

LOCAL RIGIDITY OF INVERSIVE DISTANCE CIRCLE PACKING

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ABSTRACT. A Euclidean (or hyperbolic) circle packing on a triangulated closed surface with prescribed inversive distance is locally determined by its cone angles. We prove this by establishing a variational principle.

1. INTRODUCTION

1.1. Andreev-Thurston Theorem. In his work on constructing hyperbolic metrics on 3-manifolds, W. Thurston ([23], Chapter 13) studied a Euclidean (or hyperbolic) circle packing on a triangulated closed surface with prescribed intersection angles. Thurston's work generalizes Andreev's result of circle packing on a sphere [1, 2]. The special case of tangent circle packing (every intersection angle is zero) on a sphere was obtained by Koebe [16].

Suppose (Σ, T) is a triangulated closed surface so that V, E, F are sets of all vertices, edges and triangles in T . We identify vertices of T with indexes, edges of T with pairs of indexes and triangles of T with triples of indexes. This means $V = \{1, 2, \dots, |V|\}$, $E = \{ij \mid i, j \in V\}$ and $F = \{\Delta ijk \mid i, j, k \in V\}$. Fix a vector $\Theta \in \mathbb{R}^{|E|}$ indexed by the set of edges E such that $\Theta_{ij} \in [0, \frac{\pi}{2}]$ for each $ij \in E$. This vector is called a *weight* on (Σ, T) .

A circle packing on a weighted triangulated closed surface (Σ, T, Θ) is a configuration $\{c_i, i \in V\}$ of circles such that the intersection angle of c_i and c_j is Θ_{ij} , which is the prescribed weight on the edge ij . To obtain a Euclidean (or hyperbolic) circle packing for (Σ, T, Θ) , we start with a *radius vector* $r = (r_1, r_2, \dots, r_{|V|}) \in \mathbb{R}_{>0}^{|V|}$ which assigns each vertex $i \in V$ a positive number r_i . A radius vector r produces a Euclidean (or hyperbolic) cone metric on the surface as follows.

In Figure 1, consider a topological triangle Δijk . One can construct a Euclidean (or hyperbolic) triangle Δijp such that the edges ip, jp have lengths r_i, r_j respectively and the angle at p is $\pi - \Theta_{ij}$. Let l_{ij} be the length of the edge ij in the Euclidean (or hyperbolic) triangle Δijp which is a function of r_i, r_j . In fact, in Euclidean geometry

$$l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_i r_j \cos \Theta_{ij}}$$

or in hyperbolic geometry

$$l_{ij} = \cosh^{-1}(\cosh r_i \cosh r_j + \cos \Theta_{ij} \sinh r_i \sinh r_j).$$

Similarly, we obtain l_{jk}, l_{ki} .

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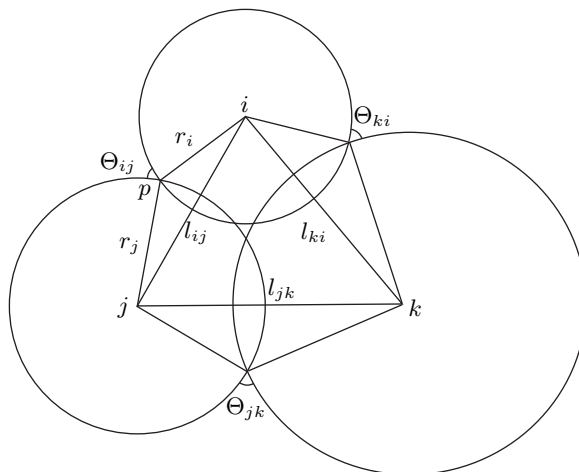


FIGURE 1. Andreev-Thurston circle packing

As observed by Thurston [23], under the assumption that $\Theta_{ij} \in [0, \frac{\pi}{2}]$ for each $ij \in E$, l_{ij}, l_{jk} and l_{ki} can be realized as edge lengths of a Euclidean (or hyperbolic) triangle $\triangle ijk$ in F . Gluing all of these Euclidean (or hyperbolic) triangles in F produces a Euclidean (or hyperbolic) cone metric on (Σ, T) with possible cones at vertices of T . On this surface with a cone metric, drawing a circle centered at vertex i with radius r_i for each vertex $i \in V$, we obtain a circle packing with prescribed intersection angles.

At each vertex $i \in V$, the cone angle a_i is the sum of all inner angles having vertex i . Thus cone angles $(a_1, a_2, \dots, a_{|V|})$ are functions of the radius vector $r = (r_1, r_2, \dots, r_{|V|})$. We try to find a radius vector when cone angles are given. The Andreev-Thurston Theorem answers the question about the existence and uniqueness of radius vectors when cone angles are given.

Theorem 1 (Andreev-Thurston). *For any weighted triangulated closed surface (Σ, T, Θ) with $\Theta_{ij} \in [0, \frac{\pi}{2}]$ for each $ij \in E$, a Euclidean (or hyperbolic) circle packing for (Σ, T, Θ) is determined by its cone angles up to Euclidean similarity (or hyperbolic isometry). Furthermore, the set of all possible cone angles form an open convex polytope in $\mathbb{R}^{|V|}$.*

For a proof, see Thurston [23], Marden-Rodin [19], Colin de Verdière [8], He [14], Chow-Luo [9], and Stephenson [22].

1.2. Inversive distance. H. S. M. Coxeter [7] introduced the notion of inversive distance to describe the relationship between two circles in a Möbius plane. The notion of inversive distance generalizes the notion of an intersection angle of two circles.

The inversive distance between two circles is independent of geometry, i.e., the inversive distance of two circles is the same under spherical, Euclidean and hyperbolic metrics. This generalizes the fact that the intersection angle of two circles is the same under spherical, Euclidean and hyperbolic metrics.

In the Euclidean plane, consider two circles c_1, c_2 with radii r_1, r_2 respectively. In the relevant case of this paper, we assume that c_i does not contain c_j for $\{i, j\} =$

$\{1, 2\}$. If the distance between their center is l , the inversive distance between c_1, c_2 is given by the formula

$$I(c_1, c_2) = \frac{l^2 - r_1^2 - r_2^2}{2r_1r_2}.$$

In hyperbolic plane, consider two circles c_1, c_2 with radii r_1, r_2 respectively. We assume that c_i does not contain c_j for $\{i, j\} = \{1, 2\}$. If the distance between their center is l , the inversive distance between c_1, c_2 is given by the formula

$$I(c_1, c_2) = \frac{\cosh l - \cosh r_1 \cosh r_2}{\sinh r_1 \sinh r_2}.$$

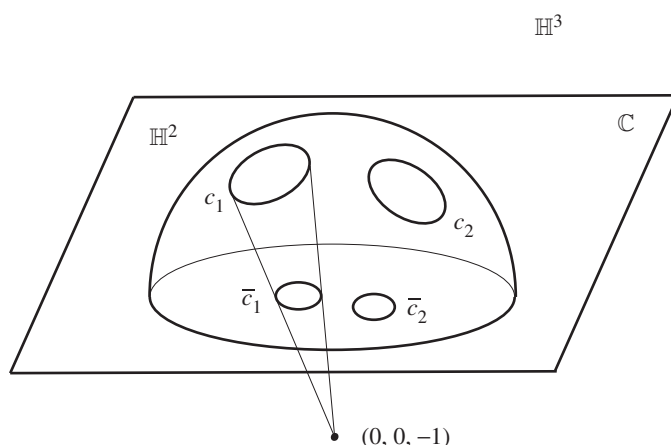


FIGURE 2. The stereographic projection

These two formulas are related by a stereographic projection. In Figure 2, consider the upper half space model of the hyperbolic space \mathbb{H}^3 . The hyperbolic plane \mathbb{H}^2 is realized as a totally geodesic plane in \mathbb{H}^3 , where it is a unit hemisphere perpendicular to the infinite boundary \mathbb{C} . The stereographic projection is the projection starting from point $(0, 0, -1)$, which sends objects in \mathbb{H}^2 into \mathbb{C} . Let c_1, c_2 be two circles in \mathbb{H}^2 with radii r_1, r_2 and the distance between centers l . The stereographic projection produces two circles \bar{c}_1, \bar{c}_2 on the Euclidean plane \mathbb{C} with radii \bar{r}_1, \bar{r}_2 and the distance between centers \bar{l} . Then we have

$$I(c_1, c_2) = \frac{\cosh l - \cosh r_1 \cosh r_2}{\sinh r_1 \sinh r_2} = \frac{\bar{l}^2 - \bar{r}_1^2 - \bar{r}_2^2}{2\bar{r}_1\bar{r}_2} = I(\bar{c}_1, \bar{c}_2).$$

We say that the stereographic projection preserves the inversive distance.

When $I(c_1, c_2) \in (-1, 0)$, the circles c_1, c_2 intersect with an intersection angle $\arccos I(c_1, c_2) \in (\frac{\pi}{2}, \pi)$. When $I(c_1, c_2) \in [0, 1)$, the circles c_1, c_2 intersect with an intersection angle $\arccos I(c_1, c_2) \in (0, \frac{\pi}{2}]$. When $I(c_1, c_2) = 1$, the circles c_1, c_2 are tangent to each other. When $I(c_1, c_2) \in (1, \infty)$, the circles c_1, c_2 are separated.

1.3. Inversive distance circle packing. Motivated by the application to discrete conformal mappings, Bowers-Stephenson [5] introduced inversive distance circle packing which generalizes Andreev-Thurston’s intersection angle circle packing. See Stephenson [22] (page 331) and Bowers-Hurdal [4] for more information. One of the

important applications of inversive distance circle packing is the conformal mapping of the human brain; see, for example, Hurdal-Stephenson [15].

