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RECONSTRUCTING POTENTIALS FROM ZEROS OF ONE EIGENFUNCTION

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ABSTRACT. We study an inverse nodal problem, concerning the reconstruction of a potential of a Sturm-Liouville operator, by using zeros of one eigenfunction as input. We propose three methods for the reconstruction, one of which is the Tikhonov regularization method. The explicit error bounds are calculated for all three methods. In case there is measurement error, the Tikhonov regularization method is still convergent. The study is motivated by physical considerations.

1. Introduction

In this paper, we are concerned about the reconstruction of a potential function

$$q \in \mathbf{Q} := \left\{ p \in H^1([0,1]) \;\Big|\; \int_0^1 p(x)dx = 0 \right\}$$

from the set of zeros of an eigenfunction to the one-dimensional Schrödinger equation

$$\begin{cases} \ddot{u} = (q-\lambda)u \quad \text{in} \quad (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where the dots denote differentiation. Denote by

$$\{\lambda_n, u_n\}_{n=1}^{\infty} = \{(\lambda(n,q), u(n,q,\cdot)\}_{n=1}^{\infty}$$

all the eigenpairs of (1.1), where $\lambda_n < \lambda_{n+1}$ for all $n \geq 1$. Then the set

$$\left\{ x \in [0,1] \mid u_n(x) = 0 \right\} = \left\{ z_0, z_1, \cdots, z_n \right\}$$

has exactly $n+1$ elements and depends only on q. We shall also normalize the eigenfunctions u_n by $\dot{u}_n(0) = 1$. We use the notation

$$\begin{aligned} \mathbf{z}(n,q) &:= (z_0, \cdots, z_n) \in \mathbf{X}(n), \\ \mathbf{X}(n) &:= \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \mid 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1\}. \end{aligned}$$

We call each $\mathbf{z}(n,q)$ $(n = 1, 2, \cdots)$ a **nodal set of** q, whereas the coordinates of $\mathbf{z}$ will be regarded as points in $[0,1]$.

In many applications certain nodal set(s) of a potential can be measured. It would be very helpful if one can calculate the potential (inhomogeneity) from the

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nodal set(s). In general, one may measure a set $\{x_0, \cdots, x_n\}$ of zeros of an oscillatory solution u to $\ddot{u} = Vu$ in an interval $[x_0, x_n] \subset \mathbb{R}$ and want to calculate the potential V restricted on $[x_0, x_n]$. After a linear scaling that maps x_0 to 0 and x_n to 1, u will be a solution to (1.1) with $q = V - \bar{V}$ and $\lambda = -\bar{V}$, where the bar stands for average. It is then natural to solve the following.

Problem 1. Given positive integers $\{n_i\}_{i=1}^m$ where $1 \leqslant n_1 < \cdots < n_m$ and sets $\mathbf{x}(n_i) \in \mathbf{X}(n_i)$, $i = 1, \cdots, m$, find $q \in \mathbf{Q}$ such that its nodal set $\mathbf{z}(n_i, q) = \mathbf{x}(n_i)$ for every $i = 1, \cdots, m$.

Such a problem is nonlinear and is often called the **inverse nodal problem**. A typical one involves the reconstruction of coefficients p, ρ or q of the Sturm-Liouville type operators

$$\mathcal{L}_1[\lambda, u] = (pu')' + \lambda\, u, \qquad \mathcal{L}_2[\lambda, u] = u'' + \lambda\, \rho\, u,$$
$$\mathcal{L}_3[\lambda, u] = u'' + (\lambda - q)u, \qquad \mathcal{L}_4[\lambda, u] = (pu')' + \lambda\, \rho\, u$$

from the zeros of its eigenfunction(s), referred to as the **nodal data**. Inverse nodal problems were first studied by McLaughlin [11] and Shen [12]. Many reconstruction formulas have since been derived and analyzed; see, for example, Hald-McLaughlin [5] and Law-Shen-Yang [8]. In particular, in [5], Hald-McLaughlin provided, among others, two numerical algorithms for the reconstruction of q. One of the algorithms can be induced from (1.4) below, while the other needs the information about the eigenvalues as well and so is not of our interest for this paper.

For a physicist, solving Problem 1 has many potential applications and therefore is quite important; for a pure mathematician, the problem can be considered as solved since many uniqueness and reconstruction results have been proved [3, 4, 5, 6, 10, 8, 9, 11, 12, 13, 14]. However, from an applied point of view, there are still a number of issues to be resolved.

First of all, the problem could be over-determined; namely, given a sequence $\{\mathbf{x}(n_i)\}_{i=1}^\infty$, there may not exist a potential q such that $\mathbf{x}(n_i) = \mathbf{z}(q, n_i)$ for all i, although it is well known that such a potential, if it exists, is unique; see [9] and the references therein. Indeed, a very surprising result of Yang [14] (see also [4]) says that a dense subset of all the nodes in $(0, \frac{1}{2} + \epsilon)$ (ϵ arbitrarily small but positive) is sufficient to determine q over the whole interval $[0, 1]$ (although no reconstruction formula is known so far to recover the part of the potential in $(\frac{1}{2} + \epsilon, 1)$). From this, it is easy to realize that the inverse problem is highly unstable if one intends to find a potential that matches exactly the given nodal data for a large number of eigenfunctions.

So far all the known reconstruction formulas rely on an accurate measurement of the nodal data; in other words, the potential sensitively depends on the nodal set. Indeed, to calculate q from $\mathbf{x}(n) = \{x_0, \cdots, x_n\}$, the accuracy needed for each point in $\mathbf{x}(n)$ should be of order $o(1)n^{-3}$. This can be seen from the asymptotic expansion, assuming that q is smooth,

$$x_{k+1} - x_k = \frac{1}{n} + \frac{q(x_{k+1}) + q(x_k)}{4\pi^2 n^3} + \frac{O(1)}{n^5} \quad \forall\, k = 0, \cdots, n-1. \tag{1.2}$$

For example, suppose $n = 100$. If the position of x_{10} is measured as 0.1000001 instead of 0.1000000, then it will produce an error of $10^{-7} \times 4\pi^2 n^3 \approx 4$ toward the potential; see Figure 1. That is to say, in deriving various kinds of asymptotic formulas (as $n \to \infty$) for the reconstruction of q (and its derivatives), one has to pay attention to the physical requirement on the accuracy needed for the measurement of the nodal set.

We remark that (1.2), to be proven at the end of this paper, is indeed the core of many known reconstruction formulas.

In practice, the problem of possible over-determination can be solved by restricting the number of nodal sets used. As a start, in this paper we shall take only one nodal set (i.e., $m = 1$ in Problem 1). We shall introduce a technique that can handle, in a certain degree, the possible noises introduced in the measurement. Though the problem is not completely solved in this paper, the main goal here is to introduce a new direction that deserves attention in the study of inverse nodal problems.

For the reader's convenience, we provide a few more details about the Inverse Problem 1.

1. When $m = 1$, the problem is always solvable. For example, given $n \geqslant 1$ and $\mathbf{x} = (x_0, \cdots, x_n) \in \mathbf{X}(n)$, we can obtain a potential $q = P$ of the form

$$P = V + \lambda, \qquad \lambda = -\int_0^1 V(x)dx, \qquad V(x) = \zeta_0(x) - \sum_{i=1}^{n} \gamma_i \, \zeta_i(x) \, \chi_{[x_{i-1}, x_i)}(x),$$

where $\zeta_0, \cdots, \zeta_n$ are arbitrarily chosen nontrivial and nonnegative functions, and $\gamma_1, \cdots, \gamma_n$ are parameters to be determined from a system of n algebraic equations. Here and in the sequel, χ_A is the characteristic function of the set A, i.e.,

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

In particular, an explicit solution is given by $q = P$, where

$$(1.3)\ P = V + \lambda, \qquad V = -\sum_{i=0}^{n-1} \Big(\frac{\pi}{x_{i+1} - x_i}\Big)^2 \chi_{[x_i, x_{i+1})}, \quad \lambda = \sum_{i=0}^{n-1} \frac{\pi^2}{x_{i+1} - x_i}.$$

2. When $m \geqslant 2$, it follows from the Sturm comparison theorem that the following necessary condition is needed:

For each $1 \leqslant i < j \leqslant m$, between any neighboring points in $\mathbf{x}(n_i)$, there is at least one point in $\mathbf{x}(n_j)$.

It is our conjecture that such a condition is also sufficient for Problem 1 to be solvable.

3. Suppose $m = \infty$. In a series of works [8, 9, 10], it is shown that a potential, if it exists, is unique and can be obtained from the following limit: writing $\mathbf{x}(n) = (x_0^{(n)}, \cdots, x_n^{(n)}) \in \mathbf{X}(n)$ for $n = n_1, n_2, \cdots$,

$$(1.4) \quad q(x) = \lim_{n \to \infty} 2\pi^2 n^2 \{nL_n(x) - 1\}, \qquad L_n := \sum_{i=0}^{n-1} (x_{i+1}^{(n)} - x_i^{(n)}) \, \chi_{[x_i^{(n)}, x_{i+1}^{(n)})}.$$

We remark that in [8, 9, 10], the function L_n was expressed as

$$L_n(x) := x_{j+1}^{(n)} - x_j^{(n)}, \qquad \text{where } j = j_n(x) := \max\{k | \ x_k^{(n)} \leq x\}.$$

Here we use one of its equivalent alternatives.

One also observes that in using the above limit for the reconstruction of a reliable q, a sufficient accuracy is needed for the measurement of the nodal set $\mathbf{x}(n)$; namely, the error of measurement should be controlled at an order of $o(1)n^{-3}$, where $o(1)$ is small.

There are many ways to modify the formula (1.4) so that the needed accuracy of measurement can be relaxed. For example, one can count zeros by groups; that is, for some $0 = k_0 < k_1 < \cdots < k_J = n$, replace L_n by

$$\tilde{L}_n = \sum_{j=1}^{J} \frac{x_{k_j}^{(n)} - x_{k_{j-1}}^{(n)}}{k_j - k_{j-1}} \, \chi_{[x_{k_{j-1}}^{(n)}, x_{k_j}^{(n)})}. \tag{1.5}$$

We shall provide error estimates for the reconstruction formulas (1.3), (1.4), and (1.5). Furthermore we shall focus on the following:

Problem 2. Design an algorithm $\mathcal{A}$ that produces an output $P = \mathcal{A}(n, \mathbf{x})$ for each input consisting of a positive integer n and a nodal set $\mathbf{x} \in \mathbf{X}(n)$.

