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BIFURCATION FROM CODIMENSION ONE RELATIVE HOMOCLINIC CYCLES

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Abstract. We study bifurcations of relative homoclinic cycles in flows that are equivariant under the action of a finite group. The relative homoclinic cycles we consider are not robust, but have codimension one. We assume real leading eigenvalues and connecting trajectories that approach the equilibria along leading directions. We show how suspensions of subshifts of finite type generically appear in the unfolding. Descriptions of the suspended subshifts in terms of the geometry and symmetry of the connecting trajectories are provided.

1. Introduction

This paper contains a study of codimension one bifurcations from nonrobust heteroclinic networks, in ordinary differential equations that are equivariant under the action of a finite group. An example that gives an impression of possible geometry and of occurring dynamics and bifurcations is contained in [31] by Matthies on a Takens-Bogdanov bifurcation with $\mathbb{D}_3$-symmetry. The bifurcations studied in [31] occur in systems for three coupled oscillators and for mode interactions in convection problems [14]. Matthies computes a normal form for differential equations on $\mathbb{C}^2$ that, truncated at second order terms, has the expression

$$\begin{aligned} \dot{v} &= w, \\ \dot{w} &= \mu_1 v + \mu_2 w + \bar{v}^2 - \bar{v}\bar{w}. \end{aligned}$$

In the bifurcation scenario, varying parameters μ_1, μ_2, a $\mathbb{D}_3$-symmetric configuration of three $\mathbb{Z}_2$-invariant homoclinic trajectories to the same ($\mathbb{D}_3$-invariant) equilibrium arises. In the unfolding Matthies found a suspended topological Markov chain.

To discuss the setting of our bifurcation study and to connect it to existing works on homoclinic and heteroclinic cycles in equivariant flows, we start with some generalities. Consider an ordinary differential equation $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$. Given two equilibria p_- and p_+, a heteroclinic trajectory (or connecting trajectory) $\{\gamma(t)\}_{t\in\mathbb{R}}$ is a solution that converges to $p_\pm$ as $t \to \pm\infty$. A heteroclinic cycle consists of disjoint equilibria $p_1, \ldots, p_k$ (with $k \geq 1$) and heteroclinic trajectories $\gamma_1, \ldots, \gamma_k$ such that $\lim_{t\to-\infty} \gamma_i(t) = p_{i-1}$ and $\lim_{t\to\infty} \gamma_i(t) = p_i$ with indices taken modulo k. This definition includes a homoclinic loop: the case with $k = 1$ of a single trajectory asymptotic to the same equilibrium for positive and negative time (a homoclinic

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trajectory). Heteroclinic networks are connected sets that can be written as a finite union of heteroclinic cycles.

For general differential equations, stable and unstable manifolds of hyperbolic equilibria with the same index (dimension of the unstable manifold) will typically not intersect. Indeed, by the Kupka-Smale theorem [33] the set of C^k differential equations with such a heteroclinic trajectory forms a set of Baire first category. Transversality arguments show that in one parameter families of differential equations one can expect an intersection to occur persistently at an isolated parameter value. As a homoclinic trajectory is obviously a connecting trajectory between equilibria of the same index, homoclinic trajectories can be expected to occur persistently in one parameter families.

To make this more precise, suppose p_-, p_+ are hyperbolic equilibria with indices $\operatorname{ind}(p_-)$ and $\operatorname{ind}(p_+)$. Heteroclinic trajectories from p_- to p_+ can occur persistently only if p_- has a larger index than p_+. In fact, the set of heteroclinic connections forms a manifold of dimension $\operatorname{ind}(p_-) - \operatorname{ind}(p_+)$ if the unstable manifold of p_- intersects the stable manifold of p_+ transversally. If the index of p_- is smaller or equal to the index of p_+, heteroclinic connections from p_- to p_+ can only be found persistently (at isolated parameter values) in k-parameter families of differential equations for $k = \operatorname{ind}(p_+) - \operatorname{ind}(p_-) + 1$. What is needed is a transverse intersection of the stable and unstable manifolds of the equilibria in the product $\mathbb{R}^n \times \mathbb{R}^k$ of state space and parameter space. The number k is called the codimension of the heteroclinic trajectory.

The situation is markedly different for differential equations that possess a discrete symmetry. The context we will assume in this paper is of a parameter-dependent differential equation

$$\dot{x} = f(x, \lambda), \tag{1.1}$$

with $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, that is equivariant under the linear action (representation) of a finite group G [13]. Thus $x(t)$ is a solution of (1.1) precisely if $gx(t)$ is a solution of (1.1) $\forall g \in G$ or, equivalently,

$$gf(x, \lambda) = f(gx, \lambda), \qquad \forall g \in G. \tag{1.2}$$

We recall the notions of isotropy group and fixed point space. The isotropy group G_q of a point q is defined by

$$G_q = \{g \in G \mid gq = q\}.$$

Note that each point of a trajectory γ has the same isotropy subgroup, so it makes sense to speak of the isotropy subgroup G_γ of γ. The fixed point space of a subgroup $H \subset G$ is defined as

$$\operatorname{Fix} H = \{x \in \mathbb{R}^n \mid gx = x \text{ for } g \in H\}.$$

Recall further that a (real) linear representation is called absolutely irreducible if the set of linear maps commuting with this representation is isomorphic to $\mathbb{R}$ [13].

Within a fixed point subspace, equilibria can have different indices even if they possess the same indices in $\mathbb{R}^n$, thus altering the codimension of the heteroclinic trajectory. As a consequence, heteroclinic connections between equilibria of equal index may occur robustly in equivariant flows; see dos Reis [8], Field [9], Guckenheimer and Holmes [15]. The geometry that is often considered is where inside a fixed point space there is a heteroclinic trajectory from an equilibrium with one-dimensional unstable manifold to an asymptotically stable equilibrium. Such connections are clearly robust. There has been much interest in the existence, asymptotic stability and also bifurcations of robust heteroclinic cycles or networks; see [10, 11, 26, 5, 29, 36, 1, 22, 2] and the references therein.

In this paper, we discuss the occurrence of heteroclinic networks Γ of codimension one, i.e. networks which typically appear persistently in a one parameter family of (equivariant) differential equations. We focus on generic codimension one heteroclinic networks for which none of the constituting heteroclinic connections are robust (i.e. of codimension zero). Such a heteroclinic network is a relative homoclinic cycle: there is a heteroclinic trajectory γ connecting equilibria p and hp for some $h \in G$, such that

$$\Gamma = G\overline{\gamma}, \tag{1.3}$$

the group orbit of the closure of a single heteroclinic trajectory. The equilibria will be assumed hyperbolic. The most important restriction we assume is the condition that the heteroclinic trajectories are tangent to the leading directions at the equilibria (the notions of leading eigenvalues and leading directions are given in Section 2). Although this is the typical case for general systems, it does not always hold for systems with symmetry. An important manifestation of symmetry is that it may enforce the linearization at a symmetric equilibrium point to have multiple leading eigenvalues. Symmetry can also force the simultaneous occurrence of several heteroclinic trajectories, all related by symmetry. We show how Lin's method, an analytic tool for the derivation of bifurcation equations for heteroclinic bifurcations [30, 34, 25], can be applied to study the dynamics near codimension one relative homoclinic cycles. While Lin's method was developed for simple leading eigenvalues, we apply the techniques to heteroclinic bifurcations with multiple leading eigenvalues by using the fact that they arise in a semisimple way.

The common picture in the bifurcations we encounter is the following. The relative homoclinic cycle exists at an isolated parameter value. When breaking the relative homoclinic cycle by varying the parameter, a recurrent set appears. The recurrent set, or at least subsets of it, can be described through a conjugation with a topological Markov chain (or a subshift of finite type; the reader can consult [21] for generalities on topological Markov chains). The dynamics on both sides of the bifurcation value differ and are described by different topological Markov chains. We explain the constructions of the topological Markov chains which describe the changes in the recurrent set. There is a natural way to use symbolic dynamics in the description of the recurrent set, that is, by using the language of topological Markov chains mentioned above. Each heteroclinic connection is thereto assigned a symbol (e.g. a unique integer), and trajectories near the relative homoclinic cycle are assigned a list of symbols describing near which connection the trajectory traverses. In this context, we recall the notion of the connectivity matrix $C = (c_{ij})$ of a heteroclinic network with heteroclinic trajectories γ_i, where $c_{ij} = 1$ if the endpoint (the ω-limit $\omega(\gamma_i)$) of heteroclinic connection γ_i is equal to the starting point (the α-limit $\alpha(\gamma_j)$) of heteroclinic connection γ_j.

Below we state a general bifurcation theorem that makes this scenario more precise. The complete constructions and conditions can be found in Section 2, where the main bifurcation theorem, Theorem 2.4, is formulated. A detailed account of relative homoclinic cycles consisting of homoclinic loops and their bifurcations will be given for differential equations with dihedral symmetry; see Section 5. This generalizes the motivating example in [31] on a Takens-Bogdanov bifurcation with $\mathbb{D}_3$-symmetry.

We introduce notation for topological Markov chains. Let

$$\Sigma_k = \{1, \ldots, k\}^{\mathbb{Z}} \tag{1.4}$$

denote the set of double infinite sequences $\kappa : \mathbb{Z} \to \{1, \ldots, k\}$, $i \mapsto \kappa_i$, equipped with the product topology. Let $A = (a_{ij})_{i,j \in \{1,\ldots,k\}}$ be a 0-1 matrix, that is, $a_{ij} \in \{0, 1\}$. By Σ_A we denote the topological Markov chain defined by A,

$$\Sigma_A = \{\kappa \in \Sigma_k \mid a_{\kappa_i \kappa_{i+1}} = 1\}.$$

Let σ be the left shift operating on Σ_k,

$$\sigma : \Sigma_k \to \Sigma_k, \quad (\sigma\kappa)_i = \kappa_{i+1}.$$

Observe that Σ_A is σ-invariant; we also write σ for the left shift restricted to Σ_A.

Theorem 1.1. *Let $\dot{x} = f(x, \lambda)$ be a one parameter family of differential equations equivariant with respect to a finite group G, with the following properties:*

(1) *At $\lambda = 0$, there is a codimension one relative homoclinic cycle Γ with hyperbolic equilibria.*
(2) *The connecting trajectories in Γ are nondegenerate (as formulated by Hypothesis* (H 6) *below).*
(3) *The isotropy subgroup G_p of an equilibrium p in Γ acts absolutely irreducibly on the leading stable eigenspace at p, and the leading stable eigenvalues of the linearized vector field about p are closest of all eigenvalues to the imaginary axis (see Hypothesis* (H 3) *below).*
(4) *The connecting trajectories in Γ approach the equilibria along the leading stable directions (a nonorbit-flip condition). The connecting trajectories in Γ satisfy a noninclination-flip condition (as formulated by Hypothesis* (H 5) *below).*

Write $\gamma_1, \ldots, \gamma_k$ for the connecting trajectories that constitute Γ. There is an explicit construction of $k \times k$ matrices A_- and A_+ with coefficients in $\{0, 1\}$ and the nonzero coefficients in mutually disjoint positions so that the following holds for any generic family unfolding a relative homoclinic cycle as above.

Take cross sections S_i transverse to γ_i and write Π_λ for the first return map on the collection of cross sections $\bigcup_{j=1}^k S_j$. For $\lambda > 0$ small enough, there is an invariant set $\mathcal{D}_\lambda \subset \bigcup_{j=1}^k S_j$ for Π_λ such that for each $\kappa \in \Sigma_{A_+}$ there exists a unique $x \in D_\lambda$ with $\Pi_\lambda^i(x) \in S_{\kappa_i}$. Moreover, $(\mathcal{D}_\lambda, \Pi_\lambda)$ is topologically conjugate to (Σ_{A_+}, σ). An analogous statement holds for $\lambda < 0$ with Σ_{A_+} replaced by Σ_{A_-}.

This above description of the dynamics provides a complete picture of the local nonwandering dynamics near Γ if and only if

$$A_+ + A_- = C, \tag{1.5}$$

where C denotes the connectivity matrix of the relative homoclinic cycle.

Example 1.2. We illustrate the result of Theorem 1.1 in the case of homoclinic bifurcation to a hyperbolic equilibrium of saddle type with $\mathbb{D}_3$-symmetry, where the relative homoclinic cycle consists of three connecting trajectories γ_i, $i = 1, 2, 3$, that each have isotropy equal to $\mathbb{Z}_2$. A complete treatment for relative homoclinic cycles with $\mathbb{D}_m$-symmetry is in Section 5.2. We furthermore assume that the leading (say, stable) eigenvalue at the equilibrium is real. As all three connecting trajectories are connecting to the same equilibrium, the connectivity matrix C is given by

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The different ways in which this bifurcation may take place are characterized by the (generically) absolutely irreducible action of the group $\mathbb{D}_3$ on the leading eigenspace. This irreducible representation is one-dimensional and trivial or is two-dimensional and acting as the symmetry group of the equilateral triangle.

In case the representation of $\mathbb{D}_3$ on the leading eigenspace is trivial, the leading eigenspace will generically be one-dimensional so that the connecting trajectories γ_i $(i = 1, 2, 3)$ come into the equilibrium in the same direction (tangent to each other). We will see that the matrices A_-, A_+ are given by

$$A_- = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad A_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that there is a nonwandering set that is topologically conjugate to the suspension of a full shift on three symbols if $\lambda < 0$ and no nontrivial nonwandering dynamics near the relative homoclinic cycle if $\lambda > 0$.

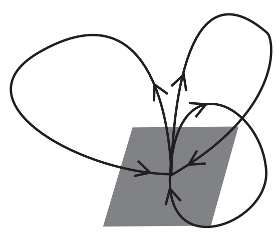


FIGURE 1. Illustration of three homoclinic trajectories approaching p when $\dim(E^s_p) = 2$. Note that the state space has to be at least four-dimensional.

In case the representation of $\mathbb{D}_3$ on the leading eigenspace is nontrivial, the leading eigenspace will generically be two-dimensional with the connecting trajectories γ_i, $i = 1, 2, 3$, coming into the equilibrium in three different directions (each separated by an angle of $2\pi/3$); see Figure 1. This configuration appears in the previously mentioned study by Matthies. It turns out that the matrices A_-, A_+ are given by

$$A_- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_+ = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

If $\lambda < 0$ the nonwandering set consists of three periodic solutions, shadowing the individual connecting trajectories γ_i $(i = 1, 2, 3)$. If $\lambda > 0$ the nonwandering dynamics is more complicated, with all trajectories avoiding shadowing twice the same connecting trajectory.

It should be noted that (1.5) is often, but not always, satisfied. In the latter case, the topological Markov chain described above describes part of the recurrent set. It is however possible to describe the situations where we need to confine ourselves as describing invariant subsets of the recurrent set. At this place we just comment that depending on the group and its representations on leading eigenspaces, the obstructions are either always avoided, or generically avoided, or enforced. We finally note that we do not prove hyperbolicity of the recurrent sets.

For an extension to codimension two bifurcations involving relative homoclinic cycles with resonance conditions among the leading eigenvalues, giving rise to saddle-node and period-doubling bifurcations of suspended topological Markov chains, see [20, 17].

The remainder of this paper is organized as follows. In the next section we start with the standing assumptions for our study and give a precise formulation of our main results. A more technical section, Section 3, follows, in which the techniques developed by Lin [30] and Sandstede [34] to derive bifurcation equations for trajectories in the recurrent set are adapted to the present context. In Section 4 these techniques are applied to prove the main theorem, Theorem 2.4. A catalogue of bifurcation scenarios for relative homoclinic cycles in systems with dihedral symmetry is derived in Section 5.

2. Setting and main results

In this section we introduce conditions on the action of the symmetry and the geometry of the flow, and present the general bifurcation theorem, Theorem 2.4. We will focus on bifurcations from relative homoclinic cycles. In this context, we note the following lemma.

Lemma 2.1. *Let γ be a heteroclinic trajectory connecting equilibria p and hp for some $h \in G$. Then $G\overline{\gamma}$ is connected, and thus a relative homoclinic cycle, if and only if*

$$G = \langle h, G_p \rangle.$$

Proof. Note that $\Gamma_1 = \langle h \rangle \overline{\gamma}$ is trivially connected. Define inductively $\Gamma_{i+1} = \bigcup_{q \in \Gamma_i} G_q \Gamma_i$, where the union is over equilibria in Γ_i. Each Γ_i is obviously connected. Since G is finite this process terminates and yields the relative homoclinic cycle Γ. We remark that for Abelian groups G, the isotropy groups G_p are identical for all equilibria p in Γ, and thus $\Gamma = \Gamma_2$. In general, the construction shows that isotropy groups of equilibria in Γ are conjugate to G_p via elements of $\langle h, G_p \rangle$. Therefore, $\Gamma = \langle h, G_p \rangle \overline{\gamma}$. □

To summarize, we assume the following to hold:

(H 1) Γ is a relative homoclinic cycle equal to the group orbit of the closure of a heteroclinic trajectory γ connecting hyperbolic equilibria p to hp for some $h \in G$.

By hyperbolicity of the equilibria in Γ, we may assume that their positions do not depend on the parameter λ, for λ close to 0.

As formulated by the following hypothesis, inside $\operatorname{Fix} G_\gamma$ the connection is assumed to be of codimension one (note that $\gamma \subset \operatorname{Fix} G_\gamma$). Write $\operatorname{ind}_{\operatorname{Fix} G_\gamma}(p)$ for the index $\dim(W^u(p) \cap \operatorname{Fix} G_\gamma)$ of p inside $\operatorname{Fix} G_\gamma$.

(H 2) $\operatorname{ind}_{\operatorname{Fix} G_\gamma}(p) = \operatorname{ind}_{\operatorname{Fix} G_\gamma}(hp)$.

The leading eigenvalues at a hyperbolic equilibrium p are the eigenvalues of $Df(p,\lambda)$ nearest to the imaginary axis, either with positive or negative real parts. Leading stable eigenvalues are the leading eigenvalues with negative real part; leading unstable eigenvalues have positive real part. The leading (stable or unstable) directions are the corresponding sums of generalized eigenspaces. By E^s_p and E^u_p we denote the leading stable and unstable directions at the equilibrium p.

In the codimension one bifurcation problem we are considering, it is readily verified that the action of G on E^s_{hp} and E^u_p is generically irreducible or is the direct sum of two isomorphic absolutely irreducible representations; cf [13].

As a spectral condition, we assume that (by changing the direction of time, if necessary) for $\lambda = 0$:

(H 3) The leading stable eigenvalue μ^s is real, and the isotropy subgroup G_{hp} acts absolutely irreducibly on E^s_{hp}. Moreover, $|\mu^s| < \mathrm{Re}(\mu^u)$ for each leading unstable eigenvalue μ^u.