We will use the same notation as in Andreev-Thurston's intersection angle circle packing. For a triangulated closed surface (Σ, T) , fix a weight $I \in \mathbb{R}^{|E|}$ with $I_{ij} \in [0, \infty)$ for each $ij \in E$. In this paper, we will not consider the case of obtuse intersection angles. (Rivin [20] and Leibon [17] obtained results for certain circle patterns on surfaces allowing obtuse intersection angles.) An inversive distance circle packing for (Σ, T, I) is a configuration of circles $\{c_i, i \in V\}$ such that the inversive distance of two circles c_i, c_j is the prescribed number I_{ij} for the edge ij . To obtain a Euclidean (or hyperbolic) circle packing for (Σ, T, I) , we start with a radius vector $r = (r_1, r_2, \dots, r_{|V|}) \in \mathbb{R}_{>0}^{|V|}$ which assigns each vertex $i \in V$ a positive number r_i . A radius vector r produces a Euclidean (or hyperbolic) cone metric on the surface as follows.

Consider a topological triangle $\Delta ijk \in F$. Assign a number l_{ij} to the edge $ij \in E$, where l_{ij} is the distance of two circles with radii r_i, r_j and inversive distance I_{ij} . Therefore l_{ij} is a function of r_i, r_j . In fact, in Euclidean geometry

$$l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_i r_j I_{ij}}$$

or in hyperbolic geometry

$$l_{ij} = \cosh^{-1}(\cosh r_i \cosh r_j + I_{ij} \sinh r_i \sinh r_j).$$

Similarly, we obtain l_{jk}, l_{ki} .

To guarantee that l_{ij}, l_{jk}, l_{ki} satisfy the triangle inequality for each triangle $\Delta ijk \in F$, a vector $r = (r_1, r_2, \dots, r_{|V|})$ must satisfy certain conditions. For details, see sections 2.1 and 3.1.

For a radius vector r , if l_{ij}, l_{jk}, l_{ki} can be realized as edge lengths of a Euclidean (or hyperbolic) triangle for each $\Delta ijk \in F$, gluing all of these Euclidean (or hyperbolic) triangles in F produces a Euclidean (or hyperbolic) cone metric on (Σ, T) with possible cones at vertices of T . On this surface with a cone metric, drawing a circle centered at vertex i with radius r_i for each vertex $i \in V$, we obtain a circle packing with prescribed inversive distance.

The cone angles $(a_1, a_2, \dots, a_{|V|})$ are functions of the radius vector $r = (r_1, r_2, \dots, r_{|V|})$. We are interested in the solutions of a radius vector when cone angles are given.

Our main theorem is the local rigidity of inversive distance circle packing.

Theorem 2. *For any weighted triangulated closed surface (Σ, T, I) with $I_{ij} \in [0, \infty)$ for each $ij \in E$, a Euclidean (or hyperbolic) inversive distance circle packing for (Σ, T, I) is locally determined by its cone angles up to Euclidean similarity (or hyperbolic isometry).*

The local rigidity of the Andreev-Thurston Theorem is generalized to the case of inversive distance circle packing. In our case, we are not able to obtain either the global rigidity or the description of the space of cone angles as in the Andreev-Thurston Theorem. For discussion, see sections 2.1 and 3.1.

In section 2, we prove Theorem 2 in Euclidean geometry. In section 3, we prove Theorem 2 in hyperbolic geometry.

1.4. Variational principle. Theorem 2 is proved by establishing a variational principle. The energy function of this variational principle is derived from the derivative of the cosine law of a Euclidean (or hyperbolic) triangle. Andreev and Thurston obtained their results without using variational principles. The variational approach to circle packing was first introduced by Colin de Verdière [8]. Since then, many works on variational principles on circle packing or circle patterns have appeared. For example, see Brägger [3], Rivin [20], Leibon [17], Chow-Luo [9], Bobenko-Springborn [6], Guo [12], Springborn [21], Luo [18], Guo-Luo [13] and others.

Applying a variational principle, Colin de Verdière [8] proved the Andreev-Thurston Theorem in the special case of tangent circle packing, i.e., $\Theta_{ij} = 0$ for each $ij \in E$. Chow-Luo [9] proved the Andreev-Thurston Theorem of intersection angle circle packing, i.e., $\Theta_{ij} \in [0, \frac{\pi}{2}]$ for each $ij \in E$. Variational principles for polyhedral surfaces including the topic of circle packing are studied systematically in Luo [18]. Many energy functions are derived from the cosine law and its derivative. Tangent circle packing is generalized to tangent circle packing with a family of discrete curvature. For exposition of this work, see also Dai-Gu-Luo [10]. We follow this variational approach to prove Theorem 2 of inversive distance circle packing, i.e., $I_{ij} \in [0, \infty)$ for each $ij \in E$.

2. CIRCLE PACKING IN EUCLIDEAN GEOMETRY

2.1. Space of radius vectors. As stated in the Introduction, to obtain a Euclidean cone metric, we need conditions on a radius vector. In this section we investigate the space of radius vectors for one triangle in Euclidean geometry. Consider a topological triangle $\Delta ijk \in F$. Assign a number l_{ij} to the edge $ij \in E$, where l_{ij} is the distance of the centers of two circles with radii r_i, r_j and inversive distance I_{ij} . Therefore l_{ij} is a function of r_i, r_j . In fact, in Euclidean geometry

$$l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_i r_j I_{ij}}.$$

Similarly, we obtain l_{jk}, l_{ki} .

Lemma 3. *The set $\mathcal{R}_E^{ijk} := \{(r_i, r_j, r_k) \in \mathbb{R}_{>0}^3 \mid l_{ij}, l_{jk}, l_{ki} \text{ satisfy triangle inequality}\}$ is a connected and simply connected open subset in $\mathbb{R}_{>0}^3$.*

Proof. Let $\rho_E : \mathbb{R}_{>0}^3 \rightarrow \mathbb{R}_{>0}^3$ be the map sending (r_i, r_j, r_k) to (l_{ij}, l_{jk}, l_{ki}) . We claim that ρ_E is a smooth embedding. First the determinant of the Jacobian matrix of ρ_E is nonzero. Next, we will show ρ_E is one-to-one.

This means we need to show that if the following equations have a solution (r_i, r_j, r_k) , then the solution is unique:

$$\begin{aligned} r_j^2 + r_k^2 + 2I_{jk}r_jr_k &= l_{jk}^2, \\ r_k^2 + r_i^2 + 2I_{ki}r_kr_i &= l_{ki}^2, \\ r_i^2 + r_j^2 + 2I_{ij}r_i r_j &= l_{ij}^2. \end{aligned}$$

Solving the first equation for r_j , we get $r_j = -I_{jk}r_k \pm \sqrt{(I_{jk}^2 - 1)r_k^2 + l_{jk}^2}$. Since $I_{jk} > 0$ and $r_j > 0$, we must have $r_j = -I_{jk}r_k + \sqrt{(I_{jk}^2 - 1)r_k^2 + l_{jk}^2}$. Hence r_j is uniquely determined by r_k . By the same argument $r_i = -I_{ki}r_k + \sqrt{(I_{ki}^2 - 1)r_k^2 + l_{ki}^2}$.

After substituting r_i, r_j in the third equation, we need to show it has a unique solution for r_k . Consider the function

$$f(t) = r_i^2(t) + r_j^2(t) + 2I_{ij}r_i(t)r_j(t) - l_{ij}^2$$

which is obtained by replacing r_k by t for simplicity of notation. Now

$$f'(t) = (2r_i + 2I_{ij}r_j)r'_i(t) + (2r_j + 2I_{ij}r_i)r'_j(t).$$

Also, we have

$$r'_i(t) = -I_{ki} + \frac{(I_{ki}^2 - 1)t}{\sqrt{(I_{ki}^2 - 1)t^2 + l_{ki}^2}}.$$

If $I_{ki} \in [0, 1]$, then each term in $r'_i(t)$ is nonpositive. Thus $r'_i(t) < 0$.

If $I_{ki} \in (1, \infty)$, then

$$r'_i(t) < -I_{ki} + \frac{(I_{ki}^2 - 1)t}{\sqrt{(I_{ki}^2 - 1)t^2}} = -I_{ki} + \sqrt{I_{ki}^2 - 1} < 0.$$

By the same argument, we have $r'_j(t) < 0$. Since $2r_i + 2I_{ij}r_j > 0$ and $2r_j + 2I_{ij}r_i > 0$, then $f'(t) < 0$ holds. Therefore $f(t) = 0$ has a unique solution.

