

IF YOU CAN HIDE BEHIND IT, CAN YOU HIDE INSIDE IT?

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ABSTRACT. Let K and L be compact convex sets in \mathbb{R}^n . Suppose that, for a given dimension $1 \leq d \leq n - 1$, every d -dimensional orthogonal projection of L contains a translate of the corresponding projection of K . Does it follow that the original set L contains a translate of K ? In other words, if K can be translated to “hide behind” L from any perspective, does it follow that K can “hide inside” L ?

A compact convex set L is defined to be d -decomposable if L is a direct Minkowski sum (affine Cartesian product) of two or more convex bodies each of dimension at most d . A compact convex set L is called d -reliable if, whenever each d -dimensional orthogonal projection of L contains a translate of the corresponding d -dimensional projection of K , it must follow that L contains a translate of K .

It is shown that, for $1 \leq d \leq n - 1$:

- (1) d -decomposability implies d -reliability.
- (2) A compact convex set L in \mathbb{R}^n is d -reliable if and only if, for all $m \geq d + 2$, no m unit normals to regular boundary points of L form the outer unit normals of an $(m - 1)$ -dimensional simplex.
- (3) Smooth convex bodies are not d -reliable.
- (4) A compact convex set L in \mathbb{R}^n is 1-reliable if and only if L is 1-decomposable (i.e. a parallelotope).
- (5) A centrally symmetric compact convex set L in \mathbb{R}^n is 2-reliable if and only if L is 2-decomposable.

However, there are non-centered 2-reliable convex bodies that are not 2-decomposable.

As a result of (5) above, the only reliable centrally symmetric covers in \mathbb{R}^3 from the perspective of 2-dimensional shadows are the affine convex cylinders (prisms). However, in dimensions greater than 3, it is shown that 3-decomposability is only sufficient, and not necessary, for L to cover reliably with respect to 3-shadows, even when L is assumed to be centrally symmetric.

Consider two compact convex subsets K and L of n -dimensional Euclidean space. Suppose that, for a given dimension $1 \leq d < n$, every d -dimensional orthogonal projection (shadow) of L contains a translate of the corresponding projection of K . Does it follow that the original set L contains a translate of K ? In other words, if K can be translated to “hide behind” L from any perspective, does it follow that K can “hide inside” L ?

In dimension 2 it is easy to see that the answer is *no*. For example, if an equilateral triangle Δ is inscribed in a disc D of unit diameter, the slightly larger triangle $(1 + \epsilon)\Delta$ still has less than unit width in every direction (provided $\epsilon > 0$ is sufficiently small), but no longer fits inside D . The same construction works for any set K inscribed in D and having strictly less than unit diameter. Another

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counterexample arises from comparing Δ and the dilated and reflected triangle $-(1 + \epsilon)\Delta$ for small $\epsilon > 0$.

Although the details are less obvious, counterexamples also exist in higher dimensions. Let B denote the Euclidean 3-ball of unit radius, and let T denote the regular tetrahedron having edge length $\sqrt{3}$. Jung's Theorem [3, p. 84], [31, p. 320] implies that every 2-projection of T is covered by a translate of the unit disk. But a simple computation shows that B cannot cover the tetrahedron T . An analogous construction yields a similar result for higher dimensional simplices and Euclidean balls. One might say that, although T can be translated within a fixed distance from B (i.e. without moving far away) to hide behind B from any observer's perspective, this does not imply that T can hide *inside* B .

Indeed, for $1 \leq d \leq n - 1$, it is shown in [17] that, if K is a compact convex set in \mathbb{R}^n having at least $d + 2$ exposed points, then there exists another compact convex set L such that every d -dimensional orthogonal projection (shadow) of L contains a translate of the corresponding projection of K , while L does not contain a translate of K . In certain cases one can even find examples where K also has *larger volume* than L (and so certainly could not fit inside L). For a detailed example of this volume phenomenon, see [16].

This leads to the question: under what additional conditions on either of the sets K or L does covering of the shadows of K by translates of the shadows of L imply covering of the original set K by a translate of the set L ?

This question is easily answered when a sufficient degree of symmetry is imposed. For example, a support function argument implies that shadow covering implies actual covering if *both* of the bodies K and L are centrally symmetric. It is also not difficult to show that, if every d -projection of K (for some $1 \leq d < n$) can be translated inside the corresponding shadow of an orthogonal n -dimensional box C , then C contains a translate of K , since one needs only to check that the widths are compatible in the n edge directions of C . Related special cases occur if C is a parallelotope (an affine image of a box), or even a cylinder (the product of an $(n - 1)$ -dimensional compact convex set with a line segment).

In [22] Lutwak uses Helly's theorem to prove that, if every n -simplex containing L also contains a translate of K , then L contains a translate of K . In the present article we generalize Lutwak's theorem in order to reduce questions about shadow covering to questions about circumscribing simplices and simplicial cylinders. A compact convex set L will be called *d-decomposable* if L is a *direct* Minkowski sum (affine Cartesian product) of two or more convex bodies each of dimension at most d (see Section 1). A compact convex set L will be called *d-reliable* if, whenever each d -shadow of L contains a translate of the corresponding d -shadow of K , it follows that L contains a translate of K (see Section 3). It will be shown that, for $1 \leq d \leq n - 1$:

- (1) d -decomposability implies d -reliability. (Theorem 1.3)
- (2) A compact convex set L in \mathbb{R}^n is d -reliable if and only if, for all $m \geq d + 2$, no m unit normals to regular boundary points of L form the outer unit normals of an $(m - 1)$ -dimensional simplex. (Theorem 3.4)
- (3) Smooth convex bodies are *not* d -reliable. (Corollary 3.6)
- (4) A compact convex set L is 1-reliable if and only if L is 1-decomposable (i.e. a parallelotope). (Corollary 3.7)

- (5) A *centrally symmetric* compact convex set L is 2-reliable if and only if L is 2-decomposable (Theorem 4.1). However, there are non-centered 2-reliable convex bodies that are not 2-decomposable. (Corollary 3.8)

As a result of (5) above, the only reliable centrally symmetric covers in \mathbb{R}^3 from the perspective of 2-dimensional shadows are the affine convex cylinders (prisms). However, in dimensions greater than 3, it will be seen (at the end of Section 4) that 3-decomposability is only sufficient, and not necessary, for L to cover reliably with respect to 3-shadows, even when L is assumed to be centrally symmetric.

