TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 363, Number 9, September 2011, Pages 4997–5020
S 0002-9947(2011)05328-6
Article electronically published on April 14, 2011

# BLOCK DIAGONALIZATION AND 2-UNIT SUMS OF MATRICES OVER PRÜFER DOMAINS

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Abstract. We show that matrices over a large class of Prüfer domains are equivalent to "almost diagonal" matrices, that is, to matrices with all the nonzero entries congregated in blocks along the diagonal, where both dimensions of the diagonal blocks are bounded by the size of the class group of the Prüfer domain. This result, a generalization of a 1972 result of L. S. Levy for Dedekind domains, implies that, for $n$ sufficiently large, every $n \times n$ matrix is a sum of two invertible matrices. We also generalize from Dedekind to certain Prüfer domains a number of results concerning the presentation of modules and the equivalence of matrices presenting them, and we uncover some connections to combinatorics.

## 0. Introduction

This article deals with a number of topics: block diagonalization of matrices, writing matrices as a sum of two invertibles, combinatorial problems arising from these, and presentations of modules by matrices, mostly over Prüfer domains.

The original motivation for this investigation was the *unit sum number problem* for matrix rings: Given an associative ring $R$ with identity, not necessarily commutative, determine the *unit sum number* of $R$—the least positive integer $k$, if one exists, so that every element of $R$ is a sum of $k$ units. This problem has been investigated in a substantial number of articles dating back to the 1950s. A brief summary of the relevant history of the unit sum number problem appears in [29].[1] Rings of unit sum number 2 are called "2-good". In 1954, D. Zelinsky showed that the ring of linear transformations of a vector space $V$ of any dimension is 2-good *unless* $V$ is the field of two elements [31]. The ring $\mathbb{Z}$ of integers does *not* have a finite unit sum number, but the ring of $n \times n$ *proper* (i.e., $n \geq 2$) matrices over $\mathbb{Z}$ is 2-good; cf. Fact 4.1. Moreover, for every associative ring, every proper $n \times n$

Received by the editors September 18, 2009 and, in revised form, October 21, 2009 and February 18, 2010.

2010 *Mathematics Subject Classification.* Primary 13F05, 13F07, 13G05, 13A15, 13C05, 13C20, 15A21, 16U60.

*Key words and phrases.* Matrices, sums of units, Prüfer domains, monoids, Picard group, class group.

The authors thank the Universities of Exeter and Nebraska for their hospitality and the University of Nebraska Research Council and the London Mathematical Society for providing funds so that the first author could spend about one month at Nebraska in the fall of 2005, when this research was begun. We thank Roger Wiegand and Lee Klingler for their helpful suggestions.

[1]Some references in addition to those cited here are [1], [6], [8], [10], [11], [12], [15], [25], [26], [27] and [28].

matrix is a sum of 3 invertible matrices, by a result of Henriksen and Kaplansky; cf. Fact 4.2, [14].[2] The question now is: Which matrix rings are 2-good?

In his 1972 paper concerning the "almost diagonalization" of matrices [22], L. S. Levy shows that every matrix over a Dedekind domain with finite class group of size $b$ is *equivalent* to a *block diagonal* matrix $B$: All entries of $B$ are zero except for rectangular blocks along a "block diagonal", where both dimensions of the nonzero blocks are at most $b$ (these terms are defined more precisely in (1.1)). Our original plan was to show that sufficiently large block diagonal matrices are sums of two units, and then use Levy's result to obtain for a Dedekind domain with finite class group (e.g. a ring of algebraic integers) that sufficiently large matrix rings are 2-good.

In trying to understand Levy's result we ended up streamlining and generalizing it to a wider class of integral domains, namely *Prüfer* domains with the *stacked bases* and $1\frac{1}{2}$-*generator* properties, terms defined in Definitions 2.1 and 2.3. The association of a module to a matrix — now familiar material in graduate algebra courses — goes back to the work of Steinitz [27, 28] and Krull [19] in the early twentieth century and was later given more explicitly by Fitting in [7]; for a modern treatment see [30]. This interplay between matrices and finitely presented modules "named" (or "presented") by the matrices is used in Sections 2 and 3 to solve our matrix problems. Where Levy used invariants arising from the primary decomposition we use invariants from a canonical form for finitely presented modules over our Prüfer domains, as well as the additive invariant $\mathrm{K}_0$ extended to finitely presented modules. For our domains the association between matrices and modules is quite precise: If $r$ and $c$ are nonnegative integers, then every pair of $r \times c$ matrices that name isomorphic modules are equivalent; cf. Theorem 3.5. We prove that an *Invariant Factor Theorem* holds for our Prüfer domains; cf. Corollary 3.4. This is related to a 1987 question of Brewer and Klingler concerning what conditions imply or are implied by such an Invariant Factor Theorem; cf. Remark 3.3, [3].

One of our main results, the *Matrix Naming Theorem*, Theorem 3.8, a generalization of Levy's "separated divisor theorem", answers the question: For which dimensions $r, c$, does there exist an $r \times c$ matrix that names a particular finitely presented module? Our other main theorem is the *Bounded Block Decomposition Theorem*, Theorem 3.12: If a Prüfer domain $D$ has the stacked bases and $1\frac{1}{2}$-generator properties and if the order of its class group is $b$, where $b < \infty$, then every matrix over $D$ is equivalent to a block diagonal matrix such that both dimensions of each nonzero block are at most $b$.

In Section 4 we return to our original problem of writing a matrix as a sum of two units. Diagonal proper matrices can always be written as a sum of two units. Our Lemma 4.3 gives a generalization: A square matrix of size at least two that avoids a permutation matrix is also a sum of two units. The combinatorial problem of finding a permutation matrix that avoids a given matrix has been researched extensively, starting with the work of Kaplansky and Riordan [18] and Joni and Rota [16]. We give a flavor of this in our Propositions 4.4 and 5.7. By applying Lemma 4.3 to the block diagonal form of the bounded block decomposition theorem, we finally obtain the result we first sought, but now for our class of Prüfer domains: If the class group has finite order $b$ and if $n \geq 2b$, then every $n \times n$ matrix is a sum of two units (Theorem 4.7). Levy's paper provides examples of Dedekind rings

[2]See also [29] for a different constructive proof.

with class group of size $b$ and indecomposable matrices of size equal to the order of the class group; thus the bound of our Theorem 3.12 is sharp for the sizes of indecomposable blocks. Is the bound sharp for sums of two units? We do not know. Conceivably, by appropriate use of such indecomposable blocks, we could obtain square matrices of sizes less than $2b$ that are not sums of two units.

Our final section, Section 5, gives pointers to open problems in this area and collects a few additional results. We describe in Part I a connection between unit sum numbers for square matrices and additive monoids. For each ring $R$, the set $\mathrm{USM}(R) := \{n > 0 \mid \text{ every } n \times n \text{ matrix is a sum of 2 units } \} \cup \{0\}$ is an *additive monoid*—an additively closed subset of $\mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the natural numbers joined with 0. These structures have been attracting increased interest recently. Question: Which monoids can occur for $R$ a ring of algebraic integers, a Dedekind or a Prüfer ring? The matrix results stated above give some information on the monoid for our class of Prüfer rings. Part II concerns *banded* matrices—all nonzero entries are congregated in a diagonal band around the main diagonal. An analogous lower bound on the size of a banded square matrix ensures it is a sum of two invertible matrices. We describe a relationship between banded and blocked matrices.

As we outline above, Section 1 contains basic definitions and terminology regarding matrices, and some basic properties and results about Prüfer domains and finitely presented modules properties are in Section 2. Our main results are in Sections 3 and 4.

All rings are associative with identity and modules are left modules. In Sections 2 and 3, and most of Section 4, the rings are commutative integral domains.

## 1. Matrix definitions and associated modules

For our purposes in this paper, we use the following matrix terminology:

**Definitions 1.1.** Let $R$ be an associative ring and let $A$ be an $r \times c$ matrix ($r$ rows, $c$ columns), where $r$, $c \in \mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$.

- The *type* of $A$ is the pair $(r, c)$, and the *size* of $A$ is $\max(r, c)$.
- A matrix with no rows or no columns ($r = 0$ or $c = 0$) is called an *empty matrix* of type $(r, c)$ and may be denoted by $[\varnothing]_{r\times c}$ or just $\varnothing$.
- Two $r \times c$ matrices $A$ and $B$ over $R$ are *equivalent*, written $A \sim B$, if there exist invertible matrices $P_{r\times r}$ and $Q_{c\times c}$ over $R$ so that $PAQ = B$.
- Let $A_1$ and $A_2$ be matrices of types $(r_1, c_1)$ and $(r_2, c_2)$, respectively. The *block diagonal sum* of $A_1$ and $A_2$ is the *block diagonal* matrix
$$\mathrm{diag}(A_1, A_2) = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$
of type $(r_1 + r_2, c_1 + c_2)$. For $t \geq 2$, the block diagonal sum is written as
$$\mathrm{diag}(A_1, \dots, A_t).$$
- A matrix of positive size is *indecomposable* if it is not equivalent to the diagonal sum of two matrices of positive sizes; otherwise it is *decomposable*.

*Remarks* 1.2. (1) Adding an empty matrix as a block diagonal summand to a matrix has the effect of adding zero rows or zero columns to this matrix; cf. Example 1.3(2) below.

(2) Every matrix of positive size is equivalent to a block diagonal sum of indecomposable matrices.
(3) Matrices of type $(0,1),(1,0)$, and those of type $(1,1)$ with a nonzero entry, are indecomposable.
(4) $[0]_{1\times 1} = \operatorname{diag}([\varnothing]_{0\times 1}, [\varnothing]_{1\times 0})$, and so it is decomposable. (This $1\times 1$ zero matrix is the result of filling out with zeroes a block of no rows and 1 column and a block of 1 row and no columns.)

**Examples 1.3.** For $A = \operatorname{diag}(A_1, \dots, A_t)$, the diagonal entries of the matrix $A$ are not necessarily within the "diagonal blocks", even if the matrix is square.

(1) The matrix $A$ below is a block diagonal sum of blocks of size $\leq 2$:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} = \operatorname{diag}(A_1, A_2, A_3),$$

where $A_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{2\times 1}$, $A_2 = [1]$ and $A_3 = \begin{bmatrix} 2 & 1 \end{bmatrix}_{1\times 2}$.

(2) For $A_1$ and $A_3$ as above and $[\varnothing]_{0\times 2}$ an empty matrix,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \end{bmatrix} = \operatorname{diag}(A_1, A_3, [\varnothing]_{0\times 2}).$$

It is easiliy seen that for matrices $A_1, A_2$ the block diagonal sums $\operatorname{diag}(A_1, A_2)$ and $\operatorname{diag}(A_2, A_1)$ are equivalent via suitable permutations of rows and columns. This leads to the following result, the formal proof of which can be safely left to the reader.

**Lemma 1.4.** *Let $t > 1$ be an integer, $\pi$ a permutation on $\{1, \dots, t\}$ and let $A_1, \dots, A_t$ be matrices. Then the block diagonal sums $X = \operatorname{diag}(A_1, \dots, A_t)$ and $X_\pi = \operatorname{diag}(A_{\pi(1)}, \dots, A_{\pi(t)})$ are equivalent. In fact, there exist permutation matrices $P$ and $Q$ (necessarily invertible) such that $PXQ = X_\pi$.* □

**Definitions and Remarks 1.5.** If $M$ is an $R$-module and $r$ an integer, then $M^{(r)}$ denotes the direct sum of $r$ copies of $M$ if $r > 0$ and the zero module if $r \leq 0$. For an associative ring $R$, there is a bijection from the set of matrices of finite size with entries in $R$ to the set of homomorphisms between finitely generated free $R$-modules: An $r \times c$ matrix $A$ defines a homomorphism

$$\alpha\colon R^{(r)} \longrightarrow R^{(c)},$$

via right multiplication by $A$. If $\boldsymbol{x} \in R^{(r)}$, then $x$ is a $1 \times r$ row matrix, and so $\alpha\colon \boldsymbol{x} \longrightarrow \boldsymbol{x}A \in R^{(c)}$. Every homomorphism $R^{(r)} \longrightarrow R^{(c)}$ is given by a matrix.