To determine the image $\rho_E(\mathbb{R}_{>0}^3)$, we consider the image of the boundary of $\mathbb{R}_{\geq 0}^3$. If $r_i = 0$, then $r_k = l_{ki}, r_j = l_{ij}$. Thus

$$l_{jk}^2 = r_j^2 + r_k^2 + 2I_{jk}r_jr_k = l_{ij}^2 + l_{ki}^2 + 2I_{jk}l_{ij}l_{ki}.$$

On the other hand, if $r_i > 0$, then $r_k < l_{ki}, r_j < l_{ij}$. We have

$$l_{jk}^2 < l_{ij}^2 + l_{ki}^2 + 2I_{jk}l_{ij}l_{ki}.$$

Therefore the image $\rho_E(\mathbb{R}_{>0}^3)$ is the subset of vectors (l_{ij}, l_{jk}, l_{ki}) satisfying

$$\begin{aligned} l_{jk}^2 &< l_{ij}^2 + l_{ki}^2 + 2I_{jk}l_{ij}l_{ki}, \\ l_{ki}^2 &< l_{jk}^2 + l_{ij}^2 + 2I_{ki}l_{jk}l_{ij}, \\ l_{ij}^2 &< l_{ki}^2 + l_{jk}^2 + 2I_{ij}l_{ki}l_{jk}. \end{aligned}$$

In fact, $\rho_E(\mathbb{R}_{>0}^3)$ is a cone bounded by three surfaces. In $\mathbb{R}_{\geq 0}^3$, the intersection of any of the two surfaces is a straight line. The three lines of intersection are

$$\begin{aligned} l_{ij} &= 0 \text{ and } l_{jk} = l_{ki}, \\ l_{jk} &= 0 \text{ and } l_{ki} = l_{ij}, \\ l_{ki} &= 0 \text{ and } l_{ij} = l_{jk}. \end{aligned}$$

Let \mathcal{L} be the subset of $\mathbb{R}_{>0}^3$ formed by vectors (l_{ij}, l_{jk}, l_{ki}) satisfying the triangle inequality. Therefore \mathcal{L} is a cone bounded by three planes. The three lines of intersection of the three planes are the same as that of the boundary surfaces of $\rho_E(\mathbb{R}_{>0}^3)$.

We claim that $\rho_E(\mathbb{R}_{>0}^3) \cap \mathcal{L}$ is a cone. Therefore it is connected and simply connected.

In fact, if $I_{ij} \in [0, 1]$, then in $\mathbb{R}_{>0}^3$ we have

$$\{(l_{ij}, l_{jk}, l_{ki}) \mid l_{ij}^2 < l_{ki}^2 + l_{jk}^2 + 2I_{ij}l_{ki}l_{jk}\} \subseteq \{(l_{ij}, l_{jk}, l_{ki}) \mid l_{ij} < l_{ki} + l_{jk}\}.$$

If $I_{ij} \in (1, \infty)$, then in $\mathbb{R}_{>0}^3$ we have

$$\{(l_{ij}, l_{jk}, l_{ki}) \mid l_{ij}^2 < l_{ki}^2 + l_{jk}^2 + 2I_{ij}l_{ki}l_{jk}\} \supseteq \{(l_{ij}, l_{jk}, l_{ki}) \mid l_{ij} < l_{ki} + l_{jk}\}.$$

To determine the shape of $\rho_E(\mathbb{R}_{>0}^3) \cap \mathcal{L}$, there are four cases to consider.

If $I_{ij}, I_{jk}, I_{ki} \in (1, \infty)$, then $\rho_E(\mathbb{R}_{>0}^3) \cap \mathcal{L} = \mathcal{L}$.

If $I_{ij} \in [0, 1], I_{jk}, I_{ki} \in (1, \infty)$, then $\rho_E(\mathbb{R}_{>0}^3) \cap \mathcal{L}$ is a cone bounded by one surface and two planes.

If $I_{ij}, I_{jk} \in [0, 1], I_{ki} \in (1, \infty)$, then $\rho_E(\mathbb{R}_{>0}^3) \cap \mathcal{L}$ is a cone bounded by two surfaces and one plane.

If $I_{ij}, I_{jk}, I_{ki} \in [0, 1]$, then $\rho_E(\mathbb{R}_{>0}^3) \cap \mathcal{L} = \rho_E(\mathbb{R}_{>0}^3)$.

To finish the proof of the lemma, by definition, $\mathcal{R}_E^{ijk} = \rho_E^{-1}(\rho_E(\mathbb{R}_{>0}^3) \cap \mathcal{L})$. Since ρ_E is a smooth embedding, \mathcal{R}_E^{ijk} is connected and simply connected. \square

In the construction of the energy function, we will use a natural variable $u = (u_1, u_2, \dots, u_{|V|})$, where $u_i = \ln(r_i)$. Let $\tau_E : \mathbb{R}^3 \rightarrow \mathbb{R}_{>0}^3$ be the map sending (u_i, u_j, u_k) to (r_i, r_j, r_k) . Since τ_E is a diffeomorphism, we have

Corollary 4. *The set $\mathcal{U}_E^{ijk} := \tau_E^{-1}(\mathcal{R}_E^{ijk})$ is a connected and simply connected open subset in \mathbb{R}^3 .*

To prove Theorem 2, we construct a concave energy function on \mathcal{U}_E^{ijk} for each triangle $\triangle ijk \in F$. We obtain the local rigidity of inversive distance circle packing from this energy function. We cannot prove the global rigidity due to the fact that \mathcal{U}_E^{ijk} is not always convex in this general setting. In fact, l_{ij}, l_{jk}, l_{ki} satisfying the triangle inequality is equivalent to

$$(1) \quad (l_{ij} + l_{jk} + l_{ki})(l_{jk} + l_{ki} - l_{ij})(l_{jk} + l_{ij} - l_{ki})(l_{ij} + l_{jk} - l_{ki}) > 0.$$

Substituting the functions l_{ij}, l_{jk}, l_{ki} in terms of u_i, u_j, u_k into inequality (1), we see that \mathcal{U}_E^{ijk} is defined by the inequality

$$(2) \quad \frac{1 - I_{jk}^2}{e^{2u_i}} + \frac{1 - I_{ki}^2}{e^{2u_j}} + \frac{1 - I_{ij}^2}{e^{2u_k}} + \frac{2(I_{jk}I_{ki} + I_{ij})}{e^{u_i+u_j}} + \frac{2(I_{ki}I_{ij} + I_{jk})}{e^{u_j+u_k}} + \frac{2(I_{ij}I_{jk} + I_{ki})}{e^{u_k+u_i}} > 0.$$

For example, when $I_{ij} = I_{jk} = I_{ki} = 2$, the inequality (2) determines a subset which is not convex.

In the Andreev-Thurston Theorem, the space of cone angles is determined by a continuity argument. Since we cannot prove the global rigidity, we are not able to determine the space of cone angles in the case of inversive distance circle packing.

2.2. Energy function. To prove Theorem 2, we will construct an energy function on the space of radius vectors. First, we construct an energy function on the space for one triangle. To simplify notation, consider the triangle $\triangle 123 \in F$ and the map

$$\begin{aligned} \mathcal{U}_E^{123} &\rightarrow \mathcal{R}_E^{123} &\rightarrow \mathcal{L} &\rightarrow \mathbb{R}_{>0}^3, \\ (u_1, u_2, u_3) &\mapsto (r_1, r_2, r_3) &\mapsto (l_1, l_2, l_3) &\mapsto (\alpha_1, \alpha_2, \alpha_3), \end{aligned}$$

where $r_i = e^{u_i}$ for $i = 1, 2, 3$. For three fixed positive numbers I_{12}, I_{23}, I_{31} , we have $l_i^2 = r_j^2 + r_k^2 + 2I_{jk}r_jr_k$ for $\{i, j, k\} = \{1, 2, 3\}$. For simplicity of notation, we set $(l_1, l_2, l_3) = (l_{23}, l_{31}, l_{12})$. Also, $\alpha_1, \alpha_2, \alpha_3$ are inner angles of a Euclidean triangle with edge lengths l_1, l_2, l_3 . By the cosine law, $\cos \alpha_i = \frac{-l_i^2 + l_j^2 + l_k^2}{2l_jl_k}$ for $\{i, j, k\} = \{1, 2, 3\}$. Therefore $\alpha_1, \alpha_2, \alpha_3$ are functions in terms of u_1, u_2, u_3 .

Lemma 5. *The Jacobian matrix of functions $\alpha_1, \alpha_2, \alpha_3$ in terms of u_1, u_2, u_3 is symmetric.*

Proof. Due to the cosine law of a Euclidean triangle, $\alpha_1, \alpha_2, \alpha_3$ are functions of l_1, l_2, l_3 . The following formula can be proved by direct calculation. For $\{i, j, k\} = \{1, 2, 3\}$,

$$\begin{pmatrix} d\alpha_1 \\ d\alpha_2 \\ d\alpha_3 \end{pmatrix} = \frac{-1}{\sin \alpha_i l_j l_k} \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{pmatrix} \begin{pmatrix} -1 & \cos \alpha_3 & \cos \alpha_2 \\ \cos \alpha_3 & -1 & \cos \alpha_1 \\ \cos \alpha_2 & \cos \alpha_1 & -1 \end{pmatrix} \begin{pmatrix} dl_1 \\ dl_2 \\ dl_3 \end{pmatrix}.$$

Next, by the construction $l_i^2 = r_j^2 + r_k^2 + 2I_{jk}r_jr_k$, for $\{i, j, k\} = \{1, 2, 3\}$, where I_{jk} is a constant. Differentiating the two sides of the equality, we obtain

$$dl_i = \frac{l_i^2 + r_j^2 - r_k^2}{2l_i r_j} dr_j + \frac{l_i^2 + r_k^2 - r_j^2}{2l_i r_k} dr_k.$$

Finally, $r_i = e^{u_i}$ implies $r_i du_i = dr_i$, for $i = 1, 2, 3$.