The containment and covering problems addressed in this article are special cases of the following question: under what conditions will a compact convex set necessarily contain a translate or otherwise congruent copy of another? Progress on different aspects of this general question also appears in the work of Gardner and Volčič [10], Groemer [12], Hadwiger [13, 14, 15, 18, 26], Jung [3, 31], Lutwak [22], Rogers [25], Soltan [30], Steinhagen [3, p. 86], Zhou [34, 35], and others (see also [8]).

These questions are also motivated in part by dramatic progress on two notorious related problems: the Shephard Problem [29] and the Busemann-Petty Problem [5]. Both address properties of bodies K and L that are assumed to be *centrally symmetric* about the origin (for which these particular questions become non-trivial). The Shephard Problem asks: if each orthogonal $(n-1)$ -projection of K has smaller $(n-1)$ -volume than the corresponding projection of L , does it follow that K has smaller n -volume than L ? The Busemann-Petty Problem addresses an analogous question by comparing the volumes of $(n-1)$ -dimensional cross-sections of K and L through the origin. The Shephard Problem was solved independently by Petty [24] and Schneider [27], who showed that, while the answer in general is *no* for dimensions $n \geq 3$, the answer is *yes* when the convex set L is a projection body; that is, a zonoid. The solution to the Busemann-Petty Problem proved more subtle and elusive. After decades of partial results by several researchers (see, for example, [1, 4, 6, 7, 9, 11, 19, 20, 23, 32, 33]) the surprising answer was revealed to be *no* for bodies of dimension $n \geq 5$ and *yes* for bodies of dimension $n \leq 4$. Moreover, in analogy to the Petty-Schneider theorem, the answer is always *yes* when the set L is an *intersection body* [21]. A more complete discussion of these and related problems (some of which remain open) can be found in the comprehensive book by Gardner [8].

0. BACKGROUND

Denote n -dimensional Euclidean space by \mathbb{R}^n , and let \mathbb{S}^{n-1} denote the set of unit vectors in \mathbb{R}^n ; that is, the unit $(n-1)$ -sphere centered at the origin.

Denote by \mathcal{K}_n the set of compact convex subsets of \mathbb{R}^n . The n -dimensional (Euclidean) volume of a convex set K will be denoted $V_n(K)$. If u is a unit vector in \mathbb{R}^n , denote by K_u the orthogonal projection of a set K onto the subspace u^\perp . More generally, if ξ is a d -dimensional subspace of \mathbb{R}^n , denote by K_ξ the orthogonal projection of a set K onto the subspace ξ . The boundary of a compact convex set K will be denoted by ∂K .

Let $h_K : \mathbb{R}^n \rightarrow \mathbb{R}$ denote the support function of a compact convex set K ; that is,

$$h_K(v) = \max_{x \in K} x \cdot v.$$

If ξ is a subspace of \mathbb{R}^n then the support function h_{K_ξ} is given by the restriction of h_K to ξ . If u is a unit vector in \mathbb{R}^n , denote by K^u the support set of K in the direction of u ; that is,

$$K^u = \{x \in K \mid x \cdot u = h_K(u)\}.$$

If P is a convex polytope, then P^u is the face of P having u in its outer normal cone.

Given two $K, L \in \mathcal{K}_n$ and $a, b \geq 0$ denote

$$aK + bL = \{ax + by \mid x \in K \text{ and } y \in L\}.$$

An expression of this form is called a *Minkowski combination* or *Minkowski sum*. Because K and L are convex, the set $aK + bL$ is also convex. Convexity also implies that $aK + bK = (a + b)K$ for all $a, b \geq 0$. Support functions satisfy the identity $h_{aK+bL} = ah_K + bh_L$. (See, for example, [3, 28, 31].)

If $K \in \mathcal{K}_n$ has non-empty interior, define the *surface area measure* S_K on the $(n - 1)$ -dimensional unit sphere \mathbb{S}^{n-1} as follows: for $A \subseteq \mathbb{S}^{n-1}$ denote by $K^A = \bigcup_{u \in A} K^u$, and define $S_K(A) = \mathcal{H}_{n-1}(K^A)$, the $(n - 1)$ -dimensional Hausdorff measure of the subset K^A of the boundary of K . (See [28, p. 203].) If P is a polytope, then S_P is a pointed measure concentrated at precisely those directions u that are outer normals to the facets of P .

The measure S_K is easily shown to satisfy the property

$$(0.1) \quad \int_{\mathbb{S}^{n-1}} u \, dS_K = \vec{0},$$

that is, the mass distribution on the sphere described by S_K has center of mass at the origin. For a convex polytope P having outward facet unit normals u_1, \dots, u_m and corresponding facet areas $\alpha_1, \dots, \alpha_m > 0$, the identity (0.1) takes the simple and intuitive form:

$$(0.2) \quad \alpha_1 u_1 + \dots + \alpha_m u_m = \vec{0}.$$

Minkowski's Existence Theorem [3, p. 125], [28, p. 390] gives a useful converse to the identity (0.1): if μ is a non-negative measure on the unit sphere \mathbb{S}^{n-1} such that μ has center of mass at the origin, and if μ is not concentrated on any great (equatorial) $(n - 1)$ -subsphere, then $\mu = S_K$ for some $K \in \mathcal{K}_n$. Moreover, this convex body K is *unique up to translation*.

Suppose that \mathcal{F} is a family of compact convex sets in \mathbb{R}^n . Helly's theorem [3, 28, 31] asserts that if every $n + 1$ sets in \mathcal{F} share a common point, then the entire family shares a common point. In [22] Lutwak used Helly's theorem to prove the following fundamental criterion for whether a set $L \in \mathcal{K}_n$ contains a translate of another set $K \in \mathcal{K}_n$.

Theorem 0.1 (Lutwak's containment theorem). *Let $K, L \in \mathcal{K}^n$. The following are equivalent:*

- (i) *For every simplex Δ such that $L \subseteq \Delta$, there exists $v \in \mathbb{R}^n$ such that $K + v \subseteq \Delta$.*
- (ii) *There exists $v_0 \in \mathbb{R}^n$ such that $K + v_0 \subseteq L$.*

In other words, if every n -simplex containing L also contains a translate of K , then L contains a translate of K .

1. CIRCUMSCRIBING SETS AND SHADOWS

A convex set $K \in \mathcal{K}_n$ will be called d -decomposable if there exists a subspace decomposition

$$\mathbb{R}^n = \xi_1 \oplus \cdots \oplus \xi_m,$$

where $\dim \xi_i \leq d$ for each i , and compact convex sets $K_i \subseteq \xi_i$ for each i , such that $K = K_1 + \cdots + K_m$. Decompositions of this kind will be denoted

$$K = K_1 \oplus \cdots \oplus K_m.$$

If $K = \Delta_1 \oplus \cdots \oplus \Delta_m$, where the component sets Δ_i are *simplices*, each of dimension at most d , then we will say that K is a d -decomposable simplex product. The product will be called *orthogonal* if the subspaces ξ_i are mutually orthogonal.