Surely, we hope that the algorithm has the property that if $\hat{\mathbf{x}} = \mathbf{z}(n, q)$ and $\mathbf{x}$ is close to $\hat{\mathbf{x}}$, then $P = \mathcal{A}(n, \mathbf{x})$ is close to q in a certain sense. To do this, here in this paper we propose to use the Tikhonov regularization method (cf. [7] for example). Fixing a small positive constant $\varepsilon > 0$, we define a Tikhonov functional on $\mathbf{X}(n) \times \mathbf{Q}$ by

$$\mathbf{E}(n, \varepsilon, \mathbf{x}, p) := \frac{|\mathbf{x} - \mathbf{z}(n, p)|^2}{\varepsilon} + \int_0^1 \dot{p}^2(x)\, dx \quad \forall \mathbf{x} \in \mathbf{X}(n),\ p \in \mathbf{Q},$$

where $|\cdot|$ is the Euclidean distance in $\mathbb{R}^{n+1}$. Consider the following minimization problem.

Problem 3. Given a real $\varepsilon > 0$, an integer $n \geqslant 1$ and $\mathbf{x} \in \mathbf{X}(n)$, find p_ε such that

$$p_\varepsilon \in \mathbf{Q}, \qquad \mathbf{E}(n, \varepsilon, \mathbf{x}, p_\varepsilon) = \min_{p \in \mathbf{Q}} \mathbf{E}(n, \varepsilon, \mathbf{x}, p). \tag{1.6}$$

Other choices of Tikhonov functionals can also be used. For example,

$$\mathbf{E}_1(n, \varepsilon, \mathbf{x}, p) := \frac{|\delta\mathbf{x} - \delta\mathbf{z}(n, p)|^2}{\varepsilon} + \int_0^1 \dot{p}^2(x)\, dx \quad \forall \mathbf{x} \in \mathbf{X}(n),\ p \in \mathbf{Q}.$$

Here and in the sequel, we use the following notation: for $\mathbf{y} = (y_0, y_1, \cdots, y_n) \in \mathbb{R}^{n+1}$,

$$\delta y_i := y_{i+1} - y_i, \quad \delta^2 y_i := y_{i+1} + y_{i-1} - 2y_i,$$
$$\delta\mathbf{y} := (\delta y_0, \cdots, \delta y_{n-1}) \in \mathbb{R}^n, \quad \delta^2\mathbf{y} := (\delta^2 y_1, \cdots, \delta^2 y_{n-1}) \in \mathbb{R}^{n-1}.$$

When $\varepsilon = 0$, both $\mathbf{E}$ and $\mathbf{E}_1$ reduce to the functional

$$\mathbf{E}_2(p) := \|\dot{p}\|^2 := \int_0^1 \dot{p}^2(x)\, dx, \qquad p \in \mathbf{Q}^{\mathbf{x}} := \{q \in \mathbf{Q} \mid \mathbf{z}(n, q) = \mathbf{x}\}.$$

Our main results are the following.

Theorem 1.1. *Given $\mathbf{x} \in \mathbf{X}(n)$, define P as in* (1.3) *and L_n as in* (1.4) *or* (1.5)*. If $q \in \mathbf{Q}$ satisfies $\mathbf{z}(n, q) = \mathbf{x}$ and $\int_0^1 |q(x)| dx < n\pi^2/4$, then, for all $r \geq 1$,*

$$\|q - P\|_{L^r} \ \leqslant\ 4\|q - \tilde{q}\|_{L^r}\ ,$$
$$\|2\pi^2 n^2(nL_n - 1) - q\|_{L^r} \ \leqslant\ \frac{10}{3}\|q - \tilde{q}\|_{L^r} + \frac{31 \max_k \overline{|q|}_k}{2n^2\pi^2}\|q\|_{L^r}\ ,$$

where $L^r = L^r((0,1))$ is the L^r-norm,

$$\bar{q}_k := \frac{1}{\delta x_k}\int_{x_k}^{x_{k+1}} q(x)dx\ , \qquad \overline{|q|}_k := \frac{1}{\delta x_k}\int_{x_k}^{x_{k+1}} |q(x)|dx, \tag{1.7}$$

and $\tilde{q} = \sum_{k=1}^{n} \bar{q}_k \chi_{[x_{k-1},x_k)}$ *is the projection of* q *over the set of piecewise constant functions.*

A similar estimate for $\tilde{L}_n$ defined in (1.5) will also be provided.

For the Tikhonov regularization method, we shall prove the following:

Theorem 1.2. (a) *For any* $\mathbf{x} \in \mathbf{X}(n)$ *and* $\varepsilon > 0$*, there is at least a minimizer* p_ε *of* $\mathbf{E}(n, \varepsilon, \mathbf{x}; \cdot)$*.*

(b) *If* $q \in \mathbf{Q}$ *satisfies* $\mathbf{x} = \mathbf{z}(n, q)$ *and* $\int_0^1 |\dot{q}|^2 \leq (\frac{\pi}{2})^6\, n^2$*, then*

$$\|p_\varepsilon - q\|^2 \leq C \left(\frac{1}{n^2} + \varepsilon n^5 \right) \int_0^1 \dot{q}^2 dx.$$

Here C *is a universal constant.*

(c) *Furthermore if* $q^\alpha \in \mathbf{Q}$ *satisfies* $\mathbf{x}^\alpha = \mathbf{z}(n, q^\alpha)$ *with* $\|x - x^\alpha\| < \alpha$*, and* $\int_0^1 |\dot{q}^\alpha|^2 \leq (\frac{\pi}{2})^6\, n^2$*, then* p_ε^α*, the minimizer of* $\mathbf{E}(n, \varepsilon, x^\alpha, \cdot)$*, satisfies, with the universal constant* C *given above,*

$$\|p_\varepsilon^\alpha - q\|^2 \leq 2Cn^5\alpha^2 + \frac{2C}{n^2}\|\dot{q}\|^2 + C\left(\frac{3}{n^2} + \varepsilon n^5\right)\|\dot{q}^\alpha\|^2 \ .$$

Remark 1.3. Hence if $\|\dot{q}\|^2, \|\dot{q}^\alpha\|^2 < M < (\frac{\pi}{2})^6 n^2$, and $\varepsilon < n^{-7}$, then

$$\|p_\varepsilon - q\| \leq \frac{CM}{n^2}.$$

However if we take into account the accuracy of the measurement and assume that $|x - x^\alpha| < \alpha$, then letting $n = \alpha^{-2/7}$, we have

$$\begin{aligned} \|p_\varepsilon^\alpha - q\|^2 &\leq C\left(\frac{5}{n^2} + \varepsilon n^5\right) M + 2Cn^5\alpha^2 \\ &\leq 8CM\alpha^{4/7} \ . \end{aligned}$$

This shows the convergence of this Tikhonov regularization method when there is an error in the measurement of $\mathbf{x}$ for $\mathbf{z}(n, q)$.

Remark 1.4. The error estimate for V is exactly the same using the same argument (cf. the proofs of Theorem 4.1 and Theorem 2). So

$$\|V^\alpha - q\| \leq Cn^5\alpha^2 + \frac{C}{n^2}(\|\dot{q}\|^2 + \|\dot{q}^\alpha\|^2).$$

Although currently we cannot show the uniqueness of solutions to our regularization method, it is expected that any solution will provide a reasonable potential P for a target unknown q, even when there are errors in the measurement $\mathbf{x}$ for $\mathbf{z}(n, q)$. We illustrate our point of view by a numerical simulation demonstrated in Figure 1. In fact, the figure shows that the regularization method is even more accurate than the other methods. We shall pursue this in another paper.

In [2], Barnes also considered a variational formulation, in fact the least square method, in solving the inverse eigenvalue problem with a finite number of eigenvalues. There he used weaker topologies for q instead of a penalty term. The existence of solutions was shown and numerical conditioning of the scheme was studied.

In Section 2, we shall show the existence of solutions to the minimizers of different Tikhonov functionals and derive the associated Euler-Lagrange equations. In Section 3, we shall prove Theorem 1.1 giving explicit error bounds for the schemes (1.3) and (1.5). A new modified Prüfer substitution is introduced for the analysis.

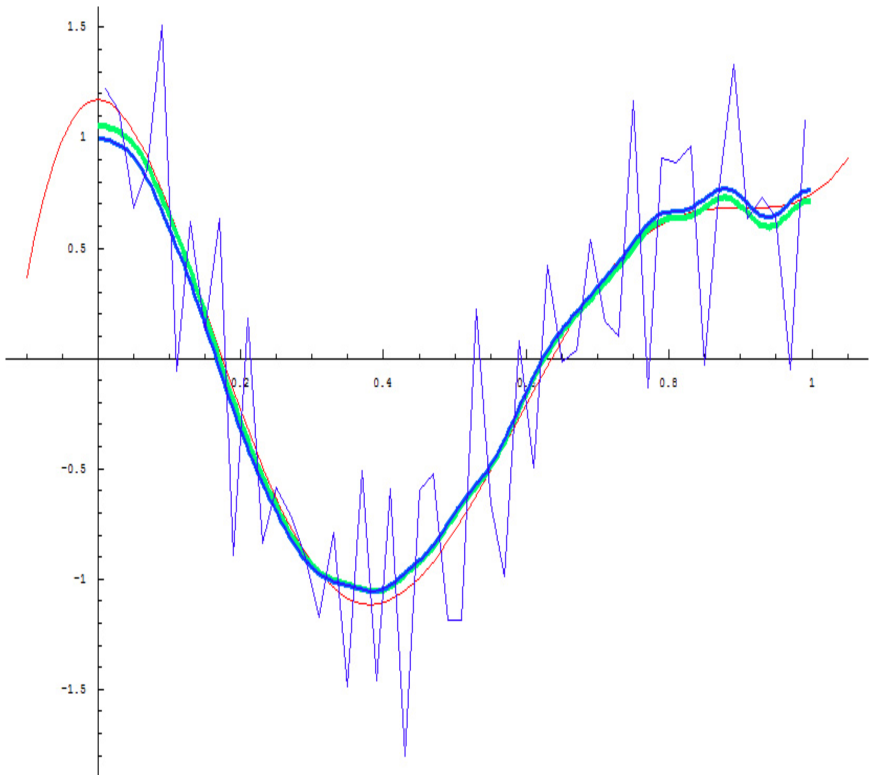

FIGURE 1. Comparison of the schemes: (i) The smooth curve is the true potential p; (ii) the thin zigzagged curve is a reconstruction using (1.4) from a noised measurement $\mathbf{x}$ of the true nodal set $\hat{\mathbf{x}}$ with random error of size 5×10^{-8}; (iii) the dark thick curve and light thick curve are reconstructions using the minimizers of $\mathbf{E}_2$ and $\mathbf{E}_1$ (overlapped), and $\mathbf{E}$, respectively, with the same noised measurement $\mathbf{x}$ as in (ii).