Combining the three relations, we have, for $\{i, j, k\} = \{1, 2, 3\}$,

$$\begin{pmatrix} d\alpha_1 \\ d\alpha_2 \\ d\alpha_3 \end{pmatrix} = \frac{-1}{\sin \alpha_i l_j l_k} \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{pmatrix} \begin{pmatrix} -1 & \cos \alpha_3 & \cos \alpha_2 \\ \cos \alpha_3 & -1 & \cos \alpha_1 \\ \cos \alpha_2 & \cos \alpha_1 & -1 \end{pmatrix} \\ \times \begin{pmatrix} 0 & \frac{l_1^2 + r_2^2 - r_3^2}{2l_1 r_2} & \frac{l_1^2 + r_3^2 - r_2^2}{2l_1 r_3} \\ \frac{l_2^2 + r_1^2 - r_3^2}{2l_2 r_1} & 0 & \frac{l_2^2 + r_3^2 - r_1^2}{2l_2 r_3} \\ \frac{l_3^2 + r_1^2 - r_2^2}{2l_3 r_1} & \frac{l_3^2 + r_2^2 - r_1^2}{2l_3 r_2} & 0 \end{pmatrix} \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}.$$

We write the above formula as

$$\begin{pmatrix} d\alpha_1 \\ d\alpha_2 \\ d\alpha_3 \end{pmatrix} = \frac{-1}{\sin \alpha_i l_j l_k} N \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}.$$

To show that the Jacobian matrix of functions $\alpha_1, \alpha_2, \alpha_3$ in terms of u_1, u_2, u_3 is symmetric is equivalent to showing that the matrix N is symmetric.

By the cosine law, we have

$$4N = \begin{pmatrix} -2l_1^2 & l_1^2 + l_2^2 - l_3^2 & l_3^2 + l_1^2 - l_2^2 \\ l_1^2 + l_2^2 - l_3^2 & -2l_2^2 & l_2^2 + l_3^2 - l_1^2 \\ l_3^2 + l_1^2 - l_2^2 & l_2^2 + l_3^2 - l_1^2 & -2l_3^2 \end{pmatrix} \begin{pmatrix} \frac{1}{l_1^2} & 0 & 0 \\ 0 & \frac{1}{l_2^2} & 0 \\ 0 & 0 & \frac{1}{l_3^2} \end{pmatrix} \\ \times \begin{pmatrix} 0 & l_1^2 + r_2^2 - r_3^2 & l_1^2 + r_3^2 - r_2^2 \\ l_2^2 + r_1^2 - r_3^2 & 0 & l_2^2 + r_3^2 - r_1^2 \\ l_3^2 + r_1^2 - r_2^2 & l_3^2 + r_2^2 - r_1^2 & 0 \end{pmatrix}.$$

To simplify notation, let $a := l_1^2, b := l_2^2, c := l_3^2, x := \frac{r_2^2 - r_3^2}{a}, y := \frac{r_3^2 - r_1^2}{b}, z := \frac{r_1^2 - r_2^2}{c}$. Note that $ax + by + cz = 0$. Thus

$$4N = \begin{pmatrix} -2a & a + b - c & c + a - b \\ a + b - c & -2b & b + c - a \\ c + a - b & b + c - a & -2c \end{pmatrix} \begin{pmatrix} 0 & 1 + x & 1 - x \\ 1 - y & 0 & 1 + y \\ 1 + z & 1 - z & 0 \end{pmatrix} \\ = \begin{pmatrix} 2a - (a + b - c)y + (c + a - b)z & c - a - b - 2ax - (c + a - b)z & b - a - c + 2ax + (a + b - c)y \\ c - a - b + 2by + (b + c - a)z & 2b - (b + c - a)z + (a + b - c)x & a - b - c - 2by - (a + b - c)x \\ b - c - a - 2cz - (b + c - a)y & a - b - c + 2cz + (c + a - b)x & 2c - (c + a - b)x + (b + c - a)y \end{pmatrix}.$$

This matrix is symmetric due to the equality $ax + by + cz = 0$. □

Glickenstein [11] gave a new proof of the above lemma. The new proof puts the result in a general context and reveals its geometric meaning.

Lemma 6. *The Jacobian matrix of functions $\alpha_1, \alpha_2, \alpha_3$ in terms of u_1, u_2, u_3 has one zero eigenvalue with associated eigenvector $(1, 1, 1)$ and two negative eigenvalues.*

Proof. Using the notation in Lemma 5, we need to show that the matrix $4N$ has one zero eigenvalue and two positive eigenvalues. Since $4N$ is symmetric and the sum of entries in each row is zero, we can write $4N$ as

$$\begin{pmatrix} -B_2 - B_3 & B_3 & B_2 \\ B_3 & -B_3 - B_1 & B_1 \\ B_2 & B_1 & -B_1 - B_2 \end{pmatrix},$$

where B_1, B_2, B_3 are functions of a, b, c, x, y, z .

The characteristic equation of $4N$ is

$$\lambda(\lambda^2 + 2(B_1 + B_2 + B_3)\lambda + 3(B_1B_2 + B_2B_3 + B_3B_1)) = 0.$$

There is a zero eigenvalue, and the associated eigenvector is $(1, 1, 1)$. We claim that $B_1 + B_2 + B_3 < 0$ and $B_1B_2 + B_2B_3 + B_3B_1 > 0$. These two inequalities imply that the matrix $4N$ has two positive eigenvalues. In the following we verify the two inequalities.

First,

$$B_1 + B_2 + B_3 < 0 \\ \iff \text{trace of } 4N = -2(B_1 + B_2 + B_3) > 0 \\ \iff a + b + c + (b - c)x + (c - a)y + (a - b)z > 0 \\ \iff l_1^2 + l_2^2 + l_3^2 + (l_2^2 - l_3^2)\frac{r_2^2 - r_3^2}{l_1^2} + (l_3^2 - l_1^2)\frac{r_3^2 - r_1^2}{l_2^2} + (l_1^2 - l_2^2)\frac{r_1^2 - r_2^2}{l_3^2} > 0 \\ \iff l_1^2 + l_2^2 + l_3^2 \\ + \frac{l_1^2l_2^2 + l_1^2l_3^2 - l_2^4 - l_3^4}{l_2^2l_3^2}r_1^2 + \frac{l_2^2l_1^2 + l_2^2l_3^2 - l_1^4 - l_3^4}{l_1^2l_3^2}r_2^2 + \frac{l_3^2l_1^2 + l_3^2l_2^2 - l_1^4 - l_2^4}{l_1^2l_2^2}r_3^2 > 0 \\ \iff: l_1^2 + l_2^2 + l_3^2 + \Xi_1r_1^2 + \Xi_2r_2^2 + \Xi_3r_3^2 > 0.$$

For $\{i, j, k\} = \{1, 2, 3\}$, since $l_i^2 = r_j^2 + r_k^2 + 2I_{jk}r_jr_k$ and $I_{ij} \in [0, \infty)$, we have $l_i > r_j$.

Without loss of generality, we may assume $l_1 \geq l_2 \geq l_3$. Therefore $\Xi_1 \geq 0$ and $\Xi_3 \leq 0$.

Case 1. If $\Xi_2 \geq 0$, then

$$\begin{aligned} & l_1^2 + l_2^2 + l_3^2 + \Xi_1 r_1^2 + \Xi_2 r_2^2 + \Xi_3 r_3^2 \\ & \geq l_1^2 + l_2^2 + l_3^2 + \Xi_3 r_3^2 \\ & > l_1^2 + l_2^2 + l_3^2 + \Xi_3 l_2^2 \\ & = l_1^2 + l_2^2 + l_3^2 + \frac{l_3^2 l_1^2 + l_3^2 l_2^2 - l_1^4 - l_2^4}{l_1^2 l_2^2} l_2^2 \\ & = \frac{1}{l_1^2} ((l_1^2 - l_2^2) l_2^2 + 2l_1^2 l_3^2 + l_2^2 l_3^2) > 0 \end{aligned}$$

due to the assumption $l_1 \geq l_2$.

Case 2. If $\Xi_2 < 0$, then

$$\begin{aligned} & l_1^2 + l_2^2 + l_3^2 + \Xi_1 r_1^2 + \Xi_2 r_2^2 + \Xi_3 r_3^2 \\ & > l_1^2 + l_2^2 + l_3^2 + \Xi_2 r_2^2 + \Xi_3 r_3^2 \\ & > l_1^2 + l_2^2 + l_3^2 + \Xi_2 l_3^2 + \Xi_3 l_2^2 \\ & = l_1^2 + l_2^2 + l_3^2 + \frac{l_2^2 l_1^2 + l_2^2 l_3^2 - l_1^4 - l_3^4}{l_1^2 l_3^2} l_3^2 + \frac{l_3^2 l_1^2 + l_3^2 l_2^2 - l_1^4 - l_2^4}{l_1^2 l_2^2} l_2^2 \\ & = \frac{1}{l_1^2} (2l_1^2 l_2^2 + 2l_2^2 l_3^2 + 2l_3^2 l_1^2 - l_1^4 - l_2^4 - l_3^4) \\ & = \frac{1}{l_1^2} (l_1 + l_2 + l_3)(l_1 + l_2 - l_3)(l_3 + l_1 - l_2)(l_2 + l_3 - l_1) > 0 \end{aligned}$$

due to the triangle inequality.

Thus $B_1 + B_2 + B_3 < 0$ holds.