If the absolutely irreducible representation has dimension m, then consequently μ^s has multiplicity m (and is semisimple). Note that $D_1f(q,\lambda)$ is conjugate to $D_1f(p,\lambda)$ for each q in the group orbit Gp, and so has identical spectrum. Denote by $\mu^s(\lambda)$ the leading stable eigenvalues of $D_1f(p,\lambda)$ for λ near 0, assuming without loss that the above hypothesis applies for all λ. By the smoothness of the vector field, $\mu^s(\lambda)$ depends smoothly on λ.

As further clarified by Lemma 3.3, we assume that γ approaches hp along a leading stable direction:

(H 4) $e^s_{hp} = \lim_{t\to\infty} \gamma(t)/\|\gamma(t)\| \in E^s_{hp}$.

We note that the isotropy subgroup of γ is a subgroup of the isotropy group of e^s_{hp}: $G_\gamma \subset G_{e^s_{hp}}$. Consequently, we have $e^s_{hp} \in \mathrm{Fix}\, G_\gamma$.

The demand that e^s_{hp} lies inside the leading direction leaves out an interesting class of codimension one relative homoclinic cycles where the symmetry (to be precise, the representations on leading eigenspaces) forces the connections to approach the equilibria along nonleading directions (compare the orbit flip in [34]).

In order to avoid a geometric configuration of manifolds similar to that arising at an inclination flip in systems without symmetry [16], we require that

(H 5) $\mathrm{Fix}\, G_\gamma \cap E^s_p \neq \{0\}$.

Again, we need to emphasize that there exist certain equivariant codimension one bifurcations where (H 5) is not satisfied. For the group $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2)^2$ we give an example of a network in $\mathbb{R}^5$ for which $\mathrm{Fix}\, G_\gamma \cap E^s_p = \{0\}$; that is, Hypothesis (H 5) is violated. Let the elements of G act on $\mathbb{R}^5$ as

$$\begin{aligned} a(x_1,x_2,x_3,x_4,x_5) &= (-x_1,-x_2,x_3,x_4,x_5), \\ b(x_1,x_2,x_3,x_4,x_5) &= (x_1,x_2,-x_3,-x_4,x_5) \end{aligned}$$

for $(\mathbb{Z}_2)^2$ and

$$c(x_1,x_2,x_3,x_4,x_5) = (x_3,x_4,x_1,x_2,-x_5)$$

for the remaining $\mathbb{Z}_2$. Note that c does not commute with a or b. The fixed point spaces $\mathrm{Fix}\,\mathbb{Z}_2(a) = \{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5 \mid x_1 = x_2 = 0\}$ and $\mathrm{Fix}\,\mathbb{Z}_2(b) = \{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5 \mid x_3 = x_4 = 0\}$ are three-dimensional and are interchanged by c. Let the equilibrium p be at $(0,0,0,0,v)$; then $cp = (0,0,0,0,-v)$. Suppose that the connection γ from p to cp is contained in $\operatorname{Fix}\mathbb{Z}_2(a)$ but not in $\operatorname{Fix}\mathbb{Z}_2(b)$ and that the leading unstable eigenvector and two stable directions of $Df(p)$ lie within $\operatorname{Fix}\mathbb{Z}_2(a)$. Thus $c\gamma \subset \operatorname{Fix}\mathbb{Z}_2(b)$, and by assumption (H 4) we have $e_p^s \in \operatorname{Fix}\mathbb{Z}_2(b)$. The connection approaches the equilibrium along the leading stable eigendirection. However, $E_p^s \cap \operatorname{Fix} G_\gamma = E_p^s \cap \operatorname{Fix}\mathbb{Z}_2(a) = \{0\}$.

Continuing our list of hypotheses, we assume that the heteroclinic trajectory γ is nondegenerate:

(H 6) The intersection of the tangent spaces along the unstable manifold $W^u(p)$ of p and along the stable manifold $W^s(hp)$ of hp is equal to the tangent space along γ.

We note that we use this hypothesis in our analysis, but we cannot exclude the fact that there exist situations where equivariance forces this hypothesis to fail.

The remaining condition we formulate implies that γ is a codimension one connecting trajectory, unfolding generically with the parameter λ. The condition contains the familiar nondegeneracy condition for general flows to avoid inclination flip inside the flow invariant fixed point space $\operatorname{Fix} G_\gamma$.

A local center unstable manifold $W^{cu}(p)$ of p is a locally invariant manifold with as tangent space at p the direct sum of the unstable and the leading stable directions. Likewise, a local center stable manifold $W^{cs}(hp)$ of hp is a locally invariant manifold with as tangent space at hp the direct sum of the stable and the leading unstable directions. Local center (un)stable manifolds are not unique but possess unique tangent spaces along the (un)stable manifold. See e.g. [16] for further generalities on invariant manifolds and foliations near equilibria.

The following hypothesis concerns the vector field $\dot{x} = f(x, \lambda)$ restricted to $\operatorname{Fix} G_\gamma$, which is a flow-invariant subspace. Denote by $\mathcal{F}_x^{ss}$ the strong stable fibers inside the stable manifold of hp and let $F^{ss} = \bigcup_{x\in\gamma} \mathcal{F}_x^{ss}$. Let the subscript $\operatorname{Fix} G_\gamma$ denote a restriction to $\operatorname{Fix} G_\gamma$.

(H 7) $F^{ss}_{\operatorname{Fix} G_\gamma}$ and $W^{cu}_{\operatorname{Fix} G_\gamma}(p)$ intersect transversally along γ (in $\operatorname{Fix} G_\gamma$):

$$F^{ss}_{\operatorname{Fix} G_\gamma} \pitchfork_\gamma W^{cu}_{\operatorname{Fix} G_\gamma}(p). \tag{2.1}$$

The bifurcation unfolds generically:

$$W^{u}_{\operatorname{Fix} G_\gamma}(p) \text{ and } W^{s}_{\operatorname{Fix} G_\gamma}(hp) \text{ split with positive speed in } \lambda. \tag{2.2}$$

The transversality expressed by (2.1) implies that a tangent space $T_x\mathcal{F}_x^{ss}$ plus a tangent space $T_x W^{cu}(p)$ spans a subspace of maximal dimension.

Condition (H 3) implies a nonresonance condition for the flow restricted to $\operatorname{Fix} G_\gamma$. See [6] for homoclinic bifurcation problems in generic systems with resonant eigenvalues. Condition (H 4) excludes an orbit flip condition (see [34]), while (2.1) is a generalization of the noninclination flip condition (see [23, 19]). Note that these conditions are automatically satisfied if $\operatorname{Fix} G_\gamma$ is two-dimensional, independent of the dimension of the entire space.

Lemma 2.2. *Suppose Hypotheses* (H 1) – (H 6) *are met. Then by an arbitrary small perturbation of the differential equation, Hypothesis* (H 7) *will be met, and inside* $\operatorname{Fix} G_\gamma$*,*

$$T\mathcal{F}^{ss} \text{ is a continuous vector bundle along } \overline{\gamma}. \tag{2.3}$$

Proof. Initially, we require only a small perturbation of the vector field restricted to $\mathrm{Fix}\, G_\gamma$, the existence of which follows from the general theory without symmetry [16]. Those perturbations can be extended to a small equivariant perturbation of f in the phase space in the neighborhood of $\mathrm{Fix}\, G_\gamma$. Note that by averaging over the group a small perturbation $\tilde{f}$ can be made equivariant:

$$\frac{1}{|G|}\sum_{g\in G} g\tilde{f}g^{-1}$$

is equivariant (here $|G|$ is the cardinal number of G). The first part of the lemma follows.

Statement (2.3) is a variant of the strong lambda-lemma as found in [7]. As the statement is for $\lambda = 0$, we will write $f(x) = f(x, 0)$. Denote by φ_t the flow of f. Then $D\varphi_t(x)$ is the flow of the variational equation along $\{\varphi_t(x)\}$. Let $\mathcal{G}_{ss}(\mathbb{R}^n)$ be the Grassmannian manifold of linear subspaces in $\mathbb{R}^n$ with dimension $\dim E^{ss}$. Lift f to an induced vector field $\bar{f}$ on $\mathbb{R}^n \times \mathcal{G}_{ss}(\mathbb{R}^n)$ by defining its flow:

$$\bar{\varphi}_t(x,\alpha) = (\varphi_t(x), D\varphi_t(x)\alpha).$$

Observe that this is a skew product flow. Clearly, (p, E^{ss}) is an equilibrium of $\bar{f}$. A direct computation shows that (p, E^{ss}) is a hyperbolic equilibrium and the unstable directions include the tangent space of the fiber $\{p\} \times T_{E^{ss}}\mathcal{G}_{ss}(\mathbb{R}^n)$; see e.g. [16].

Consider $\mathcal{F}^{ss}_q$ at a point q near p. As $\mathcal{F}^{ss}_q$ is transversal to $W^{cu}(p)$, $T_q\mathcal{F}^{ss}_q$ is close to E^{ss} by standard cone field constructions; see e.g. [21]. The trajectory of $(q, T_q\mathcal{F}^{ss}_q)$ under the flow of $\bar{f}$ therefore lies in the unstable manifold of (p, E^{ss}). This implies continuity of the bundle $T_x\mathcal{F}^{ss}_x$, $x \in \gamma$. Continuity of $T_x\mathcal{F}^{uu}_x$, $x \in \gamma$, is similar. □

Consider that Γ consists of finitely many, namely $k = |G|/|G_\gamma|$, connecting trajectories $\gamma_1, \dots, \gamma_k$ (here $|H|$ denotes the cardinal number of the group H). We can choose the time parameterization of the connecting trajectories so that for each $i \in \{1, \dots, k\}$ there is a $g_i \in G$ with $\gamma_i(0) = g_i\gamma(0)$.

Because G is compact we may assume that a G-invariant $\langle\cdot,\cdot\rangle$ inner product is given. This inner product induces a structure of a Hilbert space on the state space $\mathbb{R}^n$. We may choose $\langle\cdot,\cdot\rangle$ so that $T_pW^s(p) \perp T_pW^u(p)$ for an equilibrium p in Γ and therefore by Hypothesis (H 1) for each equilibrium in Γ. Write

$$X_i = \mathrm{span}\,\{f(\gamma_j(0), 0)\} \tag{2.4}$$

for the tangent space to the trajectory $\gamma_i(t)$ at $t = 0$. Then

$$S_i = \gamma_i(0) + X_i^\perp, \quad i \in 1, \dots, k, \tag{2.5}$$

is a hyperplane intersecting γ_i transversally at $\gamma_i(0)$. Due to the G-invariance of the inner product $\langle\cdot,\cdot\rangle$, we have $GS_1 = \{S_i \mid i = 1, \dots, k\}$. As usual, denote by $\alpha(\gamma_i)$, $\omega(\gamma_i)$ the α and ω limits of γ_i, which are equilibria here. Let

$$Z_i = \left(T_{\gamma_i(0)}W^s(\omega(\gamma_i)) + T_{\gamma_i(0)}W^u(\alpha(\gamma_i))\right)^\perp. \tag{2.6}$$

Note that the subspaces Z_i, $i = 1, \dots, k$, are g-images of each other. Further, $Z_i \subset X_i^\perp$, and so $\gamma_i(0) + Z_i \subset S_i$. As a consequence of Hypothesis (H 6),

$$\dim Z_i = 1$$

for all i. We can thus take a unit vector $\psi_1 \in Z_1$ spanning Z_1. Take further vectors $\psi_i \in Z_i$ belonging to the G-orbit of $\psi_1 \in Z_1$. Due to the invariance of the inner product, all ψ_i have norm 1.

Lemma 2.3. *For each λ close to 0 there are unique pairs $(\gamma_i^+(\lambda)(\cdot), \gamma_i^-(\lambda)(\cdot))$ of trajectories of* (1.1) *such that:*

- *(i)* $\gamma_i^\pm(\cdot)(0)$ *smooth and* $\gamma_i^\pm(0)(0) = \gamma_i(0)$,
- *(ii)* $\gamma_i^+(\lambda)(0) \in S_i \cap W^s(\omega(\gamma_i), \lambda), \quad \gamma_i^-(\lambda)(0) \in S_i \cap W^u(\alpha(\gamma_i), \lambda)$,
- *(iii)* $|\gamma_i^+(\lambda)(t) - \gamma_i(t)|$ *small* $\forall t \in \mathbb{R}^+$ *and* $|\gamma_i^-(\lambda)(t) - \gamma_i(t)|$ *small* $\forall t \in \mathbb{R}^-$,
- *(iv)* $\gamma_i^+(\lambda)(0) - \gamma_i^-(\lambda)(0) \in Z_i$.

Proof. The statements are a direct consequence of transversality, for fixed λ, of the intersection of $\bigcup_{x \in W^s(\omega(\gamma_i),\lambda)} (x + Z_i)$ with $W^u(\alpha(\gamma_i), \lambda)$ inside S_i near $\gamma_i(0)$. That the intersection is transversal follows from Hypothesis (H 6). Observe that the intersection is a single point, which we denote by $\gamma_i^-(\lambda)(0)$. The point $\gamma_i^+(\lambda)(0)$ is given as $\left(\gamma_i^-(\lambda)(0) + Z_i\right) \cap W^s(\omega(\gamma_i), \lambda)$. $\square$

In [34, 25] integral expressions are derived for the difference $\gamma_i^+(\lambda)(0) - \gamma_i^-(\lambda)(0)$.

For $\kappa \in \Sigma_k$, we denote a trajectory of (1.1) by $x(\lambda, \kappa)(\cdot)$ if there is a monotonically increasing sequence $(\tau_i)_{i \in \mathbb{Z}}$ such that

$$x(\lambda, \kappa)(\tau_i) \in S_{\kappa_i}, \quad x(\lambda, \kappa)(t) \notin \bigcup_{j=1}^k S_j, \text{ if } t \notin \{\tau_i, i \in \mathbb{Z}\}.$$

We call κ the itinerary of $x(\lambda, \kappa)(\cdot)$. Let Π_λ be the first return map defined on $\bigcup_{j=1}^k S_j$ (in fact the domain of Π_λ is only a subset of $\bigcup_{j=1}^k S_j$):

$$\Pi_\lambda : \bigcup_{j=1}^k S_j \to \bigcup_{j=1}^k S_j, \quad \Pi_\lambda(x(\lambda, \kappa)(\tau_i)) = x(\lambda, \kappa)(\tau_{i+1}). \tag{2.7}$$

The main result below describes the way recurrent sets change through the bifurcation. The following definitions serve to define the topological Markov chains to which the first return map on (subsets of) the recurrent set is conjugate. Define

$$e_i^s = \lim_{t \to \infty} \gamma_i(t)/\|\gamma_i(t)\|.$$

To elucidate the symmetrical relation with definitions of further vectors e_j^- below, we provide an equivalent definition via the variational equation. Let $\zeta_i(t) = \dot{\gamma}_i(t)$ be the solution to

$$\dot{v}(t) = D_1 f(\gamma_i(t), 0) v(t), \quad v(0) = -\dot{\gamma}_i(0).$$

Then

$$e_i^s = \lim_{t \to \infty} \zeta_i(t)/\|\zeta_i(t)\|.$$

By Hypothesis (H 7) (more precisely, (2.2)), one can choose the unit vectors $\psi_i \in Z_i$ so that additionally

$$\partial/\partial\lambda \langle \gamma_i^+(\lambda)(0) - \gamma_i^-(\lambda)(0), \psi_i \rangle > 0. \tag{2.8}$$

That is, ψ_i is chosen such that the splitting of stable and unstable manifolds is in the direction of ψ_i. Let $\psi_j(t)$ be the solution to the adjoint variational equation

$$\dot{w}(t) = -D_1 f(\gamma_j(t), 0)^* w(t), \quad w(0) = \psi_j.$$

The initial condition guarantees that $\psi_j(t)$ converges to 0 as $t \to \pm\infty$; see Section 3. Define

$$e_j^- = \lim_{t\to-\infty} \psi_j(t)/\|\psi_j(t)\|.$$

The existence of the limit is guaranteed by Lemma 3.4. By Hypothesis (H 5), the vector e_j^- is contained in the leading unstable space E^- of $D_1 f(\alpha(\gamma_j))^*$, which equals $E^s_{\alpha(\gamma_j)}$ by the choice of inner product.

Theorem 2.4. *Assume Hypotheses* (H 1) - (H 7) *and define the matrix* $M = (m_{ij})_{i,j\in\{1,\ldots,k\}}$ *by*

$$m_{ij} = \begin{cases} 0, & \text{if } \omega(\gamma_i) \neq \alpha(\gamma_j), \\ \operatorname{sgn}\langle e_i^s, e_j^-\rangle, & \text{if } \omega(\gamma_i) = \alpha(\gamma_j). \end{cases}$$

Write

$$A_+ = 1/2(M + |M|), \quad A_- = -1/2(M - |M|).$$

For $\lambda > 0$ small enough, there is an invariant set $\mathcal{D}_\lambda \subset \bigcup_{j=1}^k S_j$ for Π_λ such that for each $\kappa \in \Sigma_{A_+}$ there exists a unique trajectory $x(\lambda,\kappa)(\cdot)$ with $x(\lambda,\kappa)(0) \in \mathcal{D}_\lambda$. Moreover, $(\mathcal{D}_\lambda, \Pi_\lambda)$ is topologically conjugated to (Σ_{A_+}, σ).

An analogous statement holds for $\lambda < 0$ with Σ_{A_+} replaced by Σ_{A_-}.

If the inner products $\langle e_i^s, e_j^-\rangle$ are different from 0 for all i, j with $\omega(\gamma_i) = \alpha(\gamma_j)$, then the recurrent set for $\lambda = 0$ is the relative homoclinic cycle Γ. In this case the above described recurrent trajectories for $\lambda > 0$ and $\lambda < 0$ provide the entire recurrent set.

Applied to a single symmetric homoclinic trajectory, Theorem 2.4 gives the symmetric equivalent to the usual blue sky catastrophe [35].

Corollary 2.5. *Assume in addition to Hypotheses* (H 1) - (H 7) *that γ is a symmetric homoclinic trajectory to a symmetric equilibrium p: $G_\gamma = G_p = G$. Then f has a symmetric periodic solution for λ on one side of* 0. *This periodic solution converges as a set to γ when $\lambda \to 0$, while developing infinite period and disappears in $\lambda = 0$.*

Proof. We get $M = (\pm 1)$. □

Remark 2.6. The structure of M depends on the isotropy group of γ and the G-action on E^s_{hp} (the eigenvalue condition in Hypothesis (H 3) is the reason that it does not depend on the G-action on E^u_p). Every column and every row in M contains the same number of equal coefficients $0, -1$ or 1.