For $A$ and $\alpha : R^{(r)} \longrightarrow R^{(c)}$ corresponding to $A$ as above, we define $M(A)$, the *finitely presented module named by* $A$, to be the cokernel of the homomorphism $\alpha$, that is, $M(A) := \operatorname{Coker} A = R^{(c)}/(R^{(r)}A)$. We have the associated exact *cokernel sequences* below, where $\theta$ is the inclusion mapping and $\delta$ is the natural surjection:

$$\begin{array}{lllll} & R^{(r)} & \xrightarrow{A_{r\times c}} R^{(c)} & \xrightarrow{\delta} M(A) := \operatorname{Coker} A & \longrightarrow 0; \\ 0 \longrightarrow & R^{(r)}A & \xhookrightarrow{\theta} R^{(c)} & \xrightarrow{\delta} \qquad M(A) & \longrightarrow 0. \end{array} \tag{1.5.1}$$

We note the following facts:

(1) Since every map $\alpha : R^{(r)} \longrightarrow R^{(c)}$ is given by a matrix, every finitely presented module is isomorphic to an $M(A)$, for some matrix $A$.
(2) If $A$ is equivalent to $B$, then $M(A) \cong M(B)$.
(3) The converse of (2) is not true. For a comprehensive account of this see Warfield [30].
(4) If $A$ and $A'$ are two matrices, then $M(\mathrm{diag}(A, A')) \cong M(A) \oplus M(A')$.
(5) The sequences $0 \longrightarrow R^{(c)}$ and $R^{(r)} \longrightarrow 0$ have associated matrices $[\varnothing]_{0\times c}$ and $[\varnothing]_{r\times 0}$, and so these empty matrices name $R^{(c)}$ and 0, respectively.

## 2. Finitely presented modules over certain Prüfer domains

In this section all rings are commutative; usually they are integral domains. The goal of this section is to obtain a class of integral domains for which the finitely presented modules are determined by a complete set of invariants.

**Definition 2.1.** A *Prüfer domain* is a commutative integral domain $R$ such that every nonzero finitely generated ideal is invertible. Equivalently, the set of finitely generated fractional ideals is a group under multiplication. A *fractional* ideal is a submodule $J$ of the quotient field of $R$ such that (a) $J \neq 0$, and (b) $cJ \subseteq R$ for some nonzero $c \in R$.

*Remarks* 2.2 ([9, Theorem 1.1, p. 91]). Let $R$ be an integral domain. Then the following statements are equivalent:

(1) $R$ is Prüfer.
(2) For every nonzero ideal $I$, $I$ is finitely generated if and only if $I$ is invertible.
(3) For every maximal ideal $P$ of $R$, the localization $R_P$ is a valuation domain.

(A *valuation ring* is one for which the ideals form a chain under inclusion.) For other properties of Prüfer domains see [9].

For the convenience of the reader, we define the special types of Prüfer domains that we need. Their properties yield useful descriptions of finitely presented modules, particularly a uniqueness of decomposition of finitely presented modules.

**Definitions 2.3** ([9, Chapter V.4]). [3] Let $R$ be a Prüfer domain.

- $R$ has the *stacked bases property* provided, for every finitely generated free module $F$ and finitely generated submodule $H$, there exist (i) a basis $x_1, \dots, x_n$ for $F$ and (ii) invertible ideals $J_1, \dots, J_n, I_1, \dots, I_s$, where $0 \le s \le n$ and $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_s \subseteq R$, such that
$$F = x_1J_1 \oplus \cdots \oplus x_nJ_n, \quad H = x_1I_1J_1 \oplus \cdots \oplus x_sI_sJ_s\,.$$
This is also known as the *simultaneous basis property.*
- The *Steinitz property* on ideals is that $I_1 \oplus \cdots \oplus I_k \cong R^{(k-1)} \oplus I_1 \cdots I_k$, for every nonempty finite set $I_1, \dots, I_k$ of invertible ideals of $R$.
- $R$ is a *Bézout domain* if every finitely generated ideal is principal.
- The $1\frac{1}{2}$*-generator property*, also called *almost Bézout*, is that, for every nonzero finitely generated ideal $I$ of $R$ and every nonzero $a \in I$, there exists $b \in I$ such that $a, b$ generate $I$.

**Proposition 2.4** ([17] or [9, V.4.10]). *Let $R$ be an integral domain and let $U_1, U_2, \dots, U_m, V_1, \dots, V_n$ be nonzero ideals of $R$.*

[3] Fuchs and Salce's book gives examples, explanations, and other related definitions.

(1) *If* $U_1 \oplus U_2 \oplus \cdots \oplus U_m \cong V_1 \oplus V_2 \oplus \cdots \oplus V_n$*, then* $U_1 \cdot U_2 \cdots U_m \cong V_1 \cdot V_2 \cdots V_n$ *and* $m = n$*.*
(2) *If* $R$ *is also a Prüfer domain with the* $1\frac{1}{2}$*-generator property,* $m = n$ *and* $U_1 \cdot U_2 \cdots U_m \cong V_1 \cdot V_2 \cdots V_n$*, then* $U_1 \oplus U_2 \oplus \cdots \oplus U_m \cong V_1 \oplus V_2 \oplus \cdots \oplus V_n$*. Thus* $R$ *has the Steinitz property in this case.*

The following theorem and proof are essentially in [9]. Although the hypotheses stated in [9] are stronger than the conditions stated below, only the latter are actually used in the proof. For the sake of completeness we include a proof here. For $M$ a finitely generated $R$-module, we let $\operatorname{gen}(M)$ denote the minimal number of generators of $\mathcal{T}(M)$.

**Theorem 2.5** (**Canonical form**, [9, Theorem V.4.12])**.** *Let* $R$ *be a Prüfer domain that has the stacked bases property and the* $1\frac{1}{2}$*-generator property. Let* $M$ *be a finitely presented* $R$*-module and let* $\mathcal{T}(M)$ *denote the torsion submodule of* $M$*; put* $\mathcal{TF}(M) = M/\mathcal{T}(M)$*, the torsion-free part of* $M$*. Then*

(1) $M \cong \mathcal{T}(M) \oplus \mathcal{TF}(M)$.
(2) *There exist invertible ideals* $I_1, \ldots, I_t$ *of* $R$ *such that* $I_1 \subseteq \cdots \subseteq I_t \neq R$ *and*
$$\mathcal{T}(M) \cong (R/I_1) \oplus \cdots \oplus (R/I_t)\ .$$
(3) *If* $\mathcal{TF}(M) \neq (0)$*, there is an invertible ideal* $J$ *such that*
$$\mathcal{TF}(M) \cong J \oplus R^{(\operatorname{rk}(M)-1)}\ .$$
(4) *The integer* $t$ *and the ideals* $I_1, \ldots, I_t$ *are unique, and the ideal* $J$ *is unique up to isomorphism.*
(5) $t = \operatorname{gen}(\mathcal{T}(M))$.

*Proof.* Let $F = R^{(c)}$ be a free module that maps onto $M$ via a map $\delta\colon F \longrightarrow M$. Then, since $M$ is finitely presented and $R$ is semi-hereditary, $H := \operatorname{Ker}\delta$ is finitely generated and projective. By the stacked bases there exist decompositions of $H$ and $F$ described by finitely generated ideals $J_1, \ldots, J_c, I_1, \ldots, I_s$ and a basis $x_1, \cdots, x_c$ for $F$, so that $0 \neq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_s \subseteq R$ and

$$\begin{aligned} R^{(c)} = F &= x_1J_1 \oplus \cdots \oplus x_sJ_s \oplus x_{s+1}J_{s+1} \oplus \cdots \oplus x_cJ_c, \\ \operatorname{Ker}\delta = H &= x_1I_1J_1 \oplus \cdots \oplus x_sI_sJ_s, \\ M \cong F/H &= J_1/I_1J_1 \oplus \cdots \oplus J_s/I_sJ_s \oplus J_{s+1} \oplus \cdots \oplus J_c\,. \end{aligned} \tag{2.5.1}$$

By the Steinitz property (from Proposition 2.4), the ideals $J_{s+1} \oplus \cdots \oplus J_c$ of the torsion-free part of $M$ can be expressed as a direct sum of some number of copies of $R$ with one ideal $J$. Applying Proposition 2.4 again, we see that the ideal $J$ is unique up to isomorphism. Next, by the $1\frac{1}{2}$-generator property, each $J_i/I_iJ_i$, for $0 \leq i \leq s$, is cyclic. Also, since $J_i$ is invertible, $\operatorname{Ann}(J_i/I_iJ_i) = I_i$, and thus $J_i/I_iJ_i \cong R/I_i$. If $I_{t+1} = \cdots = I_s = R$ and $t < s$, then the corresponding cyclic modules are zero and we can omit them from the decomposition. Thus we have

$$M \cong R/I_1 \oplus \cdots \oplus R/I_t \oplus R^{(\operatorname{rk}(M)-1)} \oplus J, \quad 0 \neq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_t \subsetneq R,\ t \leq s,$$

where there is no $J$ if $\operatorname{rk}(M) = 0$. Then $\mathcal{T}(M)$ is a direct sum of $t$ cyclic modules:

$$\mathcal{T}(M) \cong R/I_1 \oplus \cdots \oplus R/I_t, \quad I_1 \subseteq I_2 \subseteq \cdots \subseteq I_t \subsetneq R. \tag{2.5.2}$$

Now the uniqueness of the ideals $I_1, \ldots, I_t$ and the identity $\operatorname{gen}(\mathcal{T}(M)) = t$ follows from Kaplansky [17, Theorem 9.3 and Lemma 9.2]. □

**Examples 2.6.** There are many examples of Prüfer domains satisfying the hypotheses of Theorem 2.5 in the literature. We list some below; for all unexplained terms see [9]:

(1) Every Dedekind domain is a Prüfer domain with the stacked bases and $1\frac{1}{2}$-generator properties (every finitely generated module has such a decomposition).
(2) Every *almost local-global* Prüfer domain has the stacked bases property and the $1\frac{1}{2}$-generator property [3], [23], [9, Theorem V.4.9, Lemma V.4.11].
(3) If $R$ is a Prüfer domain of *finite character* (every nonzero element is contained in only a finite number of maximal ideals) or if $R$ is Prüfer with Krull dimension one, then $R$ is almost local-global [3], [23], [9, Example V.4.3].
(4) An $h$-local Prüfer domain has finite character and thus is almost local-global; cf. [9, Example 4.3].

*Remarks* 2.7. (1) If $R$ is a Bézout domain with the stacked bases property, then every matrix over $R$ is equivalent to a diagonal matrix [9, V.4.6, V.4.8].

(2) It is not known whether the stacked bases property is true for every Prüfer domain. If so, then, by Remark 2.7(1) above, every Bézout domain would be an elementary divisor ring. This would solve the long-standing problem of whether or not every Bézout domain is an elementary divisor ring.