The 2-decomposable sets in \mathbb{R}^3 (as well as products of $(n - 1)$ -dimensional sets with line segments in \mathbb{R}^n) are often called *cylinders* or *prisms*.

If the circumscribing simplices are replaced by circumscribing simplex products for L , then the following generalization of Lutwak's theorem 0.1 is obtained.

Theorem 1.1 (Prismatic containment theorem). *Let $K, L \in \mathcal{K}^n$. The following are equivalent:*

- (i) *For every d -decomposable set $C \in \mathcal{K}_n$ such that $L \subseteq C$, there exists $v \in \mathbb{R}^n$ such that $K + v \subseteq C$.*
- (ii) *For every d -decomposable orthogonal simplex product C such that $L \subseteq C$, there exists $v \in \mathbb{R}^n$ such that $K + v \subseteq C$.*
- (iii) *For every d -dimensional subspace $\xi \subseteq \mathbb{R}^n$, there exists $w \in \xi$ such that $K_\xi + w \subseteq L_\xi$.*

In other words, if every d -decomposable (simplex) product C containing L also contains a translate of K , then every d -shadow L_ξ contains a translate of the corresponding shadow K_ξ , and vice versa.

The following proposition will simplify the proof of Theorem 1.1.

Proposition 1.2. *Let $K, L \in \mathcal{K}_n$. Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a non-singular linear transformation. Then L_u contains a translate of K_u for all unit directions u if and only if $(\psi L)_u$ contains a translate of $(\psi K)_u$ for all u .*

This proposition implies that nothing is gained (or lost) by allowing more general (possibly non-orthogonal) linear projections.

Proof. For $S \subseteq \mathbb{R}^n$ and a non-zero vector u , let $\mathcal{L}_S(u)$ denote the set of straight lines in \mathbb{R}^n parallel to u and meeting the set S . The projection L_u contains a translate of K_u for each unit vector u if and only if, for each u , there exists v_u such that

$$(1.1) \quad \mathcal{L}_{K+v_u}(u) \subseteq \mathcal{L}_L(u).$$

But $\mathcal{L}_{K+v_u}(u) = \mathcal{L}_K(u) + v_u$ and $\psi \mathcal{L}_K(u) = \mathcal{L}_{\psi K}(\psi u)$. It follows that (1.1) holds if and only if $\mathcal{L}_K(u) + v_u \subseteq \mathcal{L}_L(u)$, if and only if

$$\mathcal{L}_{\psi K}(\psi u) + \psi v_u \subseteq \mathcal{L}_{\psi L}(\psi u) \quad \text{for all unit } u.$$

Set

$$\tilde{u} = \frac{\psi u}{|\psi u|} \quad \text{and} \quad \tilde{v} = \psi v_u.$$

The relation (1.1) now holds if and only if, for all \tilde{u} , there exists \tilde{v} such that

$$\mathcal{L}_{\psi K}(\tilde{u}) + \tilde{v} \subseteq \mathcal{L}_{\psi L}(\tilde{u}),$$

which holds if and only if $(\psi L)_{\tilde{u}}$ contains a translate of $(\psi K)_{\tilde{u}}$ for all \tilde{u} . \square

Proof of Theorem 1.1. To begin, note that **(i)** implies **(ii)** trivially.

Suppose that **(ii)** holds. Given a d -subspace $\xi \subseteq \mathbb{R}^n$, let T be a simplex in ξ that circumscribes L_ξ . Let u_{d+1}, \dots, u_n be an orthonormal basis for ξ^\perp , and let C' be a cube in ξ^\perp with edges parallel to the directions u_i and large enough so that $L_{\xi^\perp} \subseteq C'$. Now let $C = T \oplus C'$. Since $L \subseteq C$, it follows from **(ii)** that there exists $v \in \mathbb{R}^n$ such that $K + v \subseteq C$. This implies that $K_\xi + v_\xi \subseteq C_\xi = T$. On applying Lutwak's theorem 0.1 in the subspace ξ it follows that $K_\xi + w \subseteq L_\xi$ for some $w \in \xi$. Therefore, **(ii)** implies **(iii)**.

Next, suppose that **(iii)** holds. If $L \subseteq C = C_1 \oplus \dots \oplus C_m$, where each $\dim \xi_i \leq d$, then let ψ be a non-singular linear operator on \mathbb{R}^n such that the subspaces $\psi(\xi_i)$ are mutually orthogonal. By Proposition 1.2, the condition **(iii)** also holds for ψK and ψL . For each i we obtain $v_i \in \psi \xi_i$ such that

$$(\psi K)_{\psi \xi_i} + v_i \subseteq (\psi L)_{\psi \xi_i} \subseteq (\psi C)_{\psi \xi_i} = \psi C_i.$$

Let $v = v_1 + \dots + v_m$. Since the subspaces $\psi \xi_i$ are mutually orthogonal, we have $\psi K + v \subseteq \psi C$, so that $K + \psi^{-1}v \subseteq C$. Therefore, **(iii)** implies **(i)**, and the three assertions are equivalent. \square

It is worth noting the following special case of Theorem 1.1.

Theorem 1.3. *Let $K, C \in \mathcal{K}^n$, where C is d -decomposable.*

Suppose that, for each d -dimensional subspace $\xi \subseteq \mathbb{R}^n$, there exists $w \in \xi$ such that $K_\xi + w \subseteq C_\xi$. Then there exists $v \in \mathbb{R}^n$ such that $K + v \subseteq C$.

When $d = n - 1$, Theorem 1.3 says that if you can hide behind a cylinder from any perspective (and without rotating), then you can also hide inside the cylinder.

More consequences of Theorem 1.1 are explored in [16].

2. SIMPLICIAL FAMILIES OF UNIT NORMALS

Theorem 1.3 motivates a converse question: if L is *not* d -decomposable, does there necessarily exist K such that every d -shadow of L contains a translate of the corresponding d -shadow of K , while L itself does not contain a translate of K ? The answer is *not necessarily*. We will show in a later section (see Corollary 3.8) that if L is a square pyramid (the convex hull of a square in \mathbb{R}^3 with a point above its center), then no K can hide behind L unless K can also hide inside L . However, the square pyramid is not 2-decomposable. In other words, the condition of being d -decomposable is sufficient, but not necessary.

