of uncountable subsets of $\mathcal{F}$ such that

$$\|\mu^d_{f_1-f_2}\| < \varepsilon_n^2, \ \ \|f_1 - f_2\|_\infty < \varepsilon_n \quad \text{and } \|\mu_{f_1} - \mu_{f_2}\| < \varepsilon_n, \tag{29}$$

for all $n \in \mathbb{N}$ and $f_1, f_2 \in \mathcal{F}_n$.

Given the above construction, we pick for each n, $f_1^n \neq f_2^n$ in $\mathcal{F}_n$. Notice that since $(\mathcal{F}_n)_n$ is decreasing, by (29) we have that $\|\mu_{f_1^n} - \mu_{f_1^{n+1}}\| < \varepsilon_n$ and so, as $\sum_n \varepsilon_n < \infty$, the sequence $(\mu_{f_1^n})_n$ is norm-converging to a $\mu_1 \in \mathcal{M}[0,1]$. Similarly $(\mu_{f_2^n})_n$ converges to a $\mu_2 \in \mathcal{M}[0,1]$. Let $F_n = f_1^n - f_2^n$. Applying for each $n \in \mathbb{N}$, Proposition 29 with F_n in place of f, we obtain $G_n \in X$ satisfying the following:

(i) $\delta < \|G_n\|_{V_2} \leq 2\|F_n\|_{V_2} \leq 4$;
(ii) $\|G_n\|_\infty \leq 4\|F_n\|_\infty + \varepsilon_n$;
(iii) for every $\mathcal{I} \in \mathcal{A}$, $v_2^2(G_n, \mathcal{I}) \leq 4\mu_{F_n}(\bigcup \mathcal{I}) + 32\|F_n\|_{V_2}\sqrt{\|\mu^d_{F_n}\|} + \varepsilon_n$.

By (29), $\|F_n\|_\infty < \varepsilon_n$ and therefore (ii) gives

(iv) $\lim_n \|G_n\|_\infty = 0$.

Moreover $\|\mu^d_{F_n}\| < \varepsilon_n^2$, $\|F_n\|_{V_2} \leq 2$ and setting $\mu_n = 2(\mu_{f_1^n} + \mu_{f_2^n})$, by Proposition 18, $\mu_{F_n} \leq \mu_n$. Replacing in (iii) we get that for each $n \in \mathbb{N}$,

(v) for all $\mathcal{I} \in \mathcal{A}$, $v_2^2(G_n, \mathcal{I}) \leq 4\mu_n(\bigcup \mathcal{I}) + 65\varepsilon_n$, that is, G_n is $(4, 65\varepsilon_n)$-dominated by μ_n.

By (i), (iv), (v) and since $(\mu_n)_n$ is norm-convergent to $\mu_1 + \mu_2$, the assumptions of Proposition 33 are fulfilled and so there is a subsequence of $(G_n)_n$ equivalent to the c_0-basis. □

6. On the embedding of S^2 into subspaces of V_2^0

This final section includes the main results of the paper. We divide this section into three subsections. In the first subsection we define the S^2-*systems* and we show that their existence in a subspace X of V_2^0 leads to the embedding of S^2 into X. A key ingredient is Lemma 41, which is of independent interest. In the next subsection we define the S^2-generating systems which provide the frame for building S^2-systems. Finally in the third subsection we show that every subspace X of V_2^0 with $\mathcal{M}_{X^{**}}$ non-separable contains an S^2-generating system and thus by the preceding results the space S^2 is embedded into X. We also show that S^2 is contained into TF.

6.1. **S^2-systems.** In this subsection we will define certain structures closely related with the embedding of the space S^2 into V_2^0. We start with the definition of a *system*.

Definition 38. Let M, Λ, θ be positive constants and $(\varepsilon_n)_{n=0}^\infty$ be a sequence of positive real numbers. An $(\varepsilon_n)_n-$system with constants (M, Λ, θ) is a family of the form

$$((G_s, \nu_s, \mathcal{I}_s)_{s\in 2^{<\mathbb{N}}}, (\mathcal{Q}_n)_{n\in\mathbb{N}}),$$

where $(G_s)_{s\in 2^{<\mathbb{N}}}$ is a family of functions of V_2^0, $(\nu_s)_{s\in 2^{<\mathbb{N}}}$ is a family of positive Borel measures on $[0, 1]$, $(\mathcal{I}_s)_{s\in 2^{<\mathbb{N}}}$ is a family in $\mathcal{A}$, and $(\mathcal{Q}_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite subsets of $[0, 1]$, satisfying the following properties.

(1) For every $s \in 2^{<\mathbb{N}}$, $\|G_s\|_{V_2} \leq M$ and $\|\nu_s\| \leq \Lambda$.
(2) For every $n \geq 0$ and $s \in 2^n$, $\|G_s\|_\infty \leq \varepsilon_n$.
(3) For every $n \geq 0$, the set $\mathcal{Q}_n$ ε_n-determines the quadratic variation of $\langle\{G_s : s \in 2^n\}\rangle$.
(4) For every $n \geq 0$, $s \in 2^n$ and every $\mathcal{I} \in \mathcal{F}(\mathcal{Q}_n)$, $v_2^2(G_s, \mathcal{I}) \leq \nu_s(\bigcup \mathcal{I}) + \varepsilon_n$.
(5) For every $s \perp t$, $(\mathcal{I}_s, \mathcal{I}_t)$ is a disjoint pair.
(6) For every $s \in 2^{<\mathbb{N}}$, $v_2^2(G_s, \mathcal{I}_s) > \theta$.

Remark 10. Notice that by property (2), we have that $\lim \|G_{\sigma|n}\|_\infty = 0$ and therefore by Proposition 9, for every family $(\varepsilon_s)_s$ of positive scalars there is a dyadic subtree $(t_s)_s$ such that $(G_{t_s})_s$ is $(\varepsilon_s)_s$-biorthogonal. Moreover by (3) and (4) we have that for every $s \in 2^n$, the function G_s is $(1, 2\varepsilon_n)-$dominated by ν_s.

Definition 39. An $(\varepsilon_n)_n-$ S^2 system with constants (M, Λ, θ) is an $(\varepsilon_n)_n-$ system $((G_s, \nu_s, \mathcal{I}_s)_{s\in 2^{<\mathbb{N}}}, (\mathcal{Q}_n)_n)$, with the same constants satisfying in addition the following property. For every $0 \leq n \leq m$, $s \in 2^n$, $t \in 2^m$ with $s \sqsubseteq t$ and $\mathcal{I} \in \mathcal{F}(\mathcal{Q}_n)$,

$$|\nu_t(\bigcup \mathcal{I}) - \nu_s(\bigcup \mathcal{I})| < \varepsilon_n. \tag{30}$$

Remark 11. Suppose that $(\varepsilon_n)_n$ is a null sequence. Then, as $\mathcal{Q}_n$ is increasing, by (30) we get that for every $\sigma \in 2^{\mathbb{N}}$ and every $\mathcal{I} \in \mathcal{F}(\bigcup_n \mathcal{Q}_n)$, the sequence $(\nu_{\sigma|n}(\bigcup \mathcal{I}))_n$ is Cauchy.