We use the following setting in the rest of this section and in the next section.

*Setting and Notation* 2.8. Let $R$ be a Prüfer domain with the stacked bases and $1\frac{1}{2}$-generator properties. Let $K$ be the field of fractions of $R$. For a finitely presented $R$-module $M$, let $\mathcal{T}(M)$ and $\mathcal{TF}(M)$ refer to the torsion and torsion-free parts of $M$, respectively. By Theorem 2.5, we have

$$M \cong \mathcal{T}(M) \oplus \mathcal{TF}(M) \cong (R/I_1) \oplus \cdots \oplus (R/I_t) \oplus J \oplus R^{(\mathrm{rk}(M)-1)}, \tag{2.8.1}$$

where $0 \neq I_1 \subseteq \cdots \subseteq I_t \neq R$, $0 \neq J \subsetneq R$. There is no $J$ if $\mathrm{rk}(M) = 0$, and the expression is unique as described in Theorem 2.5.

We now introduce the invariants we use for finitely presented modules derived from the $\mathrm{K}_0$-group of $R$, $\mathrm{K}_0(R)$, in the setting as given above. Recall that for an integral domain $R$, the Picard group of $R$, $\mathrm{Pic}(R)$, is defined as the isomorphism classes of invertible fractional ideals under multiplication, or equivalently as the factor group

$$\mathrm{Pic}(R) := \mathcal{I}/\mathcal{P},$$

where $\mathcal{I} := \{\text{invertible fractional ideals of } R\} = \{\text{finitely generated fractional ideals}\}$ and $\mathcal{P} := \{\text{principal fractional ideals of } R\}$. For integral domains the Picard group is the same as the class group [21, Corollary 2.21]. For $J$ an invertible fractional ideal of $R$, let $\mathrm{cl}(J)$ denote the equivalence class of $J$ in $\mathrm{Pic}(R)$. In our setting of 2.8 above, the structure of $\mathrm{K}_0(R)$ turns out to be analogous to that of a Dedekind domain. To show this, consider a nonzero finitely generated projective $R$-module $P$. Then $P$ is finitely presented and torsion-free, and so, by equation (2.8.1) above, $P$ has the unique form $P \cong J \oplus R^{(\mathrm{rk}(P)-1)}$ (up to isomorphism), where $J$ is an invertible ideal. Thus the class of $J$, $\mathrm{cl}(J) \in \mathrm{Pic}(R)$, and the rank of $P$, $\mathrm{rk}(P)$, completely determine the isomorphism class of $P$. Set $\mathrm{cl}(P) := \mathrm{cl}(J) \in \mathrm{Pic}(R)$.

Next, if $P_1$ and $P_2$ are two projective $R$-modules, then, thanks to the Steinitz property, we see that $\mathrm{cl}(P_1 \oplus P_2) = \mathrm{cl}(P_1)\,\mathrm{cl}(P_2)$. This shows that the map

$$\varphi\colon \{\text{projective } R\text{-modules}\} \longrightarrow Z \oplus \mathrm{Pic}(R),\ P \mapsto (\mathrm{rk}(P), \mathrm{cl}(P))$$

induces an isomorphism of groups $\mathrm{K}_0(R) \cong Z \oplus \mathrm{Pic}(R)$. (As is customary, the group operation on $\mathrm{K}_0(R)$ is denoted by $+$. In this case it is addition (of integers) in the first coordinate and multiplication (of ideal classes) in the second coordinate.)

Since a Prüfer domain is semi-hereditary, we extend the map $\varphi$ to finitely presented modules. To this end let $M$ be a finitely presented $R$-module. Then $M$ has a projective resolution:

$$0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow M \longrightarrow 0\ ,$$

with $P_1, P_2$ projective modules. We define $\varphi(M) := \varphi(P_2) - \varphi(P_1) \in \mathrm{K}_0(R)$. Then

$$\varphi\colon \{\text{finitely presented } R\text{-modules}\} \longrightarrow \mathrm{K}_0(R)$$

is well defined and additive over short exact sequences. This follows by the standard technique of "reduction by resolution" in K-theory (see e.g. [2]); in our case this is essentially just Schanuel's Lemma. Finally we extend the map $\mathrm{cl}(\cdot)$ to finitely presented modules as the map $\varphi$ followed by the projection to $\mathrm{Pic}(R)$. Note that the map $\mathrm{rk}(\cdot)$ is already defined for any module as torsion-free rank. To sum up we have the following result.

**Proposition 2.9.** *Assume* Setting and Notation 2.8 *and let $M$ be a finitely presented $R$-module. Then the maps*

$$\begin{aligned} &\mathrm{cl} : \{\textit{finitely presented } R\textit{-modules}\} \longrightarrow \mathrm{Pic}(R) \quad \textit{and} \\ &\varphi : \{\textit{finitely presented } R\textit{-modules}\} \longrightarrow \mathrm{K}_0(R), \quad M \mapsto (\mathrm{rk}(M), \mathrm{cl}(M)) \end{aligned} \tag{2.9.1}$$

*depend only on the isomorphism class of $M$. The map* cl *is multiplicative and $\varphi$ is additive over short exact sequences. Moreover, $\varphi$ induces an injection between the isomorphism classes of projective $R$-modules and $\mathrm{K}_0(R)$; that is, $\mathrm{K}_0(R)$ is a complete invariant for projective $R$-modules.* □

As an illustration and for future reference we list below a few calculations with our invariants $\varphi$ and cl.

*Facts* 2.10. Let $R$ and $M$ be as in Setting 2.8 with $I, I_1, I_2, \ldots, I_t$ nonzero ideals of $R$, and let $\varphi$ be as in Proposition 2.9. Then:

(1) $\varphi(R) = [R] = (1, \mathrm{cl}(R)) = (1, 1_{\mathrm{Pic}(R)})$.
(2) $\varphi(R/I) = \varphi(R) - \varphi(I) = (0, (\mathrm{cl}(I))^{-1}) = (0, \mathrm{cl}(I^{-1}))$.
(3) $\varphi(R/I_1 \oplus \cdots \oplus R/I_t) = \varphi(R/I_1) + \varphi(R/I_2) + \cdots + \varphi(R/I_t)$
$= (0, \mathrm{cl}((I_1 \cdots I_t)^{-1}))$.
(4) If $\mathcal{T}(M) \cong (R/I_1) \oplus \cdots \oplus (R/I_t)$ from equation (2.8.1), then
$\mathrm{cl}(\mathcal{T}(M)) = \mathrm{cl}((I_1 \cdots I_t)^{-1})$.

## 3. Modules to matrices and the main theorem

In this section we investigate the "inverse" of the map from matrices to finitely presented modules. We assume Setting 2.8.

*Remarks* 3.1. Lemma 3.2 below is critical for the converse to Remark 1.5(2). That is, if matrices of the same type name isomorphic modules, then the matrices are equivalent; cf. Theorem 3.5. Since $R$ has the stacked bases property, the finitely

generated projective $R$-modules $R^{(r)}A$ and $R^{(c)}$ in equation (1.5.1) can be decomposed in a "stacked" way as in the proof of Theorem 2.5 and we get as in (2.8.1)

$$M \cong R^{(c)}/R^{(r)}A \cong R/I_1 \oplus \cdots \oplus R/I_t \oplus J \oplus R^{(s)}, \tag{3.1.1}$$

where $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_t$. We show in Lemma 3.2 that the stacked decomposition for $R^{(c)}$ and $R^{(r)}A$ that give rise to the decomposition of $M(A)$ can be of the form:

$$\begin{aligned} R^{(c)} &= R \oplus R \oplus \cdots \oplus R \oplus J^{-1} \oplus J \oplus R^{(s)}, \\ R^{(r)}A &= I_1 \oplus I_2 \oplus \cdots \oplus I_{t-1} \oplus J^{-1}I_t, \end{aligned}$$

where $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_t$ are finitely generated ideals of $R$ and $J$ is a finitely generated fractional ideal of $R$.

The proof of the lemma is an adaptation of Kaplansky's proof of Proposition 2.4 from [9, V.4.10].

**Lemma 3.2.** *Let $R$ be a Prüfer domain with the $1\frac{1}{2}$-generator property and quotient field $K$. Let $I_1, \ldots, I_t$ be invertible ideals with $0 \neq I_1 \subseteq \cdots \subseteq I_t \subseteq R$.*

(1) *Suppose that $H_1, \ldots, H_c$ are fractional invertible ideals of $R$ and that*

$$\begin{aligned} Y &:= H_1x_1 \oplus H_2x_2 \oplus \cdots \oplus H_{t-1}x_{t-1} \oplus H_tx_t \oplus \cdots \oplus H_cx_c, \\ X &:= I_1H_1x_1 \oplus I_2H_2x_2 \oplus \cdots \oplus I_{t-1}H_{t-1}x_{t-1} \oplus I_tH_tx_t, \end{aligned}$$

*where each $x_i \in KY$. Then there exist $z_1, \ldots, z_c \in KY$ and invertible fractional ideals $J$ and $J'$ with*

$$\begin{aligned} Y &= Rz_1 \oplus Rz_2 \oplus \cdots \oplus Rz_{t-1} \oplus Jz_t \oplus Rz_{t+1} \cdots \oplus Rz_{c-1} \oplus J'z_c, \\ X &= I_1z_1 \oplus I_2z_2 \oplus \cdots \oplus I_{t-1}z_{t-1} \oplus I_tJz_t, \end{aligned}$$

*with the convention that $J'$ is not present if $c = t$ and $J$ is not present if $t = 0$. That is, with an appropriate change of basis, all but possibly one or two of the ideals $H_1, H_2, \ldots, H_c$ can be taken to be $R$.*

(2) *Suppose that $H_1, \ldots, H_c$, $K_1, \ldots, K_c$ are fractional invertible ideals of $R$ and*

$$\begin{aligned} Y &:= H_1x_1 \oplus H_2x_2 \oplus \cdots \oplus H_{t-1}x_{t-1} \oplus H_tx_t \oplus \cdots \oplus H_cx_c, \\ X_1 &:= I_1H_1x_1 \oplus I_2H_2x_2 \oplus \cdots \oplus I_{t-1}H_{t-1}x_{t-1} \oplus I_tH_tx_t, \\ Y &= K_1y_1 \oplus K_2y_2 \oplus \cdots \oplus K_{t-1}y_{t-1} \oplus K_ty_t \oplus \cdots \oplus K_cy_c, \\ X_2 &:= I_1K_1y_1 \oplus I_2K_2y_2 \oplus \cdots \oplus I_{t-1}K_{t-1}y_{t-1} \oplus I_tK_ty_t, \end{aligned}$$

*where the $x_i, y_i \in KY$. Then there exists an automorphism $\alpha$ of $Y$ such that $\alpha(X_1) = X_2$.*

(3) *If $Y \cong R^{(n)}$ in part (1) and $t < n$, then $J'$ may be replaced by $J^{-1}$.*

*Proof.* For (1), we can always adjust $H_1, H_2, \ldots$ by multiplying in each coordinate by a nonzero element of the quotient field $K$ (and then adjusting the basis elements by dividing); this carries over to $X$. Thus we can suppose that each of the $H_i$ is contained in $R$. Since $H_{t+1}x_{t+1} \oplus \cdots \oplus H_cx_c \cong R^{(c-t+1)} \oplus H_{t+1} \cdots H_c$ by (2.4), it suffices to prove the statement for $c = t$. If $t \leq 1$, there is nothing to prove. Furthermore, if the statement holds for $c = t = 2$, then it holds for all larger $c = t$,

by considering the summands two at a time. Thus the essential part of this proof is the following claim:

*Claim.* If $R$ is a Prüfer domain with the $1\frac{1}{2}$-generator property, $H_1, H_2, I_1, I_2$ are invertible ideals of $R$ with $I_1 \subseteq I_2 \subseteq R$, and $X := I_1H_1 \oplus I_2H_2 \subseteq F := H_1 \oplus H_2$ are finitely generated projective modules, then there exists an isomorphism $\alpha\colon F \longrightarrow H_1H_2 \oplus R$ such that $\alpha(X) = I_1H_1H_2 \oplus I_2$.

