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AVERAGES OVER STARLIKE SETS, STARLIKE MAXIMAL FUNCTIONS, AND HOMOGENEOUS SINGULAR INTEGRALS

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ABSTRACT. We improve some of the results in our 1999 paper concerning weighted norm estimates for homogeneous singular integrals with rough kernels. Using a representation of such integrals in terms of averages over starlike sets, we prove a two-weight L^p inequality for $1 < p < 2$ which we were previously able to obtain only for $p \geq 2$. We also construct examples of weights that satisfy conditions which were shown in our earlier paper to be sufficient for one-weight inequalities when $1 < p < \infty$.

1. INTRODUCTION AND MAIN RESULTS

Homogeneous singular integrals on $\mathbf{R}^d$, $d \geq 2$, are most commonly defined in the form

$$T_\Omega f(x) = p.v. \int_{\mathbf{R}^d} f(x-y)\frac{\Omega(y)}{|y|^d}dy,$$

where Ω is homogeneous of degree 0 and integrable on the unit sphere $\mathbf{S}^{d-1}$, with integral 0. In [WW], the authors derived an alternative representation in terms of averages over the set $S = S(\Omega)$ defined by

$$\begin{aligned} S &= \left\{ x \,:\, |\Omega(x)|/|x|^d \geq 1 \right\} \\ &= \left\{ x : x = r\theta,\ \ \theta \in \mathbf{S}^{d-1},\ 0 \leq r \leq |\Omega(\theta)|^{1/d} \right\}. \end{aligned} \tag{1.1}$$

This set is starlike with respect to the origin, and $|S| < \infty$ iff $\Omega \in L^1(\mathbf{S}^{d-1})$. Also define dilates $tS = \{tx : x \in S\}$, $t > 0$, and averages

$$\begin{aligned} A_t f(x) =& \frac{1}{t^d}\int_{tS} f(x-y)\,\mathrm{sgn}\,\Omega(y)dy \\ =& \int_S f(x-ty)\,\mathrm{sgn}\,\Omega(y)dy, \end{aligned}$$

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where $\operatorname{sgn}\Omega(y) = \Omega(y)/|\Omega(y)|$ is the complex sign of Ω (arbitrarily defined when $\Omega(y) = 0$ or ∞).

From [WW, Theorem 1.1] we have the representation formula

$$\mathrm{T}_\Omega f(x) = c_\Omega \cdot f(x) + d\int_0^\infty A_t f(x)\frac{dt}{t}, \tag{1.2}$$

which holds under the original conditions of the Calderón-Zygmund method of rotations [CZ], i.e., if Ω is odd the formula holds when $\Omega \in L^1(\mathbf{S}^{d-1})$ with $c_\Omega = 0$, and it holds for any $\Omega \in L\log L(\mathbf{S}^{d-1})$ with

$$c_\Omega = \frac{1}{d}\int\limits_{\mathbf{S}^{d-1}} \Omega(\theta)\log|\Omega(\theta)|d\theta.$$

Formula (1.2) is valid in the pointwise sense when f is in a class of functions which includes the Schwartz class $\mathcal{S}$, and it also holds in the sense of convergence in L^2 operator norm for any $f \in L^2(\mathbf{R}^d)$.

The representation is used in [WW] to prove various one- and two-weight inequalities for T_Ω when $\Omega \in L\log L(\mathbf{S}^{d-1})$ with integral 0. Many of the inequalities so derived are described fairly precisely in terms of a covering of S by rectangles centered at the origin, in large part adapting methods used in [CWW] for fractional integral and maximal operators associated with starlike sets.

Of particular interest to this paper is the following result of [WW]. We call any nonnegative locally integrable function a "weight".

Theorem A ([WW, Theorem 5.2]). *If $\Omega \in L\log L(\mathbf{S}^{d-1})$ and Ω has integral 0 over $\mathbf{S}^{d-1}$, then for $p \geq 2$ and $r > 1$ there is a constant $C_{p,r}$ so that for all weights v and all $f \in \mathcal{S}$,*

$$\int |\mathrm{T}_\Omega f|^p v\,dx \leq C_{p,r}\int |f|^p\{\mathrm{M}\,G^*\,\mathrm{M}(v^r)\}^{1/r}\,dx. \tag{1.3}$$

Here M is the Hardy–Littlewood maximal operator and G^* is a positive operator bounded on $L^p(\mathbf{R}^d)$ for $1 < p \leq \infty$ which we will define later. For now, we will just say that G^* is constructed from S, which was in turn constructed from Ω.

When $v = 1$, since both M and G^* are bounded on $L^\infty(\mathbf{R}^d)$, (1.3) reduces to a classical result of A. P. Calderón and A. Zygmund, although that result is valid for the larger p range $1 < p < \infty$. The methods of [WW] were insufficient to prove inequality (1.3) for $1 < p < 2$ under the conditions of Theorem A, although the stronger integrability requirement that $\Omega \in L(\log L)^\gamma(\mathbf{S}^{d-1})$ for *all* $\gamma > 0$ was shown to suffice. This situation is undesirable, not only for esthetic reasons but also because of results which are corollaries of the estimate for the full range of p values. For example, vector-valued inequalities can be derived as a consequence of two-weight inequalities such as (1.3).

In this paper, using somewhat different methods, we will prove the following result.

Theorem I. *Under the same conditions on Ω as in Theorem A,* (1.3) *also holds for $1 < p < 2$.*

With stronger assumptions on Ω than those in Theorem A, there is a history of estimates such as (1.3), but with weights which are smaller than $\{\mathrm{M}\,G^*\,\mathrm{M}(v^r)\}^{1/r}$ on the right side. For example, see [CF] for cases when the weight on the right can be $\{\mathrm{M}(v^r)\}^{1/r}$, $r > 1$, and see [Wi] and [P] for cases when it can be an appropriate iterate of M. As alluded to above, Theorem I (together with Theorem A when $p \geq 2$) has the following consequence:

Corollary. *For each $j \in \mathbf{Z}$, suppose that Ω_j has integral 0 on $\mathbf{S}^{d-1}$, and define $\Omega_*(\theta) = \sup_{j\in\mathbf{Z}} |\Omega_j(\theta)|$. If $\Omega_* \in L\log L(\mathbf{S}^{d-1})$, then the vector–valued inequalities*

$$\|\,\Big\{\sum_j |\,\mathrm{T}_{\Omega_j} f_j|^q\Big\}^{1/q}\,\|_p \leq C_{p,q}\|\,\Big\{\sum_j |f_j|^q\Big\}^{1/q}\,\|_p \tag{1.4}$$

hold for $1 < p, q < \infty$.

We will also derive two-weight norm estimates for an analogue of the Hardy-Littlewood maximal function which is related to a starlike set. Given any set S that is starlike with respect to the origin (and not necessarily derived from a function Ω with mean zero as above), the starlike maximal operator relative to S, denoted M_S, is defined by

$$\mathrm{M}_S\, f(x) = \sup_{t>0} t^{-d} \int_{tS} |f(x-y)|dy, \tag{1.5}$$

where $tS = \{\, ty : y \in S\,\}$. If χ_E denotes the characteristic function of E, then M_S can be rewritten as

$$\mathrm{M}_S\, f(x) = \sup_{t>0} t^{-d} \chi_{tS} * |f|(x).$$

Note that $||\,\mathrm{M}_S\, f||_\infty \leq |S|\,||f||_\infty$; in fact, M_S is bounded on unweighted $L^p(\mathbf{R}^d)$ if $1 < p \leq \infty$ and $|S| < \infty$ (see the comments at the beginning of the proof of Theorem 3.8 in §3).

We associate with S the radial boundary function ρ defined on $\mathbf{S}^{d-1}$ by

$$\rho(\theta) = \sup\{r \mid r \geq 0 \text{ and } r\theta \in S\}.$$

Even though the boundary of S may not have measure 0, it is not hard to see from the starlike structure of S and Fubini's theorem that $|S| < \infty$ is equivalent to $\rho \in L^d(\mathbf{S}^{d-1})$. In the case where S is derived from Ω as in (1.1), we have $\rho(\theta) = |\Omega(\theta)|^{1/d}$, and then $|S| < \infty$ if and only if $\Omega \in L^1(\mathbf{S}^{d-1})$. In fact, if S is a starlike set about the origin and Ω is *any* function with $|\Omega(\theta)| = \rho(\theta)^d$, then $|S| < \infty$ if and only if $\Omega \in L^1(\mathbf{S}^{d-1})$; moreover $\rho \in L^d \log L(\mathbf{S}^{d-1})$ if and only if $\Omega \in L\log L(\mathbf{S}^{d-1})$.

M_S satisfies a two-weight inequality similar to (1.3). This is stated in the next theorem.

Theorem II. *Let S be a starlike set about the origin whose radial boundary function $\rho \in L^d \log L(\mathbf{S}^{d-1})$. Then for $1 < p < \infty$ and $r > 1$ there is a constant $C_{p,r}$ so that*

$$\int |\,\mathrm{M}_S\, f|^p v\, dx \leq C_{p,r} \int |f|^p [\{\mathrm{M}(v^r)\}^{1/r} + \mathrm{M}\, G^* v]\, dx \tag{1.6}$$

for all weights v.

Here the operator G^* is constructed from S just as in Theorem A. In the case where S is constructed from an Ω as in (1.1), then of course the condition $\rho \in L^d \log L(\mathbf{S}^{d-1})$ is the same as $\Omega \in L\log L(\mathbf{S}^{d-1})$. Unweighted estimates for M_S again follow easily by choosing $v = 1$ and using the fact that G^* and M are bounded on $L^\infty(\mathbf{R}^d)$.

It is well known that $\mathrm{M}\, v \geq v$ for any v, and it can be shown (see [WW]) that $G^* v \geq cv$ a.e. Combining these inequalities and observing that M and G^* are positive operators, it is not difficult to see that

$$\{\mathrm{M}(v^r)\}^{1/r} + \mathrm{M}\, G^* v \leq c\{\mathrm{M}\, G^*\, \mathrm{M}(v^r)\}^{1/r} \quad \text{a.e.}$$

Hence, inequality (1.6) implies that (1.3) holds with M_S in place of T_Ω. The weight on the right of (1.6) can be much smaller than the weight on the right of (1.3), so Theorem II improves upon the maximal operator results of [WW], since all inequalities proved there for M_S corresponded to ones holding for T_Ω.

In the case where S is the unit ball, $\mathrm{M}_S\, f$ reduces to the Hardy–Littlewood maximal function. In this case, the weight on the right of (1.6) can be the smaller weight $\mathrm{M}\, v$ by a classical result of [S].

Given $r > 1$ and a starlike set S about the origin for which $\rho \in L^d \log L(\mathbf{S}^{d-1})$, it is possible to construct many weights v that satisfy

$$\{\mathrm{M}\, G^*\, \mathrm{M}(v^r)(x)\}^{1/r} \leq c\, v(x) \tag{1.7}$$

for all x; see the last remark at the end of the paper. Consequently, by Theorems I and A, if $\Omega \in L \log L(\mathbf{S}^{d-1})$ and Ω has integral zero, then T_Ω is a bounded operator on $L^p(v)$ for such v and all $1 < p < \infty$. Similarly, by Theorem II, M_S is bounded on $L^p(v)$ for such v if $1 < p < \infty$ and $\rho \in L^d \log L(\mathbf{S}^{d-1})$.

We also consider results which guarantee that T_Ω is a bounded operator on $L^p(w)$ for a particular value of $p, 1 < p < \infty$, and a particular weight w. In fact, assuming only that $\Omega \in L \log L(\mathbf{S}^{d-1})$ with integral 0, a condition on a weight w which implies that T_Ω is bounded on $L^p(w)$ for a particular p with $1 < p < \infty$ is given in [WW]. We do not know if it is true that a weight v which satisfies (1.7) also satisfies the condition in [WW]. However, in §3 of this paper we will construct examples of weights which satisfy the requirements in [WW], provided the associated starlike set S satisfies a condition that measures, roughly speaking, the "branching" of S. For the starlike maximal operator M_S, a sufficient condition for boundedness of M_S on a particular space $L^p(w)$ is also derived in [WW], as well as in [CWW], provided only that $\rho \in L^d(\mathbf{S}^{d-1})$. We will also construct examples of such weights w in §3.