Lemma 40. *Let $((G_s, \nu_s, \mathcal{I}_s)_{s\in 2^{<\mathbb{N}}}, (\mathcal{Q}_n)_{n\in\mathbb{N}})$, be an $(\varepsilon_n)_n - S^2$ system with constants (M, Λ, θ). Assume also that $(\varepsilon_n)_n$ is a null sequence. Then there exists a family of positive Borel measures $(\nu_\sigma)_{\sigma\in 2^{\mathbb{N}}}$ on $[0,1]$ such that $\sup_\sigma \|\nu_\sigma\| \leq \Lambda$, and for all $\sigma \in 2^{\mathbb{N}}$, $(G_{\sigma|n})_n$ is $(1, (3\varepsilon_n)_n)$-dominated by ν_σ.*

Proof. Let $\sigma \in 2^{\mathbb{N}}$. Since $(\nu_{\sigma|n})_n$ is a bounded sequence in $\mathcal{M}[0,1]$, there exist a subsequence $(\nu_{\sigma|n})_{n\in L}$ and a positive Borel measure ν_σ on $[0,1]$, such that $(\nu_{\sigma|n})_{n\in L}$ is w$^*-$ convergent to ν_σ. For the following, fix $k \geq 0$ and a finite family of intervals $\mathcal{I} \in \mathcal{A}$. By condition (3) of Definition 38, there exists $\widetilde{\mathcal{I}} \in \mathcal{F}(\mathcal{Q}_k)$ with $\widetilde{\mathcal{I}} \preceq \mathcal{I}$ such that $|\upsilon_2^2(G_{\sigma|k}, \mathcal{I}) - \upsilon_2^2(G_{\sigma|k}, \widetilde{\mathcal{I}})| < \varepsilon_k$. Moreover by condition (4) of Definition 38 and (30), we get that for every $m > k$,

$$\upsilon_2^2(G_{\sigma|k}, \mathcal{I}) \leq \upsilon_2^2(G_{\sigma|k}, \widetilde{\mathcal{I}}) + \varepsilon_k \leq \nu_{\sigma|k}(\bigcup \widetilde{\mathcal{I}}) + 2\varepsilon_k \leq \nu_{\sigma|m}(\bigcup \widetilde{\mathcal{I}}) + 3\varepsilon_k. \tag{31}$$

By Remark 11, we have that $\lim_m \nu_{\sigma|m}(\bigcup \widetilde{\mathcal{I}})$ exists and so (31) implies that

$$\upsilon_2^2(G_{\sigma|k}, \mathcal{I}) \leq \lim_m \nu_{\sigma|m}(\bigcup \widetilde{\mathcal{I}}) + 3\varepsilon_k = \lim_{n\in L} \nu_{\sigma|n}(\bigcup \widetilde{\mathcal{I}}) + 3\varepsilon_k. \tag{32}$$

As the set $\bigcup \widetilde{\mathcal{I}}$ is a closed subset of $[0,1]$ and $\text{w}^* - \lim_{n\in L} \nu_{\sigma|n} = \nu_\sigma$, by Portmanteau's theorem (see [12], page 111) and the monotonicity of ν_σ, we have that

$$\lim_{n\in L} \nu_{\sigma|n}(\bigcup \widetilde{\mathcal{I}}) \leq \nu_\sigma(\bigcup \widetilde{\mathcal{I}}) \leq \nu_\sigma(\bigcup \mathcal{I}). \tag{33}$$

By (32) and (33) we conclude that for every $k \geq 0$ and every $\mathcal{I} \in \mathcal{A}$,

$$\upsilon_2^2(G_{\sigma|k}, \mathcal{I}) \leq \nu_\sigma(\bigcup \mathcal{I}) + 3\varepsilon_k; \tag{34}$$

that is, the sequence $(G_{\sigma|n})_n$ is $(1, (3\varepsilon_n)_n)$-dominated by the measure ν_σ. Finally since ν_σ is in the w*-closure of $\{\nu_s\}_{s\in 2^{<\mathbb{N}}}$, $\|\nu_\sigma\| \leq \sup_n \|\nu_{\sigma|n}\| \leq \Lambda$. □

Remark 12. Notice that if $\sum_n \varepsilon_n = \epsilon < \infty$, the above lemma yields that for every $\sigma \in 2^{\mathbb{N}}$ and every disjoint family $(\mathcal{I}_n)_n$ in $\mathcal{A}$, we have that

$$\sum_n \upsilon_2^2(G_{\sigma|n}, \mathcal{I}_n) \leq \Lambda + 3\epsilon. \tag{35}$$

The next lemma concerns an inequality for tree families of positive numbers which is critical for the embedding of S^2 into subspaces of V_2^0 and could be useful elsewhere.

Lemma 41. *Let $(\alpha_s)_{s\in 2^{<\mathbb{N}}}$, $(\lambda_s)_{s\in 2^{<\mathbb{N}}}$ be two families of non-negative real numbers and let $n \geq 0$. Then there exists a maximal antichain A of $2^{\leqslant n}$ and a family of branches $(b_t)_{t\in A}$ of $2^{\leqslant n}$ such that $\sum_{s\in 2^{\leqslant n}} \lambda_s\alpha_s \leq \sum_{t\in A}(\sum_{s\in b_t} \alpha_s)\lambda_t$ and $t \in b_t$, for all $t \in A$. Therefore if $\sum_{n=1}^{\infty} \alpha_{\sigma|n} \leq C$, for all $\sigma \in 2^{\mathbb{N}}$, then for each $n \geq 0$ there is an antichain A of $2^{\leqslant n}$ such that $\sum_{s\in 2^{\leqslant n}} \lambda_s\alpha_s \leq C\sum_{s\in A} \lambda_s$.*

Proof. We shall use induction on $n \geq 0$. The lemma trivially holds for $n = 0$. Assuming that it is true for some n, we show the $n+1$ case. For each $j \in \{0,1\}$, let $\mathcal{D}_j = \{t \in 2^{n+1} : t(1) = j\}$. Then $\mathcal{D}_j$ is order isomorphic to $2^{\leqslant n}$ and so by our inductive assumption there is an antichain $A_j \subseteq \mathcal{D}_j$, and a family of branches $\{b_t^j : t \in A_j\} \subseteq \mathcal{D}_j$ with $t \in b_t^j$, for each $t \in A_j$ and $\sum_{s\in\mathcal{D}_j} \lambda_s\alpha_s \le \sum_{t\in A_j}(\sum_{s\in b_t^j}\alpha_s)\lambda_t$. Hence we easily get that

(36)
$$\sum_{s\in 2^{\leqslant n+1}} \lambda_s\alpha_s = \lambda_\emptyset\alpha_\emptyset + \sum_{t\in\mathcal{D}_0\cup\mathcal{D}_1}\lambda_t\alpha_t \le \lambda_\emptyset\alpha_\emptyset + \sum_{t\in A_0}(\sum_{s\in b_t^0}\alpha_s)\lambda_t + \sum_{t\in A_1}(\sum_{s\in b_t^1}\alpha_s)\lambda_t.$$

We distinguish two cases.

Case 1. $\lambda_\emptyset \le \sum_{t\in A_0}\lambda_t + \sum_{t\in A_1}\lambda_t$. Then let $A = A_0\cup A_1$ and $b_t = b_t^j \cup\{\emptyset\}$, for each $t\in A_j$. Obviously A is a maximal antichain in $2^{\leqslant n+1}$ and $\{b_t : t\in A\}$ is a family of branches with $t\in b_t$ for each $t\in A$. Moreover as $\lambda_\emptyset \le \sum_{t\in A}\lambda_t$, by (36) we obtain that
$$\sum_{s\in 2^{\leqslant n+1}}\lambda_s\alpha_s \le \sum_{t\in A}\Big(\sum_{s\in b_t}\alpha_s\Big)\lambda_t.$$

Case 2. $\sum_{t\in A_0}\lambda_t + \sum_{t\in A_1}\lambda_t < \lambda_\emptyset$. Then we set $A=\{\emptyset\}$ and $b_\emptyset = \{\emptyset\}\cup b_{t_0}^{j_0}$ where $\sum_{s\in b_{t_0}^{j_0}}\alpha_s = \max\bigcup_{j=0}^1\{\sum_{s\in b_t^j}\alpha_s : t\in A_j\}$. By (36), we get that
$$\sum_{s\in 2^{\leqslant n+1}}\lambda_s\alpha_s \le \lambda_\emptyset\Big(\alpha_\emptyset + \sum_{s\in b_{t_0}^{j_0}}\alpha_s\Big) \le (\sum_{s\in b_\emptyset}\alpha_s)\lambda_\emptyset. \qquad \square$$

Proposition 42. *Let $((G_s,\nu_s,\mathcal{I}_s)_{s\in 2^{<\mathbb{N}}},(\mathcal{Q}_n)_n)$, be an $(\varepsilon_n)_n - S^2$ system with constants (M,Λ,θ). Suppose that $(\varepsilon_n)_n$ is a summable sequence of positive real numbers. Then there is a dyadic subtree $(t_s)_{s\in 2^n}$ of $2^{<\mathbb{N}}$ such that $(G_{t_s})_{s\in 2^{<\mathbb{N}}}$ is equivalent to the S^2-basis.*

Proof. Let $(\varepsilon_s)_{s\in 2^{<\mathbb{N}}}$ be a family of positive real numbers such that $\sum_s \varepsilon_s = \varepsilon < \infty$ and $\theta - (\varepsilon+2M)\varepsilon > 0$. As we have already mentioned (see Remark 10), there is a dyadic subtree $(t_s)_{s\in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ such that $(G_{t_s})_{s\in 2^{<\mathbb{N}}}$ is $(\varepsilon_s)_{s\in 2^{<\mathbb{N}}}$-biorthogonal. We will show that $(G_{t_s})_{s\in 2^{<\mathbb{N}}}$ is equivalent to the S^2-basis. To this end, fix a sequence of real numbers $(\lambda_s)_{|s|\le n}$.

