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ON THE CR–OBATA THEOREM AND SOME EXTREMAL PROBLEMS ASSOCIATED TO PSEUDOSCALAR CURVATURE ON THE REAL ELLIPSOIDS IN $\mathbb{C}^{n+1}$

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Abstract. This paper studies the CR-version of Obata theorem on a pseudo-Hermitian CR-manifold (M,θ). The main result of the paper is proving that CR–Obata theorem holds on real ellipsoid $E(A)$ with contact form $\theta = \frac{1}{2i}(\partial\rho_A - \bar{\partial}\rho_A)$, where $\rho_A(z) = |z|^2 + \mathrm{Re}\sum_{j=1}^n A_j z_j^2 - 1$ with $A_j \in (-1,1)$.

1. Introduction

Let (M,θ) be a $(2n+1)$-dimensional strictly pseudoconvex pseudo-Hermitian CR-manifold with CR-dimension n in the sense of Webster [26]. Here θ is a real nowhere-vanishing one-form on M, which is the so-called Hermitian form or the contact form on M which annihilates the holomorphic tangent bundle $H(M)$ of M. Let $\{\theta^1,\cdots,\theta^n\}$ be a local basis for the holomorphic cotangent bundle $H(M)^*$ so that

$$d\theta = i\sum_{\alpha,\beta=1}^n h_{\alpha\overline{\beta}}\theta^\alpha\wedge\theta^{\overline{\beta}}, \tag{1.1}$$

where $(h_{\alpha\overline{\beta}})$ is the positive definite $n\times n$ matrix on M, which is uniquely determined by the Levi form $L_\theta(u,\overline{v}) = -id\theta(u,\overline{v})$ for $u,v\in H(M)$. Let Δ_{sb} be the sub-Laplacian with respect to θ (see [8], [22] and [6] for references) and let μ_1 be the first positive eigenvalue of Δ_{sb} on (M,θ). Let $R_{\alpha\overline{\beta}}$ be the Webster pseudo-Ricci curvature, R be the pseudoscalar curvature and Tor be the pseudotorsion defined in [26]. Under the condition:[1]

$$\mathrm{Ric}_z(w,\overline{w}) - \frac{n+1}{2}\mathrm{Tor}_z(w,w) \geq kh(w,\overline{w}), \quad w\in H_z(M), z\in M, \tag{1.2}$$

where h is the metric given by $(h_{\alpha\overline{\beta}})$ and k is a positive constant, it was proved by Greenleaf [8] (for $n\geq 3$), and by Li and Luk [22] (for $n\geq 2$) that $\mu_1 \geq \frac{n}{n+1}k$. For the case $n=1$, the same estimate was proved by Li and Luk [22] with an extra condition which involves a distinguished covariant derivative of the torsion in addition to (1.2), and by Chiu [5] with the assumption that the CR Paneitz operator

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[1]The factor $-\frac{n+1}{2}$ in (1.2) rather than $\frac{n}{2}$ is the correct one (see remark in [6]).

is positive coupling with (1.2). This is the CR-version of Lichnerowicz's theorem. Conversely, whether the CR-version of Obata's theorem holds is still open. Namely, if $\mu_1 = \frac{nk}{n+1}$ and the condition (1.2) holds, does this imply that M is CR-equivalent to the unit sphere in $\mathbb{C}^{n+1}$? Recently, it was proved by Chang and Chiu [6] (for $n = 1$) and by Li [21] (for $n \geq 2$) that the CR-Obata theorem holds when (M, θ) is torsion-free.[2] One of the main purposes of this paper is to provide an example of pseudo-Hermitian manifolds which are not torsion-free, but the CR-Obata theorem holds. Specifically, we consider the real ellipsoid in $\mathbb{C}^k$, which is a typical example in the theory of pseudo-Hermitian CR-manifolds:

$$E(a,b) = \Big\{ z = (x_1 + iy_1, \cdots, x_k + iy_k) : \sum_{j=1}^{k} a_j x_j^2 + b_j y_j^2 = 1 \Big\}, \tag{1.3}$$

where a_j, b_j are positive numbers for $1 \leq j \leq k$. After an affine transformation $z_j' = \sqrt{\frac{a_j+b_j}{2}} z_j$, the ellipsoid can be rewritten as

$$E(A) = \Big\{ z \in \mathbb{C}^k : \rho^A(z) = |z|^2 + \sum_{j=1}^{k} A_j \operatorname{Re} z_j^2 - 1 = 0 \Big\}, \quad A_j = \frac{a_j - b_j}{a_j + b_j}. \tag{1.4}$$

If it is necessary, we make a change of variable by $z_j' = iz_j$ and do a permutation on the indices of the variables. Then without loss of generality, we may assume that

$$0 \leq A_1 \leq A_2 \leq \cdots \leq A_k. \tag{1.5}$$

Let

$$\theta = \frac{1}{2i}(\partial \rho^A(z) - \overline{\partial} \rho^A(z)), \quad z \in E(A). \tag{1.6}$$

It is easy to show that

$$h(w, \overline{w}) = |w|^2, \quad w \in H_z(E(A)). \tag{1.7}$$

Let

$$k_0 = \min \Big\{ \mathrm{Ric}_z(w, \overline{w}) - \frac{n+1}{2} \mathrm{Tor}_z(w,w) : w \in H_z(E(A)) \cap \partial B_k, z \in E(A) \Big\}. \tag{1.8}$$

Then we shall prove the following theorem.

Theorem 1.1. *Let $E(A)$ be the ellipsoid in $\mathbb{C}^{n+1}$ with $A = (A_1, \cdots, A_{n+1}) \in [0,1)^{n+1}$. Let μ_1 be the first positive eigenvalue for the sub-Laplacian Δ_{sb} on $(E(A), \theta)$. Then the following two statements hold:*

(a) $k_0 = \Big((n+1)(1 - A_{n+1}) - A_{n+1}^2 \Big) / (1 + A_n)$.

(b) For $n \geq 2$, if $A_{n+1} < \frac{1}{2}\Big(\sqrt{(n+1)^2 + 4(n+1)} - (n+1) \Big)$, then $\mu_1 \geq \frac{nk_0}{n+1} > 0$ and $\mu_1 = \frac{nk_0}{n+1}$ if and only if $E(A) = S^{2n+1}$.

For the case $n = 1$, much effort is needed to verify whether the vector A associated to the ellipsoid $E(A)$ satisfies the extra condition in [22] or in [5] except when $A = \mathbf{0}$. Instead, we shall prove the following proposition.

[2] After we submitted this paper, the authors found that this result was proved independently by Chang and Chiu in [7].

Proposition 1.2. *Let $E(A)$ be the ellipsoid in $\mathbb{C}^2$ with $A \in [0,1)^2$, and let μ_1 be the first positive eigenvalue for the sub-Laplacian Δ_{sb} on $(E(A), \theta)$. Then*

$$\mu_1 \leq \frac{1 - A_2^2}{1 - \frac{1}{3}A_2^2 + \frac{2}{3}A_2}$$

and the equality holds if and only if $E(A) = S^3$, the unit sphere in $\mathbb{C}^2$.

Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^{n+1}$ with defining function ρ. Let $\theta = \frac{1}{2i}(\partial\rho - \overline{\partial}\rho)$. Then $(\partial D, \theta)$ becomes a strictly pseudoconvex pseudo-Hermitian manifold in the sense of Webster [26]. Under the assumption that $\log J(\rho)$ is harmonic in the metric $U_{i\bar{j}} dz_i \otimes d\overline{z}_j$ in D with $U = -\log(-\rho)$, Li [20] proves that if the pseudoscalar curvature R_θ is constant on ∂D, then D is biholomorphic to the unit ball B_{n+1}. This is particularly true when $U = -\log(-\rho)$ is the potential function for the Kähler-Einstein metric for D given by Cheng and Yau [4]. It is not difficult to show that $\log J(\rho^A)$ is not harmonic in the metric $U_{i\bar{j}} dz_i \otimes d\overline{z}_j$ with $U = -\log(-\rho^A(z))$. However, the second purpose of this note is to prove a stronger theorem on $(E(A), \theta_A)$ than the one in [20], where $\theta_A = \frac{1}{2i}(\partial\rho^A - \overline{\partial}\rho^A)$. Let R_A denote the pseudoscalar curvature on $E(A)$. Let s_A and r_A be chosen so that

$$s_A^2(1 + A_{n+1}) = 1, \quad r_A^2(1 - A_{n+1}) = 1, \quad e_{n+1} = (0, 0, \cdots, 0, 1). \tag{1.9}$$

Then we shall prove the following theorem.

Theorem 1.3. *Under the assumptions* (1.5) *and* (1.9), *if $R_A(s_A e_{n+1}) = R_A(i r_A e_{n+1})$, then $E(A) = S^{2n+1}$.*

The paper is arranged as follows: In Section 2, we will find the exact value of k_0 and prove Theorem 1.1. The pseudoscalar curvature will be calculated and Theorem 1.3 will be proved in Section 3. Finally, in Section 4, we will prove Proposition 1.2.

2. Proof of Theorem 1.1

Let

$$\rho(z) = \rho^A(z) = |z|^2 + \mathrm{Re}\sum_{j=1}^{n+1} A_j z_j^2 - 1, \quad A_j = \frac{a_j - b_j}{a_j + b_j} \in [0,1) \tag{2.1}$$

and let the real ellipsoid associated to A be

$$E(A) = \{z \in \mathbb{C}^{n+1} : \rho(z) = 0\}, \quad A = (A_1, \cdots, A_{n+1}) \in [0,1)^{n+1}. \tag{2.2}$$

We will assume (1.5) from now on. It is easy to see that

$$H(\rho)(z) = I_{n+1} \tag{2.3}$$

and $E(\mathbf{0}) = S^{2n+1}$, the unit sphere in $\mathbb{C}^{n+1}$. Let $J(u)$ be the Fefferman functional defined as follows:

$$J(u) = -\det \begin{bmatrix} u & \overline{\partial} u \\ (\overline{\partial} u)^* & H(u) \end{bmatrix}. \tag{2.4}$$

Then one can find that

$$
\begin{aligned}
J(\rho)(z) &= \det H(\rho)(z)\Big[-\rho(z)+\sum_{j=1}^{n+1}|\partial_j\rho(z)|^2\Big] \\
&= -\rho(z)+\sum_{j=1}^{n+1}|\overline{z}_j+A_jz_j|^2 \\
&= -\rho(z)+|z|^2+2\,\mathrm{Re}\sum_{j=1}^{n+1}A_jz_j^2+\sum_{j=1}^{n+1}A_j^2|z_j|^2 \\
&= 1+\mathrm{Re}\sum_{j=1}^{n+1}A_jz_j^2+\sum_{j=1}^{n+1}A_j^2|z_j|^2.
\end{aligned}
\tag{2.5}
$$

Let

$$
\theta=\frac{1}{2i}(\partial\rho(z)-\overline{\partial}\rho(z)) \tag{2.6}
$$

and let k_0 be defined as in (1.8). Then we shall prove the following proposition.

Proposition 2.1. *Under the assumption (1.5), one has*

$$
k_0=\frac{(n+1)(1-A_{n+1})-A_{n+1}^2}{1+A_n}. \tag{2.7}
$$

Proof. According to the formula given by Li and Luk [23] for the pseudo-Ricci curvatures, for any $w\in H_z(E(A))$, the Webster pseudo-Ricci curvature of $(E(A),\theta)$ is given by

$$
\begin{aligned}
\mathrm{Ric}_z(w,\overline{w}) &= -\sum_{i,j=1}^{n+1}\frac{\partial^2\log J(\rho)}{\partial z_i\partial\overline{z}_j}w_i\overline{w}_j+(n+1)\frac{\det H(\rho)}{J(\rho)}\sum_{i,j=1}^{n+1}\frac{\partial^2\rho(z)}{\partial z_i\partial\overline{z}_j}w_i\overline{w}_j \\
&= \frac{1}{J(\rho)^2}\Big|\sum_{j=1}^{n+1}w_j\partial_jJ\Big|^2+\frac{1}{J(\rho)}\sum_{i,j=1}^{n+1}\frac{\partial^2((n+1)\rho(z)-J(\rho))}{\partial z_i\partial\overline{z}_j}w_i\overline{w}_j \\
&= \frac{1}{J(\rho)^2}\Big|\sum_{j=1}^{n+1}w_j\partial_jJ\Big|^2+\frac{1}{J(\rho)}\sum_{j=1}^{n+1}(n+1-A_j^2)|w_j|^2.
\end{aligned}
$$

According to the formula given by Li and Luk [24] for the pseudotorsion, for any $w\in H_z(E(A))$, one has that

$$
\mathrm{Tor}_z(w,w)=\frac{2}{J(\rho)}\,\mathrm{Re}\sum_{i,j=1}^{n+1}w_iw_j(\overline{N}-\det H(\rho))\rho_{ij}=-\frac{2}{J(\rho)}\,\mathrm{Re}\sum_{j=1}^{n+1}A_jw_j^2,
$$

where $\overline{N}f$ is the complex normal derivative of f.