In this section we develop some tools for constructing necessary and sufficient conditions for when shadow covering implies actual covering. These tools are applied in later sections.

A set of unit vectors $\{u_0, \dots, u_d\} \subseteq \mathbb{S}^{n-1}$ will be called a *d -simplicial family*, or *d -simplicial*, if u_0, \dots, u_d span a d -dimensional subspace of \mathbb{R}^n , and if there exist real numbers $c_0, \dots, c_d > 0$, such that

$$c_0 u_0 + c_1 u_1 + \dots + c_d u_d = o.$$

Equivalently, u_0, \dots, u_d are the outer unit normals of some d -dimensional simplex. Note that a d -simplicial family contains exactly $d + 1$ unit vectors.

It will be seen in Sections 3 and 4 that certain translative covering properties of a compact convex set L hinge in the existence of simplicial families of unit normals to regular points of L . The next three propositions will be used in that context. (Readers in a hurry may wish to scan Sections 3 and 4 and return to these technical points later on.)

Proposition 2.1. *Suppose that $A = \{u_1, \dots, u_m\} \subseteq \mathbb{S}^{n-1}$ contains no simplicial families of size 3 or greater, and that*

$$(2.1) \quad c_1 u_1 + \dots + c_m u_m = o$$

for some $c_1, \dots, c_m > 0$. Then $m = 2s$ for some integer s , and there exist linearly independent vectors $v_1, \dots, v_s \in \mathbb{S}^{n-1}$, where $s \leq n$, such that

$$A = \{\pm v_1, \dots, \pm v_s\}.$$

Proof. By (2.1) the set A must have at least 2 elements, and if A has size 2 then the proposition is trivial.

Suppose that the proposition fails for some set A of minimal size m , where $m > 2$. By (2.1) there exists a minimal subfamily $\{u_{i_1}, \dots, u_{i_k}\} \subseteq A$ such that

$$a_1 u_{i_1} + \dots + a_k u_{i_k} = o$$

for some $a_1, \dots, a_k > 0$. Let $d = \dim(\text{Span}\{u_{i_1}, \dots, u_{i_k}\})$. Since the u_{i_j} are dependent, we have $k \geq d + 1$. If $k > d + 1$ then Carathéodory's theorem [28, p. 3] (applied in the span of the $\{u_{i_j}\}$) implies that the origin o lies in the convex hull of a sub-subfamily of size at most $d + 1 < k$ of the u_{i_j} , violating the minimality of k . Therefore $k = d + 1$, and $\{u_{i_1}, \dots, u_{i_k}\}$ is a simplicial set. By the original assumption on simplicial families in A , it follows that $k = 2$, so that $u_i = -u_j$ for some $i \neq j$.

Without loss of generality, suppose that $u_1 = -u_2$ and that $c_1 \geq c_2$. It now follows from (2.1) that

$$o = (c_1 - c_2)u_1 + c_3 u_3 + \dots + c_m u_m.$$

Suppose $c_1 - c_2 > 0$. The minimality of m implies that the proposition holds for the set $\{u_1, u_3, \dots, u_m\}$, so that $m - 1$ is even and these remaining vectors u_1, u_3, \dots, u_m can be partitioned into distinct antipodal pairs. Since $u_1 = -u_2$, this would violate the original assumption that the u_i are distinct. Therefore $c_1 = c_2$, and

$$o = c_3 u_3 + \dots + c_m u_m.$$

Once again the minimality of m implies that the proposition holds for the set u_3, \dots, u_m , so that $m - 2$ is even (and therefore m is even), and the remaining u_i can be separated into distinct antipodal pairs $\pm v_2, \dots, \pm v_s$, where the v_i are linearly independent.

It remains to show that u_1 (and similarly u_2) is linearly independent from the vectors v_i . If u_1 lies in the span of v_2, \dots, v_s , then u_1 lies in the span of a minimal subset of v_2, \dots, v_k of size at least 2, since u_1 is distinct from each $\pm v_i$. The resulting linear dependence relation violates the non-existence of simplicial subsets of size 3 or greater inside A .

Setting $v_1 = u_1$ now completes the proof of the proposition. \square

Proposition 2.2. *Suppose that $A \subseteq \mathbb{S}^{n-1}$ contains no simplicial sets of size 3 or greater, and that the origin lies in the interior of the convex hull of A . Then there exist linearly independent vectors $v_1, \dots, v_n \in \mathbb{S}^{n-1}$, such that*

$$A = \{\pm v_1, \dots, \pm v_n\}.$$

Proof. By Carathéodory's theorem there exists a finite subfamily u_1, \dots, u_m of A , such that (2.1) holds. Since o lies in the interior of the convex hull of A , we can take m large enough so that u_1, \dots, u_m spans \mathbb{R}^n . By Proposition 2.1, this subfamily has the form $\{\pm v_1, \dots, \pm v_s\} \subseteq A$, where v_1, \dots, v_s are linearly independent and span \mathbb{R}^n . It follows that $s = n$.

If $w \in A$ and $w \neq \pm v_i$, then w lies in the span of some v_{i_1}, \dots, v_{i_k} , where $k \geq 2$ is minimal. This linear dependence relation violates the non-existence of simplicial sets of size 3 or greater inside the set A .

It follows that $\{\pm v_1, \dots, \pm v_n\} = A$. □

Proposition 2.3. *Suppose that $A \subseteq \mathbb{S}^{n-1}$ is symmetric under reflection through the origin; that is, $A = -A$. Suppose also that A contains no simplicial sets of size 4 or greater, and that the origin lies in the interior of the convex hull of A .*

Then there exists a subspace direct sum decomposition

$$\mathbb{R}^n = W_1 \oplus \dots \oplus W_k,$$

where each $\dim W_i \leq 2$, and such that $A \subseteq W_1 \cup \dots \cup W_k$.

Proof. Since $A = -A$, the set A is composed of antipodal pairs $\pm v$ of unit vectors. Moreover, since the convex hull of A has interior and is centrally symmetric, there exist at least n pairs $\pm u_1, \dots, \pm u_n$ in A whose n directions are linearly independent. If $A = \{\pm u_1, \dots, \pm u_n\}$, then \mathbb{R}^n is a direct sum of the lines spanned by each $\pm u_i$, and the proposition follows.

If, instead, $\pm v$ is another antipodal pair in A , where $v \neq \pm u_i$ for all i , then without loss of generality (relabeling the signs on $\pm u_i$ as needed), we have

$$-v = c_1 u_1 + \dots + c_d u_d$$

for some $c_1, \dots, c_d > 0$, where d is minimal. If $d \geq 3$, then the relation

$$v + c_1 u_1 + \dots + c_d u_d = 0$$

implies that $\{v, u_1, \dots, u_d\}$ form a simplicial family in A of size at least 4, contradicting hypothesis. Meanwhile, since $v \neq \pm u_i$, we must have $d > 1$. The remaining possibility is $d = 2$, so that v lies in the span of $\{u_1, u_2\}$.