Let us briefly recall the sufficient condition given in [WW] for boundedness of T_Ω on $L^p(w)$ if $1 < p \leq 2$. More details, including the case when $p > 2$, are given in §3. The condition is expressed in terms of a collection of rectangles $\{R_{m,k}\}$ centered at the origin with orientations chosen so that S is efficiently covered by the rectangles. The index m corresponds to the subset $S_m \subset S$ where the radial boundary function $\rho \approx 2^m$, and the collection $\{R_{m,k} : 1 \leq k < k_m\}$, with k_m possibly infinite, consists of rectangles that cover this part of S. For each m, k_m essentially measures how much the set S is "branched out" at distance 2^m from the origin. For example, consider the simple case when $d = 2$ and $\Omega(\theta)$ is the even, 2π-periodic function equal to $\theta^{-\alpha}, 0 < \alpha < 1$, when $0 < \theta < \pi/2$ and equal to $-(\pi - \theta)^{-\alpha}$ when $\pi/2 < \theta < \pi$. In this situation, $\rho(\theta) = \theta^{\alpha/2}$ when $0 < \theta < \pi/2$, and consequently each S_m can be covered by a single long thin rectangle $R_{m,1}$ along the abscissa with dimensions essentially 2^m and $2^{-m(\frac{2}{\alpha}-1)}$.

For any $\Omega \in L \log L(\mathbf{S}^{d-1})$ with integral zero, it is proved in [WW, Theorem 1.5] that if $1 < p \leq 2$, then

$$\int_{\mathbf{R}^d} |T_\Omega f|^p\, w\, dx \leq C \int_{\mathbf{R}^d} |f|^p\, w\, dx \tag{1.8}$$

for all $f \in \mathcal{S}$, with C independent of f, provided there exists $r > 1$ so that

$$\left(\frac{1}{|R|}\int_R w\,dx\right)^{\frac{1}{p}}\left(\frac{1}{|R|}\int_R w^{-rp'/p}\,dx\right)^{\frac{1}{rp'}} \le \frac{c_{m,k}}{|R_{m,k}|} \tag{1.9}$$

for all k, m and all rectangles R that are translates and/or dilates (but not rotations) of $R_{m,k}$, and provided the constants $c_{m,k}$ satisfy

$$\sum_{m,k}(m+1)c_{m,k} < \infty. \tag{1.10}$$

Moreover, in case $p = 2$, it is possible to choose $r = 1$ in (1.9).

For example, since (see §3)

$$\sum_{m,k}(m+1)|R_{m,k}| < \infty \quad \text{if and only if } \Omega \in L\log L(\mathbf{S}^{d-1}),$$

it follows by choosing $c_{m,k}$ to be a fixed multiple of $|R_{m,k}|$ that (1.8) holds for a given p with $1 < p \le 2$ if

$$\left(\frac{1}{|R|}\int_R w\,dx\right)^{\frac{1}{p}}\left(\frac{1}{|R|}\int_R w^{-rp'/p}\,dx\right)^{\frac{1}{rp'}} \le C$$

for all m, k and R as above, with C independent of m, k and R.

We will construct an example of a weight w which satisfies (1.9) and (1.10) if the starlike cover of S satisfies the additional condition that for some $\epsilon > 0$,

$$\sum_{m,k}(m+1)|R_{m,k}|^{1-\epsilon} < \infty. \tag{1.11}$$

The exact statement is given in Theorem 3.8. As we will show in §3, condition (1.11) holds for any starlike set S which satisfies $\rho \in L^s(\mathbf{S}^{d-1})$ for some $s > d$, provided S has bounded branching, i.e., provided k_m is bounded in m. In general, condition (1.11) is a geometric condition related not only to the size of $|\rho|$ but also to how the size is distributed. We do not know how to characterize it in terms of just a simple integrability condition, although we will show in §3 that (1.11) implies $\rho \in L^d \log L(\mathbf{S}^{d-1})$.

We thank the referee for a careful reading of the manuscript of this paper and for helpful comments, including an alternative way (presented in §2) to derive boundedness of the operators G, G^* on $L^p(\mathbf{R}^d), 1 < p \le \infty$, by using the method of rotations.

2. Proof of the theorems

We will use the notation of [WW] whenever possible. We first recall from [WW] the definition of the operator G^*. Let S be starlike about the origin and ρ be the radial boundary function of S defined earlier. Decompose ρ at heights 2^m, $m \ge 0$, by letting

$$\Theta_0 = \left\{\, \theta \in \mathbf{S}^{d-1} : \rho(\theta) \le 1 \,\right\},$$

and for $m > 0$,

$$\Theta_m = \left\{\, \theta \in \mathbf{S}^{d-1} : 2^{m-1} < \rho(\theta) \le 2^m \,\right\}.$$

Let S_m be the starlike set whose boundary function is $\rho\chi_{\Theta_m}$. Then S is the disjoint (except for the origin) union of $\left\{S_m\right\}_{m=0}^{\infty}$.

We define the unsigned averages A_{t+} relative to S by

$$A_{t+}f(x) = \frac{1}{t^d}\int\limits_{tS} f(x-y)dy = \int\limits_{S} f(x-ty)dy,$$

and we define the unsigned averages A^m_{t+} relative to S_m by

$$A^m_{t+}f(x) = \frac{1}{t^d}\int\limits_{tS_m} f(x-y)dy = \int\limits_{S_m} f(x-ty)dy.$$

Thus,

$$M_S f(x) = \sup_{t>0} A_{t+}(|f|)(x) \quad \text{and} \quad M_{S_m} f(x) = \sup_{t>0} A^m_{t+}(|f|)(x).$$

In the case where ρ arises from a function Ω as in Theorem I (so that $\rho(\theta) = |\Omega(\theta)|^{1/d}$), we also define the signed averages A^m_t by

$$A^m_t f(x) = \frac{1}{t^d}\int\limits_{tS_m} f(x-y)\operatorname{sgn}\Omega(y)dy = \int\limits_{S_m} f(x-ty)\operatorname{sgn}\Omega(y)dy.$$

The averages A^m_t will be used later; they are just the averages A_t defined in the introduction with S_m replacing S. We now use the averages A^m_{t+} to define

$$Gf(x) = \sum_{m\geq 0}(m+1)\,\mathrm{M}_{S_m} f(x) = \sum_{m\geq 0}(m+1)\sup_{t>0} A^m_{t+}(|f|)(x).$$

Observation 4.3 of [WW] showed that G is a bounded operator on $L^p(\mathbf{R}^d)$ if $1 < p \leq \infty$ and $\rho \in L^d \log L(\mathbf{S}^{d-1})$. Alternatively, this can be seen by the method of rotations, since by polar coordinates,

$$\begin{aligned} A^m_{t+}(|f|)(x) &= \int_{\Theta_m}\int_0^{\rho(\theta)} |f(x-tr\theta)| r^{d-1} dr d\theta \\ &\leq \int_{\Theta_m} \rho(\theta)^d M_\theta f(x) d\theta, \end{aligned}$$

where M_θ is the one-dimensional Hardy-Littlewood maximal operator in the direction $\theta \in \mathbf{S}^{d-1}$. Taking the supremum in $t > 0$, we see that this inequality holds for M_{S_m} in place of A^m_{t+}, and since $\log_+ \rho(\theta) = \log(\max\{1, \rho(\theta)\}) > m-1$ on Θ_m, then

$$Gf(x) \leq \int_{\mathbf{S}^{d-1}} \rho(\theta)^d (2 + \log_+ \rho(\theta)) M_\theta f(x) d\theta,$$

and so the $L^p(\mathbf{R}^d)$ boundedness of G follows from the uniform $L^p(\mathbf{R}^d)$-boundedness of M_θ.

Observing that $A_{t+} = \sum_{m\geq 0} A^m_{t+}$, then

$$\begin{aligned} \mathrm{M}_S f(x) &= \sup_{t>0}\sum_{m\geq 0} A^m_{t+}(|f|)(x) \leq \sum_{m\geq 0}\sup_{t>0} A^m_{t+}(|f|)(x) \\ &= \sum_{m\geq 0} M_{S_m} f(x) \leq Gf(x). \end{aligned}$$

Consequently, M_S is bounded on $L^p(\mathbf{R}^d)$ for $1 < p < \infty$ whenever G is bounded. In fact, a method of rotations argument similar to the one above gives L^p-bounds assuming only that $\rho \in L^d(\mathbf{S}^{d-1})$, or equivalently $|S| < \infty$. Boundedness of M_S in the case where $|S| < \infty$ was originally proved in [C]. It was also proved by different methods in [CWW].

We define A^{m*}_{t+}, the adjoint of A^m_{t+}, by replacing S_m with $-S_m = \{-x : x \in S_m\}$ in the definition of A^m_{t+}, and we define G^* to be the operator obtained by making the corresponding replacements in the definition of G:

$$G^* f(x) = \sum_{m\geq 0} (m+1)\, \mathrm{M}_{-S_m} f(x) = \sum_{m\geq 0} (m+1) \sup_{t>0} A^{m*}_{t+}(|f|)(x). \tag{2.1}$$

Then $G^* f \geq M_{-S} f$.

We shall prove Theorem I by using the $p = 2$ result of [WW]. It suffices to prove the corresponding weak–type (p, p) inequalities, i.e., to prove that for $r > 1$, $1 < p < 2$ and any weight v,

$$v\{x \,:\, |\,\mathrm{T}_\Omega f(x)| > \lambda\} \leq \frac{C}{\lambda^p} \int |f|^p \{MG^*M(v^r)\}^{1/r}\, dx, \qquad \lambda > 0,$$

(where $v(E) = \int_E v(x)\, dx$) with a constant C that does not depend on v, f or λ. Once this estimate is established, the strong–type inequality (1.3) follows from the Marcinkiewicz interpolation theorem by using this weak-type estimate and the inequality for $p = 2$ from [WW]. We will prove the weak-type (p, p) estimate by using a Calderón-Zygmund decomposition, which will again use the $p = 2$ inequality, as well as Fourier transform decay estimates from [WW]. The argument is an adaptation of one from [W5], which derives two-weight inequalities for operators associated with general measures. A principal difference between the result in [W5] and the present one is that the measures in [W5] are required to satisfy a support condition that starlike sets do not have. In addition, the decomposition in [W5] is complicated by the fact that it is carried out in a general homogeneity setting not needed here.

Lemma 2.2 [Weighted Calderón-Zygmund decomposition]. *Let u be a non-negative measurable function that is not 0 on some set of positive measure, and let $1 \leq p < \infty$ and $f \in L^p(\mathrm{M}\, u)$. Then, given $\lambda > 0$, there is a family $\mathcal{Q}$ of disjoint dyadic cubes and a decomposition $f = g + b = g + \sum_{Q\in\mathcal{Q}} b_Q$ such that*

$$b_Q(x) = \left(f(x) - \frac{1}{|Q|} \int_Q f\, dy \right) \chi_Q(x) \text{ for each } Q \in \mathcal{Q}, \tag{2.3}$$

$$|g(x)| \leq c_1 \lambda \qquad a.e., \tag{2.4}$$

$$\lambda < \frac{1}{|Q|} \int_Q |f|\, dy \leq c_2 \lambda \qquad \text{for each } Q \in \mathcal{Q}, \tag{2.5}$$

$$\sum_{Q\in\mathcal{Q}} u(Q^*) \leq \frac{c_3}{\lambda^p} \int |f|^p\, \mathrm{M}\, u\, dy, \tag{2.6}$$

$$\int |b|^q u\, dy \leq \frac{C_{p,q}}{\lambda^{p-q}} \int_{\mathbf{R}^d} |f|^p\, \mathrm{M}\, u\, dy \qquad \text{if} \quad 1 \leq q \leq p, \tag{2.7}$$

$$\int |g|^p u\, dy \leq c_4 \int_{\mathbf{R}^d} |f|^p\, \mathrm{M}\, u\, dy. \tag{2.8}$$

Here Q^* is the standard "double" of Q, i.e., Q^* is the cube with the same center as Q and twice the edgelength, and c_1, c_2, c_3, c_4, $C_{p,q}$ are structural constants independent of u, f and λ.