Thus, for $w \in H_z(E(A)) \cap S^{2n+1}$ and $z \in E(A)$, we have

$$\begin{aligned}
(2.8)\quad & \mathrm{Ric}_z(w,\overline{w}) - \frac{n+1}{2}\mathrm{Tor}_z(w,w) \\
&= \frac{1}{J(\rho)}\sum_{j=1}^{n+1}[(n+1-A_j^2)|w_j|^2 + (n+1)\,\mathrm{Re}(A_j w_j^2)] + \frac{1}{J(\rho)^2}\left|\sum_{j=1}^{n+1} w_j \partial_j J\right|^2 \\
&= \frac{1}{J(\rho)}\sum_{j=1}^{n+1}[(n+1-A_j^2)|w_j|^2 + (n+1)\,\mathrm{Re}(A_j w_j^2)] + \frac{1}{J(\rho)^2}\left|\sum_{j=1}^{n+1} A_j w_j \rho_{\overline{j}}\right|^2 \\
&= \frac{K_z(w)}{J(\rho)(z)},
\end{aligned}$$

where

$$(2.9)\quad K_z(w) = (n+1)\left(1 + \sum_{j=1}^{n+1} A_j \,\mathrm{Re}\, w_j^2\right) - \sum_{j=1}^{n+1} A_j^2 |w_j|^2 + \frac{1}{J(\rho)}\left|\sum_{j=1}^{n+1} A_j w_j \rho_{\overline{j}}\right|^2$$

for $z \in E(A)$ and $w \in H_z(E(A)) \cap \partial B_{n+1}$. Let

$$(2.10)\quad k_0(z) = \min\left\{\mathrm{Ric}_z(w,\overline{w}) - \frac{n+1}{2}\mathrm{Tor}_z(w,w) : w \in H_z(E(A)) \cap \partial B_{n+1}\right\}.$$

Since $|w_{n+1}|^2 + \sum_{j=1}^n |w_j|^2 = 1$, one has

(2.11)

$$\begin{aligned}
K_z(w) &\geq (n+1) - (n+1)\sum_{j=1}^{n+1} A_j |w_j|^2 - \sum_{j=1}^{n+1} A_j^2 |w_j|^2 + \frac{1}{J(\rho)}\left|\sum_{j=1}^{n+1} A_j \rho_{\overline{j}} w_j\right|^2 \\
&= (n+1)(1-A_{n+1}) - A_{n+1}^2 + (n+1)\sum_{j=1}^{n}(A_{n+1} - A_j)|w_j|^2 \\
&\quad + \sum_{j=1}^{n}(A_{n+1}^2 - A_j^2)|w_j|^2 + \frac{1}{J(\rho)}\left|\sum_{j=1}^{n+1} A_j \rho_{\overline{j}} w_j\right|^2 \\
&= (n+1)(1-A_{n+1}) - A_{n+1}^2 + G_z(w),
\end{aligned}$$

where

$$(2.12)\quad G_z(w) = \sum_{j=1}^{n}[(n+1)(A_{n+1} - A_j) + A_{n+1}^2 - A_j^2]|w_j|^2 + \frac{|\sum_{j=1}^{n+1} A_j \rho_{\overline{j}} w_j|^2}{J(\rho)}.$$

Therefore,

$$\begin{aligned}
(2.13)\qquad \frac{K_z(w)}{J(\rho)(z)} &\geq \frac{(n+1)(1-A_{n+1}) - A_{n+1}^2}{J(\rho)(z)} + \frac{G_z(w)}{J(\rho)(z)} \\
&= \frac{(n+1)(1-A_{n+1}) - A_{n+1}^2}{1+A_n} + \frac{n+1}{(1+A_n)J(\rho)(z)}\mathcal{E}(z,w),
\end{aligned}$$

where

$$(2.14)\quad \mathcal{E}(z,w) = \left[1 - A_{n+1} - \frac{1}{n+1}A_{n+1}^2\right](1 + A_n - J(\rho)(z)) + \frac{(1+A_n)}{n+1}G_z(w).$$

For any $z \in E(A)$, we have

$$
\begin{aligned}
J(\rho)(z) &= 2 - \sum_{j=1}^{n+1}(1-A_j^2)|z_j|^2 \\
&= 2 - \sum_{j=1}^{n+1}(1-A_j)(1+A_j)|z_j|^2 \\
&\le 2 - \sum_{j=1}^{n+1}(1-A_j)(|z_j|^2 + A_j \operatorname{Re} z_j^2) \\
&= 1 + A_n - \sum_{j=1}^{n+1}(A_n - A_j)(|z_j|^2 + A_j \operatorname{Re} z_j^2) \\
&\le 1 + A_n + (A_{n+1} - A_n)(|z_{n+1}|^2 + \operatorname{Re}(A_{n+1} z_{n+1}^2)).
\end{aligned} \tag{2.15}
$$

Moreover, since $2A_{n+1} < \sqrt{(n+1)^2 + 4(n+1)} - (n+1)$, we have $(1 - A_{n+1}) - \frac{A_{n+1}^2}{n+1} > 0$. Thus by (2.15),

$$
\begin{aligned}
\mathcal{E} &= \left[(1-A_{n+1}) - \frac{A_{n+1}^2}{n+1}\right](1 + A_n - J(\rho)(z)) + (1+A_n)\frac{G_z(w)}{n+1} \\
&\ge -\left[(1-A_{n+1}) - \frac{A_{n+1}^2}{n+1}\right](A_{n+1} - A_n)(|z_{n+1}|^2 + A_{n+1} \operatorname{Re} z_{n+1}^2) \\
&\quad + (1+A_n)(A_{n+1} - A_n)\left(1 + \frac{A_{n+1} + A_n}{n+1}\right)\sum_{j=1}^{n}|w_j|^2 \\
&= (A_{n+1} - A_n)\Bigg[- \left[1 - A_{n+1} - \frac{A_{n+1}^2}{n+1}\right](|z_{n+1}|^2 + A_{n+1} \operatorname{Re} z_{n+1}^2) \\
&\quad + (1+A_n)\left(1 + \frac{A_n + A_{n+1}}{n+1}\right)\sum_{j=1}^{n}|w_j|^2\Bigg],
\end{aligned} \tag{2.16}
$$

and the equality holds if and only if

$$
\sum_{j=1}^{n}(A_n - A_j)(|z_j|^2 + A_j \operatorname{Re} z_j^2) = 0 \tag{2.17}
$$

and

$$
A_j|z_j|^2 = A_j \operatorname{Re} z_j^2 \quad \text{for } j = 1, 2, \cdots, n+1. \tag{2.17a}
$$

Since

$$
\sum_{j=1}^{n+1}(|z_j|^2 + \operatorname{Re} A_j z_j^2) = 1
$$

and

$$
|z_j|^2 + \operatorname{Re} A_j z_j^2 \ge |z_j|^2 - A_j|z_j|^2 = (1-A_j)|z_j|^2 \ge 0 \quad \text{for } j = 1, 2, \cdots, n+1,
$$

we have

$$
|z_j|^2 + \operatorname{Re} A_j z_j^2 \le 1 \quad \text{for } j = 1, 2, \cdots, n+1.
$$

Therefore, by (2.15), one has

$$
J(\rho)(z) \le 1 + A_{n+1}, \quad z \in E(A). \tag{2.18}
$$

In particular, we also need the following estimate:

$$J(\rho)(z) \le 1 + A_n \quad \text{if} \quad \operatorname{Re}(A_{n+1} z_{n+1}^2) < -A_{n+1}^2 |z_{n+1}|^2. \tag{2.19}$$