Finally, we shall have stability estimates in Section 4, culminating in the proof of Theorem 1.2. A proof of (1.2) will appear in the Appendix, again using the modified Prüfer substitution.

2. The Tikhonov regularization method

In this section, we briefly study the minimization problem for the Tikhonov functionals $\mathbf{E}, \mathbf{E}_1$ and $\mathbf{E}_2$. In particular, we prove the following:

Theorem 2.1. *Given a positive integer n and $\mathbf{x} \in \mathbf{X}(n)$, $\mathbf{E}_2(\cdot)$ in $\mathbf{Q}^{\mathbf{x}}$ admits a minimizer and for every $\varepsilon > 0$, both $\mathbf{E}(n, \varepsilon, \mathbf{x}; \cdot)$ and $\mathbf{E}_1(n, \varepsilon, \mathbf{x}; \cdot)$ in $\mathbf{Q}$ admit minimizers.*

In addition, let p_ε be a minimizer to $\mathbf{E}(n, \varepsilon, \mathbf{x}; \cdot)$ or to $\mathbf{E}_1(n, \varepsilon, \mathbf{x}; \cdot)$ in $\mathbf{Q}$. Then along a sequence of $\varepsilon \searrow 0$, $p_\varepsilon \longrightarrow p_0$, where $p_0 \in \mathbf{Q}^{\mathbf{x}}$ is a minimizer of $\mathbf{E}_2(\cdot)$ in $\mathbf{Q}^{\mathbf{x}}$.

2.1. **Existence of a minimizer.** We consider only the minimization problem for $\mathbf{E}$. The same problems as those for $\mathbf{E}_1$ or $\mathbf{E}_2$ can be considered in a similar manner and are omitted.

For notational simplicity, we write $\mathbf{E}(p) = \mathbf{E}(n, \varepsilon, \mathbf{x}; p)$. Since $\mathbf{E}$ is a nonnegative well-defined functional on $H^1([0,1]) \supset \mathbf{Q}$, there exists a minimizing sequence $\{p_i\}_{i=1}^{\infty}$, i.e.,

$$p_i \in \mathbf{Q}\ \forall\, i, \qquad \lim_{i\to\infty} \mathbf{E}(p_i) = \inf_{p\in\mathbf{Q}} \mathbf{E}(p).$$

This implies that $\{p_i\}$ is a bounded family in $H^1([0,1])$. So, by the weak compactness of any closed and bounded subset of $\mathbf{Q}$, there exists $p_\varepsilon \in \mathbf{Q}$ such that along a subsequence $j \to \infty$,

$$p_j \quad \longrightarrow \quad p_\varepsilon \quad \text{weakly in } H^1((0,1)) \text{ and uniformly in } C([0,1]).$$

Hence, by the weak lower semicontinuity of the functional $\int_0^1 \dot{p}^2$, we have

$$\int_0^1 \dot{p}_\varepsilon^2(x)dx \le \liminf_{j\to\infty} \int_0^1 \dot{p}_j^2(x)dx.$$

Furthermore,

$$\lim_{j\to\infty} \mathbf{z}(n, p_j) = \mathbf{z}(n, p_\varepsilon), \qquad \lim_{j\to\infty} \lambda(n, p_j) = \lambda(n, p_\varepsilon),$$

so that

$$\mathbf{E}(p_\varepsilon) \le \lim_{j\to\infty} \mathbf{E}(p_j) = \inf_{p\in\mathbf{Q}} \mathbf{E}(p).$$

Therefore, $p_\varepsilon \in \mathbf{Q}$ is a minimizer of $\mathbf{E}(n, \varepsilon, \mathbf{x}; \cdot)$ in $\mathbf{Q}$.

2.2. **The singular limit.** For every $\varepsilon > 0$, let p_ε be a minimizer of $\mathbf{E}(n, \varepsilon, \mathbf{x}; \cdot)$. We consider the asymptotic limit, as $\varepsilon \searrow 0$, of the minimizer p_ε.

Using the construction mentioned in Section 1, there exists $q \in \mathbf{Q}$ such that $\mathbf{x} = \mathbf{z}(n, q)$. It then follows that

$$\mathbf{E}(n, \varepsilon, \mathbf{x}; p_\varepsilon) = \frac{|\mathbf{x} - \mathbf{z}(n, p_\varepsilon)|^2}{\varepsilon} + \int_0^1 |\dot{p}_\varepsilon|^2 \leqslant \mathbf{E}(n, \varepsilon, \mathbf{x}; q) = \int_0^1 |\dot{q}|^2 dx.$$

Hence, using $\|\cdot\|$ for the $L^2((0,1))$ norm,

$$\|\dot{p}_\varepsilon\| \leqslant \|\dot{q}\|, \quad |\mathbf{x} - \mathbf{z}(n, p_\varepsilon)|^2 \le \varepsilon \|\dot{q}\|^2 \qquad \forall \varepsilon > 0.$$

Similar to the above, there exists $p_0 \in \mathbf{Q}$ such that along a sequence $\varepsilon \searrow 0$,

$$\begin{aligned} p_\varepsilon \quad &\longrightarrow \quad p_0 \quad \text{weakly in } H^1((0,1)) \text{ and uniformly in } C([0,1]) \\ \mathbf{z}(n, p_\varepsilon) \quad &\longrightarrow \quad \mathbf{x} = \mathbf{z}(n, p_0). \end{aligned}$$

Next we show that the limit $p_0 \in \mathbf{Q}^{\mathbf{x}}$ is a minimizer of $\mathbf{E}_2$ in $\mathbf{Q}^{\mathbf{x}}$. Indeed, for any $q \in \mathbf{Q}$ satisfying $\mathbf{z}(n, q) = \mathbf{x}$, from the previous step, we see that $\|\dot{p}_\varepsilon\| \le \|\dot{q}\|$, so

$$\|\dot{p}_0\| \le \liminf_{\varepsilon\to 0} \|\dot{p}_\varepsilon\| \leqslant \|\dot{q}\|.$$

Thus, p_0 is a minimizer of $\mathbf{E}_2$ in $\mathbf{Q}^{\mathbf{x}}$.

In a similar manner, one can study the asymptotic limit of minimizers of $\mathbf{E}_1(n, \varepsilon, \mathbf{x}; \cdot)$. We only point out the following: for every positive integer n and $\mathbf{x}, \mathbf{y} \in \mathbf{X}(n)$,

$$2|\mathbf{x} - \mathbf{y}|\ \sin\frac{\pi}{2n} \leqslant |\delta\mathbf{x} - \delta\mathbf{y}| \leqslant 2|\mathbf{x} - \mathbf{y}| \cos\frac{\pi}{2n}.$$

2.3. **The Euler-Lagrange equation.** Here we shall derive the Euler-Lagrange equation for a minimizer. For this, we recall a well-known result from the elliptic pde theory: Suppose $f \in L^r((0,1))$ ($r \geq 1$) and $q \in H^1([0,1])$. Then

$$\int_0^1 (\dot{q}\dot{\zeta} + \zeta f)dx = 0 \ \forall \zeta \in C^\infty([0,1]) \iff \begin{cases} \ddot{q} = f & \text{in } L^r((0,1)), \\ \dot{q}(0) = \dot{q}(1) = 0. \end{cases}$$

Now let $\varepsilon > 0$, integer $n \geq 1$, and $\mathbf{x} = (x_0, \cdots, x_n) \in \mathbf{X}(n)$ be given and fixed. We write $\mathbf{E}(\cdot) = \mathbf{E}(n, \varepsilon, \mathbf{x}; \cdot)$. For $p_\varepsilon \in \mathbf{Q}$ and $\zeta \in H^1((0,1))$, we want to calculate the first variation of $\mathbf{E}(\cdot)$ at p_ε in the direction $\zeta - \bar{\zeta} \in \mathbf{Q}$, where $\bar{\zeta} = \int_0^1 \zeta(x)dx$; namely, we calculate

$$\langle \nabla \mathbf{E}(p_\varepsilon), \zeta - \bar{\zeta} \rangle := \lim_{t \to 0} \frac{\mathbf{E}(p_\varepsilon + t[\zeta - \bar{\zeta}]) - \mathbf{E}(p_\varepsilon)}{t}.$$

1. For $t \in \mathbb{R}$, set $p(t) = p_\varepsilon + t[\zeta - \bar{\zeta}]$, $E(t) = \mathbf{E}(p(t))$, $Z(t) = (Z_0(t), \cdots, Z_n(t)) := \mathbf{z}(n, p(t))$, $\Lambda(t) = \lambda(n, p(t))$ and $U(t, \cdot) = u(n, p(t); \cdot)$. Using the notation $' = \frac{d}{dt}$ and $\dot{} = \frac{d}{dx}$, we also let $V(t, \cdot) := U'(t, \cdot)$. Note that $Z(0) = \mathbf{z}(n, p_\varepsilon)$, $\Lambda(0) = \lambda$ and $U(0, \cdot) = u(\cdot)$.