Examples studied in Section 5 provide several examples of the matrix M. Dynamics near relative homoclinic cycles in systems that are equivariant with respect to an action of a dihedral group $G = \mathbb{D}_m$ are classified in Section 5.

3. Lin's method

The following concept for analyzing the dynamics near a heteroclinic chain (an infinite sequence of consecutive heteroclinic trajectories) is due to Lin [30]; the ideas for estimating the jumps have been introduced by Sandstede [34]. See also [25] for a presentation of Lin's method.

In our presentation we restrict ourselves to chains which are related to the bifurcation problem stated in Section 2. Recall that heteroclinic trajectories $\gamma_1, \ldots, \gamma_k$

are given. Let $\kappa \in \Sigma_k$, where Σ_k is the set if double infinite sequences on k symbols (see (1.4)), be fixed. A heteroclinic chain γ^κ is a double infinite sequence of connecting trajectories γ_{κ_i}, $i \in \mathbb{Z}$, so that $\omega(\gamma_{\kappa_{i-1}}) = \alpha(\gamma_{\kappa_i})$. Write

$$p_{\kappa_i} = \omega(\gamma_{\kappa_{i-1}}) = \alpha(\gamma_{\kappa_i}).$$

Thus for each fixed index $i \in \mathbb{Z}$, γ_{κ_i} lies in the intersection of the unstable manifold $W^u(p_{\kappa_i})$ of p_{κ_i} and the stable manifold $W^s(p_{\kappa_{i+1}})$ of $p_{\kappa_{i+1}}$.

3.1. **Lin trajectories.** Recall that S_i is a cross section transverse to γ_i and the subspace Z_i gives the direction perpendicular to the stable and unstable manifolds $W^s(\omega(\gamma_i))$, $W^u(\alpha(\gamma_i))$ at $\gamma_i(0)$; see (2.5), (2.6). Given a sequence $\boldsymbol{\omega} = (\omega_i)_{i\in\mathbb{Z}}$ of sufficiently large transition times $\omega_i > 0$, one can prove the existence of a unique piecewise continuous trajectory $x = (x_i)_{i\in\mathbb{Z}}$ with the following properties:

1. Each x_i is a trajectory of the vector field, starting at a point on $S_{\kappa_{i-1}}$, staying close to $\gamma_{\kappa_{i-1}}$ until it reaches a neighborhood of p_{κ_i}, and then continuing close to γ_{κ_i} until it reaches S_{κ_i} at time $2\omega_i$.
2. The jump Ξ_i, defined as the difference between the initial point of x_{i+1} and the final point of x_i, belongs to the subspace Z_{κ_i}.

Figure 2 visualizes the trajectories with jumps in the cross sections. In what follows we will refer to such trajectories with jumps as Lin trajectories.

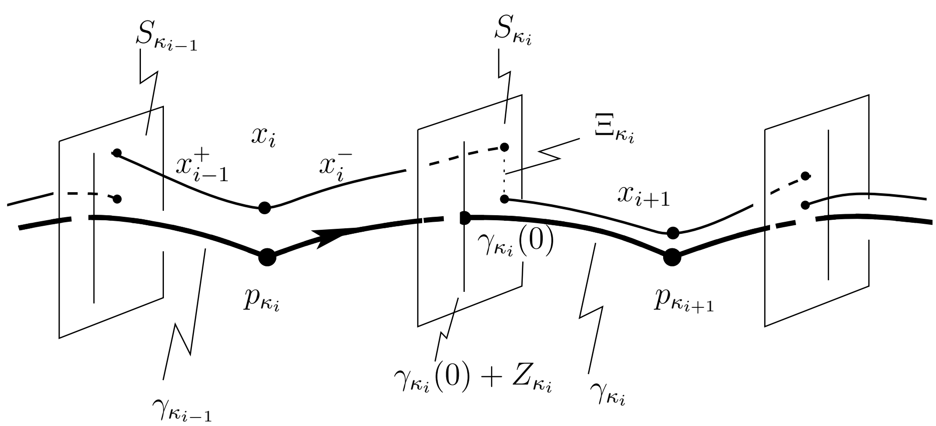


FIGURE 2. Lin's method involves the construction of piecewise continuous orbits following a heteroclinic chain with jumps in a fixed direction at points in cross sections.

Both the Lin trajectory x and the corresponding jump $\boldsymbol{\Xi} = (\Xi_i)_{i\in\mathbb{Z}}$ depend on $\boldsymbol{\omega}$, λ and $\kappa \in \Sigma_k$:

$$x = x(\boldsymbol{\omega}, \lambda, \kappa), \qquad \boldsymbol{\Xi} = \boldsymbol{\Xi}(\boldsymbol{\omega}, \lambda, \kappa).$$

In order to obtain an actual trajectory which stays for all time close to the heteroclinic chain γ^κ, one has to set the jumps equal to zero, yielding the bifurcation equation

$$\boldsymbol{\Xi}(\boldsymbol{\omega}, \lambda, \kappa) = 0.$$

The single jumps take the form

$$\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = \xi^\infty_{\kappa_i}(\lambda) + \xi_i(\boldsymbol{\omega}, \lambda, \kappa),$$

where $\xi^{\infty}_{\kappa_i}(\lambda)$ measures the splitting of the stable and unstable manifolds of $p_{\kappa_{i+1}}$ and p_{κ_i}, respectively. The expression $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$ is shown to be exponentially small as $\boldsymbol{\omega}$ tends to infinity; also, explicit expressions are obtained for the leading terms of $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$.

In [34, 25] the estimates of the leading terms of ξ_i are derived for simple leading eigenvalues μ^s and μ^u. In the present paper leading eigenvalues are in general semisimple. Although the estimates here can be attained in the same way as for simple leading eigenvalues, we will present the main steps in their derivation. For that purpose we must in some detail treat the construction of Lin trajectories.

Actually $x_{i+1}(\cdot)$ will be composed of trajectories $x_i^+(\cdot)$ and $x_{i+1}^-(\cdot)$ which are defined on $[0, \omega_{i+1}]$ and $[-\omega_{i+1}, 0]$, respectively. This requires coupling conditions

$$x_i^+(\omega_{i+1}) = x_{i+1}^-(-\omega_{i+1}) \tag{3.1}$$

and jump conditions

$$\Xi_i = x_i^+(0) - x_i^-(0) \in Z_{\kappa_i}, \tag{3.2}$$

for $i \in \mathbb{Z}$. We look for solutions of the form

$$x_i^{\pm}(t) = \gamma^{\pm}_{\kappa_i}(t) + v_i^{\pm}(t), \tag{3.3}$$

where $\gamma^{\pm}_{\kappa_i}$ are given by Lemma 2.3. We also write e.g. $\gamma^{\pm}_{\kappa_i}(\lambda)(t)$ to include dependence on λ in the notation.

The following proposition ensures the existence of Lin trajectories for each given sequence of transition times.

Proposition 3.1. *Let $\kappa \in \Sigma_k$ be fixed. There is $C > 0$, so that for each sequence $\boldsymbol{\omega} = (\omega_i)_{i\in\mathbb{Z}}$ with $\sup_{i\in\mathbb{Z}} \omega_i > C$ and each λ there is a unique Lin trajectory: there are unique functions $v_i^{\pm}(\boldsymbol{\omega}, \lambda, \kappa)(\cdot)$ such that $x_i^{\pm}$ (defined in accordance with* (3.3)*) satisfy both the coupling condition* (3.1) *and the jump condition* (3.2)*.*

Proof. We give merely an outline of the arguments, providing some statements for later use. Detailed proofs can be found in [38, 34, 24].

First we define appropriate direct sum decompositions and projections. Assigned to $\gamma_j(0)$, $j \in \{1, \dots, k\}$, consider the following orthogonal direct sum decompositions of $\mathbb{R}^n$ (recall (2.4)):

$$\mathbb{R}^n \quad = \quad X_j \oplus W_j^+ \oplus W_j^- \oplus Z_j, \tag{3.4}$$

where $W_j^+ = T_{\gamma_j(0)} W^s(\omega(\gamma_j)) \cap X_j^{\perp}$ and $W_j^- = T_{\gamma_j(0)} W^u(\alpha(\gamma_j)) \cap X_j^{\perp}$. Consider the variational equations

$$\dot{v} = D_1 f(\gamma_j^{\pm}(\lambda)(t), \lambda) v,$$

defined on $\mathbb{R}^{\pm}$, and the corresponding transition matrices $\Phi_j^{\pm}(\lambda, \cdot, \cdot)$. One can define projections $P_j^{\pm}$ satisfying

$$\ker P_j^+(\lambda, 0) = T_{\gamma_j^+(\lambda)(0)} W^s(\omega(\gamma_j)), \quad \operatorname{im} P_j^+(\lambda, 0) = W_j^- \oplus Z,$$
$$\ker P_j^-(\lambda, 0) = T_{\gamma_j^-(\lambda)(0)} W^u(\alpha(\gamma_j)), \quad \operatorname{im} P_j^-(\lambda, 0) = W_j^+ \oplus Z$$

that are commuting with $\Phi_j^{\pm}(\lambda, \cdot, \cdot)$,

$$P_j^{\pm}(\lambda, t) \Phi_j^{\pm}(\lambda, t, s) = \Phi_j^{\pm}(\lambda, t, s) P_j^{\pm}(\lambda, s).$$

These projections are related to the exponential dichotomies of the variational equations. In the limit $t \to \pm\infty$, $P_j^{\pm}(\lambda, t)$ converge to projections onto stable and unstable subspaces at the equilibria,

$$\begin{aligned} &\ker \lim_{t\to\infty} P_j^+(\lambda,t) = T_{\omega(\gamma_j^+)} W^s(\omega(\gamma_j)), \quad &&\operatorname{im} \lim_{t\to\infty} P_j^+(\lambda,t) = T_{\omega(\gamma_j^+)} W^u(\omega(\gamma_j)), \\ &\ker \lim_{t\to-\infty} P_j^-(\lambda,t) = T_{\alpha(\gamma_j^-)} W^u(\alpha(\gamma_j)), \quad &&\operatorname{im} \lim_{t\to-\infty} P_j^-(\lambda,t) = T_{\alpha(\gamma_j^-)} W^s(\alpha(\gamma_j)). \end{aligned} \tag{3.5}$$

For a given sequence of transition times $\boldsymbol{\omega}$ we denote by $\mathcal{V}_{\boldsymbol{\omega}}$ the space of all sequences $\mathbf{v} = ((v_i^+, v_i^-))_{i\in\mathbb{Z}}$ with $(v_i^+, v_i^-) \in C[0, \omega_{i+1}] \times C[-\omega_i, 0]$. Endowed with the norm $\|\mathbf{v}\| := \max\{\sup_{i\in\mathbb{Z}} \|v_i^+\|, \sup_{i\in\mathbb{Z}} \|v_i^-\|\}$, $\mathcal{V}_{\boldsymbol{\omega}}$ is a Banach space.

The $v_i^{\pm}$ introduced in (3.3) solve the (nonlinear) variational equation along $\gamma_{\kappa_i}^{\pm}$:

$$\dot v_i^{\pm}(t) = D_1 f(\gamma_{\kappa_i}^{\pm}(\lambda)(t), \lambda) v_i^{\pm}(t) + h_{\kappa_i}^{\pm}(t, v_i^{\pm}(t), \lambda), \tag{3.6}$$

where

$$h_{\kappa_i}^{\pm}(t, v, \lambda) = f(\gamma_{\kappa_i}^{\pm}(\lambda)(t) + v, \lambda) - f(\gamma_{\kappa_i}^{\pm}(\lambda)(t), \lambda) - D_1 f(\gamma_{\kappa_i}^{\pm}(\lambda)(t), \lambda) v.$$

Here $v_i^{\pm}$ must satisfy certain boundary conditions stipulated by coupling and jump conditions on $x_i^{\pm}$:

$$v_i^+(\omega_{i+1}) - v_{i+1}^-(-\omega_{i+1}) = \gamma_{\kappa_{i+1}}^-(\lambda)(-\omega_{i+1}) - \gamma_{\kappa_i}^+(\lambda)(\omega_{i+1}) =: d_{i+1}(\boldsymbol{\omega}, \lambda, \kappa), \tag{3.7}$$

$$v_i^+(0) - v_i^-(0) \in Z_{\kappa_i}. \tag{3.8}$$

Write $\mathbf{d}(\boldsymbol{\omega}, \lambda, \kappa) = (d_i(\boldsymbol{\omega}, \lambda, \kappa))_{i\in\mathbb{Z}}$.

In order to solve the boundary value problem (3.6)–(3.8) we perform several steps. First we "linearize" equation (3.6) by replacing $h_{\kappa_i}^+$ and $h_{\kappa_i}^-$ by some $g_i^+ \in C[0, \omega_{i+1}]$ and $g_i^- \in C[-\omega_i, 0]$, respectively:

$$\dot v_i^{\pm}(t) = D_1 f(\gamma_{\kappa_i}^{\pm}(\lambda)(t), \lambda) v_i^{\pm}(t) + g_i^{\pm}(t, \lambda). \tag{3.9}$$

Simultaneously, we replace the boundary condition (3.7) by prescribing projections a_i^+ and a_i^- of $v_{i-1}^+(\omega_i)$ and $v_i^-(-\omega_i)$:

$$a_i^+ = (id - P_{\kappa_{i-1}}^+(\lambda, \omega_i)) v_{i-1}^+(\omega_i), \quad a_i^- = (id - P_{\kappa_i}^-(\lambda, -\omega_i)) v_i^-(-\omega_i). \tag{3.10}$$

The corresponding sequences we denote by $\mathbf{g} = (g_i^+, g_i^-)_{i\in\mathbb{Z}}$ and $\mathbf{a} = (a_i^+, a_i^-)_{i\in\mathbb{Z}}$. The variational equation (3.9) with boundary conditions (3.8), (3.10) has a unique solution $\bar{\mathbf{v}}(\boldsymbol{\omega}, \lambda, \mathbf{g}, \mathbf{a}, \kappa) = (\bar v_i^+, \bar v_i^-)_{i\in\mathbb{Z}}$. We remark that the quantities $\bar v_i^{\pm}$ do not depend on the entire sequences $\mathbf{a}$ and κ, but only on a_{i+1}^+, a_i^- and κ_i:

$$\bar v_i^{\pm} = \bar v_i^{\pm}(\boldsymbol{\omega}, \lambda, \mathbf{g}, (a_{i+1}^+, a_i^-), \kappa_i). \tag{3.11}$$

Next we consider (3.9) with the original boundary conditions (3.7), (3.8). We claim that there exists $\mathbf{a}(\boldsymbol{\omega}, \lambda, \mathbf{g}, \mathbf{d}, \kappa)$ so that

$$\hat{\mathbf{v}}(\boldsymbol{\omega}, \lambda, \mathbf{g}, \mathbf{d}, \kappa) = \bar{\mathbf{v}}(\boldsymbol{\omega}, \lambda, \mathbf{g}, \mathbf{a}(\boldsymbol{\omega}, \lambda, \mathbf{g}, \mathbf{d}, \kappa), \kappa) \tag{3.12}$$

solves (3.9) with boundary conditions (3.7), (3.8). By (3.5), for ω_i large enough, $\operatorname{im}(id - P_{\kappa_{i-1}}^+(\lambda, \omega_i)) \oplus \operatorname{im}(id - P_{\kappa_i}^-(\lambda, -\omega_i)) = \mathbb{R}^n$. Hence, combining the boundary conditions (3.7), (3.10), $\mathbf{a}$ is obtained from

$$a_i^+ - a_i^- = d_i - P_{\kappa_{i-1}}^+(\lambda, \omega_i) \bar v_{i-1}^+(\omega_i) + P_{\kappa_i}^-(\lambda, -\omega_i) \bar v_i^-(-\omega_i). \tag{3.13}$$

Note that $a_i^{\pm}$ do not depend on the entire sequences $\mathbf{d}$ and κ but only on d_i and κ_{i-1}, κ_i:

$$a_i^{\pm} = a_i^{\pm}(\boldsymbol{\omega}, \lambda, \mathbf{g}, d_i, (\kappa_{i-1}, \kappa_i)). \tag{3.14}$$

The original boundary value problem (3.6) with boundary conditions (3.7), (3.8) is now equivalent to the following fixed point equation in $\mathcal{V}_{\boldsymbol{\omega}}$:

$$\mathbf{v} = \hat{\mathbf{v}}(\boldsymbol{\omega}, \lambda, \mathcal{H}(\mathbf{v}, \lambda, \kappa), \mathbf{d}(\boldsymbol{\omega}, \lambda, \kappa), \kappa), \tag{3.15}$$

with

$$\mathcal{H}(\mathbf{v}, \lambda, \kappa) = (H_i^+(\mathbf{v}, \lambda, \kappa), H_i^-(\mathbf{v}, \lambda, \kappa))_{i\in\mathbb{Z}}, \quad H_i^{\pm}(\mathbf{v}, \lambda, \kappa)(t) = h_{\kappa_i}^{\pm}(t, v_i^{\pm}(t), \lambda).$$

This equation has a unique solution $\mathbf{v}(\boldsymbol{\omega}, \lambda, \kappa)$; the mapping $\mathbf{v}(\cdot, \cdot, \kappa)$ is differentiable. □

Remark 3.2. It follows from the derivation that $\hat{\mathbf{v}}(\boldsymbol{\omega}, \lambda, \cdot, \cdot, \kappa)$ is linear.

In the following section we discuss asymptotic expansions for $v_i^{\pm}(\boldsymbol{\omega}, \lambda, \kappa)(\cdot)$. These will lead to bifurcation equations.

3.2. **Reference trajectories.** In the derivation of bifurcation equations certain reference trajectories play a central role. The reference trajectories are defined either for positive or negative time. The trajectories $\gamma_i^{\pm}(t)$ constructed in Lemma 2.3 are reference trajectories. The others are solutions $\psi_i^{\pm}$ to adjoint variational equations along $\gamma_i^{\pm}(t)$. They are given as solutions to the following equations. Let $\psi_i^-(t)$ be the solution for $t \leq 0$ to

$$\dot{w}(t) = -D_1 f(\gamma_i^-(t), \lambda)^* w(t), \quad w(0) = \psi_i.$$

Likewise, let $\psi_i^+(t)$ be the solution for $t \geq 0$ to

$$\dot{w}(t) = -D_1 f(\gamma_i^+(t), \lambda)^* w(t), \quad w(0) = \psi_i.$$

In this section we present asymptotic expansions for these reference trajectories; see Lemmas 3.5 and 3.6 below.

We first give two general lemmas providing asymptotic expansions for solutions of certain nonlinear and autonomous linear equations involving semisimple leading eigenvalues. The estimates are related to [34, Lemma 1.7], which however is formulated for simple leading eigenvalues.