If $w \in A$ and $w \neq \pm v, \pm u_1, \dots, \pm u_n$, then w lies in the span of 2 of the u_i by a similar argument. But if $w = a_1 u_1 + a_3 u_3$, say, where $a_1, a_3 > 0$, then

$$w = \frac{a_1}{c_1}(-c_2 u_2 - v) + a_3 u_3 = \frac{a_1 c_2}{c_1}(-u_2) + \frac{a_1}{c_1}(-v) + a_3 u_3.$$

Since every 3 of the 4 vectors v, w, u_2, u_3 are linearly independent, we obtain a simplicial set of size 4, another contradiction. Therefore, either w also lies in the span of $\{u_1, u_2\}$ or in the span of $\{u_i, u_j\}$ for $j > i > 2$. An iteration of this argument implies that \mathbb{R}^n is decomposed into a direct sum $\mathbb{R}^n = W_1 \oplus \dots \oplus W_{\lfloor \frac{n+1}{2} \rfloor}$ of subspaces W_i each having dimension at most 2, and where every $v \in A$ also lies in some W_i . □

We will also need the following proposition, which clears up ambiguities regarding when shadows cover inside a larger ambient space.

Proposition 2.4. *Suppose that ξ is a linear flat in \mathbb{R}^n . Let K and L be compact convex sets in ξ . Suppose that, for each d -subspace $\eta \subseteq \xi$, the projection L_η contains a translate of K_η . Then L_η contains a translate of K_η for every d -subspace $\eta \subseteq \mathbb{R}^n$.*

In other words, embedding K and L in a higher-dimensional space does not change whether or not every d -shadow of L contains a translate of the corresponding d -shadow of K (even though there are now more shadow directions to verify).

Proof. Suppose that η is a d -subspace of \mathbb{R}^n . Let $\hat{\eta}$ denote the orthogonal projection of η into ξ . Since $\dim(\hat{\eta}) \leq \dim(\eta) = d$, we can translate K and L inside ξ so that $K_{\hat{\eta}} \subseteq L_{\hat{\eta}}$. Let us assume this translation has taken place. Note that, for $v \in \hat{\eta}$, we now have $h_K(v) \leq h_L(v)$.

If $u \in \eta$, then express $u = u_\xi + u_{\xi^\perp}$. Since $K \subseteq \xi$,

$$h_K(u) = \max_{x \in K} x \cdot u = \max_{x \in K} x \cdot u_\xi = h_K(u_\xi),$$

and similarly for L . But since $u \in \eta$, we have $u_\xi \in \hat{\eta}$, so that

$$h_K(u) = h_K(u_\xi) \leq h_L(u_\xi) = h_L(u).$$

In other words, $K_\eta \subseteq L_\eta$. □

3. WHEN CAN A CONVEX SET CONCEAL WITHOUT COVERING?

We now address the possibility of a converse to Theorem 1.3.

Definition 3.1. Suppose that $1 \leq d \leq n - 1$. A compact convex set L in \mathbb{R}^n is said to be a d -reliable cover, or d -reliable, if whenever $K \in \mathcal{K}_n$ and every d -shadow L_ξ contains a translate of the corresponding shadow K_ξ , it follows that L contains a translate of K .

Evidently, if L is d -reliable, then L is also m -reliable for all $m > d$.

Theorem 1.3 asserts that if L is d -decomposable, then L is also d -reliable. However, we will see that a square pyramid gives a counterexample to the converse assertion. It is 2-reliable, but not 2-decomposable (Corollary 3.8).

The next two theorems describe a *necessary and sufficient* condition for L to be a d -reliable cover. Recall that a point x on the boundary of a compact convex set L is said to be *regular* if the outward normal cone to L at x contains exactly one unit vector.

Theorem 3.2. *Suppose that L has regular boundary points x_0, \dots, x_{d+1} , whose corresponding unit normals u_0, \dots, u_{d+1} are a simplicial family. Then there exists a polytope S such that L_ξ contains a translate of S_ξ for each d -subspace ξ of \mathbb{R}^n , while L does not contain a translate of S . In particular, L is not d -reliable.*

Proof. First, note that, by Proposition 2.4, it is sufficient to prove this theorem for the case in which L has interior. For if L lacks interior, we simply restrict our attention to the affine hull of L . Once the theorem is verified in this case, one can apply Proposition 2.4 to verify the theorem when L is re-embedded in a higher-dimensional space. So let us now assume that L has interior.

Suppose that L has regular boundary points x_0, \dots, x_{d+1} , as in the hypothesis of the theorem. Let S be the convex hull of $\{x_0, \dots, x_{d+1}\}$. Evidently $S \subseteq L$. Since $\{u_0, \dots, u_{d+1}\}$ is a simplicial family, there exist $c_i > 0$ such that

$$(3.1) \quad c_0 u_0 + \dots + c_{d+1} u_{d+1} = o.$$

Moreover, every $d + 1$ of the u_i are linearly independent, so that no subfamily of the u_i contains the origin in its convex hull, and the property (3.1) does not hold for any subfamily.

Since $S \subseteq L$, we have $h_S \leq h_L$. Moreover, by our choices of x_i and u_i , $h_L(u_i) = u_i \cdot x_i \leq h_S(u_i)$. Therefore, $h_L(u_i) = h_S(u_i)$ for each i .

Let ξ be a d -dimensional subspace of \mathbb{R}^n . Let π_ξ denote the orthogonal projection of \mathbb{R}^n onto ξ . If $\pi_\xi(x_i)$ lies on the boundary of L_ξ , then $h_{L_\xi}(w) = x_i \cdot w$ for some unit $w \in \xi$. Since h_{L_ξ} is given by the restriction of h_L to the subspace ξ , it follows that $h_L(w) = x_i \cdot w$. By the regularity of the boundary point x_i , we have $w = u_i$, so that $u_i \in \xi$.

Similarly, if $u_i \in \xi$, then

$$h_{L_\xi}(u_i) = h_L(u_i) = x_i \cdot u_i = \pi_\xi(x_i) \cdot u_i,$$

so that $\pi_\xi(x_i)$ lies on the boundary of L_ξ , with outward unit normal u_i .