In fact, since $A_j \le A_n$ if $j \le n$, $|z_{n+1}|^2 + \operatorname{Re} A_{n+1} z_{n+1}^2 \le (1 - A_{n+1}^2)|z_{n+1}|^2$ and $\sum_{j=1}^{n+1} |z_j|^2 + \operatorname{Re} A_j z_j^2 = 1$ on $E(A)$, with the computation in (2.15), one has

$$\begin{aligned} J(\rho)(z) &\le 2 - (1 - A_n) \sum_{j=1}^{n} (|z_j|^2 + A_j \operatorname{Re} z_j^2) - (1 - A_{n+1}^2)|z_{n+1}|^2 \\ &= 1 + A_n + (1 - A_n)(|z_{n+1}|^2 + \operatorname{Re}(A_{n+1} z_{n+1}^2)) - (1 - A_{n+1}^2)|z_{n+1}|^2 \\ &\le 1 + A_n + (1 - A_n)(1 - A_{n+1}^2)|z_{n+1}|^2 - (1 - A_{n+1}^2)|z_{n+1}|^2 \\ &= 1 + A_n - A_n(1 - A_{n+1}^2)|z_{n+1}|^2 \\ &\le 1 + A_n. \end{aligned}$$

Since $w \in H_z(E(A)) \cap \partial B_{n+1}$, we have

$$|\rho_{n+1}(z)|^2 |w_{n+1}|^2 = \left| \sum_{j=1}^{n} \rho_j(z) w_j \right|^2 \le \left(\sum_{j=1}^{n} |\rho_j|^2 \right) \left(\sum_{j=1}^{n} |w_j|^2 \right).$$

Notice that $|w_{n+1}|^2 = 1 - \sum_{j=1}^{n} |w_j|^2$ and $J(\rho)(z) = \sum_{j=1}^{n+1} |\rho_j(z)|^2$. Thus,

$$|\rho_{n+1}(z)|^2 \le J(\rho)(z) \sum_{j=1}^{n} |w_j|^2.$$

This implies that

$$\sum_{j=1}^{n} |w_j|^2 \ge \frac{|\rho_{n+1}|^2}{J(\rho)(z)}. \tag{2.20}$$

Since

$$|\rho_{n+1}(z)| = |\overline{z}_{n+1} + A_{n+1} z_{n+1}| \ge |z_{n+1}|(1 - A_{n+1}), \tag{2.21}$$

one has

$$|z_{n+1}|^2 + A_{n+1} \operatorname{Re} z_{n+1}^2 = \operatorname{Re}(z_{n+1} \rho_{n+1}(z)) \le \frac{|\rho_{n+1}|^2}{1 - A_{n+1}}. \tag{2.22}$$

Moreover,

$$|\rho_{n+1}| \ge |z_{n+1}|(1 - A_{n+1}^2) \quad \text{if} \quad \operatorname{Re} z_{n+1}^2 \ge -A_{n+1} |z_{n+1}|^2. \tag{2.23}$$

Thus

$$|z_{n+1}|^2 + A_{n+1} \operatorname{Re} z_{n+1}^2 \le \frac{|\rho_{n+1}|^2}{1 - A_{n+1}^2} \text{ if } \operatorname{Re} z_{n+1}^2 \ge -A_{n+1} |z_{n+1}|^2. \tag{2.24}$$

Now we continue estimating $\mathcal{E}$.

Case 1. If $\mathrm{Re}(z_{n+1}^2) \geq -A_{n+1}|z_{n+1}|^2$, then by (2.16), (2.18), (2.20), and (2.24), one has

$$\begin{aligned}\mathcal{E} &\geq (A_{n+1}-A_n)\Big[-\Big[1-A_{n+1}-\frac{A_{n+1}^2}{n+1}\Big]\frac{|\rho_{n+1}(z)|^2}{1-A_{n+1}^2} \\ &\qquad + (1+A_n)\Big(1+\frac{A_n+A_{n+1}}{n+1}\Big)\frac{|\rho_{n+1}(z)|^2}{J(\rho)(z)}\Big] \\ &\geq (A_{n+1}-A_n)\Big[-\frac{|\rho_{n+1}(z)|^2}{1+A_{n+1}}+\frac{A_{n+1}^2}{(n+1)(1-A_{n+1}^2)}|\rho_{n+1}(z)|^2 \\ &\qquad + (1+A_n)\Big(1+\frac{A_n+A_{n+1}}{n+1}\Big)\frac{|\rho_{n+1}(z)|^2}{1+A_{n+1}}\Big] \\ &\geq (A_{n+1}-A_n)\Big[\frac{A_n|\rho_{n+1}(z)|^2}{1+A_{n+1}}+\frac{A_{n+1}^2|\rho_{n+1}(z)|^2}{(n+1)(1-A_{n+1}^2)}\Big] \\ &\geq 0.\end{aligned} \tag{2.25}$$

Case 2. If $\mathrm{Re}(z_{n+1}^2) < -A_{n+1}|z_{n+1}|^2$, then by (2.16), (2.19), (2.20) and (2.22), one has

$$\begin{aligned}\mathcal{E} &\geq (A_{n+1}-A_n)\Big[-\Big[1-A_{n+1}-\frac{A_{n+1}^2}{n+1}\Big]\frac{|\rho_{n+1}|^2}{1-A_{n+1}}+\Big(1+\frac{A_n+A_{n+1}}{n+1}\Big)|\rho_{n+1}|^2\Big] \\ &\geq |\rho_{n+1}(z)|^2(A_{n+1}-A_n)\Big[-1+\frac{A_{n+1}^2}{(n+1)(1-A_{n+1})}+1+\frac{A_n+A_{n+1}}{n+1}\Big] \\ &= |\rho_{n+1}(z)|^2(A_{n+1}-A_n)\Big[\frac{A_{n+1}^2}{(n+1)(1-A_{n+1})}+\frac{A_n+A_{n+1}}{n+1}\Big] \\ &\geq 0.\end{aligned} \tag{2.26}$$

Therefore, we have

$$k_0(z)=\min\Big\{\frac{K_z(w)}{J(\rho)(z)} : w\in H_z(E(A))\cap\partial B_{n+1}\Big\}\geq\frac{(n+1)(1-A_{n+1})-A_{n+1}^2}{1+A_n}. \tag{2.27}$$

Let

$$s_j=\frac{1}{\sqrt{1+A_j}},\quad \mathbf{e}_j=(0,\cdots,0,1,0,\cdots,0)\in\partial B_{n+1}. \tag{2.28}$$

Then

$$z(j)=s_j\mathbf{e}_j\in E(A),\quad 1\leq j\leq n+1, \tag{2.29}$$

and

$$J(\rho)(z(j))=2-(1-A_j^2)|s_j|^2=2-\frac{1-A_j^2}{1+A_j}=1+A_j. \tag{2.30}$$

If $A_j=0$, then we let $w(j)=\mathbf{e}_j$. Otherwise, we let

$$w(j)=\sqrt{-1}\,\mathbf{e}_j. \tag{2.31}$$

Then $w(j) \in H_{z(k)}(E(A)) \cap \partial B_{n+1}$ for all $j \neq k$. By (2.8), (2.9) and (2.10), it is not hard to see that