*Proof of the Claim.* First we reduce to a more specific case. The ideal $I_2$ can be chosen to be $R$, because if we show that, for $I_2^{-1}X = I_2^{-1}I_1H_1 \oplus H_2$, we get an isomorphism $\alpha\colon F \longrightarrow H_1H_2 \oplus R$ such that $\alpha(I_2^{-1}X) = I_2^{-1}I_1H_1H_2 \oplus R$, then it follows that $\alpha(X) = I_1H_1H_2 \oplus I_2R$. Thus we suppose that $X := IH_1 \oplus H_2 \subseteq F := H_1 \oplus H_2$, where $H_1, H_2, I$ are invertible ideals contained in $R$. We show that $F \cong H_1H_2 \oplus R$ and that the isomorphism takes $X$ to $IH_1H_2 \oplus R$.

Now let $0 \neq a \in H_2$. Then $aH_2^{-1} \subseteq R$. Therefore

$$aH_2^{-1}IH_1 \subseteq aH_2{}^{-1} \subseteq R.$$

Thus, by the $1\frac{1}{2}$-generator property, there exists $b \in R$ with

$$bR + aH_2{}^{-1}IH_1 = aH_2{}^{-1}.$$

Multiply by $a^{-1}H_2$, and obtain

$$(ba^{-1}H_2) + IH_1 = R, \text{ and so also } (ba^{-1}H_2) + H_1 = R.$$

Now we have that

$$(ba^{-1}H_2) \cap H_1 = (ba^{-1}H_2) \cdot H_1; \quad (ba^{-1}H_2) \cap IH_1 = (ba^{-1}H_2) \cdot IH_1.$$

It follows that the following sequences are split short exact sequences:

$$0 \longrightarrow (ba^{-1}H_2) \cdot H_1 \longrightarrow H_1 \oplus (ba^{-1}H_2) \longrightarrow R = H_1 + (ba^{-1}H_2) \longrightarrow 0,$$
$$0 \longrightarrow (ba^{-1}H_2) \cdot IH_1 \longrightarrow IH_1 \oplus (ba^{-1}H_2) \longrightarrow R = IH_1 + (ba^{-1}H_2) \longrightarrow 0.$$

Thus

$$H_1 \oplus H_2 \cong H_1 \oplus (ba^{-1}H_2) \cong (ba^{-1}H_2)H_1 \oplus R \cong H_1H_2 \oplus R$$

via an isomorphism $\alpha$ indicated by

$$X = IH_1 \oplus H_2 \longrightarrow IH_1 \oplus (ba^{-1}H_2) \longrightarrow (ba^{-1}H_2)(IH_1) \oplus R \longrightarrow IH_1H_2 \oplus R,$$

since multiplication by $ba^{-1}$ and its inverse are coordinatewise. Now the claim is proved, and so is part (1).

Part (2) follows from part (1), because part (1) implies that there exists an isomorphism $\alpha_1$ such that

$$\alpha_1\colon Y \xrightarrow{\cong} R \oplus R \oplus \cdots \oplus R \oplus J \oplus R^{(c-t-1)} \oplus J',$$
$$\alpha_1(X_1) = I_1 \oplus I_2 \oplus \cdots \oplus I_{t-1} \oplus I_tJ$$

and that there is a similar isomorphism $\alpha_2$ from $Y$ that takes $X_2$ to $\alpha(X_1)$.

For (3), by Proposition 2.4,

$$R^{(c)} \cong R^{(c-2)} \oplus J \oplus J' \cong R^{(c-1)} \oplus JJ' \implies JJ' \cong R.$$

□

*Remark* 3.3. In the terminology of [3, p. 540] and [23], a ring with the property that every pair $X \subseteq Y$ of finitely generated projective modules can be simultaneously decomposed as in the conclusion of part (1), with $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{t-1} \subseteq I_t$, is said to satisfy the *Invariant Factor Theorem*. Levy proves that it holds for Prüfer domains of finite character [23]. Brewer and Klingler show that an almost local-global Prüfer domain has this property, and they ask the following questions. Does the Invariant Factor Theorem imply the $1\frac{1}{2}$-generator property? Does the $1\frac{1}{2}$-generator property alone imply the Invariant Factor Theorem? Does the stacked bases property plus the Steinitz property imply the Invariant Factor Theorem? In Corollary 3.4 we show that the Invariant Factor Theorem does hold under the assumption of stacked bases and the $1\frac{1}{2}$-generator property. The referee of this paper has kindly pointed out that the corollary below establishes the following implications:

$$\begin{aligned}\text{stacked bases property} + 1\tfrac{1}{2}\text{-generator property} &\implies \text{Invariant Factor Theorem}\\ &\implies \text{stacked bases property} + \text{Steinitz property.}\end{aligned}$$

This leads to the interesting question of whether either of these arrows is reversible, which is a rephrasing of two questions asked in [3, p. 545].

**Corollary 3.4** (Invariant Factor Theorem)**.** *Let $R$ be a Prüfer domain having the stacked bases and $1\frac{1}{2}$-generator properties. Suppose that $X$ and $Y$ are finitely generated projective modules of ranks $t$ and $c$, respectively, with $X \subseteq Y$, and so $t \leq c$. Then $X$ and $Y$ can be simultaneously decomposed as*

$$\begin{aligned} Y &= Rz_1 \oplus Rz_2 \oplus \cdots \oplus \quad Rz_{t-1} \oplus \quad Jz_t \oplus Rz_{t+1} \cdots \oplus Rz_{c-1} \oplus J'z_c,\\ X &= I_1 z_1 \oplus I_2 z_2 \oplus \cdots \oplus I_{t-1} z_{t-1} \oplus I_t J z_t, \end{aligned}$$

*where $z_1, \ldots, z_c \in Y$, $J$ and $J'$ are invertible fractional ideals of $R$ and $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{t-1} \subseteq I_t$ are invertible ideals of $R$.*

*Proof.* The proof is clear from Lemma 3.2(1) and Definitions 2.3. □

We now have the necessary machinery to prove in Setting 2.8 the matrix equivalence corresponding to isomorphic finitely presented modules, a converse to Remark 1.5(2).

**Theorem 3.5.** *Let $R$ be a Prüfer domain with the stacked bases and $1\frac{1}{2}$-generator properties. Suppose that $A$ and $B$ are $r \times c$ matrices over $R$, for fixed $r, c \in \mathbb{N}_0$. Then $M(A) \cong M(B)$ if and only if $A \sim B$.*

*Proof.* If $A \sim B$, then $M(A) \cong R^{(c)}/R^{(r)}A \cong R^{(c)}/R^{(r)}B \cong M(B)$, where $R^{(r)}A$ and $R^{(r)}B$ are the images of right multiplication by $A$ and $B$ (the row spaces), as described in (1.5).

For the other direction, assume that $M(A) \cong M(B)$. We have exact sequences:

$$\begin{aligned} 0 &\longrightarrow R^{(r)}A \hookrightarrow R^{(c)} \xrightarrow{\delta} M(A) \longrightarrow 0,\\ 0 &\longrightarrow R^{(r)}B \hookrightarrow R^{(c)} \xrightarrow{\tau} M(B) \longrightarrow 0. \end{aligned} \tag{3.5.1}$$

We apply the stacked bases property in Definitions 2.3, as modified in Lemma 3.2(3), to the pairs $R^{(r)}A \subseteq R^{(c)}$ and $R^{(r)}B \subseteq R^{(c)}$. Then there exist two sets of new bases for $R^{(c)}$; denote them by $e_1, \ldots, e_c$ and $f_1, \ldots, f_c \in R^{(c)}$, respectively.

Also there are finitely generated fractional ideals $J_1$ and $J_2$ and finitely generated ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_t$ and $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_s$ of $R$ such that

$$\begin{aligned}
&R^{(c)} = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_{t-1} \quad \oplus J_1^{-1}e_t \oplus Re_{t+1} \cdots \oplus Re_{c-1} \oplus J_1 e_c,\\
&R^{(r)}A = I_1e_1 \oplus I_2e_2 \oplus \cdots \oplus I_{t-1}e_{t-1} \oplus I_tJ_1^{-1}e_t,\\
&M(A) \cong R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_t \oplus R^{(c-t-1)} \oplus J_1, \qquad (3.5.2)\\
&R^{(c)} = Rf_1 \oplus Rf_2 \oplus \cdots \oplus Rf_{s-1} \oplus J_2^{-1}f_s \oplus Rf_{s+1} \cdots \oplus Rf_{c-1} \oplus J_2 f_c,\\
&R^{(r)}B = L_1f_1 \oplus L_2f_2 \oplus \cdots \oplus L_{s-1}f_{s-1} \oplus L_sJ_2^{-1}f_s,\\
&M(B) \cong R/L_1 \oplus R/L_2 \oplus \cdots \oplus R/L_s \oplus R^{(c-s-1)} \oplus J_2\,. \qquad (3.5.3)
\end{aligned}$$

By the uniqueness of decomposition of $M(A) \cong M(B)$ given in Theorem 2.5, we have that

$$s = t, \quad I_i = L_i, \text{ for every } i, \quad J_1 \cong J_2$$

in (3.5.2) and (3.5.3). Thus by Lemma 3.2(2) there exists an isomorphism

$$\beta : Re_1 \oplus Re_2 \oplus \ldots \oplus J_1e_c \longrightarrow Rf_1 \oplus Rf_2 \oplus \ldots \oplus J_2f_c$$

that induces an isomorphism $\overline{\beta} : R^{(r)}A \longrightarrow R^{(r)}B$ and, since $R^{(r)}A = \operatorname{Ker}\delta$ and $R^{(r)}B = \operatorname{Ker}\tau$ in diagram (3.5.1), we have an isomorphism $\alpha : M(A) \longrightarrow M(B)$. Therefore diagram (3.5.4) (without $\gamma$) commutes:

$$(3.5.4) \qquad \begin{tikzcd}
R^{(r)} \arrow[r, "A"] \arrow[d, dotted, "\exists\gamma"'] & R^{(r)}A \arrow[r, "\subseteq"] \arrow[d, "\overline{\beta}"'] & R^{(c)} \arrow[r, "\delta"] \arrow[d, "\beta"] & M(A) \arrow[r] \arrow[d, "\alpha"] & 0 \\
R^{(r)} \arrow[r, "B"'] & R^{(r)}B \arrow[r, "\subseteq"'] & R^{(c)} \arrow[r, "\tau"'] & M(A) \arrow[r] & 0\,.
\end{tikzcd}$$