Before beginning the proof, we note that there are known Calderón-Zygmund decompositions which are obtained as consequences of a Whitney decompostion argument, but that such decompositions do not give the first inequality in (2.5). Since we need both inequalities in (2.5) in order to derive (2.6) and (2.7), we will instead use a "stopping time" argument to derive the lemma. We also note that the collection $\mathcal{Q}$ in the lemma may be empty. This would be the case for example if u is not locally integrable, since then $\mathrm{M}\,u$ is identically $+\infty$ and consequently f is identically 0.

Proof. Let us assume for now the existence of a disjoint dyadic collection $\mathcal{Q}$ which satisfies (2.3)–(2.5). These are standard results of a typical Calderón-Zygmund decomposition argument once a minor technical point is established. We demonstrate how (2.6)–(2.8) then follow. For the purposes of this argument, we assume that the Hardy–Littlewood maximal operator M is the uncentered maximal operator relative to all cubes, i.e.,

$$\mathrm{M}\,f(x) = \sup_{\{Q:Q\ni x\}} \frac{1}{|Q|} \int\limits_Q |f|\,dy.$$

From this we see that

$$\inf_{\{x:\ x\in Q\}} \mathrm{M}\,u(x) \geq \frac{u(Q')}{|Q'|} \tag{2.9}$$

for any cube Q' containing Q. One consequence of this is the equality

$$\frac{1}{|Q|} \int_Q |h|\,dy \leq \frac{1}{u(Q)} \int_Q |h|\,\mathrm{M}\,u\,dx \tag{2.10}$$

which holds for any cube Q. Another consequence of (2.9) is that if $Q \in \mathcal{Q}$, then

$$\begin{aligned}
\int\limits_Q |f|^p\,\mathrm{M}\,u\,dy &\geq \frac{u(Q^*)}{|Q^*|} \int\limits_Q |f|^p\,dy \\
&= \frac{u(Q^*)}{2^d|Q|} \int\limits_Q |f|^p\,dy \\
&\geq \frac{u(Q^*)}{2^d} \Big(\frac{1}{|Q|} \int\limits_Q |f|\,dy\Big)^p \qquad \text{by Jensen's inequality} \\
&\geq \frac{\lambda^p}{2^d} u(Q^*) \qquad \text{by (2.5).}
\end{aligned}$$

Because the cubes in $\mathcal{Q}$ are pairwise disjoint, we therefore have

$$\begin{aligned}
\sum_{Q\in\mathcal{Q}} u(Q^*) &\leq \frac{2^d}{\lambda^p} \sum_{Q\in\mathcal{Q}} \int\limits_Q |f|^p\,\mathrm{M}\,u\,dy \\
&\leq \frac{2^d}{\lambda^p} \int\limits_{\mathbf{R}^d} |f|^p\,\mathrm{M}\,u\,dy.
\end{aligned}$$

This gives (2.6). To show (2.7), we only have to establish the inequality for $q = 1$ and $q = p$ since the whole inequality then follows by convexity. We begin with the inequality for $q = p$. With b_Q defined by (2.3), write $b = \sum_{Q\in\mathcal{Q}} b_Q = b_1 - b_2$, where

$$b_1(x) = \sum_{Q\in\mathcal{Q}} f(x)\chi_Q(x), \qquad b_2(x) = \sum_{Q\in\mathcal{Q}} (av_Q f)\,\chi_Q(x),$$

and

$$av_Q f = \frac{1}{|Q|}\int\limits_Q f\,dy.$$

Observe by Jensen's inequality and (2.10) that

$$\big|av_Q f\big| \le \Big(\frac{1}{|Q|}\int\limits_Q |f|^p\,dy\Big)^{1/p} \le \Big(\frac{1}{u(Q)}\int\limits_Q |f|^p\,\mathrm{M}\,u\,dy\Big)^{1/p}.$$

Thus

$$\begin{aligned}\int |b_2|^p u\,dy &= \sum_{Q\in\mathcal{Q}} |av_Q f|^p u(Q)\\ &\le \int\limits_{\mathbf{R}^d} |f|^p\,\mathrm{M}\,u\,dy.\end{aligned}$$

Additionally, since $u(x) \le \mathrm{M}\,u(x)$ a.e. and $b_1 = f$ on the support of b_1, then

$$\int |b_1|^p u\,dy \le \int\limits_{\mathbf{R}^d} |f|^p\,\mathrm{M}\,u\,dy.$$

Combining this inequality with the one for b_2 gives

$$\|\,b\,\|_{L^p(u)} \le 2\|\,f\,\|_{L^p(\mathrm{M}\,u)},$$

which yields (2.7) for the case $q = p$. We note for later use that this inequality does not require any of the Calderón-Zygmund properties of b except for the fact that b takes the form (2.3) for a disjoint collection of cubes $\mathcal{Q}$, and therefore the inequality

$$\|\sum_{Q\in\mathcal{Q}'} (h - av_Q h)\,\chi_Q\,\|_{L^p(u)} \le 2\|\,h\,\|_{L^p(\mathrm{M}\,u)}, \quad 1 \le p < \infty, \tag{2.11}$$

is valid for any collection $\mathcal{Q}'$ of disjoint cubes, any weight u, and any measurable function h. We further note that this inequality holds for the limiting case $p = \infty$ with the natural interpretation $\|\,h\,\|_{L^\infty(u)} = \|\,h\chi_F\,\|_{L^\infty(u)}$ for $F = \{\,x\,:\,u(x) \ne 0\}$.

To show (2.7) for $q = 1$, we first note that

$$\begin{aligned}\int |b_2|u\,dy &= \sum_{Q\in\mathcal{Q}} |av_Q f|u(Q)\\ &\le c_2\lambda\sum_{Q\in\mathcal{Q}} u(Q) \qquad \text{by (2.5)}\\ &\le \frac{c_2c_3}{\lambda^{p-1}}\int\limits_{\mathbf{R}^d} |f|^p\,\mathrm{M}\,u\,dy \qquad \text{by (2.6)}.\end{aligned}$$

For the remaining part, we write $b_1 = f \cdot \sum_{Q\in\mathcal{Q}} \chi_Q$ and use Hölder's inequality to get

$$\begin{aligned}\int |b_1| u\,dy &\le \Big(\int\limits_{\mathbf{R}^d} |f|^p\,u\,dy\Big)^{1/p} \Big(\sum_{Q\in\mathcal{Q}} u(Q)\Big)^{1/p'}\\ &\le \Big(\int\limits_{\mathbf{R}^d} |f|^p \operatorname{M} u\,dy\Big)^{1/p} \Big(\frac{c_3}{\lambda^p} \int\limits_{\mathbf{R}^d} |f|^p \operatorname{M} u\,dy\Big)^{1/p'}\\ &= \frac{C}{\lambda^{p-1}} \int\limits_{\mathbf{R}^d} |f|^p \operatorname{M} u\,dy.\end{aligned}$$

The second inequality is obtained by using $u \le \operatorname{M} u$ in the first factor and (2.6) in the second. Combining the inequalities for b_1 and b_2 gives (2.7) for $q = 1$, and so also for $1 \le q \le p$. Next, observe that $g = f$ outside the support of b_2 and $g = b_2$ on the support of b_2, so the inequality (2.8) follows using the estimate for b_2 on the support of b_2 and using $\operatorname{M} u \ge u$ otherwise.

In order to complete the proof of Lemma 2.2, we will show that $\mathbf{R}^d$ can be covered by disjoint dyadic cubes Q satisfying

$$\frac{1}{|Q|} \int\limits_Q |f|\,dy \le \lambda. \tag{2.12}$$

Once this is verified, the remaining parts of Lemma 2.2, i.e., (2.3)–(2.5), will follow by the standard Calderón-Zygmund construction. That is to say, starting with a collection of disjoint cubes which satisfy (2.12), we subdivide each cube in the collection into its 2^d "daughter" dyadic subcubes and check if (2.12) holds for the subcubes. When (2.12) holds, we continue the subdivision process, but if at some stage (2.12) fails for a subcube Q, then Q becomes a member of $\mathcal{Q}$ and the subdivision process is not continued for Q. This yields (2.5) for each $Q \in \mathcal{Q}$ by the choice of Q as the first daughter subcube for which (2.12) fails. If we define b_Q by (2.3) and let $b = \sum_{Q\in\mathcal{Q}} b_Q$, $g = f - b$, we obtain (2.4) for $x \in Q \in \mathcal{Q}$ by (2.5), and it holds for $x \notin \bigcup\limits_{Q\in\mathcal{Q}} Q$ by differentiation theory.

We now verify the initial step of the selection process. Since u is nontrivial, then $u(Q) > 0$ for some cube Q centered at the origin, and therefore

$$\operatorname{M} u(x) \ge \frac{C_u}{(1+|x|)^d}, \quad x \in \mathbf{R}^d.$$

Consequently, if $f \in L^p(\operatorname{M} u)$, then

$$\int \frac{|f(x)|^p}{(1+|x|)^d} dx < \infty.$$

Now, for each integer $j \ge 0$, let $\widehat{Q}_j$ be the cube consisting of the union of the 2^d dyadic cubes of edgelength 2^j with disjoint interiors which have the origin as a vertex, so that $\widehat{Q}_j$ is centered at the origin and has edgelength 2^{j+1}. If $j \ge 1$, then

$$\frac{1}{|\widehat{Q}_j|} \int_{\widehat{Q}_j} |f(x)|^p dx = \frac{1}{|\widehat{Q}_j|} \int_{\widehat{Q}_j \setminus \widehat{Q}_{j-1}} |f(x)|^p dx + \frac{1}{2^d |\widehat{Q}_{j-1}|} \int_{\widehat{Q}_{j-1}} |f(x)|^p dx.$$

Summing over j and regrouping terms, we obtain

$$\sum_{j=0}^{\infty}\frac{1}{|\widehat{Q}_j|}\int\limits_{\widehat{Q}_j}|f(x)|^p dx \approx \frac{1}{|\widehat{Q}_0|}\int\limits_{\widehat{Q}_0}|f(x)|^p dx \;+\; \sum_{j=1}^{\infty}\frac{1}{|\widehat{Q}_j|}\int\limits_{\widehat{Q}_j\setminus\widehat{Q}_{j-1}}|f(x)|^p dx$$
$$\lessapprox \int \frac{|f(x)|^p}{(1+|x|)^d}dx \;<\infty.$$

Therefore, for any $\epsilon>0$ we can find j_0 so that

$$\frac{1}{|\widehat{Q}_j|}\int\limits_{\widehat{Q}_j}|f(x)|^p dx \le \epsilon \qquad \text{for all } j\ge j_0.$$

Given $\lambda>0$, find j_0 for $\epsilon=(\lambda/4^d)^p$. If Q is any of the 2^d dyadic subcubes of $\widehat{Q}_{j_0}$ with edgelength 2^{j_0}, Jensen's inequality gives

$$\begin{aligned}\frac{1}{|Q|}\int\limits_Q|f(x)|dx &\le \left(\frac{1}{|Q|}\int\limits_Q|f(x)|^p dx\right)^{1/p}\\ &\le \left(\frac{2^d}{|\widehat{Q}_{j_0}|}\int\limits_{\widehat{Q}_{j_0}}|f(x)|^p dx\right)^{1/p}\\ &\le \frac{\lambda}{2^d}.\end{aligned}$$