(2.32)
$$\begin{aligned} k_0(z(n)) &= \frac{1}{J(\rho)(z(n))}\left[(n+1)\left(1+\sum_{j=1}^{n+1} A_j \operatorname{Re} w(n+1)_j^2\right) - \sum_{j=1}^{n+1} A_j^2 |w(n+1)_j|^2\right] \\ &= \frac{(n+1)(1-A_{n+1}) - A_{n+1}^2}{1+A_n}. \end{aligned}$$

Therefore, $k_0 = k_0(z(n))$ and (2.7) holds. The proof of the proposition is complete. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1**.**

Proof. Part (a) of Theorem 1.1 follows directly from the previous proposition. Next we prove Part (b). By Part (a) and under the assumption of Part (b), one can easily verify that $k_0 > 0$. For any $w \in H_z(E(A))$, one has

$$h(w, \overline{w}) = |w|^2.$$

Thus, for $n \geq 2$ and $z \in E(A)$,

$$\operatorname{Ric}_z(w, \overline{w}) - \frac{n+1}{2}\operatorname{Tor}_z(w, w) \geq k_0 h(w, \overline{w}), \quad w \in H_z(E(A)). \tag{2.33}$$

By the lower bound estimate for μ_1 in [22] (for $n \geq 2$) and in [8] (for $n \geq 3$), one has

$$\mu_1 \geq \frac{n}{n+1} k_0. \tag{2.34}$$

When $E(A) = S^{2n+1}$, then $\mu_1 = \frac{nk_0}{n+1}$. To complete the proof of Part(b), we assume that $\mu_1 = \frac{n}{n+1}k_0$. Then for any real-valued eigenfunction f of Δ_{sb} belonging to the eigenvalue μ_1 with $\int_{E(A)} |f|^2 dv = 1$ and $\int_{E(A)} f dv = 0$, if we write $w(z) = \overline{\tilde{\partial} f} \in H_z(E(A))$, where $\tilde{\partial} f$ is the pseudoholomorphic gradient of f, then by the argument of the estimation for the lower bound of μ_1 in [22], one has

$$\operatorname{Ric}(w, \overline{w}) - \frac{n+1}{2}\operatorname{Tor}(w, w) = k_0 h(w, \overline{w}) = k_0 |w|^2 \quad \text{on } E(A). \tag{2.35}$$

This implies that $k_0(z) = k_0$ on $S_f = \{z \in E(A) : \tilde{\partial} f(z) \neq \mathbf{0}\}$. Since

$$\int_{E(A)} |\tilde{\partial} f(z)|^2 dv(z) \approx \mu_1 > 0, \tag{2.36}$$

S_f contains an open set in $E(A)$.

Letting $w(z) = \overline{\tilde{\partial}(f)(z)}/|\tilde{\partial} f(z)|$, we rewrite it as a vector in $H_z(E(A)) \cap S^{2n+1}$. By (2.10)–(2.14), (2.16), (2.25) and (2.26), the fact that $k_0(z) = k_0$ on S_f implies that

$$\mathcal{E}(z, w(z)) = 0, \quad z \in S_f. \tag{2.37}$$

Since S_f contains an open set in $E(A)$, there exists $\tilde{z} \in S_f$ such that $\tilde{z}_1 \neq 0$. By (2.17) and the fact that $\mathcal{E}(\tilde{z}, w(\tilde{z})) = 0$, one has

$$A_n - A_1 = 0 \quad \text{or} \quad A_n = A_1. \tag{2.38}$$

Combining this with (1.5), one has

$$A_1 = A_2 = \cdots = A_n. \tag{2.39}$$

Next we choose $z_0 \in S_f$ so that $\rho_{n+1}(z_0) \neq 0$. Thus $|\rho_{n+1}(z_0)| > 0$ and $\mathcal{E}(z_0, w(z_0)) = 0$. Then (2.25) and (2.26) imply that $A_n = A_{n+1}$. This, together with (2.39), gives

$$A_1 = A_2 = \cdots = A_{n+1}. \tag{2.40}$$

Notice that, since $\mathcal{E}(z, w(z)) = 0$ for $z \in S_f$, all inequalities in (2.15) and (2.16) become equalities on S_f. Thus (2.17a) holds for all $z \in S_f$. That is,

$$A_j|z_j|^2 = A_j \operatorname{Re} z_j^2 \text{ for } j = 1, \cdots, n+1 \text{ and } z = (z_1, \cdots, z_{n+1}) \in S_f. \tag{2.41}$$

If we choose $\tilde{z} \in S_f$ so that $\operatorname{Im} \tilde{z}_1 \neq 0$, then (2.41) implies $A_1 = 0$. So by (2.40) we have $A_1 = \cdots = A_{n+1} = 0$, and $E(A) = S^{2n+1}$. Therefore, the proof of Part (b) of Theorem 1.1 is complete. □

3. Proof of Theorem 1.3

In order to prove Theorem 1.3, we first calculate the pseudoscalar curvature R for $(E(A), \theta)$, where

$$\theta = \frac{1}{2i}(\partial\rho - \overline{\partial}\rho), \quad \rho(z) = |z|^2 + \sum_{j=1}^{n+1} A_j \operatorname{Re} z_j^2 - 1. \tag{3.1}$$

By (2.43) in [23], at $z \in E(A)$ with $\rho_{n+1}(z) \neq 0$ we have

$$R_{\alpha\overline{\beta}} = -\mathcal{D}_{\alpha\overline{\beta}} \log J(\rho) + (n+1)\frac{\det H(\rho)}{J(\rho)}\mathcal{D}_{\alpha\overline{\beta}}(\rho), \tag{3.2}$$

where

$$\mathcal{D}_{\alpha\overline{\beta}} = \frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta} - \frac{\rho_\alpha}{\rho_{n+1}}\frac{\partial^2}{\partial z_{n+1}\partial \overline{z}_\beta} - \frac{\rho_{\overline{\beta}}}{\rho_{\overline{n+1}}}\frac{\partial^2}{\partial z_\alpha \partial \overline{z}_{n+1}} + \frac{\rho_\alpha \rho_{\overline{\beta}}}{|\rho_{n+1}|^2}\frac{\partial^2}{\partial z_{n+1}\partial \overline{z}_{n+1}}. \tag{3.3}$$

Thus

$$h_{\alpha\overline{\beta}} = \mathcal{D}_{\alpha\overline{\beta}}(\rho) = \delta_{\alpha\beta} + \frac{\rho_\alpha \rho_{\overline{\beta}}}{|\rho_{n+1}|^2} \tag{3.4}$$

and

$$h^{\gamma\overline{\delta}} = \delta_{\gamma\delta} - \frac{\rho_\delta \rho_{\overline{\gamma}}}{|\partial\rho|^2}, \quad |\partial\rho|^2 = \sum_{j=1}^{n+1} |\partial_j \rho(z)|^2 \tag{3.5}$$

since

$$\begin{aligned}
\sum_{\delta=1}^{n} h^{\gamma\overline{\delta}} h_{\alpha\overline{\delta}} &= \sum_{\delta=1}^{n}\left(\delta_{\gamma\delta} - \frac{\rho_\delta \rho_{\overline{\gamma}}}{|\partial\rho|^2}\right)\left(\delta_{\alpha\delta} + \frac{\rho_\alpha \rho_{\overline{\delta}}}{|\rho_{n+1}|^2}\right) \\
&= \delta_{\alpha\gamma} + \frac{\rho_\alpha \rho_{\overline{\gamma}}}{|\rho_{n+1}|^2} - \frac{\rho_\alpha \rho_{\overline{\gamma}}}{|\partial\rho|^2} - \frac{\rho_\alpha \rho_{\overline{\gamma}} \sum_{\delta=1}^{n} |\rho_\delta|^2}{|\partial\rho|^2 |\rho_{n+1}|^2} \\
&= \delta_{\gamma\alpha}.
\end{aligned}$$