From the definition of $\mathbf{E}$, we see that

$$\langle \nabla \mathbf{E}(p_\varepsilon), \zeta - \bar{\zeta} \rangle = E'(0), \qquad E'(t) = \frac{2(Z - \mathbf{x}) \cdot Z'}{\varepsilon} + 2 \int_0^1 (\dot{p}_\varepsilon + t\dot{\zeta})\dot{\zeta} \ dx.$$

Also, differentiating $U(t, Z_i(t)) = 0$ in t, we obtain

$$Z_i'(t) = -\frac{V(t, Z_i(t))}{\dot{U}(t, Z_i(t))} \ \ \forall \, i.$$

2. Differentiating the equation $\ddot{U} = (p - \Lambda)U$ with respect to t gives

$$\ddot{V} = (p - \Lambda)V + (\zeta - \bar{\zeta} - \Lambda')U \ \text{ in } \ (0,1), \qquad V(t,0) = V(t,1) = 0.$$

Note that the Wronskian $W = \dot{U}V - \dot{V}U$ satisfies $\dot{W} = -(\zeta - \bar{\zeta} - \Lambda')U^2$, so that

$$\dot{U}V - \dot{V}U = -\int_0^x \Big\{ \zeta - \bar{\zeta} - \Lambda' \Big\} U^2.$$

The conditions $U(t,1) = V(t,1) = 0$ give

$$\Lambda' = -\bar{\zeta} + \frac{\int_0^1 \zeta U^2}{\int_0^1 U^2}.$$

3. It then follows, writing $z_i = Z_i(0)$, $u(\cdot) = U(0, \cdot)$, that

$$\begin{aligned} Z_i'(0) &= -\frac{V(0, z_i)}{\dot{u}(z_i)} = \frac{\int_0^{z_i} (\zeta - \bar{\zeta} - \Lambda'(0))u^2}{\dot{u}^2(z_i)} \\ &= \frac{1}{\dot{u}^2(z_i)} \Big\{ \int_0^{z_i} \zeta u^2 - \frac{\int_0^{z_i} u^2}{\int_0^1 u^2} \int_0^1 \zeta u^2 \Big\} \\ &= \frac{1}{\dot{u}^2(z_i)} \int_0^1 \zeta u^2 \Big\{ \chi_{[0, z_i)} - \frac{\int_0^{z_i} u^2}{\int_0^1 u^2} \Big\} dx \\ &= \frac{1}{\dot{u}^2(z_i)} \int_0^1 \zeta u^2 \Big\{ \frac{\int_{z_i}^1 u^2}{\int_0^1 u^2} - \chi_{[z_i, 1]} \Big\} dx. \end{aligned}$$

Hence

$$\langle \nabla \mathbf{E}(p_\varepsilon), \zeta - \bar{\zeta} \rangle = 2 \int_0^1 \left(\dot{p}_\varepsilon \, \dot{\zeta} + \zeta u^2 \sum_{i=1}^{n-1} \frac{z_i - x_i}{\varepsilon \, \dot{u}^2(z_i)} \Big\{ \frac{\int_{z_i}^1 u^2}{\int_0^1 u^2} - \chi_{(z_i,1]} \Big\} \right) dx.$$

Thus the first variation of $\mathbf{E}$ at p_ε in any direction $\zeta - \bar{\zeta} \in \mathbf{Q}$ is zero, so that

$$\dot{p}_\varepsilon(0) = \dot{p}_\varepsilon(1) = 0, \quad \ddot{p}_\varepsilon = u^2 \sum_{i=1}^{n-1} \frac{z_i - x_i}{\varepsilon \, \dot{u}^2(z_i)} \Big\{ \frac{\int_{z_i}^1 u^2}{\int_0^1 u^2} - \chi_{(z_i,1]} \Big\} \quad \text{in} \ \ (0,1).$$

Set $q_\varepsilon = p_\varepsilon - \lambda$ and $\mathbf{a} = (a_0, \cdots, a_{n-1})$, where

$$a_0 = \frac{1}{\varepsilon} \sum_{i=1}^{n-1} \frac{(z_i - x_i)}{\dot{u}^2(z_i)} \frac{\int_{z_i}^1 u^2}{\int_0^1 u^2}, \tag{2.1}$$

$$a_k = a_0 + \frac{1}{\varepsilon} \sum_{i=1}^{k} \frac{x_i - z_i}{\dot{u}^2(z_i)} \quad \forall \, k = 1, \cdots, n-1. \tag{2.2}$$

The minimization problem thus can be formulated as follows:

Problem 4. Given an $\varepsilon > 0$, an integer $n \geqslant 1$ and $\mathbf{x} = (x_0, \cdots, x_n) \in \mathbf{X}(n)$, find

$$u, q_\varepsilon \in C^1([0,1]), \ \mathbf{a} \in \mathbb{R}^n, \ \mathbf{z} \in \mathbf{X}(n)$$

such that

$$\begin{cases} \ddot{u} = u \, q_\varepsilon \qquad \text{in} \ \ (0,1), \\ \qquad \dot{u}(0) = 1, \quad \{x \in [0,1] \mid u(x) = 0\} = \{z_0, \cdots, z_n\}, \\ \ddot{q}_\varepsilon = u^2 \sum_{k=0}^{n-1} a_k \chi_{[z_k, z_{k+1})} \qquad \text{in} \ \ (0,1), \\ \qquad \dot{q}_\varepsilon(0) = \dot{q}_\varepsilon(1) = 0, \\ a_k = a_{k-1} + \dfrac{1}{\varepsilon} \, \dfrac{x_k - z_k}{\dot{u}^2(z_k)} \quad \forall \, k = 1, \cdots, n, \\ a_0 \text{ is given by (2.1).} \end{cases} \tag{2.3}$$

Note that the solvability condition

$$\begin{aligned} 0 &= \int_0^1 \ddot{q}_\varepsilon \, dx = \sum_{i=0}^{n-1} \int_{z_i}^{z_{i+1}} a_i u^2 \sum_{i=0}^{n-1} a_i \Big(\int_{z_i}^1 u^2 \, dx - \int_{z_{i+1}}^1 u^2 \, dx \Big) \\ &= \sum_{i=1}^{n-1} (a_i - a_{i-1}) \int_{z_i}^1 u^2 \, dx + a_0 \int_0^1 u^2 \, dx \end{aligned}$$

and the definition of $a_i - a_{i-1}$ in (2.3) automatically give the required formula for a_0 in (2.1). The inductive definition for a_i in (2.3) then also gives the required formula for a_i in (2.2) for each $i = 1, \cdots, n-1$. Hence, we have proved the following:

Theorem 2.2. *Let $\varepsilon > 0, n \geqslant 1$, and $\mathbf{x} \in \mathbf{X}(n)$ be given and p_ε be a minimizer to* (1.6). *Then $(\mathbf{z}, \mathbf{a}, u, q_\varepsilon)$ solves* (2.3), *where $q_\varepsilon = p_\varepsilon - \lambda$, (λ, u) is the nth eigenpair to* (1.1), *$\mathbf{z}$ is the nodal set of u, and $\mathbf{a} = (a_0, \cdots, a_{n-1})$ as in* (2.1), (2.2).

Thus, a minimizer can be obtained by first solving (2.3) for q_ϵ and then defining

$$\lambda := -\int_0^1 q_\varepsilon(x) \, dx, \qquad p_\varepsilon = \lambda + q_\varepsilon.$$

In a similar manner, we can obtain the Euler-Lagrange equation for minimizers of $\mathbf{E}_1$. Define $\delta a_i := a_{i+1} - a_i$. Then

$$\begin{aligned}\frac{d}{dt}|\delta\mathbf{x} - \delta Z|^2 &= 2\sum_{i=1}^{n-1}\Big\{[Z_i - Z_{i-1}] - [x_i - x_{i-1}]\Big\}(Z_i' - Z_{i-1}') \\ &= 2\sum_{i=1}^{n-1}\Big\{[x_{i+1} - 2x_i + x_{i-1}] - [Z_{i+1} - 2Z_i + Z_{i+1}]\Big\}Z_i'.\end{aligned}$$

Hence, using the notation

$$\delta^2\mathbf{y}_i := y_{i+1} - 2y_i + y_{i-1}, \quad i = 1, \cdots, n-1, \quad \forall \mathbf{y} \in \mathbf{X}(n), \quad n \geq 2,$$

we obtain a similar Euler-Lagrange equation for any minimizer of $\mathbf{E}_1$ in $\mathbf{Q}$, with the only change that the induction relation among $\{a_i\}_{i=0}^{n-1}$ is replaced by

$$\delta a_i := \frac{1}{\varepsilon}\frac{\delta^2 z_i - \delta^2 x_i}{\dot{u}(z_i)^2}, \quad i = 1, \cdots, n-1.$$

Similarly, the Euler-Lagrange equation for a minimizer p_0 of $\mathbf{E}_2$ in $\mathbf{Q}^{\mathbf{x}}$ is the following, for an unknown $(u, p_0, \lambda, \mathbf{a}) \in C^2([0,1]) \times C^1([0,1]) \times \mathbb{R} \times \mathbb{R}^n$:

$$(2.4) \qquad \begin{cases} \ddot{u} = (p_0 - \lambda)\, u & \text{in } (0,1), \\ \ddot{p}_0 = u^2 \sum_{i=0}^{n-1} a_i \chi_{[x_i, x_{i+1})} & \text{in } (0,1), \\ \dot{u}(0) = 1, \ \dot{p}_0(0) = \dot{p}_0(1) = 0, \\ \{x \in [0,1] \mid u(x) = 0\} = \{x_0, x_1, \cdots, x_n\}. \end{cases}$$

Also we remark that (2.3) or (2.4) can be solved numerically by a finite difference method, as shown in Figure 1.

Currently, we do not know if the minimizers are unique. Of course, when $n = 1$, the minimizers for $\mathbf{E}, \mathbf{E}_1$, or $\mathbf{E}_2$ are unique, given by $p \equiv 0, \lambda = -\pi^2$. Also the system of equation (2.3) appears to be unstable as ϵ appears in the denominator. However, from the regular numerical solution in the figure, we believe that the system is stable and will leave it to a future study.

3. L^r ERROR ESTIMATES FOR SOME RECONSTRUCTION FORMULAS

Given $\mathbf{x} = (x_0, \cdots, x_n) \in \mathbf{X}(n)$, it contains $(n-1)$ independent pieces of information. If we want to recover a potential possessing $\mathbf{x}$ as a nodal set, we can get at most $n-1$ pieces of independent quantified information from $\mathbf{x}$. In this section, we shall show that such information could be the average over each $[x_i, x_{i+1}]$ of the potential. Our motivation is the following observation:

Theorem 3.1. *Given an integer $n \geqslant 1$ and $\mathbf{x} = (x_0, \cdots, x_n) \in \mathbf{X}(n)$, among all functions which have zero mean and are constants on each (x_i, x_{i+1}), $i = 0, \cdots, n-1$, there is one and only one that possesses $\mathbf{x}$ as its nodal set, given by*

$$(3.1) \quad P(\mathbf{x};\cdot) := V(\cdot) - \bar{V}, \quad V(x) := -\sum_{i=0}^{n-1}\frac{\pi^2\, \chi_{[x_i,x_{i+1})}(x)}{(x_{i+1} - x_i)^2}, \quad \bar{V} = \int_0^1 V(x)dx.$$

The assertion follows by the fact that any eigenfunction on $[x_i, x_{i+1}]$ must be a constant multiple of $\sin[\pi(x - x_i)/(x_{i+1} - x_i)]$ on $[x_i, x_{i+1}]$.