We show there exists a $\gamma$ as shown by the dotted arrow. Since $R^{(r)} \longrightarrow R^{(r)}A$ is an epimorphism and $R^{(r)}A$ is projective, the sequence

$$0 \longrightarrow P := \operatorname{Ker} A \longrightarrow R^{(r)} \longrightarrow R^{(r)}A \longrightarrow 0$$

splits. Hence $R^{(r)} \cong R^{(r)}A \oplus P$, where $R^{(r)}A$ and $P$ are projective. Similarly $R^{(r)} \cong R^{(r)}B \oplus Q$, where $Q$ is projective. Since $R^{(r)}B \cong R^{(r)}A$ and $R$ has the $1\frac{1}{2}$-generator property, we have cancellation for projectives by Proposition 2.4. Thus $P \cong Q$ and $\overline{\beta}$ extends to an isomorphism $\gamma : R^{(r)} \longrightarrow R^{(r)}$ that also induces the isomorphism $\overline{\beta}$ on $R^{(r)}A$, and, with $\gamma$ inserted, diagram (3.5.4) still commutes. It follows that $A \sim B$ via the matrices that describe $\beta$ and $\gamma$. □

*Remarks* 3.6. Fitting [7] showed that if two matrices $A$ and $B$ (not necessarily having the same types) over an arbitrary ring name isomorphic modules, then by adding, as a diagonal sum, appropriately sized identity and empty matrices of type $(1,0)$ to $A$ and $B$, the resulting matrices $A'$ and $B'$ as shown below are equivalent in the standard (of Definitions 1.1) sense:

$$A' = \begin{bmatrix} A & 0 \\ 0 & I_k \\ 0 & 0 \end{bmatrix} \qquad B' = \begin{bmatrix} B & 0 \\ 0 & I_l \\ 0 & 0 \end{bmatrix};$$

see also [20, Corollary 1.16]. (Empty matrix is our terminology. Hitherto, the effect of inserting empty matrices has been described as "adding extra zero rows or columns".) In general, this padding with identity and empty matrices is needed

even if the matrices are of the same type. Warfield [30] gave bounds on the sizes of these identity and empty matrices in terms of the stable range of the ring. Following the work of Steinitz and Krull, Levy proved in [22] that over a Dedekind domain, for matrices of the same type, no such padding is needed. (For a ring of algebraic integers he attributes this result to Steinitz [27, 28].) Thus Theorem 3.5 above extends this result to our Prüfer domains. It also tells us what the "kernel" of the map: {matrices of a given type} $\longrightarrow$ {modules} is. For an interesting and detailed account of the history of this topic, see [24, Section 3.6].

Next we consider the inverse image of a module under the mapping from matrices to modules. More precisely, we ask:

**Question 3.7.** Given a finitely presented $R$-module $M$, for which $r$ and $c$ is there an $r \times c$ matrix $A$ naming $M$, so that $M \cong M(A)$?

The following theorem, the *Matrix Naming Theorem*, gives explicit criteria regarding this question. As usual, the rank of an $r \times c$ matrix $A\colon R^{(r)} \longrightarrow R^{(c)}$ is defined to be the rank of its image, $\mathrm{rk}(R^{(r)}A)$. Note that $A$ and the transpose of $A$ have the same rank.

**Theorem 3.8** (Matrix Naming Theorem (MNT))**.** *Let $R$ and $M$ be as in* Setting 2.8 *and* (2.8.1)*, and recall that $\mathcal{T}(M)$ denotes the torsion part of $M$. Let $r, c \in \mathbb{N}_0$. Then there is an $r \times c$ matrix $A$ over $R$ naming $M$, so that $M \cong M(A)$ if and only if*

$$\mathrm{gen}(\mathcal{T}(M)) + \mathrm{rk}(M) \le c \le r + \mathrm{rk}(M) \quad \textit{and, additionally,} \tag{3.8.1a}$$

$$\textit{if } c = r + \mathrm{rk}(M), \quad \textit{then} \quad \mathrm{cl}(M) = 1_{\mathrm{Pic}(R)}. \tag{3.8.1b}$$

*Every $r \times c$ matrix naming $M$ must have rank equal to $c - \mathrm{rk}(M)$.*

*Proof.* First, suppose that $A$ is an $r \times c$ matrix with $M \cong M(A)$. Consider the short exact sequence below, where $R^{(r)}A$ is a projective module of rank $\mathrm{rk}(A)$,

$$0 \longrightarrow R^{(r)}A \overset{\subseteq}{\hookrightarrow} R^{(c)} \xrightarrow{\delta} M(A) \longrightarrow 0\,, \quad \mathrm{Ker}\,\delta = R^{(r)}A. \tag{3.8.2}$$

From (3.8.2) we readily see that $\mathrm{rk}(A) = c - \mathrm{rk}(M)$, and so the last statement holds. Moreover, this implies that

$$r \ge \mathrm{rk}(A) = c - \mathrm{rk}(M),$$

and so the second inequality of (3.8.1a) holds.

From equation (2.8.1) we have a representation of $M(A)$ involving nonzero proper ideals $I_1 \subseteq \cdots \subseteq I_t$ and a fractional ideal $J$ (all finitely generated):

$$M(A) \cong R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_t \oplus R^{(\mathrm{rk}(M)-1)} \oplus J\,, \tag{3.8.3}$$

with no $J$ if $\mathrm{rk}(M) = 0$. Then $t = \mathrm{gen}(\mathcal{T}(M))$. By the stacked bases property, $R^{(r)}A$ decomposes as the direct sum of $\mathrm{rk}(R^{(r)}A) = \mathrm{rk}(A) = s \ge t$ nonzero ideals; cf. the proof of Theorem 2.5 and equation (2.5.1). Thus $t = \mathrm{gen}(\mathcal{T}(M)) \le \mathrm{rk}(A) = c - \mathrm{rk}(M)$, and so the first inequality of (3.8.1a) follows.

For (3.8.1b), the equality given and the last statement together imply $r = c - \mathrm{rk}(M) = \mathrm{rk}(A)$. Then in the exact sequence

$$R^{(r)} \xrightarrow{A_{r\times c}} R^{(c)} \longrightarrow M \longrightarrow 0$$

the first map is injective, and so $[M] = [R^{(c)}] - [R^{(r)}]$ in $\mathrm{K}_0(R)$. Thus, in particular, $\mathrm{cl}(M) = \mathrm{cl}(R^{(c)})/\,\mathrm{cl}(R^{(r)}) = \mathrm{cl}(R)/\,\mathrm{cl}(R) = 1_{\mathrm{Pic}(R)}$, as required.

Conversely, assume that $M$ is a finitely presented module in the form of (3.8.3) and $r, c \in \mathbb{N}_0$, satisfying (3.8.1a) and (3.8.1b). Put $\rho := c - \mathrm{rk}(M)$ and $\sigma := \mathrm{rk}(M)$. By (3.8.1a), $t = \mathrm{gen}(\mathcal{T}(M)) \leq \rho$. If $t < \rho$, then we can add zero modules to $\mathcal{T}(M)$ of the form $R/R$; that is, we let $I_{t+1} = \cdots = I_\rho = R$. Now we can decompose $R^{(r)}$ and $R^{(c)}$ in the following way:

$$\begin{aligned}
R^{(r)} &= I_1 \oplus \cdots \oplus I_{\rho-1} \oplus I_\rho J^{-1} \oplus \ (I_1 \cdots I_\rho)^{-1} J \oplus R^{(r-c+\sigma-1)} && \text{if } \rho < r.\\
R^{(r)} &= I_1 \oplus \cdots \oplus I_{\rho-1} \oplus I_\rho J^{-1} && \text{if } \rho = r.\\
R^{(c)} &= R^{(\rho-1)} \oplus J^{-1} \oplus J \oplus R^{(\sigma-1)} && \text{if } \rho < c.\\
R^{(c)} &= R^{(c-1)} \oplus J^{-1} && \text{if } \rho = c.
\end{aligned}$$

Note, in the $\rho = r$ case, that $\mathrm{cl}(I_1 \cdots I_\rho)\,\mathrm{cl}(J^{-1}) = 1 = \mathrm{cl}(R)$, by (3.8.1b). In the $\rho = c$ case, $J = J^{-1} = R$, by Theorem 2.5.

Now $A$ is given by the map

$$\begin{array}{ccccccc}
R^{(r)} & \xrightarrow{\text{projection}} & I_1 & \oplus \cdots \oplus & I_{\rho-1} & \oplus & I_\rho J^{-1}\\
 & & \subseteq\big\downarrow & & \subseteq\big\downarrow & & \subseteq\big\downarrow\\
 & & R & \oplus \cdots \oplus & R & \oplus & J^{-1} \oplus J \oplus R^{(\sigma-1)} = R^{(c)}.
\end{array}$$

Thus clearly $M(A) \cong R/I_1 \oplus \cdots \oplus R/I_{\rho-1} \oplus J^{-1}/I_\rho J^{-1} \oplus J \oplus R^{(\sigma-1)}$.

Since $J^{-1}/I_\rho J^{-1} \cong R/I_\rho$, we get $M(A) \cong M$, as required. □

Before going on to the main application of the MNT we give a quick illustration of its use.

**Corollary 3.9.** *Let $R$ be a Prüfer domain having the stacked bases and $1\frac{1}{2}$-generator properties. Then a finitely presented $R$-module $M$ is named by a square matrix if and only if either*

(i) $\mathcal{TF}(M) \neq (0)$ *or* (ii) $M = \mathcal{T}(M)$ *and* $\mathrm{cl}(M) = 1_{\mathrm{Pic}(R)}$.

*In either case there is a square matrix of size $r$ naming $M$, for every $r$ with $r \geq \mathrm{gen}(\mathcal{T}(M)) + \mathrm{rk}(M)$.*

*Proof.* If $\mathrm{rk}(M) > 0$ and $r \geq \mathrm{gen}(\mathcal{T}(M)) + \mathrm{rk}(M)$, then $r$ and $c = r$ satisfy (3.8.1a) and the second inequality of (3.8.1a) is strict ($r < r + \mathrm{rk}(M)$), so that (3.8.1b) holds vacuously. That is, every finitely presented $R$-module $M$ with $\mathrm{rk}(M) > 0$ is named by a square matrix of size $r$, for every $r \geq \mathrm{gen}(\mathcal{T}(M)) + \mathrm{rk}(M)$.

Now suppose that $M$ is finitely presented with $\mathrm{rk}(M) = 0$. Then, for every $r \geq \mathrm{gen}(\mathcal{T}(M))$, (3.8.1a) holds with $r = c$. Thus, for $r \geq \mathrm{gen}(\mathcal{T}(M))$, every finitely presented $R$-module $M$ with $\mathrm{rk}(M) = 0$ is named by a square matrix of size $r$ if and only if (3.8.1b) holds, that is, $\mathrm{cl}(M) = 1_{\mathrm{Pic}(R)}$, and so the result holds. □

Our main application of the MNT is to obtain criteria for a matrix to be decomposable. These are set out in the following proposition.