Since the u_i form a simplicial family of size $d + 2$, at most d of these vectors u_i can lie in ξ . It follows that S_ξ meets the boundary of L_ξ at $j + 1$ points, for some $j \leq d - 1$, having outward normals u_0, \dots, u_j (without loss of generality). Since o does not lie in the convex hull of u_0, \dots, u_j , there exists a unit vector $v \in \xi$ such that $v \cdot u_i < 0$ for $i \leq j$.

Let $y_i = \pi_\xi(x_i)$. If $i \leq j$, then y_i is a regular point of the boundary ∂L_ξ with outward unit normal u_i . Since each $v \cdot u_i < 0$ in this case, there exists $\epsilon > 0$ such that each $y_i + \epsilon v$ lies in the relative interior of L_ξ .

Meanwhile, if $i > j$, then y_i lies in the relative interior of L_ξ already, so that $y_i + \epsilon v$ lies in the relative interior of L_ξ as well, provided we have chosen $\epsilon > 0$ small enough. In other words, there exists $\epsilon > 0$ so that $y_i + \epsilon v$ lies in the relative interior of L_ξ for all i .

Let $T = S_\xi + \epsilon v$. Since the polytope T is the convex hull of the points $y_i + \epsilon v$, it follows that T lies in the relative interior of L_ξ . Therefore, there exists $a_\xi > 1$ such that L_ξ contains a translate of $a_\xi T$, whence L_ξ contains a translate of $a_\xi S_\xi$.

Since the set of all d -subspaces of \mathbb{R}^n is compact, there exists $\alpha > 1$, independent of ξ , such that some translate of αS_ξ lies inside L_ξ for each ξ .

On the other hand, if $\alpha S + w \subseteq L$ for some w , then

$$h_L(u_i) \geq h_{\alpha S + w}(u_i) = \alpha h_S(u_i) + u_i \cdot w = \alpha h_L(u_i) + u_i \cdot w > h_L(u_i) + u_i \cdot w,$$

so that $u_i \cdot w < 0$ for all i . This strict inequality contradicts (3.1). \square

To prove the converse to Theorem 3.2, we first consider the polytope case. Recall that a *facet* of a polytope Q is a face of co-dimension 1 in the affine hull of Q .

Theorem 3.3. *Let $K \in \mathcal{K}_n$, and let Q be a convex polytope in \mathbb{R}^n . Suppose that Q_ξ contains a translate of K_ξ for every d -subspace ξ , and that Q does not contain a translate of K . Then there exists a simplicial family of facet unit normals $\{u_0, \dots, u_m\}$ to Q , for some $m \geq d + 1$.*

In other words, if a polytope Q is *not* d -reliable, then Q has a simplicial family of facet unit normals of size at least $d + 2$.

Proof. As in the previous proof, Proposition 2.4 makes it sufficient to verify the case in which Q has interior.

Suppose that K_ξ can be translated inside Q_ξ for each d -subspace ξ , while K cannot be translated inside Q . Without loss of generality, translate K so that the origin o lies inside the relative interior of K . This implies that $h_K \geq 0$.

Since Q has interior, there exists $\epsilon > 0$ such that ϵK can be translated inside Q . Since Q is compact we may assume ϵ to be maximal. Evidently $\epsilon < 1$, since no translate of K fits inside Q . Without loss of generality, translate Q so that $\epsilon K \subseteq Q$.

Denote the facets of Q by F_0, \dots, F_q , having outward unit normals u_0, \dots, u_q . Suppose that ϵK meets facets F_0, \dots, F_m , and misses the others.

If the convex hull of $\{u_0, \dots, u_m\}$ does not contain the origin o , then there exists a vector v such that $v \cdot u_i < 0$ for $i = 0, \dots, m$. This implies that, for sufficiently small δ , the translate $\epsilon K + \delta v$ lies in the interior of Q . This violates the maximality of ϵ . Therefore, there exist $a_0, \dots, a_m \geq 0$ such that

$$a_0 u_0 + \dots + a_m u_m = o.$$

Renumbering the facets as necessary, we have

$$(3.2) \quad c_0 u_0 + \dots + c_s u_s = o,$$

where each $c_i > 0$ and s is minimal, so that $\{u_0, \dots, u_s\}$ is a simplicial family.

If $s \leq d$, then the $s + 1$ unit vectors u_i lie inside a d -subspace ξ . Since ϵK meets each of the facets F_0, \dots, F_s , we have

$$(3.3) \quad \epsilon h_K(u_i) = h_{\epsilon K}(u_i) = h_Q(u_i) = h_{Q_\xi}(u_i)$$

for each $i = 0, \dots, s$. Since Q_ξ contains a translate of K_ξ , there exists $w \in \xi$ so that $K_\xi + w \subseteq Q_\xi$, and

$$\epsilon h_K(u_i) = h_{Q_\xi}(u_i) \geq h_{K_\xi}(u_i) + w \cdot u_i = h_K(u_i) + w \cdot u_i$$

for each $i = 0, \dots, s$. After summing over i , it follows from (3.2) that

$$\epsilon \sum_{i=0}^s c_i h_K(u_i) \geq \sum_{i=0}^s c_i h_K(u_i) + w \cdot \sum_{i=0}^s c_i u_i = \sum_{i=0}^s c_i h_K(u_i).$$

Recall that $h_K \geq 0$ and each $c_i > 0$. Since $\epsilon < 1$, it follows that

$$\sum_{i=0}^s c_i h_K(u_i) = 0,$$

so that each $h_K(u_i) = 0$. Therefore, each $h_Q(u_i) = 0$, by (3.3). It now follows from (3.2) and the sublinearity of the support function h_Q that the projection of Q onto the span of $\{u_0, \dots, u_s\}$ is a single point. This is a contradiction, since Q has interior. It follows that $s \geq d + 1$.

Therefore, there exists a simplicial family of facet unit normals u_0, \dots, u_s to Q , where $s \geq d + 1$. □

Putting Theorems 3.2 and 3.3 together, we obtain the following.

Theorem 3.4 (Reliability theorem). *Let $L \in \mathcal{K}_n$. Then L is a d -reliable cover if and only if every simplicial family of normals to regular boundary points of L has size at most $d + 1$.*

Proof. Suppose a simplicial family of unit normals to regular boundary points of L has size $d + 2$ or greater. By Theorem 3.2, L is not d -reliable.

To prove the converse, suppose that L is not d -reliable. Then there exists $K \in \mathcal{K}_n$ such that L_ξ contains a translate of K_ξ for every d -subspace ξ , while L does not contain a translate of K .