We remark that the advantage of V in (3.1) over $2\pi^2 n^3 L_n(x)$ in (1.4) is that the former is local; i.e., it does not depend on the total number n, which is usually hard to count.

Now suppose $q \in \mathbf{Q}$ is a generic potential that has the nodal set $\mathbf{x}$. We would like to estimate the difference between q and $P(\mathbf{x};\cdot)$. Note that a good representative of q is the following **piecewise constant mean-value approximation** (cf. (1.7)):

$$\tilde{q} := \sum_{i=0}^{n-1} \bar{q}_i \chi_{[x_i, x_{i+1})}(x).$$

In this section, we shall perform two estimates. First we estimate the location of the nodal set $\mathbf{z}(n,q)$ for a given $q \in \mathbf{Q}$ and large enough n. Next we estimate the distance between $P(\mathbf{x};\cdot)$ and the mean-value approximation $\tilde{q}$ for any $q \in \mathbf{Q}$ satisfying $\mathbf{x} = \mathbf{z}(n,q)$.

Theorem 3.2. *Suppose $q \in \mathbf{Q}$ and n is an integer satisfying*

$$\int_0^1 |q| dx \le \frac{n\pi^2}{4}. \tag{3.2}$$

Let $\lambda = \lambda(n,q)$ be the nth eigenvalue and $\mathbf{x} = \mathbf{z}(n,q)$ be the nodal set. Then

(3.3)

$$\Big|\lambda - (n\pi)^2\Big| \leqslant \min\left\{ \tfrac{5}{4}\int_0^1 |q| dx,\ \int_0^1 |q-\tilde{q}| dx + \frac{4\max_i\{\overline{|q|}_i\} + \int_0^1 |q| dx}{(4n-5/4)n\pi^2}\int_0^1 |q| dx \right\},$$

$$\Big|(x_i - x_j) - \frac{i-j}{n}\Big| \leqslant \frac{1}{\lambda}\int_0^1 |q| dx < \frac{1}{4n-5/4} \quad \forall\, i,\ j, \tag{3.4}$$

$$\Big|\bar{q}_i - \lambda + \frac{\pi^2}{(\delta x_i)^2}\Big| \leqslant \frac{3}{2\delta x_i}\int_{x_i}^{x_{i+1}} |q - \bar{q}_i|\, dx \quad \forall\, i. \tag{3.5}$$

Theorem 3.3. *Let $\mathbf{x} \in \mathbf{X}(n)$ be given. Define P, V and $\bar{V}$ as in* (3.1). *Then any q in $\mathbf{Q}$ satisfying $\mathbf{z}(n,q) = \mathbf{x}$ and* (3.2) *must be close to P in the following sense: For $\lambda = \lambda(n,q)$, $\tilde{q}$ as in* (1.7) *and every $r \in [1,\infty)$,*

$$|\lambda + \bar{V}| \leqslant \|\lambda + V - \tilde{q}\|_{L^r} \leqslant \tfrac{3}{2}\|q - \tilde{q}\|_{L^r}, \qquad \|q - P\|_{L^r} \leqslant 4\|q - \tilde{q}\|_{L^r}. \tag{3.6}$$

Proof of Theorem 3.2. We divide the proof into the following steps.

1. We use a modified Prüfer substitution:

$$u(x) = R(x)\sin\theta(x), \qquad \dot{u}(x) = m\, R(x)\cos\theta(x)$$

where $m > 0$ is a constant chosen at our convenience. Then the equations for u give

$$\begin{cases} m\,\dot{\theta} = m^2 + (\lambda - q - m^2)\sin^2\theta \quad \text{on} \quad [0,1], \\ \theta(x_k) = k\pi \quad \forall\, k = 0, \cdots, n. \end{cases} \tag{3.7}$$

We point out that $R(\cdot) > 0$ and the solution $\theta(\cdot)$ depends on m.

2. Integrating the differential equation in (3.7) over $[0,1]$ and using $\int_0^1 q\, dx = 0$ and $\sin^2\theta = (1-\cos 2\theta)/2$, we obtain

$$(\lambda - m^2)\int_0^1 \sin^2\theta\, dx = mn\pi - m^2 - \int_0^1 \frac{q\cos 2\theta}{2}\, dx.$$

Taking $m = n\pi/2$ gives $\lambda - m^2 > 0$, i.e., $\sqrt{\lambda} > n\pi/2$. Next setting $m = \sqrt{\lambda}$ we obtain

$$\begin{aligned} n\pi &= \sqrt{\lambda} + \frac{1}{2\sqrt{\lambda}} \int_0^1 q \cos 2\theta \, dx, \\ \Big|(n\pi)^2 - \lambda\Big| &= \Big|\Big\{1 + \frac{1}{4\lambda}\int_0^1 q\cos 2\theta dx\Big\} \int_0^1 q \cos 2\theta dx\Big| \\ &\le \tfrac{5}{4}\Big|\int_0^1 q\cos 2\theta dx\Big| \le \tfrac{5}{16}\, n\pi^2 \end{aligned}$$

by using $\lambda > (n\pi)^2/4$ and $\int_0^1 |q| dx \le n\pi^2/4$.

One can further estimate $|\lambda - (n\pi)^2|$ by using the following:

$$\begin{aligned} \int_0^1 q\cos 2\theta dx &= \sum_i \Big(\bar{q}_i \int_{x_i}^{x_{i+1}} \cos 2\theta dx + \int_{x_i}^{x_{i+1}} (q - \bar{q}_i)\cos 2\theta dx\Big), \\ \int_{x_i}^{x_{i+1}} \cos 2\theta dx &= \int_{x_i}^{x_{i+1}} \Big(\frac{\dot{\theta}}{\sqrt{\lambda}} + \frac{q\sin^2\theta}{\lambda}\Big)\cos 2\theta \, dx = \frac{1}{\lambda}\int_{x_i}^{x_{i+1}} q \sin^2\theta \cos 2\theta \, dx. \end{aligned}$$

Using $|\sin^2\theta\cos 2\theta| \le 1$, we then obtain

$$\Big|\lambda - (n\pi)^2\Big| \le \frac{1}{4\lambda}\Big(\int_0^1 |q|dx\Big)^2 + \frac{\max_i\{\overline{|q|}_i\}}{\lambda}\int_0^1 |q|dx + \int_0^1 |q - \tilde{q}|dx.$$

Finally, using $\lambda > \pi^2 n(n - \frac{5}{16})$ we obtain (3.3).

3. Setting $m = \sqrt{\lambda}$ and integrating (3.7) over $[x_i, x_{i+1}]$ and $[0,1]$ respectively, we obtain

$$\delta x_i = \frac{\pi}{\sqrt{\lambda}} + \frac{1}{\lambda}\int_{x_i}^{x_{i+1}} q\sin^2\theta dx, \quad \frac{1}{n} = \frac{\pi}{\sqrt{\lambda}} + \frac{1}{n\lambda}\int_0^1 q\sin^2\theta \, dx.$$

Certain linear combinations give

$$\Big|(x_k - x_l) - \frac{k-l}{n}\Big| = \frac{1}{\lambda}\Big|\Big(\int_{x_l}^{x_k} - \frac{k-l}{n}\int_0^1\Big) q\sin^2\theta\Big| \le \frac{1}{\lambda}\int_0^1 |q|dx < \frac{1}{4n - 5/4}$$

for every integer $l < k$ in $[0, n]$. Thus (3.4) follows.

4. For (3.5), we first notice that

$$\lambda\delta x_i = \pi\sqrt{\lambda} + \int_{x_i}^{x_{i+1}} q\sin^2\theta \, dx = \pi\sqrt{\lambda} + \bar{q}_i\delta x_i - \int_{x_i}^{x_{i+1}} q\cos^2\theta \, dx.$$

This implies that $\min\{\lambda\delta x_i, (\lambda - \bar{q}_i)\delta x_i\} \ge \frac{\sqrt{\lambda}\pi}{2}$, so that $\sqrt{\lambda - \bar{q}_i} \ge \pi/(2\delta x_i)$.

Next setting $m = \sqrt{\lambda - \bar{q}_i}$ and integrating (3.7) over $[x_i, x_{i+1}]$ we obtain

$$\sqrt{\lambda - \bar{q}_i}\pi = (\lambda - \bar{q}_i)\delta x_i + \int_{x_i}^{x_{i+1}} \frac{(q - \bar{q}_i)\cos 2\theta}{2} dx.$$

Solving this equation for $m = \sqrt{\lambda - \bar{q}_i}$ (recalling $\sqrt{\lambda - \bar{q}_i} \ge \pi/(2\delta x_i)$), we obtain

$$\sqrt{\lambda - \bar{q}_i} = \frac{\pi}{2\delta x_i}\left\{1 + \Big(1 - \frac{2\delta x_i}{\pi^2}\int_{x_i}^{x_{i+1}} (q - \bar{q}_i)\cos 2\theta \, dx\Big)^{1/2}\right\}.$$

The estimate (3.5) then follows by squaring both sides and using

$$\left|(1+\sqrt{1-s})^2-4\right| = \frac{|s|(3+\sqrt{1-s})}{1+\sqrt{1-s}} \le 3|s| \qquad \forall s\in[-1,1].$$

This completes the proof of Theorem 3.2. □

Remark 3.4. The modified Prüfer substitution used here is an extension of the one employed in [1, 10]. In fact, it is equivalent to the transformation

$$u(x)=R(x)\sin(\sqrt{\lambda}\varphi(x)), \qquad \dot u(x)=mR(x)\cos(\sqrt{\lambda}\varphi(x)),$$

where the scaled phase $\varphi:=\theta/\sqrt{\lambda}$ satisfies

$$\dot\varphi=\frac{m}{\sqrt{\lambda}}+\frac{\lambda-q-m^2}{m\sqrt{\lambda}}\sin^2(\sqrt{\lambda}\varphi).$$

The Prüfer substitution used in [1, 10] indeed corresponds to the choice of $m=\sqrt{\lambda}$, which is usually sufficient for the analysis.