**Proposition 3.10.** *Let $R$ be a Prüfer domain having the stacked bases and $1\frac{1}{2}$-generator properties and let $A$ be an $r \times c$ matrix with $r \geq 1, c \geq 1$ over $R$ naming the module $M = M(A)$. Then $A$ is decomposable if one of the following holds:*

(i) *$R$ is a proper summand of $M$ (e.g. if $\mathrm{rk}(M) > 1$), or $r - \mathrm{rk}(A) > 1$ or $|r - c| > 1$.*

(ii) *There exists a nonzero torsion proper summand $U$ of $M$ such that* $\mathrm{cl}(U) = 1_{\mathrm{Pic}(R)}$ *and* $\mathrm{gen}(\mathcal{T}(M)) = \mathrm{gen}(U) + \mathrm{gen}(V)$*, where* $\mathcal{T}(M) = U \oplus V$*.*

(iii) $\min(r, c) \geq 2$ *and* $c > \mathrm{gen}(\mathcal{T}(M)) + \mathrm{rk}(M)$*.*

*Proof.* For the first part of (i), let $M \cong M_1 \oplus R$, where $M_1 \neq 0$. Since the $r \times c$ matrix $A$ names $M$, we have that $r$ and $c$ satisfy (3.8.1a) and (3.8.1b) in the Matrix Naming Theorem (MNT), Theorem 3.8. Thus $c - 1 \geq 0$ from (3.8.1a). In order to see that an $r \times (c-1)$ matrix names $M_1$, we check that $r$ and $c-1$ also satisfy conditions (3.8.1a) and (3.8.1b). For this, we compute $\mathrm{rk}(M_1) = \mathrm{rk}(M) - 1$ and $\mathrm{cl}(M_1) = \mathrm{cl}(M)$; thus $\mathcal{T}(M_1) = \mathcal{T}(M)$. Now we have

$$\mathrm{gen}(\mathcal{T}(M_1))+\mathrm{rk}(M_1) = \mathrm{gen}(\mathcal{T}(M))+\mathrm{rk}(M)-1 \leq c-1 \leq r+\mathrm{rk}(M)-1 = r+\mathrm{rk}(M_1),$$

and so (3.8.1a) is satisfied for $r, c-1$ and $M_1$. If the last inequality above is an equality, then $c = r + \mathrm{rk}(M)$ also, and so by MNT, $1 = \mathrm{cl}(M) = \mathrm{cl}(M_1)$. Thus (3.8.1b) also holds, and so there is an $r \times (c-1)$ matrix $A_1$ naming $M_1$. Also the $0 \times 1$ matrix $A_2$ names $R$, and so $\mathrm{diag}(A_1, A_2)$ names $M_1 \oplus R \cong M$. Now, by Theorem 3.5, $A \sim \mathrm{diag}(A_1, A_2)$, and so $A$ is decomposable.

For the second part of (i), let $M^\perp$ be the module named by the transpose $A^\perp$ of $A$, which has $r$ columns. Now $\mathrm{rk}(M^\perp) = r - \mathrm{rk}(A^\perp) = r - \mathrm{rk}(A) > 1$. From the first part, $A^\perp$ is decomposable, whence so is $A$. Finally, if $|r - c| > 1$, then, since $\mathrm{rk}(A) \leq \min(r, c)$, we must have that either $r - \mathrm{rk}(A) > 1$ or $c - \mathrm{rk}(A) = \mathrm{rk}(M) > 1$. Thus $A$ is again decomposable by the previous cases of part (i).

Turning to part (ii), assume that $M$ has a torsion proper summand $U$ as described in part (ii) of this proposition, so that $\mathcal{T}(M) = U \oplus V$ and $M = U \oplus V \oplus \mathcal{TF}(M)$. Put $k := \mathrm{gen}(U) \geq 1$ and $M_1 = \mathcal{TF}(M) \oplus V \neq 0$, and note that $\mathrm{rk}(M_1) = \mathrm{rk}(M)$ and $\mathrm{gen}(\mathcal{T}(M_1)) = \mathrm{gen}(V) = \mathrm{gen}(\mathcal{T}(M)) - k$. We wish to show that there is an $(r-k) \times (c-k)$ matrix $A_1$ naming $M_1$, using the MNT in the same fashion as in part (i) above. To this end we compute

$$\begin{aligned}\mathrm{gen}(\mathcal{T}(M_1)) + \mathrm{rk}(M_1) &= \mathrm{gen}(\mathcal{T}(M)) - k + \mathrm{rk}(M) \leq c - k \\ &\leq r + \mathrm{rk}(M) - k = r - k + \mathrm{rk}(M_1),\end{aligned}$$

and so (3.8.1a) is satisfied for $r-k, c-k$ and $M_1$. If the second inequality above were an equality, then also $c = r+\mathrm{rk}(M)$, whence $\mathrm{cl}(M) = 1$ by MNT for $r, c$ and $M$. But then $1 = \mathrm{cl}(M) = \mathrm{cl}(M_1)\,\mathrm{cl}(U) = \mathrm{cl}(M_1)$, and so condition (3.8.1b) is satisfied for $r - k, c - k$ and $M_1$. Also note that since $M_1 \neq 0$ and $0 < \mathrm{gen}(\mathcal{T}(M_1)) + \mathrm{rk}(M_1) \leq c - k$, we have that $c - k > 0$ and, again from the displayed inequality above, $r - k \geq \mathrm{gen}(\mathcal{T}(M_1)) \geq 0$. Now both $c - k$ and $r - k$ are nonnegative integers and they cannot both be zero. We conclude that there is indeed an $(r - k) \times (c - k)$ matrix $A_1$ naming $M_1$.

Since $\mathrm{rk}(U) = 0$, $\mathrm{gen}(\mathcal{T}(U)) = \mathrm{gen}(U) = k$, and $\mathrm{cl}(U) = 1$, there is a $k \times k$ matrix $A_2$ naming $U$ by Corollary 3.9. We now have that $A_1$ names $M_1$ and $A_2$ names $U$, with $A_2$ having positive dimensions. Thus $A \sim \mathrm{diag}(A_1, A_2)$, by Theorem 3.5, and so $A$ is decomposable as required.

To prove part (iii), we have $r-1 \geq 1$, $c-1 \geq 1$ and, since $r, c, M$ satisfy the MNT, so do $r - 1, c - 1, M$, since we still have $\mathrm{gen}(\mathcal{T}(M)) + \mathrm{rk}(M) \leq c - 1$. Hence there is an $(r-1) \times (c-1)$ matrix $A_1$ of positive dimensions naming $M$. Since the $1 \times 1$ identity matrix $(\mathrm{Id})_1$ names the zero module, $\mathrm{diag}(A_1, (\mathrm{Id})_1)$ names $M \oplus 0 \cong M$. Thus $A \sim \mathrm{diag}(A_1, (\mathrm{Id})_1)$ by Theorem 3.5, and so $A$ is decomposable. $\square$

*Remark* 3.11. Note that an empty matrix of size greater than 1 is decomposable; cf. Remarks 1.2.

Next we prove the main theorem of this section.

**Theorem 3.12** (Bounded block decomposition)**.** *Let $R$ be a Prüfer domain with the stacked bases property and the $1\frac{1}{2}$-generator property. Assume that the group* $\mathrm{Pic}(R)$ *is finite. Then*

(1) *Every matrix of positive size over $R$ is equivalent to a block diagonal sum of indecomposable matrices of size at most $|\mathrm{Pic}(R)\,|$.*
(2) *Every indecomposable matrix of rank $\rho$ over $R$ is of the type*

$$\rho \times \rho, \quad (\rho+1)\times(\rho+1), \quad \rho\times(\rho+1), \quad \text{or} \quad (\rho+1)\times\rho,$$

*where $\rho \le |\mathrm{Pic}(R)\,|$ for the $\rho\times\rho$ case and $\rho \le |\mathrm{Pic}(R)\,|-1$ for the other cases.*

*Proof.* Recall that every matrix of positive size is equivalent to a block diagonal sum of indecomposable matrices; cf. Remarks 1.2(2). We may assume that $|\mathrm{Pic}(R)\,| \ge 2$. Otherwise $R$ is Bézout, and then the blocks have size one (i.e. $R$ is an elementary divisor ring), by Remarks 2.7(1). Since the second part will follow from Proposition 3.10(i), it suffices to prove that the size of an indecomposable matrix $A$ is $\le |\mathrm{Pic}(R)\,|$. For this, we adapt the proof from [22, Theorem 2.2, p. 94].

Suppose that $A$ is an $r\times c$ matrix, where $r$ or $c$ is strictly larger than $|\mathrm{Pic}(R)\,|$, which in turn is larger than 1. Then we may assume, by Proposition 3.10(i), that

$$r = \mathrm{rk}(A) \text{ or } \mathrm{rk}(A)+1 \text{ and } c = \mathrm{rk}(A) \text{ or } \mathrm{rk}(A)+1,$$

and so $\min(r,c)\ge 2$; otherwise, from above, the size of $A$ would be at most 2. We show that $A$ can be decomposed. By changing to the transpose of $A$ if necessary, we may further assume that $c > |\mathrm{Pic}(R)|$. Let $M = M(A)$ be the module named by $A$. By the MNT, $\mathrm{gen}(\mathcal{T}(M)) + \mathrm{rk}(M) \le c$. If we had strict inequality here, then $A$ would decompose by Proposition 3.10(iii). Thus we may now further assume that $\mathrm{gen}(\mathcal{T}(M)) + \mathrm{rk}(M) = c$. Notice that if $\mathrm{rk}(M)\ge 2$, then, by Proposition 3.10(i), $A$ is already decomposable. Hence we assume $\mathrm{rk}(M)\le 1$. Since $c > |\mathrm{Pic}(R)|$, this shows that $t = \mathrm{gen}(\mathcal{T}(M)) \ge |\mathrm{Pic}(R)|$; if this is an equality, then $\mathrm{rk}(M) = 1$. Therefore we may assume, using Theorem 2.5, that $M$ has the form

$$M \cong \mathcal{T}(M) \oplus \mathcal{TF}(M) \cong (R/I_1)\oplus\cdots\oplus(R/I_t)\oplus J,$$

where $0 \ne I_1 \subseteq \cdots \subseteq I_t \ne R$, $t \ge |\mathrm{Pic}(R)|$, and this inequality is strict if $\mathrm{rk}(M) = 0$, in which case $J = 0$ also. Let $b = |\mathrm{Pic}(R)| \le t$. Consider the following $|\mathrm{Pic}(R)|$ number of elements of $\mathrm{Pic}(R)$:

$$\mathrm{cl}(I_1),\ \mathrm{cl}(I_1)\,\mathrm{cl}(I_2),\ \mathrm{cl}(I_1)\,\mathrm{cl}(I_2)\,\mathrm{cl}(I_3),\ \ldots,\mathrm{cl}(I_1)\,\mathrm{cl}(I_2)\cdots\mathrm{cl}(I_b).$$

Either these are all distinct, in which case one of them is the identity of $\mathrm{Pic}(R)$, or there is a repetition: $\Pi_{i=1}^{k}\mathrm{cl}(I_i) = \Pi_{i=1}^{l}\mathrm{cl}(I_i)$, $1\le k<l\le b$. In the latter case we cancel the product of the first $k$ of them, getting $1 = \Pi_{i=k+1}^{l}\mathrm{cl}(I_i)$. Thus, in either case, we have $1\le l-k\le b$ consecutive ideals whose product is in the identity class in $\mathrm{Pic}(R)$. Let $U$ denote the direct sum of the corresponding cyclic modules: $U = R/I_{k+1}\oplus\cdots\oplus R/I_l$, and so $\mathrm{cl}(U) = 1$. Let $V$ be the direct sum of the remaining cyclics $R/I_i$ in $\mathcal{T}(M)$. Since all of these ideals are totally ordered by inclusion, $\mathrm{gen}(U)+\mathrm{gen}(V) = \mathrm{gen}(\mathcal{T}(M))$. If $V\ne 0$, then $U$ is a proper summand. If $V = 0$, then $t = b = |\mathrm{Pic}(R)|$, but then $\mathrm{rk}(M) = 1$ and $J\ne 0$, and so $U$ is

again a proper summand of $M$. Therefore $U$ satisfies the conditions of part (ii) of Proposition 3.10, and so $A$ is decomposable. This completes the proof of the theorem. □

For readers who prefer nonempty matrices, we have the following remarks.