Lemma 3.3. *Let $x = 0$ be an asymptotically stable equilibrium of a smooth family of vector fields $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. We assume that spec$(D_1 f(0, \lambda)) = \{\mu^s(\lambda)\} \cup$ spec$^{ss}(\lambda)$, where* Re *$\mu < \alpha^{ss} < \mu^s(\lambda) < \alpha^s < 0$ for all $\mu \in$ spec$^{ss}(\lambda)$. For all λ the (real) leading stable eigenvalue $\mu^s(\lambda)$ is assumed to be semisimple. We choose α^s such that $2\alpha^s < \mu^s(\lambda)$ for sufficiently small λ. Let $E^s(\lambda)$ and $E^{ss}(\lambda)$ be the (generalized) eigenspaces assigned to $\mu^s(\lambda)$ and spec$^{ss}(\lambda)$, respectively, and let $P_s(\lambda)$ be the projection on $E^s(\lambda)$ along $E^{ss}(\lambda)$. Then there is a $\delta > 0$ such that for all trajectories $x(\cdot)$ of $\dot{x} = f(x, \lambda)$ with $\|x(0)\| < \delta$, there exists the limit $\eta(x(0), \lambda) = \lim_{t\to\infty} e^{-D_1 f(0,\lambda)t} P_s(\lambda)x(t) \in E^s(\lambda)$. Furthermore, there is a constant c such that*

$$\|x(t) - e^{D_1 f(0,\lambda)t}\eta(x(0), \lambda)\| \leq ce^{\max\{\alpha^{ss}, 2\alpha^s\}t}. \tag{3.16}$$

Proof. In order to prove this lemma we use the fact that any trajectory within the stable manifold tends exponentially fast towards the equilibrium. More precisely (assuming that $x = 0$ is the equilibrium), for each $\alpha^s < 0$ which is larger than the (real) leading stable eigenvalue, there is a C such that $\|x(t)\| < Ce^{\alpha^s t}$; see [39].

The vector field allows a representation $f(x,\lambda) = D_1 f(0,\lambda)x + g(x,\lambda)$, where $g(0,\lambda) = D_1 g(0,\lambda) = 0$. Then $x(\cdot)$ is a trajectory of $\dot{x} = f(x,\lambda)$ if and only if it solves

$$x(t) = e^{D_1 f(0,\lambda)(t-s)} x(s) + \int_s^t e^{D_1 f(0,\lambda)(t-\tau)} g(x(\tau),\lambda) d\tau. \tag{3.17}$$

Therefore, because $P_s(\lambda)$ and $e^{D_1 f(0,\lambda)}$ commute,

$$e^{-D_1 f(0,\lambda)t} P_s(\lambda)x(t) = e^{-D_1 f(0,\lambda)s} P_s(\lambda)x(s) + \int_s^t e^{-D_1 f(0,\lambda)\tau} P_s(\lambda) g(x(\tau),\lambda) d\tau.$$

Hence there is a $K > 0$ such that for sufficiently small $\|x(s)\|$,

$$\begin{aligned} &\|e^{-D_1 f(0,\lambda)t_1} P_s(\lambda)x(t_1) - e^{-D_1 f(0,\lambda)t_2} P_s(\lambda)x(t_2)\| \\ &\qquad\qquad \leq \textstyle\int_{t_2}^{t_1} \|e^{-D_1 f(0,\lambda)\tau} P_s(\lambda)\| \, \|g(x(\tau),\lambda)\| d\tau. \end{aligned}$$

The right-hand side of the last inequality can be estimated by

$$\int_{t_2}^{t_1} \|e^{-D_1 f(0,\lambda)\tau} P_s(\lambda)\| \, \|g(\tau,\lambda)\| d\tau \leq \int_{t_2}^{t_1} K e^{-\mu^s(\lambda)\tau} e^{2\alpha^s \tau}.$$

This shows that $\lim_{t\to\infty} e^{-D_1 f(0,\lambda)t} P_s(\lambda)x(t)$ indeed exists.

Next we turn towards the estimate (3.16). For that we write (3.17) as a system

$$\begin{aligned} P_s(\lambda)x(t) &= e^{D_1 f(0,\lambda)(t-s)} P_s(\lambda)x(s) + \int_s^t e^{D_1 f(0,\lambda)(t-\tau)} P_s(\lambda) g(x(\tau),\lambda) d\tau, \\ (id - P_s(\lambda))x(t) &= e^{D_1 f(0,\lambda)(t-s)} (id - P_s(\lambda))x(s) \\ &\quad + \int_s^t e^{D_1 f(0,\lambda)(t-\tau)} (id - P_s(\lambda)) g(x(\tau),\lambda) d\tau. \end{aligned}$$

In the first equation the limit $s \to \infty$ does exist (see the first part of the proof), and we get

$$P_s(\lambda)x(t) = e^{D_1 f(0,\lambda)t} \eta(x(0),\lambda) - \int_t^\infty e^{D_1 f(0,\lambda)(t-\tau)} P_s(\lambda) g(x(\tau),\lambda) d\tau.$$

Therefore we find

$$\begin{aligned} \|x(t) - e^{D_1 f(0,\lambda)t} \eta(x(0),\lambda)\| &\leq \|e^{D_1 f(0,\lambda)(t-s)} (id - P_s(\lambda))x(s)\| \\ &\quad + \| \int_t^\infty e^{D_1 f(0,\lambda)(t-\tau)} P_s(\lambda) g(x(\tau),\lambda) d\tau\| \\ &\quad + \| \int_s^t e^{D_1 f(0,\lambda)(t-\tau)} (id - P_s(\lambda)) g(x(\tau),\lambda) d\tau\|. \end{aligned}$$

The single terms on the right-hand side of this inequality can be estimated as follows:

$$\|e^{D_1 f(0,\lambda)(t-s)} (id - P_s(\lambda))x(s)\| \leq c_1 e^{\alpha^{ss}(t-s)} \|x(s)\| \leq c_2 e^{\alpha^{ss} t},$$

$$\| \textstyle\int_t^\infty e^{D_1 f(0,\lambda)(t-\tau)} P_s(\lambda) g(x(\tau),\lambda) d\tau\| \leq \int_t^\infty e^{\alpha^s (t-\tau)} \|x(\tau)\|^2 \leq c_3 e^{2\alpha^s t},$$

$$\| \textstyle\int_s^t e^{D_1 f(0,\lambda)(t-\tau)} (id - P_s(\lambda)) g(x(\tau),\lambda) d\tau\| \leq \int_s^t e^{\alpha^{ss}(t-\tau)} \|x(\tau)\|^2 \leq c_4 (e^{\alpha^{ss} t} + e^{2\alpha^s t}).$$

This finally gives the desired estimate (3.16). □

The lemma (although it is only formulated for asymptotically stable equilibria) shows that any trajectory x in the stable manifold of an equilibrium has an asymptotic expansion

$$x(t) = e^{D_1 f(0,\lambda)t}\eta(x(0),\lambda) + \mathcal{O}(e^{\max\{\alpha^{ss},2\alpha^s\}t}).$$

Moreover, $\eta(x(0),\lambda) = 0$ if and only if $x(0)$ belongs to the strong stable manifold.

A corresponding assertion for linear nonautonomous differential equations is contained in the next lemma. The proof runs along the same lines as the proof of the previous lemma.

Lemma 3.4. *Consider a smooth family of linear (nonautonomous) differential equations* $\dot{x} = (A(\lambda) + B(t,\lambda))x$*, and assume that*

(i) $spec(A(\lambda)) = spec^{ss}(\lambda) \cup \{\mu^s(\lambda)\}$*, where* $\operatorname{Re}\mu < \alpha^{ss} < \mu^s(\lambda) < \alpha^s < 0$ *for all* $\mu \in spec^{ss}(\lambda)$*.*

(ii) The leading eigenvalue $\mu^s(\lambda)$ *is for all* λ *semisimple.*

(iii) There is a $\beta < 0$ *such that* $\|B(t,\lambda)\| < e^{\beta t}$ *and* $\alpha^s + \beta < \mu^s$*.*

Let $E^s(\lambda)$ *and* $E^{ss}(\lambda)$ *be the eigenspaces of* $A(\lambda)$ *assigned to* $\mu^s(\lambda)$ *and* $spec^{ss}(\lambda)$*, respectively, and let* $P_s(\lambda)$ *be the projection on* $E^s(\lambda)$ *along* $E^{ss}(\lambda)$*. Then there exists the limit* $\eta(x(0),\lambda) = \lim_{t\to\infty} e^{-A(\lambda)t}P_s(\lambda)x(t) \in E^s(\lambda)$*. Furthermore, there is a constant* c *such that*

$$\|x(t) - e^{A(\lambda)t}\eta(x(0),\lambda)\| \le ce^{\max\{\alpha^{ss},(\alpha^s+\beta)\}t}. \tag{3.18}$$

If $\dot{x} = (A(\lambda) + B(t,\lambda))x$ has an exponential dichotomy on $\mathbb{R}^+$, then one can speak of stable and strong stable subspaces (which, of course, will depend on t) of this differential equation. Then, similar to our above comment, the lemma tells us that trajectories starting in the stable subspace (at $t = 0$) can be written as

$$x(t) = e^{A(\lambda)t}\eta(x(0),\lambda) + \mathcal{O}(e^{\max\{\alpha^{ss},(\alpha^s+\beta)\}t}),$$

and $\eta(x(0),\lambda) = 0$ if and only if $\eta(x(0),\lambda)$ belongs to the strong stable subspace (at $t = 0$).

We conclude the section with two lemmas yielding asymptotic expansions for reference trajectories, resulting from the above material.

Lemma 3.5. *There are constants* $\delta > 0$*,* $c > 0$*, so that the following holds. There are vectors* $\eta_i^s \in E^s$*,* $\eta_i^u \in E^u$ *depending smoothly on* λ *with*

$$\|\gamma_i^+(\lambda)(t) - e^{D_1 f(\omega(\gamma_i),\lambda)t}\eta_i^s(\lambda)\| \le ce^{(\alpha^s-\delta)t},$$

for $t \ge 0$ *and*

$$\|\gamma_i^-(\lambda)(t) - e^{D_1 f(\alpha(\gamma_i),\lambda)t}\eta_i^u(\lambda)\| \le ce^{(\alpha^u+\delta)t},$$

for $t \le 0$*.*

Lemma 3.6. *There are constants* $\delta > 0$*,* $c > 0$*, so that the following holds. There are vectors* $\eta_i^-(\lambda) \in E^s$*,* $\eta_i^+(\lambda) \in E^u$ *depending smoothly on* λ *with*

$$\|\psi_i^-(t) - e^{D_1 f(\alpha(\gamma_i),\lambda)^* t}\eta_i^-(\lambda)\| \le ce^{(-\alpha^s+\delta)t},$$

for $t \le 0$ *and*

$$\|\psi_i^+(t) - e^{D_1 f(\omega(\gamma_i),\lambda)^* t}\eta_i^+(\lambda)\| \le ce^{(-\alpha^u-\delta)t},$$

for $t \ge 0$*.*

3.3. **Bifurcation equations.** In this section asymptotic expansions for the bifurcation equations are given. Although we sketch steps in the construction, for the derivation of the estimates we refer to [34, 25]. From the decomposition (3.3), the jumps Ξ_i have the form

$$\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = \xi^{\infty}_{\kappa_i}(\lambda) + \xi_i(\boldsymbol{\omega}, \lambda, \kappa), \tag{3.19}$$

where

$$\xi^{\infty}_{\kappa_i}(\lambda) = \gamma^{+}_{\kappa_i}(\lambda)(0) - \gamma^{-}_{\kappa_i}(\lambda)(0), \tag{3.20}$$

and

$$\xi_i(\boldsymbol{\omega}, \lambda, \kappa) = v^{+}_i(\boldsymbol{\omega}, \lambda, \kappa)(0) - v^{-}_i(\boldsymbol{\omega}, \lambda, \kappa)(0). \tag{3.21}$$

Proposition 3.7. *There is a smooth reparameterization of the parameter λ so that for fixed $\kappa \in \Sigma_k$ the bifurcation equation $\Xi_i = 0$ is equivalent to*

$$\langle \Xi_i(\boldsymbol{\omega}, \lambda, \kappa), \psi_{\kappa_i}\rangle = \lambda - e^{2\mu^s(\lambda)\omega_i}\langle \eta^s_{\kappa_{i-1}}(\lambda), \eta^-_{\kappa_i}(\lambda)\rangle + R_i(\boldsymbol{\omega}, \lambda, \kappa) = 0, \quad i \in \mathbb{Z}, \tag{3.22}$$

with $R_i(\boldsymbol{\omega}, \lambda, \kappa) = O(e^{2\mu^s(\lambda)\omega_{i+1}\delta}) + O(e^{2\mu^s(\lambda)\omega_i\delta})$ for some $\delta > 1$.

Proof. We sketch steps in the construction of the bifurcation equations. A precise version of the following reasoning (see [34, 25]), yields the asymptotic expansions for the bifurcation equations.

First consider $\xi^{\infty}_{\kappa_i}(\lambda)$. Our symmetry assumption implies that the vectors $\xi^{\infty}_j(\cdot)$, $j \in \{1, \dots, k\}$, are g-images of each other. Because $\langle \cdot, \cdot \rangle$ is G-invariant, we may identify

$$\xi^{\infty}(\cdot) = \langle \xi^{\infty}_j(\cdot), \psi_j\rangle, \quad j \in \{1, \dots, k\}.$$

Of course $\xi^{\infty}(0) = 0$, because for $\lambda = 0$ the unstable manifold $W^u(p)$ intersects the stable manifold $W^s(hp)$ along γ. By (2.2) in Hypothesis (H 7), $D\xi^{\infty}(0) \neq 0$. We may therefore assume

$$\xi^{\infty}(\lambda) = \lambda. \tag{3.23}$$

Next we turn to $\xi_i(\boldsymbol{\omega}, \lambda, \kappa) = \langle \psi_{\kappa_i}, \xi_i(\boldsymbol{\omega}, \lambda, \kappa)\rangle \psi_{\kappa_i}$. Write

$$\xi_i(\boldsymbol{\omega}, \lambda, \kappa) = \Big(\langle \psi_{\kappa_i}, P^{+}_{\kappa_i}(\lambda, 0) v^{+}_i(\boldsymbol{\omega}, \lambda)(0)\rangle - \langle \psi_{\kappa_i}, P^{-}_{\kappa_i}(\lambda, 0) v^{-}_i(\boldsymbol{\omega}, \lambda)(0)\rangle\Big)\psi_{\kappa_i}.$$

As in [34] or [25] one can show that the leading order terms (as $\boldsymbol{\omega} \to \infty$) of $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$ are contained in the expression

$$\Big(T^1_{\kappa_i}(\omega_{i+1}, \lambda) - T^2_{\kappa_i}(\omega_i, \lambda)\Big)\psi_{\kappa_i},$$

where

$$T^1_{\kappa_i}(t, \lambda) = \langle \Psi^{+}_{\kappa_i}(\lambda, t, 0) {P^{+}_{\kappa_i}}^{*}(\lambda, 0)\psi_{\kappa_i}, Q_{\kappa_{i+1}}(\lambda, t)\gamma^{-}_{\kappa_{i+1}}(\lambda)(-t)\rangle$$

and

$$T^2_{\kappa_i}(t, \lambda) = \langle \Psi^{-}_{\kappa_i}(\lambda, -t, 0) {P^{-}_{\kappa_i}}^{*}(\lambda, 0)\psi_{\kappa_i}, (id - Q_{\kappa_i}(\lambda, t))\gamma^{+}_{\kappa_{i-1}}(\lambda)(t)\rangle.$$

Here $\Psi^{\pm}_{\kappa_i}(\lambda, \cdot, \cdot)$ is the transition matrix of $\dot v(t) = -(D_1 f(\gamma^{\pm}_{\kappa_i}(\lambda)(t), \lambda))^* v(t)$, P^* stands for the adjoint of the projection P, and $Q_{\kappa_i}(\lambda, t)$ is the projection with

$$\operatorname{im} Q_{\kappa_i}(\lambda, t) = \operatorname{im} P^{+}_{\kappa_{i-1}}(\lambda, t), \qquad \ker Q_{\kappa_i}(\lambda, t) = \operatorname{im} P^{-}_{\kappa_i}(\lambda, -t).$$

Recall from (3.5) that

$$\operatorname{im} \lim_{t\to\infty} Q_{\kappa_i}(\lambda, t) = T_{p_{\kappa_i}} W^u(p_{\kappa_i}), \qquad \ker \lim_{t\to-\infty} Q_{\kappa_i}(\lambda, t) = T_{p_{\kappa_i}} W^s(p_{\kappa_i}). \tag{3.24}$$

Let us consider $T^2_{\kappa_i}(t,\lambda)$ somewhat closer. By Lemma 3.5, $\gamma^+_{\kappa_{i-1}}(\lambda)(t)$ behaves asymptotically as $t \to \infty$ like $e^{D_1 f(p_{\kappa_i},\lambda)t}\eta^s_{\kappa_{i-1}}(\lambda)$. From (3.24) we see that this is also true for $(id - Q_{\kappa_i}(\lambda,t))\gamma^+_{\kappa_{i-1}}(\lambda)(t)$. Similarly $\Psi^-_{\kappa_i}(\lambda,-t,0)P^{-*}_{\kappa_i}(\lambda,0)\psi_{\kappa_i}$ behaves asymptotically as $t \to \infty$ like $e^{(D_1 f(p_{\kappa_i},\lambda))^T t}\eta^-_{\kappa_i}(\lambda)$, by Lemma 3.6 and

$$\operatorname{im}\lim_{t\to\infty} \Psi^-_{\kappa_i}(\lambda,-t,0)P^{-*}_{\kappa_i}(\lambda,0) = T_{p_{\kappa_i}}W^u(p_{\kappa_i})^\perp.$$

So the leading term $L^2_{\kappa_i}(t,\lambda)$ of $T^2_{\kappa_i}(t,\lambda)$ is

$$\begin{aligned} L^2_{\kappa_i}(t,\lambda) &= \langle e^{(D_1 f(p_{\kappa_i},\lambda))^T t}\eta^-_{\kappa_i}(\lambda), e^{D_1 f(p_{\kappa_i},\lambda)t}\eta^s_{\kappa_{i-1}}(\lambda)\rangle \\ &= \langle \eta^-_{\kappa_i}(\lambda), e^{D_1 f(p_{\kappa_i},\lambda)2t}\eta^s_{\kappa_{i-1}}(\lambda)\rangle. \end{aligned}$$

Since $\mu^s(\lambda)$ is semisimple,

$$L^2_{\kappa_i}(t,\lambda) = e^{2\mu^s(\lambda)t}\langle \eta^-_{\kappa_i}(\lambda), \eta^s_{\kappa_{i-1}}(\lambda)\rangle. \tag{3.25}$$

In the same way we find for the leading term $L^1_{\kappa_i}$ of $T^1_{\kappa_i}(t,\lambda)$,

$$L^1_{\kappa_i}(t,\lambda) = e^{-2\mu^u(\lambda)t}\langle \eta^+_{\kappa_i}(\lambda), \eta^u_{\kappa_{i+1}}(\lambda)\rangle. \tag{3.26}$$

Summarizing, we get

$$\begin{aligned} \langle \Xi_i, \psi_{\kappa_i}\rangle &= \lambda + L^1_{\kappa_i}(\omega_{i+1},\lambda) - L^2_{\kappa_i}(\omega_i,\lambda) + \tilde R_i(\boldsymbol{\omega},\lambda) \\ &= \lambda + e^{-2\mu^u(\lambda)\omega_{i+1}}\langle \eta^+_{\kappa_i}(\lambda), \eta^u_{\kappa_{i+1}}(\lambda)\rangle - e^{2\mu^s(\lambda)\omega_i}\langle \eta^-_{\kappa_i}(\lambda), \eta^s_{\kappa_{i-1}}(\lambda)\rangle + \tilde R_i. \end{aligned}$$

Estimates for the higher order terms $\tilde R_i$ are derived in [34, 25]:

$$\tilde R_i = \tilde R_i(\boldsymbol{\omega},\lambda,\kappa) = O(e^{-2\mu^u(\lambda)\omega_{i+1}\delta}) + O(e^{2\mu^s(\lambda)\omega_i\delta}),$$

for some $\delta > 1$. Taking the eigenvalue condition in Hypothesis (H 3) into account proves the result. □

Finally, we mention that similar estimates also hold for the derivatives of ξ_i. For that, consider $\langle \xi_i, \psi_{\kappa_i}\rangle$ as a mapping $l^\infty \times \mathbb{R} \times \Sigma_k \to \mathbb{R}$. The following assertion can be taken from [34] or [25].