Since regular points are dense on the boundary of L (see [28, p. 73]), there exists a countable dense set of regular points on the boundary of L . By intersecting half-spaces that support L at these points, construct a sequence of polytopes P_i , decreasing with respect to set inclusion, such that $P_i \rightarrow L$ and each P_i has facet normals that are unit normals at regular points of L .

If P_i contains a translate of K for all i , then so does L , a contradiction. Therefore, there exists j such that P_j does not contain a translate of K . But each projection $L_\xi \subseteq (P_j)_\xi$, so that each projection $(P_j)_\xi$ contains a translate of K_ξ . In other words, the polytope P_j is not d -reliable. By Theorem 3.3, there are facet unit normals u_0, \dots, u_m for the polytope P_j that form a simplicial family, for some $m \geq d + 1$. Since the facet normals of P_j were taken from unit normals to regular points of L , this completes the proof. \square

Recall that a simplex T circumscribes L if $L \subseteq T$ and if aT contains no translate of L when $a < 1$. An n -simplex $T \supseteq L$ circumscribes L if and only if L meets every facet of T . Theorem 3.4 therefore implies the following.

Corollary 3.5. *Let $L \in \mathcal{K}_n$. Then L is $(n - 1)$ -reliable if and only if there is no circumscribing n -simplex T of L such that $\partial T \cap L$ consists of regular points of L .*

Since every boundary point of a smooth convex body is a regular boundary point, the following corollary is now immediate.

Corollary 3.6. *If L is a smooth convex body in \mathbb{R}^n , there exists an n -simplex S such that L_u contains a translate of S_u for every unit direction u , while L does not contain a translate of S .*

We can now characterize 1-reliability.

Corollary 3.7. *A convex set $L \in \mathcal{K}_n$ is a 1-reliable cover if and only if L is a parallelotope.*

Proof. If L is a parallelotope then L is 1-reliable, by Theorem 1.3.

Conversely, if L is 1-reliable, then Theorem 3.4 asserts that there are no simplicial sets of size 3 or more among the unit normals at regular points of L . By Proposition 2.4 we may assume, without loss of generality, that L has interior. In this case there exist affinely independent unit normals u_1, \dots, u_m at regular points of L , where $m \geq n + 1$, and where the u_i do not all lie in the same hemisphere. It follows that

$$0 = c_1u_1 + \dots + c_mu_m$$

for some $c_1, \dots, c_m > 0$. By Proposition 2.2, the set of regular normals of L has the form $\{\pm v_1, \dots, \pm v_n\}$, for some linearly independent set $v_1, \dots, v_n \in \mathbb{S}^{n-1}$. Let P be the unique (up to translation) parallelotope having facet unit normals $\pm v_i$ and corresponding facet areas c_i . Since the regular points of L are dense in the boundary of L , it follows that L and P must be translates. \square

Corollary 3.8. *Suppose that P is a polytope in \mathbb{R}^n . Then P is a d -reliable cover if and only if, for all $m \geq d + 2$, no m facets of P share normal directions with an $(m - 1)$ -simplex.*

Since no four facet normals of the square pyramid P in \mathbb{R}^3 contain the origin in the interior of their convex hull, any $K \in \mathcal{K}_3$ that can “hide behind” P can also “hide inside” P . In other words, P is 2-reliable, in spite of being indecomposable.

4. CENTRALLY SYMMETRIC COVERING SETS

We saw in the previous section that L is a reliable 1-cover if and only if L is 1-decomposable (i.e. a parallelotope). However, the square pyramid is 2-reliable in spite of being indecomposable.

A compact convex set L is said to be *centrally symmetric* if L and $-L$ are translates. For equivalence of 2-reliability and 2-decomposability to hold, we must restrict our attention to centrally symmetric bodies.

Theorem 4.1. *A centrally symmetric set $L \in \mathcal{K}_n$ is 2-reliable if and only if L is 2-decomposable.*

The 3-dimensional case of Theorem 4.1 has the following especially simple form.

Corollary 4.2. *A centrally symmetric set $L \in \mathcal{K}_3$ is 2-reliable if and only if L is a cylinder.*

The proof of Theorem 4.1 will use the following auxiliary results.

Proposition 4.3. *Let P be a convex polytope in \mathbb{R}^n with non-empty interior. Suppose that ξ is a proper subspace of \mathbb{R}^n , and suppose that each facet unit normal of P lies either in ξ or in ξ^\perp .*

Then there exist polytopes $P_1 \subseteq \xi$ and $P_2 \subseteq \xi^\perp$ such that $P = P_1 \oplus P_2$.

Proof. Suppose that the facet unit normals of P are given by

$$\{u_1, \dots, u_p, v_1, \dots, v_q\},$$

where $u_1, \dots, u_p \in \xi$ and $v_1, \dots, v_q \in \xi^\perp$. Suppose that each facet of P with normal u_i has area a_i and each facet with normal v_j has area b_j . By the Minkowski condition,

$$a_1u_1 + \dots + a_pu_p + b_1v_1 + \dots + b_qv_q = o.$$

It follows from the independence of ξ and ξ^\perp that

$$a_1u_1 + \dots + a_pu_p = o \quad \text{and} \quad b_1v_1 + \dots + b_qv_q = o.$$

By the Minkowski Existence Theorem [2, 28] there exists a polytope $Q_1 \subseteq \xi$ having facet normals u_i and corresponding facet areas a_i . Similarly, there exists a polytope $Q_2 \subseteq \xi^\perp$ having facet normals v_j and corresponding facet areas b_j .

Let $d = \dim \xi$, so that $\dim \xi^\perp = n - d$. For $x, y > 0$, the Minkowski sum $xQ_1 + yQ_2$ has the same unit normals as P and has corresponding facets $xQ_1^{u_i} + yQ_2$ and $xQ_1 + yQ_2^{v_j}$, having the respective facet areas $x^{d-1}y^{n-d}V_{n-d}(Q_2)a_i$ and $x^d y^{n-d-1}V_d(Q_1)b_j$. Set

$$x = \left(\frac{V_{n-d}(Q_2)^{n-d-1}}{V_d(Q_1)^{n-d}} \right)^{\frac{1}{n-1}} \quad \text{and} \quad y = \left(\frac{V_d(Q_1)^{d-1}}{V_{n-d}(Q_2)^d} \right)^{\frac{1}{n-1}},$$

and let $P_1 = xQ_1$ and $P_2 = yQ_2$. The polytope $P_1 \oplus P_2$ now has the same facet normals and the same corresponding facet areas as P . It follows from the uniqueness assertion of the Minkowski Existence Theorem that P and $P_1 \oplus P_2$ must be translates. \square

Proposition 4.4. *Let $K \in \mathcal{X}_n$ have non-empty interior. Suppose that there is a subspace decomposition $\mathbb{R}^n = \xi \oplus \xi'$ such that each unit normal at a regular point of K lies either in ξ or in ξ' .*

Then there is a subspace decomposition $\mathbb{R}^n = \eta \oplus \eta'$, where $\dim \eta = \dim \xi$ and $\dim \eta' = \dim \xi'$, and compact convex sets $K_1 \subseteq \eta$ and $K_2 \subseteq \eta'$ such that $K = K_1 \oplus K_2$.