Notice that

$$
\begin{aligned}
&J(\rho)^2\mathcal{D}_{\alpha\overline{\beta}}\log J(\rho)(z)\\
&= J(\rho)\mathcal{D}_{\alpha\overline{\beta}}J(\rho)(z) - \Big[\partial_\alpha J(\rho)\partial_{\overline{\beta}}J(\rho) - \frac{\rho_\alpha}{\rho_{n+1}}\partial_{n+1}J(\rho)\partial_{\overline{\beta}}J(\rho)\\
&\qquad - \frac{\rho_{\overline{\beta}}}{\rho_{\overline{n+1}}}\partial_\alpha J(\rho)\partial_{\overline{n+1}}J(\rho) + \frac{\rho_\alpha\rho_{\overline{\beta}}}{|\rho_{n+1}|^2}|\partial_{n+1}J(\rho)|^2\Big]\\
&= J(\rho)\Big(A_\alpha^2\delta_{\alpha\beta} + A_{n+1}^2\frac{\rho_\alpha\rho_{\overline{\beta}}}{|\rho_{n+1}|^2}\Big) - \Big[A_\alpha A_\beta\rho_{\overline{\alpha}}\rho_\beta - \frac{\rho_\alpha}{\rho_{n+1}}A_{n+1}A_\beta\rho_{\overline{n+1}}\rho_\beta\\
&\qquad - \frac{\rho_{\overline{\beta}}}{\rho_{\overline{n+1}}}A_\alpha A_{n+1}\rho_{\overline{\alpha}}\rho_{n+1} + \frac{\rho_\alpha\rho_{\overline{\beta}}}{|\rho_{n+1}|^2}A_{n+1}^2|\rho_{n+1}|^2\Big]\\
&= J(\rho)A_\alpha^2\delta_{\alpha\beta} - \Big[A_\alpha A_\beta\rho_{\overline{\alpha}}\rho_\beta - \frac{\rho_\alpha}{\rho_{n+1}}A_{n+1}A_\beta\rho_{\overline{n+1}}\rho_\beta - \frac{\rho_{\overline{\beta}}}{\rho_{\overline{n+1}}}A_\alpha A_{n+1}\rho_{\overline{\alpha}}\rho_{n+1}\Big]\\
&\quad + \Big(\frac{J(\rho)}{|\rho_{n+1}|^2} - 1\Big)\rho_\alpha\rho_{\overline{\beta}}A_{n+1}^2.
\end{aligned}
$$

Since $|\partial\rho|^2 = J(\rho)$ on $E(A)$, one has

$$
\begin{aligned}
&J(\rho)^2\sum_{\alpha,\beta=1}^n h^{\alpha\overline{\beta}}\mathcal{D}_{\alpha\overline{\beta}}\log J(\rho)\\
&= J(\rho)\sum_{\alpha=1}^n A_\alpha^2 - \sum_{\alpha=1}^n A_\alpha^2|\rho_\alpha|^2 + A_{n+1}2\,\mathrm{Re}\,\frac{\rho_{\overline{n+1}}}{\rho_{n+1}}\sum_{\alpha=1}^n A_\alpha\rho_\alpha(z)^2\\
&\quad + \Big(\frac{J(\rho)}{|\rho_{n+1}|^2} - 1\Big)A_{n+1}^2\sum_{\alpha=1}^n|\rho_\alpha|^2 - \Big[\sum_{\alpha=1}^n A_\alpha^2|\rho_\alpha|^2 - \frac{1}{|\partial\rho|^2}\Big|\sum_{\alpha=1}^n A_\alpha\rho_\alpha^2\Big|^2\\
&\quad + \frac{A_{n+1}}{|\partial\rho|^2}2\,\mathrm{Re}\,\frac{\rho_{\overline{n+1}}}{\rho_{n+1}}\sum_{\alpha=1}^n A_\alpha\rho_\alpha(z)^2\sum_{\beta=1}^n|\rho_\beta|^2 + \Big(\frac{J(\rho)}{|\rho_{n+1}|^2} - 1\Big)\frac{A_{n+1}^2}{|\partial\rho|^2}\Big(\sum_{\alpha=1}^n|\rho_\alpha|^2\Big)^2\Big]\\
&= J(\rho)\sum_{\alpha=1}^n A_\alpha^2 - 2\sum_{\alpha=1}^n A_\alpha^2|\rho_\alpha|^2 + A_{n+1}2\,\mathrm{Re}\,\frac{\rho_{\overline{n+1}}}{\rho_{n+1}}\sum_{\alpha=1}^n A_\alpha\rho_\alpha(z)^2\Big(1 - \frac{\sum_{\alpha=1}^n|\rho_\alpha|^2}{|\partial\rho|^2}\Big)\\
&\quad + \frac{|\sum_{\alpha=1}^n A_\alpha\rho_\alpha^2|^2}{|\partial\rho|^2} + (J(\rho) - |\rho_{n+1}|^2)\frac{A_{n+1}^2}{|\partial\rho|^2}\Big(\sum_{\alpha=1}^n|\rho_\alpha|^2\Big).
\end{aligned}
$$

Therefore, the pseudoscalar curvature at $z \in E(A)$ is

$$
R(z) \;=\; \sum_{\alpha,\beta=1}^n h^{\alpha\overline{\beta}}R_{\alpha\overline{\beta}} \;=\; \frac{n(n+1)}{J(\rho)(z)} - \sum_{\alpha,\beta=1}^n h^{\alpha\overline{\beta}}\mathcal{D}_{\alpha\overline{\beta}}\log J(\rho)(z).
$$

In particular, since $\rho_\alpha(0, z_{n+1}) = 0$ for $\alpha \le n$, one has

$$
R(0, z_{n+1}) = \frac{1}{J(\rho)(0, z_{n+1})}\Big(n(n+1) - \sum_{\alpha=1}^n A_\alpha^2\Big). \tag{3.6}
$$

If $R(s_A e_{n+1}) = R(ir_A e_{n+1})$, then $J(\rho)(s_A e_{n+1}) = J(\rho)(ir_A e_{n+1})$. This implies that $A_{n+1} = 0$. By (1.5), one has $A_1 = A_2 = \cdots = A_{n+1} = 0$, and so $E(A) = S^{2n+1}$. Therefore, the proof of Theorem 1.3 is complete. □