Lemma 3.8. *For fixed κ the mapping $\xi_i(\cdot,\cdot,\kappa)$ is smooth, and for $j \in \{1,2\}$ one has for some $\delta > 1$,*

$$\begin{aligned} D_j\langle \xi_i(\boldsymbol{\omega},\lambda,\kappa), \psi_{\kappa_i}\rangle = D_j &\left(e^{2\mu^s(\lambda)\omega_i}\langle \eta^s_{\kappa_{i-1}}(\lambda), \eta^-_{\kappa_i}(\lambda)\rangle\right) \\ &+ O(e^{2\mu^s(\lambda)\omega_{i+1}\delta}) + O(e^{2\mu^s(\lambda)\omega_i\delta}). \end{aligned}$$

4. Proof of the general bifurcation result

The proof of Theorem 2.4 will be given in this section, relying on expansions for the bifurcation equation from Proposition 3.7 and Lemma 3.8. The statement on the existence of a topological conjugacy between a first return map restricted to an invariant set and a topological Markov chain is proved in Section 4.2.

The matrix $M = (m_{ij})_{i,j\in\{1,\dots,k\}}$ is given, as in the statement of Theorem 2.4, by

$$m_{ij} = \operatorname{sgn}\langle \eta^s_i(0), \eta^-_j(0)\rangle \tag{4.1}$$

for $\omega(\gamma_i) = \alpha(\gamma_j)$. Due to (H 4), (H 5) and (2.1), both $\eta^s_i(0)$ and $\eta^-_j(0)$ are different from zero. Therefore, m_{ij} depend only on the relative position of the vectors $\eta^s_i(0)$

and $\eta_j^-(0)$. Note that, for $\kappa \in \Sigma_{A_{+/-}}$ and $\boldsymbol{\omega}$ large, a necessary condition to fullfill (3.22) is

$$\operatorname{sgn} \lambda = \operatorname{sgn} \langle \eta^s_{\kappa_{i-1}}(\lambda), \eta^-_{\kappa_i}(\lambda) \rangle.$$

This gives rise to the following definition:

Definition 4.1. For $\lambda > 0/\lambda < 0$ a sequence $\kappa \in \Sigma_k$ is *admissible* if $\kappa \in \Sigma_{A_{+/-}}$. A heteroclinic chain γ^κ is *admissible* if κ is admissible.

In Section 4.1 we will show that for $\lambda \neq 0$, the bifurcation equation (3.22) can be uniquely solved for $\boldsymbol{\omega} = \boldsymbol{\omega}(\lambda, \kappa)$ for all admissible κ. Given the solution $\boldsymbol{\omega}(\lambda, \kappa)$, let $x(\boldsymbol{\omega}(\lambda, \kappa), \lambda, \kappa)(\cdot)$ be the corresponding trajectory of (1.1). Then

$$x(\boldsymbol{\omega}(\lambda, \kappa), \lambda, \kappa)(\tau_n) \in S_{\kappa_n}, \quad \text{for} \quad \tau_n = \begin{cases} \sum_{i=1}^n 2\omega_i(\lambda, \kappa), & n \in \mathbb{N}, \\ 0, & n = 0, \\ \sum_{i=n+1}^0 -2\omega_i(\lambda, \kappa), & n \in -\mathbb{N}. \end{cases} \tag{4.2}$$

Define the set D_λ in the union of cross sections $\bigcup_{1 \le i \le k} S_i$ by

$$D_\lambda = \{x(\boldsymbol{\omega}(\lambda, \kappa), \lambda, \kappa)(\tau_n) \mid n \in \mathbb{Z},\ \kappa \text{ admissible}\}.$$

Uniqueness of $\boldsymbol{\omega}$ implies

$$\boldsymbol{\omega}(\lambda, \sigma\kappa) = \hat{\sigma}\boldsymbol{\omega}(\lambda, \kappa),$$

where $\hat{\sigma}$ is the left shift on the set of $\boldsymbol{\omega}$-sequences which is defined in the same way as σ: $(\hat{\sigma}\boldsymbol{\omega})_i = \omega_{i+1}$. Note that the set $\Sigma_{A_{+/-}}$ of admissible sequences κ is σ-invariant. Hence

$$D_\lambda = \{x(\boldsymbol{\omega}(\lambda, \kappa), \lambda, \kappa)(0) \mid \kappa \text{ admissible}\}.$$

The first return map Π_λ on $\bigcup_{1 \le i \le k} S_i$ leaves D_λ invariant and

$$\Pi_\lambda : D_\lambda \to D_\lambda, \quad x(\boldsymbol{\omega}(\lambda, \kappa), \lambda, \kappa)(0) \mapsto x(\boldsymbol{\omega}(\lambda, \kappa), \lambda, \kappa)(\tau_1) = x(\boldsymbol{\omega}(\lambda, \sigma\kappa), \lambda, \sigma\kappa)(0).$$

The map

$$\Phi_\lambda : \Sigma_{A_{+/-}} \to D_\lambda, \quad \kappa \mapsto x(\boldsymbol{\omega}(\lambda, \kappa), \lambda, \kappa)(0) \tag{4.3}$$

is one-to-one. Therefore

$$\Pi_\lambda \circ \Phi_\lambda = \Phi_\lambda \circ \sigma,$$

that is, the systems (D_λ, Π_λ) and $(\Sigma_{A_{+/-}}, \sigma)$ are conjugated. For topological conjugacy the continuity of Φ_λ and Φ_λ^{-1} must also be established.

In the following two subsections we first solve the bifurcations equations, thus showing that Φ_λ is well defined, and we then prove that Φ_λ is a homeomorphism, to conclude that Φ_λ defines the topological conjugacy of Π_λ on D_λ with the subshift.

4.1. **Solving the bifurcation equations.** In order to prove Theorem 2.4 it remains to demonstrate the solvability of the bifurcation equation (3.22) and to prove topological conjugacy. Here we solve bifurcation equation (3.22). Without loss of generality we take $\lambda > 0$. The following hypothesis ensures that κ is admissible:

(H 8) $\operatorname{sgn} \langle \eta^s_{\kappa_{i-1}}(0), \eta^-_{\kappa_i}(0) \rangle = 1.$

Rewrite (3.22) for fixed $\kappa \in \Sigma_{A_+}$ by introducing the notation

$$\hat{r}_i = e^{2\mu^s(0)\omega_i}, \hat{\mathbf{r}} = (\hat{r}_i)_{i \in \mathbb{Z}}$$

into

$$\lambda - \hat{r}_i \langle \eta^-_{\kappa_i}(\lambda), \eta^s_{\kappa_{i-1}}(\lambda) \rangle + \hat{R}_i(\hat{\mathbf{r}}, \lambda, \kappa) = 0, \quad i \in \mathbb{Z}. \tag{4.4}$$

Our goal is to solve this equation near $(\lambda, \hat{r}_i) = (0,0)$ for $\hat{r}_i = \hat{r}_i(\lambda,\kappa)$. Note that only $\hat{r}_i \geq 0$ makes sense. To avoid a discussion of possible extensions of $\hat{R}_i$ to $\hat{r}_j < 0$, we introduce the rescaling

$$\lambda r_i = \hat{r}_i. \tag{4.5}$$

For convenience we write $r = \lambda$. Then, using $\mathbf{r} = (r_i)_{i\in\mathbb{Z}}$, the bifurcation equations read as

$$r - r r_i \langle \eta^-_{\kappa_i}(r), \eta^s_{\kappa_{i-1}}(r)\rangle + \hat{R}_i(r\mathbf{r}, r, \kappa) = 0. \tag{4.6}$$

Note that $\hat{R}_i(r\mathbf{r}, r, \kappa) = O(r^\delta)$, $\delta > 1$. Factoring out r yields

$$1 - r_i \langle \eta^-_{\kappa_i}(r), \eta^s_{\kappa_{i-1}}(r)\rangle + O(r^\theta) = 0, \tag{4.7}$$

with some positive θ. By (H 8) this equation can be solved for $\mathbf{r} = \mathbf{r}(r,\kappa)$ near $(r, r_i) = (0, \langle \eta^-_{\kappa_i}(0), \eta^s_{\kappa_{i-1}}(0)\rangle^{-1})$; see also Remark 4.2 below. Note that $r_i(r,\kappa) > 0$ (because $\langle \eta^-_{\kappa_i}(0), \eta^s_{\kappa_{i-1}}(0)\rangle > 0$) and hence for $r > 0$ also $r r_i > 0$. Finally we find the following expression for ω_i in terms of λ and κ:

$$\omega_i = \omega_i(\lambda,\kappa) = \frac{1}{2\mu^s(0)}\big(\ln(\lambda) + \ln r_i(\lambda,\kappa)\big). \tag{4.8}$$

Remark 4.2. In order to solve (4.7) we use the implicit function theorem. For that we consider the left-hand side of (4.7) as an operator

$$\mathcal{X} : l^\infty \times \mathbb{R} \times \Sigma_A \to l^\infty, \quad (\mathbf{r}, r, \kappa) \mapsto \mathcal{X}(\mathbf{r}, r, \kappa).$$

By construction $\mathcal{X}(\cdot,\cdot,\kappa)$ is smooth for $r > 0$ and $r_i > 0$, $i \in \mathbb{Z}$. Furthermore, there exists a differentiable extension to $r \leq 0$ (as long as the r_i stay away from zero – recall that we solve (4.7) near $(r, r_i) = (0, \langle \eta^-_{\kappa_i}(0), \eta^s_{\kappa_{i-1}}(0)\rangle^{-1}) \neq (0,0)$). Note further that due to Lemma 3.8 the partial derivative with respect to $\mathbf{r}$ of the O-term in (4.7) can be made arbitrarily small by letting r tend to zero.

Remark 4.3. For any $\lambda > 0$, $\kappa \in \Sigma_{A_+}$ there is a unique $\mathbf{r}$ satisfying (4.4). Assume namely that there are two sequences $\mathbf{r}^1$ and $\mathbf{r}^2$ satisfying this equation. Then

$$(\hat{r}^1_i - \hat{r}^2_i)\langle \eta^-_{\kappa_i}(\lambda), \eta^s_{\kappa_{i-1}}(\lambda)\rangle + \hat{R}_i(\hat{\mathbf{r}}^1, \lambda, \kappa) - \hat{R}_i(\hat{\mathbf{r}}^2, \lambda, \kappa) = 0, \quad i \in \mathbb{Z}.$$

Because the derivative of $\hat{R}_i$ with respect to $\hat{\mathbf{r}}$ becomes arbitrarily small, this equation is only fulfilled for $\hat{r}^1_i = \hat{r}^2_i$.

Summarizing: for each $\lambda > 0$ and each $\kappa \in \Sigma_{\mathscr{M}_+}$ there is a unique $\boldsymbol{\omega} = \boldsymbol{\omega}(\lambda,\kappa)$ such that

$$\Xi_i(\boldsymbol{\omega}(\lambda,\kappa), \lambda, \kappa) = 0, \quad i \in \mathbb{Z}.$$

4.2. **Topological conjugation.** Next we prove the topological conjugacy claimed in Theorem 2.4. Let $\mathbf{v} = \mathbf{v}(\boldsymbol{\omega}, \lambda, \kappa)$ be the unique solution of (3.15). Some of our estimates are based upon the asymptotic behavior of variational equations along the trajectories $\gamma^\pm_j$. For that we assume that $\inf_{i\in\mathbb{Z}} \omega_i$ is sufficiently large.

We start with an arithmetical lemma.

Lemma 4.4. *Let $(a^\pm_i)_{i\in\mathbb{Z}}$ and $(d_i)_{i\in\mathbb{Z}}$ be sequences of positive numbers such that for all $j \in \mathbb{Z}$, $a^-_j + a^+_j \leq \frac{1}{4}(a^-_{j-1} + a^+_{j+1})$. Then, for any $i \in \mathbb{N}$,*

$$a^-_j + a^+_j \leq \frac{1}{2^{i+1}}(a^-_{j-i-1} + a^+_{j+i+1}).$$

Proof. An induction argument shows $(a^-_{-i} + a^+_i) + (a^+_{-i} + a^-_i) \leq \frac{1}{2}(a^-_{-i-1} + a^+_{i+1})$. In particular,

$$a^-_{-i} + a^+_i \leq \frac{1}{2}(a^-_{-i-1} + a^+_{i+1}). \tag{4.9}$$

Follow the next procedure. Consider (4.9) for $i = 0$. Then replace $a^-_{-1} + a^+_1$ by the estimate given by (4.9). Continue replacing in each case the expression $a^-_{-j} + a^+_j$. This procedure finally gives

$$a^-_0 + a^+_0 \leq \frac{1}{2^{i+1}}(a^-_{-i-1} + a^+_{i+1}). \tag{4.10}$$

The same procedure yields a similar estimate for $a^-_j + a^+_j$. □

Lemma 4.5. *The mapping $\Sigma_A \to \mathbb{R}^n \times \mathbb{R}^n$, $\kappa \mapsto v_0(\boldsymbol{\omega}, \lambda, \kappa)(0)$ is continuous.*

Proof. Let $\kappa^1, \kappa^2 \in \Sigma_A$ be two sequences which coincide on a block of length $2N+1$ centered at $i = 0$:

$$\kappa^1_i = \kappa^2_i, \quad i \in [-N, N] \cap \mathbb{Z}.$$

We will establish that $\|v_0(\boldsymbol{\omega}, \lambda, \kappa^1)(0) - v_0(\boldsymbol{\omega}, \lambda, \kappa^2)(0)\| = O(1/2^N)$, implying the lemma.

By (3.15) we get

$$\|v_0(\boldsymbol{\omega}, \lambda, \kappa^1)(0) - v_0(\boldsymbol{\omega}, \lambda, \kappa^2)(0)\| =$$
$$\|\hat{v}_0(\boldsymbol{\omega}, \lambda, \mathcal{H}(\mathbf{v}, \lambda, \kappa^1), \mathbf{d}(\boldsymbol{\omega}, \lambda, \kappa^1), \kappa^1)(0) - \hat{v}_0(\boldsymbol{\omega}, \lambda, \mathcal{H}(\mathbf{v}, \lambda, \kappa^2), \mathbf{d}(\boldsymbol{\omega}, \lambda, \kappa^2), \kappa^2)(0)\|.$$

If $N \geq 1$, then we have, due to (3.11), (3.12) and (3.14), that

$$\hat{v}_0(\boldsymbol{\omega}, \lambda, \mathcal{H}(\mathbf{v}, \lambda, \kappa^2), \mathbf{d}(\boldsymbol{\omega}, \lambda, \kappa^2), \kappa^2) = \hat{v}_0(\boldsymbol{\omega}, \lambda, \mathcal{H}(\mathbf{v}, \lambda, \kappa^2), \mathbf{d}(\boldsymbol{\omega}, \lambda, \kappa^2), \kappa^1).$$

By linearity of $\hat{\mathbf{v}}(\omega, \lambda, \cdot, \cdot, \kappa)$, recall Remark 3.2,

$$\|v_0(\boldsymbol{\omega}, \lambda, \kappa^1)(0) - v_0(\boldsymbol{\omega}, \lambda, \kappa^2)(0)\| = \|\hat{v}_0(\boldsymbol{\omega}, \lambda, \Delta\mathcal{H}, \Delta\mathbf{d}, \kappa^1)(0)\|,$$

where $\Delta\mathcal{H} = \mathcal{H}(\mathbf{v}, \lambda, \kappa^1) - \mathcal{H}(\mathbf{v}, \lambda, \kappa^2)$ and $\Delta\mathbf{d} = \mathbf{d}(\boldsymbol{\omega}, \lambda, \kappa^1) - \mathbf{d}(\boldsymbol{\omega}, \lambda, \kappa^2)$. Note that $\triangle d_i = 0$ for $i \in [-N+1, N] \cap \mathbb{Z}$; compare (3.7).

Invoking (3.12) again we find

$$\hat{v}_0(\boldsymbol{\omega}, \lambda, \Delta\mathcal{H}, \Delta\mathbf{d}, \kappa^1)(0) = \bar{v}_0(\boldsymbol{\omega}, \lambda, \Delta\mathcal{H}, \mathbf{a}(\boldsymbol{\omega}, \lambda, \Delta\mathcal{H}, \Delta\mathbf{d}, \kappa^1), \kappa^1)(0).$$

Henceforth we will use the shorthand notation $\Delta\mathbf{a} = \mathbf{a}(\boldsymbol{\omega}, \lambda, \Delta\mathcal{H}, \Delta\mathbf{d}, \kappa^1)$.