Proof of Theorem 3.3. As the average of $\tilde q$ is zero, for any $r\geqslant 1$,

$$\left|\lambda+\bar V\right| = \left|\int_0^1(\lambda+V-\tilde q)dx\right| \leqslant \|\lambda+V-\tilde q\|_{L^1}\leqslant\|\lambda+V-\tilde q\|_{L^r}.$$

In addition, using the definition of V, $\tilde q$ and (3.5) we have

$$\begin{aligned}\|\lambda+V-\tilde q\|_{L^r}^r &= \sum_i \delta x_i\left|\bar q_i-\lambda-\frac{\pi^2}{(\delta x_i)^2}\right|^r \leqslant \sum_i\delta x_i\left(\frac{3}{2\delta x_i}\int_{x_i}^{x_{i+1}}|q-\bar q_i|dx\right)^r\\ &\leqslant \left(\frac{3}{2}\right)^r\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|q-\bar q_i|^r dx=\left(\frac{3}{2}\right)^r\|q-\tilde q\|^r.\end{aligned}$$

Thus, $\|\lambda+V-\tilde q\|_{L^r}\leqslant\frac{3}{2}\|q-\tilde q\|_{L^r}$. Finally, since $P=V-\bar V$,

$$\|q-P\|_{L^r}=\|\,(q-\tilde q)+(\tilde q-\lambda-V)+(\lambda+\bar V)\|_{L^r}\le 4\|q-\tilde q\|_{L^r}.$$

This completes the proof of Theorem 3.3. □

Remark 3.5. Since the map from q to $\lambda(n,q)$ is nonlinear, certain conditions such as (3.2) are definitely needed for any linear type estimate such as (3.5) or the first estimate in (3.6) to hold. That means that (3.2) may be relaxed, but cannot be totally removed. Consider the following example. Fix any constant $A>0$, define $h=\arctan(\coth A),\Lambda=2(h^2-A^2)$, and

$$U(y)=\begin{cases}\frac{\sinh(2[A+h]y)}{\sinh A} & \text{if } \ y\in[0,\frac{A}{2(A+h)}],\\ \frac{\cos([A+h][1-2y])}{\cos h} & \text{if } \ y\in[\frac{A}{2(A+h)},\frac12],\end{cases}$$

$$Q(y)=\begin{cases}4h[A+h] & \text{if } \ y\in[0,\frac{A}{2(A+h)}],\\ -4A[A+h] & \text{if } \ y\in[\frac{A}{2(A+h)},\frac12],\end{cases}$$

$$U(y)=-U(-y)=U(1-y), \qquad Q(-y)=Q(y)=Q(1-y), \qquad \forall y\in\mathbb{R}.$$

Note that $h\to\pi/4$ and $\Lambda\to-\infty$ as $A\to\infty$. Also

$$U_{yy}=(Q-\Lambda)U \quad \text{in } \ \mathbb{R}, \quad \{y\mid U(y)=0\}=\mathbb{Z}.$$

Fix any integer $n\ge 1$ and set

$$\mathbf{x}=(0,\tfrac1n,\tfrac2n,\cdots,\tfrac{n-1}{n},1)\in\mathbf{X}(n), \quad q(x)=n^2Q(nx), \quad u(x)=U(nx), \quad \lambda=n^2\Lambda.$$

The family $\{q\}_{A>1}$ has the property that $\mathbf{x} = \mathbf{z}(n,q)$, but

$$\lim_{A\to\infty} \frac{\lambda(n,q)}{\int_0^1 |q|dx} = \lim_{A\to\infty} \frac{2(h^2-A^2)}{8Ah} = -\infty \qquad \forall\, n = 1, 2, \cdots .$$

Since $\tilde{q} \equiv 0$, the inequality (3.5) does not hold for all $A > 1$; even the constant $\frac{3}{2}$ is replaced by any larger constant.

For the second estimate in (3.6), we do not know if it holds unconditionally when the constant "4" is replaced by a certain large number.

Remark 3.6. As a demonstration, we provide an estimate for the $L^r = L^r((0,1))$ difference between q and $\tilde{q}$:

$$\|q-\tilde{q}\|_{L^r}^r := \int_0^1 |q-\tilde{q}|^r dx = \sum_i \int_{x_i}^{x_{i+1}} \Big| \frac{1}{\delta x_i} \int_{x_i}^{x_{i+1}} (q(x)-q(y))dy \Big|^r dx \tag{3.8}$$

$$\leq \sum_i \frac{1}{\delta x_i} \int_{x_i}^{x_{i+1}} dx \int_{x_i}^{x_{i+1}} |q(x)-q(y)|^r dy. \tag{3.9}$$

Set

$$\delta^* \mathbf{x} = \max\{\delta x_0, \cdots, \delta x_{n-1}\}, \qquad \delta_* \mathbf{x} = \min\{\delta x_0, \cdots, \delta x_{n-1}\}.$$

By (3.4), we have

$$\frac{2}{3n} < \delta_* \mathbf{x} < \delta^* \mathbf{x} < \frac{4}{3n}. \tag{3.10}$$

For any exponent $r \in [1, \infty)$ we have

$$\begin{aligned}
\|q-\tilde{q}\|_{L^r}^r &\leq \sum_i \frac{1}{\delta x_i} \int_{x_i}^{x_{i+1}} \int_{x_i}^{x_{i+1}} |q(t)-q(s)|^r dtds \\
&\leq \frac{1}{\delta_* \mathbf{x}} \int_0^1 \int_{\max\{-\delta^*\mathbf{x}, -t\}}^{\min\{\delta^*\mathbf{x}, 1-t\}} |q(t)-q(t+h)|^r dhdt \\
&= \frac{1}{\delta_* \mathbf{x}} \int_{-\delta^*\mathbf{x}}^{\delta^*\mathbf{x}} \int_{\min\{-h,-x\}}^{\min\{1,1-h\}} |q(t)-q(t+h)|^r dtdh \\
&\leq \frac{2\delta^*\mathbf{x}}{\delta_*\mathbf{x}} \sup_{0<h<\delta x^*} \int_0^{1-h} |f(t)-f(t+h)|dt \\
&\leq 4 \sup_{0<h<4/(3n)} \int_0^{1-h} |q(t)-q(t+h)|^r dt.
\end{aligned}$$

Proof of Theorem 1.1. The first part was proved in Theorem 3.3. It remains to show the second part. For this, let $k_0, \cdots, k_J$ be integers satisfying $0 = k_0 < k_1 < \cdots < k_J = n$. Let L_n be defined by

$$\tilde{L}_n(x) = \sum_{j=1}^J \frac{x_{k_j}^{(n)} - x_{k_{j-1}}^{(n)}}{k_j - k_{j-1}} \, \chi_{[x_{k_{j-1}}^{(n)}, x_{k_j}^{(n)})}(x). \tag{3.11}$$

We shall estimate the L^r difference between $2n^2\pi^2(n\tilde{L}_n - 1)$ and q, where $\mathbf{x} = \mathbf{z}(n,q)$.

Let l, k be any integers satisfying $0 \leqslant l < k \leqslant n$. Setting $m = n\pi$ and integrating $1 = \frac{\dot{\theta}}{n\pi} + \frac{q+n^2\pi^2-\lambda}{n^2\pi^2} \sin^2\theta$ over $[x_l, x_k]$, we obtain

$$x_k - x_l = \frac{k-l}{n} + \frac{1}{n^2\pi^2} \int_{x_l}^{x_k} (q + n^2\pi^2 - \lambda) \sin^2\theta \; dx.$$

Denote

$$\ell := \frac{n(x_k - x_l)}{k-l}, \qquad ⨍\{\ \}dx := \frac{1}{x_k - x_l}\int_{x_l}^{x_k}\{\ \}dx$$
$$W := ⨍(q + n^2\pi^2 - \lambda)\sin^2\theta\ dx. \quad \overline{q} := ⨍ q dx.$$

We then obtain the relation

$$\ell - 1 = \frac{\ell}{n^2\pi^2}W, \quad \ell = \frac{1}{1 - W/(n^2\pi^2)}.$$

Note that (3.4) implies that

$$|\ell - 1| \le \frac{n}{(k-l)(4n - 5/4)} \leqslant \frac{1}{3} \qquad \Rightarrow \qquad \frac{|W|}{n^2\pi^2} \le \frac{1}{2}.$$

The previous relation can be expressed in the form

$$2n^2\pi^2(\ell - 1) - \overline{q} = 2\ell W - \overline{q}.$$

Using the definition of W we have

$$\begin{aligned} 2W &= 2⨍(q - \overline{q})\sin^2\theta dx \\ &\quad + 2(\overline{q} + n^2\pi^2 - \lambda)⨍\sin^2\theta\Big\{\frac{d\theta}{n\pi} + \frac{(q + n^2\pi^2 - \lambda)\sin^2\theta\, dx}{n^2\pi^2}\Big\} \\ &= \frac{\overline{q} + n^2\pi^2 - \lambda}{\ell} + ⨍(\overline{q} - q)\cos(2\theta)dx \\ &\quad + \frac{2(\overline{q} + n^2\pi^2 - \lambda)}{n^2\pi^2}⨍(q + n^2\pi^2 - \lambda)\sin^4\theta\, dx. \end{aligned}$$

Upon using the estimate $|\lambda - n^2\pi^2| \le \frac{5}{4}\int_0^1 |q|$ and $|\ell| \le \frac{4}{3}$, we then obtain

$$(3.12) \qquad |2\ell W - \overline{q}| \le |\lambda - n^2\pi^2| + \frac{4}{3}⨍|q - \overline{q}| + \frac{8}{3n^2\pi^2}\Big(\overline{|q|} + \frac{5}{4}\int_0^1 |q|dx\Big)^2.$$

Now take $l = k_{j-1}, k = k_j$. Denote the corresponding $\ell, W, \overline{q}$ by $\ell_j, W_j, \overline{q}_j$, and set

$$\hat{\ell} := \sum_{j=1}^{J} \ell_j \chi_{[x_{k_{j-1}}, x_{k_j})}(x)\ , \quad \hat{W} := \sum_{j=1}^{J} W_j \chi_{[x_{k_{j-1}}, x_{k_j})}(x).$$

Then, by (3.3),

$$\begin{aligned} |\lambda - n^2\pi^2| &\le \frac{5}{4}\int_0^1 |q|dx \le \frac{5}{4}\max \overline{|q|}_j, \\ |\lambda - n^2\pi^2| &\le \frac{5\max_j \overline{|q|}_j}{n\pi^2(4n - 5/4)}\int_0^1 |q|dx + \int_0^1 |q - \tilde{q}| \le \frac{2\max_j \overline{|q|}_j}{n^2\pi^2}\|q\|_1 + \|q - \tilde{q}\|_1. \end{aligned}$$