4. Proof of Proposition 1.2

Let

$$\theta = \frac{1}{2i}(\partial\rho - \overline{\partial}\rho), \quad \rho(z) = |z|^2 + A_1 \operatorname{Re} z_1^2 + A_2 \operatorname{Re} z_2^2. \tag{4.1}$$

It is easy to see that on $E_1 = \{z \in E(A) : \rho_{n+1}(z) \neq 0\}$, by the setting in Li and Luk [23], we have

$$h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}} + \frac{\rho_\alpha \rho_{\overline{\beta}}}{|\rho_{n+1}|^2}.$$

Since $n = 1$, one has

$$h_{1\overline{1}} = \frac{|\partial\rho|^2}{|\rho_2|^2}, \quad h^1 = \frac{\rho_{\overline{1}}}{|\partial\rho|^2}. \tag{4.2}$$

Let

$$Y_1 = \frac{\partial}{\partial z_1} - \frac{\rho_1}{\rho_2}\frac{\partial}{\partial z_2}, \quad Y_{\overline{1}} = \frac{\partial}{\partial \overline{z}_1} - \frac{\rho_{\overline{1}}}{\rho_{\overline{2}}}\frac{\partial}{\partial \overline{z}_2}, \quad Y = \frac{i}{|\partial\rho|^2}\sum_{j=1}^{n+1}(\rho_{\overline{j}}\partial_j - \rho_j\partial_{\overline{j}}). \tag{4.3}$$

Let

$$f(z) = z_2 - \overline{z}_2. \tag{4.4}$$

Then

$$\begin{aligned} &\int_{E(A)} |\tilde{\partial} f|^2\, \theta \wedge d\theta \\ &= \int_{E(A)} h^{1\overline{1}}(z) Y_1 f(z) \overline{Y_1 f(z)}\, \theta \wedge d\theta + \int_{E(A)} h^{1\overline{1}}(z) Y_{\overline{1}} f(z) \overline{Y_{\overline{1}} f(z)} \theta \wedge d\theta \\ &= 2\int_{E(A)} h^{1\overline{1}} \frac{|\rho_1|^2}{|\rho_2|^2}\, \theta \wedge d\theta \\ &= 2\int_{E(A)} \frac{|\rho_1|^2}{|\partial\rho|^2}\, \theta \wedge d\theta. \end{aligned} \tag{4.5}$$

Let

$$x_1 = a_1\rho\cos\phi\cos\theta_1, y_1 = b_1\rho\cos\phi\sin\theta_1, x_2 = a_2\rho\sin\phi\cos\theta_2, y_2 = b_2\rho\sin\phi\sin\theta_2, \tag{4.6}$$

where

$$a_j = (1 + A_j)^{-1/2}, \quad b_j = (1 - A_j)^{-1/2}, \quad \phi \in [0, \pi/2], \quad \theta_j \in [0, 2\pi]. \tag{4.7}$$

Then

$$E(A) = \{(a_1 r_1 \cos\theta_1, b_1 r_1 \sin\theta_1, a_2 r_2 \cos\theta_2, b_2 r_2 \sin\theta_2) : r_1^2 + r_2^2 = 1, \theta_j \in [0, 2\pi)\}. \tag{4.8}$$

It is easy to verify that

$$\begin{aligned} &(-(1 - A_1) y_1 dx_1 + (1 + A_1) x_1 dy_1) \wedge dx_2 \wedge dy_2 \\ &= -a_1 b_1 a_2 b_2 \sin\phi \cos^3\phi((1 - A_1)\sin^2\theta_1 + (1 + A_1)\cos^2\theta_1) d\phi d\theta_1 d\theta_2 \\ &= -a_1 b_1 a_2 b_2 \sin\phi \cos^3\phi\, (1 + A_1(\cos^2\theta_1 - \sin^2\theta_1)) d\phi d\theta_1 d\theta_2 \end{aligned}$$

and

$$\begin{aligned}
&(-(1-A_2)y_2dx_2+(1+A_2)x_2dy_2)\wedge dx_1\wedge dy_1\\
&\quad= -a_1b_1a_2b_2\sin^3\phi\cos\phi((1-A_2)\sin^2\theta_2+(1+A_2)\cos^2\theta_2)d\phi d\theta_1 d\theta_2\\
&\quad= -a_1b_1a_2b_2\sin^3\phi\cos\phi\,(1+A_2(\cos^2\theta_2-\sin^2\theta_2))d\phi d\theta_1 d\theta_2.
\end{aligned}$$

Thus

(4.9)

$$\begin{aligned}
\theta\wedge d\theta &= \frac{2}{(2i)^2}(\rho_1dz_1-\rho_{\overline{1}}d\overline{z}_1+\rho_2dz_2-\rho_{\overline{2}}d\overline{z}_2)\wedge(d\overline{z}_1\wedge dz_1+d\overline{z}_2\wedge dz_2)\\
&= \frac{2}{(2i)}((\rho_1-\rho_{\overline{1}})dx_1+i(\rho_1+\rho_{\overline{1}})dy_1)\wedge dx_2\wedge dy_2\\
&\qquad+\frac{2}{(2i)}((\rho_2-\rho_{\overline{2}})dx_2+i(\rho_2+\rho_{\overline{2}})dy_2)\wedge dx_1\wedge dy_1\\
&= 2(-(1-A_1)y_1dx_1+(1+A_1)x_1dy_1)\wedge dx_2\wedge dy_2\\
&\qquad+2(-(1-A_2)y_2dx_2+(1+A_2)x_2dy_2)\wedge dx_1\wedge dy_1\\
&\quad= -a_1b_1a_2b_2\sin(2\phi)(1+A_1\cos^2\phi\cos(2\theta_1)+A_2\sin^2\phi\cos(2\theta_2))d\phi d\theta_1 d\theta_2.
\end{aligned}$$

When performing integration on $E(A)$, we choose an orientation so that the volume form $\theta\wedge d\theta$ is given by

$$\theta\wedge d\theta = a_1b_1a_2b_2\sin(2\phi)(1+A_1\cos^2\phi\cos(2\theta_1)+A_2\sin^2\phi\cos(2\theta_2))d\phi d\theta_1 d\theta_2.$$

On $E(A)$, since

$$|\rho_1|^2=(1+A_1)^2x_1^2+(1-A_1)^2y_1^2=\cos^2\phi(1+A_1\cos(2\theta_1)) \tag{4.10}$$

and

$$\begin{aligned}
|\partial\rho|^2 &= (1+A_1)^2x_1^2+(1-A_1)^2y_1^2+(1+A_2)^2x_2^2+(1-A_2)^2y_2^2\\
&= \cos^2\phi(1+A_1\cos(2\theta_1))+\sin^2\phi(1+A_2\cos(2\theta_2))\\
&= 1+A_1\cos^2\phi\cos(2\theta_1)+A_2\sin^2\phi\cos(2\theta_2),
\end{aligned} \tag{4.11}$$

one has

$$\frac{1}{|\partial\rho|^2}\theta\wedge(d\theta)=a_1b_1a_2b_2\sin(2\phi)d\phi d\theta_1 d\theta_2. \tag{4.12}$$