Recall that $\bar{v}^\pm_0(\boldsymbol{\omega}, \lambda, \mathbf{g}, \mathbf{a}, \kappa)(\cdot)$ solves

$$\dot{v}(t) = D_1 f(\gamma^\pm_{\kappa_0}(\lambda)(t), \lambda)v(t) + g^\pm_0(t) \tag{4.11}$$

and boundary conditions (3.8) with $i = 0$ and

$$a^+_1 = (id - P^+_{\kappa_0}(\lambda, \omega_1))v^+_0(\omega_1), \quad a^-_0 = (id - P^-_{\kappa_0}(\lambda, \omega_0))v^-_0(-\omega_0).$$

Exploiting the asymptotic behavior of (4.11) we find, similar to the estimates used in the proof of [24, Lemma 5.1],

$$\|v_0(\boldsymbol{\omega}, \lambda, \kappa^1)(0) - v_0(\boldsymbol{\omega}, \lambda, \kappa^2)(0)\| \leq \|\triangle a^+_1\| + \|\triangle a^-_0\|.$$

Following the procedure of [24] we get for $i \in [-N+1, N-1]$,

$$\|\triangle a^-_i\| + \|\triangle a^+_i\| \leq 1/4(\|\triangle a^-_{i-1}\| + \|\triangle a^+_{i+1}\|).$$

Now the statement follows from Lemma 4.4 because $\Delta\mathbf{a}$ is bounded. □

The sequences $\boldsymbol{\omega}$ are elements of l^∞. On l^∞ we introduce the metric ϱ which induces the product topology by

$$\varrho(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} |\omega_i^1 - \omega_i^2|.$$

Denote by (l^∞, ϱ) the set l^∞ equipped with the metric ϱ.

Lemma 4.6. *Let $\boldsymbol{\omega}$ be the sequence of transition times which are defined by* (4.8). *Then the mapping $\Sigma_A \to (l^\infty, \varrho)$, $\kappa \mapsto \boldsymbol{\omega}(\lambda, \kappa)$ is continuous.*

Proof. Actually we prove the continuous dependence of $(r_i)_{i\in\mathbb{Z}}$ (see (4.5) and (4.8)) on κ. We confine ourselves to showing continuous dependence of r_0. More precisely, we show that for two sequences $\kappa^1, \kappa^2 \in \Sigma_A$ which coincide on a block of length $2N+1$ centered at $i = 0$, the difference $|r_0(\lambda, \kappa^1) - r_0(\lambda, \kappa^2)|$ tends to zero of order $O(1/2^N)$. The same arguments apply for r_i, $i \neq 0$.

In order to prove the assertion for r_0 we reconsider the solving process of the bifurcation equation as outlined in Section 4.1. Recall from Section 4.1 that r_0 solves the fixed point equation

$$r_0 = \frac{1}{\langle \eta^-_{\kappa_0}(r), \eta_{\kappa_{-1}}(r)\rangle} + \frac{1}{r\langle \eta^-_{\kappa_0}(r), \eta_{\kappa_{-1}}(r)\rangle} \hat{R}_0(r\mathbf{r}, r, \kappa) =: F_0(\mathbf{r}, r, \kappa).$$

Now consider

$$\begin{aligned}
&|r_0(r,\kappa^1) - r_0(r,\kappa^2)| \\
&\quad = \quad |F_0(\mathbf{r}(\kappa^1), r, \kappa^1) - F_0(\mathbf{r}(\kappa^2), r, \kappa^2)| \\
&\quad \leq \quad |F_0(\mathbf{r}(\kappa^1), r, \kappa^1) - F_0(\mathbf{r}(\kappa^2), r, \kappa^1)| + |F_0(\mathbf{r}(\kappa^2), r, \kappa^1) - F_0(\mathbf{r}(\kappa^2), r, \kappa^2)|.
\end{aligned}$$

For our further consideration we exploit that κ^1 and κ^2 coincide on a block of length $2N+1$ centered at $i = 0$. With that we find for the first term on the right-hand side of the last inequality

$$\begin{aligned}
&|F_0(\mathbf{r}(\kappa^1), r, \kappa^1) - F_0(\mathbf{r}(\kappa^2), r, \kappa^1)| \\
&\quad = \quad \theta(r,\kappa^1) |\hat{R}_0(r\mathbf{r}(\kappa^1), r, \kappa^1) - \hat{R}_0(r\mathbf{r}(\kappa^2), r, \kappa^1)| \\
&\quad \leq \quad c|r_0(r,\kappa^1) - r_0(r,\kappa^2)|,
\end{aligned}$$

for some $c < 1$. Here $\theta(r, \kappa^1) = 1/r\langle \eta^-_{\kappa^1_0}(r), \eta_{\kappa^1_{-1}}(r)\rangle$. The latter estimate is a consequence of Lemma 3.8. Hence

$$(1-c)\,|r_0(r,\kappa^1) - r_0(r,\kappa^2)| \leq |F_0(\mathbf{r}(\kappa^2), r, \kappa^1) - F_0(\mathbf{r}(\kappa^2), r, \kappa^2)|.$$

With the definitions of F_0 and $\hat{R}_0$, again using the fact that κ^1 and κ^2 coincide on a block of length $2N+1$ centered at $i = 0$, we find

$$\begin{aligned}
&(1-c)|r_0(r,\kappa^1) - r_0(r,\kappa^2)| \\
&\quad \leq \quad \theta(r,\kappa^1)|\xi_0(\omega(\kappa^2), r, \kappa^1) - \xi_0(\omega(\kappa^2), r, \kappa^2)| \\
&\quad \leq \quad 2\theta(r,\kappa^1)|v_0(\omega(\kappa^2), r, \kappa^1) - v_0(\omega(\kappa^2), r, \kappa^2)|.
\end{aligned}$$

The initially stated assertion concerning $|r_0(r,\kappa^1) - r_0(r,\kappa^2)|$ follows from the considerations in the proof of Lemma 4.5. $\square$

Lemma 4.7. *The conjugation Φ_λ introduced in* (4.3) *is a homeomorphism.*

Proof. Since $x(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)$ defined in (4.2) are trajectories, we have

$$x_0^-(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)(0) = x_0^+(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)(0).$$

Hence

$$\Phi_\lambda(\kappa) = x_0^+(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)(0) = \gamma_{\kappa_0}^+(0) + v_0^+(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)(0). \tag{4.12}$$

Now, let $\kappa^1,\kappa^2 \in \Sigma_A$ be two sequences which coincide on a block of length $2N+1$ centered at $i=0$:

$$\kappa_i^1 = \kappa_i^2, \quad i \in [-N,N]\cap\mathbb{Z}.$$

Then, because $\gamma_{\kappa_0^1}^+(0) = \gamma_{\kappa_0^2}^+(0)$,

$$\begin{aligned}\|\Phi_\lambda(\kappa^1) - \Phi_\lambda(\kappa^2)\| \;\leq\;& \|v_0^+(\boldsymbol{\omega}(\lambda,\kappa^1),\lambda,\kappa^1)(0) - v_0^+(\boldsymbol{\omega}(\lambda,\kappa^2),\lambda,\kappa^1)(0)\| \\ &+\|v_0^+(\boldsymbol{\omega}(\lambda,\kappa^2),\lambda,\kappa^1)(0) - v_0^+(\boldsymbol{\omega}(\lambda,\kappa^2),\lambda,\kappa^2)(0)\|.\end{aligned}$$

Because of Lemma 4.6 the first term on the right-hand side can be estimated by means of [30, Lemma 3.4]. On the second term we can apply Lemma 4.5. This shows that $\|\Phi_\lambda(\kappa^1) - \Phi_\lambda(\kappa^2)\|$ tends to zero as N tends to infinity.

Compactness of Σ_A implies that Φ_λ^{-1} is also continuous. □

5. Examples

In this section we consider examples of relative homoclinic cycles that satisfy the assumptions of Theorem 2.4. We provide a detailed study of bifurcations from relative homoclinic cycles with dihedral symmetry with real leading eigenvalues. We first summarize the theory from a practical point of view.

5.1. Methodology.

Following the setup in Section 1, the relative homoclinic cycle is determined by an equilibrium p with isotropy G_p and a connecting trajectory γ that is asymptotic to p as $t\to-\infty$ and to hp as $t\to\infty$, where $h\in G$ and $G=\langle h, G_p\rangle$ is the group generated by h and G_p. The case that γ is G-invariant is governed by Corollary 2.5.

The resulting relative homoclinic cycle $\Gamma = G\overline{\gamma}$ consists of $|G|/|G_p|$ equilibria and $k=|G|/|G_\gamma|$ connecting trajectories.

After deciding on a labeling of the connecting trajectories in the network, in order to find the matrix M that describes the shift dynamics, as in Theorem 2.4, we need to examine the relative position of the vectors $\eta_i^s(0)$ and $\eta_j^-(0)$ for all i and j whenever the network admits the corresponding connection (as can be found from the connectivity matrix). Because of the symmetry, we need in fact to determine one row or column, as the others follow by symmetry. Recall that e_{hp}^s is the direction inside E_{hp}^s along which γ approaches hp as $t\to\infty$. This limit is well defined by Lemma 3.3. Note that $T_pW^s(p)$, $T_pW^u(p)$, E_p^s and the strong stable subspace E_p^{ss} are all G_p-invariant. Hence, we can choose a G_p-invariant inner product $\langle\cdot,\cdot\rangle$ so that

$$T_pW^s(p)\perp T_pW^u(p) \quad\text{and}\quad E_p^s \perp E_p^{ss}. \tag{5.1}$$

From (3.16) we get the following expansion for $\gamma_i(\cdot)$:

$$\gamma_i(t) = e^{D_1 f(\omega(\gamma_i),0)t}\eta_i^s(0) + O\left(e^{\max\{\alpha^{ss},2\alpha^s\}t}\right), \quad \text{as} \quad t \to \infty. \tag{5.2}$$

We note that e_i^s and $\eta_i^s(0)$ are parallel and, moreover, that there is a positive constant k_i such that

$$e_i^s = k_i \eta_i^s(0), \quad k_i > 0. \tag{5.3}$$

Write $\eta^s(0) \in E_p^s$ so that for the reference connecting trajectory γ (5.2) reads

$$\gamma(t) = e^{D_1 f(hp,0)t}h\eta^s(0) + O\left(e^{\max\{\alpha^{ss},2\alpha^s\}t}\right), \quad \text{as} \quad t \to \infty. \tag{5.4}$$

Let $\gamma_{i(g)} = g\gamma$ and let $\gamma_{i(g)}(\cdot)$ be the corresponding solution with $\gamma_{i(g)}(0)$ in $S_g = gS$. Since $gD_1 f(p,0) = D_1 f(gp,0)g$, the expansion (5.2) yields

$$\gamma_{i(g)}(t) = e^{D_1 f(\omega(\gamma_{i(g)}),0)t}gh\eta^s(0) + O\left(e^{\max\{\alpha^{ss},2\alpha^s\}t}\right), \text{ as } t \to \infty.$$

Hence

$$\eta_{i(g)}^s(0) = gh\eta^s(0). \tag{5.5}$$

Considerations in Section 3.1 give a bounded solution ψ_j to the adjoint variational equation

$$\dot{w} = -(D_1 f(\gamma_j(t),0))^* w.$$

The vector $\eta_j^-(0)$ is obtained as the leading order term of $e^{-D_1 f(\alpha(\gamma_j),0)^* t}\psi_j(t)$ as $t \to -\infty$; $\eta_j^-(0)$ belongs to the eigenspace of the leading unstable eigenvalue of $-(D_1 f(\alpha(\gamma_j),0))^*$. Hypotheses (H 5) and (H 7) imply that

$$\eta_j^-(0) \in E_{\alpha(\gamma_j)}^s. \tag{5.6}$$

Write $\psi = \psi_{i(id)}$ and $\eta^-(0) = \eta_{i(id)}^-(0)$ (note that $\eta^-(0) \in E_p^s$). Exploiting symmetry and (3.18) we obtain

$$\eta_{i(g)}^-(0) = g\eta^-(0). \tag{5.7}$$

Finally, by the G-invariance of the inner product, it follows that

$$\langle g\eta_i^s(0), g\eta_j^-(0)\rangle = \langle \eta_i^s(0), \eta_j^-(0)\rangle. \tag{5.8}$$

This can be used to compute M given the entries in one column.

5.2. **Bifurcation with dihedral symmetry.** The dihedral group $\mathbb{D}_m$ is the symmetry group of the regular m-gon in the plane. As an abstract group, $\mathbb{D}_m$ can be written in terms of generators (a, b) and relations as

$$\langle a, b | a^m = b^2 = (ab)^2 = 1\rangle.$$

In this section we consider relative homoclinic cycles with $\mathbb{D}_m$ symmetry. We recall, for reference, that Matthies [31] discussed an example with $\mathbb{D}_3$-symmetry.

There are several different types of relative homoclinic cycles with dihedral symmetry, identifiable by the isotropy subgroup G_p of the equilibria p and its representation on leading eigenspaces and by the isotropy subgroup G_γ of the connecting trajectories γ.

Let us first consider the options for G_γ. If $G_\gamma = \mathbb{D}_k$ (with $k|m$), then, necessarily, all equilibria and connections are pointwise $\mathbb{Z}_k$-invariant. In this case the group does not act faithfully on the network as $\mathbb{D}_m$ but as $\mathbb{D}_{m/k}$. Similarly, if $G_\gamma = \mathbb{Z}_k$, with $k|m$, then all equilibria and connections are pointwise $\mathbb{Z}_k$-invariant and $\mathbb{D}_m$ acts as

$\mathbb{D}_{m/k}$. Without loss, we assume that $\mathbb{D}_m$ acts faithfully on the relative homoclinic cycle. Consequently, we are led to consider the following isotropy subgroups of the connection γ: $\{id\}$ (trivial) and $\mathbb{Z}_2$ (generated by $a^k b$, for some $k \in \{0, \ldots, m-1\}$).

We further seek combinations of isotropy G_p and some $h \in G$ so that $G_\gamma \subset G_p \cap G_{hp}$ and $G = \langle h, G_p \rangle$. If $G_p = \mathbb{D}_m$, then $h = id$. If $G_p = \mathbb{D}_k$ with $k|m$ (and $k \neq m$), then h needs to satisfy $h^2 = id$ and $k = m/2$ (and m even). If $G_p = \mathbb{Z}_k(a^{m/k})$ with $k|m$, then h needs to satisfy $h^2 = id$ and $k = m$. Finally, if $G_p = \mathbb{Z}_2(a^k b)$, then h must have order m.

We now consider the (real) irreducible representations of the dihedral group $\mathbb{D}_n$. They are all absolutely irreducible, and act in one of the following three ways:

- One-dimensional trivial representation: $\mathbb{D}_n$ acts as $\{id\}$.
- One-dimensional nontrivial representation: $\mathbb{D}_n$ acts as $\mathbb{Z}_2$. There is one of these if n is odd and three if n is even.
- Two-dimensional representations: $\mathbb{D}_n$ acts as $\mathbb{D}_k$ with $k|n$.

Finally, we consider the representations of $\mathbb{Z}_2$ and $\mathbb{Z}_n$. There are two one-dimensional representations of $\mathbb{Z}_2$: a trivial one and a nontrivial one. For $\mathbb{Z}_n$ there is always the trivial one-dimensional representation, and if m is even there is also a nontrivial one-dimensional representation isomorphic to $\mathbb{Z}_2$. Furthermore, Z_n has two-dimensional representations isomorphic to $\mathbb{Z}_k$, for all $k > 2$ a divisor of n.

A list of all the cases can be found in Tables 1, 2 and 3. We now proceed with the description of the codimension one homoclinic bifurcations in these networks (the results of which are summarized in the same tables).

5.2.1. *One-dimensional E_p^s.* If $\dim E_p^s = 1$, G_p must act on E_p^s as the trivial representation or nontrivial representation isomorphic to $\mathbb{Z}_2$. In this subsection we derive the transition matrices for all networks in Tables 1, 2 and 3 with $\dim(E_p^s) = 1$.

The tables list all possible cases. In particular, if the representation of G_p on E_p^s equals $\mathbb{Z}_2$, then G_γ cannot be $\mathbb{Z}_2$. That is, Case 10 and Case 11 are the only cases where the representation of G_p on a one-dimensional E_p^s equals $\mathbb{Z}_2$. Indeed, in this case h has order m and $G_{hp} = hG_p h^{-1}$ differs from G_p. Hence, $G_\gamma \subset G_p \cap G_{hp}$ is $\{id\}$. Also for $G_p = \mathbb{Z}_m$, $G_\gamma \subset G_p$ cannot contain $\mathbb{Z}_2(a^k b)$ and thus is $\{id\}$. That is, Case 7 and Case 8 are the only cases where the representation of G_p on a one-dimensional E_p^s equals $\mathbb{Z}_m$.

We leave a detailed discussion of Cases 12, 13, 14, 15 in Table 3, which are similar to Cases 1, 2, 3, 4 in Table 1 except for the occurrence of two equilibria in the network Γ, to the reader.

The next proposition and the text immediately below treat Cases 1, 2, 7, 10 in the tables, for which G_p acts trivially on E_p^s.

Proposition 5.1. *Assume the hypotheses of Theorem* 2.4 *and suppose that* $\dim E_p^s = 1$ *with a trivial representation for G_p on E_p^s. Then $M = C$ (with an appropriate choice of sign).*

Proof. There are $\frac{|G_p|}{|G_\gamma|}$ connections approaching p as $t \to \infty$, in the manner sketched in Figure 3. Due to (5.3) and (5.5) all homoclinic trajectories that approach p as $t \to \infty$ do so from the same direction; more precisely $\eta^s_{i(gh^{-1})}(0) = \eta^s(0)$ for all $g \in G_p$. For $g \in G \setminus G_p$, $\eta^s_{i(gh^{-1})}(0)$ is contained in a different subspace, E^s_{gp}. Notice that

$$\langle \eta^s_{i(gh^{-1})}(0), \eta^-(0) \rangle = \langle \eta^s(0), \eta^-(0) \rangle \neq 0$$

TABLE 1. Relative homoclinic cycles with $\mathbb{D}_m$ symmetry.