Now for $j\ge j_0$ we can partition $\widehat{Q}_{j+1}\setminus\widehat{Q}_j$ into 4^d-2^d dyadic subcubes of edgelength 2^j which have at least one vertex in common with a vertex of $\widehat{Q}_j$. If Q is one of those dyadic cubes, then reasoning as above we have

$$\begin{aligned}\frac{1}{|Q|}\int\limits_Q|f(x)|dx &\le \left(\frac{1}{|Q|}\int\limits_Q|f(x)|^p dx\right)^{1/p}\\ &\le \left(\frac{4^d}{|\widehat{Q}_{j+1}|}\int\limits_{\widehat{Q}_{j+1}}|f(x)|^p dx\right)^{1/p}\\ &\le \lambda.\end{aligned}$$

Thus, choosing our starting grid to be the maximal dyadic subcubes of $\widehat{Q}_{j_0}$ together with the maximal dyadic subcubes of $\widehat{Q}_{j+1}\setminus\widehat{Q}_j$ for $j\ge j_0$, we obtain a family of disjoint dyadic cubes Q which exhaust $\mathbf{R}^d$ and which have the property that

$$\frac{1}{|Q|}\int\limits_Q|f(x)|dx\le\lambda,$$

which is the last part of Lemma 2.2 that we need to show. □

Whenever $\mathcal{Q}$ is a family of disjoint dyadic cubes, define $\mathcal{Q}_j=\{Q\in\mathcal{Q}:\text{edgelength}(Q)=2^j\}$ and define an associated sequence of operators $\{\Delta_j\}_{j=-\infty}^{\infty}$ by

$$\Delta_j h(x)=\sum_{Q\in\mathcal{Q}_j}\left(h(x)-\frac{1}{|Q|}\int\limits_Q h\right)\chi_Q(x).$$

When $\mathcal{Q}$ arises from the Calderón-Zygmund decomposition of f, we clearly have

$$b = \sum_j b_j \quad \text{with } b_j = \Delta_j f,$$

so that b_j is the part of b supported on cubes of edgelength 2^j. The critical estimate needed to prove the weak type (p,p) inequality is given in the next lemma.

Lemma 2.13. *For some constants $B \ge 1$ and $\alpha, c > 0$, suppose that $\{\mu_j\}_{j=-\infty}^{\infty}$ is a sequence of signed (even complex-valued) measures on $\mathbf{R}^d$ such that for each j, μ_j is supported in $|x| < c2^j$,*

$$\| \mu_j \| \le B, \qquad \textit{and}$$

$$|\widehat{\mu}_j(\xi)| \le \frac{B}{\left(2^j|\xi|\right)^{\alpha}}. \tag{2.14}$$

Let $\mathcal{Q}$ be a family of disjoint dyadic cubes. Then there are positive constants C and δ depending on α and c, but independent of B and $\mathcal{Q}$, such that for any function h,

$$\| \mu_j * (\Delta_{j-k}h) \|_2 \le CB2^{-\delta k} \| h \|_2, \qquad j \in \mathbf{Z}, \tag{2.15}$$

$$\| \sum_{j=-\infty}^{\infty} \mu_j * (\Delta_{j-k}h) \|_2 \le CB2^{-\delta k} \| h \|_2, \tag{2.16}$$

for all nonnegative integers k.

See [W4] for a proof of this lemma. It is well known that Fourier transform decay as in the second part of (2.14) yields L^2 continuity with respect to translations, i.e., $\| h * \mu_j(\,\cdot\, + \, y) - h * \mu_j(\,\cdot\,) \|_2 \le CB(2^{-j}|y|)^{\alpha} \| h \|_2$. The lemma shows that an analogous kind of smoothness holds with respect to average differences on cubes as represented by the difference operators Δ_j.

Proof of Theorem I. We will use Lemma 2.2 as follows. Given a nonnegative measurable function v, the norm inequality (1.3) holds automatically if v vanishes a.e., so we may assume that this is not the case. Given $r > 1$, we choose

$$u = \{MG^*M(v^r)\}^{1/r}$$

in the Calderón-Zygmund decomposition. We may assume that $u \in L^1_{loc}$. Note that $u > 0$ everywhere, and the Coifman–Rochberg inequality [CR]

$$M\big(\{Mw\}^{1/r}\big) \le c_r \{Mw\}^{1/r}, \qquad r > 1, \text{ for all weights } w,$$

with $w = v^r$ gives us $\mathrm{M}\, u \le c_r u$ a.e. Since $v \le c\, G^* v$ for a constant c that is independent of v (this was shown in [WW] using a differentiation argument), then also $v \le cu$. Fix p with $1 < p < 2$. Given $\lambda > 0$ and $f \in L^p(\mathrm{M}\, u) \cap L^2 = L^p(u) \cap L^2$, we obtain from Lemma 2.2 the collection $\mathcal{Q}$ and the decomposition $f = g + b = g + \sum_j b_j$, where $b_j = \Delta_j f$. In addition, it is easy to see that both $g, b \in L^2$ since $f \in L^2$.

We wish to apply Theorem A (Theorem 5.2 of [WW]) for $p = 2$ to g. Strictly speaking, the conclusion (1.3) of Theorem A was proved in [WW] only for functions that belong to $\mathcal{S}$, but it is easy to see that it is then also valid for any $f \in L^2$. In fact, given v and u as above and $q \ge 2$, start with the inequality

$$\int |T_\Omega h|^q v \, dx \le C_{q,r} \int |h|^q u \, dx \tag{2.17}$$

(which is just (1.3) with index q) for $h \in \mathcal{S}$. To show that the same inequality holds for $h \in L^2$, it is enough to prove it for $h \in L^q(u) \cap L^2$. If $h \in L^q(u) \cap L^2$, there is a sequence of functions $h_n \in C_0^\infty$ which converges to h simultaneously in $L^q(u)$ and L^2. This can be seen as follows. First approximate h simultaneously in both norms by a function h' with compact support. Then approximate h' by h'', a standard approximation to the identity formed by convolving h' with a function in C_0^∞. Clearly h'' can be chosen so that $h' - h''$ has an arbitrarily small L^2 norm. But since u satisfies the usual A_1 condition, it also satisfies the A_q condition, and consequently by the results of [M], h'' can be chosen so that the $L^q(u)$ norm of $h' - h''$ is also arbitrarily small. Hence, a sequence h_n with the desired properties exists, and (2.17) holds for each h_n. Since h_n converges to h in L^2, $T_\Omega h_n$ converges to $T_\Omega h$ in L^2 by [CZ], and so there is a subsequence $T_\Omega h_{n_k}$ which converges pointwise a.e. to $T_\Omega h$. The validity of (2.17) for h now follows from Fatou's lemma.

By this extended version of Theorem A for $p = 2$ and Lemma 2.2, we have

$$\int\limits_{\mathbf{R}^d} |T_\Omega g(x)|^2 v\,dx \le c \int\limits_{\mathbf{R}^d} |g(x)|^2 u\,dx \le c' \lambda^{2-p} \int\limits_{\mathbf{R}^d} |g(x)|^p u\,dx$$

since $1 < p < 2$, which by Chebychev's inequality and (2.8) (using the fact that $Mu \le cu$) gives

$$\begin{aligned} v\{x : |T_\Omega g(x)| \ge \lambda/2\} &\le \frac{c}{\lambda^2} \int\limits_{\mathbf{R}^d} |T_\Omega g(x)|^2 v\,dx \\ &\le \frac{c'}{\lambda^p} \int\limits_{\mathbf{R}^d} |f|^p u\,dx. \end{aligned}$$

This is the inequality we need for $T_\Omega g$. Also, by (2.6), we have (recalling again that $v \le cu$)

$$v\Big(\bigcup_{Q \in \mathcal{Q}} Q^*\Big) \le cu\Big(\bigcup_{Q \in \mathcal{Q}} Q^*\Big) \le \frac{cc_3}{\lambda^p} \int\limits_{\mathbf{R}^d} |f|^p u\,dx,$$

which is the kind of estimate we need for the exceptional set $\bigcup_{Q \in \mathcal{Q}} Q^*$. Therefore, just as in the classical Calderón-Zygmund argument, we are left with estimating $T_\Omega b$ on the complement of $E = \bigcup_{Q \in \mathcal{Q}} Q^*$, and we will be done once we show that

$$v\{x \notin E : |T_\Omega b(x)| \ge \lambda/2\} \le \frac{c}{\lambda^p} \int\limits_{\mathbf{R}^d} |f|^p u\,dx. \tag{2.18}$$

Using our representation formula (1.2), let

$$\mathrm{T}_\Omega{}'\, b(x) = \mathrm{T}_\Omega\, b(x) - c_\Omega \cdot b(x) = d \int_0^\infty A_t b(x) \frac{dt}{t},$$

with convergence valid in the L^2 sense since $b \in L^2$. Since $v \le cu$, we have

$$v\{x : |b(x)| > c\lambda\} \le \frac{1}{c\lambda} \int_{\mathbf{R}^d} |b| u\,dy \le \frac{C}{\lambda^p} \int_{\mathbf{R}^d} |f|^p u\,dy$$

by (2.7) with $q = 1$. It is then enough to estimate $\mathrm{T}_\Omega{}'\, b$ on the complement of E.

As in [WW], we define operators

$$B_j^m f(x) = \int_{2^{j-1}}^{2^j} A_t^m f(x) \frac{dt}{t},$$

for $m \geq 0$, $j \in \mathbf{Z}$, so that

$$\mathrm{T}_\Omega{}' \, b(x) = d \sum_{\substack{m \geq 0 \\ j \in \mathbf{Z}}} B_j^m b(x) = d \sum_{\substack{m \geq 0 \\ j,k \in \mathbf{Z}}} B_j^m b_{j-k}(x).$$

Now note that for any function h, $B_j^m h(x) = \beta_j^m * h(x)$, where

$$\beta_j^m(x) = \operatorname{sgn} \Omega(x) \int_{2^{j-1}}^{2^j} \chi_{tS_m}(x) \frac{dt}{t^{d+1}} \tag{2.19}$$

is a function supported in $|x| < 2^{j+m}$. Thus if Q is a cube of edgelength 2^k and h is supported in Q, then $B_j^m h$ will be supported in Q^* if $j+m+1 < k$. Therefore, since b_{j-k} is supported in cubes of edgelength 2^{j-k}, then if $m+1 < -k$, $B_j^m b_{j-k}$ is supported in the doubles of those cubes, and hence in E. Consequently, if $x \notin E$, then $B_j^m b_{j-k}(x) = 0$ if $k < -m-1$ and

$$\begin{aligned} d^{-1}\, \mathrm{T}_\Omega{}' \, b(x) &= \sum_{m \geq 0} \sum_{k=-m-1}^{\infty} \sum_j B_j^m b_{j-k}(x) \\ &= \sum_{\substack{m \geq 0 \\ -m-1 \leq k \leq -1 \\ j \in \mathbf{Z}}} B_j^m b_{j-k}(x) + \sum_{\substack{m \geq 0 \\ 0 \leq k \leq Nm \\ j \in \mathbf{Z}}} B_j^m b_{j-k}(x) + \sum_{\substack{m \geq 0 \\ k \geq Nm+1 \\ j \in \mathbf{Z}}} B_j^m b_{j-k}(x) \\ &= \quad I(x) \quad + \quad II(x) \quad + \quad III(x), \end{aligned}$$

where N is a sufficiently large integer to be determined. We will show appropriate inequalities for each of these terms.