Then we have, for each $r \geq 1$, from (3.12),

$$
\begin{aligned}
&\|2n^2\pi^2[n\tilde{L}(x)-1]-\tilde{q}\|_r \\
&\quad = \left\|2\hat{\ell}\hat{W}-\tilde{q}\right\|_r \\
&\quad \leq |\lambda - n^2\pi^2| + \frac{4}{3}\Big\|\sum_{j=1}^{J}\frac{\chi_{[x_{k_{j-1}},x_{k_j})}(x)}{\delta x_{k_{j-1}}}\int_{x_{k_{j-1}}}^{x_{k_j}}|q-\overline{q}_j|\Big\|_r \\
&\qquad + \frac{8}{3n^2\pi^2}\Big\|\Big(\overline{|q|}+\frac{5}{4}\int_0^1|q|dx\Big)^2\Big\|_r \\
&\quad \leq \frac{2\max_j\overline{|q|}_j}{n^2\pi^2}\|q\|_1 + \|q-\tilde{q}\|_1 + \frac{4}{3}\|q-\tilde{q}\|_r + \frac{8}{3n^2\pi^2}\left(\frac{81}{16}\max_j\overline{|q|}_j\|q\|_r\right) \\
&\quad \leq \frac{7}{3}\|q-\tilde{q}\|_r + \frac{31\max_j\overline{|q|}_j}{2n^2\pi^2}\|q\|_r\ .
\end{aligned}
$$

In the next to last line above, the following estimate is used. This estimate is itself a consequence of Hölder's inequality.

$$
\begin{aligned}
\left\|\overline{|q|}^2\right\|_r &= \left[\sum_{j=1}^{J}\left(\frac{1}{\delta x_{k_{j-1}}}\int_{x_{k_{j-1}}}^{x_{k_j}}|q|\right)^{2r}\delta x_{k_{j-1}}\right]^{\frac{1}{r}} \\
&\leq \max_j\overline{|q|}_j\left[\sum_{j=1}^{J}\left(\frac{1}{\delta x_{k_{j-1}}}\int_{x_{k_{j-1}}}^{x_{k_j}}|q|\right)^{r}\delta x_{k_{j-1}}\right]^{\frac{1}{r}} \\
&\leq \max_j\overline{|q|}_j\left[\sum_{j=1}^{J}\int_{x_{k_{j-1}}}^{x_{k_j}}|q|^r\right]^{\frac{1}{r}} \\
&= \max_j\overline{|q|}_j||q||_r.
\end{aligned}
$$

This completes the proof. □

We remark that the quadratic term cannot be removed because when q is piecewise constant, $2\pi^2n^2(n\tilde{L}_n-1)$ is not the exact solution.

4. Error estimates for Tikhonov regularization method

We first study a variation of a potential with respect to its nodal set; that is, we want to estimate the difference q_1-q_2 in terms of the difference $\mathbf{z}(n,q_1)-\mathbf{z}(n,q_2)$. Then we shall use it to prove Theorem 2. For definiteness, we shall use the L^2 space for potentials and the Euclidean distance for nodal sets (points in $\mathbb{R}^{n+1}$). Since it is impossible to determine completely a potential from a single given nodal set, we supply the rest of the information by an a priori bound on the H^1 norms of the potential.

Given $q_1, q_2 \in \mathbf{Q}$ and n a positive integer such that

$$\int_0^1|q_1|dx \leq \frac{n\pi^2}{4}, \qquad \int_0^1|q_2|dx \leq \frac{n\pi^2}{4},$$

set $\mathbf{x} := \mathbf{z}(n, q_1)$, $\mathbf{y} := \mathbf{z}(n, q_2)$ and

$$P_i = V_i - \bar{V}_i, \quad \text{where } V_1 = -\sum_{i=0}^{n-1} \frac{\pi^2 \chi_{[x_i, x_{i+1})}}{(\delta x_i)^2}, \quad V_2 = -\sum_{n=0}^{n-1} \frac{\pi^2 \chi_{[y_i, y_{i+1})}}{(\delta y_i)^2}.$$

Set $|\delta(\mathbf{x}-\mathbf{y})|^2 := \sum_i |\delta x_i - \delta y_i|^2 = \sum_i |(x_{i+1}-x_i) - (y_{i+1}-y_i)|^2$. Then we have

Theorem 4.1. *There is a universal constant $C > 0$ such that*

$$\int_0^1 |q_1 - q_2|^2 \leq C\Big\{n^5 |\delta(\mathbf{x}-\mathbf{y})|^2 + \frac{1}{n^2}\int_0^1 (\dot{q}_1^2 + \dot{q}_2^2)\,dx\Big\}, \tag{4.1}$$

$$\|V_1 - V_2\|^2_{L^2(0,1)} \leq C\Big\{n^5 |\delta(\mathbf{x}-\mathbf{y})|^2 + \frac{1}{n^2}\int_0^1 (\dot{q}_1^2 + \dot{q}_2^2)\,dx\Big\}, \tag{4.2}$$

$$n^5 |\delta(\mathbf{x}-\mathbf{y})|^2 \leq C\Big\{\int_0^1 |q_1 - q_2|^2 dx + \frac{1}{n^2}\int_0^1 (\dot{q}_1^2 + \dot{q}_2^2)\,dx\Big\}. \tag{4.3}$$

Proof. First, (4.1) follows from (3.6) and (4.2). Indeed, writing $\|\cdot\|_{L^2}$ as $\|\cdot\|$,

$$\|q_1 - q_2\| \leq \|q_1 - \tilde{q}_1\| + \|q_2 - \tilde{q}_2\| + \|\tilde{q}_2 - P_2\| + \|\tilde{q}_1 - P_1\| + \|P_1 - P_2\|.$$

The term $\|P_1 - P_2\|$ can be bounded by $\|V_1 - V_2\|$, whereas the term $\|q_i - \tilde{q}_i\|$ can be bounded by $\|\dot{q}_i\|_{L^2}$, using the following Poincaré inequality:

$$\int_a^b |q - \overline{q}|^2 dx \leq \frac{4(b-a)^2}{\pi^2}\int_a^b \dot{q}^2(x)dx \quad \forall\, q \in H^1((a,b)), a < b.$$

For (4.2), we use the fact that $\max_i\{|x_i - i/n|, |y_i - i/n|\} \leq 1/(4n - 5/4)$ which implies $[x_i, x_{i+1}] \subset [y_{i-1}, y_{i+2}]$ for each i (here for simplicity $y_{-1} := -1/n, y_{n+1} := 1 + 1/n$). Denote

$$\begin{aligned}
&\delta_i^+ = \max\{0, y_i - x_i\}, \qquad \delta_i^- = \max\{0, x_i - y_i\}, \quad \delta_i = |x_i - y_i|,\\
&\delta^* = \max\{\delta^*\mathbf{x}, \delta^*\mathbf{y}\} = \max_i\{\delta x_i, \delta y_i\},\\
&\delta_* = \min\{\delta_*\mathbf{x}, \delta_*\mathbf{y}\} = \min_i\{\delta x_i, \delta y_i\}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{1}{\pi^4}\|V_1 - V_2\|^2_{L^2} &= \sum_i \int_{x_i}^{x_{i+1}} \Big|\frac{1}{\delta x_i^2} - \sum_{j=i-1}^{i+1} \frac{\chi_{[y_j, y_{j+1})}}{\delta y_j^2}\Big|^2 dx\\
= \quad & \sum_i \Big|\frac{1}{\delta x_i^2} - \frac{1}{\delta y_i^2}\Big|^2 (\delta x_i - \delta_i^+ - \delta_{i+1}^-)\\
&+ \sum_i \Big|\frac{1}{\delta x_i^2} - \frac{1}{\delta y_{i-1}^2}\Big|^2 \delta_i^+ + \sum_i \Big|\frac{1}{\delta y_{i+1}^2} - \frac{1}{\delta x_i^2}\Big|^2 \delta_{i+1}^-\\
\leq \quad & \sum_i \Big|\frac{1}{\delta x_i^2} - \frac{1}{\delta y_i^2}\Big|^2 (\delta x_i + \delta_i^+ + \delta_{i+1}^-)\\
&+ 2\sum_i \Big|\frac{1}{\delta y_i^2} - \frac{1}{\delta y_{i-1}^2}\Big|^2 \delta_i^+ + 2\sum_i \Big|\frac{1}{\delta y_{i+1}^2} - \frac{1}{\delta y_i^2}\Big|^2 \delta_{i+1}^-\\
\leq \quad & 2\sum_i \Big|\frac{1}{\delta x_i^2} - \frac{1}{\delta y_i^2}\Big|^2 \delta x_i + 2\sum_i \Big|\frac{1}{\delta y_{i-1}^2} - \frac{1}{\delta y_i^2}\Big|^2 \delta_i
\end{aligned}$$

by using $(a+b)^2 \leq 2(a^2 + b^2)$. The first part can be easily estimated by

$$\sum_i \Big|\frac{1}{\delta x_i^2} - \frac{1}{\delta y_i^2}\Big|^2 \delta x_i \leq \frac{4}{\delta_*^5}\sum_i |\delta x_i - \delta y_i|^2.$$

To estimate the second term we use the Sturm comparison theorem on zeros to obtain

$$\min_{[y_i,y_{i+1}]}(\lambda(n,q_2)-q_2) \le \frac{\pi^2}{\delta y_i^2} \le \max_{[y_i,y_{i+1}]}(\lambda(n,q_2)-q_2).$$

Thus

$$\begin{aligned}\Big|\frac{\pi^2}{\delta y_{i-1}^2}-\frac{\pi^2}{\delta y_i^2}\Big|^2 &\le \Big|\max_{[y_{i-1},y_{i+1}]} q_2 - \min_{[y_{i-1},y_{i+1}]} q_2\Big|^2\\ &\le \Big(\int_{y_{i-1}}^{y_{i+1}}|\dot q_2|dy\Big)^2 \le (y_{i+1}-y_{i-1})\int_{y_{i-1}}^{y_{i+1}}\dot q_2^2 dy.\end{aligned}$$

Therefore, $\delta_i \le \delta^*$, and by (3.10),

$$\begin{aligned}\|V_1-V_2\|^2_{L^2(0,1)} &\le \frac{8\pi^4}{\delta_*^5}|\delta(\mathbf{x}-\mathbf{y})|^2 + 8\delta^{*2}\int_0^1|\dot q_2|^2dy\\ &\le \frac{(3n)^5}{4}|\delta(\mathbf{x}-\mathbf{y})|^2+\frac{128}{9n^2}\int_0^1|\dot q_2|^2dy.\end{aligned}$$

This proves (4.2).