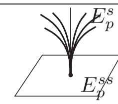
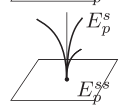
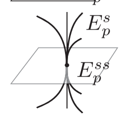
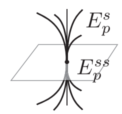
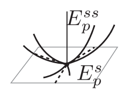
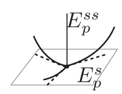
Case	G_p	$\frac{\lvert G\rvert}{\lvert G_p\rvert}$	$\dim(E_p^s)$	$\mathrm{Rep}(G_p)$ on E_p^s	Orbits approaching p	G_γ	$\frac{\lvert G\rvert}{\lvert G_\gamma\rvert}$	Shift dynamics	Notes
1	$\mathbb{D}_m$	1	1	Trivial		$\{id\}$	$2m$	M is $2m \times 2m$ with $m_{ij} = 1$ for all i, j	
2	$\mathbb{D}_m$	1	1	Trivial		$\mathbb{Z}_2$	m	M is $m \times m$ with $m_{ij} = 1$ for all i, j	
3	$\mathbb{D}_m$	1	1	Isomorphic to $\mathbb{Z}_2$		$\{id\}$	$2m$	M is $2m \times 2m$, $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	
4	$\mathbb{D}_m$	1	1	Isomorphic to $\mathbb{Z}_2$		$\mathbb{Z}_2$	m	M is $m \times m$, $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	For m even only
5	$\mathbb{D}_m$	1	2	Isomorphic to $\mathbb{D}_k$ for $k\|m$		$\{id\}$	$2m$	M is $2m \times 2m$, depends on angles of η^s and η^- to $\mathrm{Fix}\,\mathbb{Z}_2$ in E_p^s	
6	$\mathbb{D}_m$	1	2	Isomorphic to $\mathbb{D}_k$ for $k\|m$		$\mathbb{Z}_2$	m	M is $m \times m$ with entries $\frac{m}{k} \times \frac{m}{k}$ block matrices $m_{ij} = \begin{cases} 1, & \lvert i-j\rvert < m/4, \\ 0, & \lvert i-j\rvert = m/4, \\ -1, & \lvert i-j\rvert > m/4. \end{cases}$	Zeros in M when m is a multiple of 4

TABLE 2. Continued: Relative homoclinic cycles with $\mathbb{D}_m$ symmetry. The matrices are in block form, each block is of the same size, and the vertices in the graphs, one for each of $\lambda > 0$ and $\lambda < 0$, increase anticlockwise from 1 which is marked.

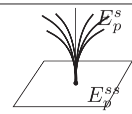
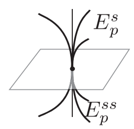
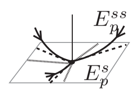
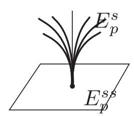
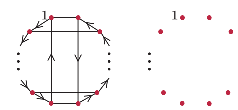
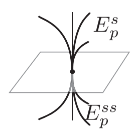
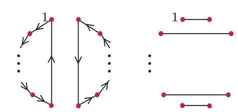
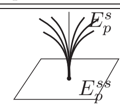
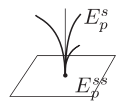
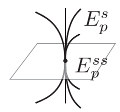
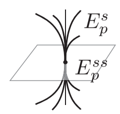
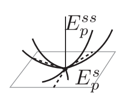
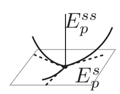
Case	G_p	$\frac{\lvert G\rvert}{\lvert G_p\rvert}$	$\dim(E_p^s)$	Rep(G_p) on E_p^s	Orbits approaching p	G_γ	$\frac{\lvert G\rvert}{\lvert G_\gamma\rvert}$	Shift dynamics	Notes
7	$\mathbb{Z}_m$	2	1	Trivial		$\{id\}$	$2m$	M is $2m \times 2m$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	
8	$\mathbb{Z}_m$	2	1	Isomorphic to $\mathbb{Z}_2$		$\{id\}$	$2m$	M is $2m \times 2m$, $\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$	For m even only
9	$\mathbb{Z}_m$	2	2	Isomorphic to $\mathbb{Z}_k$ for $k\|m$		$\{id\}$	$2m$	M is $2m \times 2m$, depends on the angles between η^s and η^-	
10	$\mathbb{Z}_2$	m	1	Trivial		$\{id\}$	$2m$	M is $2m \times 2m$, 	Labelling given by (5.9)
11	$\mathbb{Z}_2$	m	1	Isomorphic to $\mathbb{Z}_2$		$\{id\}$	$2m$	M is $2m \times 2m$, 	Labelling given by (5.9)

TABLE 3. Continued: Relative homoclinic cycles with $\mathbb{D}_m$ symmetry.

Case	G_p	$\frac{\lvert G\rvert}{\lvert G_p\rvert}$	$\dim(E_p^s)$	Rep(G_p) on E_p^s	Orbits approaching p	G_γ	$\frac{\lvert G\rvert}{\lvert G_\gamma\rvert}$	Shift dynamics	Notes
12	$\mathbb{D}_{m/2}$	2	1	Trivial		$\{id\}$	$2m$	M is $2m \times 2m$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	For m even only
13	$\mathbb{D}_{m/2}$	2	1	Trivial		$\mathbb{Z}_2$	m	M is $m \times m$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	For m even only
14	$\mathbb{D}_{m/2}$	2	1	Isomorphic to $\mathbb{Z}_2$		$\{id\}$	$2m$	M is $2m \times 2m$, $\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$	For m even only
15	$\mathbb{D}_{m/2}$	2	1	Isomorphic to $\mathbb{Z}_2$		$\mathbb{Z}_2$	m	M is $m \times m$, $\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$	For m a multiple of 4 only
16	$\mathbb{D}_{m/2}$	2	2	Isomorphic to $\mathbb{D}_k$ for $k\lvert\frac{m}{2}$		$\{id\}$	$2m$	M is $2m \times 2m$, depends on angles of η^s and η^- to Fix $\mathbb{Z}_2$ in E_p^s	For m even only
17	$\mathbb{D}_{m/2}$	2	2	Isomorphic to $\mathbb{D}_k$ for $k\lvert\frac{m}{2}$		$\mathbb{Z}_2$	m	M is $m \times m$, $\begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}$ with N having entries $\frac{m}{2k} \times \frac{m}{2k}$ block matrices $m_{ij} = \begin{cases} 1, & \lvert i-j\rvert < m/4, \\ 0, & \lvert i-j\rvert = m/4, \\ -1, & \lvert i-j\rvert > m/4. \end{cases}$	For m even only, zeros in M (at entries which are nonzero in C) when m is a multiple of 4

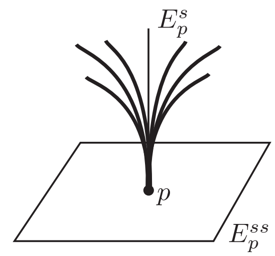


FIGURE 3. Sketch of the homoclinic connections approaching p when $\dim E^s_p = 1$ and the representation of G_p on E^s_p is trivial, with $G = G_p = \mathbb{D}_3$ and $G_\gamma = \{id\}$.

for all $g \in G_p$. Using (4.1), this gives all the entries in one column of M. These are nonzero if these entries for C are nonzero. The remaining entries of M follow by symmetry, using (5.8), leading to a matrix with all nonzero entries being identical. □

It is important to note that Proposition 5.1 applies in the case of any finite symmetry group, and not just dihedral ones. Any relative homoclinic cycle that satisfies the assumptions of Proposition 5.1 features on one side of the bifurcation a suspended horseshoe (a shift whose transition matrix is the connectivity matrix) and no recurrent dynamics on the other side (apart from the equilibrium). We note that previously, full shifts were well known to occur in bifurcations from homoclinic bellows [37, 18].

The transition matrices $A_\pm$ may also be represented by a Markov graph where each vertex represents a connecting trajectory. For fixed parameter λ there is a directed edge from vertex A to vertex B if the ABth entry in M is of the same sign as λ. We adopt the convention that an edge carries an arrow if it is uni-directional and no arrow if it is bi-directional. Note that both graphs are mutually complementary with respect to the full shift, as are the matrices.

For example, in Case 10 of Table 2 it is more convenient to represent the Markov chain by its graph than by its matrix. To construct the Markov graphs given in Table 2 we use the following labels; the equilibria are p, ap,..., $a^{m-1}p$, and we label the connections such that

$$\begin{aligned} \gamma_j &= a^{j-1}\gamma, \quad \text{for } j = 1, ..., m, \\ \gamma_j &= ba^{j-1}\gamma, \quad \text{for } j = m+1, ..., 2m. \end{aligned} \tag{5.9}$$

This gives the action of a and b on the set of heteroclinic trajectories as, in cycle notation,

$$\begin{aligned} a &= (1...m)(2m...m+1), \\ b &= (1\,m+1)(2\,m+2)...(m\,2m). \end{aligned}$$

The situation with $G = \mathbb{D}_3$, $G_p = \mathbb{Z}_2$ and $G_\gamma = \{id\}$ is sketched in Figure 4.

We now consider the situation when the representation of G_p on E^s_p is nontrivial (isomorphic to $\mathbb{Z}_2$). These are the remaining Cases 3, 4, 8, 11 in the tables.

Proposition 5.2. *Assume the hypotheses of Theorem* 2.4 *and suppose* $\dim E^s_p = 1$ *with nontrivial representation of* G_p *on* E^s_p *isomorphic to* $\mathbb{Z}_2$*. Then* M *contains equal numbers of entries* 1 *and* -1*.*

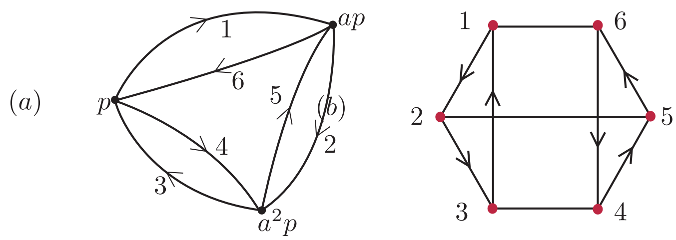


FIGURE 4. (a) Schematic representation of the connections of a relative homoclinic cycle with $G = \mathbb{D}_3$, $G_p = \mathbb{Z}_2$ and $G_\gamma = \{id\}$. (b) Markov graph representing the nonwandering shift dynamics that exists on one side of the homoclinic bifurcation, in the case of trivial representation of G_p on E^s_p (Case 10 of Table 2).

Proof. The way the homoclinic trajectories approach p as $t \to \infty$ is illustrated in Figure 5; this figure is for the case $G_p = G = \mathbb{D}_3$ and $G_\gamma = \{id\}$. The proof is

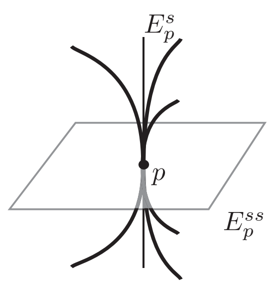


FIGURE 5. Sketch of the homoclinic connections approaching p when $\dim(E^s_p) = 1$ with nontrivial representation of G_p isomorphic to $\mathbb{Z}_2$, with $G = G_p = \mathbb{D}_3$ and $G_\gamma = \{id\}$.

similar to that of Proposition 5.1, but here the connecting trajectories come in pairs: a trajectory and its $\mathbb{Z}_2$-image which approach p along E^s_p in opposite directions, by the $\mathbb{D}_m$-action. Let g^r be an element of G that acts nontrivially on E^s_p. Then for each i, γ_i and $\gamma_i^r := g^r\gamma_i$ form such a pair at an equilibrium. In this way the connecting trajectories in Γ approaching each equilibrium are divided into two groups characterized by the direction from which they approach the equilibrium in positive time. By definition,

$$\lim_{t\to\infty} \frac{\gamma_i(t)}{\|\gamma_i(t)\|} = -\lim_{t\to\infty} \frac{\gamma_i^r(t)}{\|\gamma_i^r(t)\|}.$$

By equations (5.3) and (5.5) we find

$$\eta_i^s = -\eta^s$$

for exactly half of the connecting trajectories at p.

Given the sign of $\langle \eta^s(0), \eta^-(0)\rangle$ and then using the above we obtain for each $g \in G_p$ the signs of $\langle \eta^s_{i(gh^{-1})}(0), \eta^-(0)\rangle$ giving, using (4.1), one column of M. Using equation (5.8) then gives the remaining entries. $\square$

Proposition 5.2 applies in the case of any finite group. In the dihedral case, there exists a choice of labels of the connections, such that M admits a block structure where the nonzero blocks take the form

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

where blocks 1 and -1 represent $\bar{m} \times \bar{m}$ matrices, $\bar{m} = \frac{|G_p|}{2|G_\gamma|}$, with all entries equal to 1 and -1, respectively. Details of the corresponding Markov chains for relative homoclinic cycles with dihedral symmetry are provided in Tables 1 and 2, and Cases 3, 4, 8 and 11. We continue to discuss these in more detail.

In Cases 3 and 4, we number the connections in such a way that $\gamma_1, \dots, \gamma_{\bar{m}}$ approach p from the same direction and their pairs are ordered such that γ_i and $\gamma_{i+\bar{m}}$ form a pair. If $\lambda > 0$, say, then each of the sets $\gamma_1, \dots, \gamma_{\bar{m}}$ and $\gamma_{\bar{m}+1}, \dots, \gamma_{2\bar{m}}$ generate dynamics similar to that described in Case 1: each one exhibiting a full shift of finite type among its connections. If $\lambda < 0$, there is a single transitive recurrent set. If $m = 1$, this bifurcation is precisely the gluing bifurcation near a"figure-8" configuration with $\mathbb{Z}_2$ symmetry, as discussed in [12], with periodic solutions replaced by more general transitive sets.

In Case 4, the representation of G_p on E^s_p, $\mathbb{Z}_2$, is generated by $a^k b$, for some k. We need to distinguish between the cases when m is odd and m is even.

We first consider the situation when m is odd. When m is odd there is only one nontrivial one-dimensional representation of $\mathbb{D}_m$ isomorphic to $\mathbb{Z}_2$. Since $\dim(E^s_p) = 1$, the fixed point space of the action of this nontrivial representation on E^s_p is the origin. As the representation on E^s_p thus does not have a nontrivial intersection with $\mathrm{Fix}\, G_{h^{-1}\gamma}$, $\eta^s(0)$ cannot lie in E^s_p. Consequently, with this configuration the homoclinic connections cannot approach along the leading eigenspace, so that Hypothesis (H 4) is not satisfied. Thus m needs to be even.

When m is even, there are three irreducible representations of $\mathbb{D}_m$ which act as $\mathbb{Z}_2$. Since G_p acts as $\mathbb{Z}_2$ on E^s_p, two of these representations act trivially on E^s_p (this includes the representation of $\mathbb{Z}_2(a^{m/2})$ and one of $\mathbb{Z}_2(ab)$ or $\mathbb{Z}_2(b)$). If G_γ acts nontrivially on E^s_p we have the same problem as in the case when m is odd, but in the other two cases E^s_p lies entirely inside the fixed point space of G_γ so that Hypothesis (H 4) can be satisfied. With $G_\gamma = \mathbb{Z}_2(ab)$ or $G_\gamma = \mathbb{Z}_2(b)$, the homoclinic connections lie pairwise in fixed point subspaces of $a^n b$ with n odd or n even, respectively.

In Case 11, when $G_p = \mathbb{Z}_2$ and $G_\gamma = \{id\}$, we define the labeling of connections as in Case 10 (see (5.9)) and obtain the Markov graphs given in Table 2. As an illustration, we present in Figure 6 the Markov graphs for Case 11 with $G = \mathbb{D}_3$ (with connections as in Figure 4(a)). Before and after the bifurcation, the shift dynamics here consist of a finite set of periodic solutions. Notice also that the two Markov graphs add up to the Markov graph Figure 4(b) of Case 10, reflecting the equality (1.5) of Theorem 1.1.

5.2.2. *Two-dimensional E^s_p.* The remaining cases in Tables 1 and 2, and Cases 5, 6 and 9, are described by the following three propositions. The remaining cases in Table 3 where G_p acts on E^s_p as $\mathbb{D}_{m/2}$ instead of $\mathbb{D}_m$, and Cases 16 and 17, are similar and will not be considered further.

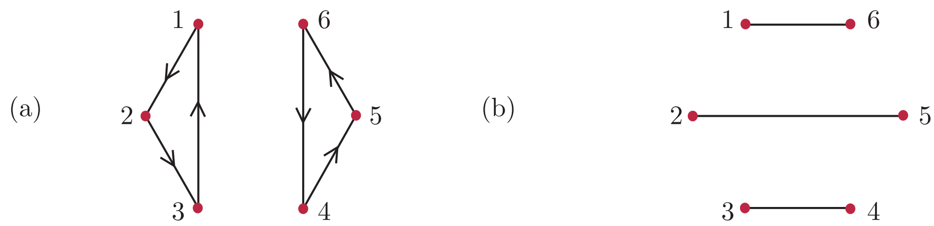


FIGURE 6. Markov graphs representing the nonwandering shift dynamics on either side ((a) and (b)) of the bifurcation point in Case 11 of Table 2, with $G = \mathbb{D}_3$.

The following proposition is listed as Case 6 in Table 1, under the further restriction that G_p acts as $\mathbb{D}_m$ on E_p^s. The discussion of a representation as a smaller group $\mathbb{D}_k$ for some $k|m$ follows after the proof.

Proposition 5.3. *Assume the hypotheses of Theorem* 2.4 *and suppose that* $G_p = \mathbb{D}_m$*,* $\dim E_p^s = 2$*, the representation of* G_p *on* E_p^s *is isomorphic to* $\mathbb{D}_m$*, and* $G_\gamma = \mathbb{Z}_2$ *(generated by* $a^k b$*, for some* k*). Then, up to a relabeling of the homoclinic connections, the elements* m_{ij} *of* M *are given by*

$$m_{ij} = \begin{cases} 1, & |i-j| < m/4, \\ 0, & |i-j| = m/4, \\ -1, & |i-j| > m/4. \end{cases} \tag{5.10}$$

The difference $i - j$ *is calculated in* $\mathbb{Z}_m$*, and* $|i-j| := \min\{i-j, j-i\}$*.*

Proof. The way the homoclinic connections approach p as $t \to \infty$ is depicted in Figure 7. The fixed point subspace $\operatorname{Fix} G_\gamma$ is invariant under the flow of the differ-

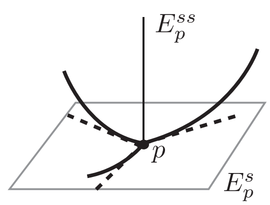


FIGURE 7. Sketch of the homoclinic connections approaching p in Case 6, with $G = \mathbb{D}_3$.

ential equation (1.1). For that reason we have $\psi \in \operatorname{Fix} G_\gamma$. Within this fixed point subspace e_p^s is the only weak stable direction. Hence both η^s and η^- are parallel to e_p^s. Without loss of generality we assume that their scalar product is positive.

Now, let $g \notin G_\gamma$. Then $G_{g\gamma} = gG_\gamma g^{-1}$ and hence $\operatorname{Fix} G_{g\gamma} = g\operatorname{Fix} G_\gamma$. Therefore $\eta_{i(g)}^s$ and $\eta_{i(g)}^-$ are parallel to ge_p^s, and their scalar product is positive.