First, if we define $\widetilde{\beta}_j^m$ by $\widetilde{\beta}_j^m(x) = \beta_j^m(-x)$, then it follows easily from the definitions that $|\widetilde{\beta}_j^m| * |f|(x) \leq \big(\int_{2^{j-1}}^{2^j} dt/t \big) \sup_{t>0} A_{t+}^{m*}(|f|)(x) = (\ln 2) \sup_{t>0} A_{t+}^{m*}(|f|)(x)$, so

$$\begin{aligned} \int_{\mathbf{R}^d} |I| v \, dx &\leq \sum_{\substack{m \geq 0 \\ -m-1 \leq k \leq -1 \\ j \in \mathbf{Z}}} \int_{\mathbf{R}^d} (|\beta_{j+k}^m| * |b_j|) \, v \, dx \\ &= \sum_{\substack{m \geq 0 \\ -m-1 \leq k \leq -1 \\ j \in \mathbf{Z}}} \int_{\mathbf{R}^d} |b_j| \, (|\widetilde{\beta}_{j+k}^m| * v) \, dx \\ &\leq (\ln 2) \sum_j \int_{\mathbf{R}^d} |b_j| \big(\sum_{m \geq 0} (m+1) \sup_{t>0} A_{t+}^{m*} v \big) \, dx \\ &= (\ln 2) \int_{\mathbf{R}^d} |b| \, G^* v \, dx. \end{aligned}$$

In [WW, (4.9)], it is shown that applying Hölder's inequality to the averages which define G^* yields the pointwise inequality $G^*v \le c_r\{G^*(v^r)\}^{1/r}$ a.e. Therefore $G^*v \le c_r\{\mathrm{M}\, G^*\, \mathrm{M}(v^r)\}^{1/r} = c_r u$ a.e. also. Using this in the above inequality, together with part (2.6) of the weighted Calderón-Zygmund decomposition, we obtain

$$\int_{\mathbf{R}^d} |I| v\, dx \le c_r(\ln 2) \int_{\mathbf{R}^d} |b| u\, dx \le \frac{c}{\lambda^{p-1}} \int_{\mathbf{R}^d} |f|^p u\, dx,$$

and this with Chebychev's inequality gives the desired estimate for the term I, i.e.

$$v\{x\,:\,|I(x)| > \lambda\} \le \frac{C}{\lambda^p} \int |f|^p \{MG^*M(v^r)\}^{1/r}\, dx, \qquad \lambda > 0.$$

The term II is handled almost identically since

$$\begin{aligned} \int_{\mathbf{R}^d} |II| v\, dx &\le \sum_j \int_{\mathbf{R}^d} |b_j| \Big(\sum_{\substack{m\ge 0 \\ 0\le k\le Nm}} |\widetilde{\beta}^m_{j+k}| * v \Big)\, dx \\ &\le (\ln 2) \sum_j \int_{\mathbf{R}^d} |b_j| \Big(\sum_{m\ge 0} (Nm+1) \sup_{t>0} A^{m*}_{t+} v \Big)\, dx \\ &\le N(\ln 2) \int_{\mathbf{R}^d} |b|\ G^* v\, dx \end{aligned}$$

for any choice of $N > 0$, and arguing as for the term I, we then get the desired distributional estimate for II, but this time with a constant which grows like N, i.e.,

$$v\{x\,:\,|II(x)| > \lambda\} \le \frac{cN}{\lambda^p} \int_{\mathbf{R}^d} |f|^p u.$$

The choice of N will be determined by our estimate for III.

To estimate III, we first need some estimates for β^m_j. We have

$$\begin{aligned} &\|\, \beta^m_j \,\|_1 \le (\ln 2)|S_m| \le (\ln 2)\|\, \Omega \,\|_1, \\ &\beta^m_j \ \text{ is supported in } \ |x| \le 2^{j+m}, \\ &|\beta^{m\,\widehat{}}_j(\xi)| \le \frac{c_\alpha 2^{dm}}{|2^{j+m}\xi|^\alpha}, \qquad 0 < \alpha < 1. \end{aligned} \tag{2.20}$$

The first of these estimates is easily seen, the second was already noted after (2.19), and a simple proof of the last estimate can be found in [WW, (2.34)]. For the proof of Theorem II we will need to know that the same estimates hold for the functions $\beta^m_{j+} = |\beta^m_j|$ (which amounts to replacing $\operatorname{sgn}\Omega$ with 1 in the definition of β^m_j). This is true by exactly the same arguments used for β^m_j. It follows that for $m \ge 0$ the functions $\{\beta^m_{j-m}\}_j$ satisfy the requirements of Lemma 2.13 for the measures μ_j, and so we have the norm estimate

$$\begin{aligned} \|\sum_{j=-\infty}^{\infty} B^m_j(\Delta_{j-k} h)\,\|_2 &= \|\sum_{j=-\infty}^{\infty} \beta^m_{j-m} * (\Delta_{j-m-k} h)\,\|_2 \\ &\le C 2^{dm} 2^{-\delta(m+k)} \|\, h \,\|_2, \end{aligned} \tag{2.21}$$

for some $\delta > 0$ and all nonnegative integers k. We will interpolate this estimate with a crude weighted one to obtain weighted estimates with a convergence factor that are useful for III. To obtain a crude weighted estimate, we first observe that for $m \geq 0$ and any weight w,

$$|\widetilde{\beta}_j^m| * (w^r) \leq \frac{\ln 2}{m+1} G^*(w^r) \leq cG^*(w^r) \quad \text{and}$$

$$\int_{\mathbf{R}^d} \sum_{j=-\infty}^{\infty} |\Delta_j h| w \, dx \leq 2 \int_{\mathbf{R}^d} |h| \, \mathrm{M}\, w \, dx.$$

This last inequality follows from (2.11) with $p = 1$. Putting these together, we see that for any $r > 1$,

$$\begin{aligned} \int_{\mathbf{R}^d} \Big| \sum_{j=-\infty}^{\infty} B_j^m(\Delta_{j-k}h)\Big| v^r \, dx &= \int_{\mathbf{R}^d} \Big| \sum_{j=-\infty}^{\infty} \beta_{j+k}^m * (\Delta_j h)\Big| v^r \, dx \\ &\leq \int_{\mathbf{R}^d} \sum_{j=-\infty}^{\infty} |\Delta_j h| \big(\big|\widetilde{\beta}_{j+k}^m\big| * (v^r) \big) \, dx \\ &\leq c \int_{\mathbf{R}^d} \Big(\sum_{j=-\infty}^{\infty} |\Delta_j h| \Big) G^*(v^r) \, dx \\ &\leq c \int_{\mathbf{R}^d} |h| \, \mathrm{M}\, G^*(v^r) \, dx, \end{aligned} \tag{2.22}$$

and since $\mathrm{M}\, G^*(v^r) \leq \mathrm{M}\, G^* \, \mathrm{M}(v^r) = u^r$, we therefore have

$$\int_{\mathbf{R}^d} \Big| \sum_{j=-\infty}^{\infty} B_j^m(\Delta_{j-k}h)\Big| v^r \, dx \leq c \int_{\mathbf{R}^d} |h| \, u^r \, dx. \tag{2.23}$$

Note. The last inequality worsens the weight in the sense that we could have kept just $\mathrm{M}\, G^*(v^r)$ on the right. We do this because the estimate that we already applied for $\mathrm{T}_\Omega \, g$ requires the use of a fact for $p \geq 2$ that we were only able to prove for the weight u. The argument just given would allow us to replace the weight u with the weight $\{\mathrm{M}\, G^*(v^r)\}^{1/r}$ if the corresponding estimate for $p \geq 2$ could also be proven.

We will use (2.23) twice. First, taking $v = 1$ gives the unweighted estimate

$$\int_{\mathbf{R}^d} \Big| \sum_{j=-\infty}^{\infty} B_j^m(\Delta_{j-k}h)\Big| \, dx \leq c \int_{\mathbf{R}^d} |h| \, dx,$$

and interpolating between this and (2.21) gives

$$\Big\| \sum_{j=-\infty}^{\infty} B_j^m(\Delta_{j-k}h) \Big\|_t \leq C 2^{\kappa_t m} 2^{-\delta_t k} \| h \|_t$$

for $1 < t \leq 2$, with κ_t, $\delta_t > 0$. We can then interpolate with change of measures between the last estimate and (2.23) for an appropriate choice of t depending on p and r to see that there exists q with $1 < q \leq p$ so that

$$\Big\| \sum_{j=-\infty}^{\infty} B_j^m(\Delta_{j-k}h) \Big\|_{L^q(v)} \leq C_{p,r} 2^{\kappa m} 2^{-\delta k} \| h \|_{L^q(u)}$$

for some positive κ and δ depending on p and r. In fact, given p and r, consider any q with

$$1 < q \le \min\left\{p, 1 + \frac{1}{r'}\right\}$$

and choose $t = 1 + r'(q-1)$ above. Then $q < t \le 2$, and the desired estimate follows from [WW, Theorem [SW1], p. 4145].

We are now in a position to complete our estimate of III. Taking $h = b$ and recalling that $\Delta_j b = b_j$, the last estimate gives

$$\| \sum_{j=-\infty}^{\infty} B_j^m b_{j-k} \|_{L^q(v)} \le C_{p,r} 2^{\kappa m} 2^{-\delta k} \| b \|_{L^q(u)},$$

and so

$$\begin{aligned} \| III \|_{L^q(v)} &\le \sum_{m\ge 0} \sum_{k=Nm+1}^{\infty} \| \sum_{j=-\infty}^{\infty} B_j^m b_{j-k} \|_{L^q(v)} \\ &\le C_{p,r} \| b \|_{L^q(u)} \sum_{m \ge 0} \sum_{k=Nm+1}^{\infty} 2^{\kappa m} 2^{-\delta k}. \end{aligned}$$

The double sum is finite if N is chosen so that $\delta N > \kappa$, and then

$$\begin{aligned} \int_{\mathbf{R}^d} |III|^q v\, dx &\le C \int_{\mathbf{R}^d} |b|^q u\, dx \\ &\le \frac{C'}{\lambda^{p-q}} \int_{\mathbf{R}^d} |f|^p u\, dx \end{aligned}$$

by (2.6). Using this estimate, together with Chebychev's inequality, we obtain

$$\begin{aligned} v\{x : |III(x)| \ge \lambda\}\, dx &\le \frac{1}{\lambda^q} \int_{\mathbf{R}^d} |III|^q v\, dx \\ &\le \frac{C'}{\lambda^p} \int_{\mathbf{R}^d} |f|^p u\, dx, \end{aligned}$$

and Theorem I is proved. □

Proof of Theorem II. Let S be a starlike set with respect to the origin whose radial boundary function ρ satisfies $\rho \in L^d \log L(\mathbf{S}^{d-1})$. As mentioned near the beginning of §2, this guarantees that G^* is bounded on $L^p(\mathbf{R}^d)$ for $1 < p \le \infty$. We first reduce M_S to a discrete operator. We begin by defining averaging operators

$$B_{j+} f(x) = \int_{2^{j-1}}^{2^j} A_{t+} f(x) \frac{dt}{t} = \beta_{j+} * f(x),$$

$$B_{j+}^m f(x) = \int_{2^{j-1}}^{2^j} A_{t+}^m f(x) \frac{dt}{t} = \beta_{j+}^m * f(x),$$

where A_{t+} and A_{t+}^m are as we have described earlier. Note that $\beta_{j+}^m = |\beta_j^m|$, where β_j^m is defined by (2.19), and that $\beta_{j+} = \sum_{m\ge 0} \beta_{j+}^m$. Thus the operators B_{j+}^m, B_{j+}

are like the operators B_j^m, $\sum_{m\geq 0} B_j^m$, respectively, except that they are taken with respect to unsigned averages over dilates of S.