First we show the upper S^2-estimate. Let $\mathcal{I}\in\mathcal{A}$. By Lemma 11 we have that

$$v_2^2\Big(\sum_{|s|\le n}\lambda_s G_{t_s},\mathcal{I}\Big) \le \sum_{|s|\le n}|\lambda_s|^2 v_2^2(G_{t_s},\mathcal{I}^{(t_s)}) + \max_{|s|\le n}|\lambda_s|^2(2M+\varepsilon)\varepsilon. \tag{37}$$

By Remark 12, we have that $\sum_n v_2^2(G_{t_{\sigma|n}},\mathcal{I}^{(t_{\sigma|n})}) \le \Lambda + 3\epsilon$, where $\epsilon = \sum_n \varepsilon_n$. Hence by Lemma 41 (with $v_2^2(G_{t_s},\mathcal{I}^{(t_s)})$ and $|\lambda_s|^2$ in place of α_s and $|\lambda_s|$, respectively), we obtain an antichain $A\subseteq 2^{\leqslant n}$ such that

$$\sum_{|s|\le n}|\lambda_s|^2 v_2^2(G_{t_s},\mathcal{I}^{(t_s)}) \le (\Lambda+3\epsilon)\sum_{s\in A}|\lambda_s|^2. \tag{38}$$

By (37) and (38), we get that
$$\begin{aligned}v_2^2\Big(\sum_{|s|\le n}\lambda_s G_{t_s},\mathcal{I}\Big) &\le (\Lambda+3\epsilon)\sum_{s\in A}|\lambda_s|^2 + \max_{|s|\le n}|\lambda_s|^2(2M+\varepsilon)\varepsilon\\ &\le \Big(\Lambda+3\epsilon+(2M+\varepsilon)\varepsilon\Big)\Big\|\sum_{|s|\le n}\lambda_s e_s\Big\|_{S^2}^2.\end{aligned}$$

This yields that there is $C>0$ such that $\|\sum_{|s|\le n}\lambda_s G_{t_s}\|_{V_2} \le C\|\sum_{|s|\le n}\lambda_s e_s\|_{S^2}$.

We now proceed to show the lower S^2-estimate. Let A be an antichain of $2^{\leqslant n}$ such that

$$\Big\| \sum_{|s|\leq n} \lambda_s e_s \Big\|_{S^2} = \Big(\sum_{s\in A} |\lambda_s|^2 \Big)^{1/2}. \tag{39}$$

Since $(\mathcal{I}_{t_s})_{s\in A}$ is a disjoint family, we get that $\mathcal{I} = \bigcup_{s\in A} \mathcal{I}_{t_s} \in \mathcal{A}$. By Lemma 11 and (39), we have that

$$\begin{aligned} \upsilon_2^2\Big(\sum_{|s|\leq n} \lambda_s G_{t_s}, \mathcal{I}\Big) &\geq \sum_{s\in A} |\lambda_s|^2 \upsilon_2^2(G_{t_s}, \mathcal{I}^{(t_s)}) - \max_{|s|\leq n} |\lambda_s|^2 \Big(\sum_{|s|\leq n} \varepsilon_s \Big) 2M \\ &\geq \sum_{s\in A} |\lambda_s|^2 \upsilon_2^2(G_{t_s}, \mathcal{I}^{(t_s)}) - \Big(\sum_{s\in A} |\lambda_s|^2 \Big) 2M\varepsilon. \end{aligned} \tag{40}$$

By the properties of the S^2-system, we have that for every $s \in A$,

$\theta < \upsilon_2^2(G_{t_s}, \mathcal{I}_{t_s}) \leq \upsilon_2^2(G_{t_s}, \mathcal{I}) = \upsilon_2^2(G_{t_s}, \mathcal{I}^{(t_s)}) + \upsilon_2^2(G_{t_s}, \mathcal{I}\setminus\mathcal{I}^{(t_s)}) \leq \upsilon_2^2(G_{t_s}, \mathcal{I}^{(t_s)}) + \varepsilon^2$ and therefore $\upsilon_2^2(G_{t_s}, \mathcal{I}^{(t_s)}) \geq \theta - \varepsilon^2$. Hence by (40), we obtain that

$$\upsilon_2^2\Big(\sum_{|s|\leq n} \lambda_s G_{t_s}, \mathcal{I}\Big) \geq \Big(\theta - (\varepsilon + 2M)\varepsilon\Big) \Big(\sum_{s\in A} |\lambda_s|^2 \Big),$$

which gives that there is $c > 0$ such that $\| \sum_{|s|\leq n} \lambda_s G_{t_s} \|_{V_2} \geq c \, \| \sum_{|s|\leq n} \lambda_s e_s \|_{S^2}$. □

6.2. S^2-generating systems.

Definition 43. An $(\varepsilon, (\varepsilon_n)_n) - S^2$-generating system with constants (M, Λ, θ) is an $(\varepsilon_n)_n$-system $((H_s, \mu_s, \mathcal{J}_s)_{s\in 2^{<\mathbb{N}}}, (\mathcal{P}_n)_n)$, with the same constants satisfying in addition the following properties.

(i) For every $m > n \geq 0$, $s \in 2^n$ and $\{s_0, s_1\} \subseteq 2^m$ such that $s^\frown 0 \sqsubseteq s_0$ and $s^\frown 1 \sqsubseteq s_1$ and every $\mathcal{I} \in \mathcal{F}(\mathcal{P}_n)$, we have that

$$\Big|\frac{\mu_{s_0} + \mu_{s_1}}{2}\Big(\bigcup \mathcal{I}\Big) - \mu_s\Big(\bigcup \mathcal{I}\Big)\Big| < \varepsilon_n. \tag{41}$$

(ii) For every $n \geq 1$, the sequence $(H_s)_{s\in 2^n}$ is $(\varepsilon_s)_{s\in 2^n}$-biorthogonal, where $\varepsilon = \sum_{s\in 2^n} \varepsilon_s > 0$.

In the following we present a partition of $2^{<\mathbb{N}}$ in continuum many almost disjoint subtrees. (Recall that a family of countable sets is called *almost disjoint* if the intersection of any two members of the family is finite.) This partition is induced by the canonical bijection between $2^{<\mathbb{N}}$ and $2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$.

Definition 44. Let $s \in 2^{<\mathbb{N}}$. If $s = \emptyset$, then let $L_\emptyset = \{\emptyset\}$. If $\emptyset \neq s = (s(1), ..., s(n))$, let $L_s = \{t \in 2^{2n} : \; t(2i) = s(i), \text{ for all } 1 \leq i \leq n\}$.

Remark 13. It is easy to see that for each $\sigma \in 2^{\mathbb{N}}$, $T_\sigma = \bigcup_n L_{\sigma|n}$ is a dyadic subtree of $2^{<\mathbb{N}}$ and $(T_\sigma)_{\sigma\in 2^{\mathbb{N}}}$ is an almost disjoint family and hence their bodies $([T_\sigma])_{\sigma\in 2^{\mathbb{N}}}$ are disjoint. (Recall that the *body* of a tree T in $2^{<\mathbb{N}}$ is defined to be the set $[T] = \{\sigma \in 2^{\mathbb{N}} : \forall n(\sigma|n \in T)\}$ (see also [12], page 5).)

The following properties of $(L_s)_{s\in 2^{<\mathbb{N}}}$ are easily established.

(L1) For all $n \geq 0$ and $s \in 2^n$, $L_s \subseteq 2^{2n}$ and $|L_s| = 2^n$, where $|L_s|$ is the cardinality of L_s.