Thus

$$\begin{aligned}
&\frac{1}{a_1b_1a_2b_2}\int_{E(A)}\frac{|\rho_1|^2}{|\partial\rho|^2}\,\theta\wedge d\theta\\
&\qquad=\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi/2}\cos^2\phi(1+A_1\cos(2\theta_1))\sin(2\phi)d\phi d\theta_1 d\theta_2\\
&\qquad=\int_0^{2\pi}\int_0^{2\pi}\int_0^{1}t^2(1+A_1\cos(2\theta_1))2\,t\,dt d\theta_1 d\theta_2\\
&\qquad=2\pi^2.
\end{aligned} \tag{4.13}$$

Moreover,

$$
\begin{aligned}
&\frac{1}{a_1b_1a_2b_2}\int_{E(A)}|z_2|^2\theta\wedge d\theta\\
&= \int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi/2}\Big[\sin^2\phi(a_2^2\cos^2\theta_2+b_2^2\sin^2\theta_2)\\
&\qquad\qquad\cdot\sin(2\phi)(1+A_1\cos^2\phi\cos(2\theta_1)+A_2\sin^2\phi\cos(2\theta_2))\Big]d\phi d\theta_1 d\theta_2\\
&= \int_0^{2\pi}\int_0^{2\pi}\int_0^{1}2t^3(a_2^2\cos^2\theta_2+b_2^2\sin^2\theta_2)\\
&\qquad\qquad\cdot(1+A_1(1-t^2)\cos(2\theta_1)+A_2t^2\cos(2\theta_2))dtd\theta_1d\theta_2\\
&= \int_0^{2\pi}\int_0^{2\pi}\int_0^{1}2t^3(a_2^2\cos^2\theta_2+b_2^2\sin^2\theta_2)(1+A_2t^2\cos(2\theta_2))dtd\theta_1d\theta_2\\
&= \pi\int_0^{2\pi}(a_2^2\cos^2\theta_2+b_2^2\sin^2\theta_2)\left(1+A_2\frac{2}{3}\cos(2\theta_2)\right)d\theta_2\\
&= \pi\left[\pi(a_2^2+b_2^2)+\frac{2A_2}{3}\int_0^{2\pi}\left(\frac{a_2^2+b_2^2}{2}+\frac{a_2^2-b_2^2}{2}\cos(2\theta_2)\right)\cos(2\theta_2)d\theta_2\right]\\
&= \pi\left[\pi(a_2^2+b_2^2)+\frac{2A_2}{3}\frac{a_2^2-b_2^2}{2}\pi\right]\\
&= \pi^2\left[\frac{2}{1-A_2^2}+\frac{A_2}{3}\frac{-2A_2}{1-A_2^2}\right]\\
&= 2\pi^2\left[\frac{3-A_2^2}{3(1-A_2^2)}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{a_1b_1a_2b_2}\int_{E(A)}\mathrm{Re}(z_2^2)\,\theta\wedge d\theta\\
&= \int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi/2}\Big[\sin^2\phi(a_2^2\cos^2\theta_2-b_2^2\sin^2\theta_2)\\
&\qquad\qquad\cdot\sin(2\phi)(1+A_1\cos^2\phi\cos(2\theta_1)+A_2\sin^2\phi\cos(2\theta_2))\Big]d\phi d\theta_1 d\theta_2\\
&= \int_0^{2\pi}\int_0^{2\pi}\int_0^{1}2t^3(a_2^2\cos^2\theta_2-b_2^2\sin^2\theta_2)\\
&\qquad\qquad\cdot(1+A_1(1-t^2)\cos(2\theta_1)+A_2t^2\cos(2\theta_2))dtd\theta_1d\theta_2\\
&= \int_0^{2\pi}\int_0^{2\pi}\int_0^{1}2t^3(a_2^2\cos^2\theta_2-b_2^2\sin^2\theta_2)(1+A_2t^2\cos(2\theta_2))dtd\theta_1d\theta_2\\
&= \pi\int_0^{2\pi}(a_2^2\cos^2\theta_2-b_2^2\sin^2\theta_2)\left(1+A_2\frac{2}{3}\cos(2\theta_2)\right)d\theta_2\\
&= \pi\left[\pi(a_2^2-b_2^2)+\frac{2A_2}{3}\int_0^{2\pi}\left(\frac{a_2^2-b_2^2}{2}+\frac{a_2^2+b_2^2}{2}\cos(2\theta_2)\right)\cos(2\theta_2)d\theta_2\right]\\
&= \pi\left[\pi(a_2^2-b_2^2)+\frac{2A_2}{3}\frac{a_2^2+b_2^2}{2}\pi\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \pi^2\left[\frac{-2A_2}{1-A_2^2}+\frac{A_2}{3}\frac{2}{1-A_2^2}\right]\\
&= -\frac{4\pi^2}{3}\frac{A_2}{1-A_2^2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
(4.14)\quad & \frac{1}{a_1b_1a_2b_2}\int_{E(A)}|f|^2\theta\wedge(d\theta)\\
&= \frac{2}{a_1b_1a_2b_2}\int_{E(A)}|z_2|^2\theta\wedge(d\theta)-\frac{2}{a_1b_1a_2b_2}\int_{E(A)}\operatorname{Re}z_2^2\,\theta\wedge(d\theta)\\
&= 4\pi^2\frac{3-A_2^2}{3(1-A_2^2)}+\frac{4\pi^2}{3}\frac{2A_2}{1-A_2^2}.
\end{aligned}
$$

Combining (4.5), (4.13) and (4.14), one has

$$
\mu_1 \le \frac{4\pi^2}{4\pi^2[\frac{3-A_2^2}{3(1-A_2^2)}+\frac{2A_2}{3(1-A_2^2)}]} = \frac{1-A_2^2}{1-\frac{1}{3}A_2^2+\frac{2}{3}A_2}.
$$

If $\mu_1 = \frac{1-A_2^2}{1-\frac{1}{3}A_2^2+\frac{2}{3}A_2}$, then $f = z_2 - \overline{z}_2$ is an eigenfunction of Δ_{sb}. Straightforward computation shows that $(-\Delta_{sb})f = \frac{\rho_{\overline{2}}-\rho_2}{|\partial\rho|^2}$. Thus on $E(A)$,

$$
\mu_1(z_2-\overline{z}_2) = \frac{\rho_{\overline{2}}(z)-\rho_2(z)}{|\partial\rho|^2} = \frac{1}{|\partial\rho|^2}(1-A_2)(z_2-\overline{z}_2).
$$

This implies $\mu_1|\partial\rho|^2 = 1-A_2$ on $E(A)$. In particular, for $z = \left(0, \sqrt{\frac{1}{1+A_2}}\right) \in E(A)$, one has

$$
\mu_1(1+A_2) = 1-A_2.
$$

Solving for A_2 in the above equation, we obtain $A_2 = 0$. Therefore, $A_1 = A_2 = 0$, i.e., $E(A) = S^3$. The proof of Proposition 1.2 is complete. □

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