From this we define a maximal operator

$$\mathcal{B}f(x) = \sup_{j\in\mathbf{Z}} |B_{j+}f(x)|,$$

which will be the operator we compare to M_S. Clearly, $\mathcal{B}f(x) \leq (\ln 2)\,\mathrm{M}_S\, f(x)$. Let us now show that $\mathrm{M}_S\, f(x) \leq c\mathcal{B}(|f|)(x)$. Observe that if f is nonnegative and $R/2 \leq r \leq R \leq t \leq 2R$, $R > 0$, then

$$A_{r+}f(x) \leq 4^d A_{t+}f(x),$$

so that

$$\sup_{R/2\leq r\leq R} A_{r+}f(x) \leq \frac{4^d}{\ln 2}\int_R^{2R} A_{t+}f(x)\frac{dt}{t}.$$

Using this for $R = 2^j$ and taking the supremum of both sides over all integers j, we then see

$$\mathrm{M}_S\, f(x) \leq \frac{4^d}{\ln 2}\mathcal{B}f(x), \qquad f \geq 0,$$

and so the inequality (1.6) will follow once we prove the same inequality (up to a constant multiple) for $\mathcal{B}$, i.e.,

$$\int |\mathcal{B}f|^p v\, dx \leq C_{p,r}\int |f|^p\,[\{\mathrm{M}(v^r)\}^{1/r} + \mathrm{M}\, G^*v]\, dx. \tag{2.24}$$

The proof of (2.24) will parallel that of Theorem I. Before we can discuss the similarities, we will need one more inequality. Since S is measurable, if f is bounded and measurable, then for all $x \in \mathbf{R}^d$ we have

$$|A_{t+}f(x)| \leq |S|\,\|\, f\,\|_\infty,$$

and then we easily deduce from above that

$$\|\, \mathcal{B}f(x)\,\|_\infty \leq (\ln 2)|S|\,\|\, f\,\|_\infty. \tag{2.25}$$

This inequality will be used in much the same way that a weighted L^2 inequality for T_Ω was used in the proof of Theorem I. That is, we will first prove a weak-type (p, p) inequality corresponding to (2.24), namely

(2.26)
$$v\{x\, :\, \mathcal{B}f(x) > \lambda\} \leq \frac{C}{\lambda^p}\int |f|^p\,[\{\mathrm{M}(v^r)\}^{1/r} + \mathrm{M}\, G^*v]\, dx, \quad \lambda > 0,\ 1 < p \leq 2.$$

Together with (2.25) this gives the result we seek, because once we have established (2.26) for an exponent $p = p_1 > 1$, the Marcinkiewicz interpolation theorem and (2.25) then gives (2.24) for $p_1 < p < \infty$.

We shall also use (2.25) to establish (2.26), to whose proof we now turn. Given measurable f, weight v and $\lambda > 0$, we wish to apply the Calderón-Zygmund decomposition $f = g + b$ of Lemma 2.2, but in the following way: first, we will take $u = G^*v$ in the decomposition, and second, instead of applying the lemma at height λ, we instead apply it at height $\lambda_1 = \lambda/(2c_1|S|\ln 2)$, where c_1 is the constant appearing in (2.4). This latter requirement has the effect of replacing (2.4) with the inequality

$$|g(x)| \leq \frac{\lambda}{2|S| \ln 2} \qquad \text{a.e.} \tag{2.27}$$

and similarly rescales the other inequalities of Lemma 2.2 by constants.

The reason for this change is that (2.27) with (2.25) gives

$$\mathcal{B}g(x) \leq \lambda/2,$$

so that

$$\{x : \mathcal{B}f(x) > \lambda\} \subseteq \{x : \mathcal{B}b(x) > \lambda/2\},$$

and consequently (2.26) follows from the inequality

$$v\{x : \mathcal{B}b(x) > \lambda/2\} \leq \frac{C}{\lambda^p} \int |f|^p [\{\mathrm{M}(v^r)\}^{1/r} + \mathrm{M}\, G^* v]\, dx, \quad \lambda > 0. \tag{2.28}$$

The proof of (2.28) is similar to the proof of the corresponding inequality in Theorem I. Recalling that $v \leq c\, G^* v$ a.e. (see the beginning of the proof of Theorem I) and that we are applying Lemma 2.2 to the weight $u = G^* v$, we then form an exceptional set $E = \bigcup_{Q \in \mathcal{Q}} Q^*$ and see as a consequence of (2.6) that we have the desired estimate for the size of $v(E)$, and so it will suffice to estimate $\mathcal{B}b(x)$ for x in the complement of E. Writing $B_{j+} = \sum_{m \geq 0} B^m_{j+}$ and using $b = \sum_{k=-\infty}^{\infty} b_{j-k}$ for every integer j, we observe that

$$\begin{aligned}\mathcal{B}b(x) &= \sup_j \Big| \sum_{m \geq 0} \sum_{k=-\infty}^{\infty} B^m_{j+} b_{j-k}(x) \Big| \\ &\leq \sum_{m \geq 0} \sum_{k=-\infty}^{\infty} \sup_j \big| B^m_{j+} b_{j-k}(x) \big|,\end{aligned}$$

and as in Theorem I, $B^m_{j+} b_{j-k}$ is supported in E if $k < -(m+1)$, so if $x \notin E$, then

$$\begin{aligned}\mathcal{B}b(x) &\leq \sum_{m \geq 0} \sum_{k=-m-1}^{\infty} \sup_j \big| B^m_{j+} b_{j-k}(x) \big| \\ &= \sum_{\substack{m \geq 0 \\ -m-1 \leq k \leq -1}} \sup_j \big| B^m_{j+} b_{j-k}(x) \big| + \sum_{\substack{m \geq 0 \\ 0 \leq k \leq Nm}} \sup_j \big| B^m_{j+} b_{j-k}(x) \big| \\ &\quad + \sum_{\substack{m \geq 0 \\ k \geq Nm+1}} \sup_j \big| B^m_{j+} b_{j-k}(x) \big| \\ &\leq \sum_{\substack{m \geq 0 \\ -m-1 \leq k \leq -1 \\ j \in \mathbf{Z}}} \big| B^m_{j+} b_{j-k}(x) \big| + \sum_{\substack{m \geq 0 \\ 0 \leq k \leq Nm \\ j \in \mathbf{Z}}} \big| B^m_{j+} b_{j-k}(x) \big| \\ &\quad + \sum_{\substack{m \geq 0 \\ k \geq Nm+1}} \Big\{ \sum_{j=-\infty}^{\infty} \big| B^m_{j+} b_{j-k}(x) \big|^p \Big\}^{\frac{1}{p}} \\ &= \quad I(x) \quad + \quad II(x) \quad + \quad III(x),\end{aligned}$$

for an integer N to be determined. In the last inequality we dominated the supremum by a sum for the first two terms and by an ℓ^p sum in the third term. Our estimates for the terms I and II are virtually identical to the corresponding estimates in the proof of Theorem II. First, arguing exactly as in Theorem I, we have

$$\int_{\mathbf{R}^d} |I| v\, dx \le \ln 2 \int_{\mathbf{R}^d} |b|\, G^* v\, dx \qquad \text{and}$$
$$\int_{\mathbf{R}^d} |II| v\, dx \le N \ln 2 \int_{\mathbf{R}^d} |b|\, G^* v\, dx,$$

while by (2.6) and our choice of $u = G^* v$, we have

$$\int_{\mathbf{R}^d} |b| G^* v dx \le \frac{C_{p,1}}{\lambda^{p-1}} \int_{\mathbf{R}^d} |f|^p M G^* v dx.$$

Using Chebychev's inequality with this inequality and each of the two inequalities preceding it gives us the estimates we seek for I, II, namely

$$v\{x : |I(x)| > \lambda\} \le \frac{c}{\lambda^p} \int_{\mathbf{R}^d} |f|^p M G^* v,$$
$$v\{x : |II(x)| > \lambda\} \le \frac{cN}{\lambda^p} \int_{\mathbf{R}^d} |f|^p M G^* v.$$

The estimation of the term III in some ways parallels the corresponding estimate in Theorem I but requires further explanation. Using (2.11) with $u = 1$ gives the inequality

$$\| \Delta_j h \|_q \le 2 \| h \|_q, \qquad 1 \le q \le \infty, \quad j \in \mathbf{Z}.$$

Using this with Young's integral inequality and the observation that B^m_{j+} is given by convolution against an integrable function β^m_{j+} satisfying $\| \beta^m_{j+} \|_1 \le c$ uniformly in j and m, we therefore have the crude estimate

$$\| B^m_{j+} \Delta_{j-k} h \|_q \le c' \| h \|_q, \qquad 1 \le q \le \infty,\ j, k \in \mathbf{Z},\ m \ge 0. \tag{2.29}$$

A better estimate will follow from the observation following (2.20), namely, the same estimates as in (2.20) also hold for β^m_{j+}, so that we may apply (2.15) of Lemma 2.13 to get

$$\| B^m_{j+} \Delta_{j-k} h \|_2 \le C 2^{dm - \delta(m+k)} \| h \|_2, \tag{2.30}$$

for some $\delta > 0$ (cf. (2.21)). Here we are using (2.15), whereas in the proof of Theorem I we instead used (2.16). Using the Riesz-Thorin theorem to interpolate between this inequality and (2.29) with $q = 1$ gives the inequality

$$\| B^m_{j+} \Delta_{j-k} h \|_p \le C 2^{[dm - \delta(m+k)]2/p'} \| h \|_p, \tag{2.31}$$

for $1 < p \le 2$ and $k \ge -(m+1)$. Although we only need this inequality for p between 1 and 2, we observe that interpolation between (2.30) and (2.29) with $q = \infty$ gives a similar inequality for $2 \le p < \infty$. We will use (2.31) to obtain a good weighted estimate, this time by interpolation with change of measures with a crude weighted estimate. First, we note that (2.11) gives the inequality

$$\| \Delta_k h \|_{L^p(u)} \le 2 \| h \|_{L^p(\mathrm{M}\, u)}, \quad 1 \le p < \infty. \tag{2.32}$$

Next, since $\beta^m_{j+}(x)$ is supported in $|x| \le 2^{j+m}$ and $|\beta^m_{j+}(x)| \le c_d 2^{-dj}$, we have the pointwise inequality

$$|B^m_{j+} h(x)| \le c'_d 2^{dm}\, \mathrm{M}\, h(x). \tag{2.33}$$

We recall the Fefferman-Stein inequality

$$\int |\,\mathrm{M}\, f|^p v\, dx \le C_p \int |f|^p\, \mathrm{M}\, v\, dx \tag{2.34}$$

derived in [FS]. Using this with (2.33) and then applying (2.32) gives

$$\begin{aligned} \|\, B^m_{j+}\Delta_{j-k} h\,\|_{L^p(v)} &\le c2^{dm}\|\,\Delta_{j-k}h\,\|_{L^p(\mathrm{M}\, v)} \\ &\le c'2^{dm}\|\,h\,\|_{L^p(\mathrm{M}\,\mathrm{M}\, v)}. \end{aligned} \tag{2.35}$$

Let $M_s v$ be defined by $M_s v = \{M(v^s)\}^{1/s}$ for $1 \le s < \infty$. Since $\mathrm{M}\, v \le \mathrm{M}_s\, v$ for $s > 1$ by Jensen's inequality, then the Coifman-Rochberg inequality gives $\mathrm{M}\,\mathrm{M}\, v \le c_s\, \mathrm{M}_s\, v$, and combining this with (2.35) gives

$$\|\, B^m_{j+}\Delta_{j-k} h\,\|_{L^p(v)} \le c'2^{dm}\|\,h\,\|_{L^p(\mathrm{M}_s\, v)}, \tag{2.36}$$

which is the crude weighted estimate we seek. Interpolating with change of measures between this estimate and (2.31), we obtain a better weighted estimate, namely, if $0 < \theta < 1$, then

$$\|\, B^m_{j+}\Delta_{j-k} h\,\|_{L^p(v^\theta)} \le C2^{\sigma m-\gamma k}\|\,h\,\|_{L^p((\mathrm{M}_s\, v)^\theta)}, \tag{2.37}$$

with positive constants σ, γ depending on θ. We now rewrite (2.37) by replacing h with $\Delta_{j-k}h$ (observing that $\Delta_{j-k}\Delta_{j-k}h = \Delta_{j-k}h$) and also replacing v with $v^{1/\theta}$. Setting $s/\theta = r$ and using $(\mathrm{M}_s(v^{1/\theta}))^\theta = \mathrm{M}_r\, v$ gives

$$\|\, B^m_{j+}\Delta_{j-k} h\,\|_{L^p(v)} \le C2^{\sigma m-\gamma k}\|\,\Delta_{j-k}h\,\|_{L^p(\mathrm{M}_r\, v)}, \qquad k \ge -(m+1),\ r > 1, \tag{2.38}$$

for positive constants C, σ and γ depending on p and r.