(L2) For $s_1 \sqsubseteq s_2$, $L_{s_1} = L_{s_2}|n_1 = \{t|n_1 : \; t \in L_{s_2}\}$, where $n_1 = 2|s_1|$.

(L3) For $s_1 \perp s_2$, $L_{s_1} \perp L_{s_2}$.
(L4) For all $n \geq 0$, $2^{2n} = \bigcup_{s\in 2^n} L_s$.

Given an $(\varepsilon, (\varepsilon_n)_n) - S^2$-generating system $((H_s, \mu_s, \mathcal{J}_s)_{s\in 2^{<\mathbb{N}}}, (\mathcal{P}_n)_n)$, with constants (M, Λ, θ), we set $G_s = 2^{-n/2}\sum_{t\in L_s} H_t$, $\nu_s = 2^{-n}\sum_{t\in L_s}\mu_t$, $\mathcal{I}_s = \bigcup_{t\in L_s}\mathcal{J}_t$, $\mathcal{Q}_n = \mathcal{P}_{2n}$ and $\varepsilon'_n = \frac{\theta}{2^{n/2}} + 2^{n/2}\varepsilon_{2n}$.

Proposition 45. *If $\theta' = \theta - (2M+\varepsilon)\varepsilon > 0$, then the system $((G_s, \nu_s, \mathcal{I}_s)_{s\in 2^{\mathbb{N}}}, (\mathcal{Q}_n)_n)$ is an $(\varepsilon'_n)_n - S^2$-system with constants $(M+\varepsilon, \Lambda, \theta')$.*

Proof. Let $n \geq 1$ and $s \in 2^n$. Since $L_s \subseteq 2^{2n}$, $|L_s| = 2^n$ and $(H_t)_{t\in 2^{2n}}$ is $(\varepsilon_t)_{t\in 2^{2n}}$-biorthogonal, with $\sum_{t\in 2^{2n}}\varepsilon_t = \varepsilon < 1$, by Lemma 11, we get that for every $\mathcal{I} \in \mathcal{A}$,

$$v_2^2(G_s, \mathcal{I}) < |L_s|^{-1}\sum_{t\in L_s} v_2^2(H_t, \mathcal{I}^{(t)}) + 2^{-n}(2M+\varepsilon)\varepsilon \leq (M+\varepsilon)^2$$

and so $\|G_s\|_{V_2^0} \leq M + \varepsilon$.

Moreover $\|G_s\|_\infty \leq \sqrt{|L_s|}\varepsilon_{2n} = \sqrt{2^n}\varepsilon_{2n} < \varepsilon'_n$. We now show that the quadratic variation of $\langle\{G_s\}_{s\in 2^n}\rangle$ is ε'_n-determined by $\mathcal{Q}_n$. Let $\mathcal{I} \in \mathcal{A}$. Since $\mathcal{P}_{2n}$ ε_{2n}-determines the quadratic variation of $\langle\{H_t\}_{t\in 2^{2n}}\rangle$, there is $\widetilde{\mathcal{I}} \preceq \mathcal{I}$ in $\mathcal{F}(\mathcal{P}_{2n})$ such that for all $(\mu_t)_{t\in 2^{2n}}$,

$$|v_2^2(\sum_{t\in 2^{2n}}\mu_t H_t, \mathcal{I}) - v_2^2(\sum_{t\in 2^{2n}}\mu_t H_t, \widetilde{\mathcal{I}})| \leq (\sum_{t\in 2^{2n}}|\mu_t|^2)\varepsilon_{2n}.$$

Notice that for every sequence of scalars $(\lambda_s)_{s\in 2^n}$, $\sum_{s\in 2^n}\lambda_s G_s = \sum_{t\in 2^{2n}}\mu_t H_t$, where for each $t \in 2^{2n}$, $\mu_t = |L_s|^{-1/2}\lambda_s$, for $t \in L_s$. Therefore

$$\begin{aligned}|v_2^2(\sum_{s\in 2^n}\lambda_s G_s, \mathcal{I}) - v_2^2(\sum_{s\in 2^n}\lambda_s G_s, \widetilde{\mathcal{I}})| &= |v_2^2(\sum_{t\in 2^{2n}}\mu_t H_t, \mathcal{I}) - v_2^2(\sum_{t\in 2^{2n}}\mu_t H_t, \widetilde{\mathcal{I}})| \\ &\leq (\sum_{t\in 2^{2n}}|\mu_t|^2)\varepsilon_{2n} \leq (\sum_{s\in 2^n}|\lambda_s|^2)\varepsilon'_n.\end{aligned}$$

We proceed to show property (4) of Definition 38. Let $\mathcal{I} \in \mathcal{F}(\mathcal{Q}_n)$. Then $\mathcal{I} \in \mathcal{F}(\mathcal{P}_{2n})$, $\mathcal{I} = \bigcup_{t\in 2^{2n}}\mathcal{I}^{(t)}$ and for all $t \in 2^{2n}$, $v_2^2(H_t, \mathcal{I}^{(t)}) \leq \mu_t(\bigcup\mathcal{I}^{(t)}) + \varepsilon_{2n}$. Notice also that for every $t \in L_s$, $\mu_t \leq |L_s|\nu_s$. Hence, by Lemma 11, we get that

$$\begin{aligned}v_2^2(G_s, \mathcal{I}) &\leq |L_s|^{-1}\sum_{t\in L_s}\mu_t(\bigcup\mathcal{I}^{(t)}) + \varepsilon_{2n} + 2^{-n}(2M+\varepsilon)\varepsilon \\ &\leq \sum_{t\in L_s}\nu_s(\bigcup\mathcal{I}^{(t)}) + \varepsilon_{2n} + 3\theta/(4\sqrt{2^n}) \leq \nu_s(\bigcup\mathcal{I}) + \varepsilon'_n.\end{aligned}$$

We now show (30) of Definition 39. Let $m \geq 1$ and $t = (t(1), ..., t(m)) \in 2^m$. Let $0 \leq n < m$ and $s = t|n$. For each $w \in L_s$, let $w_0 = w^\frown 0^\frown t(n+1)$ and $w_1 = w^\frown 1^\frown t(n+1)$. Also let $u = (u(1), ..., u(m-n-1))$, where $u(i) = t(n+1+i)$, for all $1 \leq i \leq m-n-1$. Then $L_t = \bigcup_{w\in L_s}(w_0^\frown L_u \cup w_1^\frown L_u)$ and so $\nu_t = 2^{-m}\sum_{w\in L_s}\sum_{v\in L_u}(\mu_{w_0^\frown v} + \mu_{w_1^\frown v})$. Therefore,

$$\begin{aligned}\nu_t - \nu_s &= 2^{-m}\sum_{w\in L_s}\sum_{v\in L_u}(\mu_{w_0^\frown v} + \mu_{w_1^\frown v}) - 2^{-n}\sum_{w\in L_s}\mu_w \\ &= 2^{-m}\sum_{w\in L_s}\sum_{v\in L_u}(\mu_{w_0^\frown v} + \mu_{w_1^\frown v} - 2\mu_w).\end{aligned}$$

Moreover by (41), for every $\mathcal{I} \in \mathcal{F}(\mathcal{Q}_n) = \mathcal{F}(\mathcal{P}_{2n})$, and all $v \in L_u$, we have that

$$|(\mu_{w_0^\frown v} + \mu_{w_1^\frown v} - 2\mu_w)(\bigcup \mathcal{I})| \leq 2\varepsilon_{2n}.$$

Hence $|\nu_t(\bigcup \mathcal{I}) - \nu_s(\bigcup \mathcal{I})| \leq 2^{-m}2^n 2^{m-n-1} 2\varepsilon_{2n} = \varepsilon_{2n} \leq \varepsilon_n'$.