The proof of (4.3) follows from the observation that

$$\frac{4}{(\delta^*)^5}\|\delta(\mathbf{x}-\mathbf{y})\|^2 \le \sum\Big|\frac{1}{\delta x_i^2}-\frac{1}{\delta y_i^2}\Big|^2\delta x_i \le \|V_1-V_2\|^2\max_i\frac{\delta x_i}{\delta x_i-\delta_i^+-\delta_{i+1}^-}.$$

Also,

$$\begin{aligned}\|V_1-V_2\| \le\ &\|q_1-q_2\|+|\lambda_1-(n\pi)^2|+|\lambda_2-(n\pi)^2|\\ &+\|\lambda_1+V_1-\tilde q_1\|+\|\lambda_2+V_2-\tilde q_2\|+\|q_1-\tilde q_1\|+\|q_2-\tilde q_2\|.\end{aligned}$$

By Theorems 3.2 and 3.3, each term on the right-hand side except the first term can be bounded by $\|\dot q_1\|+\|\dot q_2\|$. □

Proof of Theorem 1.2. Recall that $\|\dot p_\varepsilon\| \le \|\dot q\|$. Then notice that, since the average of q is zero,

$$\int_0^1|q|dx \le \Big(\int_0^1 q^2dx\Big)^{1/2} \le \Big(\frac{4}{\pi^2}\int_0^1\dot q^2dx\Big)^{1/2} \le \frac{n\pi^2}{4}.$$

The same estimates also hold for p_ε. Note that $\|\delta(\mathbf{x}-\mathbf{y})\|^2 \le 2|\mathbf{x}-\mathbf{y}|^2$. Thus, we can apply (4.1) to conclude that

$$\begin{aligned}\int_0^1|p_\varepsilon-q|^2dx &\le Cn^5|\mathbf{z}(n,p_\varepsilon)-\mathbf{x}|^2+\frac{C}{n^2}\int_0^1(\dot p_\varepsilon^2+\dot q^2)dx\\ &= Cn^5\Big(\varepsilon\mathbf{E}(\mathbf{x},p_\varepsilon)-\varepsilon\int_0^1\dot p_\varepsilon^2\Big)+\frac{C}{n^2}\int_0^1(\dot p_\varepsilon^2+\dot q^2)dx\\ &\le \frac{C}{n^2}\Big\{\big(n^7\varepsilon+1\big)\int_0^1\dot q^2dx+\big(1-n^7\varepsilon\big)\int_0^1\dot p_\varepsilon^2dx\Big\}\\ &\le C\left(\frac{1}{n^2}+\varepsilon n^5\right)\int_0^1\dot q^2dx.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|p_\varepsilon^\alpha-q\|^2 &\le 2\|p_\varepsilon^\alpha-q^\alpha\|^2+2\|q^\alpha-q\|^2\\ &\le C\Big(\frac{1}{n^2}+\varepsilon n^5\Big)\|\dot q^\alpha\|^2+2\|q^\alpha-q\|^2,\end{aligned}$$

where

$$\begin{aligned}\|q^\alpha - q\|^2 &\le C\Big(n^5|\mathbf{z}(n,q^\alpha)-\mathbf{x}|^2 + \frac{1}{n^2}(\|\dot q^\alpha\|^2 + \|\dot q\|^2)\Big)\\ &\le C\Big(n^5\alpha^2 + \frac{1}{n^2}(\|\dot q^\alpha\|^2 + \|\dot q\|^2)\Big).\end{aligned}$$

Therefore,

$$\|p^\alpha_\varepsilon - q\|^2 \le 2Cn^5\alpha^2 + \frac{2C}{n^2}\|\dot q\|^2 + C\left(\frac{3}{n^2}+\varepsilon n^5\right)\|\dot q^\alpha\|^2.$$

□

5. Appendix

Here we derive the asymptotic expansion (1.2). Consider the equation $\ddot u + Vu = 0$, where V is positive and large. Suppose $\mathbf{x} = (x_0, x_1, \cdots, x_n)$ is the nodal set of u. The modified Prüfer transformation $u = R(x)\sin\theta(x), \dot u = mR(x)\sin\theta(x)$, where m is a large positive constant to be chosen at our convenience, gives the equivalent form

$$1 = \frac{\dot\theta}{m} + \frac{p(x)\,\sin^2\theta}{m^2}, \qquad \theta(x_k) = k\pi\ \ \forall k, \qquad p(x) := m^2 - V(x).$$

In each interval $I_k = [x_k, x_{k+1}]$, we shall take the constant m to be close to $\sqrt{V(x_k)}$, so m is a large constant. Then p is not very large in I_k, so the equation can be regarded as a small perturbation from the equation $\theta' = m$. We shall first derive an expansion for a general m, and then take a particular m to make the leading error term vanish.

Integrating the equation over $I_k := [x_k, x_{k+1}]$ and using $dx = \frac{d\theta}{m} + \frac{p\,\sin^2\theta\,dx}{m^2}$ we obtain

$$\begin{aligned}
x_{k+1} - x_k - \frac{\pi}{m} &= \int_{I_k} \frac{p\,\sin^2\theta}{m^2}dx = \int_{I_k}\frac{p\,\sin^2\theta}{m^2}\Big\{\frac{d\theta}{m} + \frac{p\,\sin^2\theta dx}{m^2}\Big\}\\
&= \frac{[2\theta-(2k+1)\pi - \sin(2\theta)]p}{4m^3}\Big|_{x_k}^{x_{k+1}} + \int_{I_k}\frac{p^2\sin^4\theta}{m^4}\,dx\\
&\quad + \int_{I_k}\frac{[(2k+1)\pi - 2\theta + \sin(2\theta)]\dot p}{4m^3}\Big\{\frac{d\theta}{m} + \frac{p\,\sin^2\theta dx}{m^2}\Big\}\\
&= \frac{[p(x_k)+p(x_{k+1}]\pi}{4m^3} + \frac{1}{m^4}\int_{I_k}\Big\{p^2\sin^4\theta - \tfrac14[(\theta-k\pi)([k+1]\pi-\theta)+\sin^2\theta]\ddot p\Big\}dx\\
&\quad - \frac{1}{m^5}\int_{I_k}[2\theta-(2k+1)\pi-\sin(2\theta)]\dot p\,p\,\sin^2\theta dx\\
&= \frac{[p(x_k)+p(x_{k+1}]\pi}{4m^3} + \frac{p^2(\frac12[x_k+x_{k+1}])}{m^5}\int_0^\pi \sin^4\theta\,d\theta\\
&\quad - \frac{\ddot p(\frac12[x_k+x_{k+1}])}{4m^5}\int_0^\pi\Big\{(\theta-k\pi)([k+1]\pi-\theta)+\sin^2\theta\Big\}d\theta + \frac{O(1)}{m^6}\\
&= \frac{[p(x_k)+p(x_{k+1})]\pi}{4m^3} + \frac{3\pi}{8m^5}\Big\{p^2(\tfrac12[x_k+x_{k+1}]) - \tfrac{3+\pi^2}{9}\ddot p(\tfrac12[x_k+x_{k+1}])\Big\} + \frac{O(1)}{m^6}.
\end{aligned}$$

Here $O(1)$ depends on the bounds of $p := m^2 - V$ and its derivatives.

1. Now we take $m = m_k$ such that $p(x_k) + p(x_{k+1}) = 0$. This yields

$$x_k - x_{k-1} = \frac{\pi}{m_k} + \frac{O(1)}{m_k^5}, \qquad m_k := \sqrt{\frac{V(x_k) + V(x_{k+1})}{2}}.$$

We remark that if one takes other choices of m^2, say $m^2 = V(\frac{1}{2}[x_k + x_{k+1}])$ or the average of V over $[x_k, x_{k+1}]$, then the remainder term in general has only the order of $O(1)m^{-4}$.

Then the previous expansion can be written as

$$\sqrt{\frac{2\pi^2}{V(x_k) + V(x_{k+1})}} = (x_{k+1} - x_k)\Big\{1 + O(1)(x_{k+1} - x_k)^4\Big\}.$$

This provides a reconstruction formula

$$\frac{V(x_k) + V(x_{k+1})}{2} = \Big(\frac{\pi}{x_{k+1} - x_k}\Big)^2 \Big\{1 + O(1)(x_{k+1} - x_k)^4\Big\} \qquad \forall k.$$

2. Suppose $V = \lambda - q$, where λ is a large constant and q is a smooth and bounded function. Then the above gives the expansion

$$x_{k+1} - x_k = \frac{\pi}{\sqrt{\lambda}}\Big\{1 + \frac{q_k + q_{k+1}}{4\lambda} + \frac{O(1)}{\lambda^2}\Big\}.$$

Assume further that $x_n - x_0 = 1$ and $\int_{x_0}^{x_n} q\,dx = 0$. Taking the sum of the above expansions from $k = 0$ to $n-1$ we then obtain successively the following information:

$$1 = \frac{n\pi}{\sqrt{\lambda}}\Big\{1 + o(1)\Big\}, \qquad 1 = \frac{n\pi}{\sqrt{\lambda}}\Big\{1 + \frac{O(1)}{n^2}\Big\}, \quad x_k - x_{k-1} = \frac{1}{n} + \frac{O(1)}{n^3},$$

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{q(x_k) + q(x_{k+1})}{2n} &= \sum_{k=0}^{n-1} (x_{k+1} - x_k)\frac{q(x_k) + q(x_{k+1})}{2} + \sum_{k=0}^{n-1} \frac{O(1)}{n^3} \\ &= \int_{x_0}^{x_n} q(x)dx + \sum_{k=0}^{n-1} O([x_{k+1} - x_k]^3) + \frac{O(1)}{n^2} = \frac{O(1)}{n^2}. \end{aligned}$$

It then follows that

$$1 = \frac{n\pi}{\sqrt{\lambda}}\Big\{1 + \frac{O(1)}{n^4}\Big\}, \qquad \sqrt{\lambda} = n\pi\Big\{1 + \frac{O(1)}{n^4}\Big\}.$$

Consequently, we have the following asymptotic expansions:

$$\begin{aligned} x_{k+1} - x_k &= \frac{1}{n} + \frac{q(x_k) + q(x_{k+1})}{2n^3\pi^2} + \frac{O(1)}{n^5} \\ &= \frac{1}{n} + \frac{q(\frac{1}{2}[x_k + x_{k+1}])}{4n^3\pi^2} + \frac{O(1)}{n^5} \\ &= \frac{1}{n} + \frac{1}{2n^3\pi^2}\Big(\frac{1}{x_{k+1} - x_k}\int_{x_k}^{x_{k+1}} q(x)dx\Big) + \frac{O(1)}{n^5}. \end{aligned}$$

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