To prove $B_1 B_2 + B_2 B_3 + B_3 B_1 > 0$, we substitute

$$\begin{aligned} B_1 &= 4N_{23} = a - b - c - 2by - (a + b - c)x, \\ B_2 &= 4N_{31} = b - c - a - 2cz - (b + c - a)y, \\ B_3 &= 4N_{12} = c - a - b - 2ax - (c + a - b)z \end{aligned}$$

to obtain

$$\begin{aligned} & B_1 B_2 + B_2 B_3 + B_3 B_1 \\ & = 2ab + 2bc + 2ac - a^2 - b^2 - c^2 \\ & \quad + (b^2 - c^2 - 3a^2 + 6ab + 4ac)xy \\ & \quad + (c^2 - a^2 - 3b^2 + 6bc + 4ab)yz \\ & \quad + (a^2 - b^2 - 3c^2 + 6ac + 4bc)zx \\ & \quad + 2a(a + b - c)x^2 + 2b(b + c - a)y^2 + 2c(c + a - b)z^2 \\ & \quad + 2(a + b + c)(ax + by + cz) \\ & = (2ab + 2bc + 2ac - a^2 - b^2 - c^2)(xy + yz + zx + 1) \\ & \quad + 2(ax + by + cz)((a + b - c)x + (b + c - a)y + (c + a - b)z) \\ & \quad + 2(a + b + c)(ax + by + cz) \\ & = (2ab + 2bc + 2ac - a^2 - b^2 - c^2)(xy + yz + zx + 1) \end{aligned}$$

due to the fact that $ax + by + cz = 0$.

Since

$$\begin{aligned} & 2ab + 2bc + 2ac - a^2 - b^2 - c^2 \\ &= 2l_1^2l_2^2 + 2l_2^2l_3^2 + 2l_3^2l_1^2 - l_1^4 - l_2^4 - l_3^4 \\ &= (l_1 + l_2 + l_3)(l_1 + l_2 - l_3)(l_3 + l_1 - l_2)(l_2 + l_3 - l_1) > 0, \end{aligned}$$

we have

$$\begin{aligned} & B_1B_2 + B_2B_3 + B_3B_1 > 0 \\ \iff & xy + yz + zx + 1 > 0 \\ (3) \iff & (r_1^2 - r_2^2)(r_3^2 - r_1^2)l_1^2 + (r_2^2 - r_3^2)(r_1^2 - r_2^2)l_2^2 + (r_3^2 - r_1^2)(r_2^2 - r_3^2)l_3^2 + l_1^2l_2^2l_3^2 > 0. \end{aligned}$$

To verify inequality (3), first note that, since $l_i^2 = r_j^2 + r_k^2 + 2I_{jk}r_jr_k$ and $I_{jk} \in [0, \infty)$, we have $l_i^2 \geq r_j^2 + r_k^2$ for $\{i, j, k\} = \{1, 2, 3\}$.

Since the coefficient of l_1^2 on the left hand side of (3) satisfies

$$\begin{aligned} (r_1^2 - r_2^2)(r_3^2 - r_1^2) + l_2^2l_3^2 &\geq (r_1^2 - r_2^2)(r_3^2 - r_1^2) + (r_1^2 + r_3^2)(r_1^2 + r_2^2) \\ &= 2r_1^2r_2^2 + 2r_1^2r_3^2 > 0, \end{aligned}$$

by replacing l_1^2 by $r_2^2 + r_3^2$, we see that the left hand side of (3) is not less than

$$(4) \quad (r_1^2 - r_2^2)(r_3^2 - r_1^2)(r_2^2 + r_3^2) + (r_2^2 - r_3^2)(r_1^2 - r_2^2)l_2^2 + (r_3^2 - r_1^2)(r_2^2 - r_3^2)l_3^2 + (r_2^2 + r_3^2)l_2^2l_3^2.$$

Since the coefficient of l_2^2 in (4) satisfies

$$\begin{aligned} (r_2^2 - r_3^2)(r_1^2 - r_2^2) + (r_2^2 + r_3^2)l_3^2 &\geq (r_2^2 - r_3^2)(r_1^2 - r_2^2) + (r_2^2 + r_3^2)(r_1^2 + r_2^2) \\ &= 2r_1^2r_2^2 + 2r_2^2r_3^2 > 0, \end{aligned}$$

by replacing l_2^2 by $r_1^2 + r_3^2$, we see that (4) is not less than

$$(5) \quad (r_1^2 - r_2^2)(r_3^2 - r_1^2)(r_2^2 + r_3^2) + (r_2^2 - r_3^2)(r_1^2 - r_2^2)(r_1^2 + r_3^2) + (r_3^2 - r_1^2)(r_2^2 - r_3^2)l_3^2 + (r_2^2 + r_3^2)(r_1^2 + r_3^2)l_3^2.$$

Since the coefficient of l_3^2 in (8) is positive, by replacing l_3^2 by $r_1^2 + r_2^2$, we see that (5) is not less than

$$(6) \quad (r_1^2 - r_2^2)(r_3^2 - r_1^2)(r_2^2 + r_3^2) + (r_2^2 - r_3^2)(r_1^2 - r_2^2)(r_1^2 + r_3^2) + (r_3^2 - r_1^2)(r_2^2 - r_3^2)(r_1^2 + r_2^2) + (r_2^2 + r_3^2)(r_1^2 + r_3^2)(r_1^2 + r_2^2).$$

Since (6) is equal to $8r_1^2r_2^2r_3^2$, we see that $B_1B_2 + B_2B_3 + B_3B_1 > 0$. □

For the special case of intersection angle circle packing, Lemma 6 was proved in Chow-Luo [9] (Lemma 3.1). In that case, by a geometric reasoning due to Thurston [23], the angle α_i is a decreasing function of r_i and an increasing function of r_j for $j \neq i$. That means $\frac{\partial \alpha_i}{\partial r_i} < 0$ and $\frac{\partial \alpha_i}{\partial r_j} > 0$ for $j \neq i$. Therefore the diagonal entries of the matrix N in Lemma 5 are positive and the off-diagonal entries of N are negative. By a theorem in linear algebra, this property implies Lemma 6. But Chow-Luo's proof does not work in the general case of inversive distance circle packing, because it is not always true that the angle α_i is a decreasing function of r_i and an increasing

function of r_j for $j \neq i$. For example, when $l_1 = 2, l_2 = 2, l_3 = 3, r_1 = r_2 = r_3 = 1$, the matrix N is

$$\begin{pmatrix} 2 & \frac{1}{4} & -\frac{9}{4} \\ \frac{1}{4} & 2 & -\frac{9}{4} \\ -\frac{9}{4} & -\frac{9}{4} & \frac{9}{2} \end{pmatrix}$$

with eigenvalue $0, \frac{27}{4}, \frac{7}{4}$. This shows that $\frac{\partial \alpha_1}{\partial r_2} < 0$, while $\frac{\partial \alpha_1}{\partial r_3} > 0$ for some value of l_i 's and r_i 's.

Since, by Corollary 4, the space \mathcal{U}_E^{123} of vectors (u_1, u_2, u_3) is connected and simply connected, Lemma 5 and Lemma 6 imply

Corollary 7. *The differential 1-form $\sum_{i=1}^3 \alpha_i du_i$ is closed. For any $c \in \mathcal{U}_E^{123}$, the integration $w(u_1, u_2, u_3) = \int_c^{(u_1, u_2, u_3)} \sum_{i=1}^3 \alpha_i du_i$ is a concave function on \mathcal{U}_E^{123} and satisfies, for $i = 1, 2, 3$,*

$$(7) \quad \frac{\partial w}{\partial u_i} = \alpha_i.$$

Proof of Theorem 2 in Euclidean geometry. Let's prove the local rigidity of inverse distance circle packing in Euclidean geometry. For a weighted triangulated closed surface (Σ, T, I) with $I_{ij} \in [0, \infty)$ for each $ij \in E$, let \mathcal{U}_E be the open subset of $\mathbb{R}^{|V|}$ formed by the vectors $u = (u_1, u_2, \dots, u_{|V|})$ satisfying $(u_i, u_j, u_k) \in \mathcal{U}_E^{ijk}$ whenever $\Delta ijk \in F$.

For a radius vector $r = (r_1, r_2, \dots, r_{|V|})$, we may rescale it to get cr , while the cone angles remain the same. Under the rescaling, u changes to $u + \ln(c)(1, 1, \dots, 1)$. Therefore we only consider the intersection $\mathcal{U}_E \cap P$ where P is the hyperplane defined by $\sum_{i=1}^{|V|} u_i = 0$.

Let a_i be the cone angle at vertex i under the Euclidean cone metric. Then $\sum_{i=1}^{|V|} a_i =$ the sum of all inner angles $= \pi|F|$. Therefore the space of possible cone angles is contained in the intersection $\mathbb{R}_{>0}^{|V|} \cap Q$, where Q is the hyperplane defined by $\sum_{i=1}^{|V|} a_i = \pi|F|$. We have a map $\xi : \mathcal{U}_E \cap P \rightarrow \mathbb{R}_{>0}^{|V|} \cap Q$.

By Corollary 7, for each triangle $\Delta ijk \in F$ there is a function $w(u_i, u_j, u_k)$. Define a function $W : \mathcal{U}_E \rightarrow \mathbb{R}$ by

$$W(u_1, u_2, \dots, u_{|V|}) = \sum_{\Delta ijk \in F} w(u_i, u_j, u_k),$$

where the sum is over all triangles in F .

By (7), $\frac{\partial W}{\partial u_i}$ equals the sum of inner angles having vertex i , i.e., the cone angle at vertex i . Therefore, when restricted on $\mathcal{U}_E \cap P$, the gradient ∇W is the map $\xi : \mathcal{U}_E \cap P \rightarrow \mathbb{R}_{>0}^{|V|} \cap Q$.

We claim that the restriction of W on $\mathcal{U}_E \cap P$ is strictly concave. By the definition of W , we get the relation between the Hessian matrixes

$$H(W) = \sum_{\Delta ijk \in F} H(w(u_i, u_j, u_k)),$$

where we think of $w(u_i, u_j, u_k)$ as a function on \mathcal{U}_E .