We apply the last inequality with $h = b$, observing that $\Delta_{j-k}b = b_{j-k}$, and if we choose N to be large enough so that $N\gamma > \sigma$, we may then interchange the order of summation and integration to see that

$$\begin{aligned} \|\, III\,\|_{L^p(v)} &\le \sum_{\substack{m\ge 0\\ k\ge Nm+1}} \Big\{ \sum_{j=-\infty}^{\infty} ||B^m_{j+} b_{j-k}||^p_{L^p(v)} \Big\}^{1/p} \\ &\le C\Big(\sum_{\substack{m\ge 0\\ k\ge Nm+1}} 2^{\sigma m-\gamma k} \Big\{ \sum_{j=-\infty}^{\infty} ||b_{j-k}||^p_{L^p(\mathrm{M}_r\, v)} \Big\}^{1/p} \Big) \\ &= C\Big(\sum_{\substack{m\ge 0\\ k\ge Nm+1}} 2^{\sigma m-\gamma k} \Big) \Big\{ \sum_{j=-\infty}^{\infty} ||b_j||^p_{L^p(\mathrm{M}_r\, v)} \Big\}^{1/p} \\ &\le C'\|\,\mathrm{M}\, f\,\|_{L^p(\mathrm{M}_r\, v)} \le C''\|\,f\,\|_{L^p(\mathrm{M}_r\, v)}, \end{aligned}$$

where we have used (2.38) to get the second line, the independence on k in the inner sum to get the third line, the fact that $|b_j(x)| \le 2\,\mathrm{M}\, f(x)$ a.e. and is supported on $\mathcal{Q}_j$ to get the penultimate inequality, and (2.34) and the Coifman-Rochberg inequality to get the last inequality. Used with Chebyschev's inequality, this estimate gives us the inequality we seek for III. This completes the proof of Theorem II. $\square$

Proof of (1.4). The proof of (1.4) is an extremely straightforward adaptation of A. Cordoba's and C. Fefferman's derivation [CF] of vector-valued inequalities from two-weight inequalities for smooth kernel singular integral operators. We include a sketch of the proof for the reader's convenience.

First, under the assumptions of the Corollary, construct G^* relative to Ω_* rather than Ω, and note that under these assumptions our proofs of Theorem A and Theorem I show that (1.3) holds for each Ω_j with the new G^*, with a constant $C_{p,r}$ that is uniform in j, for $1 < p < \infty$.

This immediately gives (1.4) for $p = q$: just take $v = 1$ and interchange the order of summation and integration for the pth power of the left side, observing that the weight on the right side is the constant $(\sum_{m\geq 0}(m+1)|S_m|)^{1/r} < \infty$. For the case $1 < q < p$, since $p/q > 1$ we have

$$\begin{aligned}\| \{\sum_j |\,\mathrm{T}_{\Omega_j} f_j|^q\}^{1/q} \|_p^p &= \int_{\mathbf{R}^d} \{\sum_j |\,\mathrm{T}_{\Omega_j} f_j|^q\}^{p/q} dx \\ &= \Big(\sup_{\substack{v\geq 0,\\ \| v \|_{(p/q)'}\leq 1}} \int_{\mathbf{R}^d} \{\sum_j |\,\mathrm{T}_{\Omega_j} f_j|^q\} \cdot v\, dx\Big)^{p/q}.\end{aligned} \tag{2.39}$$

Here $(p/q)'$ is the dual exponent to p/q, i.e., the exponent such that $1/(p/q) + 1/(p/q)' = 1$. For the quantity inside the supremum, interchanging the order of summation and integration and applying (1.3) followed by Hölder's inequality with exponents $p/q, (p/q)'$ gives, for $1 < r < \infty$,

$$\begin{aligned}\int_{\mathbf{R}^d} \{\sum_j |\,\mathrm{T}_{\Omega_j} f_j|^q\} v dx &\leq C_{q,r}^q \int_{\mathbf{R}^d} \{\sum_j |f_j|^q\}\{\mathrm{M}\, G^*\, \mathrm{M}(v^r)\}^{1/r}\, dx \\ &\leq C_{q,r}^q \| \{\sum_j |f_j|^q\} \|_{p/q} \cdot \| \{\mathrm{M}\, G^*\, \mathrm{M}(v^r)\}^{1/r} \|_{(p/q)'} \\ &= C_{q,r}^q \| \{\sum_j |f_j|^q\}^{1/q} \|_p^q \cdot \| \{\mathrm{M}\, G^*\, \mathrm{M}(v^r)\}^{1/r} \|_{(p/q)'}.\end{aligned} \tag{2.40}$$

If we choose r with $1 < r < (p/q)'$, the operator $v \mapsto \{\mathrm{M}\, G^*\, \mathrm{M}(v^r)\}^{1/r}$ is bounded on $L^{(p/q)'}(\mathbf{R}^d)$, so for v as given in the supremum in (2.39), the last term in (2.40) is bounded by a constant, giving (1.4) for the case $p > q$.

The case $p < q$ follows from the $p > q$ case by duality (since $p' > q'$), observing that the dual operators simply replace $\Omega_j(\theta)$ with $\overline{\Omega}_j(-\theta)$, which also satisfy the requirements of the corollary. □

3. Examples

As mentioned in the introduction, a condition on a weight function w that guarantees the boundedness of T_Ω on $L^p(w)$ for any particular p, $1 < p < \infty$, is given in [WW]. Similar results for the starlike maximal operator M_S are also proved in [WW], as well as in [CWW]. As usual, the starlike set S associated with T_Ω is the one in (1.1); then $\rho = |\Omega|^{1/d}$ by definition. Recall that for *any* function Ω with $|\Omega(\theta)| = \rho(\theta)^d$, the condition $\Omega \in L^1(\mathbf{S}^{d-1})$ is the same as $\rho \in L^d(\mathbf{S}^{d-1})$, i.e., is the same as $|S| < \infty$, and the condition $\Omega \in L\log L(\mathbf{S}^{d-1})$ is the same as $\rho \in L^d \log L(\mathbf{S}^{d-1})$. We now construct examples of such weights in case the starlike set S has a starlike cover which satisfies condition (1.11). We first give some more details about starlike covers of S.

Given $\rho \in L^d(\mathbf{S}^{d-1})$ and the associated starlike set S, [CWW] and [WW] show that there is a collection of rectangles $\{R_{m,k}\}$ centered at the origin with orientations chosen so that S is covered efficiently by $\bigcup_{m,k} R_{m,k}$. Here, the index m satisfies $m \geq 0$ and corresponds to the sets S_m and Θ_m defined in §2, and S_m is covered by $\{R_{m,k} : 1 \leq k < k_m\}$ with k_m finite or infinite. For $m > 0$, S_m is the starlike set generated by the set Θ_m where $\rho(\theta) \approx 2^m$, so we may think of the value of k_m as determining how many branches of length 2^m the set S has. To be more precise, for any set S which is starlike about the origin, there are rectangles $\{R_{m,k}\}_{m,k}$ centered at the origin with

$$\begin{gathered} S_m \subset \bigcup_k R_{m,k} \quad \text{for } m \geq 0, \\ \text{the longest edgelength of } R_{m,k} \approx 2^m, \text{ and} \\ \sum_k |R_{m,k}| \leq c|S_m| \quad \text{for } m \geq 0, \end{gathered} \tag{3.1}$$

where c depends only on the dimension d. Such covers always exist by [WW], and any one is called a *stratified starlike cover* of S. If S contains a neighborhood of the origin, the cover can be chosen with the additional property that

$$c_0\, R_{m,k} \subset S_m \tag{3.2}$$

for an appropriate positive constant c_0 that is independent of m and k. In any case,

$$\begin{gathered} \sum_{m,k} |R_{m,k}| \leq c\,|S| < \infty \quad \text{if and only if } \rho \in L^d(\mathbf{S}^{d-1}), \text{ and} \\ \sum_{m,k} (m+1)\,|R_{m,k}| < \infty \quad \text{if and only if } \rho \in L^d \log L(\mathbf{S}^{d-1}). \end{gathered}$$

In fact, to obtain the last equivalence, note that

$$\begin{aligned} \int_{\mathbf{S}^{d-1}} \rho(\theta)^d \big(1 + \log^+ \rho(\theta)\big)\, d\theta &= \sum_{m=0}^{\infty} \int_{\Theta_m} \cdots \\ &\approx \sum_m (m+1)\, 2^{md} |\Theta_m| \\ &\approx \sum_m (m+1)\, |S_m| \quad \text{since } |S_m| = \int_{\Theta_m} \rho^d \approx 2^{md} |\Theta_m| \\ &\approx \sum_{m,k} (m+1)\, |R_{m,k}| \quad \text{by (3.1)}. \end{aligned}$$

Let $\mathcal{B}(R_{m,k})$ denote the collection of all rectangles R that are obtained from $R_{m,k}$ by translation or dilation. It is proved in [WW, Theorem 1.5] that if $1 < p < \infty$, $\Omega \in L \log L(\mathbf{S}^{d-1})$ and Ω has integral zero, then

$$\int_{\mathbf{R}^d} |T_\Omega f|^p\, w\, dx \leq C \int_{\mathbf{R}^d} |f|^p\, w\, dx \tag{3.3}$$

for all $f \in \mathcal{S}$, with C independent of f, provided there exists $r > 1$ so that for all k, m and all $R \in \mathcal{B}(R_{m,k})$,

$$\left(\frac{1}{|R|}\int_R w\,dx\right)^{\frac{1}{p}}\left(\frac{1}{|R|}\int_R w^{-rp'/p}\,dx\right)^{\frac{1}{rp'}} \le \frac{c_{m,k}}{|R_{m,k}|} \quad \text{if } 1 < p \le 2, \tag{3.4}$$

$$\left(\frac{1}{|R|}\int_R w^r\,dx\right)^{\frac{1}{rp}}\left(\frac{1}{|R|}\int_R w^{-p'/p}\,dx\right)^{\frac{1}{p'}} \le \frac{c_{m,k}}{|R_{m,k}|} \quad \text{if } 2 \le p < \infty, \tag{3.5}$$

and provided that the constants $c_{m,k}$ satisfy

$$\sum_{m,k}(m+1)\,c_{m,k} < \infty. \tag{3.6}$$

Moreover, r can be chosen to be 1 in the case where $p = 2$. Also, the analogue of (3.3) with T_Ω replaced by the starlike maximal operator M_S is valid for $1 < p < \infty$ and $\rho \in L^d(\mathbf{S}^{d-1})$, provided condition (3.4) holds and the constants satisfy

$$\sum_{m,k} c_{m,k} < \infty. \tag{3.7}$$

(See [CWW] or [WW].)

The next theorem shows that there exist weights w satisfying both (3.4) and (3.5), provided the coefficients $c_{m,k}$ satisfy $c_{m,k} \le C|R_{m,k}|^{1-\epsilon}$ for some $\epsilon \in (0,1)$.