Proof. To begin, suppose that $\xi' = \xi^\perp$, so that $\mathbb{R}^n = \xi \oplus \xi'$ is an orthogonal decomposition. Since regular points are dense on the boundary of K (see [28, p. 73]), there exists a countable dense set of regular points on the boundary of K . By intersecting half-spaces that support K at these points, construct a sequence of polytopes P_i , decreasing with respect to set inclusion, such that $P_i \rightarrow K$ and each P_i has facet normals that are unit normals at regular points of K .

By Proposition 4.3, each $P_i = Q_i \oplus Q'_i$, where $Q_i \subseteq \xi$ and $Q'_i \subseteq \xi'$. Since projections are continuous, the $Q_i = (P_i)_\xi$ converge to K_ξ , and similarly $Q'_i \rightarrow K_{\xi'}$. Therefore $K = \lim_i P_i = K_\xi \oplus K_{\xi'}$.

More generally, if ξ and ξ' are not orthogonal complements, then let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a non-singular linear transformation such that $\psi^{-T}\xi \perp \psi^{-T}\xi'$, where ψ^{-T} denotes the inverse transpose of ψ . Let $\eta = \psi^{-1}\psi^{-T}\xi$ and $\eta' = \psi^{-1}\psi^{-T}\xi'$.

Recall that $(\psi\xi)^\perp = \psi^{-T}(\xi^\perp)$. Therefore, if each unit normal at a regular point of K lies either in ξ or in ξ' , then each unit normal at a regular point of ψK lies either in $\psi^{-T}\xi$ or in $\psi^{-T}\xi'$. Since these subspaces form an orthogonal decomposition, the previous argument implies that $\psi K = L_1 \oplus L_2$, where $L_1 \subseteq \psi^{-T}\xi$ and $L_2 \subseteq \psi^{-T}\xi'$. It follows that $K = K_1 \oplus K_2$, where $K_1 = \psi^{-1}L_1 \subseteq \eta$ and $K_2 = \psi^{-1}L_2 \subseteq \eta'$. \square

Proof of Theorem 4.1. If L is 2-decomposable then L is 2-reliable by Theorem 1.3.

For the converse, suppose that L is 2-reliable. Let A denote the set of unit normals at regular points of L . Since L is 2-reliable, A contains no simplicial subsets sets of size 4, by Theorem 3.4.

Since L is centrally symmetric, we have $A = -A$. By Proposition 2.3, there exists a subspace direct sum decomposition

$$\mathbb{R}^n = W_1 \oplus \cdots \oplus W_k,$$

where each $\dim W_i \leq 2$, and such that $A \subseteq W_1 \cup \cdots \cup W_k$. It follows from Proposition 4.4 that L is 2-decomposable. \square

In view of Theorem 4.1, one may be tempted to conjecture that d -reliability is equivalent to d -decomposability for centrally symmetric bodies, but this turns out to be false for $d = 3$. Consider the following 12 vectors in \mathbb{R}^4 :

$$\pm(1, 1, 0, 0), \pm(1, 0, 1, 0), \pm(1, 0, 0, 1), \pm(0, 1, 1, 0), \pm(0, 1, 0, 1), \pm(0, 0, 1, 1).$$

By Minkowski's Existence Theorem, there exists a unique 12-faceted polytope Q in \mathbb{R}^4 , centrally symmetric about the origin (i.e. $Q = -Q$), having facet normals parallel to the directions above, with each facet having unit 3-volume. One can verify that the set of vectors above contains no simplicial 5-family, so that Q is 3-reliable by Corollary 3.8. A routine linear algebra computation (using Proposition 4.4) also verifies that Q is not 3-decomposable.

5. SOME OPEN QUESTIONS

There remain several fundamental open questions about convex bodies and projections, among them the following:

- I. Under what symmetry (or other) conditions on $L \in \mathcal{K}_n$ is d -reliability equivalent to d -decomposability, for $d > 2$?

A solution to Problem I would generalize Corollary 3.7 and Theorem 4.1. For example, what happens if we assume that L is a zonoid?

Denote the n -dimensional (Euclidean) volume of $L \in \mathcal{K}_n$ by $V_n(L)$.

- II. Let $K, L \in \mathcal{K}_n$ such that $V_n(L) > 0$, and let $1 \leq d \leq n - 1$. Suppose that L_ξ contains a translate of K_ξ for every d -subspace ξ of \mathbb{R}^n .

What is the best upper bound for the ratio $\frac{V_n(K)}{V_n(L)}$?

Some partial answers to Problem II are offered in [16]. There it is shown that if K_ξ can be translated inside L_ξ for all d -dimensional subspaces ξ , then K has smaller volume than L whenever L can be approximated by Blaschke combinations of d -decomposable sets. However, there are cases in which $V_n(K) > V_n(L)$, in spite of the covering condition on shadows. For all $L \in \mathcal{K}_n$ it is also shown that, if K_u can be translated inside L_u for all unit directions u , then $V_n(K) \leq nV_n(L)$, where n is the dimension of the ambient space for K and L . However, I doubt this is the best possible bound.

- III. Let $K, L \in \mathcal{K}_n$, and let $1 \leq d \leq n - 1$. Suppose that, for each d -subspace ξ of \mathbb{R}^n , the orthogonal projection K_ξ of K can be moved inside L_ξ by some *rigid motion* (i.e. a combination of translations, rotations, and reflections). Under what simple (easy to state, easy to verify) additional conditions does it follow that K can be moved inside L by a rigid motion?

Problem III is an intuitive generalization of the questions addressed in this article. Indeed, each question can be re-phrased allowing for rotations (and reflections) as well as translations. However, the arguments presented so far rely on the observation that the set of translates of K that fit inside L , that is, the set

$$\{v \in \mathbb{R}^n \mid K + v \subseteq L\},$$

is a compact convex set in \mathbb{R}^n . By contrast, the set of rigid motions of K that fit inside L will lie in a more complicated Lie group. For this reason (at least) the questions of covering via rigid motions may be more difficult to address than the case in which only translation is allowed.

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