*Remarks* 3.13. Let $R$ be a Prüfer domain with the stacked bases property and the $1\frac{1}{2}$-generator property. Assume that the group $\mathrm{Pic}(R)$ is finite. Then

(1) Every nonempty matrix $X$ of positive size is equivalent to a matrix of the form

$$\begin{bmatrix} D & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} D \\ 0 \end{bmatrix},$$

where $D$ is a block diagonal sum of nonempty matrices of size at most $|\mathrm{Pic}(R)|$ and 0 symbolizes zero rows or columns below or right of $D$.

(2) Every square matrix of positive size over $R$ is equivalent to a block diagonal sum of nonempty matrices of size at most $|\mathrm{Pic}(R)|$.

*Proof.* Let $X$ be a nonempty matrix of size $n > 0$. By Theorem 3.12, $X$ is equivalent to one that is decomposed as a block diagonal sum of indecomposable matrices of size at most $|\mathrm{Pic}(R)|$, where possibly some are empty matrices. If none are empty, we are done with both items.

If there are empty blocks for item (1), by Lemma 1.4 the empty diagonal blocks can be collected into a block matrix $Y$ at the lower right of the matrix. If $Y$ is not empty, then it is a zero matrix; if it is an empty matrix, then its presence effectively adds zero rows or zero columns.

For item 2, we define as in the appendix the *defect* of an $r \times c$ matrix $A$ to be $r - c$. If among the diagonal blocks one is empty and indecomposable, then it must be $0 \times 1$ or $1 \times 0$. Say there is a $0 \times 1$ block of defect $-1$. Then there must be another block (possibly also empty) of defect $> 0$, and so bringing these next to each other and adding them diagonally we get a nonempty block of the same size but no longer indecomposable (if it was). Continuing this way we eliminate all the empties without increasing sizes but ruining indecomposability. □

## 4. Results for the sum of 2 units problem for matrices

We recall the following facts mentioned in the introduction. Here $\mathbf{u}(R)$ denotes the unit sum number of $R$, defined in the second paragraph of the introduction.

*Fact* 4.1. If $R$ is an associative ring and $n \geq 2$, then every diagonal matrix in $M_n(R)$ is a sum of 2 units [29, Lemma 5, Proposition 6]. It follows that if $R$ is an *elementary divisor ring* (every square matrix is equivalent to a diagonal matrix), then $\mathbf{u}(M_n(R)) \leq 2$, for every $n \geq 2$.

*Fact* 4.2. Every *proper* $n \times n$ matrix over an associative ring is a sum of 3 invertible matrices [14].

We generalize Fact 4.1 in Lemma 4.3 and Proposition 4.4. First, recall that a *permutation matrix* is a square matrix having exactly one "1" in each row and in each column, and all other entries 0. We say that a permutation matrix $P$ of size $n$ *avoids* an $n \times n$ matrix $X$ (and vice versa) if, for every $i, j$ such that the row $i$, column $j$ entry of $P$ is nonzero, the $(i, j)$-entry of $X$ is 0. For example, a diagonal proper matrix always avoids a permutation matrix corresponding to a fixed point free permutation.

**Lemma 4.3.** *Let $R$ be an arbitrary associative ring with 1. Let $n \geq 2$ and suppose that $X$ is an $n \times n$ matrix over $R$ that avoids a permutation matrix $Q$ over $R$. Then $X$ is a sum of 2 units.*

*Proof.* We show there exist $n \times n$ matrices $X_1, X_2$ such that

$$X_1 + X_2 = X, \text{ and } X_1 + Q \text{ and } X_2 - Q \text{ are both invertible.} \tag{4.3.1}$$

First we prove (4.3.1) for $Q = I$, the identity matrix. In this case, all diagonal entries of $X$ are 0. We take for $X_1$ the strictly lower triangular $n \times n$ matrix, with those entries strictly below the diagonal the same as the corresponding entries of $X$ below the diagonal; the rest of the entries of $X_1$ are all 0. Then $X_2$ can be the strictly upper triangular matrix using the (possibly nonzero) entries of $X$ above the diagonal, and the result holds.

Now suppose that $Q$ is an arbitrary $n \times n$ permutation matrix. Recall that every permutation matrix is invertible with inverse equal to its transpose. If a permutation matrix $Q$ has a "1" in the $(i, j_i)$ position, then row $i$ of $QX$ is row $j_i$ of $X$. Similarly, since $Q^\perp$, the transpose of $Q$, has a "1" in the $(j_i, i)$ position, row $j_i$ of $Q^{-1}X = Q^\perp X$ is row $i$ of $X$, with a "0" in the $j_i$ position, and so the $(j_i, j_i)$-entry of $Q^{-1}X$ is zero, for each $i$. Thus $Q^{-1}X$ has a "0" in every diagonal position. Now, by the case $Q = I$ above, there exist $X_1'$ and $X_2'$ so that $Q^{-1}X = X_1' + X_2'$, and $X_1' + I$ and $X_2' - I$ are units. This implies that $X_1 := QX_1'$ and $X_2 := QX_2'$ satisfy condition (4.3.1), as desired, and the lemma is proved. □

Next we show that a blocked matrix of an appropriate size avoids a permutation matrix. For the analogous result for banded matrices, see the proof of Proposition 5.7.

**Proposition 4.4.** *Let $B = \operatorname{diag}(B_1, B_2, \ldots, B_t)$ be an $n \times n$ matrix, where $n$ is an integer greater than 0 and each $B_\ell$ is a matrix of size at most $n/2$. Then there exists an $n \times n$ permutation matrix $Q$ that avoids (the blocks of) $B$.*

*Proof.* If $n \leq 3$, the result holds because the block size is at most 1, and so all nonzero entries are on the diagonal (see proof of Proposition 5.7). Thus we may assume that $t \geq 2$ and $n \geq 4$.

For this proof we consider the *meeting number* $m(P, B)$ of an $n \times n$ permutation matrix $P$ with $B$: $m(P, B) := |\{\text{positions } (i, j) \text{ where both } B \text{ and } P \text{ are nonzero}\}|$. Pick a permutation matrix $P$. If $m(P, B) = 0$, then we are done. If not, then we modify $P$ by a transposition to get a permutation matrix $P'$ with $m(P', B) < m(P, B)$. By continuing this process we finally obtain a permutation matrix $Q$ avoiding $B$.

Therefore let $m(P, B) > 0$ and choose a particular $(i, j)$ so that both $P$ and $B$ are nonzero in the $(i, j)$-entry. Then $(i, j)$ must be inside one of the nonempty blocks of $B$. Observe that the meeting number is unchanged if we permute the rows or columns of both $P$ and $B$ the same way. Thus we may assume (in view of Lemma 1.4) that $(i, j)$ is in the first block $B_1$. Let $(p, q)$ be the type of $B_1$; then $(i, j) \leq (p, q) \leq (\frac{n}{2}, \frac{n}{2})$. Consider $B$ partitioned into four submatrices, of which the top right section is $B_1$ and the strict lower left section determined by the $(p, q)$ position is $A$; that is, $B = \operatorname{diag}(B_1, A)$, where $A = \operatorname{diag}(B_2, \ldots, B_t)$. Also, partition

$P$ into pieces of the same type. We have

$$B = \begin{bmatrix} (B_1)_{p\times q} & 0_{p\times(n-q)} \\ 0_{(n-p)\times q} & A_{(n-p)\times(n-q)} \end{bmatrix},\ P = \begin{bmatrix} (P_1)_{p\times q} & (P_2)_{p\times(n-q)} \\ (P_3)_{(n-p)\times q} & (P_4)_{(n-p)\times(n-q)} \end{bmatrix}, \tag{4.4.1}$$

where the $(i,j)$-entry of $B_1$ is nonzero and the $(i,j)$-entry of $P_1$ is 1.

*Claim.* Referring to (4.4.1), some entry of $P_4$ is 1; that is, there exists a pair $r, c$ with $p < r \leq n$ and $q < c \leq n$ so that $P$ has a 1 in the $(r,c)$-entry (within $P_4$), corresponding to an entry of the $A$ part of $B$.

*Proof.* If no entry of $P_4$ is 1, then $P$ must have $n-p$ "1"s, one for each of the last $n-p$ rows in the bottom left $(n-p)\times q$ section of $P$ called $P_3$. These $n-p$ "1"s must fit in the first $q$ columns of $P$, but *not* in the $j^{\text{th}}$ column, since $P$ already has a "1" in the $(i,j)$-entry. That is, $n-p$ ones must fit into $q-1$ columns of $P$. Since $q$ is the number of columns of $B_1$ and therefore $q-1 < n/2$, whereas $p \leq n/2$ implies $n-p \geq n/2$, this is impossible. Thus the claim holds. □

To complete the proof of Proposition 4.4, we now suppose that $P$ has a "1" in the $(r,c)$ entry, with $p < r \leq n$ and $q < c \leq n$, as in the Claim. Denote by $E_{i,r}$ the elementary matrix that is the outcome of switching rows $i$ and $r$ applied to the identity matrix. Then $E_{i,r}$ is the permutation matrix corresponding to the transposition $(i,r)$, and $P' := E_{i,r}P$ is the result of swapping the rows $i, r$ of $P$. That is, $P'$ has "1" in positions $(r,j)$ and $(i,c)$, both of which are in the 0 blocks of $B$ shown in (4.4.1), since $p < r$, $q < c$, $i \leq p$, $j \leq q$. Otherwise $P'$ has the same "1"s as $P$, except that it does *not* have a "1" in positions $(i,j)$ and $(r,c)$. Thus $P'$ is a permutation matrix and its meeting number $m(P', B) < m(P, B)$. Thus Proposition 4.4 is true. □

*Remark* 4.5. We were surprised to learn that the existence of permutation matrices disjoint or avoiding a given matrix has been extensively studied in combinatorics in connection with a "rook" problem; see for example [18], [16].

**Corollary 4.6.** *If $B$ is an $n \times n$ matrix in block diagonal form where the blocks have size $\leq n/2$ and $n \geq 2$, then $B$ is a sum of two units.*

*Proof.* This follows from Proposition 4.4 and Lemma 4.3. □

All the pieces are now in place for the main result of this section.

**Theorem 4.7.** *Let $R$ be a Prüfer domain with the stacked bases property and the $1\frac{1}{2}$-generator property. Assume that the group $\operatorname{Pic}(R)$ is finite; let $b := |\operatorname{Pic}(R)|$. Then, for every $n \geq 2b$, every $n \times n$ matrix over $R$ is a sum of two units; that is, $M_n(R)$ is 2-good ($\mathbf{u}(M_n(R) = 2)$ for $n \geq 2b$.*

*Proof.* This follows from Theorem 3.12 and Corollary 4.6. □

**Corollary 4.8.** *Every sufficiently large matrix over a ring of algebraic integers is a sum of two units.*

*Proof.* A ring of algebraic integers has a finite Picard (class) group. The size of this group yields a bound as in the theorem above. □

It would be nice to know how sharp this bound is for a ring of algebraic integers. The following proposition is the only tool we know of that can produce matrices that are *not* sums of two units.