But the scalar product of $\eta_{i(g)}^s$ and η^- depends on g. For the computation hereof it suffices to consider elements g acting on E_p^s as a rotation by an angle $2l\pi/m$ around the origin for some $l = 1, \dots, m$. Hence $\operatorname{sgn} \langle \eta_{i(g)}^s(0), \eta^-(0) \rangle$ depends on l. This scalar product is positive if $l < m/4$, it vanishes if $l = m/4$, and it is negative if $l > m/4$. If we number the homoclinic trajectories $\gamma_l = g_l \gamma$, where g_l is assigned to the rotational angle $2l\pi/m$, then up to a multiple of -1, M is given by (5.10). $\square$

To complete the description of Case 6 in Table 1 we need to consider when the representation of G_p on E_p^s is isomorphic to $\mathbb{D}_k$ for k a divisor of m. In that case there are $\frac{m}{k}$ homoclinic trajectories approaching p along each fixed point space in E_p^s rather than just one when $k = m$. The corresponding matrix M takes the form of a $k \times k$ block matrix defined in the same way as in (5.10), but where each entry is a $\frac{m}{k} \times \frac{m}{k}$ matrix with all entries 0, 1 or -1 as indicated. We observe that $A_+ + A_- \neq C$ if m is a multiple of four. In that case the matrices $A_\pm$ may not necessarily describe the complete recurrent dynamics.

As an illustration, let us consider $G = \mathbb{D}_3$; then

$$M = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Correspondingly, if $\lambda > 0$, the nonwandering dynamics consist of three periodic solutions, sequences κ with $\kappa_i = \kappa_{i+1}$ for all i. For $\lambda < 0$ the set of nonwandering dynamics is nontrivial, consisting of trajectories that each time pass an equilibrium that is different from those previously passed. This is characterized by sequences κ satisfying $\kappa_i \neq \kappa_{i+1}$ for all i. This is precisely the bifurcation observed by Matthies [31].

Next we consider Case 5 of Table 1, again treating a representation of G_p on E_p^s as $\mathbb{D}_m$ in the proposition and representations $\mathbb{D}_k$ for some $k|m$ in the text following its proof.

Proposition 5.4. *Assume the hypotheses of Theorem* 2.4 *and suppose that* $G_p = \mathbb{D}_m$, $\dim E_p^s = 2$, *the representation of* G_p *on* E_p^s *is isomorphic to* $\mathbb{D}_m$, *and* $G_\gamma = \{id\}$. *Then, typically* M *has no zeros but the entries depend on* $\alpha^s := \sphericalangle(\varepsilon, \eta^s)$ *and* $\alpha^- := \sphericalangle(\varepsilon, \eta^-)$, *where* ε *is a basis of a one-dimensional fixed point space within* E_p^s.

Proof. The way the homoclinic trajectories approach p as $t \to \infty$ is depicted in Figure 8. Generically for $i \neq j$, $i, j = 1, \ldots, 2m$, we have

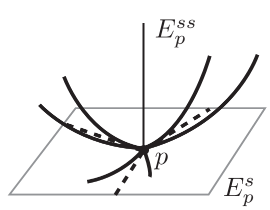


FIGURE 8. Sketch of the homoclinic connections approaching p in Case 5, with $G = \mathbb{D}_3$. The dashed lines represent the fixed point spaces of the elements $a^k b$ for $k = 1, ..., m$.

$$\lim_{t\to\infty} \frac{\gamma_i(t)}{\|\gamma_i(t)\|} \neq \lim_{t\to\infty} \frac{\gamma_j(t)}{\|\gamma_j(t)\|}.$$

That is, e_p^s will typically lie outside the fixed point subspaces of the reflections in the two-dimensional representation isomorphic to $\mathbb{D}_k$. Let the representation of

$g^r \in \mathbb{D}_m$ on E_p^s be a reflection. Let $\{\varepsilon\}$ be a basis of the fixed point subspace of g^r restricted to E_p^s. The relative homoclinic cycle Γ originates from $\{\gamma, g^r\gamma, p\}$ by rotation through angles $2l\pi/m$, $l = 1, \ldots, m-1$.

By α^s we denote the angle between η^s and ε measured counter-clockwise:

$$\alpha^s := \sphericalangle(\varepsilon, \eta^s).$$

Similarly we define

$$\alpha^- := \sphericalangle(\varepsilon, \eta^-).$$

The other η_j^- will be generated by rotations of η^- or $g^r\eta^-$, respectively, by $2l\pi/m$. It turns out that the matrix M depends on the parameters (α^s, α^-) on the torus.

The m_{ij} have been defined by (4.1). So $m_{ij} = 0$ if and only if the angle between η_i^s and η_j^- is some multiple of $\pi/2$. The above considerations give

$$\sphericalangle(\eta_i^s, \eta_j^-) = \alpha^- \pm \alpha^s + 2l\pi/m$$

for some $l = 0, \ldots, m-1$. On the two-dimensional torus parameterized by (α^s, α^-), there are finitely many simply connected open regions separated by lines so that m_{ij} are different from zero within each of these regions. The union of the closures of these regions is the entire torus. At the transition from one region to an adjacent one some of the m_{ij} change sign. □

To complete the description of Case 5 we need to consider when the representation of G_p on E_p^s is $\mathbb{D}_k$, with $k|m$. Then there are $\frac{m}{k}$ homoclinic trajectories approaching p along the same direction. In the notation of the proof of Proposition 5.4 this is $\frac{m}{k}$ homoclinic trajectories tangent to e_p^s and $\frac{m}{k}$ tangent to $g^r e_p^s$. Then rotations of $\frac{2\pi l}{k}$ for $l = 0, \ldots, k-1$ of this pair give the remaining connections. The transition matrix M can be viewed as a $2k \times 2k$ block matrix where each block is an $\frac{m}{k} \times \frac{m}{k}$ matrix with identical entries.

The following proposition and the text below treat Case 9 of Table 2.

Proposition 5.5. *Assume the hypotheses of Theorem* 2.4 *and suppose that* $G_p = \mathbb{Z}_m$, $\dim E_p^s = 2$, *the representation of* G_p *on* E_p^s *is isomorphic to* $\mathbb{Z}_m$, *and* $G_\gamma = \{id\}$. *Then* M *is a* $2m \times 2m$ *matrix which depends on* $\alpha := \sphericalangle(\eta^-, \eta^s)$. *Generically* M *has no zero entries other than those implied by the connectivity matrix.*

Proof. In this network there are $\frac{|G|}{|G_p|} = 2$ equilibria and $\frac{|G|}{|G_\gamma|} = 2m$ heteroclinic trajectories, with m approaching each equilibrium as $t \to \infty$. The connecting trajectories that approach p as $t \to \infty$ do approach the two-dimensional E_p^s as sketched in Figure 9. Now given the position of η^s in E_p^s the position of $g\eta^s$,

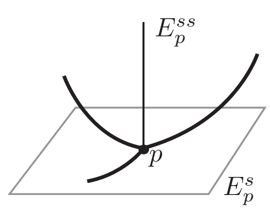


FIGURE 9. Sketch of the heteroclinic trajectories approaching p in Case 9, with $G = \mathbb{D}_3$.

$g \in G_p$, is a rotation of η^s over a multiple of $\frac{2\pi}{m}$, according to the representation of g; similarly for η^-. The relative position of η^s and η^- is free due to the trivial isotropy of the connections. Define

$$\alpha := \sphericalangle(\eta^-, \eta^s).$$

Then the signs of $\langle \eta^s, g\eta^- \rangle$ depend on α. Typically in each row and column of M there will be m nonzero entries, and up to a change of sign $\frac{m}{2}$ each of $+1$ and -1 for m even and $\frac{m-1}{2}$ $+1$ and $\frac{m+1}{2}$ -1 for m odd. The exact ordering depends on the labeling of the heteroclinic trajectories. □

Similar to Cases 5 and 6 the representation of G_p on E_p^s can be $\mathbb{Z}_k$ for k a divisor of m. Then there are $\frac{m}{k}$ heteroclinic trajectories approaching p along each direction.

To illustrate this result we consider the case when $m = 3$ and the representation of G_p on E_p^s is isomorphic to $\mathbb{Z}_3$. We choose the labeling on the heteroclinic trajectories as in (5.9). Using the $\mathbb{Z}_3$ symmetry on E_{ap}^s, rotations of η_1^s over $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ give η_2^s and η_3^s. Similarly, rotations of η_4^- give η_6^- and η_5^-. To calculate the first row of M we need to consider the positions of η_4^-, η_5^- and η_6^- relative to η_1^s. Up to a sign change there are typically two -1 and one $+1$ in each row of the matrix. On the boundaries there is a zero in each row of M where one entry changes sign between adjacent regions. There are six different regions of E_{ap}^s in which $\eta^s = \eta_1^s$ can lie, as shown in Figure 10. We illustrate one of these transitions from region I to region II marked in Figure 10. If η^s lies in region I, then $m_{14} > 0$, $m_{15} < 0$

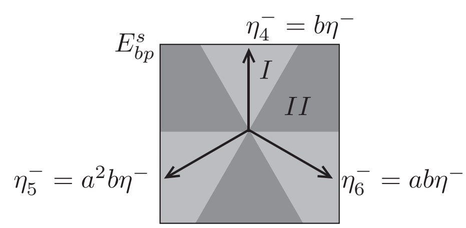


FIGURE 10. Illustration of the fact that the sign of the entries in M depends on the regions of E_p^s in which η^s lies, with $G = \mathbb{D}_3$.

and $m_{16} < 0$. If η^s lies in region II, then $m_{14} > 0$, $m_{15} > 0$ and $m_{16} < 0$. Only one sign in each row of M has changed. The remaining signs can be calculated from these using (5.8). This gives the following two situations in regions I and II:

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The graphs representing these dynamics are given in Figures 11 and 12, respectively.

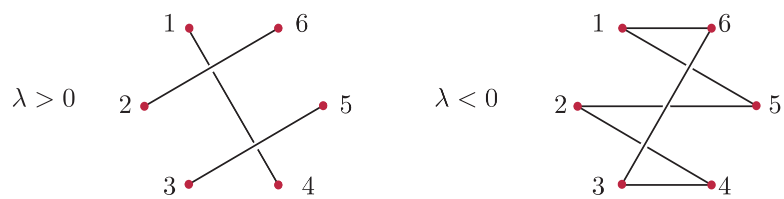


FIGURE 11. Symbolic dynamics in region I of Figure 10.

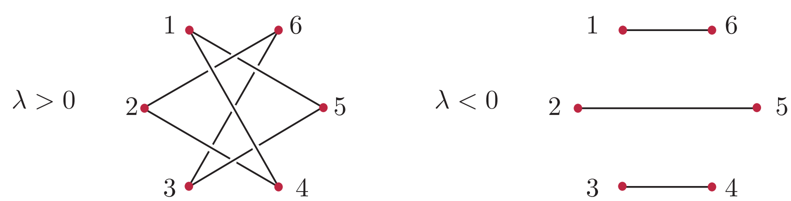


FIGURE 12. Symbolic dynamics in region II of Figure 10.

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References

1. M. A. D. Aguiar, S. B. S. D. Castro, and I. S. Labouriau. Dynamics near a heteroclinic network. *Nonlinearity*, 18(1):391–414, 2005. With multimedia enhancement available from the abstract page in the online journal. MR2109482 (2006a:37020)
2. P. Ashwin and M. J. Field. Heteroclinic networks in coupled cell systems. *Arch. Ration. Mech. Anal.*, 148(2):107–143, 1999. MR1716564 (2001d:37072)
3. S. Bochner. Compact groups of differentiable transformations. *Ann. of Math.*, 46:372–381, 1945. MR0013161 (7:114g)
4. G. E. Bredon. *Introduction to compact transformation groups*. Pure and Applied Mathematics, 46. Academic Press., New York-London, XIII, 459 pp., 1972. MR0413144 (54:1265)
5. P. Chossat and R. Lauterbach. *Methods in equivariant bifurcations and dynamical systems*, volume 15 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2000. MR1831950 (2002b:37081)
6. S.-N. Chow, B. Deng, and B. Fiedler. Homoclinic bifurcation at resonant eigenvalues. *J. Dynam. Differential Equations*, 2(2):177–244, 1990. MR1050642 (91g:58199)
7. B. Deng. The Shil′nikov problem, exponential expansion, strong λ-lemma, C^1-linearization, and homoclinic bifurcation. *J. Differential Equations*, 79(2):189–231, 1989. MR1000687 (90k:58161)
8. G. L. dos Reis. Structural stability of equivariant vector fields on two-manifolds. *Trans. Amer. Math. Soc.*, 283(2):633–643, 1984. MR737889 (85h:58095)
9. M. J. Field. Equivariant dynamical systems. *Trans. Amer. Math. Soc.*, 259(1):185–205, 1980. MR561832 (82i:58037)
10. M. J. Field. *Lectures on bifurcations, dynamics and symmetry*, volume 356 of *Pitman Research Notes in Mathematics Series*. Longman, Harlow, 1996. MR1425388 (97h:58115)
11. M. J. Field. *Dynamics and symmetry*, volume 3 of *ICP Advanced Texts in Mathematics*. World Scientific, London, 2007. MR2358722 (2009e:37021)

12. J.-M. Gambaudo, P. Glendinning, and C. Tresser. The gluing bifurcation. I. Symbolic dynamics of the closed curves. *Nonlinearity*, 1:203–214, 1988. MR928953 (89a:58080)
13. M. Golubitsky, I. Stewart, and D.G. Schaeffer. *Singularities and groups in bifurcation theory. Vol. II*, volume 69 of *Applied Mathematical Sciences*. Springer, New York, 1988. MR950168 (89m:58038)
14. M. Golubitsky, J. W. Swift, and E. Knobloch. Symmetries and pattern selection in Rayleigh-Bénard convection. *Physica D*, 10:249–276, 1984. MR763472 (86c:58027)
15. J. Guckenheimer and P. J. Holmes. Structurally stable heteroclinic cycles. *Math. Proc. Cambridge Philos. Soc.*, 103(1):189–192, 1988. MR913462 (88k:58109)
16. A. J. Homburg. Global aspects of homoclinic bifurcations of vector fields. *Mem. Amer. Math. Soc.*, 121(578):viii+128, 1996. MR1327210 (96i:58125)
17. A. J. Homburg, A. Jukes, J. Knobloch, and J. S. W. Lamb. Saddle-nodes and period-doublings of Smale horseshoes: A case study near resonant homoclinic bellows. *Bull. Belg. Math. Soc. Simon Stevin*, 15(5):833–850, 2008. MR2484136
18. A. J. Homburg and J. Knobloch. Multiple homoclinic trajectories in conservative and reversible systems. *Trans. Amer. Math. Soc.*, 358(4):1715–1740, 2006. MR2186994 (2007a:34073)
19. A. J. Homburg, H. Kokubu, and M. Krupa. The cusp horseshoe and its bifurcations in the unfolding of an inclination-flip homoclinic trajectory. *Ergodic Theory Dynam. Systems*, 14(4):667–693, 1994. MR1304138 (96a:58134)
20. A. C. Jukes. *On homoclinic bifurcations with symmetry.* Ph.D. thesis, Imperial College, 2006.
21. A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems. With a supplement by Anatole Katok and Leonardo Mendoza.* Encyclopedia of Mathematics and its Applications, 54, Cambridge University Press, Cambridge, xviii, 802 pp., 1997. MR1326374 (96c:58055)
22. V. Kirk and M. Silber. A competition between heteroclinic cycles. *Nonlinearity*, 7(6):1605–1621, 1994. MR1304441 (95j:58121)
23. M. Kisaka, H. Kokubu, and H. Oka. Bifurcations to N-homoclinic trajectories and N-periodic solutions in vector fields. *J. Dynam. Differential Equations*, 5(2):305–357, 1993. MR1223451 (94m:58158)
24. J. Knobloch. Lin's method for discrete dynamical systems. *J. of Difference Eqn. and Appl.*, 6:577–623, 2000. MR1802448 (2001k:37018)
25. J. Knobloch. Lin's method for discrete and continuous dynamical systems and applications. Habilitation thesis, Technische Universität Ilmenau, 2004.
26. M. Krupa. Robust heteroclinic cycles. *J. Nonlinear Sci.*, 7(2):129–176, 1997. MR1437986 (98g:58126)
27. M. Krupa and I. Melbourne. Asymptotic stability of heteroclinic cycles in systems with symmetry. *Ergodic Theory Dynam. Systems*, 15(1):121–147, 1995. MR1314972 (96g:58101)
28. M. Krupa and I. Melbourne. Nonasymptotically stable attractors in O(2) mode interactions. In *Normal forms and homoclinic chaos (Waterloo, ON, 1992)*, volume 4 of *Fields Inst. Commun.*, pages 219–232. Amer. Math. Soc., Providence, RI, 1995. MR1350551 (96h:58103)
29. M. Krupa and I. Melbourne. Asymptotic stability of heteroclinic cycles in systems with symmetry. II. *Proc. Roy. Soc. Edinburgh Sect. A*, 134(6):1177–1197, 2004. MR2107489 (2006e:37086)
30. X.-B. Lin. Using Melnikovs method to solve Shil′nikovs problems. *Proc. Roy. Soc. Edinburgh*, 116A:295–325, 1990. MR1084736 (92b:58195)
31. K. Matthies. A subshift of finite type in the Takens-Bogdanov bifurcation with D_3 symmetry. *Doc. Math.*, 4:463–485, 1999. MR1713483 (2001m:37105)
32. I. Melbourne. An example of a nonasymptotically stable attractor. *Nonlinearity*, 4(3):835–844, 1991. MR1124335 (92f:58093)
33. J. Palis and W. de Melo. *Geometric theory of dynamical systems. An introduction.* Transl. from the Portuguese by A. K. Manning. Springer-Verlag, New York - Heidelberg - Berlin, 1982. MR669541 (84a:58004)
34. B. Sandstede. *Verzweigungstheorie homokliner Verdopplungen.* Ph.D. thesis, Freie Universität Berlin, 1993.
35. L. P. Shil′nikov. On the generation of a periodic motion from trajectories doubly asymptotic to an equilibrium state of saddle type. *Math. USSR Sbornik*, 6:427–438, 1968.
36. N. Sottocornola. Simple homoclinic cycles in low-dimensional spaces. *J. Differential Equations*, 210(1):135–154, 2005. MR2114127 (2005h:37050)

37. D. V. Turaev. Bifurcations of a homoclinic "figure eight" of a multidimensional saddle. *Russian Math. Surveys*, 43:264–265, 1988. MR971495 (90m:58156)
38. A. Vanderbauwhede and B. Fiedler. Homoclinic period blow-up in reversible and conservative systems. *Z. Angew. Math. Phys.*, 43:292–318, 1992. MR1162729 (93f:58209)
39. F. Verhulst. *Nonlinear differential equations and dynamical systems. 2nd rev. a. exp. ed.* Springer, Berlin, viii, 306 p., 1996. MR1422255 (97g:34003)

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