For any $x \in \mathcal{U}_E \cap P$,

$$\begin{aligned} xH(W)x^T &= \sum_{\Delta ijk \in F} xH(w(u_i, u_j, u_k))x^T \\ &= \sum_{\Delta ijk \in F} (x_i, x_j, x_k)H(w(u_i, u_j, u_k))(x_i, x_j, x_k)^T \leq 0, \end{aligned}$$

since $H(w(u_i, u_j, u_k))$ is negative semi-definite for each $\Delta ijk \in F$.

Thus $xH(W)x^T = 0$ implies $(x_i, x_j, x_k)H(w(u_i, u_j, u_k))(x_i, x_j, x_k)^T = 0$ for each Δijk . By Lemma 6, (x_i, x_j, x_k) is an eigenvector of zero eigenvalue. Therefore $(x_i, x_j, x_k) = c_{ijk}(1, 1, 1)$ for some constant c_{ijk} . If two triangles share a vertex, then the corresponding constants c_{ijk} of the two triangles must be the same. In fact, consider the two triangles Δijk and $\Delta ij'k'$ which share the vertex i . For Δijk , we have $(x_i, x_j, x_k) = c_{ijk}(1, 1, 1)$; in particular, $x_i = c_{ijk}$. Also, for $\Delta ij'k'$, we have $(x_i, x_{j'}, x_{k'}) = c_{ij'k'}(1, 1, 1)$; in particular, $x_i = c_{ij'k'}$. Hence $c_{ijk} = x_i = c_{ij'k'}$. Since the surface is connected, the corresponding constants c_{ijk} of any triangles are the same. Thus c_{ijk} must be a constant c independent of the triangle Δijk . Therefore $x = c(1, 1, \dots, 1)$. Since $x \in P$, i.e., $\sum x_i = 0$, we have $c = 0$. Thus $x = (0, 0, \dots, 0)$. This shows that the Hessian $H(W)$ is negative definite when restricted on $\mathcal{U}_E \cap P$.

The local rigidity of inversive distance circle packing in Euclidean geometry follows from the following lemma in analysis

Lemma 8. *If X is an open set in \mathbb{R}^n and the Hessian matrix of f is positive definite for all $x \in X$, then the gradient $\nabla f : X \rightarrow \mathbb{R}^n$ is a local diffeomorphism. \square*

3. CIRCLE PACKING IN HYPERBOLIC GEOMETRY

3.1. Space of radius vectors. As stated in the Introduction, to obtain a hyperbolic cone metric, we need conditions on a radius vector. In this section we investigate the space of radius vectors for one triangle in hyperbolic geometry. Consider a topological triangle $\Delta ijk \in F$. Assign a number l_{ij} to the edge $ij \in E$, where l_{ij} is the distance of the centers of two circles with radii r_i, r_j and inversive distance I_{ij} . Therefore l_{ij} is a function of r_i, r_j . In fact, in hyperbolic geometry, l_{ij} is determined by $\cosh l_{ij} = \cosh r_i \cosh r_j + I_{ij} \sinh r_i \sinh r_j$. Similarly, we obtain l_{jk}, l_{ki} .

Lemma 9. *The set $\mathcal{R}_H^{ijk} := \{(r_i, r_j, r_k) \in \mathbb{R}_{>0}^3 \mid l_{ij}, l_{jk}, l_{ki} \text{ satisfy triangle inequality}\}$ is a connected and simply connected open subset in $\mathbb{R}_{>0}^3$.*

Proof. Let $\rho_H : \mathbb{R}_{>0}^3 \rightarrow \mathbb{R}_{>0}^3$ be the map sending (r_i, r_j, r_k) to (l_{ij}, l_{jk}, l_{ki}) . We claim that ρ_H is a smooth embedding. First the determinant of the Jacobian matrix of ρ_H is nonzero. Next, we will show ρ_H is one-to-one.

That means, for fixed numbers $I_{ij}, I_{jk}, I_{ki} \in [0, \infty)$ and a fixed hyperbolic triangle, there is at most one configuration of three circles centered at the vertices of the triangle and having the prescribed inversive distance I_{ij}, I_{jk}, I_{ki} .

The statement is true in Euclidean geometry as proved in the proof of Lemma 3. We use the result in Euclidean geometry to prove a similar result in hyperbolic geometry. Let's assume there are two configurations of three circles in a hyperbolic plane with the fixed triangle and inversive distance. As in Figure 2, the stereographic projection sends the two configurations to Euclidean plane \mathbb{C} . Since the

stereographic projection preserves the inversive distance, there will be two different configurations of three circles with the same Euclidean triangle and the same inversive distance. This is impossible.

To determine the image $\rho_H(\mathbb{R}_{>0}^3)$, we consider the image of the boundary of $\mathbb{R}_{\geq 0}^3$. If $r_i = 0$, then $r_k = l_{ki}, r_j = l_{ij}$. Thus

$$\cosh l_{jk} = \cosh r_j \cosh r_k + I_{jk} \sinh r_j \sinh r_k = \cosh l_{ij} \cosh l_{ki} + I_{jk} \sinh l_{ij} \sinh l_{ki}.$$

On the other hand, if $r_i > 0$, then $r_k < l_{ki}, r_j < l_{ij}$. We have

$$\cosh l_{jk} < \cosh l_{ij} \cosh l_{ki} + I_{jk} \sinh l_{ij} \sinh l_{ki}.$$

Therefore the image $\rho_H(\mathbb{R}_{>0}^3)$ is the subset of vectors (l_{ij}, l_{jk}, l_{ki}) satisfying

$$\begin{aligned} \cosh l_{jk} &< \cosh l_{ij} \cosh l_{ki} + I_{jk} \sinh l_{ij} \sinh l_{ki}, \\ \cosh l_{ki} &< \cosh l_{jk} \cosh l_{ij} + I_{ki} \sinh l_{jk} \sinh l_{ij}, \\ \cosh l_{ij} &< \cosh l_{ki} \cosh l_{jk} + I_{ij} \sinh l_{ki} \sinh l_{jk}. \end{aligned}$$

In fact, $\rho_H(\mathbb{R}_{>0}^3)$ is a cone bounded by three surfaces. In $\mathbb{R}_{\geq 0}^3$, the intersection of any of the two surfaces is a straight line. For example, if

$$\cosh l_{jk} = \cosh l_{ij} \cosh l_{ki} + I_{jk} \sinh l_{ij} \sinh l_{ki}$$

and

$$\cosh l_{ki} = \cosh l_{jk} \cosh l_{ij} + I_{ki} \sinh l_{jk} \sinh l_{ij},$$

then the sum of the two equation gives

$$\begin{aligned} 0 &= (\cosh l_{ki} + \cosh l_{jk})(\cosh l_{ij} - 1) + (I_{jk} \sinh l_{ki} + I_{ki} \sinh l_{jk}) \sinh l_{ij} \\ &= (\cosh l_{ki} + \cosh l_{jk})2 \sinh^2 \frac{l_{ij}}{2} + (I_{jk} \sinh l_{ki} + I_{ki} \sinh l_{jk})2 \cosh \frac{l_{ij}}{2} \sinh \frac{l_{ij}}{2}. \end{aligned}$$

The only possibility is $\sinh \frac{l_{ij}}{2} = 0$. Therefore $l_{ij} = 0$ and $l_{jk} = l_{ki}$.

Hence the three lines of intersection are

$$\begin{aligned} l_{ij} &= 0 \text{ and } l_{jk} = l_{ki}, \\ l_{jk} &= 0 \text{ and } l_{ki} = l_{ij}, \\ l_{ki} &= 0 \text{ and } l_{ij} = l_{jk}. \end{aligned}$$

Recall that \mathcal{L} is the subset of $\mathbb{R}_{>0}^3$ formed by vectors (l_{ij}, l_{jk}, l_{ki}) satisfying the triangle inequality. Therefore \mathcal{L} is a cone bounded by three planes. The three lines of intersection of the three planes are the same as that of the boundary surfaces of $\rho_H(\mathbb{R}_{>0}^3)$.

We claim that $\rho_H(\mathbb{R}_{>0}^3) \cap \mathcal{L}$ is a cone.

In fact, if $I_{ij} \in [0, 1]$, then in $\mathbb{R}_{>0}^3$ we have

$$\begin{aligned} \{(l_{ij}, l_{jk}, l_{ki}) \mid \cosh l_{ij} < \cosh l_{ki} \cosh l_{jk} + I_{ij} \sinh l_{ki} \sinh l_{jk}\} \\ \subseteq \{(l_{ij}, l_{jk}, l_{ki}) \mid l_{ij} < l_{ki} + l_{jk}\}. \end{aligned}$$

If $I_{ij} \in (1, \infty)$, then in $\mathbb{R}_{>0}^3$ we have

$$\begin{aligned} \{(l_{ij}, l_{jk}, l_{ki}) \mid \cosh l_{ij} < \cosh l_{ki} \cosh l_{jk} + I_{ij} \sinh l_{ki} \sinh l_{jk}\} \\ \supset \{(l_{ij}, l_{jk}, l_{ki}) \mid l_{ij} < l_{ki} + l_{jk}\}. \end{aligned}$$

Applying the same argument in Euclidean geometry, we see $\rho_H(\mathbb{R}_{>0}^3) \cap \mathcal{L}$ is a cone. Therefore $\mathcal{R}_H^{ijk} = \rho_H^{-1}(\rho_H(\mathbb{R}_{>0}^3) \cap \mathcal{L})$ is connected and simply connected. \square

In the construction of the energy function, we will use a natural variable $u = (u_1, u_2, \dots, u_{|V|})$ where $u_i = \ln \tanh \frac{r_i}{2}$. Let $\tau_H : \mathbb{R}^3 \rightarrow \mathbb{R}_{>0}^3$ be the map sending (u_i, u_j, u_k) to (r_i, r_j, r_k) . Since τ_H is a diffeomorphism, we have

Corollary 10. *The set $\mathcal{U}_H^{ijk} := \tau_H^{-1}(\mathcal{R}_H^{ijk})$ is a connected and simply connected open subset in $\mathbb{R}_{>0}^3$.*

To prove Theorem 2, we construct a strictly concave energy function on \mathcal{U}_H^{ijk} for each triangle $\triangle ijk \in F$. As in the case of Euclidean geometry, the space \mathcal{U}_H^{ijk} is not always convex (for example, when $I_{ij} = I_{jk} = I_{ki} = 2$); we cannot obtain global rigidity. Therefore we are not able to determine the space of cone angles in the case of inversive distance circle packing.