**Proposition 4.9** ([29, Proposition 10])**.** *Let $R$ be an arbitrary ring and $n \geq 2$. Let $L = Ra_1 + \cdots + Ra_n$ be a left ideal generated by the elements $a_1, \ldots, a_n \in R$. Let $X$ be the $n \times n$ matrix whose entries are all zero except for the first column, which is $(a_1, \ldots, a_n)^{\perp}$. Suppose that*

(1) *$L$ cannot be generated by fewer than $n$ elements, and*
(2) *$0$ is the only element in $L$ that is a sum of $2$ units.*

*Then $X$ is not a sum of $2$ units.* □

*Remarks* 4.10. As remarked in [29], condition (1) above is the easier to satisfy. Indeed, by a result of I. S. Cohen [5], if $R$ is a commutative Noetherian domain of Krull dimension greater than 1, then for every $n \geq 1$ there is an $n$-generated ideal of $R$ that cannot be generated by fewer than $n$ elements. If $R$ is $\mathbb{Z}[x]$ or $F[x, y]$ where $F$ is a field, then condition (2) is also satisfied by ideals not containing constants, and so by Proposition 4.9, for every $n \geq 2$, $M_n(R)$ is not 2-good. By Fact 4.2, $\mathbf{u}(M_n(R)) = 3$.

Unfortunately, for Dedekind domains and in particular for the ring of algebraic integers, this proposition is of rather limited use. The reason is that in a Dedekind domain ideals need at most 2 generators, and so condition (1) restricts us to the case of $2 \times 2$ matrices. Moreover, it is easy to see that in any ring of algebraic integers with infinite unit group, every nonzero ideal contains a nonzero sum of two units, and so in these rings we cannot satisfy condition (2) above. Among the ring of algebraic integers all this leaves us with is the non-PID complex quadratic case by virtue of Dirichlet's unit theorem. Thus we can just about squeeze out the following example from Proposition 4.9.

**Example 4.11.** Let $A$ be the ring of integers in $\mathbb{Q}[\sqrt{-d}]$, where $d > 0$, $d$ is square-free and $A$ contains an ideal $I$ that is not principal; note that the group of units is finite [13, p. 213]. (For example, in $\mathbb{Z}[\sqrt{-5}]$ the ideal $(3, 1 + \sqrt{-5})$ is not principal, and also the only units are $\pm 1$.) Thus only finitely many elements of $I$ can be sums of 2 units. Consider the sequence of powers $I^n$ of $I$: $I, I^2, \ldots$. Then $\bigcap_{i=1}^{\infty} I^i = (0)$, and so there exists a power $t$ so that no nonzero element of the form $u_1 + u_2$ with $u_1, u_2$ units is in $I^t$. In case $I^t$ is principal, some larger power $I^k$ is not principal and still contains no nonzero elements that are sums of two units. Thus by Proposition 4.9, $\mathbf{u}(M_2(A)) \neq 2$, and so by Fact 4.2, $\mathbf{u}(M_2(A)) = 3$. In the case of $A = \mathbb{Z}[\sqrt{-5}]$ the class group has order 2. Therefore it follows from Theorem 4.7 that $\mathbf{u}(M_n(A)) = 2$, if $n \geq 4$. We do not know whether $\mathbf{u}(M_3(A)) = 2$ or 3.

**Question 4.12.** What is the value of $\mathbf{u}(M_n(R))$ for $2 \leq n < 2|\mathrm{Pic}(R)|$ if $R$ is a ring of algebraic integers?

## 5. Appendix

Here we include some results related to the material in the main body of the paper. This includes the notion of the unit sum monoid and a discussion of banded matrices.

**Part I: The unit sum monoid of a ring.**

Question 4.12 is related to the larger problem concerning unit sum numbers of matrix rings for a fixed ring. This can be phrased in terms of additive submonoids of the set of nonnegative integers $\mathbb{N}_0$.

**Definition 5.1.** The *unit sum matrix monoid* of an associative ring $R$, $\mathrm{USM}(R)$, is the additive submonoid of $\mathbb{N}_0$ given by

$$\mathrm{USM}(R) := \{n \geq 1 \mid \mathbf{u}(M_n(R)) = 2\} \cup \{0\}.$$

(It is shown in [29, Corollary 8] that $\mathrm{USM}(R)$ is indeed an additive monoid.)

Shifting our focus to the monoid, we have the following questions:

**Questions 5.2.** What submonoids of $\mathbb{N}_0$ arise as $\mathrm{USM}(R)$ for some ring $R$? What if $R$ is a ring of algebraic integers?

As corollaries to our results above and in [22], we have partial answers to Questions 5.2:

**Corollary 5.3.** *The unit sum matrix monoid of a ring of algebraic integers is co-finite in $\mathbb{N}_0$. More generally:*

(1) *Let $R$ be a Prüfer domain with the stacked bases property and the $1\frac{1}{2}$-generator property and such that $b := |\mathrm{Pic}(R)| < \infty$. Then $\{0, 2b, 2b+1, \dots\} \subseteq \mathrm{USM}(R)$.*
(2) *Suppose that $R$ is a Dedekind domain and $e < \infty$ is the exponent of $\mathrm{Pic}(R)$. That is, $e$ is the least positive integer such that, for every ideal $I$ of $R$, $I^e$ is principal. Then $\{2(2e-1), 4e-1, 4e, \dots\} \subseteq \mathrm{USM}(R)$.*

*Proof.* The first sentence is just a reformulation of Corollary 4.8. Part (1) is a direct result of Theorem 4.7. For part (2), we use the result of Levy, that indecomposable matrices over such rings have size bounded by $2e-1$ [22, Proposition 2.6]. Thus, by Corollary 4.6, we see that matrices of size at least $4e-2$ are sums of two invertible matrices. □

*Remarks* 5.4. (1) If $R$ is a division ring with at least 3 elements, then $\mathrm{USM}(R) = \mathbb{N}_0$. More generally, if $R$ is an elementary divisor ring, then $\{0, 2, 3, \dots\} \subseteq \mathrm{USM}(R)$ with equality for $R = \mathbb{Z}$, by Fact 4.1. On the other hand, if $R = \mathbb{Z}[x]$ or $\mathbb{Q}[x, y]$, then $\mathrm{USM}(R) = \{0\}$, by Remarks 4.10.

(2) Since $|\mathrm{Pic}(Z[\sqrt{-5}])| = 2$, Theorem 4.7 or [22] shows that $\{0, 4, 5, 6, \dots\} \subseteq \mathrm{USM}(\mathbb{Z}[\sqrt{-5}])\}$. By Example 4.11 and [1, Theorem 6], $2 \notin \mathrm{USM}(\mathbb{Z}[\sqrt{-5}])$ and $1 \notin \mathrm{USM}(\mathbb{Z}[\sqrt{-5}])$. But our methods do not show whether or not $3 \in \mathrm{USM}(\mathbb{Z}[\sqrt{-5}])$.

**Part II: Banded matrices and the unit sum problem.**

**Definition 5.5.** A (not necessarily square) matrix $(a_{ij})$ over a ring is *$b$-banded* if all the nonzero entries lie strictly within a "band" of width $b$ around the main diagonal so that the entry $a_{ij}$ in row $i$, column $j$, is 0 whenever $|i - j| \geq b$.

*Remarks* 5.6. In applied matrix theory, the term *$(p, q)$-banded* is sometimes used. A *$(p, q)$-banded* matrix is all zeroes except possibly the terms within the $p$ positions to the *left* of every diagonal entry and the $q$ positions to the *right* of diagonal entries. With that notation, for an $n \times n$ matrix, $(n, 0)$-banded is equivalent to lower triangular and $(0, n)$-banded is equivalent to upper triangular.

In our notation, a 1-banded matrix is just a diagonal matrix; our "2-banded" matrix is sometimes referred to as a tri-diagonal matrix. Banded matrices appear often in the literature, e.g. in relation to the finite element method in numerical linear algebra. Banded matrices with thin bands are another category of "close to diagonal" matrices, as are the diagonally blocked matrices with small blocks.

The next proposition relates thinly banded matrices to the sum of units problem:

**Proposition 5.7.** (1) *Suppose that $n \geq 2b$ and that $X$ is a $b$-banded $n \times n$ matrix over an arbitrary associative ring $R$. Then $X$ is a sum of two invertible matrices.*

(2) *If a ring $R$ is such that every $n \times n$ matrix is equivalent to a $b$-banded matrix, with $n \geq 2b$, then every $n \times n$ matrix is a sum of two invertible matrices.*

*Proof.* It suffices to prove the first statement. Such a $b$-banded matrix $X$ avoids a permutation matrix $Q$ of the form below. To see this, partition $Q$ and $X$ in a like manner as shown:

$$Q_{n\times n} = \begin{bmatrix} 0_{(n-b)\times b} & I_{(n-b)\times(n-b)} \\ I_{b\times b} & 0_{b\times(n-b)} \end{bmatrix}, \quad X_{n\times n} = \begin{bmatrix} A_{(n-b)\times b} & B_{(n-b)\times(n-b)} \\ C_{b\times b} & D_{b\times(n-b)} \end{bmatrix}.$$

First note that $B$ is strictly lower triangular. That is, the entries on the diagonal and above are all zero, since for each $j \geq i$, the $(i,j)$-entry of $B$, $b_{ij}$, is the $(i, j+b)$-entry of $X$ and since $j+b \geq i+b$, we have $j+b-i \geq b$, which implies $b_{ij} = x_{i,j+b} = 0$. Similarly $C$ is strictly upper triangular, since, for $c_{ij}$ the $(i,j)$-entry of $C$ with $i \geq j$, $c_{ij}$ is the $(n-b+i, j)$-entry of $X$. Then $n-b \geq b$, $n-b+i \geq b+j$, and thus $n-b+i-j \geq b$; hence $c_{ij} = x_{n-b+i,j} = 0$. Now the proposition follows from Lemma 4.3. □

When we first began investigating diagonally blocked matrices, having observed Proposition 5.7, we wanted to show that diagonally blocked square matrices could be rearranged to form banded matrices with the same size bands as the blocks. This turned out to be not quite true. The following lemma, Lemma 5.8, is a nice related result due to Byott and Vámos [4]. Since we now have a direct proof of Proposition 4.4, Lemma 5.8 is no longer necessary for the main part of the paper. For $A$ an $r \times c$ matrix, the *defect* $f$ of $A$ is defined by $f := r - c$.

**Lemma 5.8** (cf. [4])**.** *Let $b$ and $t$ be positive integers and let $t$ matrices of sizes at most $b$ be given. Suppose that the sum of the defects of these $t$ matrices is zero, so that the block diagonal sum (in any given order) is a square matrix. Let $d$ be the maximum of the moduli of the $t$ matrix defects. Then we can arrange these matrices in a sequence $A_1, \ldots, A_t$ so that the block diagonal sum $A = \operatorname{diag}(A_1, \ldots, A_t)$ is $b + [d/2]$-banded; here $[d/2]$ denotes the integer part of $d/2$.*

This bound is sharp as the following example shows. Consider a $13 \times 13$ matrix consisting of blocks of types and defects $1 \times 4, -3$; $4 \times 3, 1$; $4 \times 3, 1$; and $4 \times 3, 1$:

$$\begin{bmatrix} \boxed{1\times 4} & & & 0 \\ & \boxed{4\times 3} & & \\ & & \boxed{4\times 3} & \\ 0 & & & \boxed{4\times 3} \end{bmatrix}_{13\times 13}.$$

Then $d = 3$, $b = 4$ and this matrix can be 5-banded as shown above, but no permutation of the blocks will yield a 4-banded matrix. However, the following corollary is immediate.

**Corollary 5.9.** *In the notation of* Lemma 5.8*, suppose that the modulus of the defect of each matrix is at most one. Then we can arrange the matrices in a sequence $A_1, \ldots, A_t$ so that the block diagonal sum* $\operatorname{diag}(A_1, \ldots, A_t)$ *is $b$-banded.* □

The result of this corollary would be sufficient for our purpose of decomposing block diagonal matrices into sums of units, since our blocks have defect at most one by Theorem 3.12.

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