That the pair $(\mathcal{I}_s, \mathcal{I}_t)$ is disjoint is straightforward by properties (5) of Definition 38 and (L3) of Definition 44. Finally, $\upsilon_2^2(H_t, \mathcal{J}_t) > \theta$ implies that $\upsilon_2^2(H_t, \mathcal{I}_s^{(t)}) > \theta - \varepsilon^2$, for all $t \in 2^{<\mathbb{N}}$. Hence by Lemma 11, we obtain that

$$\upsilon_2^2(G_s, \mathcal{I}_s) \geq \sum_{t \in L_s} |L_s|^{-1} \upsilon_2^2(H_t, \mathcal{I}_s^{(t)}) - 2^{-n} 2M\varepsilon \geq \theta - (\varepsilon + 2M)\varepsilon = \theta'. \qquad \square$$

6.3. The embedding of S^2 into X when $\mathcal{M}_{X^{**}}$ is non-separable.

Lemma 46. *Let $\{\mu_\xi\}_{\xi<\omega_1}$ be a non-separable subset of $\mathcal{M}^+[0,1]$. Then there is an uncountable subset Γ of ω_1 such that for every $\xi \in \Gamma$, $\mu_\xi = \lambda_\xi + \tau_\xi$, where λ_ξ, τ_ξ are positive Borel measures on $[0,1]$ satisfying the following properties.*

(1) *For all $\xi \in \Gamma$, $\lambda_\xi \perp \tau_\xi$ and $\|\tau_\xi\| > 0$.*
(2) *For all $\zeta < \xi$ in Γ, $\mu_\zeta \perp \tau_\xi$.*

Proof. We may suppose that for some $\delta > 0$, $\|\mu_\xi - \mu_\zeta\| > \delta$, for all $0 \leq \zeta < \xi < \omega_1$. By transfinite induction we construct a strictly increasing sequence $(\xi_\alpha)_{\alpha<\omega_1}$ in ω_1 such that for each $\alpha < \omega_1$, $\mu_{\xi_\alpha} = \lambda_{\xi_\alpha} + \tau_{\xi_\alpha}$, with $\lambda_{\xi_\alpha} \perp \tau_{\xi_\alpha}$, $\|\tau_{\xi_\alpha}\| > 0$ and $\tau_{\xi_\alpha} \perp \mu_{\xi_\beta}$ for all $\beta < \alpha$. The general inductive step of the construction is as follows. Suppose that for some $\alpha < \omega_1$, $(\xi_\beta)_{\beta<\alpha}$ has been defined. Let $(\beta_n)_n$ be an enumeration of α and set

$$\zeta_\alpha = \sup_n \xi_{\beta_n}, \;\; \nu_\alpha = \sum_n \mu_{\xi_{\beta_n}}/2^n \text{ and } N_\alpha = \{\xi < \omega_1 : \zeta_\alpha < \xi \text{ and } \mu_\xi \ll \nu_\alpha\}.$$

By the Radon-Nikodym theorem, $\{\mu_\xi\}_{\xi \in N_\alpha}$ is isometrically contained in $L_1([0,1], \nu_\alpha)$ and therefore it is norm separable. Since we have assumed that $\|\mu_\xi - \mu_\zeta\| > \delta$, for all $0 \leq \zeta < \xi < \omega_1$, we get that N_α is countable. Hence we can choose $\xi_\alpha > \sup N_\alpha$. Let $\mu_{\xi_\alpha} = \lambda_{\xi_\alpha} + \tau_{\xi_\alpha}$ be the Lebesgue analysis of μ_{ξ_α}, where $\lambda_{\xi_\alpha} \ll \nu_\alpha$ and $\tau_{\xi_\alpha} \perp \nu_\alpha$. By the definition of ν_α and ξ_α, we have that $\|\tau_{\xi_\alpha}\| > 0$, $\tau_{\xi_\alpha} \perp \mu_{\xi_\beta}$, for all $\beta < \alpha$ and the inductive step of the construction has been completed. $\square$

Lemma 47. *Let $\{\tau_\xi\}_{\xi<\omega_1}$ be an uncountable family of pairwise singular positive Borel measures on $[0,1]$. Then for every finite family $(\Gamma_i)_{i=1}^k$ of pairwise disjoint uncountable subsets of ω_1 and every $\varepsilon > 0$, there exist a family $(\Gamma_i')_{i=1}^k$ with Γ_i' an uncountable subset of Γ_i and a family $(U_i)_{i=1}^k$ of open and pairwise disjoint subsets of $[0,1]$ such that $\tau_\xi([0,1] \setminus U_i) < \varepsilon$, for all $1 \leq i \leq k$ and $\xi \in \Gamma_i'$.*

Proof. For every $\alpha < \omega_1$, we choose $(\xi_i^\alpha)_{i=1}^k \in \prod_{i=1}^k \Gamma_i$ such that for every $\alpha \neq \beta$ in ω_1 and every $1 \leq i \leq k$, $\xi_i^\alpha \neq \xi_i^\beta$. For each $0 \leq \alpha < \omega_1$ the k-tuple $(\tau_{\xi_i^\alpha})_{i=1}^k$ consists of pairwise singular measures and so we may choose a k-tuple $(U_i^\alpha)_{i=1}^k$ of open subsets of $[0,1]$ with the following properties: (a) for each i, $\tau_{\xi_i^\alpha}([0,1] \setminus U_i^\alpha) < \varepsilon$; (b) for all $i \neq j$, $U_i^\alpha \cap U_j^\alpha = \emptyset$; (c) for each i, U_i^α is a finite union of open in $[0,1]$ intervals with rational endpoints.

Since the family of all finite unions of open intervals with rational endpoints is countable, there is a k-tuple $(U_i)_i$ and an uncountable subset Γ of ω_1, such that for all $1 \leq i \leq k$ and all $\alpha \in \Gamma$, $U_i^\alpha = U_i$. For each $1 \leq i \leq k$, set $\Gamma_i' = \{\xi_i^\alpha : \alpha \in \Gamma\}$. Then for each $1 \leq i \leq k$ and all $\xi \in \Gamma_i'$, $\tau_\xi([0,1] \setminus U_i) < \varepsilon$. $\square$

Lemma 48. *Let X be a subspace of V_2^0 and suppose that X^{**} contains an uncountable family $\mathcal{F}$ such that $D_{\mathcal{F}} = \bigcup_{f\in\mathcal{F}} D_f$ is countable and $\mathcal{M}_{\mathcal{F}} = \{\mu_f\}_{f\in\mathcal{F}}$ is non-separable. Then there are constants (M,Λ,θ) such that for every $\varepsilon > 0$ and every sequence $(\varepsilon_n)_n$ of positive scalars there is an $(\varepsilon,(\varepsilon_n)_n)-S^2$-generating system $((H_s,\mu_s,\mathcal{J}_s)_{s\in 2^{<\mathbb{N}}},(\mathcal{P}_n)_n)$ with constants (M,Λ,θ) and $H_s\in X$, for all $s\in 2^{<\mathbb{N}}$.*

Proof. Since for all $f \in V_2$ and $\lambda\in\mathbb{R}$, $\mu_{\lambda f} = \lambda^2\mu_f$, we may assume that $\mathcal{F}\subseteq S_{X^{**}}$. By Lemma 46, there is a non-separable subset $\{\mu_\xi\}_{\xi<\omega_1}$ of $\mathcal{M}_{\mathcal{F}}$ such that for all $0\le\xi<\omega_1$, $\mu_\xi = \lambda_\xi+\tau_\xi$, $\lambda_\xi\perp\tau_\xi$ and for all $\zeta<\xi$, $\mu_\zeta\perp\tau_\xi$. By passing to a further uncountable subset, we may also assume that there is $\theta_0>0$ such that $\|\tau_\xi\|>\theta_0$. We fix $\varepsilon>0$ and a sequence $(\varepsilon_n)_n$ of positive real numbers. We will construct the following objects:

(1) a Cantor scheme $(\Gamma_s)_s$ of uncountable subsets of ω_1 (that is, for all $s\in 2^{<\mathbb{N}}$, $\Gamma_{s^\frown 0}\cup\Gamma_{s^\frown 1}\subseteq\Gamma_s$ and $\Gamma_{s^\frown 0}\cap\Gamma_{s^\frown 1}=\emptyset$),
(2) a family $((\xi_s^0,\xi_s^1))_s$ of pairs with $\xi_s^0<\xi_s^1$ in Γ_s, for all $s\in 2^{<\mathbb{N}}$,
(3) a Cantor scheme of open subsets $(V_s)_s$ of $[0,1]$,
(4) a family of functions $(H_s)_s$ in X,
(5) an increasing sequence $(\mathcal{P}_n)_n$ of finite subsets of $[0,1]\setminus D_{\mathcal{F}}$, and
(6) a family $(\mathcal{J}_s)_s$ in $\mathcal{A}$,

such that the following are satisfied.