3.2. Energy function. To prove Theorem 2, we will construct an energy function on the space of radius vectors. First, we construct an energy function on the space for one triangle. To simplify notation, consider the triangle $\triangle 123 \in F$ and the map

$$\begin{aligned} \mathcal{U}_H^{123} &\rightarrow \mathcal{R}_H^{123} &\rightarrow \mathcal{L} &\rightarrow \mathbb{R}_{>0}^3, \\ (u_1, u_2, u_3) &\mapsto (r_1, r_2, r_3) &\mapsto (l_1, l_2, l_3) &\mapsto (\alpha_1, \alpha_2, \alpha_3), \end{aligned}$$

where $r_i = \ln \frac{1+e^{u_i}}{1-e^{u_i}}$ for $i = 1, 2, 3$. For three fixed positive numbers I_{12}, I_{23}, I_{31} , we have $\cosh l_i = \cosh r_j \cosh r_k + I_{ij} \sinh r_j \sinh r_k$ for $\{i, j, k\} = \{1, 2, 3\}$. For simplicity of notation, we set $(l_1, l_2, l_3) = (l_{23}, l_{31}, l_{12})$. Also, $\alpha_1, \alpha_2, \alpha_3$ are inner angles of a hyperbolic triangle with edge lengths l_1, l_2, l_3 . By the cosine law,

$$\cos \alpha_i = \frac{-\cosh l_i + \cosh l_j \cosh l_k}{\sinh l_j \sinh l_k}$$

for $\{i, j, k\} = \{1, 2, 3\}$. Therefore $\alpha_1, \alpha_2, \alpha_3$ are functions in terms of u_1, u_2, u_3 .

Lemma 11. *The Jacobian matrix of functions $\alpha_1, \alpha_2, \alpha_3$ in terms of u_1, u_2, u_3 is symmetric.*

Proof. Due to the cosine law of a hyperbolic triangles, $\alpha_1, \alpha_2, \alpha_3$ are functions of l_1, l_2, l_3 . The following formula can be proved by direct calculation. For $\{i, j, k\} = \{1, 2, 3\}$,

$$\begin{aligned} \begin{pmatrix} d\alpha_1 \\ d\alpha_2 \\ d\alpha_3 \end{pmatrix} &= \frac{-1}{\sin \alpha_i \sinh l_j \sinh l_k} \begin{pmatrix} \sinh l_1 & 0 & 0 \\ 0 & \sinh l_2 & 0 \\ 0 & 0 & \sinh l_3 \end{pmatrix} \\ &\quad \times \begin{pmatrix} -1 & \cos \alpha_3 & \cos \alpha_2 \\ \cos \alpha_3 & -1 & \cos \alpha_1 \\ \cos \alpha_2 & \cos \alpha_1 & -1 \end{pmatrix} \begin{pmatrix} dl_1 \\ dl_2 \\ dl_3 \end{pmatrix}. \end{aligned}$$

Next, by the construction $\cosh l_i = \cosh r_j \cosh r_k + I_{jk} \sinh r_j \sinh r_k$, for $\{i, j, k\} = \{1, 2, 3\}$, where I_{ij} is a constant. Differentiating the two sides of the equality, we obtain

$$dl_i = \frac{-\cosh r_k + \cosh l_i \cosh r_j}{\sinh l_i \sinh r_j} dr_j + \frac{-\cosh r_j + \cosh l_i \cosh r_k}{\sinh l_i \sinh r_k} dr_k.$$

Finally, $r_i = \ln \frac{1+e^{u_i}}{1-e^{u_i}}$ implies $\sinh r_i du_i = dr_i, i = 1, 2, 3$.

Combining the three relations, we have, for $\{i, j, k\} = \{1, 2, 3\}$,

$$\begin{aligned} & \begin{pmatrix} d\alpha_1 \\ d\alpha_2 \\ d\alpha_3 \end{pmatrix} = \frac{-1}{\sin \alpha_i \sinh l_j \sinh l_k} \\ & \times \begin{pmatrix} \sinh l_1 & 0 & 0 \\ 0 & \sinh l_2 & 0 \\ 0 & 0 & \sinh l_3 \end{pmatrix} \begin{pmatrix} -1 & \cos \alpha_3 & \cos \alpha_2 \\ \cos \alpha_3 & -1 & \cos \alpha_1 \\ \cos \alpha_2 & \cos \alpha_1 & -1 \end{pmatrix} \\ & \times \begin{pmatrix} 0 & \frac{-\cosh r_3 + \cosh l_1 \cosh r_2}{\sinh l_1 \sinh r_2} & \frac{-\cosh r_2 + \cosh l_1 \cosh r_3}{\sinh l_1 \sinh r_3} \\ \frac{-\cosh r_3 + \cosh l_2 \cosh r_1}{\sinh l_2 \sinh r_1} & 0 & \frac{-\cosh r_1 + \cosh l_2 \cosh r_3}{\sinh l_2 \sinh r_3} \\ \frac{-\cosh r_2 + \cosh l_3 \cosh r_1}{\sinh l_3 \sinh r_1} & \frac{-\cosh r_1 + \cosh l_3 \cosh r_2}{\sinh l_3 \sinh r_2} & 0 \end{pmatrix} \\ & \times \begin{pmatrix} \sinh r_1 & 0 & 0 \\ 0 & \sinh r_2 & 0 \\ 0 & 0 & \sinh r_3 \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}. \end{aligned}$$

We write the above formula as

$$\begin{pmatrix} d\alpha_1 \\ d\alpha_2 \\ d\alpha_3 \end{pmatrix} = \frac{-1}{\sin \alpha_i \sinh l_j \sinh l_k} M \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix},$$

where M is a product of four matrixes.

To show that the Jacobian matrix of functions $\alpha_1, \alpha_2, \alpha_3$ in terms of u_1, u_2, u_3 is symmetric is equivalent to showing that the matrix M is symmetric.

To simplify notation, let $a := \cosh l_1, b := \cosh l_2, c := \cosh l_3, x := \cosh r_1, y := \cosh r_2, z := \cosh r_3$. By the cosine law, we have

$$M = \begin{pmatrix} 1-a^2 & ab-c & ca-b \\ ab-c & 1-b^2 & bc-a \\ ca-b & bc-a & 1-c^2 \end{pmatrix} \begin{pmatrix} \frac{1}{a^2-1} & 0 & 0 \\ 0 & \frac{1}{b^2-1} & 0 \\ 0 & 0 & \frac{1}{c^2-1} \end{pmatrix} \begin{pmatrix} 0 & ay-z & az-y \\ bx-z & 0 & bz-x \\ cx-y & cy-x & 0 \end{pmatrix}.$$

We check that M is symmetric, for example, by showing

$$M_{12} = M_{21} = z - \frac{ac-b}{c^2-1}x - \frac{bc-a}{c^2-1}y. \quad \square$$

Lemma 12. *The Jacobian matrix of functions $\alpha_1, \alpha_2, \alpha_3$ in terms of u_1, u_2, u_3 is negative definite.*

Proof. Using the notation of Lemma 11, we need to verify that the matrix M is positive definite. The determinant of M is positive since the determinant of each factor is positive. Note that the second matrix is a Gram matrix up to the negative sign. Since, by Corollary 10, the set \mathcal{U}_H^{123} is connected, to show that M is positive definite, it is enough to check that M is positive definite at one vector in \mathcal{U}_H^{123} .

It is equivalent to picking up a vector (r_1, r_2, r_3) such that the resulting l_1, l_2, l_3 satisfy the triangle inequality. We pick up $(r_1, r_2, r_3) = (s, s, s)$. In the following, we show that such kind of number s exists.

For fixed $I_{12}, I_{23}, I_{31} \in [0, \infty)$, we have $\cosh l_i = \cosh r_j \cosh r_k + I_{jk} \sinh r_j \sinh r_k = \cosh^2 s + I_{jk} \sinh^2 s \geq \cosh^2 s$, for $\{i, j, k\} = \{1, 2, 3\}$. Therefore $\sinh l_i \geq \sqrt{\cosh^4 s - 1}$ for $i = 1, 2, 3$.

The fact that l_1, l_2, l_3 satisfy the triangle inequality is equivalent to

$$\begin{aligned} & \cosh(l_i + l_j) > \cosh l_k \\ \iff & \cosh l_i \cosh l_j + \sinh l_i \sinh l_j > \cosh l_k = \cosh^2 s + I_{ij} \sinh^2 s \\ \iff & \cosh^4 s + \cosh^4 s - 1 > \cosh^2 s + I_{ij} \sinh^2 s. \end{aligned}$$

Let $x := \cosh^2 s$. The above inequality is equivalent to $(2x - I_{ij} + 1)(x - 1) > 0$. Since $x > 1$, we need $x > \frac{I_{ij}-1}{2}$.