Theorem 3.8. *Let S be a set of finite measure which is starlike with respect to the origin. Let $\{R_{m,k}\}$ be a stratified starlike cover of S satisfying* (3.1). *Then, given p, r, ϵ with $1 \le p < \infty, 1 \le r < \infty$ and $0 < \epsilon < 1$, there is a weight w such that*

$$\left(\frac{1}{|R|}\int_R w^r\,dx\right)^{\frac{1}{rp}}\left(\frac{1}{|R|}\int_R w^{-rp'/p}\,dx\right)^{\frac{1}{rp'}} \le C\,|R_{m,k}|^{-\epsilon} \tag{3.9}$$

for all $R \in \mathcal{B}(R_{m,k})$, all k and all large m, with C independent of R, k and m. In particular, if the cover satisfies

$$\sum_{m,k}(m+1)\,|R_{m,k}|^{1-\epsilon} < \infty, \tag{3.10}$$

then (3.4), (3.5) *and* (3.6) *hold with $c_{m,k} = |R_{m,k}|^{1-\epsilon}$. If the cover satisfies*

$$\sum_{m,k}|R_{m,k}|^{1-\epsilon} < \infty, \tag{3.11}$$

then (3.4), (3.5) *and* (3.7) *hold with $c_{m,k} = |R_{m,k}|^{1-\epsilon}$.*

In case $p = 1$, (3.9) means that

$$\left(\frac{1}{|R|}\int_R w^r\,dx\right)^{\frac{1}{r}} \le C|R_{m,k}|^{-\epsilon}\operatorname*{ess\,inf}_R w.$$

Clearly, any w which satisfies (3.9) for $p = 1$ also satisfies (3.9) for all $1 \le p < \infty$ with the same r and ϵ.

Remark. As mentioned in the introduction, condition (3.10) is satisfied for any ϵ with $0 < \epsilon < 1$ if $\rho \in L^s(\mathbf{S}^{d-1})$ for some $s > d$ *provided* the corresponding starlike

S has bounded branching, i.e., provided k_m is bounded in m. In fact, if there is bounded branching, then

$$\sum_{m,k}(m+1)|R_{m,k}|^{1-\epsilon} = \sum_m (m+1)\sum_k |R_{m,k}|^{1-\epsilon} \le c\sum_m (m+1)\left(\sum_k |R_{m,k}|\right)^{1-\epsilon} \le c\sum_m (m+1)|S_m|^{1-\epsilon}.$$

However, if Θ_m are the subsets of the unit sphere defined in §2, then

$$|S_m| \le c\, 2^{md}|\Theta_m|_{\mathbf{S}^{d-1}} \le \frac{c}{2^{m(s-d)}}$$

by Chebyshev's inequality since $\rho \in L^s$. Hence, if $0 < \epsilon < 1$, then since $s > d$,

$$\sum_{m,k}(m+1)|R_{m,k}|^{1-\epsilon} \le c\sum_m \frac{m+1}{2^{m(s-d)(1-\epsilon)}} < \infty.$$

Before proving Theorem 3.8, we list a corollary.

Corollary 3.12. (i) *Suppose that Ω has integral* 0 *over $\mathbf{S}^{d-1}$. Let S be the starlike set associated with Ω as in* (1.1) *and suppose* (3.10) *holds for some $\epsilon \in (0,1)$ for a stratified starlike cover of S. Given $1 < p, r < \infty$, let $w = w_{p,r,\epsilon}$ be the weight constructed in Theorem* 3.8. *Then T_Ω is bounded on $L^p(w)$.*

(ii) *Let S be a starlike set with respect to the origin which has a stratified starlike cover satisfying* (3.11) *for some $\epsilon \in (0,1)$. Given $1 < p, r < \infty$, let $w = w_{p,r,\epsilon}$ be the weight constructed in Theorem* 3.8. *Then M_S is bounded on $L^p(w)$.*

The corollary follows from Theorem 3.8 and our earlier comments. For part (i), we will show that (3.10) implies $\Omega \in L\log L(\mathbf{S}^{d-1})$. In fact, if ρ is the radial boundary function of a set S which is starlike with respect to 0 and $\{R_{m,k}\}_{m,k}$ is a stratified starlike cover of S, then $\rho \in L^d \log L(\mathbf{S}^{d-1})$ if (3.10) holds for some $\epsilon \in (0,1)$ since

$$\sum_{m,k}(m+1)|R_{m,k}| \le \big(\sup_{m,k}|R_{m,k}|\big)^{\epsilon}\sum_{m,k}(m+1)|R_{m,k}|^{1-\epsilon} \le (c|S|)^{\epsilon}\sum_{m,k}(m+1)|R_{m,k}|^{1-\epsilon} < \infty,$$

using the fact that $\sup_{m,k}|R_{m,k}| \le \sup_m c|S_m| \le c|S|$. Furthermore, for part (ii), note that (3.11) implies $|S| < \infty$.

Proof of Theorem 3.8. Recall from the discussion preceding (2.1) that M_S is bounded on (unweighted) $L^q(\mathbf{R}^d)$ for $1 < q \le \infty$. Let $\{R_{m,k}\}$ be a starlike cover of S as in (3.1), and define

$$\tilde{S} = \bigcup_{m,k} R_{m,k}.$$

Then $\tilde{S}$ is a starlike set with finite measure which contains S, and $\{R_{m,k}\}$ is a starlike cover of $\tilde{S}$ which satisfies the extra condition that $R_{m,k} \subset (\tilde{S})_m$, i.e., the analogue of (3.2) for $\tilde{S}$ with $c_0 = 1$. Hence, by replacing S with $\tilde{S}$ in the argument, we may assume that (3.2) holds for S. □

Fix constants q, A, θ and a function g with A strictly larger than the $L^q(\mathbf{R}^d)$ operator norm of M_S, $\theta > 1$ to be chosen, and $g \in L^q(\mathbf{R}^d)$. Define a nonnegative function $w(x)$ on $\mathbf{R}^d$ by

$$w^\theta = |g| + \sum_{j=1}^{\infty} \frac{\mathrm{M}_S^{[j]} g}{A^j},$$

where $\mathrm{M}_S^{[j]} g$ denotes the j-fold iterate $\mathrm{M}_S(\mathrm{M}_S(\cdots(\mathrm{M}_S g)\cdots))$. The sum converges in $L^q(\mathbf{R}^d)$ because of the choice of A, and consequently $w \in L^{\theta q}(\mathbf{R}^d)$. Note that

$$\mathrm{M}_S(w^\theta) \le \mathrm{M}_S\, g + \sum_{j=1}^{\infty} \frac{\mathrm{M}_S^{[j+1]} g}{A^j} = A \sum_{j=1}^{\infty} \frac{\mathrm{M}_S^{[j]} g}{A^j} \le A\, w^\theta \tag{3.13}$$

for all x.

Let $R_{m,k}$ be a rectangle in the starlike cover of S chosen as in (3.1). As explained above, we may also assume that (3.2) holds with $c_0 = 1$. Thus $R_{m,k} \subset S$. Now let $R \in \mathcal{B}(R_{m,k})$. Then

$$R = z + \gamma R_{m,k} \quad \text{for some } z \in \mathbf{R}^d \text{ and } \gamma > 0.$$

Pick a positive constant c_1 independent of R, m, γ and k such that for all $x \in R$,

$$R \subset x + c_1 \gamma R_{m,k},$$

and note that

$$c_1 \gamma R_{m,k} \subset c_1 \gamma\, S.$$

Therefore,

$$R \subset x + c_1 \gamma\, S \quad \text{for all } x \in R,$$

and then for $x \in R$,

$$\begin{aligned}
A\, w(x)^\theta \ge M_S(w^\theta)(x) &= \sup_{t>0} \frac{1}{t^d} \int_{x+tS} w(y)^\theta\, dy \\
&\ge \frac{1}{(c_1\gamma)^d} \int_{x+c_1\gamma\, S} w(y)^\theta\, dy \ge \frac{1}{(c_1\gamma)^d} \int_R w(y)^\theta\, dy \\
&= \frac{c}{\gamma^d |R_{m,k}|} \int_R w(y)^\theta\, dy \cdot |R_{m,k}| \\
&= c|R_{m,k}|\, \frac{1}{|R|} \int_R w(y)^\theta\, dy \quad \text{since } |R| = \gamma^d |R_{m,k}|.
\end{aligned}$$

Hence,

$$A \operatorname*{ess\,inf}_R w^\theta \ge c|R_{m,k}|\, \frac{1}{|R|} \int_R w^\theta\, dy$$

or

$$\frac{1}{|R|} \int_R w^\theta\, dy \le \frac{A}{c}\, \frac{1}{|R_{m,k}|} \operatorname*{ess\,inf}_R w^\theta \quad \text{for all } R \in \mathcal{B}(R_{m,k}). \tag{3.14}$$

Now let $p \geq 1, r \geq 1$ and $\epsilon > 0$. Pick $\theta = \max\{r, 1/(\epsilon p)\}$ and let w be defined as above. If $R \in \mathcal{B}(R_{m,k})$ and $1 \leq p < \infty$, then

$$\begin{aligned}
\left(\frac{1}{|R|}\int_R w^r\,dy\right)^{\frac{1}{pr}}\left(\frac{1}{|R|}\int_R w^{-rp'/p}\,dy\right)^{\frac{1}{p'r}} &\leq \left[\left(\frac{1}{|R|}\int_R w^r\,dy\right)^{\frac{1}{r}}\left(\operatorname*{ess\,inf}_R w\right)^{-1}\right]^{\frac{1}{p}}\\
&\leq \left[\left(\frac{1}{|R|}\int_R w^\theta\,dy\right)^{\frac{1}{\theta}}\left(\operatorname*{ess\,inf}_R w\right)^{-1}\right]^{\frac{1}{p}} \quad \text{since } \theta \geq r\\
&\leq \left[\left(\frac{c}{|R_{m,k}|}\operatorname*{ess\,inf}_R w^\theta\right)^{\frac{1}{\theta}}\left(\operatorname*{ess\,inf}_R w\right)^{-1}\right]^{\frac{1}{p}} \quad \text{by (3.14)}\\
&= c\,|R_{m,k}|^{-\frac{1}{\theta p}}.
\end{aligned}$$

But $\sup_k |R_{m,k}| \leq c|S_m| \to 0$ as $m \to \infty$, so since $\epsilon \geq 1/(\theta p)$, the last expression is bounded for large m and all k by $c|R_{m,k}|^{-\epsilon}$, which proves the theorem.

Remarks. (i) There are variants of the weight that was constructed above which have similar properties. For example, given $\theta > 1$, two weights w_1 and w_2 as above and $a, b \geq 0$, let

$$w = w_1^a w_2^{b(1-p)}.$$

Then by choosing $a, b \leq \theta/r$, it is easy to check that for $1 \leq p < \infty$ and all $R \in \mathcal{B}_{m,k}$,

$$\left(\frac{1}{|R|}\int_R w^r\,dy\right)^{\frac{1}{pr}}\left(\frac{1}{|R|}\int_R w^{-rp'/p}\,dy\right)^{\frac{1}{p'r}} \leq c|R_{m,k}|^{-\left(\frac{a}{p}+\frac{b}{p'}\right)\frac{1}{\theta}}.$$

(ii) If the radial boundary function $\rho \in L^d \log L(\mathbf{S}^{d-1})$ and $r > 1$, it is possible to construct weights v so that $\{\mathrm{M}\,G^*\,\mathrm{M}(v^r)\}^{1/r} \leq c\,v$. To do so, define v in the same way as the weight w in the proof of Theorem 3.8 with $\theta = r$, but with M_S replaced by $\mathrm{M}\,G^*\,\mathrm{M}$ and A replaced by the operator norm of $\mathrm{M}\,G^*\,\mathrm{M}$ on $L^q(\mathbf{R}^d)$. The simple argument that led to (3.13) then yields the desired estimate for v. It then follows from Theorem I that T_Ω is bounded on $L^p(v)$ if $\Omega \in L \log L(\mathbf{S}^{d-1})$ and has integral zero.

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