(i) For every $\xi\in\Gamma_s$, $\tau_\xi(V_s)>\theta_0/2$ and $\tau_\xi([0,1]\setminus V_s)<\big(\sum_{i=0}^{|s|}2^{-(i+2)}\big)\theta_0$.
(ii) The measures $\mu_{\xi_s^0}$ and $\mu_{\xi_s^1}$ are w*-condensation points of $\{\mu_\xi\}_{\xi\in\Gamma_s}$.
(iii) For every $n\ge 1$, $s\in 2^n$, $\xi\in\Gamma_s$ and $\mathcal{I}\in\mathcal{F}(\mathcal{P}_{n-1})$,

$$|(\mu_\xi-\mu_{\xi_{s^-}^{s(n)}})(\bigcup\mathcal{I})|<\frac{\varepsilon_{n-1}}{16},\text{ where } s^-=(s(1),...,s(n-1)).$$

(iv) $\|H_s\|_{V_2}\le 2$, $\bigcup\mathcal{J}_s\subseteq V_s$ and $v_2^2(H_s,\mathcal{J}_s)>\theta_0/2$.
(v) For every $s\in 2^n$, $\|H_s\|_\infty\le\varepsilon_n$ and $v_2^2(H_s,\mathcal{I})\le 8(\mu_{\xi_s^0}+\mu_{\xi_s^1})(\bigcup\mathcal{I})+\varepsilon_n$, for all $\mathcal{I}\in\mathcal{F}(\mathcal{P}_n)$.
(vi) If $(s_i)_{i=1}^{2^n}$ is the lexicographical enumeration of $\{0,1\}^n$, then $(H_{s_i})_{i=1}^{2^n}$ is $(\varepsilon/2^i)_{i=1}^{2^n}$-biorthogonal.
(vii) The set $\mathcal{P}_n$ ε_n-determines the quadratic variation of $\langle\{H_s\}_{s\in 2^{<\mathbb{N}}}\rangle$.

Given the above construction, we set $\mu_s = 8(\mu_{\xi_s^0}+\mu_{\xi_s^1})$ and we claim that the family $((H_s,\mu_s,\mathcal{J}_s)_{s\in 2^{<\mathbb{N}}},(\mathcal{P}_n)_n)$ is an $(\varepsilon,(\varepsilon_n)_n)-S^2$-generating system with constants (M,Λ,θ), where $M=2$, $\Lambda=16$ and $\theta=\theta_0/2$. We only verify condition (41) of Definition 43, since the other conditions are immediate. So let $n<m$, $s\in 2^n$ and $s_0,s_1\in 2^m$, with $s^\frown 0\sqsubseteq s_0$ and $s^\frown 1\sqsubseteq s_1$. Then $\Gamma_{s_0}\subseteq\Gamma_{s^\frown 0}$, $\Gamma_{s_1}\subseteq\Gamma_{s^\frown 1}$, and so by (iii), for all $\mathcal{I}\in\mathcal{F}(\mathcal{P}_n)$ and $j\in\{0,1\}$, we get that

$$\max\{|(\mu_{\xi_{s_0}^j}-\mu_{\xi_s^0})(\bigcup\mathcal{I})|,|(\mu_{\xi_{s_1}^j}-\mu_{\xi_s^1})(\bigcup\mathcal{I})|\}\le\frac{\varepsilon_n}{16}. \tag{42}$$

Since

$$|\frac{\mu_{s_0}+\mu_{s_1}}{2}-\mu_s|\le 4(|\mu_{\xi_{s_0}^0}-\mu_{\xi_s^0}|+|\mu_{\xi_{s_0}^1}-\mu_{\xi_s^0}|+|\mu_{\xi_{s_1}^0}-\mu_{\xi_s^1}|+|\mu_{\xi_{s_1}^1}-\mu_{\xi_s^1}|),$$

by (42), we have that for all $\mathcal{I}\in\mathcal{F}(\mathcal{P}_n)$, $|(\frac{\mu_{s_0}+\mu_{s_1}}{2}-\mu_s)(\bigcup\mathcal{I})|\le\varepsilon_n$.

We now present the general inductive step of the construction. Let us suppose that the construction has been carried out for all $s \in 2^{<n}$. For every $s = (s(1), ..., s(n)) \in 2^n$, we define

$$\Gamma_s^{(1)} = \{\xi \in \Gamma_{s^-} :\ \forall \mathcal{I} \in \mathcal{F}(\mathcal{P}_{n-1}), |(\mu_\xi - \mu_{\xi_{s^-}^{s(n)}})(\bigcup \mathcal{I})| < \varepsilon_n/16\}. \tag{43}$$

Since $\mathcal{F}(\mathcal{P}_{n-1})$ is a finite subset of $\mathcal{F}([0,1] \setminus D_{X^{**}})$ and $\mu_\xi(\partial(\cup\mathcal{I})) = 0$, for all $\mathcal{I} \in \mathcal{F}([0,1] \setminus D_{X^{**}})$ and all $\xi < \omega_1$ (where $\partial(\bigcup\mathcal{I})$ is the boundary of $\bigcup\mathcal{I}$), the set $\{\mu_\xi : \xi \in \Gamma_s^{(1)}\}$ is a relatively w*-open neighborhood of $\mu_{\xi_{s^-}^{s(n)}}$ in $\{\mu_\xi\}_{\xi\in\Gamma_{s^-}}$. By our inductive assumption, $\mu_{\xi_{s^-}^0}$ and $\mu_{\xi_{s^-}^1}$ are w*-condensation points of $\{\mu_\xi\}_{\xi\in\Gamma_{s^-}}$ and therefore for all $s \in 2^n$ the set $\Gamma_s^{(1)}$ is uncountable. Applying Lemma 47 we obtain a 2^n-tuple $(U_s)_{s\in 2^n}$ of pairwise disjoint open subsets of $[0,1]$ and a family $(\Gamma_s^{(2)})_{s\in 2^n}$ such that for each $s \in 2^n$, $\Gamma_s^{(2)}$ is an uncountable subset of $\Gamma_s^{(1)}$ and for all $\xi \in \Gamma_s^{(2)}$,

$$\tau_\xi([0,1] \setminus U_s) < \theta_0/2^{n+2}. \tag{44}$$

For every $s \in 2^n$ we set $V_s = U_s \cap V_{s^-}$. Since $\Gamma_s^{(2)} \subseteq \Gamma_s^{(1)} \subseteq \Gamma_{s^-}$, using (i), we get that for all $s \in 2^n$ and all $\xi \in \Gamma_s^{(2)}$,

$$\tau_\xi([0,1] \setminus V_s) < \big(\sum_{i=0}^{n} 2^{-(i+2)}\big)\theta_0. \tag{45}$$

Moreover as $(\sum_{i=0}^{n} 2^{-(i+2)})\theta_0 < \theta_0/2$ and $\|\tau_\xi\| > \theta_0$, we get that for all $\xi \in \Gamma_s^{(2)}$,

$$\tau_\xi(V_s) > \theta_0/2. \tag{46}$$

Since for all $\zeta < \xi < \omega_1$, we have $\mu_\xi \geq \tau_\xi$ and $\tau_\xi \perp \mu_\zeta$, by Lemma 17, we get that $\mu_{f_\xi - f_\zeta} \geq \tau_\xi$. Therefore

$$\mu_{f_\xi - f_\zeta}(V_t) > \tau_\xi(V_t) > \theta_0/2, \tag{47}$$

for all $s \in 2^n$ and $\zeta < \xi$ in $\Gamma_s^{(2)}$.