Therefore l_1, l_2, l_3 as functions of (s, s, s) satisfy the triangle inequality if $\cosh^2 s > \max\{\frac{I_{12}-1}{2}, \frac{I_{23}-1}{2}, \frac{I_{31}-1}{2}\}$.

Now it is enough to check that M is positive definite at such a vector (s, s, s) when s is sufficiently large. In this case,

$$\begin{aligned} M &= \begin{pmatrix} \sinh l_1 & 0 & 0 \\ 0 & \sinh l_2 & 0 \\ 0 & 0 & \sinh l_3 \end{pmatrix} \begin{pmatrix} -1 & \cos \alpha_3 & \cos \alpha_2 \\ \cos \alpha_3 & -1 & \cos \alpha_1 \\ \cos \alpha_2 & \cos \alpha_1 & -1 \end{pmatrix} \\ &\times \begin{pmatrix} 0 & \frac{-1 + \cosh l_1}{\sinh l_1} & \frac{-1 + \cosh l_1}{\sinh l_1} \\ \frac{-1 + \cosh l_2}{\sinh l_2} & 0 & \frac{-1 + \cosh l_2}{\sinh l_2} \\ \frac{-1 + \cosh l_3}{\sinh l_3} & \frac{-1 + \cosh l_3}{\sinh l_3} & 0 \end{pmatrix} \cosh s := M_1 \cosh s. \end{aligned}$$

The fact that M is positive definite is equivalent to M_1 being positive definite. Since M_1 only involves l_1, l_2, l_3 , the fact that M_1 is positive definite is a property of a hyperbolic triangle. In fact, the fact that M_1 is positive definite was proved in the paper of Guo-Luo [13] (Lemma 4.4) by showing that its leading principal 1×1 and 2×2 minor is positive. For completeness, we include a proof here.

That the leading principal 1×1 minor is positive is equivalent to

$$\cos \alpha_3 \frac{-1 + \cosh l_2}{\sinh l_2} + \cos \alpha_2 \frac{-1 + \cosh l_3}{\sinh l_3} > 0.$$

Replacing α 's by l 's by the cosine law, we obtain

$$(8) \quad \cosh l_1 > \frac{\cosh^2 l_2 + \cosh^2 l_3 + \cosh l_2 + \cosh l_3}{2 \cosh l_2 \cosh l_3 + \cosh l_2 + \cosh l_3}.$$

Since $l_1 > |l_2 - l_3|$, therefore $\cosh l_1 > \cosh(l_2 - l_3)$. To show that (8) holds, it is enough to check that

$$(9) \quad \cosh l_2 \cosh l_3 - \sinh l_2 \sinh l_3 \geq \frac{\cosh^2 l_2 + \cosh^2 l_3 + \cosh l_2 + \cosh l_3}{2 \cosh l_2 \cosh l_3 + \cosh l_2 + \cosh l_3}.$$

We simplify the notation by introducing $a := \cosh l_2 > 1, b := \cosh l_3 > 1$. Then (9) is rewritten as

$$(10) \quad ab - \sqrt{(a^2 - 1)(b^2 - 1)} \geq \frac{a^2 + b^2 + a + b}{2ab + a + b}.$$

(10) is equivalent to

$$(11) \quad ab - \frac{a^2 + b^2 + a + b}{2ab + a + b} \geq \sqrt{(a^2 - 1)(b^2 - 1)}.$$

Since $a > 1, b > 1$, the left hand side of (11) is positive. To show that (11) holds, we square the two sides and simplify. We have

$$\begin{aligned} & (ab^4 + a^4b - a^3b^2 - a^2b^3) + (a^4 + b^4 - 2a^2b^2) + (a^3 + b^3 - ab^2 - a^2b) \geq 0 \\ \iff & (ab + 1)(a^3 + b^3 - a^2b - ab^2) + (a^2 - b^2)^2 \geq 0 \\ \iff & (ab + 1)(a + b)(a - b)^2 + (a^2 - b^2)^2 \geq 0. \end{aligned}$$

This shows that the leading principal 1×1 minor is positive.

To show that the leading principal 2×2 minor of M_1 is positive, for simplifying notation, let $t_i := \frac{-1 + \cosh l_i}{\sinh l_i}$. Up to a diagonal matrix, M_1 is

$$\begin{pmatrix} -1 & \cos \alpha_3 & \cos \alpha_2 \\ \cos \alpha_3 & -1 & \cos \alpha_1 \\ \cos \alpha_2 & \cos \alpha_1 & -1 \end{pmatrix} \begin{pmatrix} 0 & t_1 & t_1 \\ t_2 & 0 & t_2 \\ t_3 & t_3 & 0 \end{pmatrix}.$$

That the leading principal 2×2 minor of M_1 is positive is equivalent to

$$\begin{aligned} & (t_2 \cos \alpha_3 + t_3 \cos \alpha_2)(t_1 \cos \alpha_3 + t_3 \cos \alpha_1) - (-t_2 + t_3 \cos \alpha_1)(-t_1 + t_3 \cos \alpha_2) > 0 \\ \iff & (\cos^2 \alpha_3 - 1)t_1 t_2 + (\cos \alpha_1 \cos \alpha_3 + \cos \alpha_2)t_2 t_3 + (\cos \alpha_2 \cos \alpha_3 + \cos \alpha_1)t_1 t_3 > 0 \\ \iff & -\sin^2 \alpha_3 t_1 t_2 + \cosh l_2 \sin \alpha_1 \sin \alpha_3 t_2 t_3 + \cosh l_1 \sin \alpha_2 \sin \alpha_3 t_1 t_3 > 0 \\ \iff & \cosh l_2 \sinh l_1 t_2 t_3 + \cosh l_1 \sinh l_2 t_1 t_3 > \sinh l_3 t_1 t_2 \\ \iff & \frac{\cosh l_2 \sinh l_1}{t_1} + \frac{\cosh l_1 \sinh l_2}{t_2} > \frac{\sinh l_3}{t_3} \\ \iff & \cosh l_2(1 + \cosh l_1) + \cosh l_1(1 + \cosh l_2) > 1 + \cosh l_3 \\ \iff & (\cosh l_1 + \cosh l_2 - 1) + (2 \cosh l_1 \cosh l_2 - \cosh l_3) > 0. \end{aligned}$$

The cosine law and the sine law of a hyperbolic triangle are used in the above calculation.

The last inequality is true since $\cosh l_1 + \cosh l_2 > 1$ and $2 \cosh l_1 \cosh l_2 > \cosh l_1 \cosh l_2 + \sinh l_1 \sinh l_2 = \cosh(l_1 + l_2) > \cosh l_3$. □

For the special case of intersection angle circle packing, Lemma 12 was proved in Chow-Luo [9] (Lemma 3.1) based on Thurston’s observation [23] that the angle α_i is a decreasing function of r_i and an increasing function of r_j for $j \neq i$. But this observation does not hold in the general case of inversive distance circle packing. For example, when $l_1 = 2, l_2 = 2, l_3 = 3, r_1 = r_2 = r_3 = 1$, the matrix M is

$$\begin{pmatrix} 6.08 & 0.49 & -2.94 \\ 0.49 & 6.08 & -2.94 \\ -2.94 & -2.94 & 22.11 \end{pmatrix}$$

with eigenvalue 23.15, 5.59, 5.53. This shows that $\frac{\partial \alpha_1}{\partial r_2} < 0$, while $\frac{\partial \alpha_1}{\partial r_3} > 0$ for some value of l_i and r_i .

Since, by Corollary 10, the space U_H^{123} of vectors (u_1, u_2, u_3) is connected and simply connected, Lemma 11 and Lemma 12 imply

Corollary 13. *The differential 1-form $\sum_{i=1}^3 \alpha_i du_i$ is closed. For any $c \in U_H^{123}$, the integration $w(u_1, u_2, u_3) = \int_c^{(u_1, u_2, u_3)} \sum_{i=1}^3 \alpha_i du_i$ is a strictly concave function on U_H^{123} and satisfying, for $i = 1, 2, 3$,*

$$(12) \quad \frac{\partial w}{\partial u_i} = \alpha_i.$$

Proof of Theorem 2 in hyperbolic geometry. Let's prove the local rigidity of inversive distance circle packing in hyperbolic geometry. For a weighted triangulated closed surface (Σ, T, I) with $I_{ij} \in [0, \infty)$ for each $ij \in E$, let \mathcal{U}_H be the open subset of $\mathbb{R}^{|V|}$ formed by the vectors $u = (u_1, u_2, \dots, u_{|V|})$ satisfying $(u_i, u_j, u_k) \in \mathcal{U}_H^{ijk}$ whenever $\Delta ijk \in F$.

By Corollary 13, for each triangle $\Delta ijk \in F$, there is a function $w(u_i, u_j, u_k)$. Define a function $W : \mathcal{U}_H \rightarrow \mathbb{R}$ by

$$W(u_1, u_2, \dots, u_{|V|}) = \sum_{\Delta ijk \in F} w(u_i, u_j, u_k),$$

where the sum is over all triangles in F . By Corollary 13, W is strictly concave on \mathcal{U}_H so that $\frac{\partial W}{\partial u_i}$ equals the sum of inner angles having vertex i , i.e., the cone angle at vertex i . That means the gradient of W is exactly the map sending a vector u to its cone angles.

The local rigidity of inversive distance circle packing in hyperbolic geometry holds due to Lemma 8. \square

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