Let $(s_i)_{i=1}^{2^n}$ be the lexicographical enumeration of 2^n. Using Lemma 8, Proposition 4 and a finite induction on $1 \leq i \leq 2^n$, we will choose for every $1 \leq i \leq 2^n$, the set Γ_{s_i}, the function H_{s_i}, the pair of ordinals $(\xi_{s_i}^0, \xi_{s_i}^1)$ and the family $\mathcal{J}_{s_i}$ satisfying (ii)-(vi). Suppose that for some $1 \leq k < 2^n$, $(H_{s_i})_{i\leq k}$ have been chosen so that $(H_{s_i})_{i\leq k}$ is an $((\varepsilon_i^k), (\delta_i)_{i=0}^{k-1})$-biorthogonal sequence, where $\varepsilon_i^k = (\sum_{r=1}^{k-1+1} 2^{-r})\varepsilon/2^i$. Then by Lemma 8 there are $\delta_m > 0$ and $\epsilon > 0$ such that for every $H \in V_2^0$ with $\|H\|_\infty < \epsilon$, the sequence $H_{s_1}, ..., H_{s_{m-1}}, H$ is an $((\varepsilon_i^{k+1})_{i=1}^{k+1}, (\delta_i)_{i=0}^{m})$-biorthogonal sequence. Clearly, we may suppose that $\epsilon < \varepsilon_n$. For each $\xi < \omega_1$, let $f_\xi \in \mathcal{F}$ such that $\mu_\xi = \mu_{f_\xi}$. Since $D_{\mathcal{F}}$ is countable, by Proposition 4, we have that $(\mathcal{F}, \|\cdot\|_\infty)$ is separable and so there is an uncountable subset $\Gamma_{t_{k+1}}^{(3)}$ of $\Gamma_{t_{k+1}}^{(2)}$ such that for all ζ, ξ in $\Gamma_{t_{k+1}}^{(3)}$,

$$\|f_\xi - f_\zeta\|_\infty < \epsilon/8. \tag{48}$$

Also applying Lemma 36, for the family $\mathcal{F} = \{f_\xi\}_{\xi\in\Gamma_{s_{k+1}}^{(3)}}$ we pass to a further uncountable subset $\Gamma_{s_{k+1}}^{(4)}$ of $\Gamma_{s_{k+1}}^{(3)}$ such that for every $\zeta, \xi \in \Gamma_{s_{k+1}}^{(4)}$,

$$\|\mu_{f_\xi - f_\zeta}^d\| < (\epsilon/128)^2. \tag{49}$$

We set $\Gamma_{s_{k+1}} = \Gamma^{(4)}_{s_{k+1}}$ and we choose $\xi^0_{s_{k+1}} < \xi^1_{s_{k+1}}$ in $\Gamma_{s_{k+1}}$ such that $\mu_{\xi^0_{s_{k+1}}}$ and $\mu_{\xi^1_{s_{k+1}}}$ are w*-condensation points of the set $\{\mu_\xi\}_{\xi\in\Gamma_{s_{k+1}}}$. We put $F = f_{\xi^0_{s_{k+1}}} - f_{\xi^1_{s_{k+1}}}$. Since for all $\xi < \omega_1$, $\|f_\xi\|_{V_2} = 1$, we have that $\|F\|_\infty \le 2$. Moreover by (47)-(49), we have that

$$\mu_F(V_{s_{k+1}}) > \theta_0/2, \ \ \|F\|_\infty < \epsilon/12, \ \text{ and } \ \|\mu^d_F\| < (\epsilon/128)^2. \tag{50}$$

Let $(f_n)_n$ be a sequence in X pointwise converging to F with $\|f_n\|_{V_2} \le \|F\|_{V_2}$ (see Remark 7). By Proposition 29, there exist a convex block sequence $(h_n)_n$ of $(f_n)_n$ and $\mathcal{J} \in \mathcal{A}$ such that setting $H = h_l - h_k$, for sufficiently large $k < l$, we have that

(a) $\|H\|_{V_2} \le 2\|F\|_{V_2} \le 4$ and $\|H\|_\infty \le 4\|F\|_\infty + \epsilon/2 \le \epsilon$,
(b) $\bigcup \mathcal{J} \subseteq V_{s_{k+1}}$ and $\upsilon^2_2(H,\mathcal{J}) > \theta_0/2$,
(c) for all $\mathcal{I} \in \mathcal{A}$,

$$\upsilon^2_2(H,\mathcal{I}) \le 4\mu_F(\bigcup \mathcal{I}) + 32\|f\|_{V_2}\sqrt{\|\mu^d_F\|} + \epsilon/2 \le 8(\mu_{\xi^0_{s_{k+1}}} + \mu_{\xi^1_{s_{k+1}}})(\bigcup \mathcal{I}) + \epsilon.$$

We set $H_{s_{k+1}} = H$ and $J_{s_{k+1}} = J$ and the inductive step is completed. Finally, using Proposition 1, we choose a sufficiently dense finite subset $\mathcal{P}_n \subseteq [0,1] \setminus D_{\mathcal{F}}$ determining the quadratic variation of $\langle \{H_s\}_{s\in 2^n}\rangle$ which completes the proof of the inductive step. □

Lemma 48, Proposition 45 and Proposition 42 yield the following.

Proposition 49. *Let X be a subspace of V^0_2 and suppose that X^{**} contains an uncountable family $\mathcal{F}$ such that $D_{\mathcal{F}} = \bigcup_{f\in\mathcal{F}} D_f$ is countable and $\mathcal{M}_{\mathcal{F}} = \{\mu_f\}_{f\in\mathcal{F}}$ is non-separable. Then X contains a subspace isomorphic to the space S^2.*

In the next lemma and proposition we state some results concerning the space TF and the tree families generating this space. We refer the reader to [1] for the relevant notation.

Lemma 50. *Let $\mathcal{G} = ((g_s),(I_s,J_s))_{s\in 2^{<\mathbb{N}}}$ be a tree family such that $TF = \overline{\langle\{g_s\}_s\rangle}$ and for every $n \ge 0$, let $K_n = \bigcup_{s\in 2^n} I_s$. Then for every $f \in TF^{**}$, supp $\mu_f \subseteq K$, where $K = \bigcap_{n=0}^\infty K_n$.*

Proof. Let $f \in TF^{**}$ and let $(f_m)_m$ be a sequence in $\langle\{g_s\}_s\rangle$ pointwise convergent to f and such that $\|f_n\|_{V_2} \le \|f\|_{V_2}$. For each $n \ge 0$, let $P_n : TF^{**} \to G_n$ be the natural projection onto the finite-dimensional space $G_n = \langle\{g_s\}_{|s|<n}\rangle$ (where $G_0 = \{0\}$). Also let $h^n_m = f_m - P_n(f_m)$ and $h_n = f - P_n(f)$. Since P_n is w*-w* continuous, the sequence $(h^n_m)_m$ is pointwise convergent to h_n and so *supp* $h_n \subseteq K_n$. Since $\mu_{h_n} = \mu_f$ and *supp* $\mu_{h_n} \subseteq$ *supp* h_n, we conclude that *supp* $\mu_f \subseteq K_n$ for all $n \ge 0$. □

Proposition 51. *The set $\mathcal{M}^c_{TF^{**}} = \{\mu_f : f \in TF^{**} \cap C[0,1]\}$ is a non-separable subset of $\mathcal{M}[0,1]$. Therefore the space S^2 is embedded into TF.*

Proof. Let $\mathcal{G} = ((g_s),(I_s,J_s))_{s\in 2^{<\mathbb{N}}}$ be a tree family such that $TF = \overline{\langle\{g_s\}_s\rangle}$. Also let $(T_\sigma)_{\sigma\in 2^{\mathbb{N}}}$ be the almost disjoint family of dyadic subtrees in $2^{<\mathbb{N}}$ defined in Remark 13. For each $\sigma \in 2^{\mathbb{N}}$, we set $\mathcal{G}_\sigma = ((g_s),(I_s,J_s))_{s\in T_\sigma}$. Since $\mathcal{G}_\sigma$ is also a tree family, the space $X_\sigma = \overline{\langle\{g_s\}_{s\in T_\sigma}\rangle}$ is a copy of TF. Therefore c_0 is embedded into X_σ which gives that $(X^{**}_\sigma \setminus X) \cap C[0,1] \ne \emptyset$ (cf. Remark 9). So for each $\sigma \in 2^{\mathbb{N}}$, we can pick an $f_\sigma \in X^{**}_\sigma \setminus X \cap C[0,1]$. Setting $T_\sigma = (t^\sigma_s)_{s\in 2^{\mathbb{N}}}$ and $K_\sigma = \bigcap_n \bigcup_{s\in 2^n} I_{t^\sigma_s}$, by Lemma 50 we have that *supp* $\mu_{f_\sigma} \subseteq K_\sigma$. Since $(T_\sigma)_{\sigma\in 2^{\mathbb{N}}}$ is an almost disjoint family, we get that $(K_\sigma)_{\sigma\in 2^{\mathbb{N}}}$ is a disjoint family of compact subsets of $[0,1]$ and so $\{\mu_{f_\sigma}\}_{\sigma\in 2^{\mathbb{N}}}$ consists of pairwise singular positive measures. As $\{f_\sigma\}_{\sigma\in 2^{\mathbb{N}}} \subseteq TF^{**}\cap C[0,1]$, we conclude that $\mathcal{M}^c_{TF^{**}}$ is non-separable. Finally, that S^2 is embedded into X follows by Proposition 49, for $\mathcal{F} = TF^{**}\cap C[0,1]$. □

Proposition 52. *Let X be a subspace of V_2^0 such that the space S^2 is embedded into X. Then the set $\mathcal{M}_{X^{**}} = \{\mu_f : f\in X^{**}\}$ is a non-separable subset of $\mathcal{M}[0,1]$.*

Proof. Let T be an isomorphic embedding of S^2 into X and let $f_s = T(e_s)$, where $(e_s)_s$ is the usual basis of S^2. From [3] we have that for each $\sigma\in 2^{\mathbb{N}}$, the sequence $(\sum_{k=0}^n f_{\sigma|k})_n$ is pointwise converging to a function $f_\sigma \in (X^{**}\setminus X)\cap C[0,1]$. Hence there exist an uncountable subset $\Sigma\subseteq 2^{\mathbb{N}}$ and $\delta>0$ such that for all $\sigma\in\Sigma$, $dist(f_\sigma, X) > \delta$. We will show that the set $\{\mu_{f_\sigma} : \sigma\in\Sigma\}\subseteq \mathcal{M}[0,1]$ is non-separable. Indeed, otherwise, we can choose a norm-condensation point $\mu\in\mathcal{M}[0,1]$ of $\{\mu_\sigma : \sigma\in 2^{\mathbb{N}}\}$. Also fix a positive integer $m\in\mathbb{N}$ and $\varepsilon>0$. Then for uncountably many $\sigma\in\Sigma$, we have that

$$\|\mu_{f_\sigma} - \mu\| \le \varepsilon/m. \tag{51}$$

Let $\sigma_1,...,\sigma_m\in\Sigma$ satisfy (51) and let $n_0\in\mathbb{N}$ be such that for all $n\ge n_0$ and $1\le i<j\le m$, $\sigma_i|n \perp \sigma_j|n$. We set $g_{\sigma_i} = \sum_{n\ge n_0} f_{\sigma_i|n}$. Since $f_{\sigma_i} - g_{\sigma_i} = \sum_{n<n_0} f_{\sigma_i|n}\in X$, we have that $dist(g_{\sigma_i}, X) = dist(f_{\sigma_i}, X) > \delta$ and $\mu_{f_{\sigma_i}} = \mu_{g_{\sigma_i}}$. For every $n\in\mathbb{N}$, let $F_n^i = \sum_{k=n_0}^{n_0+n} f_{\sigma_i|k}$. Then

$$\|F_n^i\|_{V_2} \le \|T\| \Big\| \sum_{k=n_0}^{n_0+n} e_{\sigma_i|k}\Big\|_{S^2} \le \|T\|.$$

Applying Proposition 34, for the continuous function $g_{\sigma_i}\in X^{**}\setminus X$, the sequence $(F_n^i)_n$ and $\varepsilon_n = \varepsilon/m2^n$, we obtain a convex block sequence $(h_n^i)_n$ of $(F_n^i)_n$ such that the functions $G_n^i = h_{2n+1}^i - h_{2n}^i$ satisfy the following. (i) $\delta < \|G_n^i\|_{V_2}\le 2\|T\|$, (ii) $\|G_n^i\|_\infty < \varepsilon/(m2^{2n})$ and (iii) for every $\mathcal{I}\in\mathcal{A}$, $v_2^2(G_n^i,\mathcal{I}) \le 4\mu_{f_{\sigma_i}}(\bigcup\mathcal{I}) + \varepsilon/(m2^{2n})$.

By (ii) and Lemma 8, we can choose $n_1<...<n_m$ such that the finite sequence $(G_{n_i}^i)_{i=1}^m$ is $\varepsilon/m-$biorthogonal. By the definition of G_n^i, we have that $G_{n_i}^i = \sum_{s\in F_i}\lambda_s f_s$, where F_i is a finite subset of $\{\sigma_i|n : n\in\mathbb{N}\}$. Hence

$$\delta < \|G_{n_i}^i\|_{V_2} \le \|T\|\Big\|\sum_{s\in F_i}\lambda_s e_s\Big\|_{S^2} \le \|T\| \max_{s\in F_i}|\lambda_s|. \tag{52}$$

Let $s_i\in F_i$ be such that $|\lambda_{s_i}| = \max_{s\in F_i}|\lambda_s|$. Then by (52), $|\lambda_{s_i}|\ge \delta/\|T\|$ and so, since the set $\{s_i : 1\le i\le m\}$ is an antichain of $2^{<\mathbb{N}}$, we get that

$$\Big\|\sum_{i=1}^m G_{n_i}^i\Big\|_{V_2} \ge \frac{1}{\|T^{-1}\|}\Big\|\sum_{i=1}^m\sum_{s\in F_i}\lambda_s e_s\Big\|_{S^2} \ge \frac{1}{\|T^{-1}\|}\sqrt{\frac{m\delta^2}{\|T\|^2}} \ge \frac{\delta\sqrt{m}}{\|T^{-1}\|\|T\|}. \tag{53}$$

By Lemma 11 and (iii), for every $\mathcal{I}\in\mathcal{A}$ we have that

$$\begin{aligned} v_2^2(\sum_{i=1}^m G_{n_i}^i,\mathcal{I}) &\le \sum_{i=1}^m v_2^2(G_{n_i}^i,\mathcal{I}^{(i)}) + (4\|T\|+1)\varepsilon \le 4\sum_{i=1}^m \mu_{f_{\sigma_i}}(\bigcup \mathcal{I}^{(i)}) + (4\|T\|+2)\varepsilon \\ &\le 4\mu(\bigcup\mathcal{I}) + \varepsilon + (4\|T\|+2)\varepsilon. \le 4\|\mu\| + (4\|T\|+3)\varepsilon. \end{aligned}$$

Therefore, letting $\varepsilon\to 0$, $4\|\mu\| \ge \Big\|\sum_{i=1}^m G_{n_i}^i\Big\|_{V_2}$ and so by (53), we get a contradiction. □

We are finally ready to prove the main results of the paper.

Theorem 53. *Let X be a subspace of V_2^0. Then the space S^2 is embedded into X if and only if $\mathcal{M}_{X^{**}}$ is non-separable.*

Proof. By Proposition 52, if S^2 is embedded into X, then $\mathcal{M}_{X^{**}}$ is non-separable. Conversely suppose that $\mathcal{M}_{X^{**}}$ is non-separable. Then we distinguish two cases. If X^* is separable, then by Proposition 2, the set $D_{X^{**}}$ is countable and hence by Proposition 49, for $\mathcal{F} = X^{**}$, we get that S^2 is embedded into X. In the case that X^* is non-separable, by [1], the space TF is embedded into X. By Proposition 51, we have that S_2 is embedded into TF and hence into X. □

Theorem 54. *Let X be a subspace of V_2^0. Then c_0 is embedded into X if and only if X^{**} is non-separable.*

Proof. Suppose that X^{**} is non-separable (the other direction is obvious). If X^* is non-separable, then as we have already mentioned the space TF and hence c_0 is embedded into X. So assume that X^* is separable. We distinguish the following cases. If $\mathcal{M}_{X^{**}}$ is non-separable, then the result follows by Theorem 53. Otherwise, $\mathcal{M}_{X^{**}}$ is separable and so by Proposition 37, c_0 is again embedded into X. □

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